THE QUADRATIC EIGENVALUE PROBLEM AND PSEUDOSPECTRA

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ABSTRACT

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In this study we investigate quadratic eigenvalue problem. It has extensive applications in engineering such as linear stability of flows in fluid mechanics, the dynamic analysis of mechanical systems in acoustics and signal processing.

Our aim is to find spectrum points of quadratic pencil $\lambda^2 A_1 + \lambda A_2 + A_3$ and investigate sensitivity of eigenvalues of quadratic pencil to perturbations. Eigenvalues of matrix pencil and eigenvalues of its factors are found and they are compared. Moreover, eigenvalues of quadratic pencil and eigenvalues of its linearization also their sensitivity are compared.

In chapter two we will give general background for linearization and factorization. Pseudospectra notion for matrices can be extended to pseudospectra of quadratic matrix polynomial. Spectrum and

pseudospectrum of matrix pencil are compared using factorization and linearization tecniques.

In the last chapter we give computed examples for the spectrum and pseudospectrum of quadratic pencils. In our examples our matrix coefficients are choosen special matrices such as Hermitian, positive definite, Toeplitz etc… We use Mathematica programming to compute examples.

Key words: Eigenvalue, eigenvector, quadratic eigenvalue problem, spectrum, pseudospectrum, positive definite matrix, Hermitian matrix, Toeplitz matrix.

ÖZET

İKİNCİ DERECEDEN OPERATÖR KATSAYILI POLİNOMLARIN ÖZDEĞER PROBLEMİ VE PSEUDOSPEKTRUM

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Bu çalışmada ikinci dereceden operatör katsayılı polinomların özdeğer problemini araştırıyoruz. İkinci dereceden operatör katsayılı polinomların özdeğer problemi, sıvı mekaniğinde akışın lineer durağanlığı, akustikte mekanik sistemlerin dinamik analizi ve sinyal işleme gibi birçok mühendislik alanında yaygın uygulamalara sahiptir.

Amacımız ikinci dereceden matris polinomlarının $\lambda^2 A_1 + \lambda A_2 + A_3$ spektrum noktalarını bulmak ve ötelemeye karşı matris polinomlarının özdeğerlerinin duyarlılığını araştırmaktır. Matris polinomunun ve çarpanlarının özdeğerleri bulunup karşılaştırıldı. Ayrıca ikinci dereceden matris polinomunun özdeğerleri ile onun lineerleştirilmişinin özdeğerleri ve duyarlılıkları karşılaştırıldı.

İkinci bölümde lineerleştirme ve çarpanlara ayırma için genel bilgiler verilmektedir. Matrisler için geçerli olan pseudospektrum kavramının ikinci dereceden operatör katsayılı polinomların pseudospektrumu için genişletilebileceği görülür. Matris katsayılı polinomun spektrum ve pseudospektrum kavramları çarpanlara ayırma ve lineerleştirme teknikleri kullanılarak karşılaştırılmaktadır.

Son bölümde ikinci dereceden operatör katsayılı polinomların spektrum ve pseudospektrumları için hesaplanmış örnekler veriyoruz. Örneklerimizde matris katsayıları Hermitian matris, pozitif tanımlı matris Toepliz matris vb. gibi özel matrislerden seçilmiştir. Örneklerimizi hesaplamak için Mathematica programını kullanıyoruz.

Anahtar kelimeler: Özdeğer, özvektör, ikinci dereceden özdeğer problemi, spektrum, pseudospektrum, pozitif tanımlı matris, Hermitian matris, Toepliz matris.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

When a structure is not disturbed by external forces its vibration rate called as natural frequency. If we disturb this structure by external forces whose frequencies are close to the natural frequencies the vibration grows and system become unstable. When we add damping effects to a model the natural modes and frequencies of a structure are the solution of a quadratic eigenvalue problems.

The quadratic eigenvalue problem (QEP) applications are very important in such dynamic analysis of mechanical systems in acoustics and linear stability of flows in fluid mechanics.

In the beginning we need to give some important definitions.

Definition 1.1.1 The polynomial eigenvalue problem is to find the solutions (x,λ) of $P(\lambda)x=0$ where

$$P(\lambda)=\lambda^m A_m+\lambda^{m-1}A_{m-1}+\ldots+A_0 \qquad (1.1.1)$$

is the matrix polynomial with $A_k\in\mathbb{C}^{nxn}$, $k=0,1,2,\ldots,m$ and $\det A_m\neq 0$. When $m=2$, $P(\lambda)$ is called *quadratic matrix polynomial* or quadratic *pencil*.

It will be

$$P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0 .$$

The QEP is to find scalars λ and nonzero vectors x, y satisfying

$$(\lambda^2 A_1 + \lambda A_2 + A_3)x = 0 ,$$

$$y^*(\lambda^2 A_1 + \lambda A_2 + A_3) = 0 ,$$

where A_1, A_2 and A_3 are nxn complex matrices and x, y are the right and left eigenvectors of eigenvalue λ. QEPs are an important class of nonlinear eigenvalue problems. There is a major algebraic difference between QEP and the standard eigenvalue problem $Ax = \lambda x$, and the generalized eigenvalue problem $Ax = \lambda Bx$. It is that the QEP has $2n$ eigenvalues with up to $2n$ right and $2n$ left eigenvectors and if these eigenvectors are more than n they do not form a linearly independent set.

Now we will give some basic definitions about what we will come across in this study.

Definition 1.1.2 Let A be an nxn matrix with real or complex coefficients; we write $A \in \mathbb{C}^{nxn}$. Let v be a nonzero real or complex column vector of length n, and let λ be a real or complex scalar; we write $v \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Then v is an *eigenvector* of A, and $\lambda \in \mathbb{C}$ is its corresponding *eigenvalue*, if

$$Av = \lambda v .$$

Even if A is real, its eigenvalues are in general complex unless A is self adjoint. The set of all eigevalues of A is the *spectrum* of A, a nonempty subset of the complex plane $\mathbb{C}$ that we denote by $\sigma(A)$. The spectrum can also be defined as the set of points $z \in \mathbb{C}$ where the *resolvent* matrix ,

$$(z - A)^{-1},$$

does not exist. $z - A$ can be used for $z\mathrm{I} - A$, where I is the identity matrix.

Definition 1.1.3 Let $P(\lambda)$ is a matrix polynomial which like (1.1.1). A scalar $\lambda_0 \in \mathbb{C}$ is called an eigenvalue of $P(\lambda)$ if the system $P(\lambda_0)v_0 = 0$ has a nonzero solution $v_0 \in \mathbb{C}^n$. v_0 is called eigenvector of $P(\lambda)$ corresponding to λ_0. The set of all eigenvalues of $P(\lambda)$ is the spectrum of $P(\lambda)$. It can be shown like

$$\sigma(P) = \left\{\lambda \in \mathbb{C} : \det(P(\lambda)) = 0,\ (P(\lambda)) \in \mathbb{C}^{nxn}\right\}.$$

Definition 1.1.4 Let A be an nxn matrix with complex entries. If A is equal to its conjugate transpose (A^*) then it is called *Hermitian matrix*. If $A^* = -A$ then it is called *skew-Hermitian*.

Definition 1.1.5 Let A be a real symmetric nxn matrix and x is non-zero column vectors of n real numbers with x^T which denotes the transpose of x. A is a *positive definite* matrix if $x^T Ax > 0$ for all x.

Definition 1.1.6 Any nxn matrix A of the form

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & \ddots & \vdots & \vdots \\ a_{-2} & a_{-1} & \ddots & a_1 & a_2 \\ \vdots & \vdots & \ddots & a_0 & a_1 \\ a_{1-n} & a_{2-n} & \cdots & a_{-1} & a_0 \end{bmatrix}$$

is a *Toeplitz Matrix*. $A_{i,j}$ denotes i, j element of A and $A_{i,j} = a_{j-i}$.

Definition 1.1.7 The Euclidean norm of a matrix A_{mxn} is

$$\|A\|_2 = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$$

where a_{ij} is the i-th row and j-th column of the matrix A.

Definition 1.1.8 The Euclidean norm of a vector $x = [x_1, x_2, ..., x_n]$ is

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2} \quad .$$

1.2 History of Pseudospectra

We know that pseudospectra has very large application area in recent years. In this section we will discuss who studied in this area for years.

In the context of Hermitian and near-Hermitian systems, mathemaical physicists have spoken of *quasimodes*, which are the same as what we would call *pseudomodes* or *pseudoeigenvectors*. An early reference

is a 1972 article, 'Modes and Quasimodes', by Arnol'd [1], and an even earlier one by Vishik and Lyusternik dates to 1957 [2].

In the nonnormal context, the earliest definition of pseudospectra we have encountered is given in J. M. Varah's 1967 thesis at Stanford University, The Computation of Bounds for the Invariant Subspaces of a General Matrix Operator [3]. Varah introduced the notion of an ε-pseudoeigenvalue under the name *r-approximate eigenvalue*. Varah incorporated a parameter, η_1, in his definition to describe floating-point precision motivated by his analysis of the accuracy of eigenpairs produced by computer implementations of the inverse iteration algorithm.

Definition 1.2.1 $\begin{Bmatrix}\lambda\\ y\end{Bmatrix}$ is an *r-approximate* $\begin{Bmatrix}\textit{eigenvalue}\\ \textit{eigenvector}\end{Bmatrix}$ of A if there exists a matrix E with $\|E\|_2 = r.\eta_1$ such that $\begin{Bmatrix}\lambda\\ y\end{Bmatrix}$ is an exact $\begin{Bmatrix}\textit{eigenvalue}\\ \textit{eigenvector}\end{Bmatrix}$ of $A+E$. *(From Varah's unpublished 1967 thesis)*

In 1975, H. J. Landau of AT&T Bell Laboratories published a paper, 'On Szego's Eigenvalue Distribution Theorem and Non-Hermitian Kernels', that indepedently introduced ε-pseudoeigenvalues under the name *ε-approximate eigenvalues* [4]. Landau applied this concept to the theory of Toeplitz matrices and associated integral operators.

Definition 1.2.2 λ is an ε-approximate eigenvalue of A, if there exists $\varphi \in P(rQ)$, with $\|\varphi\| = 1$, such that $\|A_r\varphi - \lambda\varphi\| \le \varepsilon$. We call φ an ε-approximate eigenfunction corresponding to λ. (*First published definition of pseudospectra from Landau)*

In Novosibirsk, S. K. Godunov and his colleagues conducted research related to pseudospectra throughout the 1980s. Those involved included A. G. Antonov, A. Y. Bulgakov (later Haydar Bulgak), O. P. Kirilyuk, V. I. Kostin, A. N. Malyshev, and S. I. Razzakov. Godunov, together with Ryabenkii and others, had made significant contributions in the 1960s to the study of how nonnormality affects the numerical stability of discretized differential equations.

