T.C.
İSTANBUL UNIVERSITY
INSTITUTE OF GRADUATE STUDIES IN
SCIENCE AND ENGINEERING

M.Sc. THESIS

OPTIMIZATION PROBLEM WITH SECOND ORDER SEMILINEAR DIFFERANTIAL INCLUSION

ŞÜKRAN KESKİN

Department of Mathematics

Mathematics Programme

uygundur

SUPERVISOR
Assist. Prof. Dr. Gülseren ÇİÇEK

June, 2019

İSTANBUL

This study was accepted on 10/7/2019 as a M. Sc. thesis in Department of Mathematics, Mathematics Programme by the following Committee.

Examining Committee Members

Assist. Prof. Dr. Gülseren ÇİÇEK(Supervisor)
İstanbul University
Faculty of Science

Assist. Prof. Dr. Özkan DEĞER
İstanbul University
Faculty of Science

Neşe Dernek.
Prof. Dr. Neşe DERNEK
Marmara University
Faculty of Science and Arts

Academic Title Name SURNAME
University
Faculty

Academic Title Name SURNAME
University
Faculty

FOREWORD

Along with my mastership graduate study, with her knowledge and experience, I would like to state my appreciation to Ms Gülseren ÇİÇEK who has supported me all the time during this process.

I am grateful to my dear husband Mr Abdullah KESKİN who supported me in every subject during my thesis studies and for his patience. Thank you very much to my parents who supported me and gave me confidence to finish the thesis.

June 2019 Şükran KESKİN

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LIST OF SYMBOLS AND ABBREVIATIONS

Symbol	Explanation
$\mathbb{R}$	: Real numbers field
$\mathbb{R}^n$	: $n-$dimentional Euclidean space
$\subseteq$	: Subset
$\langle \cdot,\cdot \rangle$	: Scalar product
$\\| \\|$	: Norm
conv A	: Convex hull of set A
Aff A	: Afine hull of set A
rel int A	: Set of relative interior points of set A
int A	: Set of interior points of set A
K^*	: Dual cone of cone K
dom f	: Definition set of f
epi f	: Above the graph set of f
f^*	: Dual function of f
$\partial f(x_0)$	: Subdifferantial set of f at the point x_0
cone A	: Cone of set A
$gphF$	: Graph set of F
$0^+ gphF$	: Recession cone of F
$F^*(y^*)$	: Dual function of multivalued function F
$F^*(\cdot,z)$	: Local dual function of multivalued function F
$N_F(x,x^*)$	: Hamiltonian function of multivalued function F

Abbreviations	**Explanation**
AM F^*	: Adjoint multivalued F^*
LDM	: Locally dual mapping
LDM F^*	: Locally dual mapping F^*
CTD	: Cone of tangent directions
CUA	: Convex upper approximation
DAP	: Discrete approximation problem
NSC	: Necessary and sufficient condition
CF	: Convex function
CP	: Convex problem
CMM	: Convex multivalued mapping
OP	: Optimization problem
DP	: Discrete problem
MF	: Multivalued function
CS	: Convex set
CC	: Convex cone
E-L	: Euler-Lagrange
s.s	: Sufficiently small
iff	: If and only if

ÖZET

İKİNCİ MERTEBEDEN YARIDOĞRUSAL DİFERANSİYEL İÇERMELİ OPTİMİZASYON PROBLEMİ

YÜKSEK LİSANS TEZİ

ŞÜKRAN KESKİN

İstanbul Üniversitesi

Fen Bilimleri Enstitüsü

Matematik Anabilim Dalı

Danışman : Dr. Öğr. Üyesi Gülseren ÇİÇEK

Bu tez çalışmasında, ikinci mertebeden diferansiyel içermeli yarıdoğrusal Bolza tipli optimizasyon problemi üzerinde durulmuş ve farklı sınır koşulları altında problemin çözümünün optimalliği için yeterli şartları formüle etmek amaçlanmıştır.

İlk bölümde tez için gerekli olan konveks kümeler, konveks fonksiyonlar ve subdiferansiyel gibi tanımlar ve teoremler verilmektedir. Tezin asıl kısmında ikinci mertebeden diskret ve diferansiyel içermeli yarıdoğrusal optimizasyon problemleri incelenmiş ve optimallik için gerek ve yeter koşullar formülize edilmiştir. Ayrıca discret yaklaşım probleminin optimalliği için gerek ve yeter koşullar belirlenmiştir. Limite geçilerek diferansiyel içermelerle oluşturulmuş optimal problemin yeter koşulları elde edilmiştir. Son bölümde genel değerlendirmeler ve öneriler verilmiştir.

Haziran 2019, 54 sayfa

Anahtar Kelimeler: Konveks küme, konveks fonksiyon, çokdeğerli fonksiyon, diskret ve diferansiyel içerme, ikinci dereceden diskret optimizasyon problemi, Euler-Lagrange içermesi, yerel dual fonksiyon.

SUMMARY

OPTIMIZATION PROBLEM WITH SECOND ORDER SEMILINEAR DIFFERENTIAL INCLUSION

M.Sc. THESIS

Şükran KESKİN

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In this thesis, second order semilinear discrete and differential inclusions, Bolza type optimization problems are studied. It is intended to formulate the sufficient conditions in different state conditions for the solution of optimization.

Firstly, definitions and theorems, such as convex set, convex functions, subdifferential required for the thesis are given. In the chief part of the thesis, second order discrete and differential semilinear optimization problems are studied and for the optimization, necessary and sufficient conditions are formulized. Besides, for the sake of optimization of discrete approximation problem, necessary and sufficient conditions are determined. Shifting to limit, necessary conditions of the optimal problem with differential inclusions are attained. In the last chapter, general evaluations and suggestions are submitted.

June 2019, 54 pages

Keywords: Convex set, convex function, multivalued function, discrete ve differential inclusion, second order discrete optimization problem, Euler-Lagrange inclusion, locally dual function.

1. INTRODUCTION

In this thesis the Bolza Problem for the second order semilinear differential inclusion will be analysied.

$$\min\ J\left[z(.)\right]=\int_a^b g\left(z(t),t\right)dt+\Psi\left(z(b),z'(b)\right) \tag{1}$$

$$z''(t)\in Az(t)+F\left(z(t),z'(t)\right), \qquad t\in\left[a,b\right] \tag{2}$$

$$z(a)\in M_1, \;\; z'(a)\in M_2, \tag{3}$$

here $F:\mathbb{R}^{2n}\to P(\mathbb{R}^n)$ is a multivalued function(MF), $g(.,t):\mathbb{R}^n\times\left[a,b\right]\to\mathbb{R}$ is a continuous function according to z, $\Psi:\mathbb{R}^n\to\mathbb{R}$ is proper single valued function, M_1 and M_2 are subsets of $\mathbb{R}^n$.

The intent of the problem is to detect a trajectory $\tilde{z}(t)$, convincing (2) almost everywhere (a.e) on $\left[a,b\right]$ and the boundary conditions (3) at $t=a$ that minimizes the Bolza functional $J\left[z(\cdot)\right]$. The functions, multifunction and sets in the problem are convex then the problem is named convex problem.

The studies intended to develop the optimization theory began in the 17th century. The workings that started by Descartes and Fermat were maintained in order to acquire optimals by the other mathematicians. The analysis improved by Newton and Leibnitz plays an significant role improvement of optimization theory. This study enabled the optimal conditions to emerge for the sake of mathematical function and independent factors that determine the function.

In 1947 by G.B Dantzig, the Simplex Method in terms of the solution of the Lineer optimization problem(OP) was promoted. Later on, required factors in the interest of optimal with Kuhn-Tucker conditions that H.Kuhn and A.Tucker worked on were constituted. Dynamic Programming model and solution were improved by R.Bellman and Maximum

Principle of Optimal Control by Pontryagin [1] was adapted to the problem of discrete and continuous problem. In recent years, optimization field takes interest to a great extent. Particularly, user friendly software development, high speed processor including neural networks due to fast development of computer techology is still an active investigation field. Besides maths programming methods are used in deverse fields such as agriculture, medicine, military, fields, satellite orbits, engineering, economy and radiotherapy.

Variation analysis is a branch that is involved in the OPs of maths. The knowledge regarding this subject was mentioned in Rockafellar [2]. Especially, second order problems that include discrete and differential inclusions were examined in Aitalioubrahim [3] and Benchohra [4] .In general, concerning these problems, it was studied over existence and validity [5,6,7,8,9,10]. MFs, discrete, discrete approximation and problems including differential inclusion problems that are about the comparison of ordinary and partial differential inclusion is viewed in Mahmudov [11]. Especially high order discrete and differential inclusion OPs were studied in Mahmudov [12].

