ON OSCILLATION OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Zekiye Dağdelen

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Prof. Dr. Davut Köşker
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Cenap Özel
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Davut Uğurlu
Supervisor

Examining Committee Members

Prof. Dr. Aydın Tiryaki

Prof. Dr. Ağacık Zafer

Assist. Prof. Dr. Davut Uğurlu

ABSTRACT

ON OSCILLATION OF SOLUTIONS OF SECOND ORDER

NONLINEAR DIFFERENTIAL EQUATIONS

Dağdelen, Zekiye

Master, Department of Mathematics

Supervisor: Assist. Prof. Dr. Davut Uğurlu

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In this thesis, we survey oscillation criteria for solutions of second-order nonlinear equations.

Keywords: Second-order, nonlinear differential equation, oscillation, Riccati transformation, integral averaging method.

ÖZ

İKİNCİ MERTEBEDEN LİNEER OLMAYAN DİFERANSİYEL

DENKLEMLERİN ÇÖZÜMLERİNİN SALINIMLILIĞI

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Bu tezde ikinci mertebeden lineer olmayan diferansiyel denklemlerin çözümlerinin salınımlılık kriterleri verilmiştir.

Anahtar Kelimeler: İkinci mertebe, lineer olmayan diferansiyel denklem, salınımlılık, Riccati dönüşümü, integral ortalama metodu.

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To My Family

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

In recent years, there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions for different classes of linear and nonlinear second order differential equations. This has resulted in hundreds of research papers most of which can be found in the books of Agarwal et al [1], Kreith [17] and Swanson [39].

The following definitions will be frequently used.

Definition 1.1 *Consider the second order ordinary differential equation*

$$F(t, x(t), x'(t), x''(t)) = 0, \tag{1.1}$$

defined for $t \geq t_0 \geq 0$, where $F : [t_0, \infty) \to \mathbb{R}$ is a continuous function. By a solution of (1.1) we mean a twice continuously differentiable real-valued function $x(t)$ defined on $[t_x, \infty) \subset [t_0, \infty)$ and satisfies equation (1.1) on the interval $[t_x, \infty)$. The number $t_x \geq t_0 \geq 0$ depends on the particular solution $x(t)$ under consideration.

Definition 1.2 *A nontrivial solution $x(t)$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, $x(t)$ is said to be nonoscillatory, i.e., $x(t)$ is nonoscillatory if there exists $t_1 \geq t_0$ such that $x(t) \neq 0$ for $t \geq t_1$. In other words, a nonoscillatory solution must be eventually positive or negative.*

As explained above, there have been a great number of results on the subject of oscillation criteria obtained by different methods, here we concentrate on some results obtained by the integral averaging method. This method is an important tool in studying the oscillatory behavior of linear and nonlinear differential equations of second order. This method goes back as far as the classical results of Wintner [43] for the linear equation

$$x''(t) + q(t)x(t) = 0. \tag{1.2}$$

Wintner [43] proved that if

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds = \infty, \tag{1.3}$$

then (1.2) is oscillatory. Condition (1.3) is more general than the well-known Fite-Wintner-Leighton criterion (see Fite [7] and Leighton [20]): if

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty, \tag{1.4}$$

then (1.2) is oscillatory. In the original work of Fite [7], it was assumed that $q(t) \geq 0$. Leighton [20] studied the more general linear equation

$$[p(t)x'(t)]' + q(t)x(t) = 0, \tag{1.5}$$

where $p(t)$ is a positive continuous function. The method of proof of Wintner [43] differs from that of [7] and [20] and it is the first result based on integral averages. Hartman [14] improved the result of Wintner [43] by proving that condition (1.3) can be replaced the weaker one:

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)duds < \limsup \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(s)duds \leq \infty. \tag{1.6}$$

The result of Wintner was also improved by Kamanev [16], who proved that the condition

$$\limsup_{t\to\infty} t^{1-n}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds = \infty, \quad \text{for some} \quad n > 2, \tag{1.7}$$

is sufficient for oscillation of (1.2). Kamanev's criterion extended by Philos [30], was further generalized for the damped linear equation

$$x'' + r(t)x' + q(t)x(t) = 0 \tag{1.8}$$

by Yan [50] and Yeh [53]. They proved that if r and q satisfy

$$\limsup_{t\to\infty} t^{-n}\int_{t_0}^{t}(t-s)^n s^m q(s)dsdt = \infty \tag{1.9}$$

and

$$\limsup_{t\to\infty} t^{-n}\int_{t_0}^{t}\Big[(t-s)r(s)s + ns - m(t-s)\Big]^2(t-s)^{n-2}s^{m-2}ds < \infty, \tag{1.10}$$

where $n > 1,\ \ 0 \leq m < 1$, then Eq.(1.8) is oscillatory.

In [34], Philos proved that if there exists a continuous function $H(t,s)$ from $D = \{(t,s)|\ \ t \geq s \geq t_0\}$ to $\mathbb{R}$ such that $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for all $t > s$ and that

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)}, \quad \text{for all} \quad (t,s) \in D,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[q(s)H(t,s) - \frac{1}{4}h^2(t,s) \right] ds = \infty,$$

then (1.2) is oscillatory.

Later, using a generalized Riccati transformation

$$y(t) = \exp\left(-2\int_{t_0}^{t} f(s)ds \right) p(t) \left(\frac{x'(t)}{x(t)} + f(t) \right),$$

where $f \in C^1([t_0,\infty);\mathbb{R})$ is a given function, Li [22] gave some extensions to the results of Philos for (1.8). The results of Philos and Li have been extended to the nonlinear differential equation

$$\left[p(t)\phi(x(t))x'(t) \right]' + r(t)x'(t) + q(t)f(x(t)) = 0 \tag{1.11}$$

by Grace [8].

Recently, Kong [18] proved that (1.2) is oscillatory if for each $a \geq t_0$, there exists $n > 1$ such that

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} (s-a)^n q(s)ds > \frac{n^2}{4(n-1)} \tag{1.12}$$

and

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} (t-s)^n q(s)ds > \frac{n^2}{4(n-1)}. \tag{1.13}$$

One of the most considered differential equations of sublinear and superlinear types are the Emden-Fowler equation,

$$x''(t) + q(t)|x(t)|^{\gamma}\operatorname{sgn} x(t) = 0, \tag{1.14}$$

where $\gamma > 0$ is a constant, and the generalized Emden-Fowler equation

$$x''(t) + q(t)f(x(t)) = 0 \tag{1.15}$$

under appropriate assumptions on $f \in C(\mathbb{R}, \mathbb{R}) \bigcap C^1((-\infty, 0) \bigcup (0, \infty))$. We note that according to Definitions 3.2 and 3.3 given in Chapter 3, (1.14) becomes sublinear if $0 < \gamma < 1$ and superlinear if $\gamma > 1$. Similarly, (1.15) becomes sublinear and superlinear if f satisfies the conditions of Definitions 3.2 and 3.3, respectively. Since (1.15) are prototypes of sublinear and superlinear equations, let us mention several oscillation results obtained for them.

In [3], Butler proved that Wintner's criteria for (1.2) remains true for the superlinear equation (1.14). In the sublinear case, Kamanev [15] showed that (1.3) can be relaxed to

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau) d\tau ds = \limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (t - s) q(s) ds = \infty. \tag{1.16}$$

Willet [41] proved that (1.16) alone is not sufficient for the oscillation of (1.14) when $\gamma \geq 1$. In [47], Wong proved that in addition to (1.16), the condition

$$\liminf_{t \to \infty} \int_{t_0}^{t} q(s) ds = -\nu > -\infty, \tag{1.17}$$

where $\nu > 0$ implies that the superlinear equation (1.14) is oscillatory. Wong's result also shows that (1.2) is oscillatory.

In the 1960s, efforts were made to show that oscillation criteria for the linear equation and for the Emden-Fowler equation remain valid for the generalized equation (1.15).

Philos [31, 32] and Onose [38] extended the result of Wong [47] to the general case of the superlinear differential equation (1.15). It follows from Wintner's and

Hartman's criteria, (1.16) together with the condition

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds > -\infty \tag{1.18}$$

are sufficient to prove the oscillation of (1.2). By an application of the Cauchy-Schwarz inequality, we see that (1.16) implies the condition

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} [\int_{t_0}^{s} q(\tau)d\tau]^2 ds = \infty. \tag{1.19}$$

Philos and Purnaras [37] give an affirmative answer to the question as to whether conditions (1.18) and (1.19) are sufficient for the oscillation of (1.15). In place of assumption (1.18), they used the condition

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds > -\infty \tag{1.20}$$

for some integer $n \geq 2$ in the sublinear case, and the condition

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds > -\infty, \tag{1.21}$$

for some integer $n \geq 1$ in the superlinear case. Clearly, both (1.20) and (1.21) hold if (1.18) is satisfied. Philos and Purnaras [37] established the following oscillation criteria.

Theorem 1.3 *Suppose that $xf(x) > 0$ for $x \neq 0$ and*

$$\min\left\{ \inf_{x>0} f'(x) \int_{0+}^{x} \frac{du}{f(u)}, \quad \inf_{x<0} f'(x) \int_{0-}^{x} \frac{du}{f(u)} \right\} > 0. \tag{1.22}$$

The sublinear equation (1.15) *is oscillatory if* (1.19) *and* (1.20) *are satisfied.*

Theorem 1.4 *Suppose that $xf(x) > 0$ for $x \neq 0$ and*

$$\min\left\{\inf_{x>0} f'(x)\int_{0+}^{x}\frac{du}{f(u)},\quad \inf_{x<0} f'(x)\int_{0-}^{x}\frac{du}{f(u)}\right\} > 1. \tag{1.23}$$

The sublinear equation (1.15) *is oscillatory if* (1.19) *and* (1.21) *are satisfied.*

In this thesis we study the oscillatory behavior of nontrivial solutions of the nonlinear second order ordinary differential equations of the form

$$\frac{d}{dt}\Big[p(t)\phi(x(t))\psi(x'(t))\Big] + q(t)f(x(t)) = 0, \;\; t \geq t_0, \tag{1.24}$$

where p, ϕ, ψ, q and f are given.

We note that according to the types of the nonlinearities appearing in (1.24), the equation can be classified as half-linear, sublinear or superlinear. For example, when we take $\phi(s) = 1$, $\psi(s) = f(s) = |s|^{\alpha-1}s$, $\alpha > 1$, Eq.(1.24) reduces to a *half-linear* equation. In the case where $\psi(s) = |s|^{\alpha-1}s$, $\alpha > 1$, and ϕ and f satisfy the conditions of Definitions 3.2 and 3.3 given below, respectively, Eq. (1.24) is called a *sublinear* and a *superlinear* equation.

This thesis is organized as follows.

Section 1.2 includes some definitions and fundamental facts. Specifically, a most important lemma due to Hardy, Littlewood and Polya [13] used in proving the oscillation criteria is given.

In Chapter 2, first the integral averaging method and some oscillation criteria using this method are mentioned for the linear equations. Then using averaging method several oscillation and nonoscillation theorems of Manojlovic [28] for the half-linear equations in the form

$$\Big[p(t) \mid x'(t) \mid^{\alpha-1} x'(t)\Big]' + q(t) \mid x(t) \mid^{\alpha-1} x(t) = 0,\; t \geq t_0,$$

where $p \in C^1([t_0,\infty);(0,\infty))$, $q \in C([t_0,\infty);\mathbb{R})$, and $\alpha > 0$ is a constant, are presented.

In Chapter 3, we discuss oscillation and nonoscillation criteria for second order nonlinear differential equations of sublinear and superlinear types in the form

$$[p(t)\phi(x(t))x'(t)]' + q(t)f(x(t)) = 0, \quad t \geq t_0 > 0,$$

where $p \in C^1([t_0,\infty);(0,\infty))$, $q \in C([t_0,\infty);\mathbb{R})$ is allowed the change sign on $[t_0,\infty)$, ϕ and $f \in C^1(\mathbb{R},\mathbb{R})$ such that $\phi(x) > 0$, $xf(x) > 0$, $f'(x) > 0$ for $x \neq 0$. These criteria are obtained by using integral averaging method.

Chapter 4 deals with some oscillation criteria for the nonlinear equation

$$\left[p(t)\phi(x(t)) \mid x'(t) \mid^{\alpha-2} x'(t)\right]' + q(t) \mid x(t) \mid^{\alpha-2} x(t) = 0, \; t \geq t_0,$$

where $p \in C^1([t_0,\infty);(0,\infty))$, $q \in C([t_0,\infty);\mathbb{R})$, $\alpha > 1$ is a constant, and ϕ is a continuous real valued function such that $0 < \phi(x) \leq \gamma$ for all $x \in \mathbb{R}$ and for some γ. The method of proof used in this chapter is the integral averaging method as for the previous chapters.

1.2 Definitions and fundamental facts

In this section we shall give some definitions and general facts which will be used.

Definition 1.5 *Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.*

Definition 1.6 *A second order differential equation is said to be half-linear if any constant multiple of a solution is also a solution.*

Young's inequality. Let a and b positive qualities and let p and q be positive real numbers satisfying the condition $\frac{1}{p} + \frac{1}{q} = 1$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Cauchy-Schwarz's inequality. The inequality

$$\int_a^b |u(x)v(x)|dx \leq \left[\int_a^b |u(x)|^2 dx\right]^{1/2} \left[\int_a^b |v(x)|^2 dx\right]^{1/2}$$

holds provided that the integral on the right exist.