In fact, in their 1962 monograph, Godunov and Ryabenkii introduce the *spectrum of the family of operators* $\{R_h\}$, indexed by a mesh parameter h [5, page 188]. A point $z \in \mathbb{C}$ is in this set if, for every $\varepsilon > 0$, z is an ε-pseudoeigenvalue of R_h for all sufficiently small h.

The Novosibirsk group defined the *ε-spectrum* by relative rather than absolute perturbations:

$$\sigma_\varepsilon(A) = \left\{ z \in \mathbb{C} : \ \left\| (z - A)^{-1} \right\| \ge (\varepsilon \|A\|)^{-1} \right\}.$$

In [6] and [7], computed plots of pseudospectra were presented and called ‘spectral portraits of matrices’ containing various ‘patches of spectrum’, and computations based on both contour plotting and curve

tracing were discussed. A sketch of two pseudospectra had appeared earlier in a 1985 paper by Kostin and Razzakov [8].

One of the last papers of the eminent numerical analyst J. H. Wilkinson, 'Sensitivity of Eigenvalues II' (1986), defined the ε-pseudospectrum for an arbitrary matrix norm $\|.\|$ induced by a vector norm [9]. The set was denoted by $D(\eta)$ and given no name other than 'the domain $D(\eta)$'. Wilkinson described in his usual lucid fashion how $D(\eta)$ could be interpreted equivalently in terms of matrix perturbations or the norm of the resolvent.

Pseudospectra were investigated in several papers by J. W. Demmel in the mid-1980s, appearing with the labels $S(A,\varepsilon)$ in [10] and $\sigma(\varepsilon,A)$ in [11].

Beginning in the mid-1980s, D. Hinrichsen and A. J. Pritchard wrote a number of papers on the *stability radius* of a matrix, i.e, the distance to the set of unstable matrices. In a 1992 paper [12] they introduced the term *spectral value set* and notation $\sigma(A,\rho)$ to denote the real structured ε-pseudospectrum of a nonnormal matrix. Another paper by Hinrichsen and Kelb in 1993 extended these ideas to complex perturbations and to more general structured perturbations [13]. Again in these time, F. Chatelin, later Chaitin-Chatelin, and her colleagues developed another body of early work related to pseudospectra in France. This group investigated questions of

conditioning, stability, and floating-point arithmetic with the aid of random perturbations [14, 15]. By perturbating a problem at random, they pointed out, one can acquire knowledge about the properties of the unperturbed problem.

Trefethen's first publications that mentioned pseudospectra were [16] and [17] in 1990 (the first uses the term ε-approximate eigenvalues). These were an outgrowth of a 1987 paper by Trefethen and Trummer, who found eigenvalues that were extraordinarily sensitive to perturbations but failed fully to appreciate their significance [18]. Trefethen's student Satish Reddy, who began to work with pseudospectra in 1988 and made many contributions after that date, summarized his early work on these topics in his 1991 thesis at MIT [19].

In 1992 Trefethen's paper 'Pseudospectra of Matrices' appeared, which presented the idea of pseudospectra and exhibited thirteen examples [20]. After this point that the idea began to be widely known. A later companion paper, 'Pseudospectra of Linear Operators', presented ten examples involving operators on infinite-dimensional spaces [21].

We can summarize important developments as below.

J. M. Varah	1967	r-approximate eigenvalues
	1979	ε-spectrum

H. J. Landau	1975	ε-approximate eigenvalues
S. K. Godunov et. al.	1982	spectral portrait
L. N. Trefethen	1990	ε-pseudospectrum
D. Hinrichsen and A. J. Pritchard	1992	spectral value set

CHAPTER 2

PSEUDOSPECTRA OF OUADRATIC EIGENVALUE PROBLEMS

2.1 Some Spectral Properties of Quadratic Eigenvalue Problems

We attempt to give general structure of quadratic eigenvalue problems and how we can factor them. We also give a summary of linearization.

We suppose that A_1, A_2 and A_3 are *nxn* matrices with complex entries. Thus $P(\lambda) = \lambda^2 A_1 + \lambda A_2 + A_3$ is an *nxn* matrix polynomial with degree 2. $P(\lambda)$ is called a λ-matrix. The coefficients of the matrix $P(\lambda)$ are quadratic polynomials in the scalar λ. The spectrum of $P(\lambda)$ is denoted by $\Lambda(P)$.

$$\Lambda(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$$

is the set of eigenvalues of $P(\lambda)$. If $\det P(\lambda)$ is identically zero for all values of λ, the λ-matrix $P(\lambda)$ is called nonregular. When $\det P(\lambda)$ is not identically zero then λ-matrix $P(\lambda)$ is called regular.

The spectrum of $P(\lambda)$ is symmetric to the real axis of the complex plane if the coefficient matrices are real. Thus the eigenvalues are real or complex conjugate pairs.

Theorem (The Generalized Bezout Theorem)

When the matrix polynomial $P(\lambda)$ is the divided on the right by the binomial $\lambda \mathrm{I} - A$, the remainder is $P(A)$; when it is divided on the left, the remainder is $\hat{P}(A)$.

Corollary A polynomial $P(\lambda)$ is divisible by the binomial $\lambda \mathrm{I} - A$ on the right (left) without remainder if and only if

$$P(A) = 0 \ (\hat{P}(A) = 0).$$

Now let $S \in \mathbb{C}^{nxn}$ and $P(S) = A_1 S^2 + A_2 S + A_3$. From this theorem and corollary

$$P(\lambda) - P(S) = A_1(\lambda^2 \mathrm{I} - S^2) + A_2(\lambda \mathrm{I} - S)$$

$$= (\lambda A_1 + A_1 S + A_2)(\lambda \mathrm{I} - S)$$

can be written for degree 2. A solution $S \in \mathbb{C}^{nxn}$ of the quadratic matrix equation $P(S) = 0$ is called a right solvent. If it is a solution of $S^2 A_1 + S A_2 + A_3 = 0$ then it is called left solvent. So

$$P(\lambda) = (\lambda A_1 + A_1 S + A_2)(\lambda \mathrm{I} - S)$$

can be written if S is a solvent and we derive a factorization of $P(\lambda)$. We can not change the sequence of the factors because matrix multiplication is not commutative.

For linearization we need to find an equivalent linear λ-matrix $A-\lambda B$ for $P(\lambda)=\lambda^2 A_1+\lambda A_2+A_3$. This method is also used for linearization of second-order differential equation as reduce it to first-order equation. $A_1 y''(t)+A_2 y'(t)+A_3 y(t)=f(t)$

is rewritten by an equivalent system that has twice as many unknowns $\left[y_0^T, y_1^T\right]^T$. We define $y_0=y$ and $y_1=y_0'$.

$$A_1 y_1'+A_2 y_1+A_3 y_0=f$$

$$y_1=y_0'$$

Let $E(\lambda)$ and $F(\lambda)$ are $2nx2n$ λ-matrices with constant nonzero determinants. If we write

$$\begin{bmatrix} P(\lambda) & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}=E(\lambda)(A-\lambda B)F(\lambda)$$

then a $2nx2n$ linear λ-matrix $A-\lambda B$ is a linearization of $P(\lambda)$ [22], [23]. Linearization of $P(\lambda)$ is not unique. We need to choose proper linearization due to structural properties of $P(\lambda)$ and symmetry. Eigenvalues of $P(\lambda)$ and $A-\lambda B$ must fit.

$$L_1:\begin{bmatrix} 0 & N \\ -A_3 & -A_2 \end{bmatrix}-\lambda\begin{bmatrix} N & 0 \\ 0 & A_1 \end{bmatrix}$$

is the first companion form.

$$L_2 : \begin{bmatrix} -A_3 & 0 \\ 0 & N \end{bmatrix} - \lambda \begin{bmatrix} A_2 & A_1 \\ N & 0 \end{bmatrix}$$

is the second companion form. These linearizations are very useful in practice. In here N is nonsingular nxn matrix. L_1 is shown that how it is a linearizations of $P(\lambda)$ in [31].

2.2 Basic Definitions of Pseudospectra for Matrices

In quadratic eigenvalue problems we know that our coefficients are martices. Therefore we need to give some definitions about pseudospectra of matrices before the beginning.

Definition 2.2.1 First definition of pseudospectra

Let $A \in \mathbb{C}^{nxn}$ and $\varepsilon > 0$ be arbitrary. The ε-pseudospectra $\sigma_\varepsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\left\| (z - A)^{-1} \right\| > \varepsilon^{-1}. \tag{2.2.1}$$

Definition 2.2.2 Second definition of pseudospectra

$\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that

$$z \in \sigma(A + E) \tag{2.2.2}$$

for some $E \in \mathbb{C}^{nxn}$ with $\|E\| < \varepsilon$.

Definition 2.2.3 Third definition of pseudospectra

$\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that

$$\|(z-A)v\| < \varepsilon \tag{2.2.3}$$

for some $v \in \mathbb{C}^n$ with $\|v\| = 1$.

Theorem For any matrix $A \in \mathbb{C}^{nxn}$, the three definitions above are equivalent.

Proof For $z \in \sigma(A)$ the equivalence is trivial, so assume $z \notin \sigma(A)$, implying the existence of $(z-A)^{-1}$. To prove $(2.2.2) \Rightarrow (2.2.3)$, suppose that $(A+E)v = zv$ fo some $E \in \mathbb{C}^{nxn}$ with $\|E\| < \varepsilon$ and some nonzero $v \in \mathbb{C}^n$, which we may take to be normalized, $\|v\| = 1$. Then $\|(z-A)v\| = \|Ev\| < \varepsilon$, as required. To prove $(2.2.3) \Rightarrow (2.2.1)$, suppose $(z-A)v = su$ for some $v,u \in \mathbb{C}^n$ with $\|v\| = \|u\| = 1$ and $s < \varepsilon$. Then $(z-A)^{-1}u = s^{-1}v$, so $\|(z-A)^{-1}\| \geq s^{-1} > \varepsilon^{-1}$. Finally, to prove $(2.2.1) \Rightarrow (2.2.2)$, suppose $\|(z-A)^{-1}\| > \varepsilon^{-1}$. Then $(z-A)^{-1}u = s^{-1}v$ and consequently $zv - Av = su$ for some $v,u \in \mathbb{C}^n$ with $\|v\| = \|u\| = 1$ and $s < \varepsilon$. To establish (2.2.2), it is enough to show that there exists a matrix $E \in \mathbb{C}^{nxn}$ with $\|E\| = s$ and $Ev = su$, for then v will be an eigenvector of $A+E$ with eigenvalue z. In fact, E can be taken to be a rank-1 matrix of the form $E = suw^*$ for some $w \in \mathbb{C}^n$ with $w^*v = 1$. If $\|.\|$ is the 2-norm, this is evident simply by taking $w = v$. In the case of an

arbitrary norm $\|.\|$, the existence of a vector w satisfing the required conditions can be interpreted as the existence of a linear functional L on $\mathbb{C}^n$ with $\|Lv\| = 1$ and $\|L\| = 1$, which is guaranteed by the Hahn-Banach theorem.