First valid outcomes in terms of second order differential inclusion were mentioned by Yarou and Haddad [8]. In this case, Cauchy problem is thought for the sake of infinite dimension state and secondary degree differential inclusion. In the finite dimensional case, Cernea [9] and Lupulescu [10] studied the non-convex problem. As for some cases, in Hilbert space, it was studied by İbrahim and Alkualibi [13]. Mahmudov [14] studies Bolza Problem that includes optimal control theory with second order differential inclusion. As an additional discrete approximation problem(DAP) has been studied after getting necessary and sufficient condition(NSC) of second order discrete inclusion. In order to get optimal condition, local dual condition and equivalent theorems are used.

Generally, in the investigation made so far, it has been studied over the existence and validity of the second order differential inclusion problems. Optimization of second order discrete inclusion problems were examined in Mahmudov [15]. Çiçek and Mahmudov [16] formulate Mayer problem for second order differential inclusions with preliminary boundary liminations. In article [17], Boris S. Mordukhovich has been studied optimal control of semilinear unlimited evoluation inclusions with functional constraints. Leisure time optimization with second-order differential inclusions for endpoint constraints has been introduced in [18]. Also semilinear OP with second order differential inclusions for nonlinear boundary values concerned in [19].

A mathematical programming problem occurs by means of optimization, minimization or maximization of the objective function under definite constraints. An optimization model is designed and a solution is acquired. This operator is called maths programming.

This thesis includes five chapters. In the first chapter, the history of optimality is given and the sources are concerned.

In the second chapter, convex set(CS), convex cones(CCs), convex functions(CFs), duality, subdifferential, multivalued mapping and locally dual mapping(LDM) are defined. The problem which is worked in [14] is given and the NSC for the optimality are explained. The methods in practice utilised along with this thesis are cited.

In the third chapter, OPs with second order semilinear discrete and differential inclusions are examined. Firstly, using the definition of LDM and NSC are formulated for the discrete problem(DP). Using the equivalence relations, NSC are formed for the optimization of DAP and performing the limit, optimal problem with sufficient optimality conditions defined by second order semilinear differential inclusions are established.

In the fourth chapter, the methods in chapter three are applied on non-convex problem and sufficient conditions are formulated.

In the last chapter, general evaluations and suggestions are submitted.

2. MATERIALS AND METHODS

In this chapter we study some definitions and theorems of Convex Analysis. In this thesis the vector z in $n-$dimensional real Euclidean space $Z=\mathbb{R}^n$ will be considered. We also study its dual space $Z^*=\mathbb{R}^n$. The inner product of two vectors $z\in Z$ and $z^*\in Z^*$ is denoted by

$$\langle z,z^*\rangle=\sum_{i=1}^{n} z^i z^{i^*}$$

where z^i and z^{i^*} are the componets of z and z^* [11].

2.1. CONVEX SETS AND CONVEX CONES

At that part we study some definitions and theorems for CSs and CCs.

2.1.1. Convex Sets

Definition 2.1. [11],[20] Let A be a subset of $\mathbb{R}^n$ and A is convex if

$$\alpha_1 z_1+\alpha_2 z_2\in A,\ \ \alpha_1+\alpha_2=1,\ \ \alpha_1,\alpha_2\geq 0,\ \ \forall z_1,z_2\in A.$$

Open sets and subspaces are convex.

Lemma 2.1. [11],[20] (i) Let $\forall i\in I,\ A_i\subseteq\mathbb{R}^n$ be CSs then $A=\bigcap_{i\in I}A_i$ is convex.

(ii) $A_1,...,A_t\subseteq\mathbb{R}^n$ be CSs then $\alpha_1 A_1,...,\alpha_t A_t$ are CSs if $\alpha_1,\alpha_2,...,\alpha_t$ are constants.

Definition 2.2. [11] Let B is open unit ball in $\mathbb{R}^n$, then $B=\{z:\|z\|<1\}$. z is an interior point of $A\in\mathbb{R}^n$ if there exists a number $\varepsilon>0$ such that $z+\varepsilon B\subseteq A$.

Definition 2.3. [11] If for $z_1,...,z_t$ there exists nonnegative $\alpha_1,...,\alpha_t$ such that

$$z = \alpha_1 z_1 + ... + \alpha_t z_t$$

$$\alpha_i \geq 0,\ i = 1,...,t,\ \alpha_1 + ... + \alpha_t = 1 \tag{1}$$

then z is called a convex combination of points $z_1,...,z_t$.

Definition 2.4. [11] Let $A \subseteq \mathbb{R}^n$ be a given set. The convex hull of A, $\operatorname{conv} A$ is the set that has all convex combinations of points of A, that is

$$\operatorname{conv} A = \{\alpha_1 z_1 + ... + \alpha_t z_t : \alpha_1,...,\alpha_t \geq 0,\ \alpha_1 + ... + \alpha_t = 1,\ z_1,...,z_t \in A\}.$$

Theorem 2.1. (Carathėodory Theorem). [20] Any $z \in \operatorname{conv} A$ can be represented as a convex combinations of $n+1$ or fewer elements of A.

Definition 2.5. [11] Let A be a CS. The relative interior of A is the interior of A for the topology relative to the afine hull of A and is denoted by $\operatorname{reint} A$, that is $z \in \operatorname{reint} A$ iff

$z \in \operatorname{Aff} A$ and $(\operatorname{Aff} A) \cap B(z,\delta) \subseteq A$ for a $\delta > 0$.

Lemma 2.2. [11] Let $A \subseteq \mathbb{R}$ be a CS. If $z_1 \in \operatorname{ri} A$ and $z_2 \in \overline{A}$ for all $\alpha \in [0,1]$ then we have $\alpha z_1 + (1-\alpha) z_2 \in \operatorname{ri} A$.

Definition 2.6. [11] A hyperplane is a set defined by

$$H = \{z \in \mathbb{R}^n : \langle s, z \rangle = \alpha\}$$

where $\alpha \in \mathbb{R},\ s \in \mathbb{R}^n,\ s \neq 0$.

Definition 2.7. [11] $H_A(z^*) = \sup\limits_{x} \{\langle z, z^* \rangle : z \in A\}$ is the support function of a nonempty set A in $\mathbb{R}^n$.

Theorem 2.2. [11] Let A be a closed CS. Then $z \in A$ iff

$$\langle z, z^* \rangle \le H_A(z^*) \tag{2}$$

for all $z^* \in \mathbb{R}^n$.

2.1.2. Convex Cones

Definition 2.8. [11] $T \subseteq \mathbb{R}^n$ be a CS. T is a CC if $z \in T$ and $\alpha > 0$ imply that $\alpha z \in T$.

Definition 2.9. [11] The set of vectors $z^* \in Z^* = \mathbb{R}^n$ which $\langle z, z^* \rangle \ge 0$ holds for all $z \in T$ is called the dual cone(DC) of T and is denoted by T^*. It can be written as:

$$T^* = \left\{ z^* \in Z^* : \langle z, z^* \rangle \ge 0,\ \forall z \in T \right\}.$$

Definition 2.10. [11] $A \subseteq Z$ be an arbitrary subset. A vector $\bar{z} \in Z$ is called a tangent direction of A at a point $z \in A$, if there exists a function $\varphi(\alpha)$ such that $z + \alpha \bar{z} + \varphi(\alpha) \in A$ for suffeciently small $\alpha \ge 0$ and $\alpha^{-1}\varphi(\alpha) \to 0$ as $\alpha \downarrow 0$.

Definition 2.11. [11] The set of tangent directions of A is called the tangent cone of A (CTD) denoted by $T_A(z)$. $T_A(z)$ is not unique. If A is convex, then

$$T_A(z) = \operatorname{cone}(A - z) = \left\{ z : \bar{z} = \alpha(z_1 - z),\ z_1 \in A,\ \alpha > 0 \right\} \tag{3}$$

is CTD to A at z.