Hölder's inequality. If p and q are positive real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b |u(x)v(x)|dx \leq \left[\int_a^b |u(x)|^p dx\right]^{1/p} \left[\int_a^b |v(x)|^q dx\right]^{1/q}$$

provided that the two integrals on the right exist.

The following well-known inequality which is due to Hardy, Littlewood and Polya [13] is most useful in proving the oscillation theorems.

Lemma 1.7 *If X and Y are nonnegative constants, then*

$$X^q + (q-1)Y^q - qXY^{q-1} \geq 0, \quad q > 1,$$

where equality holds if and only if $X = Y$.

CHAPTER 2

HALF LINEAR DIFFERENTIAL EQUATIONS

2.1 Introduction

In this chapter we shall discuss the oscillation behavior for second-order half-linear differential equations of the form

$$\left[p(t) \mid x'(t) \mid^{\alpha-1} x'(t)\right]' + q(t) \mid x(t) \mid^{\alpha-1} x(t) = 0, \tag{2.1}$$

where $p \in C^1([t_0, \infty); (0, \infty))$, $q \in C([t_0, \infty); \mathbb{R})$, and $\alpha > 0$ is a constant.

Recently, the study of half-linear differential equations has become an important area of research (see Agarwal et. al[1] and the references quoted therein). This is largely due to the fact that half-linear differential equations occur in a variety of real world problems such as in the study of p-Laplace equations, non-Newtonian fluid theory and the turbulent flow of a polytrophic gas in a porous medium.

2.2 Oscillation criteria for half linear equations

In this section, we shall employ the integral averaging method and the generalized Riccati technique as well as Lemma 1.7 to obtain oscillation criteria for (2.1). Here the results presented are extensions of Grace [8], Li [22] and Philos [36] to Eq.(2.1).

Definition 2.1 *By a solution of (2.1) on an interval* $[t_0, \infty)$ *we mean a nontrivial continuously differentiable function* $x(t)$ *defined on* $[t_0, \infty)$ *with* $p(t)|x'(t)|^{\alpha-1}x(t)$ *in* $C^1([t_0, \infty))$ *such that* $x(t)$ *satisfies (2.1).*

The main results are the following.

Theorem 2.2 *Let* $H : D = \{(t, s)| \quad t \geq s \geq t_0\} \to \mathbb{R}$ *be a continuous function such that*

$$H(t, t) = 0, \quad t \geq t_0, \quad H(t, s) > 0, \quad t > s \tag{2.2}$$

and

$$h(t, s) = -\frac{\partial H(t, s)}{\partial s} \tag{2.3}$$

is nonnegative continuous function for $(t, s) \in D$. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[q(s)H(t, s) - \frac{p(s)h^{\alpha+1}(t, s)}{(\alpha + 1)^{\alpha+1}H^{\alpha}(t, s)} \right] ds = \infty, \tag{2.4}$$

then Eq.(2.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(2.1). Assume that $x(t) \neq 0$, for $t \geq t_0$. Then defining

$$w(t) = \frac{p(t)|x'|^{\alpha-1}x'}{|x(t)|^{\alpha-1}x(t)}, \quad \text{for} \quad t \geq t_0, \tag{2.5}$$

and differentiating this equation we have

$$w'(t) = -q(t) - \alpha \frac{|w(t)|^{(\alpha+1/\alpha)}}{(p(t))^{1/\alpha}}. \tag{2.6}$$

Multiplying (2.6) by $H(t,s)$ integrating over the interval $[t_0, t]$ we get

$$\int_{t_0}^{t} w'(s)H(t,s)ds = -\int_{t_0}^{t} q(s)H(t,s)ds - \alpha \int_{t_0}^{t} H(t,s)\frac{|w(s)|^{(\alpha+1/\alpha)}}{(p(s))^{1/\alpha}}ds. \quad (2.7)$$

By integration by parts and (2.2) we have

$$\int_{t_0}^{t} w'(s)H(t,s)ds = -w(t_0)H(t,t_0) - \int_{t_0}^{t} w(s)\frac{\partial H(t,s)}{\partial(s)}ds. \quad (2.8)$$

Thus (2.3), (2.7) and (2.8) give

$$\begin{aligned}\int_{t_0}^{t} q(s)H(t,s)ds &\le w(t_0)H(t,t_0) + \int_{t_0}^{t} |w(s)|h(t,s)ds \quad (2.9)\\ &\quad - \alpha \int_{t_0}^{t} H(t,s)\frac{|w(s)|^{(\alpha+1/\alpha)}}{(p(s))^{1/\alpha}}ds.\end{aligned}$$

Now applying Lemma 1.7 with

$$X = (\alpha H(t,s))^{(\alpha/\alpha+1)}\frac{|w(s)|}{(p(s))^{(1/\alpha+1)}},$$
$$Y = \frac{\alpha^{\alpha/\alpha+1}}{(\alpha+1)^{\alpha}}\frac{(p(s))^{(\alpha/\alpha+1)}h^{\alpha}(t,s)}{H(t,s)^{(\alpha^2/\alpha+1)}},$$

and $q = (\alpha+1)/\alpha$ we obtain

$$|w(s)|h(t,s) - \alpha H(t,s)\frac{|w(s)|^{(\alpha+1/\alpha)}}{(p(s))^{1/\alpha}} \le \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)}ds.$$

Hence, (2.9) implies

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} q(s)H(t,s)ds \le w(t_0) + \frac{1}{\lambda H(t,t_0)}\int_{t_0}^{t} p(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}ds \quad (2.10)$$

for $t \geq t_0$, where $\lambda = (\alpha+1)^{\alpha+1}$. Consequently, a simple calculation implies

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[q(s)H(t,s) - \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \right] ds \leq w(t_0), \quad t \geq t_0.$$

Taking the upper limit as $t \to \infty$, we see that

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[q(s)H(t,s) - \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \right] ds < \infty, \ \forall\, t \geq t_0,$$

which contradicts (2.4) and hence the proof is completed.

Remark 2.3 *Taking $H(t,s) = (t-s)^{\lambda}$ for some constant $\lambda > 1$, which obviously satisfies the conditions of Theorem 2.2, Theorem 2.2 reduces to the oscillation criteria of Li and Yeh [25].*

Corollary 2.4 *If condition (2.4) in Theorem 2.2 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{p(s)h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)} ds < \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s)ds = \infty,$$

then Eq.(2.1) is oscillatory.

As an application of Theorem 2.2, we consider the following example.

Example 2.5 *Consider the differential equation*

$$(t^{-\nu}|x'(t)|^{\alpha-1}x'(t))' + t^{\lambda}\left(\lambda\frac{2-\cos t}{t} + \sin t\right)|x(t)|^{\alpha-1}x(t) = 0, \tag{2.11}$$

for $t \geq t_0$, where ν, λ, α are arbitrary positive constants and $\alpha \neq 2$.

Then, for any $t \geq t_0$, we have

$$\int_{t_0}^{t} q(s)ds = \int_{t_0}^{t} d[s^{\nu}(2-\cos s)] = t^{\nu}(2-\cos s) - t_0^{\nu}(2+\cos t_0)$$
$$= t^{\nu}(2-\cos s) - k_0 \geq t^{\nu} - k_0.$$

Taking $H(t,s) = (t-s)^2$ for $t \geq s \geq t_0$ and denoting by $\mu = (\alpha+1)^{\alpha+1}$, we have

$$\begin{aligned}
&\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left[(t-s)^2 q(s) - \frac{(t-s)^{1-\alpha}}{\mu s^{\nu}}\right]ds \\
&\qquad = \frac{1}{t^2}\int_{t_0}^{t}\left[2(t-s)\left(\int_{t_0}^{s} q(u)du\right) - \frac{(t-s)^{1-\alpha}}{\mu s^{\nu}}\right]ds \\
&\qquad \geq \frac{2}{t^2}\int_{t_0}^{t}(t-s)(s^{\lambda}-k_0)ds - \frac{1}{\mu t_0^{\nu} t^2}\int_{t_0}^{t}(t-s)^{1-\alpha}ds \\
&\qquad = \frac{2t^{\lambda}}{(\lambda+1)(\lambda+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0 - \frac{k_3}{t^{\alpha}}\left(1-\frac{t_0}{t}\right)^{2-\alpha},
\end{aligned}$$

where $k_1 = \frac{2t_0^{\lambda+2}}{\lambda+2} - k_0 t_0^2$, $k_2 = 2k_0 t_0 - \frac{2t_0^{\lambda+1}}{\lambda+1}$, $k_3 = \frac{1}{t_0^{\nu}\mu(2-\alpha)}$. It is not difficult to see that the limsup of the above expression goes to infinity. Thus, condition (2.4) in Theorem 2.2 is satisfied and hence Eq.(2.1) is oscillatory .

Theorem 2.6 *Let the functions H and h be defined as in Theorem 2.2 such that (2.2), (2.3),*

$$0 < \inf_{s \geq t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty \tag{2.12}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} p(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}ds < \infty, \tag{2.13}$$

are satisfied. If there exists a continuous function φ on $[t_0,\infty)$ such that for every $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_{T}^{t}\left[q(s)H(t,s) - \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)}\right]ds \geq \varphi(T) \tag{2.14}$$

and

$$\int_{t_0}^{\infty} \frac{\varphi_+^{(\alpha/\alpha+1)}(s)}{(p(s))^{1/\alpha}} ds = \infty, \tag{2.15}$$

where $\varphi_+(s) = max\{\varphi(s), 0\}$*, then Eq.(2.1) is oscillatory.*

Proof. We suppose that there exists a solution $x(t)$ of Eq.(2.1) such that $x(t) \neq 0$ for $t \geq t_0$. Defining the function w as in the proof of Theorem 2.2, we get (2.9) and (2.10). Then, for $t > T \geq t_0$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[q(s)H(t,s) - \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \right] ds \leq w(T).$$

Therefore, by (2.14) we have

$$\varphi(T) \leq w(T), \quad \text{for every} \quad T \geq t_0 \tag{2.16}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t q(s)H(t,s) \geq \varphi(t_0). \tag{2.17}$$

Let us define the functions

$$F(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t |w(s)| h(t,s) ds$$

and

$$G(t) = \frac{\alpha}{H(t,t_0)} \int_{t_0}^t H(t,s) \frac{|w(s)|^{\alpha+1/\alpha}}{(p(s))^{1/\alpha}} ds.$$

Then, by (2.9) and (2.17), it easy to see that

$$\begin{aligned} \liminf_{t\to\infty} [G(t) - F(t)] &\leq w(t_0) - \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t q(s)H(t,s) ds \\ &\leq w(t_0) - \varphi(t_0) < \infty. \end{aligned} \tag{2.18}$$

We shall next prove that

$$\int_{t_0}^{\infty} \frac{|w(s)|^{(\alpha/\alpha+1)}}{(p(s))^{1/\alpha}} ds < \infty. \tag{2.19}$$

If we suppose that (2.19) fails, there exists a $t_1 > t_0$ such that

$$\int_{t_0}^{t} \frac{|w(s)|^{(\alpha/\alpha+1)}}{(p(s))^{1/\alpha}} ds \geq \frac{\mu}{\alpha\xi}, \tag{2.20}$$

where μ is an arbitrarily positive number and ξ is a positive constant such that

$$\inf_{s \geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] > \xi > 0. \tag{2.21}$$

Using integration by parts and (2.20), we have

$$\begin{aligned}
G(t) &= \frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} H(t,s) d\left(\int_{t_0}^{s} \frac{|w(\tau)|^{(\alpha+1/\alpha)}}{(p(\tau))^{1/\alpha}} d\tau \right) \\
&= -\frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} \frac{\partial H}{\partial s}(t,s) \left(\int_{t_0}^{s} \frac{|w(\tau)|^{(\alpha+1/\alpha)}}{(p(\tau))^{(1/\alpha)}} d\tau \right) ds \\
&\geq -\frac{\alpha}{H(t,t_0)} \int_{t_1}^{t} \frac{\partial H}{\partial s} H_s(t,s) \left(\int_{t_0}^{s} \frac{|w(\tau)|^{(\alpha+1/\alpha)}}{(p(\tau))^{(1/\alpha)}} d\tau \right) ds \\
&\geq -\frac{\mu}{\xi H(t,t_0)} \int_{t_1}^{t} \frac{\partial H}{\partial s} H_s(t,s) ds = \frac{\mu H(t,t_1)}{\xi H(t,t_0)},
\end{aligned}$$

for all $t \geq t_1$. Also, by (2.21), there is $t_2 \geq t_1$ such that $\frac{H(t,t_1)}{H(t,t_0)} \geq \xi$ for all $t \geq t_2$, and so $G(t) \geq \mu$ for all $t \geq t_2$. Since μ is arbitrary,

$$\lim_{t\to\infty} G(t) = \infty. \tag{2.22}$$

Further, consider a sequence $\{T_n\}_{n=1}^{\infty}$ in (t_0, ∞) such that $\lim_{n\to\infty} T_n = \infty$ and

$$\lim_{n\to\infty} [G(T_n) - F(T_n)] = \liminf_{t\to\infty} [G(t) - F(t)] < \infty.$$

