Blow-up Solutions of Second Order Nonlinear Parabolic Equations

by

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Koç University

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and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

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ABSTRACT

In this thesis, we study the problem of blow-up of solutions of second order nonlinear parabolic equations and numerical analysis of blow-up of solutions. Firstly, blow-up of solutions in semilinear heat equation $\partial_t u = \Delta u + f(u)$ is explained by using three methods namely concavity method, eigenfunction method and comparison method. Secondly, the finite difference approximation to the initial boundary value problem for semilinear heat equation $\partial_t u = \partial_x^2 u + u^2$ is studied. Two convergence results are shown; the convergence of finite difference solution to the exact solution and the convergence of numerical blow-up time to the exact blow-up time.

ÖZETÇE

Bu tezde ikinci mertebeden doğrusal olmayan parabolik denklemlerin çözümlerinin sonlu zamanda patlaması ve nümerik analizi incelenmiştir. Öncelikle, yarıdoğrusal ısı denkleminin $\partial_t u = \Delta u + f(u)$ çözümlerinin sonlu zamanda patlaması problemi, isimleri "iç bükey metodu", "özfonksiyonlar metodu" ve "karşılaştırma metodu" olan üç yöntem ile açıklandı. Sonrasında, başlangıç-sınır değer probleminde sonlu fark yaklaşımı yarıdoğrusal ısı denklemi $\partial_t u = \partial_x^2 u + u^2$ için çalışıldı. İki yakınsaklık sonucu gösterildi; sonlu fark çözümünün tam çözüme yakınsaklığı ve sayısal patlama zamanının tam patlama zamanına olan yakınsaklığı.

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NOMENCLATURE

Ω :	an open connected subset of $\mathbb{R}^n$
$\overline{\Omega}$:	closure of Ω
$\partial\Omega$:	boundary of Ω
Ω_T :	$\Omega \times (0, T)$
Δ :	Laplace Operator
$\Delta u =$	$\displaystyle\sum_{i=0}^{n} \frac{\partial^2 u}{\partial {x_i}^2}$ where u is a smooth real valued function given by
	$u : \Omega \to \mathbb{R}$ and $x_1, x_2, \cdots, x_n$ are coordinates for $\Omega \subset \mathbb{R}^n$
$\partial_x u =$	$\dfrac{\partial u}{\partial x}$ partial derivative of u with respect to variable x
$\partial_x^2 u =$	$\dfrac{\partial^2 u}{\partial x^2}$

Chapter 1

PRELIMINARIES

1.1 Introduction

It is well known that solutions of initial boundary value problems for non-linear evolutionary partial differential equations (PDEs) may not exist for all time, some norm of solution (the maximum norm, L^2 norm or H^1 norm) may tend to infinity in a finite time. This is called the blow-up phenomena.

The question of blow-up of solutions of nonstationary nonlinear partial differential equations has received much attention during last decades. A number of methods are developed to study the problem of blow-up of solutions of initial boundary value problems for nonlinear parabolic equations, nonlinear hyperbolic equations, nonlinear pseudoparabolic equations, nonlinear Schrödinger equations and other nonlinear PDE's (see, e.g., [2], [6], [16], [18] and references therein). Sufficient conditions on initial data guaranteeing blow-up of solutions in a finite time of these problems are found. The main reason for investigation of this problems is not only because the problem of blow-up of solutions of nonlinear PDE's is important from theoretical point of view, but mainly because these problems are modeling many processes of thermodynamics, hydrodynamic, kinetics of chemical reactions, nonlinear optics, combustion theory.

One of the important question in the blow-up theory is the numerical study of solutions that blow-up in a finite time.

This thesis is devoted to some methods of blow-up of solutions of initial boundary value problems for nonlinear heat equations and their finite difference analogues.

The thesis is organized as follows:

In Chapter 1, we give some necessary information about functional spaces and inequalities used in the squeal.

In Chapter 2, after introducing the concept of blow-up of solutions of semi-linear parabolic equation, following the references [7], [8] and [9], we prove theorems about blow-up of solutions of initial boundary value problem for semi-linear parabolic equations of the form

$$\partial_t u = \Delta u + f(u) \qquad \text{in } \Omega_T = \Omega \times (0,T), \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega. \tag{1.3}$$

In Chapter 3, following [12], we study the problem of blow-up of solutions of finite difference approximation of the initial boundary value problem for the heat equation with quadratic nonlinearity:

$$\partial_t u = \partial_x^2 u + u^2 \qquad 0 < x < 1, \quad t > 0 \tag{1.4}$$

$$u(0,t) = u(1,t) = 0 \qquad t \geq 0 \tag{1.5}$$

$$u(x,0) = u_0(x) \qquad 0 \leq x \leq 1. \tag{1.6}$$

when the exact solution blows-up with the blowing-up time T^*. It is shown first the convergence of the finite difference solution to the exact solution in any time interval $0 < t < T$, where $T < T^*$ and secondly the convergence of numerical blowing-up time to the exact one.

We would like to note that the theory of numerical analysis of blow-up of solutions is extended to the several directions in a number of publications (see, e.g., [1], [3], [4], [13] and [17])

1.2 Function Spaces

Space of Continuous Functions

$C^0(\Omega)$ is the space of continuous functions on Ω with the supremum norm:

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$$

$C^r(\Omega)$ is the space of all functions $f(x)$ defined on Ω and which are continuous on Ω with all their derivatives up to and including order r. The norm on this set is defined as follows:

$$\|f\|_{C^r(\Omega)} = \sum_{|\alpha|\leq r} \sup_{x\in\Omega} |D^\alpha f(x)|$$

$C^\infty(\Omega)$ is the set of all functions which are infinitely differentiable.
C_c^∞ is the space of test functions, defined as :

$$C_c^\infty = \{\eta \in C^\infty : \text{supp}(\eta) \text{ is compact in } \Omega\}$$

where

$$\text{supp}(\eta) = \overline{\{\text{x} \in \Omega : \ \eta(\text{x}) \neq 0\}}$$

Lebesgue Spaces

$L^p(\Omega)$ is the space of Lebesgue integrable functions for $0 \leq p < \infty$ with the following norm:

$$\|f\|_{L^p} = \int_\Omega (|f(x)|^p dx)^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty} = \text{ess}\sup_{x\in\Omega}|f(x)|, \qquad p = \infty.$$

Note that the space $L^p(\Omega)$ is a Banach space. $L^2(\Omega)$ is a Hilbert space with the inner product

$$(f, g) = \int_\Omega f(x)g(x)dx$$

Space of Locally Integrable Functions

$L^p_{loc}(\Omega)$ is the space of locally integrable functions defined as follows:

$$L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable} : f|_K \in L^p(\Omega), \ \forall K \subset \Omega, K \text{ compact}\}$$

Sobolev Spaces $H^1(\Omega)$ and $H^1_0(\Omega)$

Definition 1.2.1 (Weak Derivative)**.** *Assume that $\alpha = (\alpha_1, .., \alpha_n)$ is a given multi-index, u and v are locally integrable functions over Ω. Assume that $\forall h \in C_c^\infty(\Omega)$ the* *following equation is satisfied:*

$$\int_\Omega u(x)D^\alpha h(x)dx = (-1)^{|\alpha|}\int_\Omega v(x)h(x)dx,$$

where

$$D^\alpha h(x) = \frac{\partial^{|\alpha|}h(x)}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}.$$

Then the function v is called α-th weak derivative of u on the domain Ω.

The Sobolev Space $H^1(\Omega)$ is a Hilbert Space of all functions belonging to $L^2(\Omega)$ with all their first order weak derivatives. The inner product and norm in $H^1(\Omega)$ is defined as follows:

$$(u,v)_{H^1(\Omega)} = \int_\Omega \left(u(x)v(x) + \sum_{i=1}^n u_{x_i}(x)v_{x_i}(x) \right) dx$$

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (u(x)^2 + |\nabla u(x)|^2) \right)^{\frac{1}{2}}$$

$H_0^1(\Omega)$ is the completion of C_c^∞ in the sense of the norm $H^1(\Omega)$. So if $u \in H_0^1(\Omega)$, then $u \in H^1(\Omega)$ and there exist a sequence $\{u_n\} \subset C_c^\infty(\Omega)$ such that

$$\|u_n - u\|_{H^1(\Omega)} \to 0 \text{ as } n \to \infty$$

The inner product and the norm in $H_0^1(\Omega)$ is defined as follows:

$$(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u(x) \cdot \nabla v(x)dx$$

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}$$

1.3 Some Useful Inequalities

Lemma 1.3.1 (Cauchy-Schwarz Inequality). *Let V be an inner product space with inner product $(\cdot,\cdot)$. Then for any $u, v \in V$, the following inequality holds:*

$$|(u,v)| \leq \|u\|\|v\|$$

Cauchy-Schwartz Inequality with ϵ: For any $\epsilon > 0$ the following inequality holds:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Lemma 1.3.2 (Hölder Inequality). *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1}(\Omega) \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

Lemma 1.3.3 (Young's Inequality). *If p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any nonnegative a and b the following inequality holds:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Young's Inequality with ϵ: For any $\epsilon > 0$ the following inequality holds:

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

Lemma 1.3.4 (Jensen Inequality). *Assume that f is a convex function in $\mathbb{R}$. Let $\psi \in L^1(\Omega)$ is a positive function and $u \in C(\overline{\Omega})$. Then the following inequality holds:*

$$\frac{\int_\Omega f(u(x))\psi(x)dx}{\int_\Omega \psi(x)dx} \geq f\left(\frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}\right)$$