Theorem 2.3. [11] Suppose $T_1, ..., T_m$ are CCs then either

$$\left(T_1 \cap T_2 \cap ... \cap T_m \right)^* = T_1^* + T_2^* ... + T_m^*$$

or $z_i^* \in T_i^*$, such that not all zero and

$$z_1^* + z_2^* + ... + z_m^* = 0.$$

2.2. CONVEX FUNCTIONS

Let $f:\mathbb{R}^n \to [-\infty,\infty]$ be defined then its domain is

$$\operatorname{dom} f = \{z : f(z) < +\infty\}.$$

The epigraph is

$$\operatorname{epi} f = \{(z, z_0) : z_0 \in \mathbb{R},\ z_0 \geq f(z)\}[11].$$

Lemma 2.3. [11] Let f be a proper function. That is let $\operatorname{dom} f \neq \varnothing$ and $f(z) < +\infty$ for $z \in \operatorname{dom} f$ then $\alpha_1 \geq 0, \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1$ such that

$$f(\alpha_1 z_1 + \alpha_2 z_2) \leq \alpha_1 f(z_1) + \alpha_2 f(z_2).$$

Lemma 2.4. [11] $\operatorname{dom} f$ is convex if f is CF.

Lemma 2.5. (Jensen's inequality) [20]. If f is a proper CF. Then for

$\alpha_i \geq 0,\ i = 1,\dots,m$ and $\alpha_1 + \dots + \alpha_m = 1$ we get

$$f(\alpha_1 z_1 + \dots + \alpha_m z_m) \leq \alpha_1 f(z_1) + \dots + \alpha_m f(z_m).$$

Lemma 2.6. [11] Let for each $i \in I$ functions $f_i(z)$ be convex then $f(z) = \sup\limits_{i \in I} f_i(z)$ is convex.

Definition 2.12. [20] A function f on $\mathbb{R}^n$ is positively homogeneous if for every z one has $f(\alpha z) = \alpha f(z),\ \alpha > 0.$

Example 2.1. [11] For $z \in \mathbb{R}^n$, $f(z) = \|z\|$ and applying the Definition 2.12

$$f(\alpha z) = \|\alpha z\| = \alpha\|z\| = \alpha f(z)$$

then we see $f(z)$ is positively homogeneous function.

Theorem 2.4. [11] Let f is a proper CF and f is connected in neighborhoods of some point $z' \in \operatorname{dom} f$. Then f is continuous at z'.

Theorem 2.5. [11] If a CF f is continuous at z' then f satisfies the Lipschitz condition in $B(z',\varepsilon)$ then

$$\left|f(z)-f(z')\right| \leq \varepsilon \left\|z-z'\right\|.$$

Theorem 2.6. [11] If f is continuous on $\operatorname{rel\,int}(\operatorname{dom} f)$, then f is a proper CF.

2.2.1. Subdifferentials

CFs generally can not be differentiated but there are many properties that such functions satisfies.

Definition 2.13. [11] The directional derivative of f at x with respect to a vector $r \in X = \mathbb{R}^n$ is defined to be the limit

$$f'(z,r) = \lim_{\alpha \downarrow 0} \frac{f(z+\alpha r)-f(z)}{\alpha}$$

if it exist.

Lemma 2.7. [11] Let f be a proper CF. If $z_0 \in \operatorname{dom} f$ then $f'(z_0,r)$ exists for $\forall r \in \mathbb{R}^n$.

Definition 2.14. [11] Let f be a proper CF. If for each $r \in \mathbb{R}^n$

$$f(z)-f(z_0) \geq \left\langle z-z_o, z^* \right\rangle$$

is satisfied then we say a vector z^* is a subgradient of f at $z_0 \in \operatorname{dom} f$.

Lemma 2.8. [11] Let f be a proper CF. Then for z^* is a subgradient of f at $z_0 \in \operatorname{dom} f$ iff for each $r \in \mathbb{R}^n$

$$f'(z_0,r) \geq \left\langle r, z^* \right\rangle.$$

Lemma 2.9. [11] $f'(z_0,r)$ is a positively homogeneous CF of r.

Definition 2.15. [11] The subgradients of f at a point z_0 form the subdifferential of f at z_0, simply denoted by $\partial f(z_0)$. Therefore

$$\partial f(z_0)=\left\{z^*: f(z)-f(z_0)\geq \left\langle z-z_0,z^*\right\rangle,\ \forall z\right\}.$$

Generally, $\partial f(z_0)\neq\varnothing$ then f is subdifferantiable at z_0.

Lemma 2.10. [11] The subdifferential set $\partial f(z_0)$ is closed and convex.

Theorem 2.7. [11] $\partial f(z_0)\neq\varnothing$ if the directional derivative $f'(x_0,r)$ is closed and

$$f'(z_0,r)=\sup_{x^*}\left\{\left\langle r,z^*\right\rangle: z^*\in\partial f(z_0)\right\}.$$

Remark 2.1. [11] The subdifferential set $\partial f(z_0)$ is nonempty bounded set for a CF f, continuous at z_0. Then Theorem 2.7 has the form

$$f'(z_0,r)=\max_{x^*}\left\{\left\langle r,z^*\right\rangle: z^*\in\partial f(z_0)\right\}.$$

Theorem 2.8. [11] The subdifferential is the single gradient vector if f is a CF differantiable at z_0, in that case

$$\partial f(z_0)=\{f'(z_0)\}.$$

Theorem 2.9. [20] (Moreau-Rockafellar) Let f_1, f_2 are proper CFs and $f=f_1+f_2$, then for each z

$$\partial f(z)\supset\partial f_1(z)+\partial f_2(z).$$

If the CFs $\operatorname{rel\,int}(\operatorname{dom} f_i), i=1,2$ intersect then for each z

$$\partial f(z)=\partial f_1(z)+\partial f_2(z).$$

Theorem 2.10. [11] Let define $A=\{z: f(z)\le 0\}$, where f is a CF. Let there exist $z_1 \in A$ such that $f(z_1)<0$. If directional derivatives at z_0 are closed and $f(z_0)=0$ then

$$[\text{cone}(A-z_0)]^* = -\text{cone}\,\partial f(z_0)$$

where $\text{cone}(A-z_0)$ is a cone generated by $A-z_0$.

2.3. MULTIVALUED LOCALLY ADJOINT MAPPINGS

2.3.1. Locally Dual Mappings to Convex Multivalued Mappings

Let X and Y are finite dimensional Euclidean spaces and let denote the $X\times Y$ by W. M be an arbitrary subset of $W=X\times Y$. Then the multivalued mapping F is defined by

$$F(z)=\{\, y:(z,y)\in M\,\}.$$

The set M is called the graph of MF F and is denoted by $gphF$,

$$gphF=\{(z,y): y\in F(z)\}$$

and later on, we get

$$dom\,F=\{\, z: F(z)\neq\varnothing\},$$

$$\|F(z)\|=\sup_y\{\,\|y\|: y\in F(z)\}\ [11].$$

Definition 2.16. [11] Let F is multivalued mapping such that

1) If $gphF$ is convex, then F is convex.

2) If $F(z)$ is convex in Y, then F is convex valued.

3) If $gphF$ is closed, then F is closed.

4) F is bounded if there exist a constant $C>0$, such that $\|F(z)\|=C(1+\|z\|)$.

2.3.2. Dual Functions

Definition 2.17. [11] The recession cone of a convex mapping F defined by

$$0^{+} gphF = \left\{(z, y) : (z + \alpha\overline{z}, y + \alpha\overline{y}) \in gphF,\ \alpha \geq 0,\ \forall (z, y) \in gphF\right\}.$$

Then, multivalued mapping F^{*} defined by

$$F^{*}(y^{*}) = \left\{z^{*} : (z^{*}, -y^{*}) \in (0^{+} gphF)^{*}\right\}$$

is the adjoint to the convex F briefly denoted by DM F^{*}.

Lemma 2.11. [11] The DM F^{*} is bounded iff $\operatorname{dom} f = X$.

Let us introduce some definition to define the locally dual mappings (LDM).

$$N(z, y^{*}) = \sup_{y} \left\{\left\langle y, y^{*}\right\rangle : y \in F(z)\right\},\ y^{*} \in Y^{*} = \mathbb{R}^{n} \tag{4}$$

is Hamiltonian function. If $F(z) = \varnothing$ we put $N(z, y^{*}) = -\infty$.