Then, there exists a constant M such that

$$G(T_n) - F(T_n) \leq M, \tag{2.23}$$

for all sufficiently large n. Since (2.22) ensures that

$$\lim_{n\to\infty} G(T_n) = \infty, \tag{2.24}$$

(2.23) implies

$$\lim_{n\to\infty} F(T_n) = \infty. \tag{2.25}$$

By taking into account (2.24), from (2.23) we derive for n sufficiently large

$$\frac{F(T_n)}{G(T_n)} - 1 \geq -\frac{M}{G(T_n)} > -\frac{1}{2}.$$

Therefore,

$$\frac{F(T_n)}{G(T_n)} > \frac{1}{2}, \quad \text{for all large n},$$

which together with (2.25) implies

$$\lim_{n\to\infty} \frac{F^{\alpha+1}(T_n)}{G^{\alpha}(T_n)} = \infty. \tag{2.26}$$

On the other hand, by Hölder's inequality, we have for every $n \in N$

$$\begin{aligned}
F(T_n) &= \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n}|w(s)|h(T_n,s)ds \\
&= \int_{t_0}^{T_n}\left(\frac{\alpha^{(\alpha/\alpha+1)}}{H^{(\alpha/\alpha+1)}(T_n,t_0)}\frac{|w(s)|H^{(\alpha/\alpha+1)}(T_n,s)}{(p(s))^{(1/\alpha+1)}}\right) \\
&\times \left(\frac{\alpha^{-(\alpha/\alpha+1)}}{H^{(1/\alpha+1)}(T_n,t_0)}\frac{h(T_n,s)(p(s))^{(1/\alpha+1)}}{H^{(\alpha/\alpha+1)}(T_n,s)}\right)ds \\
&\leq \left(\frac{\alpha}{H(T_n,t_0)}\int_{t_0}^{T_n}\frac{|w(s)|^{(\alpha+1/\alpha)}H(T_n,s)}{(p(s))^{1/\alpha}}ds\right)^{\alpha/\alpha+1} \\
&\times \left(\frac{1}{\alpha^{\alpha}H(T_n,t_0)}\int_{t_0}^{T_n}p(s)\frac{h^{\alpha+1}(T_n,s)}{H^{\alpha}(T_n,s)}ds\right)^{1/\alpha+1},
\end{aligned}$$

and so

$$\frac{F^{\alpha+1}(T_n)}{G^{\alpha}(T_n)} \leq \frac{1}{\alpha^{\alpha}H(T_n,t_0)}\int_{t_0}^{T_n}p(s)\frac{h^{\alpha+1}(T_n,s)}{H^{\alpha}(T_n,s)}ds.$$

Thus, because of (2.26), we have

$$\lim_{n\to\infty}\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n}p(s)\frac{h^{\alpha+1}(T_n,s)}{H^{\alpha}(T_n,s)}ds = \infty,$$

which gives

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}p(s)\frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)}ds = \infty,$$

contradicting (2.13). Therefore, (2.19) holds. Now, from (2.16) we obtain

$$\int_{t_0}^{\infty}\frac{\varphi_+^{(\alpha/\alpha+1)}(s)}{(p(s))^{1/\alpha}}ds \leq \int_{t_0}^{\infty}\frac{|w(s)|^{\alpha/\alpha+1}}{(p(s))^{1/\alpha}}ds < \infty,$$

which contradicts (2.15). This completes the proof.

Theorem 2.7 *Let the functions* H *and* h *be defined as in Theorem 2.2 such that (2.2), (2.3), (2.12) and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} |q(s)| H(t,s) ds < \infty, \tag{2.27}$$

are satisfied. If there exists a continuous function φ *on* $[t_0, \infty)$ *such that for every* $T \geq t_0$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[q(s)H(t,s) - \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \right] ds \geq \varphi(T), \tag{2.28}$$

and Eq.(2.15) holds, then Eq.(2.1) is oscillatory.

Proof. For the nonoscillatory solution $x(t)$ of (2.1) as in the proof of Theorem 2.2, (2.9) and (2.10) are satisfied. As in the proof of Theorem 2.6, (2.16) holds for $t > T \geq t_0$. Using (2.27), we see that

$$\limsup_{t\to\infty} [G(T) - F(T)] \leq w(t_0) - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)H(t,s)ds < \infty.$$

It follows from (2.28) that

$$\varphi(t_0) \leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \left[\int_{t_0}^{t} q(s)H(t,s)ds - \int_{t_0}^{t} \frac{p(s)h^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} ds \right] < \infty.$$

Thus, (2.27) implies

$$\liminf_{t\to\infty} \frac{1}{(\alpha+1)^{\alpha+1}H(t,t_0)} \int_{t_0}^{t} p(s) \frac{h^{\alpha+1}(t,s)}{H^{\alpha}(t,s)} < \infty.$$

Considering a sequence $\{T_n\}_{n=1}^{\infty}$ in (t_0, ∞) with $\lim\limits_{n\to\infty} T_n = \infty$ such that

$$\lim_{n\to\infty} [G(T_n) - F(T_n)] = \limsup_{t\to\infty} [G(T) - F(T)].$$

Then, using the procedure of the proof of Theorem 2.6, we conclude that (2.19) is satisfied. The remainder of the proof proceeds as in the proof of Theorem 2.6. We observe that Theorem 2.6 can be applied in some cases in which Theorem 2.2 is not applicable. Such a case is described in the following example.

Example 2.8 *Consider the differential equation*

$$\Big(t^\nu |x'(t)|^{\alpha-1}x'(t)\Big)' + t^\alpha \cos t |x(t)|^{\alpha-1}x(t) = 0, \tag{2.29}$$

for $t \geq t_0$, where ν, λ, α are constants such that $-1 < \lambda \leq 1$, $\nu < \alpha$, $\alpha \neq 2$, and $\alpha^2\lambda \geq (\alpha+1)(\nu-\alpha)$. Taking $H(t,s) = (t-s)^2$, $t \geq s \geq t_0$, we have

$$\frac{1}{(t-t_0)^2}\int_{t_0}^{t} s^\nu (t-s)^{1-\alpha} ds \leq \begin{cases} \frac{t^{\nu-\alpha}}{2-\alpha}\left(\frac{t-t_0}{t}\right)^{2-\alpha}, & \text{if } \nu > 0, \\ \frac{t_0^{\nu-\alpha}}{2-\alpha}\left(\frac{t-t_0}{t}\right)^{2-\alpha} & \text{if } \nu < 0. \end{cases}$$

Therefore, condition (2.13) is satisfied and for arbitrary small constant $\varepsilon > 0$, there exists $t_1 \geq t_0$ such that for $T \geq t_1$

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[(t-s)^2 s^\lambda \cos s - s^\nu \frac{(t-s)^{1-\alpha}}{(1+\alpha)^{1+\alpha}}\right] ds \geq -T^\lambda \cos T - \varepsilon.$$

Now set $\varphi(T) = -T^\lambda \cos T - \varepsilon$. Then, there exists an integer N such that $(2N+1)\pi - \pi/4 > t_1$ and if $n \geq N$

$$(2n+1)\pi - \frac{\pi}{4} \leq T \leq (2n+1)\pi + \frac{\pi}{4}, \quad \varphi(T) \geq \delta T^\lambda,$$

where δ is a small constant. Taking into account that $\alpha^2\lambda \geq (\alpha+1)(\nu-\alpha)$, *we obtain*

$$\int_{t_0}^{\infty} \frac{\varphi_+^{(\alpha/\alpha+1)}(s)}{(p(s))^{1/\alpha}} ds \geq \sum_{n=N}^{\infty} \delta^{\alpha/\alpha+1} \int_{(2n+1)\pi-\pi/4}^{(2n+1)\pi+\pi/4} s^{(\alpha\lambda/\alpha+1)-\nu/\alpha} ds$$
$$\geq \sum_{n=N}^{\infty} \delta^{\alpha/\alpha+1} \int_{(2n+1)\pi-\pi/4}^{(2n+1)\pi+\pi/4} \frac{ds}{s} = \infty.$$

Accordingly, all conditions of Theorem 2.2 are satisfied and hence, Eq.(2.29) is oscillatory.

Theorem 2.9 *Suppose that the functions H and h are defined as in Theorem 2.2 such that (2.2) and (2.12) hold. If there exists a positive, nondecreasing function $\rho \in C^1([t_0, \infty))$ and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[q(s)\rho(s)H(t,s) - \frac{p(s)\rho(s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \times \left(h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \right)^{\alpha+1} \Big] ds = \infty, \tag{2.30}$$

then Eq.(2.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (2.1). Without loss of generality, we assume that $x(t) \neq 0$ for $t \geq t_0$. Now we define

$$w(t) = \rho(t) \frac{p(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)}, \quad \text{for all } \ t \geq t_0.$$

Then, for every $s \geq t_0$, we obtain

$$w'(s) = -q(t)\rho(t) + \frac{\rho'(s)}{\rho(s)} w(s) - \alpha \frac{|w(s)|^{\alpha+1/\alpha}}{(p(s)\rho(s))^{1/\alpha}}. \tag{2.31}$$

If we multiply (2.31) by $H(t,s)$ for $t \geq s \geq t_0$, integrate from t_0 to t and use (2.8), we get

$$\int_{t_0}^{t} w'(s)H(t,s)ds = -\int_{t_0}^{t} q(s)\rho(s)H(t,s)ds + \int_{t_0}^{t} \frac{\rho'(s)}{\rho(s)} w(s)H(t,s)ds - \alpha \int_{t_0}^{t} H(t,s)\frac{|w(s)|^{\alpha+1/\alpha}}{(p(s)\rho(s))^{1/\alpha}} ds.$$

Using (2.8), we have

$$\begin{aligned}\int_{t_0}^{t} q(s)\rho(s)H(t,s)ds &\leq w(t_0)H(t,t_0) + \int_{t_0}^{t}\left(h(t,s) + \frac{\rho'(s)}{\rho(s)}H(t,s)\right)|w(s)|ds \\ &- \alpha \int_{t_0}^{t} H(t,s)\frac{|w(s)|^{\alpha+1/\alpha}}{(p(s)\rho(s))^{1/\alpha}} ds. \qquad (2.32)\end{aligned}$$

Furthermore, an application of Lemma 1.7 with

$$X = (\alpha H(t,s))^{\alpha/\alpha+1}\frac{|w(s)|}{(p(s)\rho(s))^{1/\alpha+1}},$$
$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\left(\frac{p(s)\rho(s)}{(\alpha H(t,s))^{\alpha}}\right)^{\alpha/(\alpha+1)}\left(h(t,s) + \frac{\rho'(s)}{\rho(s)}H(t,s)\right)^{\alpha},$$

and $q = (\alpha+1)/\alpha$ yields

$$\begin{aligned}&|w(s)|\left(h(t,s) + \frac{\rho'(s)}{\rho(s)}H(t,s)\right) - \alpha H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(p(s)\rho(s))^{1/\alpha}} \\ &\leq \frac{p(s)\rho(s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)}\left(h(t,s) + \frac{\rho'(s)}{\rho(s)}H(t,s)\right)^{\alpha+1}. \qquad (2.33)\end{aligned}$$

Thus, from (2.32) and (2.33) we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[q(s)\rho(s)H(t,s) - \frac{p(s)\rho(s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \times \left(h(t,s) + \frac{\rho'(s)}{\rho(s)}H(t,s)\right)^{\alpha+1}\right]ds \leq w(t_0) < \infty$$

which contradicts (2.30).

Corollary 2.10 *Let (2.30) in Theorem 2.9 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{p(s)\rho(s)}{H^{\alpha}(t,s)} \Big(h(t,s) + \frac{\rho'(s)}{\rho(s)} H(t,s) \Big)^{\alpha+1} ds < \infty, \tag{2.34}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} q(s)\rho(s)H(t,s)ds = \infty, \tag{2.35}$$

then the conclusion of Theorem 2.9 holds.

Example 2.11 *Consider the differential equation*

$$\Big(t^{\nu}|x'(t)|^{\alpha-1}x'(t) \Big)' + [\lambda t^{\lambda-3}(2-\cos t) + t^{\lambda-2}\sin t]|x(t)|^{\alpha-1}x(t) = 0, \tag{2.36}$$

for $t \geq t_0 > 0$, where λ is arbitrary positive constant and ν, α are constants such that $\nu < \alpha - 2$, $\alpha < 1$. Here, we choose $\rho(t) = t^2$ and $H(t,s) = (t-s)^2$ for $t \geq s \geq t_0$. Then, since $\rho(t)q(t) = \frac{d}{ds}[s^{\lambda}(2-\cos s)]$, as in Example (2.5), we get

$$\int_{t_0}^{t} \rho(s)q(s)ds \geq t^{\lambda} - k_0,$$

and therefore

$$\frac{1}{(t-t_0)^2} \int_{t_0}^{t} (t-s)^2\rho(s)q(s)ds \geq \frac{2t^{\lambda}}{(\lambda+1)(\lambda+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0,$$

where $k_1 = \frac{2t_0^{\lambda+2}}{\lambda+2} - k_0t_0^2$, $k_2 = 2k_0t_0 - \frac{2t_0^{\lambda+1}}{\lambda+1}$. Since $\lambda > 0$, the upper limit of the left-hand side of the last inequality tends to infinity, (2.35) is satisfied. On the other hand,

$$\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\frac{s^{\nu+2}}{(t-s)^{2\alpha}}\left(2(t-s)+\frac{2}{s}(t-s)^2\right)^{\alpha+1}ds = t^{\alpha-1}2^{\alpha+1}\int_{t_0}^{t}s^{\nu-\alpha+1}(t-s)^{1-\alpha}ds$$
$$\leq 2^{\alpha+1}\left(1-\frac{t_0}{t}\right)^{1-\alpha}\frac{t^{\nu-\alpha+2}-t_0^{\nu-\alpha+2}}{\nu-\alpha+2},$$

so that (2.34) is also satisfied. Consequently, Corollary 2.10 implies that Eq.(2.36) is oscillatory.