Proof. Let us consider

$$k_0 = \frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}$$

Since f is a convex function, there exist a number m such that

$$f(u) \geq f(k_0) + m(u - k_0) \tag{1.7}$$

Multiply both sides of equation (1.7) by ψ and integrate over Ω, we will get

$$\int_\Omega f(u)\psi dx \geq f(k_0) \int_\Omega \psi dx + m\left(\int_\Omega u\psi dx - k_0 \int_\Omega \psi dx\right) \tag{1.8}$$

Now, write the value of k_0 in the equation (1.8). Hence we obtain

$$\int_\Omega f(u)\psi(x)dx \geq f\left(\frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}\right) \int_\Omega \psi(x)dx$$

$\square$

Chapter 2

FINITE TIME BLOW-UP FOR PARABOLIC EQUATIONS

In this chapter, we discuss the methods and techniques for studying blow-up in a finite time of solutions to initial boundary value problems for second order parabolic equations. These methods are applicable to more general nonlinear parabolic equations under various boundary conditions, but for simplicity we study the initial boundary value problem for nonlinear heat equation under homogeneous Dirichlet's boundary condition:

$$\partial_t u(x,t) = \Delta u(x,t) + f(u) \quad \text{in } \Omega_T = \Omega \times (0,T), \tag{2.1}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.2}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega.. \tag{2.3}$$

It is well known that if the function $f(s)$ is differentiable on $\mathbb{R}$, then the problem (2.1)-(2.3) has a unique local classical solution.
We are going to discuss the question of global non-existence or blow -up of solutions of this problem. Let us first recall the definition of blow-up.

Definition 2.0.1 (L^∞ blow-up, [7]). *We say that a solution u of the problem* (2.1) *blows-up in a finite time, if there exists a finite time T^* such that*

$$\max_{x\in\overline{\Omega}} |u(x,t)| \to +\infty, \quad as \;\; t \to T^*.$$

Remark. *In the following calculations, we will use the inner product $(,)$ and the norm $\|\cdot\|$ defined as:*

$$(u,v) = \int_\Omega uv dx$$

$$\|u\|^2 = (u,u)$$

2.1 Concavity Method

The concavity method introduced by H.A.Levine in 1973, is a powerful method for proving finite time blow-up of the solutions of the Cauchy problem and initial boundary value problems for nonlinear partial differential equations. The concavity method is an energy method and its idea is based on a construction of a positive functional $\Psi(t) = \Psi(u(t))$, depending on some norm of the local solution of the corresponding problem and proving that the function $\Psi(t)$ satisfies the following inequality

$$\frac{d^2}{dt^2}\Psi^{-\alpha}(t) \leq 0, \tag{2.4}$$

for some positive α, i.e. the function $\Psi^{-\alpha}(t)$ is concave function. Integrating this inequality we find that

$$\Psi^{\alpha}(t) \geq \frac{\Psi^{\alpha+1}(0)}{\Psi(0) - t\Psi'(t)\alpha}.$$

Hence we find that the function $\Psi^{\alpha}(t)$ is bounded from below by a function that blows up in finite time if $\Psi'(0) > 0$. The blow-up must occur before or at the time

$$T^* = \frac{\Psi(0)}{\alpha\Psi'(0)} > 0.$$

So the following lemma holds true:

Lemma 2.1.1. *(see [9]) Let $\Psi(t)$ be a positive, twice differentiable function, which satisfies, for $t > 0$, the inequality*

$$\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi'(t)\right]^2 \geq 0 \tag{2.5}$$

with some $\alpha > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then there exists a time $t_1 \leq \frac{\Psi(0)}{\alpha\Psi'(0)}$ such that $\Psi(t) \to +\infty$ as $t \to t_1$.

Now, we will use Lemma(2.1.1) to illustrate the concavity method by considering the initial boundary value problem (2.1) with

$$f(u) = |u|^{p-1}u \qquad (p > 1) \tag{2.6}$$

Theorem 2.1.2 ([7]). *Let Ω be a smooth domain. If f is given by (2.6) and $u_0(x)$ satisfies*

$$-\frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx + \frac{1}{p+1}\int_\Omega |u_0(x)|^{p+1} dx \ > 0,$$

then the solution (2.1) must blow-up in finite time.

Proof. First, multiply the equation (2.1) by u and integrate over Ω, and obtain

$$\int_\Omega u(x,t)\partial_t u(x,t)dx = \int_\Omega u(x,t)\Delta u(x,t)dx + \int_\Omega |u(x,t)|^{p+1}dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx = -\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^{p+1}dx$$

The second equality gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 = \int_\Omega |u(x,t)|^{p+1}dx. \tag{2.7}$$

Next, multiplying (2.1) with $\partial_t u$ and integrating over Ω gives

$$\|\partial_t u(t)\|^2 = \int_\Omega \partial_t u(x,t)\Delta u(x,t)dx + \int_\Omega |u(x,t)|^p \partial_t u dx$$

or

$$\|\partial_t u(t)\|^2 = \frac{d}{dt}\left[-\frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{p+1}\int_\Omega |u(x,t)|^{p+1}dx\right]. \tag{2.8}$$

Let

$$J(t) = -\frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{p+1}\int_\Omega |u(x,t)|^{p+1}dx$$

Then (2.8) gives

$$J'(t) = \|\partial_t u(t)\|^2 \geq 0$$

which implies that

$$J(t) = J(0) + \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau$$

Introduce a new function

$$\Psi(t) = \int_0^t \|u(\tau)\|^2 d\tau + A$$

where $A > 0$ is to be determined below. Then

$$\Psi'(t) = \|u(t)\|^2,$$

and

$$\Psi''(t) = \frac{d}{dt}\|u(t)\|^2 = -2\|\nabla u(t)\|^2 + 2\int_\Omega |u(x,t)|^{p+1}dx \tag{2.9}$$

where we used (2.7) in deriving the equality (2.9).

By comparing the terms in $J(t)$ and (2.9), for $\delta = \frac{1}{2}(p-1) > 0$, we have

$$\Psi''(t) \geq 4(1+\delta)J(t) = 4(1+\delta)\left(J(0) + \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau\right)$$

Clearly

$$\Psi'(t) = 2\int_0^t\int_\Omega u(x,\tau)\partial_\tau u(x,\tau)dxd\tau + \int_\Omega u_0(x)^2 dx$$

It follows that,

$$\begin{aligned}(\Psi'(t))^2 \leq& 4\left(\int_0^t \|u(\tau)\|^2 d\tau\right)\left(\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau\right)\\ &+ 4\underbrace{\left(\int_0^t \|u(\tau)\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau\right)^{\frac{1}{2}}}_{\text{a}}\underbrace{\|u_0\|^2}_{\text{b}} + \|u_0\|^4\end{aligned}$$

Next we use the Cauchy Schwartz Inequality with ϵ (1.3.1) and obtain:

$$\begin{aligned}(\Psi'(t))^2 \leq& 4\int_0^t \|u(\tau)\|^2 d\tau \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau\\ &+ 4\left(\epsilon\int_0^t \|u(\tau)\|^2 d\tau\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau + \frac{1}{4\epsilon}\|u_0\|^4\right) + \|u_0\|^4\\ =& 4(1+\epsilon)\int_0^t \|u(\tau)\|^2 d\tau\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau + \left(1+\frac{1}{\epsilon}\right)\|u_0\|^4\end{aligned}$$

Combining the above estimates, we find that for $\alpha > 0$,

$$
\begin{aligned}
\Psi''(t)\Psi(t) - &(1+\alpha)\Psi'(t)^2 \\
&\geq \Psi''(t)\Psi(t) + 4(1+\alpha)\left[(1+\epsilon)\int_0^t \|u(\tau)\|^2 d\tau \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau + \left(1+\frac{1}{\epsilon}\right)\|u_0\|^4\right] \\
&\geq 4(1+\delta)\left(J(0) + \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau\right)\left(\int_0^t \|u(\tau)\|^2 d\tau + A\right) \\
&\quad - (1+\alpha)\left(4(1+\epsilon)\int_0^t \|u(\tau)\|^2 d\tau \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau\right) \\
&\quad - (1+\alpha)\left(1+\frac{1}{\epsilon}\right)\|u_0\|^2 \\
&= 4(1+\delta)J(0)\int_0^t \|u(\tau)\|^2 d\tau + 4(1+\delta)J(0)A \\
&\quad + 4(1+\delta)\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau \int_0^t \|u(\tau)\|^2 d\tau \\
&\quad + 4(1+\delta)\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau A \\
&\quad - (1+\alpha)4(1+\epsilon)\int_0^t \|u(\tau)\|^2 d\tau \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau \\
&\quad - (1+\alpha)\left(1+\frac{1}{\epsilon}\right)(\|u_0\|^2)^2
\end{aligned}
$$

Now we choose ϵ and α small enough such that

$$1+\delta \geq (1+\alpha)(1+\epsilon)$$

By our assumption, $J(0) > 0$. Thus we can choose A to be large enough so that

$$\Psi''(t)\Psi(t) - (1+\alpha)\Psi'(t)^2 > 0 \tag{2.10}$$

The result (2.10) shows that $\Psi(t) \to \infty$ by lemma (2.1.1). Below, we explain the reason for that.