Moreover,

$$R(z^{*}, y^{*}) = \inf_{(z,y)} \left\{\left\langle z, z^{*}\right\rangle - \left\langle y, y^{*}\right\rangle : (z, y) \in gphF\right\}; \tag{5}$$

it is exact that

$$R(z^{*}, y^{*}) = \inf_{x} \left\{\left\langle z, z^{*}\right\rangle - N\left\langle z, y^{*}\right\rangle\right\}; \tag{6}$$

for $\forall x \in \mathbb{R}^{n}$

$$R(z^{*}, y^{*}) \leq \left\langle z, z^{*}\right\rangle - N\left\langle z, y^{*}\right\rangle.$$

Moreover, we get

$$R(z^{*}, y^{*}) = -(-N(., y^{*}))^{*}(z^{*}).$$

Take a point $z \in gphF$ and the cone $T_{gphF}(z) = \text{cone}(gphF - z)$ or

$$T_{gphF}(z) = \left\{ \overline{z} : \overline{z} = \alpha(z_1 - z),\ \alpha > 0,\ z_1 \in gphF \right\}$$

is defined.

It is not hard to explain that

$$T_{gphF}(z) = \left\{ \overline{z} : z + \alpha\overline{z} \in gphF \textit{ for } \text{s.s } \alpha > 0 \right\}.$$

$$F(z; y^*) = \left\{ y \in F(z) : \left\langle y, y^* \right\rangle = N(z, y^*) \right\}. \tag{7}$$

is the argmaximum set of F.

In order that $gphF_z = T_{gphF}(z)$ the multivalued mapping F,

$$F_z(\overline{z}) = \left\{ \overline{y} : (\overline{z}, \overline{y}) \in T_{gphF}(z) \right\}$$

is defined [11].

Definition 2.18. [11] Let F be a CMM. The multivalued mapping F^*, from X^* into Y^* defined by

$$F^*(y^*; z) = \left\{ z^* : (z^*, -y^*) \in T^*_{gphF}(z, y) \right\}$$

is called the locally dual mapping (LDM) to F at the point $(z, y) \in gphF$. Here $T^*_{gphF}(z, y)$ is the DC of $T_{gphF}(z, y)$.

Theorem 2.11. [11] Suppose $F : X \to P(Y)$ is a CMM, in order that

$\partial_z N(z, y^*) = -\partial_z \left[-N(z, y^*) \right]$ then LDM F^* can be described as

$$F^*(y^*; (z, y)) = \begin{cases} \partial_z N(z, y^*), & y \in F(z, y^*) \\ \varnothing, & y \notin F(z, y^*) \end{cases}.$$

Lemma 2.12. [11] Let F^* be LDM of F then z^* is an element of the LDM F^* iff

$$R(z^*, y^*) = \langle z, z^* \rangle - N(z, y^*).$$

Theorem 2.12.[11] Let $F : X \to P(Y)$ be a bounded, closed, convex MF and Ψ be a proper CF such as $\operatorname{dom} \Psi = X$. Let define function

$$f(z) = \min_y \{ \Psi(z, y) : y \in F(z) \}$$

then f is CF and for an arbitrary $z_0 \in \operatorname{dom} F$ the subdifferential of function f is defined by

$$\partial f(z_0) = \{ z^* - F^*(-y^*; x_0) : (z^*, y^*) \in \partial_x \Psi(x_0) \},$$

where $x_0 = (z_0, y_0)$ and $y_0 \in F(z_0)$ is a point that make the function Ψ minimal.

If Ψ is differantiable, Ψ'_x and Ψ'_y are the vectors of partial derivatives with respect to z and y, in combination then $\partial f(z_0)$ is defined by

$$\partial f(z_0) = \Psi'_x(x_0) - F^*(-\Psi'_y(x_0); x_0)$$

Ψ is continuous on X by the Theorem 2.12 and there exists the subdifferential of $\partial_z \Psi(x)$ at an arbitrary point.

Theorem 2.13. [11] Let $\Psi : Z \to \mathbb{R}$ be a CF continuous in z on some neighborhood of z_0 and $y \in A$ then

$$f(z) = \min_y \{ \Psi(z, y) : y \in A \}$$

and suppose that the minimum is attained at $y_0 \in A$ where $z = z_0$. Then

$$\partial f(z_0) = \{ z^* : (z_0^*, y_0^*) \in \partial_x \Psi(x_0),\ y^* \in [cone\ (A - y_0)]^* \}.$$

Example 2.2. [11] Let $F(z)=Az+V$, here $V\subseteq Y$ is a CS and $N:X\to Y$ is an operator. Then

$$N(z,y^*)=\sup_y\left\{\left\langle y,y^*\right\rangle : y\in F(z)\right\}=\left\langle Az,y^*\right\rangle+\sup_v\left\{\left\langle v,y^*\right\rangle : v\in V\right\}=\left\langle Az,y^*\right\rangle+N_V(y^*)$$

and by Eq.(7) we have

$$R(z^*,y^*)=\inf_z\left\{\left\langle z,z^*\right\rangle-N(z,y^*)\right\}=\inf_z\left\{\left\langle z,z^*-A^*y^*\right\rangle-N_V(y^*)\right\}.$$

Thus,

$$R(z^*,y^*)=\begin{cases}-\infty, & A^*y^*\neq z^*,\\ N_V(y^*), & A^*y^*=z^*.\end{cases}$$

If $y_0=Az_0+v_0,\ v_0\in V$

$$F(z_0,y^*)=\left\{y_0\in Az_0+V:\left\langle y_0,y^*\right\rangle=N(z_0,y^*)\right\}=\left\{v_0\in V:\left\langle v_0,y^*\right\rangle=N_V(y^*)\right\}$$

$$=\left\{v_0\in U:-y^*\in T_V^*(v_0)\right\},$$

where $T_V^*(v_0)=\left\{\overline{v}:\overline{v}=\alpha(v-v_0),\ \forall\alpha>0,\ v\in V\right\}.$

From Theorem 2.11, we conclude that $F^*(y^*;x_0)=\begin{cases}A^*y^*, & -y^*\in\left[\text{cone }(V-v_0)\right]^*,\\ \varnothing, & -y^*\notin\left[\text{cone }(V-v_0)\right]^*.\end{cases}$

2.4. CONVEX PROGRAMMING PROBLEMS

Necessary notations can be found in [11].

2.4.1. Needed Facts and Problem Statement

Let $\mathbb{R}^n$ be an Euclidean space which is $n-$dimensional including variable z. Suppose that

$F:\mathbb{R}^{2n}\to P(\mathbb{R}^n)$ is a multivalued mapping and $\Psi:\mathbb{R}^{2n}\to\mathbb{R}^n$ is proper distict valued function where $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. M_1 and M_2 are convex subsets of $\mathbb{R}^n$. The graph of F is

$$gphF = \{(z, w, x) : z \in F(z, w)\}.$$

The domain of F is signified by $\operatorname{dom} F$ and defined as

$$\operatorname{dom} F = \{(z, w) : F(z, w) \neq \varnothing\}.$$

Let us define the Hamiltonian function multivalued mapping F

$$N_F(z, w, x^*) = \sup_z \left\{\left\langle x, x^* \right\rangle : z \in F(z, w)\right\}, \ x^* \in \mathbb{R}^n,$$

and argmaximum set for multivalued mapping F

$$F(z, w; x^*) = \sup_x \left\{x \in F(z, w) : \left\langle x, x^* \right\rangle = N_F(z, w, x^*)\right\}.$$

For convex F, we get $N_F(z, w, x^*) = -\infty$ if $F(z, w) = \varnothing$.

For $\forall (z, w, x) \in gphF$ at the point $(z_0, w_0, x_0) \in gphF$

$$T_{gphF}(z_0, w_0, x_0) = \operatorname{cone}\left[gphF - (z_0, w_0, x_0)\right]$$

$$= \{(\bar{z}, \bar{w}, \bar{z}) : \bar{z} = \alpha(z - z_0), \ \bar{w} = \alpha(w - w_0), \ \bar{x} = \alpha(x - x_0)\}$$

The LDM of convex F at the point $(z_0, w_0, x_0) \in gphF$ is defined

$$F^*(x^*; (z_0, w_0, x_0)) = \left\{(z^*, w^*) : (z^*, w^*, -x^*) \in T^*_{gphF}(z_0, w_0, x_0)\right\} \text{ [11].}$$

Now we deal with the second order discrete model given by Mahmudov [14]:

$$\min \sum_{t=2}^{T-1} g(z_t, t) \tag{8}$$

$$z_{t+2} \in F(z_t, z_{t+1}), \quad t = 0, ..., L-2 \tag{9}$$

$$z_0 \in \alpha_0, \quad z_1 \in \alpha_1 \tag{10}$$

where $z_t \in \mathbb{R}^n$, $g(.,t): \mathbb{R}^n \times [0,1] \to \mathbb{R}$ is real-valued function $F(\mathbb{R}^n) \to P(\mathbb{R}^n)$ is real valued function, t is a natural number, α_t, $t = 0,1$ fixed vectors. If F is convex MF and $g(.,t)$ is proper CF, the state problem is convex.