Following the procedure of Theorems 2.6 and 2.7, we can also prove the following two theorems.

Theorem 2.12 *Let the functions H, h and ρ be defined as in Theorem 2.2 such that (2.2), (2.3), (2.12), and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{p(s)\rho(s)}{H^\alpha(t,s)}\left(h(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s)\right)^{\alpha+1}ds<\infty$$

are satisfied. If there exists a continuous function φ on $[t_0,\infty)$ such that for every $T\geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[q(s)\rho(s)H(t,s)-\frac{p(s)\rho(s)}{(\alpha+1)^{\alpha+1}H^\alpha(t,s)}\times\left(h(t,s)\frac{\rho'(s)}{\rho(s)}H(t,s)\right)^{\alpha+1}\right]ds\geq\varphi(T),$$

and (2.15) is satisfied, then Eq.(2.1) is oscillatory.

Theorem 2.13 *Let the functions H, h and ρ be defined as in Theorem 2.2 such that (2.2), (2.3), (2.12), and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}|q(s)|\rho(s)H(t,s)ds<\infty$$

are satisfied. If there exists a continuous function φ on $[t_0, \infty)$ such that for every $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[q(s)\rho(s)H(t,s) - \frac{p(s)\rho(s)}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \right. \\ \left. \times \left(h(t,s)\frac{\rho'(s)}{\rho(s)}H(t,s) \right)^{\alpha+1} \right] ds \geq \varphi(T),$$

and (2.15) is satisfied, then (2.1) is oscillatory.

CHAPTER 3

SUBLINEAR AND SUPERLINEAR DIFFERENTIAL EQUATIONS

3.1 Introduction

In this chapter, using a general class of the parameter functions $H(t, s)$ in the averaging technique, some oscillation criteria for the second-order sublinear and superlinear differential equations in the form

$$[p(t)\phi(x(t))x'(t)]' + q(t)f(x(t)) = 0, \quad t \geq t_0 > 0, \tag{3.1}$$

where $p \in C^1([t_0, \infty); (0, \infty))$, $q \in C([t_0, \infty); \mathbb{R})$ is allowed the change sign on $[t_0, \infty)$, ϕ and $f \in C^1(\mathbb{R}, \mathbb{R})$ such that $\phi(x) > 0$, $xf(x) > 0$, $f'(x) > 0$ for $x \neq 0$ are established.

The results obtained here are extension and improvement of Theorems 1.3 and 1.4 given in Chapter 1 due to Philos and Purnaras [37].

In order to simplify notation, we set the following definitions.

Definition 3.1 *By a solution of (3.1) on an interval* $[t_0, \infty)$ *we mean a nontrivial continuously differentiable function* $x(t)$ *defined on* $[t_0, \infty)$ *with* $p(t)\phi(x(t))x'(t)$ *in* $C^1([t_0, \infty))$ *such that* $x(t)$ *satisfies (3.1).*

Definition 3.2 *Equation (3.1) called strongly sublinear in the sense that*

$$\lim_{\varepsilon \to 0^+} \int_0^{\varepsilon} \frac{\phi(u)}{f(u)} du \ \ and \ \lim_{\varepsilon \to 0^-} \int_0^{-\varepsilon} \frac{\phi(u)}{f(u)} du < \infty \quad for \quad all \quad \varepsilon > 0.$$

Definition 3.3 *Equation (3.1) is called strongly superlinear in the sense that*

$$0 < \int_{\varepsilon}^{\infty} \frac{\phi(u)}{f(u)} du \ \textit{and} \ \int_{-\varepsilon}^{\infty} \frac{\phi(u)}{f(u)} du < \infty \quad for \quad all \quad \varepsilon > 0.$$

Definition 3.4 *Let $H : \mathcal{D} = \{(t,s) : t \geq s \geq t_0\} \longrightarrow \mathbb{R}$ be a continuous and have continuous partial derivative on D with respect to s satisfying*

$$H(t,t) = 0, \quad for \quad t \geq t_0, \quad H(t,s) > 0 \quad for \quad t > s \geq t_0. \tag{3.2}$$

(i) If $H(t,s)$ satisfies

$$h(t,s) = -\frac{\partial H(t,s)}{\partial s} \geq 0, \quad for \quad (t,s) \in \mathcal{D}, \tag{3.3}$$

$$\liminf_{t \to \infty} \frac{\frac{\partial H(t,s)}{\partial s}}{H(t,s)} > -\infty, \quad for \quad s \geq t_0, \tag{3.4}$$

and

$$0 < \liminf_{s \geq t_0} \left[\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty, \tag{3.5}$$

we shall say that it has a property **H** *on $\mathcal{D}$. We denote a class of functions $H(t,s)$ with property* **H** *on $\mathcal{D}$ by $\mathcal{H}(\mathcal{D})$.*

(ii) If $H(t,s)$ satisfies (3.4) and (3.5),

$$H(t,t) = 0, \quad for \quad t \geq t_0, \quad H(t,s) \leq 0, \quad for \quad (t,s) \in \mathcal{D}, \tag{3.6}$$

and

$$\frac{\partial}{\partial s}\left(p(s)\frac{\partial H}{\partial s}(t,s)\right) \geq 0, \quad for \quad (t,s) \in \mathcal{D}, \tag{3.7}$$

we shall say that it has a property $\mathbf{H}^*_{\mathbf{a}}$ *on* $\mathcal{D}$*. We denote a class of functions* $H(t,s)$ *with property* $\mathbf{H}^*_{\mathbf{a}}$ *on* $\mathcal{D}$ *by* $\mathbf{H}^*_{\mathbf{a}}(\mathcal{D})$*.*

Here, we also impose the following additional assumptions on the function f and ϕ in both sublinear and superlinear cases. Namely, in the sublinear case, suppose that

$$\min\left\{\inf_{x>0}\frac{f'(x)}{\phi(x)}\int_{0+}^{x}\frac{\phi(u)}{f(u)}du, \quad \inf_{x<0}\frac{f'(x)}{\phi(x)}\int_{0-}^{-x}\frac{\phi(u)}{f(u)}du\right\} > 0, \tag{3.8}$$

and define

$$\Phi(x) = \int_{0+}^{x}\frac{\phi(u)}{f(u)}du, \quad \text{for} \quad x>0, \quad \Phi(x) = \int_{0-}^{-x}\frac{\phi(u)}{f(u)}du, \quad \text{for} \quad x<0,$$

while in the superlinear case suppose that

$$L_{f,\phi} = \min\left\{\inf_{x>0}\frac{f'(x)}{\phi(x)}\int_{x}^{\infty}\frac{\phi(u)}{f(u)}du, \quad \inf_{x<0}\frac{f'(x)}{\phi(x)}\int_{-x}^{-\infty}\frac{\phi(u)}{f(u)}du\right\} > 1 \tag{3.9}$$

and define

$$\Psi(x) = \int_{x}^{\infty}\frac{\phi(u)}{f(u)}du, \quad \text{for} \quad x>0, \quad \Psi(x) = \int_{-x}^{-\infty}\frac{\phi(u)}{f(u)}du, \quad \text{for} \quad x<0.$$

Obviously, for $\phi(x) \equiv 1$, (3.8) and (3.9) reduce to (1.22) and (1.23) of Theorems 1.3 and 1.4, respectively.

Before proceeding, let us introduce the following lemmas which will be used in the proof of our theorems. For that purpose, we consider an arbitrary nonoscillatory solution $x(t)$ of (3.1). In other words, by definition $x(t) \neq 0$ for

$t \geq T_0$ for some $T_0 \geq t_0$. Let us define

$$w(t) = \Phi(x(t)), \tag{3.10}$$

in the sublinear case and

$$w(t) = \Psi(x(t)), \tag{3.11}$$

in the superlinear case for every $t \in [T_0, \infty)$.

Lemma 3.5 *If we suppose that for arbitrary nonoscillatory solution $x(t)$,*

$$\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds = \infty, \tag{3.12}$$

then for the function $H(t,s)$ which satisfies Eqs.(3.2) and (3.5) and has a continuous, nonpositive partial derivative on D with respect to the second order variable, it follows that

$$\limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds = \infty. \tag{3.13}$$

Proof. According to Eq.(3.5) there exist a constant ρ such that

$$\inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] > \rho > 0, \tag{3.14}$$

so that by (3.12), there exists $T_1 \geq T_0$ such that

$$\int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds \geq \frac{\mu}{\rho}, \quad \text{for every} \quad t \geq T_1,$$

where μ is an arbitrary constant. Then we have that

$$\begin{aligned}
&\frac{1}{H(t,T_0)}\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \\
&\qquad = \frac{1}{H(t,T_0)}\int_{T_0}^{t} H(t,s)d\left(\int_{T_0}^{s}\frac{f'(x(\tau))}{\phi(x(\tau))}p(\tau)[w'(\tau)]^2 d\tau\right) \\
&\qquad = \frac{1}{H(t,T_0)}\int_{T_0}^{t}\left(-\frac{\partial H}{\partial s}(t,s)\right)\left(\int_{T_0}^{s}\frac{f'(x(\tau))}{\phi(x(\tau))}p(\tau)[w'(\tau)]^2 d\tau\right)ds \\
&\qquad \geq \frac{1}{H(t,T_0)}\int_{T_1}^{t}\left(-\frac{\partial H}{\partial s}(t,s)\right)\left(\int_{T_0}^{s}\frac{f'(x(\tau))}{\phi(x(\tau))}p(\tau)[w'(\tau)]^2 d\tau\right)ds \\
&\qquad \geq \frac{\mu}{\rho H(t,T_0)}\int_{T_1}^{t}\left(-\frac{\partial H}{\partial s}(t,s)\right)ds = \frac{\mu}{\rho}\frac{H(t,T_1)}{H(t,t_0)}.
\end{aligned}$$

Since (3.14) guarantees that

$$\liminf_{t\to\infty}\frac{H(t,T_1)}{H(t,t_0)} > \rho,$$

there exists $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \rho, \quad \text{for every} \quad t \geq T_2.$$

Consequently,

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \geq \mu, \quad \text{for every} \quad t \geq T_2$$

which shows that (3.13) holds.

Lemma 3.6 *If the following condition holds:*

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\frac{ds}{p(s)} < \infty, \tag{3.15}$$

and for an arbitrary nonoscillatory solution $x(t)$ of Eq.(3.1),

$$\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds < \infty \tag{3.16}$$

then it is

$$\lim_{t\to\infty} \frac{w(t)}{t} = 0. \tag{3.17}$$

Proof. First, we consider the sublinear equation. That is, the functions ϕ and f appearing in the equation satisfy the sublinearity condition. By Eq.(3.8), there exists a positive constant δ such that

$$\frac{f'(x(t))}{\phi(x(t))}\Phi(x(t)) = \frac{f'(x(t))}{\phi(x(t))} w(t) \geq \delta, \quad t \geq T_0. \tag{3.18}$$

Also, by (3.15), there exists a constant M such that

$$M = \sup_{t\geq T_0} \frac{1}{t} \int_{T_0}^{t} \frac{ds}{p(s)}$$

Let us consider an arbitrary number $\varepsilon > 0$. We can choose a $T_1 \geq T_0$, so that

$$\int_{T_1}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds \leq \frac{\varepsilon\delta}{4M}. \tag{3.19}$$

By using the Cauchy-Schwarz inequality, for $t \geq T_1$, we derive

$$\begin{aligned} w(t) - w(T_1) &\leq \left| \int_{T_1}^{t} w'(s) ds \right| \\ &\leq \left[\int_{T_1}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds \right]^{1/2} \left[\int_{T_1}^{t} \frac{\phi(x(s))}{f'(x(s))p(s)} ds \right]^{1/2} \end{aligned}$$

and thus by (3.19), we get

$$w(t) \leq w(T_1) + \frac{\sqrt{\varepsilon\delta}}{2\sqrt{M}} \left[\int_{T_1}^{t} \frac{\phi(x(s))}{f'(x(s))p(s)} ds\right]^{1/2} \quad \text{for} \quad t \geq T_1. \tag{3.20}$$

If we suppose that

$$\int_{T_1}^{\infty} \frac{\phi(x(s))}{f'(x(s))p(s)} ds < \infty,$$

then from (3.20), we conclude that w is bounded function on $[T_1, \infty)$, so that (3.17) is true. Thus, we restrict ourselves to the case where

$$\int_{T_1}^{\infty} \frac{\phi(x(s))}{f'(x(s))p(s)} ds = \infty.$$

In that case, there exists $T_2 > T_1$ such that

$$w(T_1) \leq \frac{\sqrt{\varepsilon\delta}}{2\sqrt{M}} \left[\int_{T_1}^{t} \frac{\phi(x(s))}{f'(x(s))p(s)} ds\right]^{1/2}, \quad \text{for} \quad t \geq T_2$$

so that from (3.18) and (3.20), it follows that for $t \geq T_2$,

$$w(t) \leq \frac{\sqrt{\varepsilon\delta}}{\sqrt{M}} \left[\int_{T_1}^{t} \frac{\phi(x(s))}{f'(x(s))p(s)} ds\right]^{1/2} \leq \frac{\sqrt{\varepsilon}}{\sqrt{M}} \left[\int_{T_1}^{t} \frac{w(s)}{p(s)} ds\right]^{1/2}. \tag{3.21}$$