The inequality (2.10) implies that

$$\frac{d}{dt}\left(\frac{\Psi'(t)}{\Psi^{\alpha+1}(t)}\right) > 0,$$

so that,

$$\frac{\Psi'(t)}{\Psi^{\alpha+1}(t)} \geq \frac{\Psi'(0)}{\Psi^{\alpha+1}(0)} \quad \text{for } t > 0$$

or

$$-\frac{1}{\alpha}\frac{d}{dt}\left[\Psi^{-\alpha}(t)\right] \geq A_0,$$

where

$$A_0 := \frac{\Psi'(0)}{\Psi^{\alpha+1}(0)}.$$

Integrating last inequality we obtain:

$$\Psi^{-\alpha}(t) \leq -A_0\alpha t + \Psi^{-\alpha}(0)$$

or

$$\Psi^{\alpha}(t) \geq \frac{1}{[\Psi^{-\alpha}(0) - A_0\alpha t]}$$

It follows from the last inequality that

$$\Psi(t) = \int_0^t \|u(\tau)\|^2 d\tau + A$$

cannot remain finite for all t. □

2.2 *Eigenvalue Method*

Eigenvalue method, introduced by S. Kaplan in 1963, (see: [8]) is another powerful method for proving finite time blow-up for nonlinear partial differential equations.

Theorem 2.2.1 ([7]). *Let Ω be a bounded domain with $\partial\Omega \in C^1$. Assume that f is convex (i.e, $f'' \geqslant 0$), and $f(u) > 0$ for $u > 0$ and the condition*

$$\int_M^\infty \frac{du}{f(u) - \lambda_1 u} < \infty, \quad \textit{where} \quad M = (u_0, \phi_1),$$

is satisfied, where λ_1 is the first eigenvalue of the Laplacian corresponding eigenfunction ϕ_1. Let $u \in C(\bar{\Omega}_T) \cap C^2(\bar{\Omega}_T)$ be a classical solution of (2.1). If $M := \int_\Omega u_0(x)\phi_1(x)dx$ is sufficiently large, then the solution u of the problem (2.1)-(2.3) blows-up in finite time.

Proof. Consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi \ \text{ in } \ \Omega, \\ \phi = 0 \quad \text{on} \ \ \partial\Omega. \end{cases} \tag{2.11}$$

Let ϕ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the problem (2.11). It is well known that

$$\lambda_1 = \inf\{\|\nabla u\|_{L^2(\Omega)}; u \in K\}, \quad K = \{u \in H_0^1(\Omega); \|u\|_{L^2(\Omega)} = 1\}.$$

Let

$$\Psi(t) = \int_\Omega u(x,t)\phi_1(x)dx$$

It is clear that $\Psi(t)$ is well defined on the existence interval of the solution u and

$$\Psi'(t) = \int_\Omega \partial_t u(x,t)\phi_1(x)dx.$$

Using equation (2.1) with the boundary condition (2.2) and the equality $-\Delta\phi_1(x) = \lambda_1\phi_1(x)$ we find that

$$\begin{aligned}
\Psi'(t) &= \int_\Omega \partial_t u(x,t)\phi_1(x)dx \\
&= \int_\Omega \Delta u(x,t)\phi_1(x)dx + \int_\Omega f(u)\phi_1(x)dx \\
&= -\int_\Omega \nabla u(x,t)\nabla\phi_1 dx + \int_\Omega f(u)\phi_1 dx \\
&= \int_\Omega u(x,t)\Delta\phi_1(x)dx + \int_\Omega f(u(x,t))\phi_1(x)dx \qquad (2.12) \\
&= -\lambda_1\Psi(t) + \int_\Omega f(u(x,t))\phi_1(x)dx
\end{aligned}$$

Since $\int_\Omega \phi_1(x)dx = 1$ and f is a convex function, we employ the Jensen inequality (1.3.4) and obtain

$$\int_\Omega f(u(x,t))\phi_1(x)dx \geq f\left(\int_\Omega u(x,t)\phi_1(x)dx\right) = f(\Psi(t)).$$

Substituting this into (2.12) , we obtain

$$\Psi'(t) \geq -\lambda_1\Psi(t) + f(\Psi(t)) \qquad (2.13)$$

which yields

$$\int_{\Psi(0)}^{\Psi(t)} \frac{ds}{f(s) - \lambda_1 s} \geq t$$

and if we let

$$\int_{\Psi(0)}^{\infty} \frac{ds}{f(s) - \lambda_1 s} = A_1$$

we get

$$A_1 \geq \int_{\Psi(0)}^{\Psi(t)} \frac{ds}{f(s) - \lambda_1 s} \geq t \tag{2.14}$$

If u remains finite for all t, then $\Psi(t)$ is defined for all t. However, we get from (2.14) that $\Psi(t)$ will blow-up in finite time as $t \to A_1$, provided $\Psi(0)$ is large enough. Therefore we conclude that $u(x,t)$ must blow-up in finite time.

As an example, let $f(u) = u^2$ in equation (2.1). For that case equation (2.13) becomes

$$\Psi'(t) \geq -\lambda_1 \Psi(t) + \Psi^2(t)$$

Define a new function $\xi = e^{\lambda_1 t}\Psi(t)$ then we get

$$\xi'(t) = e^{\lambda_1 t}\Psi'(t) + \lambda_1 e^{\lambda_1 t}\Psi(t) \geq e^{\lambda_1 t}\Psi^2(t) = e^{-\lambda_1 t}\xi^2(t)$$

for $0 \leq t \leq T$. Thus

$$\left(-\frac{1}{\xi(t)}\right)' = \frac{\xi'(t)}{\xi^2(t)} \geq e^{-\lambda_1 t}$$

hence, by integrating we find that

$$-\frac{1}{\xi(t)} \geq -\frac{1}{\xi(0)} + \frac{1 - e^{-\lambda_1 t}}{\lambda_1 t}$$

or

$$\xi(t) \geq \frac{\xi(0)\lambda_1}{\lambda_1 - \xi(0)(1 - e^{-\lambda_1 t})}$$

if $\lambda_1 - \xi(0)(1 - e^{-\lambda_1 t}) \neq 0$. Now, suppose

$$\Psi(0) = \xi(0) > 0.$$

Then we have

$$\xi(t) \to \infty \quad \text{as} \quad t \to t^*$$

where

$$t^* = -\frac{1}{\lambda_1} \log\left(\frac{\Psi(0) - \lambda_1}{\Psi(0)}\right)$$

As a conclusion we can say that if

$$\Psi(0) = \int_\Omega u(x,0)\phi_1(0)dx = \int_\Omega u_0\phi_1 dx > \lambda_1$$

then the smooth solution u of (2.1)-(2.3) cannot exist or else,

$$\int_\Omega u(x,t)\phi_1(x)dx \to \infty \quad \text{as} \quad t \to t_1$$

for some $0 < t_1 \leq t^*$. In this case we say u blows up at time $t = t_1$. □

2.3 Comparison Method

For semilinear heat equations of the form

$$\partial_t u = \Delta u + f(u)$$

blow-up can also be established by using the comparison principle. The comparison method asserts that for a classical solution $u(x,t)$ of (2.1) - (2.3) if there exist a function $v(x,t)$ such that $v(x,t) \leq u(x,t)$ on $\Omega_T = \Omega \times (0,T)$ then the solution u must blow-up if the subsolution v blows up in a finite time.

We will prove below the weak maximum principle for solutions of the second order parabolic equation and show that a solution of the equation under some restrictions on coefficients must attain its maximum or minimum only on the parabolic boundary of the domain. Weak maximum principles yield the comparison principle for solutions of nonlinear second order parabolic equations. Employing the comparison theorem we will obtain a result on blow-up of solutions in a finite time of the problem (2.1) - (2.3).

Weak maximum principle:

The idea of the comparison method is based on the following well known maximum principle

Theorem 2.3.1 (Weak Maximum Principle, [10]). *Suppose $u \in C(\overline{\Omega}_T) \cap C^2(\Omega_T)$ be a solution of the equation*

$$Lu + cu = f(x,t), \quad (x,t) \in \overline{\Omega}_T$$

in which

$$Lu = a(x,t)u_{xx} + b(x,t)u_x - u_t$$

where $\Omega_T = \Omega \times (0,T)$*,* f *and* $c \in C(\Omega_T)$*,* $c(x,t) \leq 0$*,* $f(x,t) \geq 0$ *in* $\overline{\Omega}_T$ *with* a *and* b *are continuous functions and* $a(x,t) \geq m > 0$ *for some constant* m*. Then* u *attains its maximum value on the parabolic boundary of the domain* Ω_T*, i.e. on the set*

$$B_T = \{ either \ \ x = 0, x = L, t \in [0,T], \quad or \ \ x \in (0,L), t = 0 \} .$$

Proof. Assume that the solution $u(x,t)$ attains its maximum in $\overline{\Omega}_T$. We need to show that this maximum must occur somewhere on the parabolic boundary B_T. For this purpose define an auxiliary function:

$$v(x,t) = u(x,t) - \epsilon t.$$

By our assumption u has a positive maximum in $\overline{\Omega}_T$ so v has also a positive maximum in $\overline{\Omega}_T$ if we take ϵ sufficiently small. Assume on the contrary that the positive maximum of v occurs at a point (x_0,t_0) in Ω_T. Then we have

$$v_x(x_0,t_0) = 0, \quad v_{xx}(x_0,t_0) \leq 0.$$

Now, calculate v_t :

$$\begin{aligned}
v_t &= u_t - \epsilon = a u_{xx} + b u_x + cu - f - \epsilon \\
&= a v_{xx} + b v_x + c(v + \epsilon t) - f - \epsilon \\
&\leq cv + \epsilon tc - \epsilon \quad \text{at} \ \ (x_0,t_0) \\
&\leq -\epsilon
\end{aligned}$$