2.3 Pseudospectra of Quadratic Eigenvalue Problems

In section 2.2 we defined pseudospectra for single matrices. But it can be extended to matrix polynomials. In here we will give ε-pseudospectrum definition for QEPs.

$\Delta P(\lambda)$ denotes the perturbation $\Delta P(\lambda) = \lambda^2 \Delta A_1 + \lambda \Delta A_2 + \Delta A_3$ of $P(\lambda)$. The definition of ε-pseudospectra of quadratic eigenvalue problem is

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \text{ and } \Delta P(\lambda)$$

$$\text{with } \|\Delta A_1\| \leq \varepsilon\alpha_2,\ \|\Delta A_2\| \leq \varepsilon\alpha_1,\ \|\Delta A_3\| \leq \varepsilon\alpha_0\}. \qquad (2.3.1)$$

The definition (2.3.1) and the following expression (2.3.2) are equivalent by Lemma (2.3.1).

$$\Lambda_\varepsilon(P) = \left\{\lambda \in \mathbb{C} : \left\|P(\lambda)^{-1}\right\| \geq (\varepsilon(|\lambda|^2 \alpha_2 + |\lambda|\alpha_1 + \alpha_0))^{-1}\right\}. \qquad (2.3.2)$$

We are interested in the location of the eigenvalues of the pertubed quadratic eigenvalue problem

$$\lambda^2(A_1 + \Delta A_1)x + \lambda(A_2 + \Delta A_2)x + (A_3 + \Delta A_3)x = 0 \ .$$

Lemma 2.3.1 $\Lambda_\varepsilon(P) = \left\{\lambda \in \mathbb{C} : \left\|P(\lambda)^{-1}\right\| \geq (\varepsilon p(|\lambda|))^{-1}\right\}$ where

$p(x) = \sum_{k=0}^{m} \alpha_k x_k$.

Proof Let S denote the set on the right-hand side of claimed equality. We first show that $\lambda \in \Lambda_\varepsilon(P)$ implies $\lambda \in S$. If λ is an eigenvalue of P this is immediate, so we can assume that λ is not an eigenvalue of P and hence that $P(\lambda)$ is nonsingular. Since

$$P(\lambda) + \Delta P(\lambda) = P(\lambda)(\mathrm{I} + P(\lambda)^{-1}\Delta P(\lambda))$$

is singular, we have

$$1 \leq \left\|P(\lambda)^{-1}\Delta P(\lambda\right\| \leq \left\|P(\lambda)^{-1}\right\|(\sum_{k=0}^{m}|\lambda|^k \alpha_k \varepsilon) = \left\|P(\lambda)^{-1}\right\| \varepsilon p(|\lambda|),$$

so that $\lambda \in S$.

Now we let $\lambda \in S$. Again we can assume that $P = P(\lambda)$ is nonsingular. Choose y with $\|y\| = 1$ so that $\left\|P^{-1}y\right\| = \left\|P^{-1}\right\|$ and let $x = P^{-1}y \Big/ \left\|P^{-1}\right\|$, so that $\|x\| = 1$. Then there exists a matrix H with $\|H\| = 1$ such that $Hx = y$. Let $E = -H \Big/ \left\|P^{-1}\right\|$. Then

$$(P+E)x = \frac{y}{\left\|P^{-1}\right\|} - \frac{y}{\left\|P^{-1}\right\|} = 0 \qquad \text{and} \qquad \|E\| = \frac{1}{\left\|P^{-1}\right\|} \leq \varepsilon p(|\lambda|) .$$

We now apportion E between the A_k by defining

$$\Delta A_k = sign(\lambda^k)\alpha_k p(|\lambda|)^{-1} E,$$

where for complex z we define

$$sign(z) = \begin{cases} \bar{z}/|z|, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Then

$$\Delta P(\lambda) = \sum_{k=0}^{m} \lambda^k \Delta A_k = (\sum_{k=0}^{m} |\lambda|^k \alpha_k) p(|\lambda|)^{-1} E = E$$

and $\|\Delta A_k\| \le \alpha_k \varepsilon,\ k = 0:m$. Hence $\lambda \in \Lambda_\varepsilon(P)$.

CHAPTER 3

APPLICATIONS OF SOME QUADRATIC EIGENVALUE PROBLEMS

3.1 Second-order Differential Equations

One of the applications of quadratic eigenvalue problems is a linear second-order differential equation which is used in many engineering applications. Its solution can be expressed in terms of the eigensolution of the corresponding QEP. It forms like

$$A_1 y''(t) + A_2 y'(t) + A_3 y(t) = f(t) \ . \tag{3.1.1}$$

In here A_1, A_2 and A_3 are *nxn* complex matrices and $y(t)$ is an nth-order vector.

Now we will discuss the solution of homogeneous equation (3.1.1). We consider $f(t) = 0$ and $n = 1$.

$$y''(t) + 2\zeta\omega y'(t) + \omega^2 y(t) = 0\,, \qquad \omega = \sqrt{\frac{A_3}{A_1}}\,, \qquad \zeta = \frac{A_2}{2A_1\omega}\,. \tag{3.1.2}$$

In engineering ω is called the natural frequency and ζ is a dimensionless quantity known as the viscous damping factor. The general solution of (3.1.2) is,

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\,.$$

$\lambda_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega$ are the roots of $\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0$. c_1 and c_2 are constants determined by the initial conditions $y(t_0)\,, y'(t_0)$.

We will study the case where all the eigenvalues λ_i, $i = 1,...,2n$ of $(\lambda^2 A_1 + \lambda A_3 + A_2)x = 0$ are distinct. Now we define

$$\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{2n}), \qquad X = [x_1, x_2, ..., x_{2n}], \qquad Y = [y_1, y_2, ..., y_{2n}].$$

In here x_i and y_i are right and left eigenvectors corresponding to λ_i. The general solutions of homogeneous equation (3.1.1) can be written $y(t) = \sum_{k=1}^{2n} c_k x_k e^{\lambda_k t} = X e^{\Lambda t} a$. $a = [c_1, ..., c_{2n}]^T$ is a vector of arbitrary constants.

We can control the system (3.1.1) with feedback controller of the form

$$f(t) = -K(F_{A_2} y'(t) + F_{A_3} y(t) + r(t)),$$

where $F_{A_2}, F_{A_3} \in \mathbb{C}^{mxn}$, $K \in \mathbb{C}^{nxm}$ and $r(t) \in \mathbb{C}^m$ with $m \le n$. When we replace $f(t)$ in (3.1.1) we will have

$$A_1 y''(t) + (A_2 + KF_{A_2}) y'(t) + (A_3 + KF_{A_3}) y(t) = -Kr(t)$$

and corresponding QEP

$$(\lambda^2 A_1 + \lambda(A_2 + KF_{A_2}) + A_3 + KF_{A_3})x = 0. \qquad (3.1.3)$$

F_{A_2} and F_{A_3} are the feedback gain matrices. If we consider $A_1 = C$, $A_2 + KF_{A_2} = D$ and $A_3 + KF_{A_3} = E$, equation (3.1.3) turns $P(\lambda) = C\lambda^2 + D\lambda + E$. If $P(\lambda)$ can be factored, it can be written as $P(\lambda) = (\lambda \mathrm{I} - A)(\lambda \mathrm{I} - B)$. We give an example for factorization of the analized quadratic pencil. We choose three diagonal matrices for matrix coefficients of equation

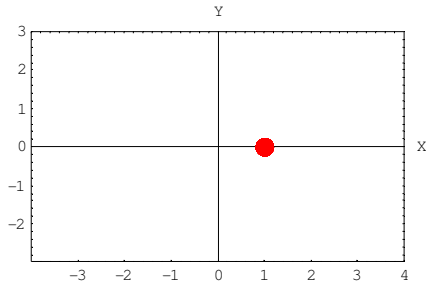
(3.1.3) specially and we assume that A, B and $P(\lambda)$ are three diagonal in the factorization of obtained pencil. We study their

pseudospectra in the examples below. While we are plotting spectrum and pseudospectrum of matrices and pencils we use programs which is written in Mathematica programming.

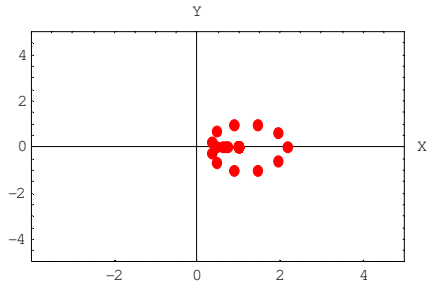
Example 3.1.1 $P(\lambda) = (\lambda \mathrm{I} - A)(\lambda \mathrm{I} - B)$

In this example $A_{120x120}$ and $B_{120x120}$ three diagonal matrices.