Thus,

$$\frac{w(t)}{p(t)} \left[\int_{T_1}^{t} \frac{w(s)}{p(s)} ds\right]^{-1/2} \leq \frac{\sqrt{\varepsilon}}{\sqrt{M} p(t)}, \quad \text{for} \quad t \geq T_2.$$

An integration of the last inequality from T_2 to t with $t \geq T_2$, gives

$$2\left[\int_{T_1}^{t} \frac{w(s)}{p(s)} ds\right]^{1/2} - 2\left[\int_{T_1}^{T_2} \frac{w(s)}{p(s)} ds\right]^{1/2} \leq \frac{\sqrt{\varepsilon}}{\sqrt{M}} \int_{T_2}^{t} \frac{ds}{p(s)} \leq \sqrt{\varepsilon M} t.$$

By setting

$$T_3 = max\left\{T_2, \frac{2}{\sqrt{\varepsilon M}} \left[\int_{T_1}^{T_2} \frac{w(s)}{p(s)} ds\right]^{1/2}\right\},$$

we obtain for $t \geq T_3$,

$$\left[\int_{T_1}^{t} \frac{w(s)}{p(s)} ds\right]^{1/2} < \frac{\sqrt{\varepsilon M}}{2} t + \left[\int_{T_1}^{T_2} \frac{w(s)}{p(s)} ds\right]^{1/2} \leq \frac{\sqrt{\varepsilon M}}{2} t + \frac{\sqrt{\varepsilon M}}{2} T_3 \leq \sqrt{\varepsilon M} t$$

Then, from (3.21), it follows that

$$w(t) < \varepsilon t, \quad \text{for every} \quad t \geq T_3,$$

which, since $\varepsilon > 0$ is arbitrary, completes the proof for the sublinear equation. In the case of the superlinear equation, (3.9) ensures that

$$\frac{f'(x(t))}{\phi(x(t))} w(t) = \frac{f'(x(t))}{\phi(x(t))} \Psi(x(t)) \geq 1, \quad \text{for} \quad t \geq T_1. \tag{3.22}$$

Next, following the same procedure as for the sublinear equation, for $\delta = 1$, we can prove (3.17).

We are now ready state and prove oscillation criteria for the sublinear and superlinear equations.

3.2 Sublinear equations

In this section we present oscillation criteria for the sublinear equations.

The main result of this section is as follows.

Theorem 3.7 *The sublinear equation (3.1) is oscillatory if (3.15) and*

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \frac{1}{p(s)} \left[\int_{t_0}^{s} q(\tau) d\tau\right]^2 ds = \infty \tag{3.23}$$

hold for some function $H \in \mathbf{H_a}^*(D)$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s)ds > -\infty \tag{3.24}$$

is fulfilled.

Proof. Assume the contrary; then there exists a nonoscillatory solution $x(t)$ of Eq.(3.1) on $T_0 \geq t_0$ such that $x(t) \neq 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. We can easily show that the function $w(t)$ defined by (3.10) satisfies the equation

$$(p(t)w'(t))' = -q(t) - \frac{f'(x(t))}{\phi(x(t))} p(t)[w'(t)]^2, \quad t \geq T_0. \tag{3.25}$$

We separate the proof two cases where

$$\int_{T_0}^{\infty} (f'(x(s))/\phi(x(s)))p(s)[w'(s)]^2 ds$$

is finite or infinite, and arrive in each case at a contradiction.

CASE 1. Suppose that

$$\int_{T_0}^{\infty} \frac{f'(x(t))}{\phi(x(t))} p(t)[w'(t)]^2 ds < \infty.$$

In this case, according to the Lemma 3.6, Eq.(3.17) is satisfied. From (3.25) we see that

$$\int_{t_0}^{t} q(s)ds = C_1 - p(t)w'(t) - \int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds > \infty$$

for all $t \geq T_0$, where

$$C_1 = \int_{t_0}^{T_0} q(s)ds + p(T_0)w'(T_0).$$

Then, we have

$$\begin{aligned}\left[\int_{t_0}^{t} q(s)ds\right]^2 &= \left[C_1 - p(t)w'(t) - \int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds\right]^2 \\ &\leq 3C_1^2 + 3[p(t)w'(t)]^2 + 3\left[\int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds\right]^2 \\ &\leq C_2 + 3[p(t)w'(t)]^2\end{aligned}$$

for every $t \geq T_0$, where

$$C_2 = 3C_1^2 + 3\left[\int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds\right]^2.$$

By taking

$$Q(t) = \int_{t_0}^{t} q(\tau)d\tau, \quad t \geq t_0 \quad \text{and} \quad \sigma = \int_{t_0}^{T_0} \frac{Q^2(s)}{p(s)} ds,$$

and using (3.18) we derive

$$\begin{aligned}\frac{1}{t}\int_{t_0}^{t} \frac{Q^2(s)}{p(s)} ds &= \frac{\sigma}{t} + \frac{1}{t}\int_{T_0}^{t} \frac{Q^2(s)}{p(s)} ds \\ &\leq \frac{\sigma}{t} + \frac{C_2}{t}\int_{T_0}^{t} \frac{ds}{p(s)} + \frac{3}{t}\int_{T_0}^{t} p(s)[w'(s)]^2 ds \\ &\leq \frac{\sigma}{t} + \frac{C_2}{t}\int_{t_0}^{t} \frac{ds}{p(s)} + \frac{3}{\delta t}\int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} w(s)p(s)[w'(s)]^2 ds \\ &\leq \frac{\sigma}{t} + \frac{C_2}{t}\int_{t_0}^{t} \frac{ds}{p(s)} + \frac{3}{\delta t}\left[\max_{T_0 \leq s \leq t} w(s)\right]\int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds\end{aligned}$$

for every $t \geq T_0$. Consequently,

$$\frac{1}{t}\int_{t_0}^{t} \frac{Q^2(s)}{p(s)} ds \leq \frac{\sigma}{t} + \frac{C_2}{t}\int_{t_0}^{t} \frac{ds}{p(s)} + \frac{C_4}{t} \max_{T_0 \leq s \leq t} w(s), \quad t \geq T_0, \tag{3.26}$$

where

$$C_4 = \frac{3}{\delta}\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds > 0.$$

On the other hand, by (3.17), we can choose $T^* \geq T_0$ such that $w(t) \leq t$ for every $t \geq T^*$. Therefore,

$$\max_{T_0 \leq s \leq t} w(s) \leq \max_{T_0 \leq s \leq T*} w(s) + t, \quad \text{for every} \quad t \geq T_0.$$

Thus, Eq.(3.26) gives

$$\frac{1}{t}\int_{t_0}^{t} \frac{Q^2(s)}{p(s)} ds \leq \frac{\sigma}{t} + \frac{C_2}{t}\int_{t_0}^{t} \frac{ds}{p(s)} + \frac{C_4}{t}\left[\max_{T_0 \leq s \leq T*} w(s) + t\right],$$

Accordingly , by taking into account (3.15), we conclude that

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \frac{1}{p(s)}\left[\int_{t_0}^{s} q(\tau)d\tau\right]^2 ds \leq C_4 + C_2 \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \frac{ds}{p(s)} < \infty,$$

which contradicts assumption (3.23).

CASE 2. Suppose that

$$\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds = \infty.$$

In this case, according to Lemma 3.5, Eq.(3.13) holds. From (3.25) we obtain

$$\int_{T_0}^{t} H(t,s)q(s)ds + \int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds = -\int_{T_0}^{t} H(t,s)(p(s)w'(s))' ds$$

for every $t \geq T_0$. Integrating by parts, using properties (3.2), (3.6) and (3.7) of the weighted function $H(t,s)$, we have that

$$\begin{aligned}
\int_{t_0}^{t} H(t,s)q(s)ds &+ \int_{t_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \\
&\leq \int_{t_0}^{T_0} H(t,s)q(s)ds + \int_{t_0}^{T_0} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \\
&+ H(t,T_0)p(T_0)w'(T_0) - \frac{\partial H}{\partial s}Hs(t,T_0)p(T_0)w(T_0) \\
&\leq H(t,t_0)\int_{t_0}^{T_0} |q(s)|ds + H(t,t_0)\int_{t_0}^{T_0} \frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \\
&+ H(t,t_0)p(T_0)|w'(T_0)| - \frac{\partial H}{\partial s}(t,t_0)p(t_0)w(T_0),
\end{aligned}$$

which implies that

$$\begin{aligned}
\limsup_{t\to\infty}&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \\
&\leq L - p(t_0)w(T_0)\limsup_{t\to\infty}\frac{\frac{\partial H}{\partial s}(t,t_0)}{H(t,t_0)} - \liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)q(s)ds,
\end{aligned}$$

where

$$L = \int_{t_0}^{T_0} |q(s)|ds + \int_{t_0}^{T_0} \frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds + p(T_0)|w'(T_0)|.$$

Thus, according to Eqs. (3.4) and (3.24), we conclude that

$$\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds < \infty \tag{3.27}$$

which contradicts (3.13). Therefore, the proof of the theorem is completed. By an application of the Cauchy-Schwarz inequality, we have

$$\left(\frac{1}{t}\int_{t_0}^{t}\frac{1}{p(s)}\int_{t_0}^{s} q(\tau)d\tau ds\right)^2 \leq \left(\frac{1}{t}\int_{t_0}^{t}\frac{1}{p(s)}\left[\int_{t_0}^{s} q(\tau)d\tau\right]^2 ds\right)\left(\frac{1}{t}\int_{t_0}^{t}\frac{ds}{p(s)}\right).$$

Consequently, by Eq.(3.15), we deduce that the condition

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau)d\tau ds = \infty \tag{3.28}$$

implies (3.23). Therefore, we have the following corollary.

Corollary 3.8 *Equation (3.1) is oscillatory if (3.15) and (3.28) hold and for the function $H : D \longrightarrow \mathbb{R}$ with a property $\mathbf{H}^*_{\mathbf{a}}$, Eq. (3.24) is satisfied in the sublinear case.*

Now, we state and prove oscillation criterion for the superlinear equation (3.1).

3.3 Superlinear equations

The purpose of this section is to present oscillation criteria for the superlinear equations.

Theorem 3.9 *The superlinear equation (3.1) is oscillatory if (3.15) and (3.23) are satisfied and for the function $H \in H(C)$ for which there exists constant $m > 1/L_{f,\phi}$ such that*

$$\frac{\partial}{\partial s}\left(p(s)\frac{\partial H}{\partial s}(t,s)\right) \geq m\frac{(\frac{\partial H}{\partial s}(t,s))^2}{H(t,s)}p(s), \quad \text{for} \quad (t,s) \in D, \tag{3.29}$$

(3.24) is fulfilled.

Proof. Assume that the differential equation (3.1) has a nonoscillatory solution $x(t) \neq 0$ for every $t \geq T_0 \geq t_0$. From (3.11), by differentiation and using (3.1), we obtain

$$(p(t)w'(t))' = q(t) + \frac{f'(x(t))}{\phi(x(t))}p(t)[w'(t)]^2 ds, \quad t \geq T_0. \tag{3.30}$$

We consider two cases there $\int_{T_0}^{\infty}(f'(x(s))/\phi(x(s)))p(s)[w'(s)]^2 ds$ is finite or infinite.

CASE I. Suppose that

$$\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds < \infty.$$

In this case, according to Lemma 3.6, we have that (3.17) is true. From (3.30), we obtain

$$\int_{t_0}^{t} q(s)ds = K_1 + p(t)w'(t) - \int_{T_0}^{t} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds, \quad \text{for} \quad t \geq T_0,$$

where

$$K_1 = \int_{t_0}^{T_0} q(s)ds - p(T_0)w'(T_0).$$

Thus, if we define

$$K_2 = 3K_1^2 + 3\left[\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds\right]^2,$$

we get

$$\begin{aligned} \left[\int_{t_0}^{t} q(s)ds\right]^2 &\leq 3K_1^2 + 3[p(t)w'(t)]^2 + 3\left[\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds\right]^2 \\ &\leq K_2 + 3[p(t)w'(t)]^2, \quad t \geq T_0. \end{aligned}$$

Condition (3.9) guarantees that (3.22) holds for every $t \geq T_0$. Accordingly, following the same procedure as in the proof of Theorem 3.7, for $\delta = 1$, we can arrive at the contradiction to (3.23).