Next, choose a number $d > 0$ such that $v_t(x_0,t) \leq -\epsilon/2$ for $t_0 - d \leq t \leq t_0$. Integration gives

$$v(x_0,t_0) - v(x_0,t_0-d) \leq \int_{t_0-d}^{t_0} -\frac{\epsilon}{2} dt = -\frac{\epsilon d}{2} < 0.$$

This result contradicts with the assumption that $v(x,t)$ has a positive maximum at (x_0,t_0). Therefore positive maximum must occur on the boundary B_T. As a final remark we can write:

$$\begin{aligned}
\max_{\overline{\Omega}} u &= \max_{\overline{\Omega}} (v + \epsilon t) \leq \max_{\overline{\Omega}} v + \epsilon T \\
&= \max_{B_T} v + \epsilon T = \max_{B_T} (u - \epsilon t) + \epsilon T \leq \max_{B_T} u + \epsilon T.
\end{aligned}$$

Since ϵ is arbitrarily small we can conclude that $\max\limits_{\overline{\Omega}} u \leq \max\limits_{B_T} u$ means the maximum must occur somewhere on the boundary B_T. □

Comparison Principle

Consider the semilinear equation

$$u_t - a(x,t)u_{xx} - b(x,t)u_x = f(x,u) \tag{2.15}$$

on the domain $\Omega_T := (0,L) \times (0,T)$. Here a and b are continuous functions on $\overline{\Omega}_T$ and $a(x,t) \geq m > 0, \quad \forall (x,t) \in \overline{\Omega}_T$. The function f is continuously differentiable on $[0,L] \times \mathbb{R}$. By using the weak maximum principle we can prove the following:

Theorem 2.3.2 (Comparison Theorem, [10]). *Suppose that $f \in C^1([0,L],\mathbb{R})$ is a given function and $u, v \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ be two solutions of the problem* (2.15) *under the assumptions written above. If $u \leq v$ on the boundary B_T then $u \leq v$ in Ω_T. [In our case, $a(x,t) = 1, b(x,t) = 0$ and $f(x,u) = f(u)$ in equation* (2.15)*.]*

Proof. Let $w = u - v$. Then $w \leq 0$ on B_T and

$$a(x,t)w_{xx} + b(x,t)w_x - w_t = f(x,v) - f(x,u) \ \text{ in } \ \Omega_T.$$

By using mean value theorem we get:

$$f(x,u) - f(x,v) = f_u(x,u^*)w, \qquad u^* = \theta u + (1-\theta)v \ \text{ for } \ 0 < \theta < 1.$$

Therefore we have

$$\begin{cases} a(x,t)w_{xx} + b(x,t)w_x + f_u w - w_t = 0 \ \text{ on } \ \Omega_T \\ w \leq 0 \ \text{ in } \ B_T \end{cases} \tag{2.16}$$

such that (2.16) has the form that we can apply the weak maximum principle. Since $u, v \in C(\overline{\Omega}_T)$ and f_u is continuous on $[0,L] \times \mathbb{R}$ the function $f_u(u^*(x,t))$ is bounded on $\overline{\Omega}_T$. First assume $f_u(u^*(x,t)) \leq 0$ on $\overline{\Omega}_T$. Then by weak maximum principle $w \leq 0$ in Ω_T. Since if w were positive at some point in Ω_T that maximum must occur on B_T, which is a contradiction. If the function $f_u(u^*(x,t))$ takes positive values in $\overline{\Omega}_T$ we take a number $\lambda > 0$ such that $\lambda \geq f_u$ on Ω_T and let $w = We^{\lambda t}$. Then W will satisfy

$$\begin{cases} a(x,t)W_{xx} + bW_x + (f_u - \lambda)W - W_t = 0 \ \text{ in } \ \Omega_T \\ W \leq 0 \ \text{ on } \ B_T \end{cases}$$

Hence by weak maximum principle $W \leq 0$ in Ω_T which implies that $w \leq 0$ in Ω_T. □

Thanks to this comparison theorem the following theorem holds true.

Theorem 2.3.3. *Suppose that $f \in C^1(\mathbb{R})$ such that the solution $y(t)$ of the problem*

$$\begin{cases} \frac{d}{dt}y(t) = f(y(t)), \\ y(0) = y_0 \end{cases} \tag{2.17}$$

such that

$$y(t) \to +\infty, \quad as \quad t \to T_0^-$$

for some $T_0 > 0$. Suppose also that $u(x,t)$ is a solution of the problem

$$\begin{cases} \partial_t u - \partial_x^2 u = f(u), \ x \in (0,L), t > 0, \\ u(0,t) = \phi_1(t), \quad u(L,t) = \phi_2(t), \ t > 0, \\ u(x,0) = \psi(x), \quad x \in (0,L). \end{cases} \tag{2.18}$$

where ϕ_1, ϕ_2, ψ are continuous functions such that

$$\phi_1(t) \geq y_0, \quad \phi_2(t) \geq y_0, \quad \forall t > 0, \ \psi(x) \geq y_0, \ \forall x \in [0,L].$$

and

$$\phi_1(0) = y_0, \quad \phi_2(L) = y_0.$$

Then there exists $T_1 \leq T_0$ such that

$$\max |u(x,t)| \to +\infty, \quad as \ \ t \to T_1.$$

Proof. It is clear that the ODE solution $v(x,t)$of (2.17) is a solution of the problem:

$$\begin{cases} \partial_t v - \partial_x^2 v = f(v) \\ v(x,0) = y_0, \quad x \in (0,L) \\ v(0,t) = v(L,t) = y_0 \end{cases}$$

□

and $v(x,t) \leq u(x,t)$ on $(0,L)$. Then we have $v(x,t) \leq u(x,t) \;\; \forall \;\; (x,t) \in (0,L) \times \mathbb{R}$ by comparison theorem. Since we are given $y(t) = v(x,t)$ blows up as $t \to T_0^-$, we conclude that $\max |u(x,t)| + \infty$ as $t \to T_1, \; T_1 \leq T_0$.

Chapter 3

BLOW-UP OF SOLUTIONS OF FINITE DIFFERENCE EQUATION

In this chapter, we study a finite difference scheme for the approximation to the semi-linear heat equation

$$\partial_t u = \partial_x^2 u + u^2, \quad 0 < x < 1, \quad t > 0 \tag{3.1}$$

subject to the Dirichlet boundary condition

$$u(0,t) = u(1,t) = 0, \quad t \geq 0 \tag{3.2}$$

and the initial condition

$$u(x,0) = a(x) \quad 0 \leq x \leq 1, \tag{3.3}$$

where $a(x)$ is a non-negative not identically zero continuous function on the interval $[0,1]$. Our main goal is to understand character of blow-up of solution of the problem (3.1) - (3.3) employing numerical method. We assume that the exact solution blows-up with the blowing-up time T^*. Here blowing-up is considered in L^2 norm.

3.1 *Blow-up in Semilinear Heat Equation*

In this section we introduce a functional $J(t)$ and illustrate the blow-up occurrences in equations (3.1) - (3.3). Let

$$J(t) = -\frac{1}{2}\|\partial_t u(t)\|^2 + \frac{1}{3}(u^3(t),1) \tag{3.4}$$

in which $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and the norm of $L^2(0,1)$:

$$(f,g) = \int_0^1 f(x)g(x)dx \qquad \text{and} \qquad \|f\|^2 = (f,f).$$

Now, we will prove the some lemmas about the properties of $J(t)$ and the blow-up solutions of (3.1) - (3.3).

Lemma 3.1.1. *The function $J(t)$ is monotonically increasing:*

Proof. Multiply the equation (3.1) by $\partial_t u$ and integrate over $0 \leq x \leq 1$

$$\int_0^1 (\partial_t u(x,t))^2 dx = \int_0^1 \partial_x^2 u(x,t)\partial_t u(x,t)dx + \int_0^1 u^2(x,t)\partial_t u(x,t)dx$$
$$= \partial_t u(x,t)\partial_x u(x,t) - \int_0^1 \partial_{xt} u(x,t)\partial_x u(x,t)dx + \int_0^1 u^2(x,t)\partial_t u(x,t)dx$$

This leads to

$$\|\partial_t u(t)\|^2 = -\int_0^1 \partial_{xt} u(x,t)\partial_x u(x,t)dx + \int_0^1 u^2(x,t)\partial_t u(x,t)dx = \frac{d}{dt}J(t) \geq 0$$

□

Lemma 3.1.2. *If $J(0) > 0$, then the solution blows up. The blowing up time T^* is bounded from above by the number*

$$T^* \leq \frac{1}{\frac{1}{3}\|u(0)\|}.$$

Proof. Multiply (3.1) by u and integrate over $0 \leq x \leq 1$

$$\int_0^1 u(x,t)\partial_t u(x,t)dx = \int_0^1 u(x,t)\partial_x^2 u(x,t)dx + \int_0^1 u^3(x,t)dx$$
$$= u(x,t)\partial_x u(x,t) - \int_0^1 \partial_x u^2(x,t)dx + \int_0^1 u^3(x,t)dx$$

This leads to

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = -\|\partial_x u(t)\|^2 + (u^3(t),1) \tag{3.5}$$

Our assumption is $J(t_0) > 0$ and by Lemma (3.1.1) we have $J(t) > J(t_0) > 0$ for $t > t_0$.