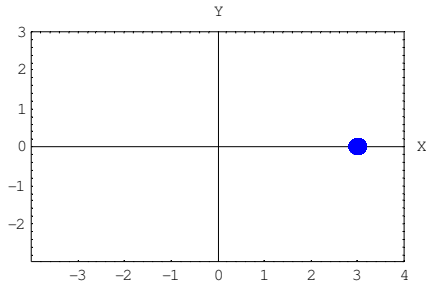
$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & 1 & 1 & 0 & \vdots \\ & \ddots & & \ddots & \ddots & \ddots & 0 \\ & & & & 1 & 1 & 1 \\ \vdots & & & \ddots & \ddots & 1 & 1 \\ 0 & & \dots & & \dots & 0 & 1 \end{bmatrix}_{120x120}, \qquad B = \begin{bmatrix} 3 & 3 & 3 & 0 & \dots & \dots & 0 \\ 0 & 3 & 3 & 3 & 0 & \dots & 0 \\ \vdots & \ddots & 3 & 3 & 3 & 0 & \vdots \\ & \ddots & & \ddots & \ddots & \ddots & 0 \\ & & & & 3 & 3 & 3 \\ \vdots & & & \ddots & \ddots & 3 & 3 \\ 0 & & \dots & & \dots & 0 & 3 \end{bmatrix}_{120x120}$$



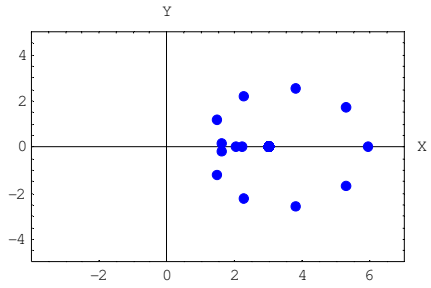
$\sigma(A)$



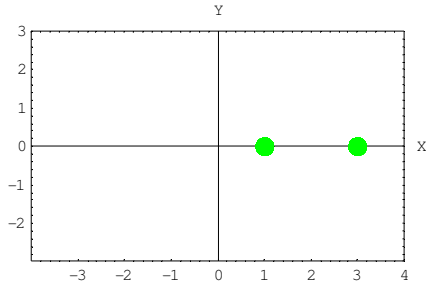
$\sigma_\varepsilon(A)$



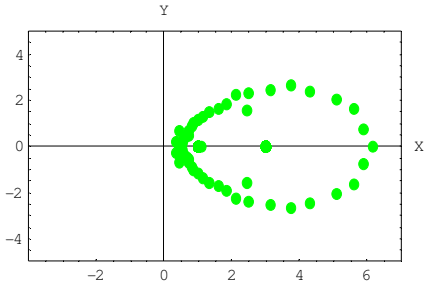
$\sigma(B)$



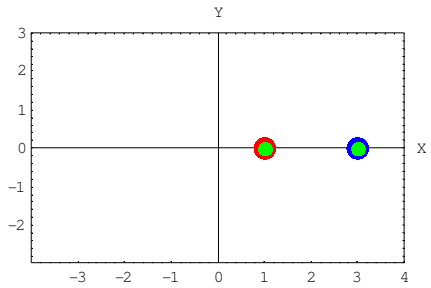
$\sigma_\varepsilon(B)$



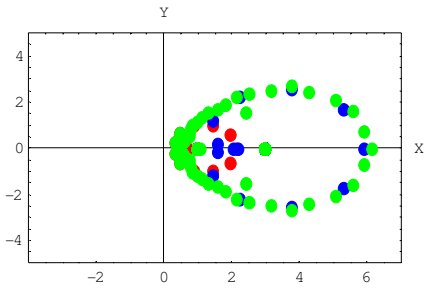
$\sigma(P)$



$\sigma_{\varepsilon}(P)$



$\sigma(A),\ \sigma(B),\ \sigma(P)$

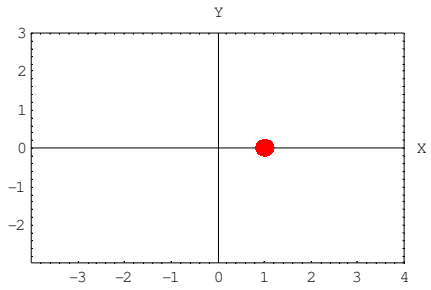


$\sigma_{\varepsilon}(A),\ \sigma_{\varepsilon}(B),\ \sigma_{\varepsilon}(P)\,,\ \varepsilon = 10^{-2}$

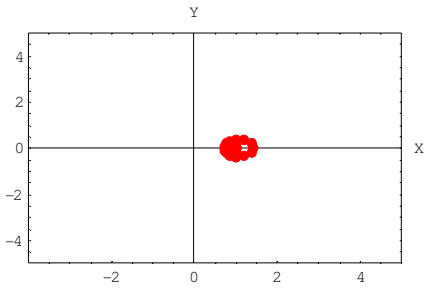
Figure 3.1.1 In the left four plots corresponding to $n = 120$ *show the spectral points of threediagonal matrices* A, B *and of quadratic matrix polynomial* $P(\lambda)$. *In the right four plots corresponding to* $n = 120$, $\|\Delta\| = \varepsilon = 10^{-2}$ *show the* ε*-pseudospectral points of perturbed matrices* $\sigma_{\varepsilon}(A) = \sigma(A + \Delta)$, $\sigma_{\varepsilon}(B) = \sigma(B + \Delta)$ *and of perturbed matrix polynomial* $\sigma_{\varepsilon}(P) = \sigma(P_{\Delta}(\lambda))$.

Example 3.1.2 $A=\begin{bmatrix} 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & 1 & 1 & 0 & \vdots \\ & \ddots & & \ddots & \ddots & \ddots & 0 \\ & & & & 1 & 1 & 1 \\ \vdots & & & \ddots & \ddots & 1 & 1 \\ 0 & & \dots & & \dots & 0 & 1 \end{bmatrix}_{240 x 240}$ and $B=\begin{bmatrix} 3 & 3 & 3 & 0 & \dots & \dots & 0 \\ 0 & 3 & 3 & 3 & 0 & \dots & 0 \\ \vdots & \ddots & 3 & 3 & 3 & 0 & \vdots \\ & \ddots & & \ddots & \ddots & \ddots & 0 \\ & & & & 3 & 3 & 3 \\ \vdots & & & \ddots & \ddots & 3 & 3 \\ 0 & & \dots & & \dots & 0 & 3 \end{bmatrix}_{240 x 240}$

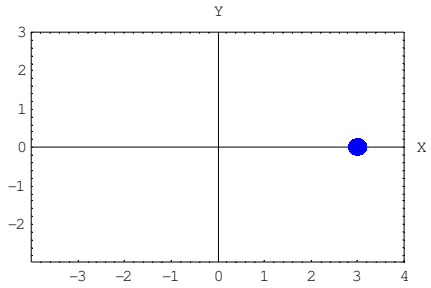
$P(\lambda)=(\lambda \mathrm{I}-A)(\lambda \mathrm{I}-B)$



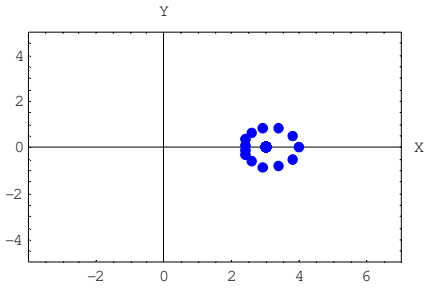
$\sigma(A)$



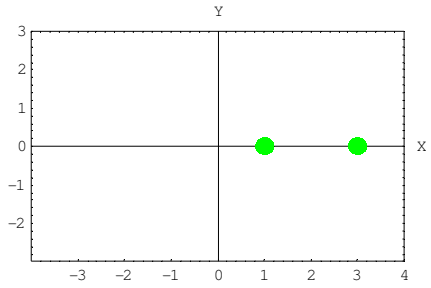
$\sigma_{\varepsilon}(A)$



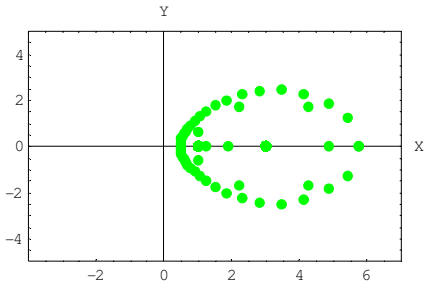
$\sigma(B)$



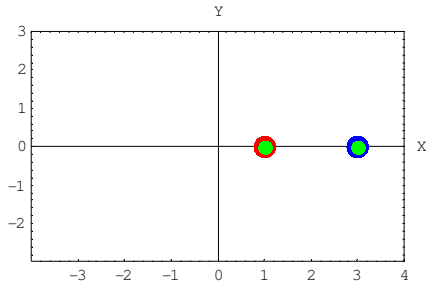
$\sigma_{\varepsilon}(B)$



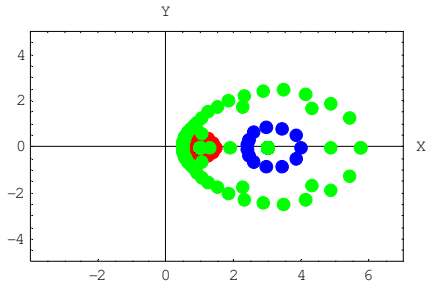
$\sigma(P)$



$\sigma_{\varepsilon}(P)$



$\sigma(A),\ \sigma(B),\ \sigma(P)$



$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(P),\ \varepsilon = 10^{-7}$

Figure 3.1.2 In the left four plots corresponding to $n = 240$ *show the spectral points of threediagonal matrices* A, B *and of quadratic matrix polynomial* $P(\lambda)$. *In the right four plots corresponding to* $n = 240$, $\|\Delta\| = \varepsilon = 10^{-7}$ *show the* ε*-pseudospectral points of perturbed matrices* $\sigma_\varepsilon(A) = \sigma(A + \Delta)$, $\sigma_\varepsilon(B) = \sigma(B + \Delta)$ *and of perturbed matrix polynomial* $\sigma_\varepsilon(P) = \sigma(P_\Delta(\lambda))$.

3.2 Vibration Analysis of Structural Systems-Modal Superposition Method

Another major area for second-order differential equations is structural mechanics. Vibration analysis intends to determine the effect of vibration on the performance, safety of systems and to control this effect. The structure and properties of the matrices defining the quadratic eigenvalue problem are important because of designing a numerical method for the solution of the eigenvalue problem.

Our equation forms like

$$A_1 y''(t) + A_2 y'(t) + A_3 y(t) = f(t) \ . \qquad (3.2.1)$$

In this equation A_1 is the mass matrix, A_2 is viscous damping matrix, A_3 is the stiffness matrix and $f(t)$ is a time-dependent external force vector. A quadratic form relates matrix A_1 to kinetic energy and matrix A_3 to strain energy. It makes them symmetric. Moreover A_1 and A_3 are positive definite and sparse for most structures.