For $\{z_t\}_{t=0}^{L} = \{z_t : t = 0,1,...,L\}$ is called the feasible trajectory for the problem (8)-(10).

At first; we approach the convex problem (8)-(10). Let indicate vector v which defined by

$v = (z_0, z_1, ..., z_L) \in \mathbb{R}^{n(L+1)}$ and describe in the $\mathbb{R}^{n(L+1)}$ the following CSs

$$S_t = \left\{ v = (z_0, z_1, ..., z_L) \middle| \; (z_t, z_{t+1}, z_{t+2}) \in gphF \right\}, \; t = 0,1,...,L-2$$

$$M_1 = \left\{ v = (z_0, z_1, ..., z_L) \middle| \; z_0 = \alpha_0 \right\}$$

$$M_2 = \left\{ v = (z_0, z_1, ..., z_L) \middle| \; z_1 = \alpha_1 \right\}$$

Principally, we estimate the DCs $T^*_{S_t}(v)$, $T^*_{M_1}(v)$ and $T^*_{M_2}(v)$ [14].

Lemma 2.13. [14] Let $T_{gphF}(z_t, z_{t+1}, z_{t+2})$ be CTD for $(z_t, z_{t+1}, z_{t+2}) \in gphF$.

Then the DC of the set S_t is

$$T^*_{S_t}(v) = \left\{ v^* = (z_0^*, ..., z_L^*) : (z_t^*, z_{t+1}^*, z_{t+2}^*) \in T^*_{gphF}(z_t, z_{t+1}, z_{t+2}), \; z_k^* = 0, \; k \neq t, t+1, t+2 \right\}.$$

Theorem 2.14. [14] Let (8)-(10) be convex problem then the trajectory $\{z_t^0\}_{t=0}^{L}$ is optimal if there exist not equal to zero vectors z_t^*, z_L^*, ω_t^*, $t = 0,...,L-1,$ and a number α that equals 0 or 1 and are fulfilled discrete E-L inclusion and transversality inclusion:

$$(i)\ (z_t^* - \omega_t^*, \omega_{t+1}^*) \in F^*\left(z_{t+2}^*; (\overline{z}_t, \overline{z}_{t+1}, \overline{z}_{t+2})\right) - \alpha\,\partial_x g(\overline{z}_t, t) \times \{0\},\ t = 0,...,L-2,\ \omega_0^* = 0,$$

$$\partial_x g(\tilde{z}_0, 0) = \partial_x g(\tilde{z}_1, 1) = 0,$$

$$(ii)\ \omega_{L-1}^* - z_{L-1}^* \in \alpha\,\partial_x g(\tilde{z}_{L-1}, L-1),\ z_L^* = 0.$$

In addition, if reqularity condition is fulfilled then the above conditions are sufficient for the trajectory $\{\tilde{z}_t\}_{t=0}^{L}$.

Now let examine the CP stated by Mahmudov [14]

$$\min J\left[z(.)\right] = \int_0^1 g\left(z(t), t\right)dt + \Psi\left(z(1)\right) \tag{11}$$

$$z''(t) \in F\left(z(t), z'(t)\right),\quad t \in [0,1] \tag{12}$$

$$z(0) = \alpha_0,\quad z'(0) = \alpha_1 \tag{13}$$

where α_0, α_1 are fixed vectors.

The intent of the CP is to formalize the optimality conditions that the feasible arc $\tilde{z}(t)$ satisfies.

Let us identify this problem with the following DAP:

$$\min J\left[z(\cdot)\right] = \sum_{t=2l,...,1-2l} l\,\mathrm{g}\left(z(t), t\right) + \Psi\left(z(1-h)\right), \tag{14}$$

$$\Delta^2 z(t) \in F\left(z(t), \Delta z(t)\right),\quad t = 0, h, 2h, ..., 1-2h, \tag{15}$$

$$z(0) = \alpha_0,\ \Delta z(0) = \alpha_1 \tag{16}$$

The problem (11)-(13) is reduced to (14)-(16). Using mapping

$Q(z,w) = 2w - z + l^2 F\left(z, \frac{w-z}{l}\right)$ newwritten as follows the problem (14)-(16) is

$$\min J_l\left[z(\cdot)\right] = \sum_{t=2l,\dots,1-2l} l\,g\left(z(t),t\right) + \Psi\left(z(1-l)\right),$$

$$z(t+2l) \in Q\left(z(t), z(t+l)\right),\ t = 0, l, \dots, 1-2l$$

$$z(0) = \alpha_0,\ z(1) = \alpha_0 + l\beta_1.$$

By the Theorem 2.16 there exist $\{\omega^*(t)\}, \{z^*(t)\}$ and a number $\alpha = \alpha_l$ that takes values 0 or 1 and satisfies

$$\begin{aligned} &\left(z^*(t) - \omega^*(t), \omega^*(t+l)\right) \in Q^*(z^*(t+2l)); \\ &\left(\tilde{z}(t), \tilde{z}(t+l), \tilde{z}(t+2l)\right) - \alpha\, l\, \partial_z g(\tilde{z}_t, t) \times \{0\},\ t = 2l, \dots, 1-2l, \end{aligned} \tag{17}$$

$$\omega^*(1-l) - z^*(1-l) \in \alpha\ \partial_z \Psi\left(\tilde{z}(1-l)\right),\ z^*(1) = 0 \tag{18}$$

Theorem 2.15. [14] $g(\cdot,t): \mathbb{R}^n \times [0,1] \to \mathbb{R}$ and Ψ are proper CF, F is a CMM of z and F is continuous throughout some feasible curve $\{z^0(t)\}$, $t = 0, l, \dots, 1$. If there exist a number α, that takes values 0 or 1 and vector pairs $\{z^*(t), \omega^*(t)\}$ that satisfies the discrete E-L and transversality inclusions

$$\begin{aligned} (i)\ &\left(\frac{z^*(t) - \omega^*(t) + \omega^*(t+l) - z^*(t+2l}{l^2}, \frac{\omega^*(t+l) - 2z^*(t+2l)}{l}\right) \\ &\in F^*\left(z^*(t+2l); (z(t), \Delta z(t), \Delta^2 z(t))\right) - \alpha\, \partial g(\tilde{z}(t), t) \times \{0\},\ t = 2l, \dots, 1-l; \end{aligned} \tag{19}$$

$$(ii)\ \omega^*(1-l) - z^*(1-l) \in \alpha\, l\, \partial \Psi\left(\tilde{z}(1-l)\right),\ z^*(1) = 0. \tag{20}$$

Respectively then feasible curve $\{\tilde{z}(t)\}$ is optimal curve for the DAP.

Theorem 2.16. [11] Let the problem (11)-(13) be convex problem. If there exist a pair of complete CFs $\{z^*(t), v^*(t)\}$, $t \in [0,1]$ that satisfies second order E-L inclusion

$$(i)\left(\frac{d^2 z^*(t)}{dt^2}+\frac{dv^*(t)}{dt}, v^*(t)\right) \in F^*\left(z^*(t);(\tilde{z}(t),\tilde{z}'(t),\tilde{z}''(t)\right) \text{ a.e. } t \in [0,1], \tag{21}$$

$$(ii)\ v^*(1)+\frac{dz^*(1)}{dt} \in \partial\Psi\left(\tilde{z}(1)\right),\ z^*(1)=0, \tag{22}$$

$$(iii)\ \frac{d^2\tilde{z}(t)}{dt^2} \in F\left(\tilde{z}(t),\tilde{z}'(t);z^*(t)\right),\ \ t \in [0,1], \tag{23}$$

then $\tilde{z}(t)$ is optimal trajectory for (11)-(13).

At the next section, our main problem is handled and optimality conditions are obtained. Utilizing the definitions in Mahmudov [11].

3. RESULTS

We begin with the CP given in Introduction of this thesis:

$$\min \; J\left[z(.)\right]=\int_a^b g\left(z(t),t\right)dt+\Psi\left(z(b),z'(b)\right) \tag{1}$$

$$z''(t)\in Az(t)+F\left(z(t),z'(t)\right),\quad t\in\left[a,b\right] \tag{2}$$

$$z(a)\in M_1,\quad z'(a)\in M_2 \tag{3}$$

The aim is to formalize optimality conditions for the feasible arc $\tilde{z}(t)$ that satisfies the conditions (1)-(3).