Since

$$J(t) = -\frac{1}{2}\|\partial_x u(t)\|^2 + \frac{1}{3}(u^3(t),1) > 0$$

we have:

$$-\|\partial_x u(t)\|^2 > -\frac{2}{3}(u^3(t), 1)$$

Employing this inequality in (3.5) we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 \geq \frac{1}{3}(u^3(t), 1), \quad t > t_0 \tag{3.6}$$

By the use of Hölder's inequality, the right-hand side of (3.6) is made bounded from below by $\|u(t)\|^3$. In fact, employing Hölder's inequality

$$\int |f(x)g(x)|dx \leq (\int |f(x)|^p dx)^{\frac{1}{p}} (\int |g(x)|^q dx)^{\frac{1}{q}}$$

for $f = u^2, \quad g = 1$ with $p = \frac{3}{2}, \quad q = 3$ and obtain

$$\int u^2(x,t)dx \leq \left(\int_0^1 u^3(x,t)dx\right)^{\frac{2}{3}} \left(\int_0^1 1dx\right)^{\frac{1}{3}} = \left(\int_0^1 u^3(x,t)dx\right)^{\frac{2}{3}}$$

So we have

$$\int_0^1 u^3(x,t)dx \geq \left(\int u^2(x,t)dx\right)^{\frac{3}{2}}$$

or

$$(u^3(t), 1) \geq \|u(t)\|^3$$

Thus (3.6) implies

$$\frac{d}{dt}\|u(t)\|^2 \geq \frac{2}{3}\|u(t)\|^3 \tag{3.7}$$

Now, put $Z(t) = \|u(t)\|^2$. Then (3.7) takes form:

$$-\frac{d}{dt}[Z(t)^{-\frac{1}{2}}] \geq \frac{1}{3}$$

Integrating this inequality we get

$$Z(0)^{-\frac{1}{2}} - Z(t)^{-\frac{1}{2}} \geq \frac{t}{3}$$

or

$$\sqrt{Z(t)} \geq \frac{3}{\frac{3}{\sqrt{Z(0)}} - t}$$

Hence

$$\|u(t)\| \to \infty, \quad \text{as} \quad t \to T_*^-,$$

where $T_* = \frac{3}{\|u_0\|}$.

□

Lemma 3.1.3. *Given a nonnegative function $\phi(x)$ that satisfies $\phi(x) \in C^1[0,1]$, and $\phi(0) = \phi(1) = 0$, take the initial value $a(x)$ as*

$$a(x) = \alpha\phi(x)$$

If

$$\alpha > \frac{1}{2}\|\phi'\|^2 \left[\frac{1}{3}(\phi^3, 1)\right]^{-1},$$

then the solution of the problem blows up in a finite time.

Proof. For this concrete type of initial data we have

$$J(0) = -\frac{1}{2}\|\partial_x u(0)\|^2 + \frac{1}{3}(u^3(0), 1) = -\frac{1}{2}\alpha^2\|\phi'\|^2 + \frac{1}{3}\alpha^3(\phi^3, 1).$$

Thus

$$\alpha^2 \left\{-\frac{1}{2}\|\phi'\|^2 + \frac{1}{3}\alpha(\phi^3, 1)\right\} > 0,$$

i.e. $J(0) > 0$. Hence by Lemma (3.1.2) the solution blows up. □

Lemma 3.1.4. *Suppose that the solution blows up with the blowing-up time $t = T^*$. Then, there exists $t_0, \quad t_0 < T^*$ such that*

$$J(t_0) > 0.$$

Proof. Thanks to Newton- Leibniz formula, we have

$$u(x,t) = \int_0^t \partial_\tau u(x,\tau)d\tau + u(x,0)$$

Hence

$$|u(x,t)| \leq \int_0^t |\partial_\tau u(x,\tau)|d\tau + |u(x,0)|$$

and

$$|u(x,t)|^2 \leq \left[\int_0^t |\partial_\tau u(x,\tau)|d\tau + |u(x,0)|\right]^2 \leq 2\left[\int_0^t |\partial_\tau u(x,\tau)|d\tau\right]^2 + 2|u(x,0)|^2.$$

By Cauchy-Schwarz Inequality (1.3.1) we have

$$\begin{aligned}|u(x,t)|^2 &\leq 2\int_0^t d\tau \int_0^t |\partial_\tau u(x,\tau)|^2 d\tau + 2|u(x,0)|^2 \\ &\leq 2T^* \int_0^t |\partial_\tau u(x,\tau)|^2 d\tau + 2|u(x,0)|^2, \quad 0 < t < T^*.\end{aligned}$$

By integrating both sides over $[0,1]$, it follows that

$$\|u(t)\|^2 \leq 2T^* \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau + 2\|u(0)\|^2 \quad 0 < t < T^* \cdot$$

Since $\|u(t)\|^2$ tends to infinity as $t \to T^*$, so does $\int_0^t \|\partial_\tau u(\tau)\|^2 d\tau$ as $t \to T^*$. Consequently,

$$J(t) = J(0) + \int_0^t \|\partial_\tau u(\tau)\|^2 d\tau \to \infty \quad as \;\; t \to T^* \cdot$$

we get $J(t_0) > 0$ for some $t_0 < T^*$. □

Lemma 3.1.5. *Solution of the problem* (3.1)-(3.3) *does not blow-up in a finite time if and only if*

$$J(t) \leq 0.$$

Proof. Lemma (3.1.2) says that if $J(t) > 0$ the solution blows up and lemma (3.1.4) states that if solution blows up then $\exists \;\; t_0 < T_\infty$ such that $J(t_0) > 0$. Hence the result. □

3.2 Finite Difference Scheme

In this section we consider the finite difference approximation for (3.1) - (3.3) with fixed grid size h and time increment Δt_n. Now, let us state the sufficient definitions.

$$x_i = i \times h (i = 0, 1, \cdots, I)$$

denote the i-th mesh point on the x-interval $[0,1]$ with the mesh size of $h = \frac{1}{I}$, and we denote by t_n the n-th mesh point on the t -axis, $t \geq 0$, defined by

$$t_0 = 0$$
$$t_n = t_{n-1} + \Delta t_{n-1} \equiv \sum_{k=0}^{n-1} \Delta t_k, \quad (\Delta t_k > 0, k = 0, 1, 2, \cdots)$$

and the parameters τ and λ defined as follows

$$\begin{cases} \tau = \max\limits_{1 \leq j \leq n} \Delta t_k \\ \lambda = \tau / h^2 \end{cases}$$

The finite difference scheme is as follows:

$$\begin{aligned} D_t v^{i,n} &= D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2 \\ v^{i,n} &= v^{I,n} = 0 \\ v^{i,0} &= a(x_i) \end{aligned} \tag{3.8}$$

where D_t denotes the forward difference in t, and D_x and $D_{\bar{x}}$ the forward and backward differences in x, respectively. That is:

$$\begin{cases} D_t v^{i,n} \equiv \frac{1}{\Delta t_n}(v^{i,n+1} - v^{i,n}) \\ D_x v^{i,n} \equiv \frac{1}{h}(v^{i+1,n} - v^{i,n}) \\ D_{\bar{x}} v^{i,n} \equiv \frac{1}{h}(v^{i,n} - v^{i-1,n}) \end{cases}$$

in which $v^{i,n} = v(x_i, t_n)$ is the finite difference approximation of the solution (3.1) at point (x_i, t_n). The solution of (3.8) satisfies:

Lemma 3.2.1. *If* $0 < \lambda \leq \frac{1}{2}$, *then* $v^{i,k} \geq 0$ *for all* i $(i = 0, 1, \cdots, I)$ *and all* k $(k = 0, 1, \cdots, n)$·

Proof. Write (3.8) $\;D_t v^{i,n} = D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2\;$ explicitly

$$
\begin{aligned}
\frac{v^{i,n+1} - v^{i,n}}{\Delta t_n} &= D_x \left(\frac{v^{i,n} - v^{i-1,n}}{h} \right) + (v^{i,n})^2 \\
&= \frac{1}{h} \left(\frac{v^{i+1,n} - v^{i,n}}{h} + \frac{-v^{i,n} + v^{i-1,n}}{h} \right) + (v^{i,n})^2 \\
&= \frac{1}{h^2} (v^{i+1,n} - 2v^{i,n} + v^{i-1,n}) + (v^{i,n})^2
\end{aligned}
$$

$$
v^{i,n+1} = \lambda_n v^{i+1,n} + (1 - 2\lambda_n) v^{i,n} + \lambda_n v^{i-1,n} + \Delta t_n (v^{i,n})^2
$$

where $\lambda_n = \Delta t_n / h^2 \cdot$ Since $0 < \lambda_n < \frac{1}{2}$ holds, $v^{i,n+1}$ assume nonnegative values if $v^{i,n}$ are nonnegative. By knowing that the initial value $a(x)$ is nonnegative by assumption($v^{i,0} = a(x)$), the proof is completed. □

3.3 Local Convergence

In this section, we will state the local convergence result in Proposition (3.3.1) which proves the convergence of finite difference solution of (3.1) to the exact solution. In this result, the parameter λ plays a key role.