CASE II. Suppose the following

$$\int_{T_0}^{\infty} \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds = \infty.$$

According to Lemma 3.5, Eq.(3.13) holds. From (3.30), it follows that

$$\int_{T_0}^{t} H(t,s)q(s)ds + \int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds = \int_{T_0}^{t} (p(s)w'(s))' ds$$
$$= -H(t,T_0)p(T_0)w'(T_0) - \int_{T_0}^{t} \frac{\partial H}{\partial s}(t,s)p(s)w'(s)ds. \qquad (3.31)$$

Let $\chi : D \to \mathbb{R}$ be continuous function satisfying

$$-\frac{\partial H}{\partial s}(t,s) = \chi(t,s)\sqrt{H(t,s)}, \quad \text{for all} \quad (t,s) \in D.$$

By (3.29) there exists constant $m > 1/L_{f,\phi}$ such that

$$\frac{\partial}{\partial s}\left(p(s)\frac{\partial H}{\partial s}(t,s)\right) \geq m\chi(t,s)p(s), \quad \text{for all} \quad (t,s) \in D.$$

Hence, we can choose real number R such that $1 < R < mL_{f,\phi}$. We claim that, for every $t^* \geq T_0$, there is a $t \geq t^*$ such that

$$\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds > R\int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds. \qquad (3.32)$$

Otherwise, there exists a $t^* \geq T_0$ such that

$$\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \leq R\int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds, \qquad (3.33)$$

for all $t \geq t^*$. According to selection of the number $L_{f,\phi}$, we have that

$$\frac{\phi(x(t))}{f'(x(t))} \leq \frac{w(t)}{L_{f,\phi}}, \quad \text{for every} \quad t \geq T_0. \qquad (3.34)$$

By using the Cauchy-Schwarz inequality and taking into account (3.33) and (3.34), for $t \geq t^*$, we obtain

$$
\begin{aligned}
0 &\leq R \int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds \\
&\leq \left(\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \right)^{1/2} \times \left(\int_{T_0}^{t} \chi^2(t,s)\frac{\phi(x(s))}{f'(x(s))}p(s)ds \right)^{1/2} \\
&\leq \frac{\sqrt{R}}{\sqrt{L_{f,\phi}}} \left(\int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds \right)^{1/2} \times \left(\int_{T_0}^{t} \chi^2(t,s)p(s)w(s)ds \right)^{1/2},
\end{aligned}
$$

so that we have for $t \geq t^*$,

$$
\int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds \leq \frac{R}{L_{f,\phi}} \int_{T_0}^{t} \chi^2(t,s)p(s)w(s)ds. \tag{3.35}
$$

On the other hand, using partial integration and assumption (3.29) we obtain

$$
\begin{aligned}
\int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds &= -\int_{T_0}^{t} \frac{\partial H}{\partial s}(t,s)p(s)w'(s)ds \\
&= w(T_0)p(T_0)\frac{\partial H}{\partial s}(t,T_0) + \int_{T_0}^{t} w(s)\frac{\partial}{\partial s}\left(p(s)\frac{\partial H}{\partial s}(t,s) \right) ds \\
&\geq (T_0)p(T_0)\frac{\partial H}{\partial s}(t,T_0) + m\int_{T_0}^{t} \chi^2(t,s)p(s)w(s)ds,
\end{aligned}
$$

so from (3.35), we find that for all $t \geq t^*$,

$$
\left(1 - \frac{R}{mL_{f,\phi}}\right) \int_{T_0}^{t} \chi(t,s)\sqrt{H(t,s)}p(s)w'(s)ds \leq -\frac{RW}{mL_{f,\phi}}\frac{\partial H}{\partial s}(t,T_0),
$$

where $W = w(T_0)p(T_0) > 0$. Therefore, from (3.33), we have for every $t \geq t^*$,

$$
\int_{T_0}^{t} H(t,s)\frac{f'(x(s))}{\phi(x(s))}p(s)[w'(s)]^2 ds \leq -\frac{R^2 W}{mL_{f,\phi} - R}\frac{\partial H}{\partial s}(t,T_0).
$$

Hence, since $R < mL_{f,\phi}$, by (3.4), we conclude that

$$\limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds < \infty,$$

which contradicts (3.13). Thus, we proved that (3.32) is satisfied, so that we can consider a sequence $(t_\nu)_{\nu\in N}$ of points in the interval $[T_0, \infty]$, such that $\lim_{\nu\to\infty} t_\nu = \infty$, and for all $\nu \in N$,

$$\int_{T_0}^{t_\nu} H(t_\nu, s) \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds > R \int_{T_0}^{t_\nu} \chi(t_\nu, s)\sqrt{H(t_\nu, s)} p(s) w'(s) ds$$

Then from (3.31), it follows that

$$\begin{aligned}
&\int_{T_0}^{t_\nu} H(t_\nu, s) q(s) ds = -\int_{T_0}^{t_\nu} H(t_\nu, s) \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds \\
&- H(t_\nu, T_0) W^* + \int_{T_0}^{t_\nu} \chi(t_\nu, s)\sqrt{H(t_\nu, s)} p(s) w'(s) ds \\
&< -H(t_\nu, T_0) W^* + \frac{1-R}{R} \int_{T_0}^{t_\nu} H(t_\nu, s) \frac{f'(x(s))}{\phi(x(s))} p(s)[w'(s)]^2 ds, \qquad (3.36)
\end{aligned}$$

where $W^* = p(T_0)w'(T_0)$. Since $R > 1$, from the previous inequality and (3.13), we derive that

$$\lim_{\nu\to\infty} \frac{1}{H(t_\nu, T_0)} \int_{T_0}^{t_\nu} H(t_\nu, s) q(s) ds = -\infty,$$

which shows that

$$\liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) q(s) ds = -\infty. \qquad (3.37)$$

Finally, for every $t \geq T_0$, we obtain

$$\begin{aligned}\int_{t_0}^{t} H(t,s)q(s)ds &\leq \int_{t_0}^{T_0} H(t,s)|q(s)|ds + \int_{T_0}^{t} H(t,s)q(s)ds \\ &\leq H(t,t_0)\int_{t_0}^{T_0} |q(s)|ds + \int_{T_0}^{t} H(t,s)q(s)ds.\end{aligned}$$

Accordingly, (3.37) implies that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s)ds = -\infty,$$

which contradicts (3.24) and therefore, proves the theorem.

Remark 3.10 *If for function $H \in H(C)$, there exists some constant $m > 1/L_{f,\phi}$ such that (3.29) holds, then (3.7) is fulfilled, and accordingly, $H \in \mathbf{H}^*_{\mathbf{a}}(D)$. Therefore, the class of the weighted functions for which (3.24) together with (3.15) and (3.23) ensures the oscillation of the superlinear equation (3.1) is smaller than the appropriate class of the weighted functions in the case of the sublinear equation (3.1).*

Example 3.11 *For the illustration let us consider the differential equation*

$$[\sqrt{t}x^{2m}(t)x'(t)]' + [\lambda t^{\lambda-1}(2-\cos t) + t^{\lambda}\sin t]|x(t)|^{\alpha} sgn x(t) = 0,$$

where $m \in N$, $\alpha, \lambda \in \mathbb{R}$, $\alpha > 1$. First we observe that $L_{f,\phi} = \lambda/(\lambda - 2m - 1) > 1$ for all $m \in N$. Also, Eq. (3.23) is satisfied. Furthermore, we get

$$\int_{t_0}^{t} q(s)ds = \int_{t_0}^{t} d[s^{\lambda}(2-\cos s)] = t^{\lambda}(2-\cos t) - k_0,$$

$$\textit{where}\ \ t_0^{\lambda}(2+\cos t_0), \quad \textit{so that}\ \ t^{\lambda} - k_0 \leq \int_{t_0}^{t} q(s)ds \leq 3t^{\lambda}, \quad \textit{for every}\ \ t \geq t_0.$$

Accordingly,

$$\frac{1}{t}\int_{t_0}^{t}\frac{1}{p(s)}\left[\int_{t_0}^{s}q(\tau)d\tau\right]^2 ds \geq \frac{t^{2\lambda-\mu}}{2\lambda-\mu+1}-\frac{2k_0t^{\lambda-\mu}}{\lambda-\mu+1}+\frac{k_0^2t^{-\mu}}{1-\mu}+\frac{C_1}{t},$$

where

$$\mu=\frac{1}{2},\quad C_1=\frac{2k_0}{\lambda-\mu+1}t_0^{\lambda-\mu+1}-\frac{t_0^{2\lambda-\mu+1}}{2\lambda-\mu+1}-\frac{k_0^2}{1-\mu}t_0^{-\mu+1}.$$

Hence, condition (3.23) is also satisfied for $\lambda > 1/2$.

We can choose $H(t,s) = (\sqrt{t}-\sqrt{s})^2, \quad t \geq s \geq t_0$. *It is easy to check that* $H \in H(C)$. *Moreover, condition (3.29) holds for* $m \leq 1/2$. *To ensure the assumption* $m > 1/L_{f,\phi}$, *it has to be* $\alpha < 4m+2$. *Besides that, we have*

$$\begin{aligned}\frac{1}{t}\int_{t_0}^{t}(\sqrt{t}-\sqrt{s})^2q(s)ds &= \frac{1}{t}\int_{t_0}^{t}(\sqrt{t}-\sqrt{s})^2 d\left(\int_{t_0}^{s}q(u)du\right)\\ &= \frac{1}{t}\int_{t_0}^{t}\left(\frac{\sqrt{t}}{\sqrt{s}}-1\right)\left(\int_{t_0}^{s}q(u)du\right)\\ &\geq \frac{1}{t}\int_{t_0}^{t}\left(\frac{\sqrt{t}}{\sqrt{s}}-1\right)(s^{\lambda}-k_0)ds\\ &\geq \frac{1}{(\lambda+1)(2\lambda+1)}t_0^{\lambda}+\frac{L_1}{t}+\frac{L_2}{\sqrt{t}}-k_0,\end{aligned}$$

where

$$L_1=\frac{t_0^{\lambda+1}}{\lambda+1}-k_0t_0,\qquad L_2=2k_0\sqrt{t_0}-\frac{2t_0^{\lambda+1/2}}{2\lambda+1}.$$

Therefore (3.23) is satisfied. So, by Theorem 3.9, the differential equation under consideration is oscillatory for $\lambda > 1/2$ *and* $\alpha < 4m+2$.

CHAPTER 4

NONLINEAR DIFFERENTIAL EQUATIONS

4.1 Introduction

In this chapter, we present a new oscillation criteria for the second-order nonlinear differential equation

$$\Big[p(t)\phi(x(t)) \mid x'(t) \mid^{\alpha-2} x'(t)\Big]' + q(t) \mid x(t) \mid^{\alpha-2} x(t) = 0,\ t \geq t_0, \tag{4.1}$$

where $p \in C^1([t_0, \infty); (0, \infty))$, $q \in C([t_0, \infty); \mathbb{R})$, $\alpha > 1$ is a constant, and ϕ is a continuous real valued function such that

$$0 < \phi(x) \leq \gamma \tag{4.2}$$

for all $x \in \mathbb{R}$, where γ is real number. Note that if $\alpha = 2$, $p(t) \equiv 1$, and $\phi(x) \equiv 1$, then Eq. (4.1) is reduced to the linear equation (1.2) given in Chapter 1. Many results guaranteeing that the solutions of (1.2) are oscillatory can be found in the literature and some of them have been quoted in the introductory chapter. However, the previously mentioned criteria of Wintner, Hartman, Kamanev, Yan, Philos and Li cannot be applied to the Euler differential equations,

$$x''(t) + \frac{\mu}{t^2}\ x(t) = 0, \quad t \geq t_0 > 0, \tag{4.3}$$

where $\mu > 0$ is a constant, which is oscillatory if $\mu > \frac{1}{4}$ and nonoscillatory if $\mu \leq \frac{1}{4}$.

The purpose of this chapter is to study the general equation (4.1) and establish some new oscillation criteria which contain some of the above-mentioned results as special cases. Here we also extend and improve the results of Li [22] and Philos [34]. Note that in 1996, Wong and Agarwal [44] established oscillation criteria for the more general equation,

$$[p(t)|x'(t)|^{\alpha-1}x'(t)]' + Q(t, x(t)) = P(t, x(t), x'(t)).$$

The authors [11], [45] and [46] also considered some special cases of this equation by using the technique which is an extension of the methods used in the works of Graef and Spikes [9] and Kwang and Wong [19] for the differential equations. But their results for Eq.(4.1) required that

$$\int_{t_0}^{\infty} \frac{dt}{p(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)dt < \infty. \tag{4.4}$$

We will also show that we do not use any restriction on the functions p and q above, which is of particular interest.

4.2 Oscillation criteria for nonlinear equations

In this section, we will establish some oscillation criteria for Eq.(4.1) which generalize and improve Li [22] and Philos [34] criteria.

Let us start with the definition.

Definition 4.1 *By a solution of (4.1) on $[t_0, \infty)$ we mean a nontrivial continuously differentiable function $x(t)$ defined on $[t_0, \infty)$ with $p(t)\phi(x(t))|x'(t)|^{\alpha-2}x'(t)$ in $C^1([t_0, \infty))$ such that $x(t)$ satisfies (4.1).*

Theorem 4.2 *Let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. Assume that $H \in C(D, \mathbb{R})$ satisfy the following two conditions:*

(i) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $(t,s) \in D_0$;

(ii) H has a continuous and nonpositive partial derivative on D with respect to the second variable.

Suppose that $h : D_0 \to \mathbb{R}$ is a continuous function with

$$-\frac{\partial H(t,s)}{\partial(s)} = h(t,s)[H(t,s)]^{\frac{1}{q}}, \quad \text{for all} \quad (t,s) \in D_0,$$

where $(1/p) + (1/q) = 1$. If there exists a function $\rho \in C^1[[t_0, \infty)]; (0, \infty)]$ such that $\rho'(t) \geq 0$ for all $t \geq t_0$ and

$$\limsup_{t \to \infty} \{X(t,t_0) - \gamma Y(t,t_0)\} = \infty, \tag{4.5}$$

where

$$X(t,t_0) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds$$

$$Y(t,t_0) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s)p(s) \left[\frac{1}{p}\left(h(t,s) + [H(t,s)]^{1/p}\frac{\rho'(s)}{\rho(s)}\right)\right]^p ds,$$

then Eq.(4.1) is oscillatory.