The general solution to the homogeneous equation is given in terms of the solution of the QEP. This solution forms as

$$(\lambda^2 A_1 + \lambda A_2 + A_3)x = 0 \, . \qquad (3.2.2)$$

A_1 and A_3 are real and $A_1 > 0$ or $A_3 > 0$. So we obtain a real definite pair (A_1, A_3). It implies the existence of a matrix $X \in R^{nxn}$ of eigenvectors which satisfy $X^T A_1 X = \mathrm{I}$, $X^T A_3 X = \Omega^2$. Ω^2 is a nonnegative diagonal matrix. It contains the inverse of the n eigenvalues of (A_1, A_3). Now we will premultiply (3.2.1) by X^T.

$$X^T A_1 y''(t) + X^T A_2 y'(t) + X^T A_3 y(t) = X^T f(t) \, .$$

Then we change $y(t) = Xz(t)$ in the last equation.

$$X^T A_1 X z''(t) + X^T A_2 X z'(t) + X^T A_3 X z(t) = X^T f(t)$$

$$z''(t) + X^T A_2 X z'(t) + \Omega^2 z(t) = X^T f(t)\,. \qquad (3.2.3)$$

The last equation (3.2.3) is called modal equation. If $A_2 = 0$, then the system is called undamped and it is a system of uncoupled second-order differential equations. We can solve it separately. If we superpose the modal solutions of (3.2.3) using $y(t) = Xz(t)$ we can obtain the complete solution. It is called the modal superposition method.

If damping which is not proportional damping is present then the differential equations in (3.2.3) are coupled. When the damping matrix satisfies the orthogonality conditions

$x_j^T A_2 x_i = 0$, $j \neq i$

it is called proportional damping. We can write a general class of matrices giving proportional damping as [24, sect 13.3]

$$A_2 = \sum_j \alpha_j A_1 (A_1^{-1} A_3)^j = \alpha_0 A_1 + \alpha_1 A_3 + \alpha_2 A_3 A_1^{-1} A_3 + \ldots.$$

A_1 is a nonsingular matrix. Rayleigh damping ($A_2 = \alpha_0 A_1 + \alpha_1 A_3$) is a case of proportional damping which is used in practice.

We need the eigensolution of the damped system (3.2.2) [25, 26] to uncouple the equations in (3.2.1) and use the modal superposition technique. We define

$$A = \begin{bmatrix} 0 & \mathrm{I} \\ A_3 & A_2 \end{bmatrix},\ B = \begin{bmatrix} -\mathrm{I} & 0 \\ 0 & A_1 \end{bmatrix},\ g(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix},\ p(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}. \qquad (3.2.4)$$

So we can reformulate the differential equation (3.2.1) into a first-order $2n$-dimensional equation

$$Ap(t) + B\dot{p}(t) = g(t) \quad . \tag{3.2.5}$$

When $g(t) = 0$, then its solution is $p(t) = ue^{\lambda t}$ which yields $Au = \lambda Bu$. If all the eigenvalues are distinct, the pair (A,B) is diagonalizable [27, prop. 9.2],

$$W^* AU = diag(\alpha_1,...,\alpha_{2n}), \qquad W^* BU = diag(\beta_1,...,\beta_{2n}).$$

W and U are $2nx2n$ matrices containing the left and right eigenvectors of (A,B). If A and B are real then $W^* = U^T$.

In this approach quadratic pencil is turned to a linear pencil. Spectrum points are found by the linear pencil in above. These points are same to the spectum points of the quadratic pencil which we are dealing. Therefore studying a quadratic pencil problem is derived by linearization instead of factorization.

We will give an example for this approach. We take our matrix coefficients A_1 and A_3 in (3.2.2) as real and positive definite.

$$A_1 = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

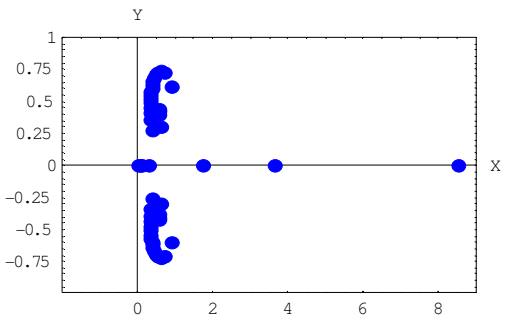
So our A and B in (3.2.4) are

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 3 & -2 & 0 \\ -1 & 2 & -1 & 0 & 3 & -2 \\ 0 & -1 & 2 & 0 & 0 & 3 \end{bmatrix}, \qquad B=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix}.$$

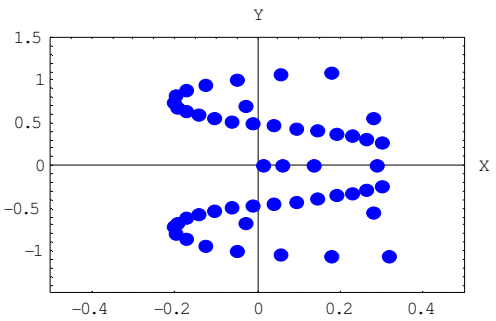
Thus our quadratic eigenvalue problem (3.2.2) turns a generalized eigenvalue problem $Au=\lambda Bu$ and if we multiply it with inverse of $B\,(B^{-1})$ we will have standard eigenvalue problem $Cu=\lambda u$. Solving $\det(C-\lambda \mathrm{I})=0$ gives our spectrum points. In following example we search the spectrum and pseudospectrum points of matrix C for three situations where $n=50,\ 100,\ 200$.

Example 3.2.1 We are studying the matrix $C=B^{-1}A$ and its spectrum points with its pseudospectrum points.

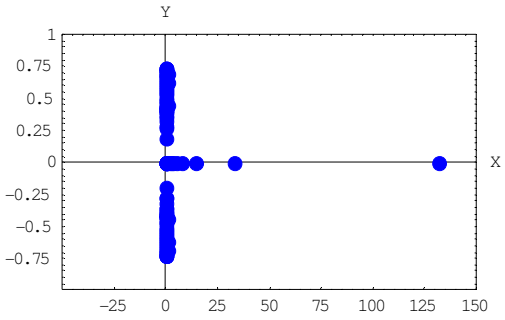
$$A=\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ 2 & -1 & 0 & 3 & -2 & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 2 & 0 & 0 & 3 \end{bmatrix}_{nxn}, \qquad B=\begin{bmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \\ & 0 & -1 & & & \\ \vdots & & 0 & 4 & -2 & 0 \\ & \vdots & \vdots & -2 & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ddots & 4 \end{bmatrix}_{nxn}$$



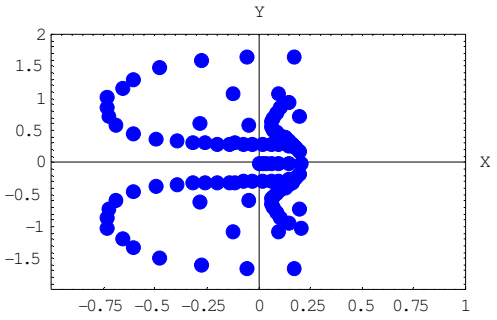
$\sigma(C)$



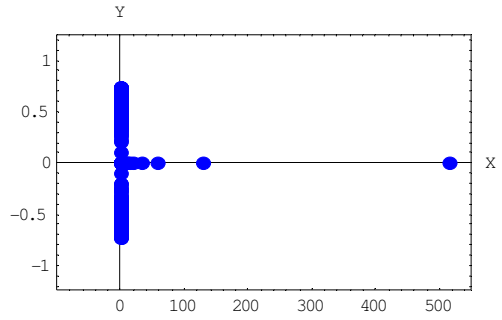
$\sigma_{\varepsilon}(C),\ \varepsilon=10^{-1}$



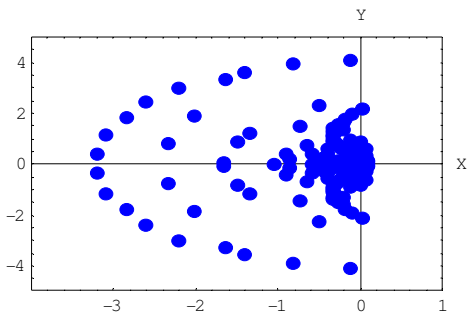
$\sigma(C)$



$\sigma_\varepsilon(C),\ \varepsilon = 10^{-1}$



$\sigma(C)$



$\sigma_\varepsilon(C),\ \varepsilon = 10^{-1}$

Figure 3.2.1 The spectrum points of the matrix C are shown respectively for $n = 50,\ 100,\ 200$ in the left column. The pseudospectrum points of the matrix C are shown respectively for $n = 50,\ 100,\ 200$ in the right column. We did not show the point $\{(34.2699,0)\}$ in $\sigma(C)$ and $\{(-3.90481x10^7,0)\}$ in $\sigma_\varepsilon(C)$ for $n = 50$, the points $\{(-119.931,-3.16933x10^7),\ (-119.931,3.16933x10^7)\}$ in $\sigma_\varepsilon(C)$ for $n = 100$ and $\{(-446.937,-5.36707x10^7),\ (-446.937,5.36707x10^7)\}$ in $\sigma_\varepsilon(C)$ for $n = 200$. By ignoring these points we can see changes in small area successfully.

3.3 Signal Processing

The signal is modeled as an autoregressive (AR) process of the form

$$x_t = -\sum_{k=1}^{p} \alpha_k x_{t-k} + \varepsilon_t , \quad t = 1,...,n$$

in the linear predictive coding of speech, biomedical signal processing and time series forecasting. n is the number of data points, p is the order of the process, ε_t is zero-mean white noise with variance σ_ε, x_t, $t = -p+1,...,0$ are given. We will estimate the parameters α_k by taking $n = p$ and solving equations

$$R\alpha = -r^x . \tag{3.3.1}$$

In here $r^x = [r_1^x,...,r_p^x]^T$ is a vector. Its components $r_k^x = E(x_1, x_{1+k})$ are the autocorrelation function of x_t. $\alpha = [\alpha_1,...,\alpha_p]^T$ is the AR parameter vector. $R \in \mathbb{R}^{pxp}$ is a symmetric Toeplitz matrix with coefficients $R_{ij} = r_{|j-i|}^x$. Durbin's algorithm [28] is being used to compute the solution in $2n^2$ flops.