Proposition 3.3.1. *Fix $\lambda = \tau / h^2$ such that $0 < \lambda \leq \frac{1}{2}$. Then the estimates*

$$
\begin{cases}
\max\limits_{0 \leq i \leq I} |v^{i,k} - u(x_i, t_k)| \leq c_0(T) \cdot h^2 \\
\max\limits_{0 \leq i \leq I} |D_{\bar{x}} v^{i,k} - \partial_x u(x_i, t_k)| \leq c_1(T) \cdot h \\
\max\limits_{0 \leq i \leq I-1} |D_x v^{i,k} - \partial_x u(x_i, t_k)| \leq c_2(T) \cdot h
\end{cases}
$$

hold for $k = 0, 1, \cdots, n-1$, if t_n lies in the interval $[0, T]$, $T < T^*$, and τ is chosen as

$$
0 < \tau < \frac{4}{R} U^2 / (4 + e^{2UT})
$$

$$
\begin{cases}
U = \max\limits_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} |u(x,t)| \\
R = \frac{1}{2} \max\limits_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} |\partial_t^2 u(x,t)| + \frac{1}{12\lambda} \max\limits_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} \left| {\partial_x}^4 u(x,t) \right|
\end{cases}
$$

Here $c_0(t), c_1(t)$, and $c_2(t)$ are some functions of t.

Proof. By Taylor expansion theorem we have

$$u_{i+1} = u(x_{i+1}, t_k) = u(x_i + h, t_k) = u(x_i, t_k) + h\partial_x u\Big|_{(x_i,t_k)} + \frac{h^2}{2!}{\partial_x}^2 u\Big|_{(x_i,t_k)} + \frac{h^3}{3!}{\partial_x}^3 u\Big|_{(x_i,t_k)} + \frac{(h)^4}{4!}{\partial_x}^4 u\big|_{(x_i+\theta_1 h,t_k)}$$

$$u_{i-1} = u(x_{i-1}, t_k) = u(x_i - h, t_k) = u(x_i, t_k) - h\partial_x u\Big|_{(x_i,t_k)} + \frac{h^2}{2!}{\partial_x}^2 u\Big|_{(x_i,t_k)} - \frac{h^3}{3!}{\partial_x}^3 u\Big|_{(x_i,t_k)} + \frac{(h)^4}{4!}{\partial_x}^4 u\big|_{(x_i-\theta_2 h,t_k)}$$

$$u^{k+1} = u(x_i, t_{k+1)} = u(x_i, t_k+\Delta t_k) = u(x_i, t_k)+\Delta t_k\partial_t u\Big|_{(x_i,t_k)} + \frac{(\Delta t_k)^2}{2!}({\partial_t}^2 u)\Big|_{(x_i,t_k+\kappa\Delta t_k}$$

which gives

$$\partial_t u\Big|_{(x_i,t_k)} = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{1}{2}\Delta t(\partial_t^2 u).$$

and

$$\partial_x^2 u\Big|_{(x_i,t_k)} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{h^2}{24}[{\partial_x}^4 u(x_i + \theta_1 h, t_k) + {\partial_x}^4 u(x_i - \theta_2 h, t_k)]$$

Then we can write

$$\begin{aligned}
\partial_t u\Big|_{(x_i,t_k)} &= D_t u(x_i, t_k) - \frac{1}{2}\Delta t_k \partial_t^2 u\Big|_{(x_i,t_k+\kappa\Delta t_k)} \\
\partial_x^2 u\Big|_{(x_i,t_k)} &= D_x D_{\bar{x}} u + \frac{1}{24}[{\partial_x}^4 u(x_i + \kappa_1 h, t_k) + \partial_x^4 u(x_i - \kappa_2 h, t_k)].
\end{aligned}$$

Define $\epsilon^{i,k}$ by

$$\epsilon^{i,k} = v^{i,k} - u(x_i, t_k)\cdot$$

The $\epsilon^{i,k}$ satisfies

$$\begin{aligned}
D_t\epsilon^{i,k} &= D_t(v^{i,k} - u(x_i, t_k)) \\
&= D_t v^{i,k} - D_t u(x_i, t_k) - r^{i,k} \\
&= D_x D_{\bar{x}} v^{i,k} + (v^{i,k})^2 - D_x D_{\bar{x}} u(x_i, t_k) - (u(x_i, t_k))^2 - r^{i,k} \\
&= D_x D_{\bar{x}}\epsilon^{i,k} + \epsilon^{i,k}(v^{i,k} + u(x_i, t_k)) - r^{i,k} \\
&= D_x D_{\bar{x}}\epsilon^{i,k} + \epsilon^{i,k}(\epsilon^{i,k} + 2u(x_i, t_k)) - r^{i,k}
\end{aligned}$$

in which

$$r^{i,k} = \frac{1}{2}\Delta t_k {\partial_t}^2 u(x_i, t_k + \theta\Delta t_k) + \frac{1}{24}[{\partial_x}^4 u(x_i + \kappa_1 h, t_k) + {\partial_x}^4 u(x_i - \kappa_2 h, t_k)]$$
$$0 < \theta < 1, \quad 0 < \kappa_1 < 1, \quad 0 < \kappa_2 < 1$$

Let $E^k = \max_{0 \le i \le I} |\epsilon^{i,k}|\cdot$ Then we have

$$\begin{cases} E^{k+1} \le E^k + \Delta t_k E^k (E^k + 2U) + \Delta t_k \cdot \tau \cdot R, \quad k = 0, 1, 2, \\ E^0 = 0. \end{cases}$$

Let us consider the Cauchy problem

$$\begin{cases} \frac{dy}{dt} = y(y + 2U) + \tau R, \\ E^0 = 0. \end{cases} \tag{3.9}$$

By induction we can show that E^k is bounded from above by the solution $y = y(t)$ of the problem (3.9) at $t = t_k$. It is not difficult to see that

$$y(t) = \frac{\tau R}{U + \sqrt{U^2 - \tau R}}(e^{2\sqrt{U^2 - \tau R}\ t} - 1)\left[1 - \frac{U - \sqrt{U^2 - \tau R}}{U + \sqrt{U^2 - \tau R}} e^{2\sqrt{U^2 - \tau R}\ t}\right]^{-1}$$

is the solution of the problem (3.9), provided that $U^2 - \tau R > 0$ and that the denominator $U + \sqrt{U^2 - \tau R} \ne 0$. Consequently we have

$$\begin{aligned} E^k \le y(t_k) &\le \frac{\tau R}{U + \sqrt{U^2 - \tau R}}(e^{2UT} - 1)\left[1 - \frac{\pi R}{4(U^2 - \pi R)} e^{2UT}\right]^{-1} \\ &\le c_0(T) \cdot h^2 \end{aligned}$$

under the condition

$$0 < \pi R < 4U^2/(4 + e^{2UT}).$$

Now, since

$$D_{\bar{x}} v^{i,k} = \frac{1}{h}(v^{i,k} - v^{i-1,k})$$

by definition, we obtain the following estimate

$$\begin{aligned}
&\left|D_{\bar{x}}v^{i,k} - \partial_x u(x_i, t_k)\right| \\
&\leq \left|\frac{1}{h}(u(x_i,t_k) - u(x_{i-1},t_k)) - \partial_x u(x_i,t_k) - \frac{1}{h}(\epsilon^{i,k} - \epsilon^{i-1,k})\right| \\
&\leq \frac{h}{2}\left|\partial_x^2 u(x_i - \theta h, t_k)\right| + \frac{1}{h}(|\epsilon^{i,k}| - |\epsilon^{i-1,k}|) \\
&\leq c_1(T)\cdot h.
\end{aligned}$$

Similarly we get,

$$D_x v^{i,k} = \frac{1}{h}(v^{i+1,k} - v^{i,k}) = \frac{1}{h}(u_{i+1} + \epsilon^{i+1,k} - u_i - \epsilon^{i,k})$$

$$\begin{aligned}
\left|D_x v^{i,k} - \partial_x u(x_i,t_k)\right| &= \left|\frac{1}{h}(u_{i+1} - u_i) - \partial_x u(x_i,t_k) + \frac{1}{h}(\epsilon^{i+1,k} - \epsilon^{i,k})\right| \\
&\leq \frac{h}{2}\left|\partial_x^2 u(x_i + \theta h, t_k)\right| + \frac{1}{h}(|\epsilon^{i+1,k}| - |\epsilon^{i,k}|) \\
&\leq c_2(T)\cdot h
\end{aligned}$$

□

3.4 Convergence of Numerical Blow-up Time

In this section, we will define differential inequality $\hat{J}^n$ as a discrete analogue of $J(t)$ and we will illustrate the numerical convergence result in Proposition (3.4.4) after explaining some useful lemmas.

$$\hat{J}^n \equiv -\frac{1}{2}\sum_{i=1}^{I} h(D_{\bar{x}}v^{i,n})^2 + \frac{1}{3}\langle (v^n)^3, 1\rangle,$$

where

$$\langle f, g\rangle := \sum_{i=0}^{I} h\cdot f^{i,n}\cdot g^{i,n}$$

with the norm

$$\|f^n\|^2 = \langle f^n, f^n\rangle.$$

Now we will show that $\hat{J}^n$ has similar properties to $J(t)$. First we prove the following lemma:

Lemma 3.4.1. *Fix λ such that $0 < \lambda \le \frac{1}{2}$. Then $\hat{J}^n$ is monotonically increasing, i.e.,*

$$D_t \hat{J}^n \equiv \frac{1}{\Delta t_n}(\hat{J}^{n+1} - \hat{J}^n) \ge 0.$$

Proof. By direct computation we get

$$\begin{aligned}
&\frac{1}{3}\sum_{i=0}^{I} h\{(v^{i,n+1})^3 - (v^{i,n})^3\} \\
&\qquad = \Delta t_n \sum_{i=0}^{I} h(D_t v^{i,n})(v^{i,n})^2 + \frac{1}{3}(\Delta t_n)^2 \sum_{i=0}^{I} h(D_t v^{i,n})^2(2v^{i,n} + v^{i,n+1})
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=0}^{I} h\{(D_{\bar{x}} v^{i,n+1})^2 - (D_{\bar{x}} v^{i,n})^2\} \\
&\qquad = -\Delta t_n \sum_{i=0}^{I} h(D_{\bar{x}} D_t v^{i,n})(D_{\bar{x}} v^{i,n}) - \frac{1}{2}(\Delta t_n)^2 \sum_{i=0}^{I} h|D_{\bar{x}} D_t v^{i,n}|^2 \\
&\qquad = \Delta t_n \sum_{i=0}^{I} h(D_t v^{i,n})(D_x D_{\bar{x}} v^{i,n}) - \frac{1}{2}(\Delta t_n)^2 \sum_{i=0}^{I} h|D_{\bar{x}} D_t v^{i,n}|^2.
\end{aligned}$$

In the last equality we employed the indentity

$$\sum_{i=0}^{I} h(D_{\bar{x}} v^i)(D_{\bar{x}} w^i) \equiv -\sum_{i=0}^{I-1} h v^i (D_x D_{\bar{x}} w^i) \tag{3.10}$$

which holds for each v^i and w^i $(i = 1, 2, \cdots, I-1)$ under the Dirichlet boundary conditions $v^0 = v^I = 0$ and $w^0 = w^I = 0$. Hence we have

$$\begin{aligned}
D_t \hat{J}^n =& \frac{1}{\Delta t_n}(\hat{J}^{n+1} - \hat{J}^n) \\
=& \frac{1}{\Delta t_n}\left(-\frac{1}{2}\sum_{i=0}^{I} h(D_{\bar{x}} v^{i,n+1})^2 + \frac{1}{3}\sum_{i=0}^{I} h(v^{i,n+1})^3 + \frac{1}{2}\sum_{i=0}^{I} h(D_{\bar{x}} v^{i,n})^2 - \frac{1}{3}\sum_{i=0}^{I} (v^{i,n})^3 \right) \\
=& \sum_{i=0}^{I-1} (D_t v^{i,n})(D_x D_{\bar{x}} v^{i,n}) - \frac{1}{2}(\Delta t_n)\sum_{i=0}^{I} h(D_{\bar{x}} D_t v^{i,n})^2 \\
&+ \Delta t_n \sum_{i=0}^{I} h(D_t v^{i,n})(v^{i,n})^2 + \frac{1}{3}(\Delta t_n)\sum_{i=0}^{I} h(D_t v^{i,n})^2 (2v^{i,n} + v^{i,n+1}) \\
=& \sum_{i=0}^{I-1} h(D_t v^{i,n})\{D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2\} \\
&+ \frac{1}{3}\Delta t_n \sum_{i=0}^{I} h(D_t v^{i,n})^2 \{2v^{i,n} + v^{i,n+1}\} \\
&- \frac{1}{2}\Delta t_n \sum_{i=0}^{I} h|D_{\bar{x}} D_t v^{i,n}|^2 \\
\geq& \sum_{i=0}^{I-1} h|D_t v^{i,n}|^2 - \frac{1}{2}\Delta t_n \sum_{i=0}^{I} h|D_{\bar{x}} D_t v^{i,n}|^2.
\end{aligned}$$

It may be proved that

$$\sum_{i=0}^{I} h|D_{\bar{x}} w^i|^2 \leq \frac{K}{h^2}\sum_{i=0}^{I} h|w^i|^2$$

holds for any $w^i \quad (i = 0, 1, 2, \cdots, I)$ with a choice of $K = 4$. Thus, $D_t \hat{J}^n$ is bounded from below as

$$D_t \hat{J}^n \geq \left(1 - \frac{\Delta t_n}{\tau}\frac{2\tau}{h^2}\right)\sum_{i=0}^{I} h|D_t v^{i,n}|^2.$$

The right-hand side is non-negative if $0 < \lambda \leqslant \frac{1}{2}$ □

Lemma (3.4.1) yields the following lemma.

Lemma 3.4.2. *Fix λ such that $0 < \lambda \leq \frac{1}{2}$. If $\hat{J}^k > 0$ for some k, then there exist a positive constant $\hat{C}$, independent of h, such that*

$$\frac{1}{2}D_t\|v^n\|^2 \geq \hat{C}\|v^n\|^3 \quad \textit{for all} \quad n \geq k.$$

Proof. Multiplying both sides of the difference equation

$$D_t v^{i,n} = D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2$$

by $\frac{1}{2}h(v^{i,n+1} + v^{i,n})$ and summing up the resulting equation over $i = 1, 2, \cdots, I-1$ we obtain

$$\frac{1}{2}\sum_{i=1}^{I-1} h(v^{i,n+1} + v^{i,n}) D_t v^{i,n} = \frac{1}{2}\sum_{i=1}^{I-1} h(v^{i,n+1} + v^{i,n})\{D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2\}$$

The left-hand side is equal to

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{I-1} h(v^{i,n+1} + v^{i,n})\frac{(v^{i,n+1} - v^{i,n})}{\Delta t} \\ &\quad = \frac{1}{2}\sum_{i=1}^{I-1} h\frac{(v^{i,n+1})^2 - (v^{i,n})^2}{\Delta t} = \frac{1}{2}D_t \|v^{i,n}\|^2 \end{aligned}$$

while the right-hand side may be rewritten as

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{I-1} h(v^{i,n+1} + v^{i,n})\{D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2\} \\ &\quad = \sum_{i=1}^{I-1} h(v^{i,n} + \frac{1}{2}\Delta t_n D_t v^{i,n})(D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2) \\ &\quad = \sum_{i=1}^{I-1} h v^n \big(D_x D_{\bar{x}} v^{i,n} + (v^{i,n})^2\big) + \frac{1}{2}\Delta t \sum_{i=1}^{I-1} h(D_t v^{i,n})^2 \\ &\quad \geq -\sum_{i=1}^{I-1} h|D_{\bar{x}} v^{i,n}|^2 + \sum_{i=1}^{I-1} h(v^{i,n})^3 \end{aligned}$$

The last inequality follows from

$$\begin{aligned} &\sum_{i=1}^{I-1} h v^{i,n} D_x D_{\bar{x}} v^{i,n} + \sum_{i=1}^{I-1} h(v^{i,n})^3 + \frac{1}{2}\Delta t \sum_{i=1}^{I-1} h(D_t v^{i,n})^2 \\ &\qquad = -\sum_{i=1}^{I-1} h(D_{\bar{x}} v^{i,n})^2 + \sum_{i=1}^{I-1} h(v^{i,n})^3 + \frac{1}{2}\Delta t \sum_{i=1}^{I-1} h(D_t v^{i,n})^2 \end{aligned} \tag{3.11}$$

and

$$\frac{1}{2}\Delta t \sum_{i=1}^{I-1} h(D_t v^{i,n})^2 \geq 0.$$

Let us note that in (3.11) we used the identity (3.10).

Hence it follows that

$$
\begin{aligned}
\frac{1}{2}D_t\|v^{i,n}\|^2 &\geq -\sum_{i=1}^{I-1} h|D_{\bar{x}}v^{i,n}|^2 + ((v^{i,n})^3, 1) \\
&\geq \frac{1}{3}\langle (v^{i,n})^3, 1\rangle, \quad n > k \\
&\geq \hat{C}\|v^{i,n}\|^3.
\end{aligned}
$$

Since

$$
\begin{aligned}
\hat{J}^k = -\frac{1}{2}\sum_{i=1}^{I} h(D_{\bar{x}}v^{i,k})^2 + \frac{1}{3}((v^{i,k})^3, 1) > 0 \quad &\text{by assumption} \\
-\sum_{i=1}^{I} h(D_{\bar{x}}v^{i,k})^2 > -\frac{2}{3}((v^{i,k})^3, 1) \quad &\text{for some } k
\end{aligned}
$$

and by Hölder's inequality (1.3.2),

$$
\begin{aligned}
\hat{C}\|v^{i,n}\|^3 &\leq \frac{1}{3}\langle (v^{i,k})^3, 1\rangle \\
\hat{C}\big(\langle v^n, v^n\rangle\big)^{\frac{3}{2}} &\leq \frac{1}{3}\langle (v^n)^3, 1\rangle \\
\hat{C}\langle v^n, v^n\rangle &\leq \langle (v^n)^3, 1\rangle^{\frac{2}{3}}
\end{aligned}
$$

□

In the following lemma we will define time increment Δt_n and the numerical blowing-up time $\hat{T}^*(\tau)$:

Lemma 3.4.3. *Suppose that there exists k for which $\hat{J}^k > 0$ and $\|v^k\| > 1$, under the condition of $0 < \lambda \leq \frac{1}{2}, \lambda = \tau/h^2(\tau = \max_{0\leq j\leq k-1} \Delta t_j)\cdot$ Choose Δt_n as*

$$
\Delta t_n = \tau/\|v^n\| \quad \textit{for} \quad n \geq k
$$

Then "the numerical blowing-up time" $\hat{T}^*(\tau)$ defined by

$$
\hat{T}^*(\tau) = \lim_{n\to\infty} t_n = \lim_{n\to\infty}\sum_{j=0}^{n-1}\Delta t_j
$$

is finite and bounded from above as

$$
\hat{T}^*(\tau) \leq t_k + \frac{1}{\hat{C}\|v^k\|}\sqrt{1+2\hat{C}\tau}\;\frac{\left(\sqrt{1+2\hat{C}\tau}+1\right)}{2}
$$

Proof. By Lemma (3.4.2) we have

$$\begin{aligned}
\frac{1}{2}D_t\|v^n\|^2 &\geq \hat{C}\|v^n\|^3 \\
\frac{1}{2}\frac{\left(\|v^{n+1}\|^2 - \|v^n\|^2\right)}{\Delta t_n} &\geq \hat{C}\|v^n\|^3 \\
\|v^{n+1}\|^2 &\geq \left(1 + 2\hat{C}\Delta t_n\|v^n\|\right)\|v^n\|^2 \\
&= (1 + 2\hat{C}\tau)\|v^n\|^2
\end{aligned}$$

which yields

$$\|v^n\| \geq \left(\sqrt{1 + 2\hat{C}\tau}\right)^{n-k}\|v^k\|, \quad n \geq k\cdot$$

Thus, the discrete time t_n is estimated as

$$\begin{aligned}
t_n &= t_k + \sum_{m=k}^{n-1}\Delta t_m \\
&\leq t_k + \frac{\tau}{\|v^k\|}\sum_{m=k}^{n-1}\left(\frac{1}{\sqrt{1 + 2\hat{C}\tau}}\right)^{m-k} \\
&\leq t_k + \frac{1}{\hat{C}\|v^k\|}\sqrt{1 + 2\hat{C}\tau}\frac{\left(\sqrt{1 + 2\hat{C}\tau} + 1\right)}{2}
\end{aligned}$$

Here the last inequality follows from

$$\begin{aligned}
\sum_{m=k}^{n-1}\left(\frac{1}{\sqrt{1 + 2\hat{C}\tau}}\right)^{m-k} &\leq \sum_{m=0}^{\infty}\left(\frac{1}{\sqrt{1 + 2\hat{C}\tau}}\right)^{m} \\
&\leq 1 + \frac{1}{1 - \frac{1}{\sqrt{1+2\hat{C}\tau}}} \\
&= \frac{\sqrt{1 + 2\hat{C}\tau}}{\sqrt{1 + 2\hat{C}\tau} - 1} \\
&= \frac{1 + 2\hat{C}\tau + \sqrt{1 + 2\hat{C}\tau}}{2\hat{C}\tau}
\end{aligned}$$

Hence

$$\begin{aligned} t_n &\leq t_k + \frac{\tau}{\|v^k\|}\sum_{m=k}^{n-1}\left(\frac{1}{\sqrt{1+2\hat{C}\tau}}\right)^{m-k} \\ &\leq t_k + \frac{\tau}{\|v^k\|}\left(\frac{1+2\hat{C}\tau+\sqrt{1+2\hat{C}\tau}}{2\hat{C}\tau}\right) \\ &= t_k + \frac{1}{\hat{C}\|v^k\|}\left(\frac{1+2\hat{C}\tau+\sqrt{1+2\hat{C}\tau}}{2}\right) \\ &\leq t_k + \frac{1}{\hat{C}\|v^k\|}\sqrt{1+2\hat{C}\tau}\frac{\left(\sqrt{1+2\hat{C}\tau}+1\right)}{2} \end{aligned}$$

□

Now, we are ready to state the result on convergence of the numerical blowing-up time.

Proposition 3.4.4. *Suppose that the solution $u(x,t)$ blows up as $t \to T^*,\ T^* < +\infty\cdot$ Let*

$$\Delta t_n = \Delta t_n(\tau) = \tau \times \min\left\{1, \frac{1}{\|v^n\|}\right\}, \quad n = 0, 1, 2, \cdots$$

Under the condition of fixed $\lambda = \frac{\tau}{h^2},\ 0 < \lambda \leq \frac{1}{2}$, the quantity $\lim_{\tau \downarrow 0} \hat{T}^*(\tau)$ exists and is equal to the exact blowing-up time T^*, that is

$$\lim_{\tau\downarrow 0}\left[\lim_{n\to\infty}\sum_{k=0}^{n-1}\Delta t_k(\tau)\right] = T^*\cdot$$

Proof. We prove the proposition by showing that

$$\begin{cases} \liminf_{\tau\downarrow 0} \hat{T}^*(\tau) \geq T^* \\ \limsup_{\tau\downarrow 0} \hat{T}^*(\tau) \leq T^*\cdot \end{cases}$$

Firstly, suppose

$$t' = \liminf_{\tau\downarrow 0} \hat{T}^*(\tau) < T^*$$

would hold. Then we could take a sequence $\{\tau_\mu\},\ \tau_\mu > 0$, such that

$$\begin{cases} \hat{T}^*(\tau_\mu) \leq t' + \delta,\ \delta = (T^* - t')/2 \\ \tau_\mu \to 0 \quad \text{as} \quad \mu \to \infty \end{cases}$$

Since

$$t_n(\tau_\mu) = \sum_{k=0}^{n-1} \Delta t_k(\tau_\mu)$$

$t_n(\tau_\mu)$ is a strictly increasing sequence, so we have

$$t_n(\tau_\mu) < \hat{T}_\infty(\tau_\mu) \leqslant t' + \delta$$

and by lemma (3.4.3), for every finite difference solution $v^{i,n} = v^{i,n}(\tau_\mu)$, we have

$$\begin{cases} t_n(\tau_\mu) \leq t' + \delta < T^* \\ \|v^n(\tau_\mu)\| \to \infty \quad \text{as} \quad n \to \infty \end{cases}$$

This contradicts Proposition (3.3.1).

Now suppose $t'' = \limsup_{\tau \downarrow 0} \hat{T}^*(\tau) > T^*$ would hold. We will first deal with the case t'' is finite. The case of $t'' = \infty$ may be treated in a similar way. Take a sequence $\{\tau_\mu\},\ \tau_\mu > 0$, such that

$$\begin{cases} \hat{T}^*(\tau_\mu) \geq T^* + (M-1)\delta, \quad \delta = (t'' - T^*)/M \\ \tau_\mu \to 0 \quad \text{as} \quad \mu \to \infty \end{cases}$$

in which M is an arbitrary number satisfying $M \geq \max\{4, 2\hat{C}(t'' - T^*)\}\cdot$ By our assumption $u(x,t)$ blows up as $t \to T^*$, so by lemma (3.1.4) $\exists\ t = T_0,\ \ T_0 < T_\infty$ so that

$$\begin{cases} \|u(T_0)\| \geq \frac{1}{\hat{C}\delta} \\ J(T_0) \geq d > 0 \quad \text{(d is some fixed constant,)} \end{cases}$$

Put $\tau_0 = \min\{(T^* - T_0)/2, \xi_0\}$, where ξ_0 is the unique positive root of

$$\sqrt{(1 + 2\hat{C}\xi)}\ \left(1 + \sqrt{1 + 2\hat{C}\xi}\right) = 3.$$

Thanks to Proposition (3.3.1), we could find a time step number k so that

$$\begin{cases} \|v^k\| > \|u(T_0)\| - \frac{1}{2\hat{C}\xi} \\ \hat{J}^k > J(T_0) - \frac{d}{2} \\ T_0 \leq t_k < T_0 + \tau_0 (< T^*) \end{cases}$$

would hold simultaneously for a sufficiently small $\tau_{\mu'}$ where $\tau_{\mu'} \in \{\tau_\mu\}$, $\tau_{\mu'} < \tau_0\cdot$ Clearly $\|v^k\| > 1$ and $\hat{J}^k > 0\cdot$ Let this number k be denoted by $k = k(\tau_{\mu'})$. In virtue of Lemma (3.4.3), it would hold that

$$\begin{aligned}
\hat{T}^*(\tau_{\mu'}) &\leq t_{k(\tau_{\mu'})} + \frac{1}{\hat{C}\|v^k\|}\sqrt{(1+2\hat{C}\xi)\frac{1+\sqrt{1+2\hat{C}\xi}}{2}} \\
&\leq t_k(\tau_\mu')\frac{1}{\hat{C}}2\hat{C}\xi\frac{3}{2} \\
&\leq T^* + 3\delta < T^* + (M-1)\delta.
\end{aligned}$$

This contradicts the supposition. $\square$

Chapter 4

CONCLUSION

In this thesis, we study the subject of blowing up solutions in a finite time of initial boundary value problem for the semilinear heat equations. We illustrate three methods for finding sufficient conditions on the sufficiently data which guarantee blow-up of corresponding solutions of initial boundary value problem for semilinear heat equations. After introducing the concept of blow-up solutions, we study the problem of blow-up of solutions of finite -differece analog of the initial boundary value problem for semilinear heat equation following [12]. First, we establish blow-up of solution of the finite -difference equation and then we studied the relation between the numerical blow-up time and the exact blow-up time. Finally we demonstrate the corresponding result on convergence of blow-up time of finite difference equaion to the blow-up time of the original problem obtained in [12].

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