Proof. Let x(t) be a nonoscillatory solution of Eq.(4.1). Without loss of generality let us assume that $x(t) > 0$ for all $t \geq T_0$ for some $T_0 \geq t_0$. The proof in the case where $x(t) < 0$ for all $t \geq T_0$ is similar. Let us define

$$w(t) = [\rho(t)] \left[\frac{p(t)\phi(x(t))\psi(x'(t))}{\psi(x(t))}\right], \quad t \geq t_0, \tag{4.6}$$

where $\psi(s) = |s|^{\alpha-2}s$. Now differentiating (4.6) and using (4.1) and (4.2) we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) - (p-1)[\gamma\rho(s)p(s)]^{1-q}|w(t)|^q, \quad \text{for all} \quad t \geq T_0. \quad (4.7)$$

Then multiplying Eq.(4.7) by $H(t,s)$ and integrating this equation from $T_0 \leq T$ to $T_0 \leq t$ and considering (ii) leads to,

$$X(t,T) \leq w(t) - J(t,T) + \gamma Y(t,T), \quad (4.8)$$

where

$$J(t,T) = \frac{1}{H(t,T)} \int_T^t \Bigg\{ \gamma\rho(s)p(s) \left[\frac{1}{p} \Bigg(h(t,s) + [H(t,s)]^{1/p}\rho(s) \Bigg) \right]^p$$
$$+(p-1)[\gamma\rho(s)p(s)]^{1-q}|w(s)|^q H(t,s) - (h(t,s)[H(t,s)]^{1/q} + \frac{\rho'(s)}{\rho(s)} H(t,s))|w(s)| \Bigg\} ds.$$

Since $q > 1$, taking

$$X = [\gamma\rho(s)p(s)]^{\frac{1}{p}} \left[\frac{1}{p} \Bigg(h(t,s) + [H(t,s)]^{\frac{1}{p}} \frac{\rho'(s)}{\rho(s)} \Bigg) \right],$$
$$Y = [H(t,s)]^{\frac{1}{p}} |w(s)|^{\frac{q}{p}} [\gamma\rho(s)p(s)]^{\frac{1-q}{p}}$$

we see that

$$pXY^{p-1} = h(t,s)[H(t,s)]^{\frac{1}{q}} + H(t,s)\frac{\rho'(s)}{\rho(s)}|w(s)|$$

and

$$X^p + (p-1)Y^p = \gamma\rho(s)p(s) \left[\frac{1}{p} \Bigg(h(t,s) + [H(t,s)]^{\frac{1}{p}} \frac{\rho'(s)}{\rho(s)} \Bigg) \right]^p$$
$$+ (p-1)[\gamma\rho(s)p(s)]^{1-q}|w(s)|^q H(t,s)$$

and hence using Lemma 1.7 we get

$$\gamma\rho(s)p(s)\left[\frac{1}{p}\left(h(t,s)+[H(t,s)]^{\frac{1}{p}}\frac{\rho'(s)}{\rho(s)}\right)\right]^{p}+(p-1)[\gamma\rho(s)p(s)]^{1-q}|w(s)|^{q}H(t,s)$$
$$-(h(t,s)[H(t,s)]^{\frac{1}{p}}+H(t,s)\frac{\rho'(s)}{\rho(s)})|w(s)|\geq 0,$$

for all $t>s\geq T_0$. Also, by (4.8) we have

$$\begin{aligned}H(t,T_0)[X(t,T_0)-\gamma Y(t,T_0)] &\leq H(t,T_0)w(T_0)\\ &\leq H(t,T_0)|w(T_0)|\\ &\leq H(t,t_0)|w(T_0)|.\end{aligned}$$

Therefore,

$$\begin{aligned}H(t,t_0)[X(t,t_0)-\gamma Y(t,t_0)] &= H(T_0,t_0)[X(T_0,t_0)-\gamma Y(T_0,t_0)]\\ &+H(t,T_0)H(t,t_0)[X(t,T_0)-\gamma Y(t,T_0)]\\ &\leq H(t,t_0)\int_{t_0}^{T_0}|q(s)|\rho(s)ds+H(t,t_0)|w(T_0)|\\ &= H(t,t_0)\left\{\int_{t_0}^{T_0}|q(s)|\rho(s)ds+|w(T_0)|\right\},\end{aligned}$$

for all $t\geq T_0$. This gives,

$$\limsup_{t\to\infty}X(t,t_0)-\gamma Y(t,t_0)\leq\int_{t_0}^{T_0}|q(s)|\rho(s)ds+|w(T_0)|<\infty$$

which contradicts Eq.(4.5). This completes to proof of the theorem.

Example 4.3 *As a simple example consider the Euler differential equation (4.3) for* $\gamma>1/4$ *and let* $H(t,s)=(t-s)^{\lambda}$ *and* $\rho(s)=s$ *for* $t\geq s\geq 1$*, where* $\lambda>1$.

Then, using the inequality [13]

$$(t-s)^{\lambda} \geq t^{\lambda} - \lambda s t^{\lambda-1}, \quad t \geq s \geq 1$$

we can show that

$$\begin{aligned} X(t,t_0) - \gamma Y(t,t_0) &= \frac{1}{(t-s)^{\lambda}} \int_{t_0}^{t} (t-s)^{\lambda} \frac{\gamma}{s} - s[\frac{1}{2}(\lambda(t-s)^{\frac{\lambda}{2}-1} - \frac{1}{s}(t-s)^{\frac{\lambda}{2}})]^2 ds \\ &\geq (\gamma - \frac{1}{4}) \limsup_{t\to\infty} \log t + \frac{1}{2} - \lambda = \infty. \end{aligned}$$

Therefore, all the hypothesis of Theorem 4.2 are fulfilled and hence (4.3) is oscillatory when $\gamma > 1/4$.

Example 4.4 *Consider the equation,*

$$[t^2(1+e^{-|x(t)|})|x'(t)|x'(t)]' + t^{-1}|x(t)|x(t) = 0, \quad t \geq t_0 \geq 1.$$

Clearly

$$\phi(x) = 1 + e^{-|x|} \leq 2 = \gamma \ \ \textit{for all} \ \ x \in \mathbb{R}.$$

Let us apply Theorem 4.2, with $\alpha = 3$, $p(t) = t^2$, $q(t) = \frac{1}{t}$, $H(t,s) = (t-s)^3$, *so that* $h(t,s) = 3$ *and* $\rho(s) = 1$ *so that* $\rho'(s) = 0$. *A straightforward computation yields*

$$\begin{aligned} X(t,t_0) - \gamma Y(t,t_0) &= \frac{1}{(t-t_0)^3}\left(\int_{t_0}^{t} \frac{(t-t_0)^3}{s} ds - 2\int_{t_0}^{t} s^2 ds\right) \\ &= \limsup_{t\to\infty} \frac{1}{(t-t_0)^3}\left[t^3 \ln s - 3t^2 s + \frac{3ts^2}{2} - s^3\right]_{s=t_0}^{t} \\ &= \infty. \end{aligned}$$

Thus, Eq.(4.5) holds, and we conclude by Theorem 4.2 that equation is oscillatory.

Here we also observe that (4.4) does not hold. Therefore, as stated before the results of [11], [44] cannot be applied to this example.

Example 4.5 *Consider the differential equation,*

$$[t^{-3}(1+e^{-|x(t)|})|x'(t)|x'(t)]' + t^{-1}|x(t)|^2 x(t) = 0, \quad t \geq t_0 \geq 1.$$

Let us take $H(t,s) = (t-s)^3$ *and* $\rho(s) = s^2$*. Observe that* $\phi(x) = 1 + e^{-|x|}$ *so that* $0 < \varphi(x) \leq 2 = \gamma, \quad \forall x \in \mathbb{R}$*. Then, an easy calculation gives*

$$X(t,t_0) = \frac{1}{(t-t_0)^3}\int_{t_0}^{t}(t-s)^3 s ds \quad and \quad Y(t,t_0) = \frac{1}{(t-t_0)^3}\int_{t_0}^{t}\frac{(s+2t)^3}{27s^4}ds$$

so that

$$\begin{aligned}\limsup_{t\to\infty}\Big[X(t,t_0) - \gamma Y(t,t_0)\Big] &= \limsup_{t\to\infty}\Big[X(t,t_0) - 2Y(t,t_0)\Big] \\ &= \limsup_{t\to\infty}\frac{1}{(t-t_0)^3}\int_1^t\left((t-s)^3 s - 2\frac{(s+2t)^3}{27s^4}\right)ds \\ &= \infty.\end{aligned}$$

Therefore, by Theorem 4.2, the equation given above is oscillatory.

We also see that the conditions in (4.4) are not satisfied so that the results obtained by Hong [11] and Wong [44] cannot be applied to this example.

A close look at the proof Theorem 4.2 reveals that (4.5) may be replaced by the conditions

$$\limsup_{t\to\infty} X(t,t_0) = \infty, \tag{4.9}$$

and

$$\limsup_{t\to\infty} Y(t,t_0) < \infty. \tag{4.10}$$

This leads to the following result.

Corollary 4.6 *Let the conditions of Theorem 4.2 be satisfied except that (4.5) is now replaced by (4.9) and (4.10), then Eq.(4.1) is oscillatory. In Theorem 4.2, the condition;*

$$\limsup_{t\to\infty} X(t,t_0) = \infty$$

is necessary in the remainder of this paper, we do not require this condition, but will have some other conditions instead thus, the following result provides an essentially new oscillation for Eq.(4.1).

Theorem 4.7 *Let H and h be as in Theorem 4.2, and let*

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty, \tag{4.11}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \gamma\rho(s)p(s)\left(h(t,s) + [H(t,s)]^{1/p}\frac{\rho'(s)}{\rho(s)}\right)^{p} ds < \infty. \tag{4.12}$$

Suppose there exist functions $A([t_0,\infty))$ and $\rho \in C^1([[t_0,\infty);(0,\infty)])$ such that $\rho'(t) \geq 0$, $t \geq t_0$,

$$\int_{t_0}^{\infty} A_+^q ds = 0 \tag{4.13}$$

and for every $T \geq t_0$,

$$\liminf_{t\to\infty}\{X(t,T) - \gamma Y(t,T)\} \geq A(T) \tag{4.14}$$

hold, where $A_+(s) = \max\{A(s),0\}, \quad s \geq t_0$. Then Eq.(4.1) is oscillatory.

Proof. Let x(t) be a nonoscillatory solution of (4.1) without loss of generality, we may assume that $x(t) > 0$ on $[T_0,\infty)$, defining $w(t)$ as in the proof of

Theorem 4.2, (4.8) holds then

$$X(t,t_0) - \gamma Y(t,t_0) \le w(t) - J(t,T)$$

for $t > T \ge T_0$. Consequently,

$$\liminf_{t\to\infty}\{X(t,T) - \gamma Y(t,T)\} \le w(t) - \limsup_{t\to\infty} J(t,T) \tag{4.15}$$

for all $T \ge T_0$ thus by (4.14),

$$w(t) \ge A(T) + \limsup_{t\to\infty} J(t,T) \tag{4.16}$$

for all $T \ge T_0$. This shows that,

$$w(t) \ge A(T) \tag{4.17}$$

and

$$\limsup_{t\to\infty} J(t,T) < \infty \tag{4.18}$$

for all T$\ge T_0$. Let

$$f(t) = \frac{(p-1)}{H(t,T_0)} \int_{T_0}^{t} [\gamma\rho(s)p(s)]^{1-q} H(t,s)|w(s)|^q ds$$

and

$$g(t) = -\frac{1}{H(t,T_0)} \int_{t_0}^{t} \left[h(t,s)[H(t,s)]^{1/q} + H(t,s)\frac{\rho'(s)}{\rho(s)}\right]|w(s)|ds$$

for all $t \geq T_0$ then it follows from (4.18),

$$\limsup_{t\to\infty}[f(t)+g(t)] = \limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} \Bigg[(p-1)[\gamma\rho(s)p(s)]^{1-q}H(t,s)|w(s)|^q - \left(h(t,s)[H(t,s)]^{1/q} + H(t,s)\frac{\rho'(s)}{\rho(s)}\right)|w(s)|\Bigg]ds$$
$$\leq \limsup_{t\to\infty} J(t,T_0) < \infty. \tag{4.19}$$

Now, we claim that

$$\int_{T_0}^{\infty} |w(s)|^q < \infty. \tag{4.20}$$

Suppose to the contrary that,

$$\int_{T_0}^{\infty} |w(s)|^q ds = \infty. \tag{4.21}$$

By (4.11), there is a positive constant η satisfying

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} \geq \eta > 0. \tag{4.22}$$