Our modeled signal is observed with noise [29], [30]. The observed time series are

$$y_t = x_t + w_t , \quad t = 1,...,n .$$

w_t is an uncorrelated white observation noise with unknown variance σ_w. The Yule-Walker estimates of the parameters of the model can produce wrong results in this case. So we utilize Davila's subspace approach [29] in

this problem. In this approach the noise-compensated Yule-Walker equations are used. It is defined by

$$(S-\lambda B)v=0\,. \tag{3.3.2}$$

$S\in\mathbb{R}^{(p+n)x(p+1)}$ with $n\geq p$ is rectangular matrix. Its coeffcients $S_{ij}=r^{y}_{i-j+1}$ are defined from the autocorrelation functions of y_t. With *pxp* identity matrix I_p, $B=\begin{bmatrix}0 & \mathrm{I}_p\\ 0 & 0\end{bmatrix}\in\mathbb{R}^{(p+n)x(p+1)}$. In (3.3.2) $v=[1,\alpha_1,...,\alpha_p]^T$ and λ , an estimation of the unknown variance σ_w^2, are unknown. We will premultiply the equation (3.3.2) on the left by $(S-\lambda B)^T$

$$(S-\lambda B)^T\,(S-\lambda B)v=0$$

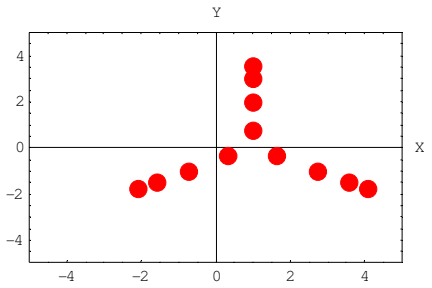
$$(S^TS-\lambda S^TB-\lambda B^TS+\lambda^2B^TB)v=0\,.$$

When we show $A_1=B^TB$, $A_2=-(S^TB+B^TS)$ and $A_3=S^TS$ then we derive our $(p+1)x(p+1)$ symmetric QEP

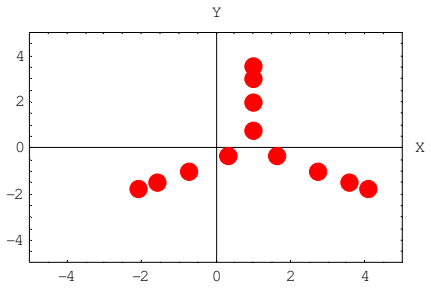
$$(A_3+\lambda A_2+\lambda^2A_1)v=0\,.$$

Now we will give examples for $n=12,\ 48,\ 72$. Our coefficient matrices are Toeplitz matrices and we assume that $P(\lambda)$ can be factored like $P(\lambda)=(\lambda\mathrm{I}-A)(\lambda\mathrm{I}-B)$.

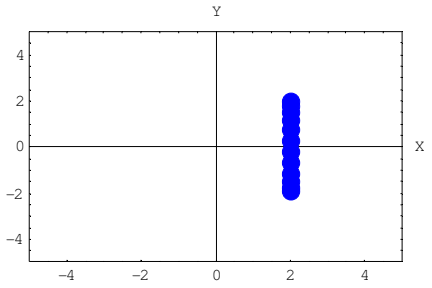
Example 3.3.1 $A=\begin{bmatrix} 1 & 2i & 0 & \cdots \\ 0 & 1 & \ddots & 0 \\ 2i & \ddots & \ddots & 2i \\ 0 & 2i & 0 & 1 \end{bmatrix}_{nxn}$, $B=\begin{bmatrix} 2 & i & 0 & \cdots \\ i & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & i \\ \vdots & \ddots & i & 2 \end{bmatrix}_{nxn}$, $n=12$



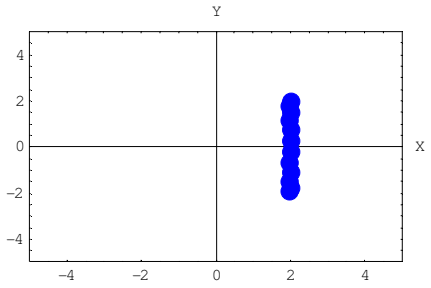
$\sigma(A)$



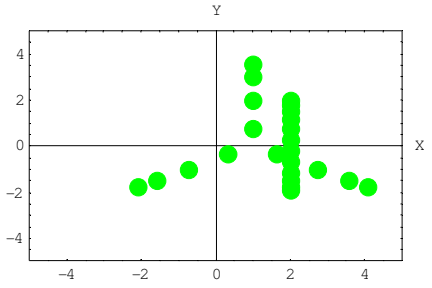
$\sigma_\varepsilon(A)$



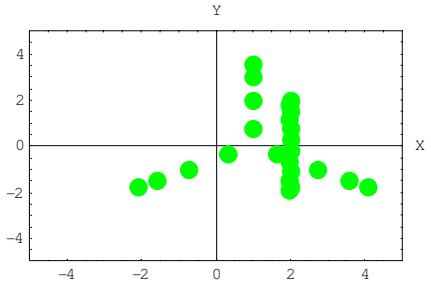
$\sigma(B)$



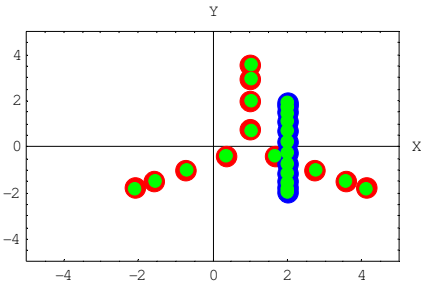
$\sigma_\varepsilon(B)$



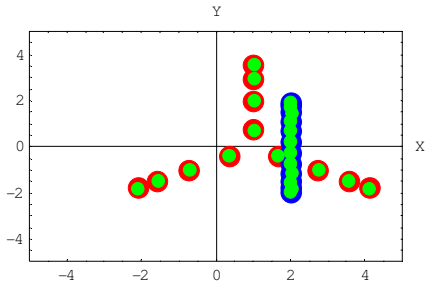
$\sigma(P)$



$\sigma_\varepsilon(P)$



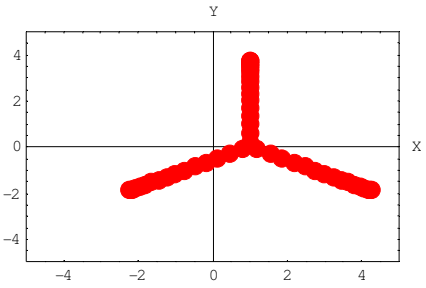
$\sigma(A),\ \sigma(B),\ \sigma(P)$



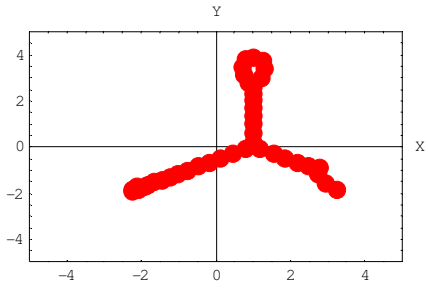
$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(P),\ \varepsilon = 10^{-1}$

Figure 3.3.1 In the left four plots corresponding to $n = 12$ we show spectrum points of $A,\ B,\ P(\lambda)$ where A and B are Toeplitz matrices. Pseudospectrum points of $A,\ B,\ P(\lambda)$ are shown in the right column.

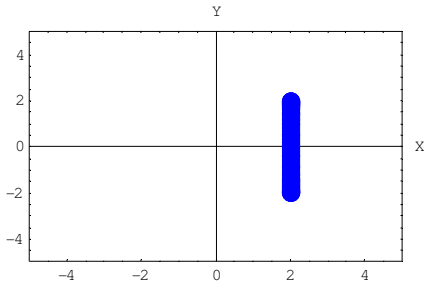
Example 3.3.2 $A = \begin{bmatrix} 1 & 2i & 0 & \cdots \\ 0 & 1 & \ddots & 0 \\ 2i & \ddots & \ddots & 2i \\ 0 & 2i & 0 & 1 \end{bmatrix}_{nxn}$, $B = \begin{bmatrix} 2 & i & 0 & \cdots \\ i & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & i \\ \vdots & \ddots & i & 2 \end{bmatrix}_{nxn}$, $n = 48$



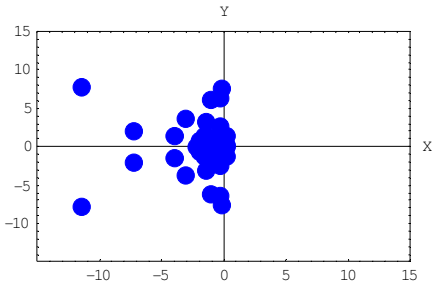
$\sigma(A)$



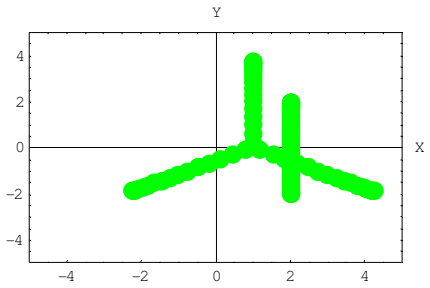
$\sigma_\varepsilon(A)$



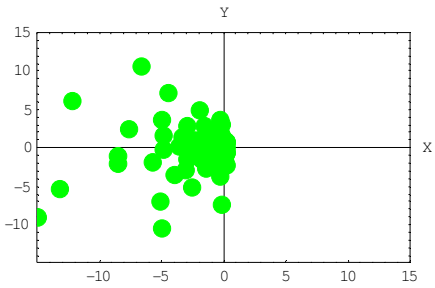
$\sigma(B)$



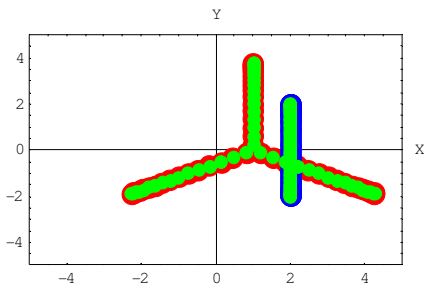
$\sigma_\varepsilon(B)$



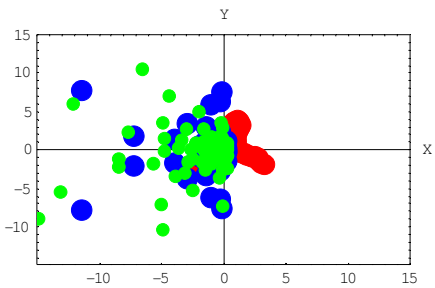
$\sigma(P)$



$\sigma_\varepsilon(P)$



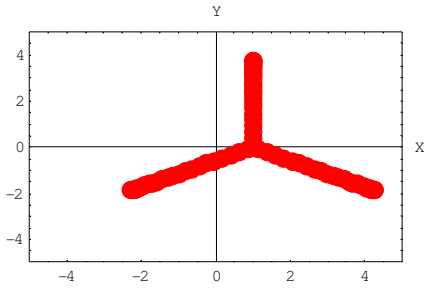
$\sigma(A),\ \sigma(B),\ \sigma(P)$



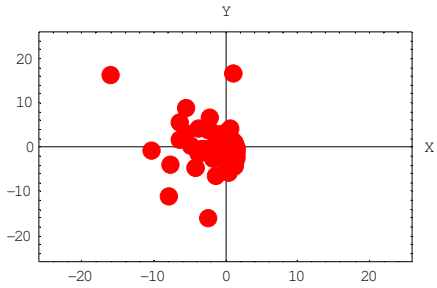
$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(P),\ \varepsilon = 10^{-1}$

Figure 3.3.2 In the left four plots corresponding to $n = 48$ we show spectrum points of A, B, $P(\lambda)$ where A and B are Toeplitz matrices. Pseudospectrum points of A, B, $P(\lambda)$ are shown in the right column.