3.1. CONDITIONS OF OPTIMALITY

Let now study the following DP

$$\min \sum_{t=2,..,L-1} g(z_t,t) \tag{4}$$

$$z_{t+2}\in F(z_t,z_{t+1}),\quad t=0,...,L-2 \tag{5}$$

$$z_0\in M_1,\quad z_1-z_0\in M_2 \tag{6}$$

and L is a constant natural number, $\{z_t\}_{t=0}^{L}=\{z_t : t=0,...,L\}$ is a feasible solution for the DP (4)-(6).

Condition 3.1. [14] If one of the conditions

$(i)\ (z_t^{\,0},z_{t+1}^0,z_{t+2}^{\,0})\in \text{rel int}\,(gphF),\ z_t^0\in \text{rel int}\ (\text{dom}\,g(.,t)),$

$(ii)\ (z_t^{\,0},z_{t+1}^0,z_{t+2}^{\,0})\in \text{int}\,gphF,\ t=0,...,L-2$

is fulfilled by $z_t^0\in\mathbb{R}^n$ we say steady condition is supplied for (4)-(6).

Now let us obtain NSC for second order DP.

Let us define a vector $v \in \mathbb{R}^{n(L+1)}$ and following CSs

$S_t = \left\{ v = (z_0, ..., z_L) \middle| \ (z_t, z_{t+1}, z_{t+2}) \in gphF \right\}, \ t = 0,1,...L-2.$

$\tilde{M}_1 = \left\{ v = (z_0, ..., z_L) \middle| \ z_0 \in M_1 \right\}.$

$\tilde{M}_2 = \left\{ v = (z_0, ..., z_L) \middle| \ z_1 - z_0 \in M_2, \ z_0 \in M_1 \right\}.$

At first we should estimate the cones $T^*_{S_t}(v), \ T^*_{\tilde{M}_1}(v)$ and $T^*_{\tilde{M}_2}(v).$

Lemma 3.1. [14] Let $T_{gphF}(z_t, z_{t+1}, z_{t+2})$ be CTD, where

$(z_t, z_{t+1}, z_{t+2}) \in gphF.$ Then,

$$T^*_{S_t}(v) = \left\{ v^* = (z^*_0, ..., z^*_L) \middle| (z^*_t, z^*_{t+1}, z^*_{t+2}) \in T^*_{gphF}(z_t, z_{t+1}, z_{t+2}), \ z^*_k = 0, \ k \neq t, t+1, t+2 \right\}.$$

Lemma 3.2. [16] Let $T_{M_1}(z_0)$ be the CTD at the point $z_0 \in M_1$ to the set M_1 and $T_{M_2}(z_1 - z_0)$ be the CTD at point $z_1 - z_0 \in M_2$ to the set M_2. Then

$T^*_{\tilde{M}_1}(v) = \left\{ v^* = (z^*_0, ..., z^*_L) \middle| \ z^*_0 \in T^*_{M_1}(z_0), \ z^*_t = 0, \ t = 1, ..., L \right\}$ and

$T^*_{\tilde{M}_2}(v) = \left\{ v^* = (z^*_0, ..., z^*_L) \middle| z^*_0 + z^*_1 \in T^*_{M_1}(z_0), \ z^*_1 \in T^*_{M_2}(z_1 - z_0), \ z^*_t = 0, \ t \neq 0,1 \right\}.$

Proof . [16] Since $v + \alpha \overline{v} \in \tilde{M}_1$ iff $\overline{z}_0 \in T_{M_1}(z_0)$, then we have

$$T_{\tilde{M}_1}(v) = \left\{ \overline{v} = (\overline{z}_0, ..., \overline{z}_L) \middle| \ \overline{z}_0 \in T_{M_1}(z_0) \right\} \text{ and hence}$$

$$T^*_{\tilde{M}_1}(v) = \left\{ v^* = (z^*_0, ..., z^*_L) \middle| \ z^*_0 \in T^*_{M_1}(z_0), \ z^*_t = 0, \ t = 1, ..., L \right\}.$$

Let $v + \alpha \overline{v} \in \tilde{M}_2$ for s.s $\alpha > 0$, then $(z_1 + \alpha \overline{z}_1) - (z_0 + \alpha \overline{z}_0) \in M_2$ and

$(z_0+\alpha\bar{z}_0)\in M_1$. So $\bar{z}_0\in T_{M_1}(z_0)$ and $\bar{z}_1-\bar{z}_0\in T_{M_2}(z_1-z_0)$ since $(z_1-z_0)+\alpha(\bar{z}_1-\bar{z}_0)\in M_2$.

We obtain

$$T_{\tilde{M}_2}(v)=\left\{\bar{v}=(\bar{z}_0,...,\bar{z}_L)\middle|\ \bar{z}_0\in T_{M_1}(z_0),\ \bar{z}_1-\bar{z}_2\in T_{M_2}(z_1-z_0)\right\}.$$

By the Definition 2.9, $v^*\in T^*_{M_2}(v)$ iff

$$\left\langle v^*,\bar{v}\right\rangle=\sum_{k=0}^{T}\left\langle z_k^*,\bar{z}_k\right\rangle\geq 0,\forall\bar{v}\in T_{\tilde{M}_2}(v).$$

Let add and subtract $\left\langle z_1^*,\bar{z}_0\right\rangle$ to the last inequality. Then after some calculations we get the inequality

$$\left\langle z_0^*+z_1^*,\bar{z}_0\right\rangle+\left\langle z_1^*,\bar{z}_1-\bar{z}_0\right\rangle+\left\langle z_2^*,\bar{z}_2\right\rangle+...+\left\langle z_L^*,\bar{z}_L\right\rangle\geq 0,$$

for all $\bar{z}_0\in T_{M_1}(z_0)$ and $\bar{z}_1-\bar{z}_0\in T_{M_2}(z_1-z_0)$ and $\bar{z}_k$ arbitrary for $k\neq 0,1$. By using the arbitrariness of components $\bar{z}_k,\ k\neq 0,1$ we have $z_k^*=0,\ k\neq 0,1$ and the inequality takes the form

$\left\langle z_0^*+z_1^*,\bar{z}_0\right\rangle+\left\langle z_1^*,\bar{z}_1-\bar{z}_0\right\rangle\geq 0,\ \ \bar{z}_0\in T_{M_1}(z_0)$ and $\bar{z}_1-\bar{z}_0\in T_{M_2}(z_1-z_0)$. The last expression gives us that $z_0^*+z_1^*\in T^*_{M_1}(z_0)$ and $z_1^*\in T^*_{M_2}(z_1-z_0)$. Then we derive

$$T^*_{\tilde{M}_2}(v)=\left\{v^*=(z_0^*,...,z_L^*)\middle|\ z_0^*+z_1^*\in T^*_{M_1}(z_0),\ z_1^*\in T^*_{M_2}(z_1-z_0),\ z_t^*=0,\ t\neq 0,1\right\}.$$

Theorem 3.1. Let (9)-(11) be CP then the trajectory $\left\{\tilde{z}_t^{\,0}\right\}_{t=0}^{L}$ is optimal if there exist not all equal to zero vectors $z_t^*,z_L^*,\omega_t^*,\ t=0,...,L-1,$ and a number α that equals 0 or 1 and are fulfilled discrete E-L inclusion and transversality inclusion:

$(i)\ (z_t^*-\omega_t^*,\omega_{t+1}^*)\in F^*\left(z_{t+2}^*;(\tilde{z}_t,\tilde{z}_{t+1},\tilde{z}_{t+2})\right)-\alpha\,\partial_x\,g(\tilde{z}_t,t)\times\{0\},\ t=0,...,L-2,$

$\partial_z\,g(\tilde{z}_0,0)=\partial_z\,g(\tilde{z}_1,1)=0,$

$(ii)\ \omega_{L-1}^{*} - z_{L-1}^{*} \in \alpha\ \partial_z\ g(\tilde{z}_{L-1}, L-1),\ \ z_L^{*} = 0$

$(iii)\ \omega_0^{*} \in T_{M_1}^{*}(\tilde{z}_0),\ -z_0^{*} - z_1^{*} \in T_{M_1}^{*}(\tilde{z}_0),\ -z_1^{*} \in T_{M_2}^{*}(\tilde{z}_1 - \tilde{z}_0),$

respectively.

In addition, if steady condition is fulfilled then these conditions are sufficient.