Let μ be any arbitrary positive number. Then it follows from (4.21) that there exist a $T_1 > T_0$ such that,

$$\int_{T_0}^{\infty} |w(s)|^q \geq \frac{\eta}{\mu}$$

for all $t \geq T_1$. Therefore,

$$\begin{aligned}
f(t) &= \frac{(p-1)}{H(t,T_0)} \int_{T_0}^{t} [\gamma\rho(s)p(s)]^{1-q} H(t,s) (\int_{T_0}^{s} |w(s)|^q d\tau) \\
&= \frac{(p-1)}{H(t,T_0)} \int_{T_0}^{t} (\int_{T_0}^{s} |w(\tau)|^q d\tau)(-\frac{\partial}{\partial s}[H(t,s)[\gamma\rho(s)p(s)]^{1-q}])ds \\
&\geq \frac{(p-1)}{H(t,T_0)} \int_{T_1}^{t} (\int_{T_0}^{s} |w(\tau)|^q d\tau)(-\frac{\partial}{\partial s}[H(t,s)[\gamma\rho(s)p(s)]^{1-q}])ds \\
&\geq \frac{(p-1)\mu}{H(t,T_0)\eta} \int_{T_1}^{t} (-\frac{\partial}{\partial s}[H(t,s)[\gamma\rho(s)p(s)]^{1-q}])ds \\
&= (p-1)\frac{\mu H(t,T_1)}{\eta H(t,T_0)}[\gamma\rho(T_1)p(T_1)]^{1-q},
\end{aligned}$$

for all $t \geq T_1$. By (4.22), there is a $T_2 \geq T_1$ such that,

$$\frac{H(t,T_1)}{H(t,T_0)} \geq \eta$$

for all $t \geq T_2$, which implies

$$f(t) \geq (p-1)\mu[\gamma\rho(T_1)p(T_1)]^{1-q}$$

for all $t \geq T_2$. Since μ is arbitrary,

$$\lim_{t\to\infty} f(t) = \infty. \tag{4.23}$$

Next, let us consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0, ∞) with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\lim_{n\to\infty} [f(t_n) + g(t_n)] = \limsup_{t\to\infty} [f(t) + g(t)]$$

It follows from (4.19) that there exist a constant M such that,

$$f(t_n) + g(t_n) \leq M \quad for \quad n = 1, 2, 3, ... \tag{4.24}$$

Furthermore, (4.23) guarantees that,

$$\lim_{n\to\infty} f(t_n) = \infty \tag{4.25}$$

and hence (4.24) gives,

$$\lim_{n\to\infty} g(t_n) = -\infty. \tag{4.26}$$

Then, by (4.24) and (4.25),

$$1 + \frac{g(n)}{f(t_n)} \leq \frac{M}{f(t_n)} < \frac{1}{2}$$

for n large enough. Thus,

$$\frac{g(t_n)}{f(n)} < -\frac{1}{2}$$

for n large enough. Thus and (4.26) imply that

$$\lim_{n\to\infty} \frac{|g(t_n)|^q}{f(t_n)} = \infty. \tag{4.27}$$

On the other hand, by the *Hölder* inequality, we have

$$
\begin{aligned}
|g(t_n)|^q &= \left|\frac{-1}{H(t_n,T_0)}\int_{T_0}^{t_n}\left(h(t_n,s)|H(t_n,s)|^{1/q}+H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right)w(s)ds\right|^q \\
&= \left|\frac{-1}{H(t_n,T_0)}\int_{T_0}^{t_n}\left(h(t_n,s)|H(t_n,s)|^{1/q}+H(t_n,s)\frac{\rho'(s)}{\rho(s)}\right)[H(t_n,s)]^{-1/q}[H(t_n,s)]^{1/q}\right|^q \\
&\times \left|[\gamma\rho(s)p(s)]^{q-1/q}[\gamma\rho(s)p(s)]^{1-q/q}w(s)ds\right|^q \\
&= \left|\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\left(h(t_n,s)+\frac{\rho'(s)}{\rho(s)}[H(t_n,s)]^{1/p}\right)\right|^q \\
&\times \left|[\gamma\rho(s)p(s)]^{q-1/q}[\gamma\rho(s)p(s)]^{1-q/q}[H(t_n,s)]^{1/q}w(s)ds\right|^q \\
&\leq \frac{1}{p-1}\left(\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\left[h(t_n,s)+\frac{\rho'(s)}{\rho(s)}[H(t_n,s)]^{1/p}\right]^p\gamma\rho(s)p(s)\right)^{q-1} \\
&\times \left(\frac{p-1}{H(t_n,T_0}\int_{T_0}^{t_n}[\gamma\rho(s)p(s)]^{1-q}[H(t_n,s)]|w(s)|^q ds\right) \\
&\leq \frac{f(t_n)}{p-1}\left(\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\left[h(t_n,s)+\frac{\rho'(s)}{\rho(s)}[H(t_n,s)]^{1/p}\right]^p\gamma\rho(s)p(s)\right)^{q-1}
\end{aligned}
$$

for any positive integer n. Consequently,

$$
\frac{|g(t_n)|^q}{f(t_n)} \leq \frac{1}{p-1}\left(\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\left[h(t_n,s)+\frac{\rho'(s)}{\rho(s)}[H(t_n,s)]^{1/p}\right]^p\gamma\rho(s)p(s)\right)^{q-1}
$$

for n large enough. But (4.22) guarantees that,

$$
\liminf_{t\to\infty}\frac{H(t,T_0)}{H(t,t_0)} > \eta
$$

and hence, there exists a $T_3 \geq T_0$ such that,

$$
\frac{H(t,T_0)}{H(t,t_0)} \geq \eta
$$

for every $t \geq T_3$. Thus,

$$
\frac{H(t_n,T_0)}{H(t_n,t_0)} \geq \eta
$$

for n large enough and therefore,

$$\frac{|g(t_n)|^q}{f(t_n)} \leq \frac{1}{p-1}\left(\frac{1}{\eta H(t_n,t_0)}\int_{t_0}^{t_n}\left[h(t_n,s)+\frac{\rho'(s)}{\rho(s)}[H(t_n,s)]^{1/p}\right]^p \gamma\rho(s)p(s)\right)^{q-1}.$$

It follows from (4.27) that,

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n}\left[h(t,s)+\frac{\rho'(s)}{\rho(s)}[H(t_n,s)]^{1/p}\right]^p \gamma\rho(s)p(s)ds = \infty \tag{4.28}$$

This gives

$$\lim_{n\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t_n}\left[h(t,s)+\frac{\rho'(s)}{\rho(s)}[H(t,s)]^{1/p}\right]^p \gamma\rho(s)p(s) = \infty$$

which contradicting (4.12). Then (4.20) hold. Hence by (4.27),

$$\int_{T_0}^{\infty} A_+^q(s)ds \leq \int_{T_0}^{\infty} |w(s)|^q ds < \infty$$

which contradicts (4.13). This completes the proof of the theorem.

Example 4.8 *As a particular case, consider the differential equation*

$$[t^{-1/2}(1+e^{-|x(t)|})x'(t)]' + t^{-3/2}x(t) = 0, \quad t \geq t_0 > 1.$$

Let us choose $H(t,s)=(t-s)^2$ *and* $\rho(s)=1$ *so that we have* $h(t,s)=2$, $\rho'(s)=0$ *and*

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int \gamma\rho(s)p(s)\left(h(t,s)+[H(t,s)]^{1/p}\frac{\rho'(s)}{\rho(s)}\right)^p ds\\ &= \liminf_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t} 8s^{-1/2}ds = 0 < \infty.\end{aligned}$$

It can be easily shown that

$$\liminf_{t\to\infty}\left[X(t,T)-\gamma Y(t,T)\right] = \liminf_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\left[(t-s)^2 s^{-3/2}-\frac{s^{-1/2}}{2}\right]ds$$
$$= \liminf_{t\to\infty}\frac{2t^2T^{-1/2}}{(t-T)^2}\geq T^{-1/2},$$

from which we see that $A(T) = T^{-1/2}$. *Since* $\int_{T_0}^{\infty} A_+^2 ds = \infty$, *all hypotheses of Theorem 4.7 are satisfied and hence the given equation is oscillatory.*

The following theorems are, respectively generalizations of Theorems 3 and 4 in [22] for (4.1). The proofs are similar to that of Theorem 4.7 and hence omitted.

Theorem 4.9 *Let* H *and* h *be as in Theorem 4.2, and suppose that*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t_n}[h(t,s)+\frac{\rho'(s)}{\rho(s)}[H(t,s)]^{1/p}]^p\gamma\rho(s)p(s)ds<\infty \tag{4.29}$$

and (4.11) hold. If there exist functions $A\in C[t_0,\infty)$, *and* $\rho\in C^1[[t_0,\infty);(0,\infty)]$, *such that* $\rho'(t)\geq 0$, $t\geq t_0$, *(4.13), and for every* $T\geq t_0$.

$$\limsup_{t\to\infty}\{X(t,T)-\gamma Y(t,T)\}\geq A(T) \tag{4.30}$$

hold, then Eq.(4.1) is oscillatory.

Example 4.10 *Consider the differential equation,*

$$[t^{\nu}(1+e^{-|x(t)|})|x'(t)|^{p-2}x'(t)]'+t^{\lambda}\cos t|x(t)|^{p-2}x(t)=0,\quad t\geq t_0>1$$

where, ν, λ, p *are constants such that;* $\lambda<0$, $p>1$, $p\neq 3$, *and* $\nu<p-1$. *Moreover, taking,*

$$H(t,s)=(t-s)^2,\quad for\quad t\geq s\geq t_0\quad and\quad \rho(t)=1,$$

and

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\Rightarrow\infty}\frac{(t-s)^2}{(t-t_0)^2}\right] = 1 < \infty$$

is satisfied. We have,

$$\frac{1}{(t-t_0)^2}\int_{t_0}^{t} s^{\nu}(t-s)^{2-p}ds \leq \begin{cases} \frac{t^{\nu}(t-t_0)^{3-p}}{3-p} & \text{if } \nu < 0, \\ \frac{t_0^{\nu}(t-t_0)^{3-p}}{3-p} & \text{if } \nu > 0. \end{cases}$$

Since $\nu < p-1$, it is not difficult to see that

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t} s^{\nu}(t-s)^{2-p}ds = 0$$

and so (4.29) is satisfied. In this case,

$$X(t,T) = \frac{1}{(t-T)^2}\int_{T}^{t}(t-s)^2 s^{\lambda}\cos s ds$$

and

$$Y(t,T) = \frac{1}{(t-T)^2}\int_{T}^{t} s^{\nu}\left[\frac{1}{p}\left(2(t-s)^{1-2/q}\right)\right]^{p}.$$

Then for arbitrary small constant $\epsilon > 0$, there exists $t_1 \geq t_0$ such that for $T \geq t_1$, we have

$$\begin{aligned}\limsup_{t\to\infty}\Big[X(t,T) - \gamma Y(t,T)\Big] &= \limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_{T}^{t}[(t-s)^2 s^{\lambda}\cos s - 2s^{\nu}(t-s)^{2-p}]ds \\ &\geq -T^{\lambda}\sin t - \varepsilon = A(T).\end{aligned}$$

Consider an integer N such that

$$2N\pi + \frac{5\pi}{4} \geq \max\{t_1, (1+\sqrt{2}\varepsilon)^{1/\lambda}\}.$$

Then for all integers $n \geq N$ we have,

$$A(T) \geq \frac{1}{\sqrt{2}}, \quad \textit{for every} \quad T \in [2n\pi + \frac{5\pi}{4}, 2n\pi + \frac{7\pi}{4}]$$

Taking into account that $\nu < p-1$, we obtain,

$$\begin{aligned}\int_{T_0}^{\infty} A_+^q(s)ds &\geq \sum_{n=N}^{\infty}(\sqrt{2}^{-q})\int_{2n\pi+\frac{5\pi}{4}}^{2n\pi+\frac{7\pi}{4}} \\ &\geq (\sqrt{2})^{-q}\sum_{n=N}^{\infty}\int_{2n\pi+\frac{5\pi}{4}}^{2n\pi+\frac{7\pi}{4}} ds = \infty.\end{aligned}$$

Accordingly, all conditions of Theorem 4.9 are satisfied, and hence the above equation is oscillatory.

Remark 4.11 *In Theorem 3, let $\phi(x) = 1$, $p(t) = 1$, and $\rho(t) = 1$. If*

$$H(t,s) = (t-s)^2 \quad for \quad t \geq s \geq t_0, \quad where \quad \lambda > 1$$

is a constant, then we can easily verify Corollary 5, which contains Kamanev's criterion [16] and improved the results of Yan [51] in [22] with $p = 2$. Hence, we have improved the oscillation result of Yan [51] for Eq.(4.2).

Theorem 4.12 *Let $H(t,s)$ and $h(t,s)$ be as in Theorem 4.2 and suppose that*

$$\liminf_{t\longrightarrow\infty} X(t,t_0) < \infty$$

and (4.11) hold. If there exist functions $A \in C([t_0,\infty))$ and $\rho \in C^1([t_0,\infty);(0,\infty])$ such that $\rho'(t) \geq 0$, $t \geq t_0$, (4.13) and (4.14) hold, then (4.1) is oscillatory.

Remark 4.13 *If $\phi(x) \equiv 1$, $p(t) \equiv 1$, and $\rho(t) \equiv 1$, then Theorem 4.2, Theorem 4.9 and Theorem 4.12 reduce to Theorem 2-4 in [22], respectively. Furthermore,* *let*

$$H(t,s) = \left(\int_s^t \frac{dx}{k(x)} \right)^{\lambda}, \quad t \geq s \geq t_0$$

where $\lambda > p-1$ *is a constant and* $k(x)$ *is a positive continuous function on* $[t_0, \infty)$ *such that;*

$$\int_{t_0}^{\infty} \frac{dx}{k(x)} = \infty.$$

By applying Theorem 4.2, Theorem 4.9 and Theorem 4.12 in the special case considered, we derive four new oscillation criteria for (4.1).

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