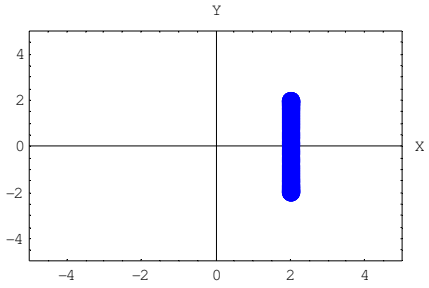
Example 3.3.3 $A=\begin{bmatrix} 1 & 2i & 0 & \cdots \\ 0 & 1 & \ddots & 0 \\ 2i & \ddots & \ddots & 2i \\ 0 & 2i & 0 & 1 \end{bmatrix}_{nxn}$, $B=\begin{bmatrix} 2 & i & 0 & \cdots \\ i & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & i \\ \vdots & \ddots & i & 2 \end{bmatrix}_{nxn}$, $n=72$



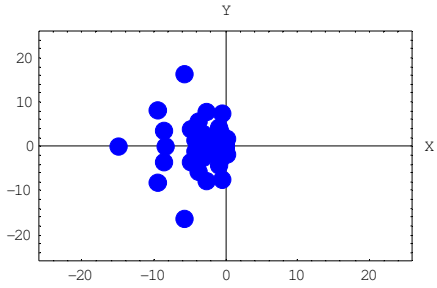
$\sigma(A)$



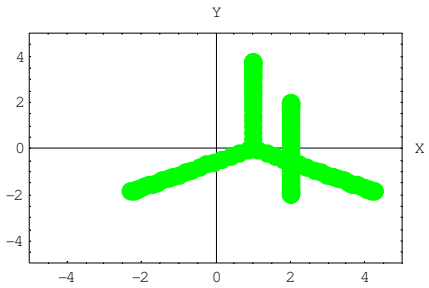
$\sigma_\varepsilon(A)$



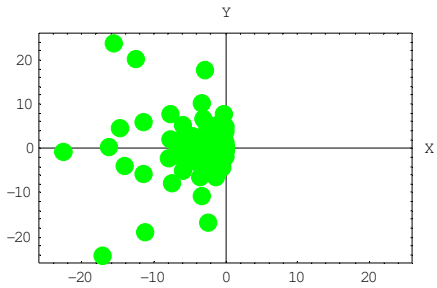
$\sigma(B)$



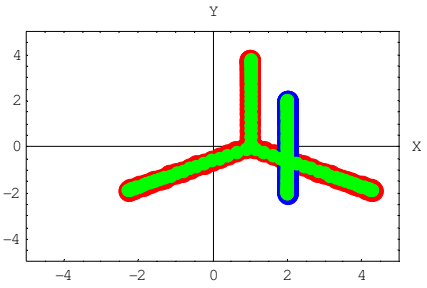
$\sigma_\varepsilon(B)$



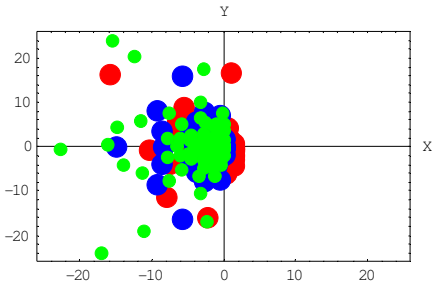
$\sigma(P)$



$\sigma_\varepsilon(P)$



$\sigma(A),\ \sigma(B),\ \sigma(P)$



$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(P),\ \varepsilon = 10^{-1}$

Figure 3.3.3 In the left four plots corresponding to $n = 72$ we show spectrum points of $A,\ B,\ P(\lambda)$ where A and B are Toeplitz matrices. Pseudospectrum points of $A,\ B,\ P(\lambda)$ are shown in the right column.

3.4 Gyroscopic Systems

Another example for QEP is gyroscopic systems. We defined λ-matrix as

$$P(\lambda) = \lambda^2 A_1 + \lambda A_2 + A_3 .$$

In gyroscopic system we will denote $P(\lambda)$ as $G(\lambda)$. Our matrix coefficients A_1 and A_3 are Hermitian, $A_2 = -A_2^*$ skew-Hermitian and A_1 is positive definite. $G(\lambda)$ is associated with a second-order differential equation

$$A_1 y''(t) + A_2 y'(t) + A_3 y(t) = f(t) .$$

We know that $G(\lambda) = \lambda^2 A_1 + \lambda A_2 + A_3$ can be factorized as

$$G(\lambda) = (\lambda A_1 + A_1 S + A_2)(\lambda \mathrm{I} - S)$$

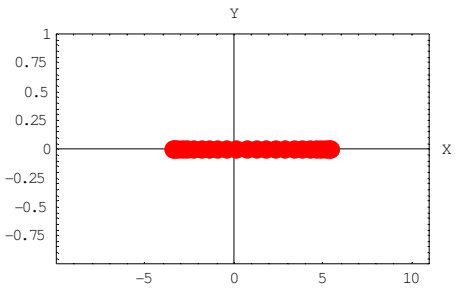
if and only if S is a solution of

$$A_1 S^2 + A_2 S + A_3 = 0 .$$

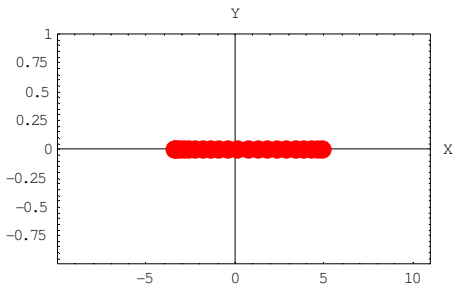
We will give examples for $n = 24,\ 48,\ 72$ about the situation which A and B are complex Hermitian matrices in $G(\lambda) = (\lambda \mathrm{I} - A)(\lambda \mathrm{I} - B)$.

Example 3.4.1

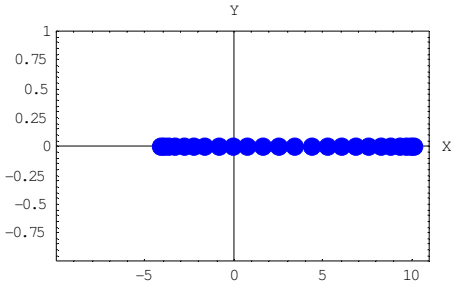
$$A = \begin{bmatrix} 1 & 1+2i & 0 & \cdots \\ 1-2i & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & 1+2i \\ \vdots & 0 & 1-2i & 1 \end{bmatrix}_{nxn}, \quad B = \begin{bmatrix} 3 & 2+3i & 0 & \cdots \\ 2-3i & 3 & \ddots & 0 \\ 0 & \ddots & \ddots & 2+3i \\ \vdots & 0 & 2-3i & 3 \end{bmatrix}_{nxn}, \quad n = 24$$



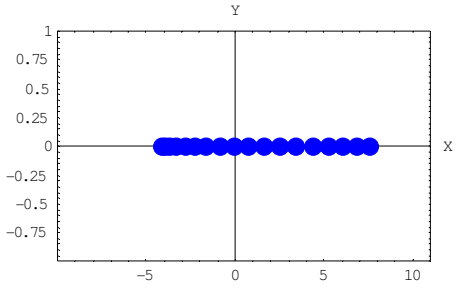
$\sigma(A)$



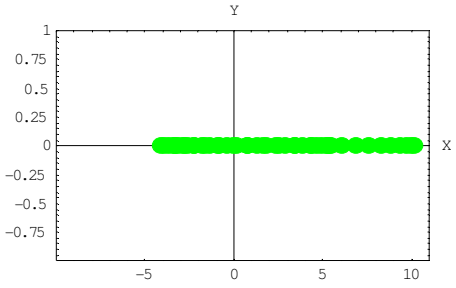
$\sigma_\varepsilon(A)$



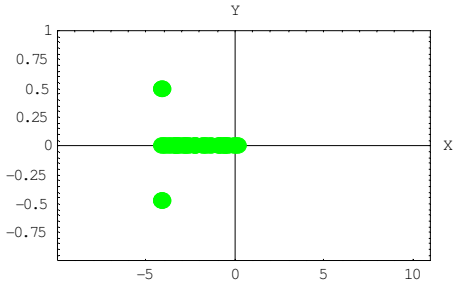
$\sigma(B)$



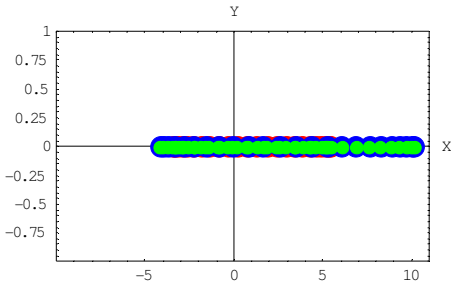
$\sigma_\varepsilon(B)$



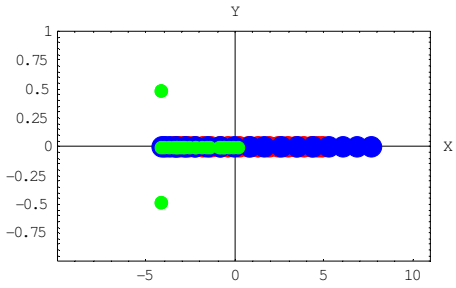
$\sigma(G)$



$\sigma_\varepsilon(G)$



$\sigma(A),\ \sigma(B),\ \sigma(G)$

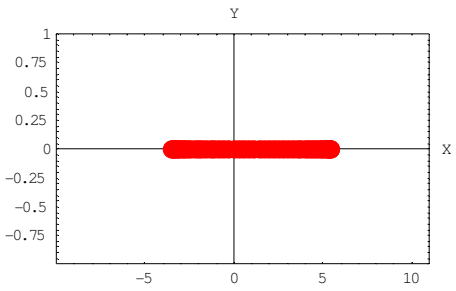


$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(G), \varepsilon = 10^{-1}$

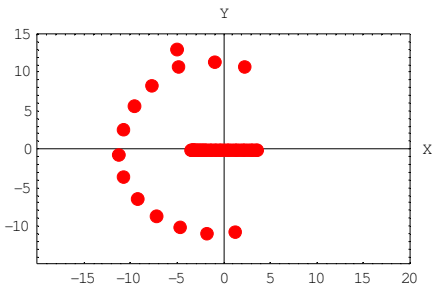
Figure 3.4.1 In the left four plots corresponding to $n = 24$ we show spectrum points of $A, B,\ G(\lambda)$ where A and B are Hermitian matrices. Pseudospectrum points of $A, B, G(\lambda)$ are show in the right column.