Proof. Stating $f(v) = g(z_{L\text{-}1}, z_L)$ and CS P, let handle this problem as mathematical programming problem.

$$\begin{gathered} \text{minimize}\, f(v) \\ \text{subject to } P = (\bigcap_{t=0}^{L-2} S_t) \cap \tilde{M}_1 \cap \tilde{M}_2. \end{gathered} \tag{7}$$

Since $\left\{\tilde{z}_t^{\ 0}\right\}_{t=0}^{L}$ is optimal therefore $\tilde{v} = (\tilde{z}_0, ..., \tilde{z}_L)$ is a solution of (7). By the Theorem 3.4 in [11] there exist

$$v^{*}(t) \in T_{S_t}^{*}(\tilde{v}),\ t = 0,1,...,L-2,\ v_0^{*} \in T_{\tilde{M}_1}^{*}(\tilde{v}),\ v_1^{*} \in T_{\tilde{M}_2}^{*}(\tilde{v}),$$

not all zero, and the number $\alpha \in \{0,1\}$, such that

$$\alpha v^{0*} = \sum_{t=0}^{T-2} v^{*}(t) + v_0^{*} + v_1^{*},\ v^{0*} \in \partial_v\ f(\tilde{v}), \tag{8}$$

where $v^{0*} = \left(0, 0, \overline{z}_2^{*}, \overline{z}_3^{*}, ..., \overline{z}_{L-1}^{*}, \overline{z}_L^{*}\right),\ \overline{z}_t^{*} \in \partial_z\ g(\tilde{z}_t, t).$

By the Lemmas given above we get

$$v^{*}(t) = (0, 0, ..., 0, z_t^{*}, z_{t+1}^{*}, z_{t+2}^{*}, 0, ..., 0),$$

$$(z_t^{*}(t), z_{t+1}^{*}(t), z_{t+2}^{*}(t) \in T_{gphF}^{*}(\tilde{z}_t, \tilde{z}_{t+1}, \tilde{z}_{t+2}),\ t = 0,1,...,L-2, \tag{9}$$

$$v_0^{*} = (z_p^{*}, 0, ..., 0),\ z_p^{*} \in T_{M_1}^{*}(\tilde{z}_0);$$

$$v_1^* = (z_r^*, z_s^*, 0, ..., 0);\ \ z_r^* + z_s^* \in T_{M_1}^*(\tilde{z}_0),\ \ z_s^* \in T_{M_2}^*(\tilde{z}_1 - \tilde{z}_0).$$

Representing (8) component-wise we derive

$$0 = z_p^* + z_r^* + z_0^*(0)$$

$$0 = z_s^* + z_1^*(1) + z_1^*(0)$$

$$\alpha \bar{z}_2^* = z_2^*(2) + z_2^*(1) + z_2^*(0)$$

$$\vdots$$

$$\alpha \bar{z}_t^* = z_t^*(t) + z_t^*(t-1) + z_t^*(t-2)$$

$$\vdots$$

$$\alpha \bar{z}_{L-2}^* = z_{L-2}^*(L-2) + z_{L-2}^*(L-3) + z_{L-2}(L-4)$$

$$\alpha \bar{z}_{L-1}^* = z_{L-1}^*(L-2) + z_{L-1}^*(L-3)$$

$$\alpha \bar{z}_L^* = z_L^*(L-2).$$

From the definition of LDM and using (9), we get

$$(z_t^*(t), z_{t+1}^*(t)) \in F^*(-z_{t+2}^*(t); (\tilde{z}_t, \tilde{z}_{t+1}, \tilde{z}_{t+2}),\ \ t = 0,1,...,L-2. \tag{10}$$

Determine the new formulaes $-z_{t+2}^*(t) \equiv z_{t+2}^*$ and $z_{t+1}^* \equiv \omega_{t+1}^*$, $t = 0,1,...,L-2,$ as a result

$$(-z_p^* - z_r^*, \omega_1^*) \in F^*(z_2^*; (\tilde{z}_0, \tilde{z}_1, \tilde{z}_2))$$

$$(-z_s^* - \omega_1^*, \omega_2^*) \in F^*(z_3^*; (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)). \tag{11}$$

We generalize the formula (11) as

$$(z_t^* - \omega_t^*, \omega_{t+1}^*) \in F^*\left(z_{t+2}^*; (\tilde{z}_t, \tilde{z}_{t+1}, \tilde{z}_{t+2})\right),\ t = 0,1,...,L-2,$$

where $\omega_0^* \in T_{M_1}^*(\tilde{z}_0),\ -z_0^* - z_1^* \in T_{M_1}^*(\tilde{z}_0),\ -z_1^* \in T_{M_2}^*(\tilde{z}_1 - \tilde{z}_0).$

In the end, for $t = L-1$ and $t = L$ we get $\alpha \overline{z}_{L-1}^* = z_{L-1}^*(L-2) + z_{L-1}^*(L-3),\ \ \alpha \overline{z}_L^* = z_L^*(L-2),$

or on the regarded notations

$$\begin{aligned} \alpha \overline{z}_{L-1}^* &= \omega_{L-1}^* - z_{L-1}^* \\ \alpha \overline{z}_L^* &= -z_L^*. \end{aligned} \tag{12}$$

Therefore $(\omega_{L-1}^* - z_{L-1}^*, -z_L^*) \in \alpha\, \partial\, g(\tilde{z}_{L-1}, \tilde{z}_L).$

The proof is completed.

Let l be a scale on the $t-$axis and $z(t) = z_l(t)$ be a uniform net function on $[a,b]$. The first and second order difference operators are as follows

$$\Delta z(t) = \frac{z(t+l) - z(t)}{l} \text{ and}$$

$$\Delta^2 z(t) = \frac{\Delta z(t+l) - \Delta z(t)}{l} = \frac{z(t+2l) - 2z(t+l) + z(t)}{l^2}$$

Now we write the following second order DAP to formulae NSC of optimality.

$$\min\ J_l\left[z(.)\right] = \sum_{t=2l,...,1-2l} l\, g\left(z(t),t\right) + \Psi\left(z(b-l), \Delta z(b-l)\right) \tag{13}$$

$$\Delta^2 z(t) \in Az(t) + F\left(z(t), \Delta z(t)\right), \quad t = a, a+l, a+2l, ..., b-2l \tag{14}$$

$$z(a) \in M_1, \quad \Delta z(a) \in M_2 \tag{15}$$

Here, by (14) we obtain

$$\frac{z(t+2l) - 2z(t+l) + z(t)}{l^2} \in Az(t) + F(z(t), \Delta z(t)),$$

$$z(t+2l)-2z(t+l)+z(t)\in l^2Az(t)+l^2F\left(z(t),\frac{z(t+l)-z(t)}{l}\right),$$

$$z(t+2l)\in -z(t)+l^2Az(t)+2z(t+l)+l^2F\left(z(t),\frac{z(t+l)-z(t)}{l}\right),$$

$$z(t+2l)\in (l^2A-\mathrm{I})z(t)+2z(t+l)+l^2F\left(z(t),\frac{z(t+l)-z(t)}{l}\right).$$

Let us denote

$(l^2A-\mathrm{I})z(t)+2z(t+l)+l^2F\left(z(t),\frac{z(t+l)-z(t)}{l}\right)=Q\big(z(t),z(t+l)\big)$ then,

we get $z(t+2l)\in Q(z(t),z(t+l).$

If we write $z(t)=z,\ z(t+l)=w,\ z(t+2l)=x$ and $x\in Q(z,w)$ we obtain

$$x\in Q(z,w)=(l^2A-\mathrm{I})z+2\mathrm{I}w+l^2F\left(z,\frac{w-z}{l}\right)$$

Now we write the following second-order DAP

$$\min\ J_l\big[z(.)\big]=\sum_{t=a+2l,\dots,b-2l} l\,g\big(z(t),t\big)+\Psi\big(z(b-l),\Delta z(b-l)\big) \tag{16}$$

$$z(t+2l)\in Q\big(z(t),z(t+l)\big),\quad t=a,a+l,a+2l,\dots,b-2l \tag{17}$$

$$z(a)\in M_1,\ \ z(a+l)-z(a)\in lM_2 \tag{18}$$

By the Theorem 3.1 if there exist, not all equal to zero, a pair $\{\omega^*(t),z^*(t)\}$and a number α taking values 0 or 1 that satisfies the conditions

$$\begin{aligned}&(z^*(t)-\omega^*(t),\ \omega^*(t+l))\in Q^*\big(z^*(t+2l);\ \big(\tilde z(t),\tilde z(t+l),\tilde z(t+2l)\big)\big)\\&-\alpha\, l\,\partial g\big(\tilde z(t),t\big)\times\{0\},\ t=a,a+l,a+2l,\dots,b-2l,\end{aligned} \tag{19}$$

$$\omega^*(a) \in T^*_{M_1}(\tilde{z}^*(a)),\ \ -z^*(a)-z^*(a+l) \in T^*_{M_1}(\tilde{z}^*(a)),\ \ -z^*(a+l) \in T^*_{lM_2}(\tilde{z}(a+l)-\tilde{z}(a)), \tag{20}$$

$$\left(\omega^*(b-l)-z^*(b-l),-z^*(b)\right) \in \alpha\, l\, \partial\, g\left(\tilde{z}^*(b-l),\tilde{z}^*(b)\right), \tag{21}$$

then the curve $\{\tilde{z}(t)\}=\{\tilde{z}(t): t=a,a+l,...,b\}$is optimal for the problem (16)-(18).