Example 3.4.2

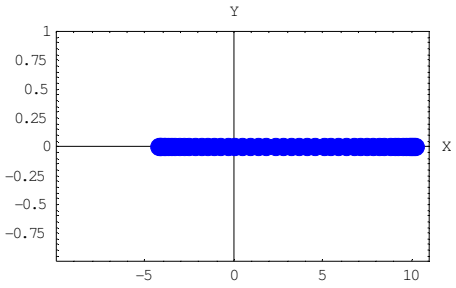
$$A=\begin{bmatrix} 1 & 1+2i & 0 & \cdots \\ 1-2i & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & 1+2i \\ \vdots & 0 & 1-2i & 1 \end{bmatrix}_{nxn}, \quad B=\begin{bmatrix} 3 & 2+3i & 0 & \cdots \\ 2-3i & 3 & \ddots & 0 \\ 0 & \ddots & \ddots & 2+3i \\ \vdots & 0 & 2-3i & 3 \end{bmatrix}_{nxn}, \quad n=48$$



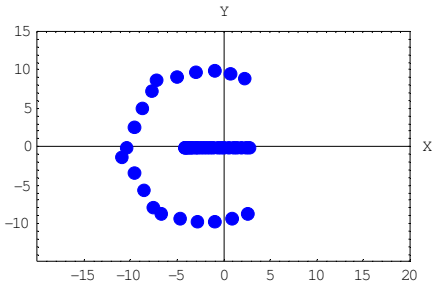
$\sigma(A)$



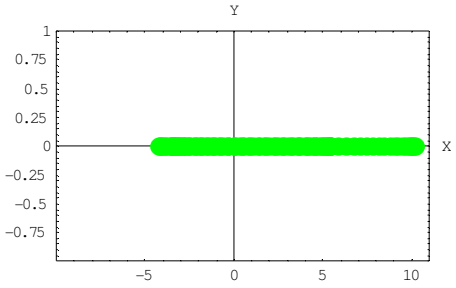
$\sigma_\varepsilon(A)$



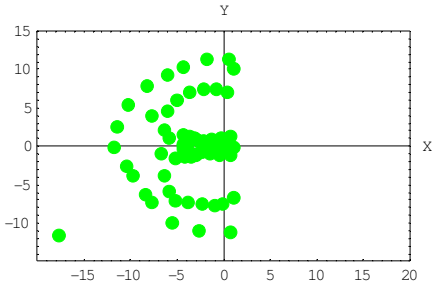
$\sigma(B)$



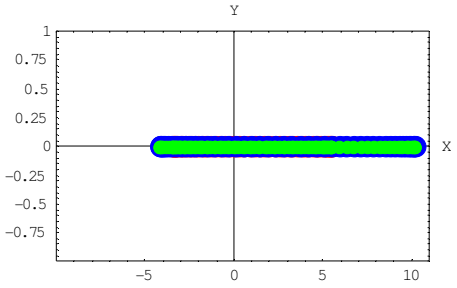
$\sigma_\varepsilon(B)$



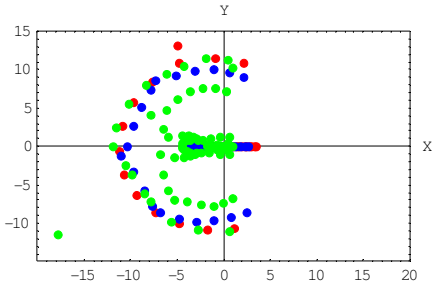
$\sigma(G)$



$\sigma_\varepsilon(G)$



$\sigma(A),\ \sigma(B),\ \sigma(G)$

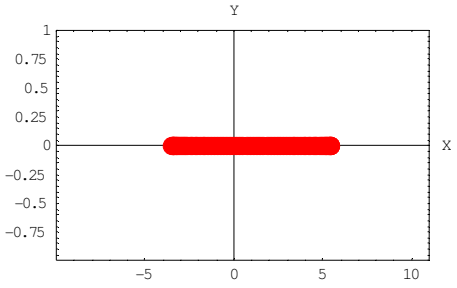


$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(G),\ \varepsilon = 10^{-1}$

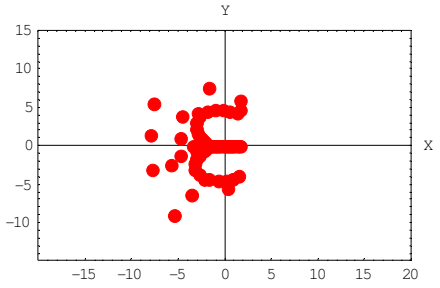
Figure 3.4.2 In the left four plots corresponding to $n = 48$ we show spectrum points of $A, B, G(\lambda)$ where A and B are Hermitian matrices. Pseudospectrum points of $A, B, G(\lambda)$ are show in the right colum.

Example 3.4.3

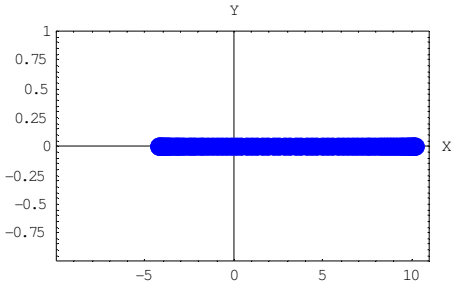
$$A = \begin{bmatrix} 1 & 1+2i & 0 & \cdots \\ 1-2i & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & 1+2i \\ \vdots & 0 & 1-2i & 1 \end{bmatrix}_{nxn}, \quad B = \begin{bmatrix} 3 & 2+3i & 0 & \cdots \\ 2-3i & 3 & \ddots & 0 \\ 0 & \ddots & \ddots & 2+3i \\ \vdots & 0 & 2-3i & 3 \end{bmatrix}_{nxn}, \quad n = 72$$



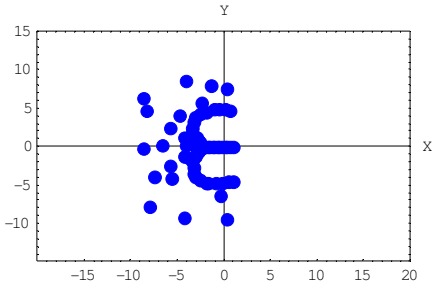
$\sigma(A)$



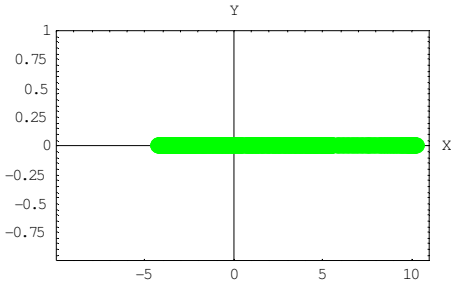
$\sigma_\varepsilon(A)$



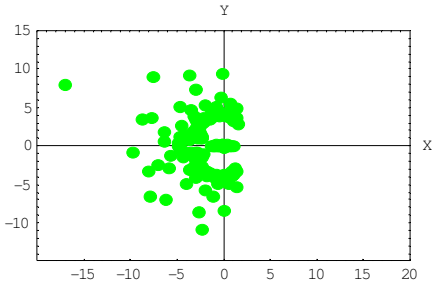
$\sigma(B)$



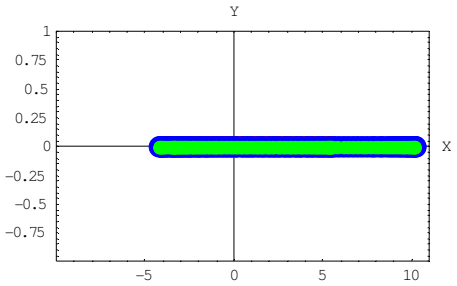
$\sigma_\varepsilon(B)$



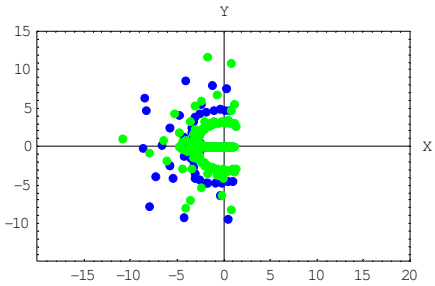
$\sigma(G)$



$\sigma_\varepsilon(G)$



$\sigma(A),\ \sigma(B),\ \sigma(G)$



$\sigma_\varepsilon(A),\ \sigma_\varepsilon(B),\ \sigma_\varepsilon(G), \varepsilon = 10^{-1}$

Figure 3.4.3 In the left four plots corresponding to $n = 72$ we show spectrum points of $A, B,\ G(\lambda)$ where A and B are Hermitian matrices. Pseudospectrum points of $A, B, G(\lambda)$ are show in the right colum.

CONCLUSIONS

In this study we take quadratic eigenvalue problem applications in engineering. We assume that the quadratic pencils can be factorized like $P(\lambda) = (\lambda \mathrm{I} - A)(\lambda \mathrm{I} - B)$ and we take matrices A, B as Hermitian, Toeplitz and diagonal. We investigate ε-pseudospectrum of these pencils. Also we compute an example for linearization and we take our matrices positive definite. We use Mathematica programming to plot ε-pseudospectrum for $n = 12,\ 24,\ 48,\ 50,\ 72,\ 100,\ 200,\ 240$. We don't plot ε-pseudospectrum for $n = 300,\ 400,\ 500$ or higher dimensions because we don't have a computer with delicate CPU to compute our problems.

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