The function Ψ in (13)-(15) and (16)-(18) gets into

$$\begin{aligned} &\Psi\left(z(b-l)-\Delta z(b-l)\right)=l\,\mathrm{g}\left(z(b-l),z(b)\right) \\ &\left(\omega^*(b-l)-z^*(b-l),-z^*(b)\right) \in \alpha\,\partial\,\overline{\Psi}\left(\tilde{z}(b-l),\tilde{z}(b)\right), \end{aligned} \tag{22}$$

where $\overline{\Psi}\left(z(b-l),z(b)\right) \equiv \Psi\left(z(b-l),\Delta z(b-l)\right)$. (23)

Lemma 3.3. [16] Let $\overline{\Psi}(.,.)$ be proper CF written by (23) in other words

$\overline{\Psi}(z,w) \equiv \Psi\left(z,\dfrac{w-z}{l}\right)$. Then we get the following equivalent inclusions

$$(\overline{z}^*,\overline{w}^*) \in \partial_{z,w}\Psi(z^0,w^0),(z^0,w^0) \in dom\Psi \tag{24}$$

$$(\overline{z}^*+\overline{w}^*,l\overline{w}^*) \in \partial\Psi\left(z^0,\frac{w^0-z^0}{l}\right). \tag{25}$$

Proof [16].

Let relate Q^* in (19) to F^*.

Lemma 3.4. Let $F:\mathbb{R}^n\times\mathbb{R}^n \to P(\mathbb{R}^n)$ be a CMM and Q be mapping defined as

$$Q(z,w)=(l^2A-I)z+2Iw+l^2F\left(z,\frac{w-z}{l}\right).$$

Then the following inclusions are equivalent

$$(i)\left(\frac{z^*+w^*-x^*}{l^2}-A^*z^*,\frac{z^*-2w^*}{l}\right)\in F\left(x^*;\left(z,\frac{w-z}{l},\frac{x-(l^2A-I)z-2Iw}{l^2}\right)\right),$$

$$\frac{z+w-x}{l^2}\in F\left(z,\frac{w-z}{l};z^*\right),\ z^*\in\mathbb{R}^n, \tag{26}$$

$$(ii)\ (z^*,w^*)\in Q^*(x^*;(z,w,x),\ x\in Q(z,w;x^*), \tag{27}$$

where $Q(z,w;x^*)$ is the argmaximum set of mapping Q

$$Q(z,w,x^*)=\left\{x\in Q(z,w)\middle|\left\langle x,x^*\right\rangle=N_Q(z,w,x^*)\right\}.$$

Proof. Let $\overline{\eta}=(\overline{z},\overline{w},\overline{x})\in T_{gphQ}(t),$ be given then from the definition of CTD

$\eta+\alpha\overline{\eta}\in gphQ$ or

$(z+\alpha\overline{z},w+\alpha\overline{w},x+\alpha\overline{x})\in gphQ,\quad x+\alpha\overline{x}\in Q(z+\alpha\overline{z},w+\alpha\overline{w})$

$$x+\alpha\overline{x}\in(l^2A-I)(z+\alpha\overline{z})+2I(w+\alpha\overline{w})+l^2F\left(z+\alpha\overline{z},\frac{w+\alpha\overline{w}-z-\alpha\overline{z}}{l}\right)$$

$$x+\alpha\overline{x}-(l^2A-I)(z+\alpha\overline{z})-2I(w+\alpha\overline{w})\in l^2F\left(z+\alpha\overline{z},\frac{w+\alpha\overline{w}-z-\alpha\overline{z}}{l}\right)$$

$$\left[x-(l^2A-I)z-2Iw\right]+\left[\alpha\overline{x}-(l^2A-I)\alpha\overline{z}-2I\alpha\overline{w}\right]\in l^2F\left(z+\alpha\overline{z},\frac{w-z}{l}+\frac{\overline{w}-\overline{z}}{l}\right),\text{ then}$$

$$\left[x-(l^2A-I)z-2Iw\right]+\alpha\left[\overline{x}-(l^2A-I)\overline{z}-2I\overline{w}\right]\in l^2F\left(z+\alpha\overline{z},\frac{w-z}{l}+\alpha\frac{\overline{w}-\overline{z}}{l}\right),$$

dividing both side by l^2, we get

$$\left[\frac{x-(l^2A-I)z-2Iw}{l^2}\right]+\alpha\left[\frac{\overline{z}-(l^2A-I)\overline{z}-2I\overline{w}}{l^2}\right]\in F\left(z+\alpha\overline{z},\frac{w-z}{l}+\alpha\frac{\overline{w}-\overline{z}}{l}\right).$$

Then $\left(z+\alpha\bar{z},\dfrac{w-z}{l}+\alpha\dfrac{\bar{w}-\bar{z}}{l},\dfrac{x-(l^2A-I)z-2Iw}{l^2}+\alpha\dfrac{\bar{x}-(l^2A-I)\bar{z}-2I\bar{w}}{l^2}\right)\in gphF$

In other words

$$\left(\bar{z},\frac{\bar{w}-\bar{z}}{l},\frac{\bar{x}-(l^2A-I)\bar{z}-2I\bar{w}}{l^2}\right)\in T_{gphF}\left(z,\frac{w-z}{l},\frac{x-(l^2A-I)z-2Iw}{l^2}\right) \qquad (28)$$

here $T_{gphF}\left(z,\dfrac{w-z}{l},\dfrac{x-(l^2A-I)z-2Iw}{l^2}\right)$ is a CTD to $gphF$.

Conversely, we get $(\bar{z},\bar{w},\bar{x})\in T_{gphQ}(z,w,x)$ from the inclusion (28) (29)

so, (28) and (29) are equivalent.

Let $(z^*,w^*)\in Q^*(x^*;(z,w,x)),\ \ x\in Q(z,w;x^*)$, that is (27) holds.

On the other hand $(z^*,w^*,-x^*)\in T^*_{gphQ}\ (z,w,x)$ means that

$$\langle\bar{z},z^*\rangle+\langle\bar{w},w^*\rangle-\langle\bar{x},x^*\rangle\geq 0,\ (\bar{z},\bar{w},\bar{x})\in T_{gphQ}(z,w,x). \qquad (30)$$

Let be $(\gamma_1^*,\gamma_2^*,-\gamma^*)\in T^*_{gphF}\left(z,\dfrac{w-z}{l},\dfrac{x-(l^2A-I)z-2w}{l^2}\right)$ then from definition of DC

$$\langle\bar{z},\gamma_1^*\rangle+\left\langle\frac{\bar{w}-\bar{z}}{l},\gamma_2^*\right\rangle-\left\langle\frac{\bar{x}-(l^2A-I)\bar{z}-2\bar{w}}{l^2},\gamma^*\right\rangle\geq 0$$

$$\langle\bar{z},l^2\gamma_1^*-l\gamma_2^*\rangle+\langle\bar{w},l\gamma_2^*\rangle-\langle\bar{x},\gamma^*\rangle+\langle(l^2A-I)\bar{z},\gamma^*\rangle+\langle 2\bar{w},\gamma^*\rangle\geq 0$$

$\langle\bar{z},l^2\gamma_1^*-l\gamma_2^*+l^2A^*\gamma^*-\gamma^*\rangle+\langle\bar{w},l\gamma_2^*+2\gamma^*\rangle-\langle\bar{x},\gamma^*\rangle\geq 0,\ \ (\bar{z},\bar{w},\bar{x})\in T_{gphQ}(z,w,x)$ then by (30)

$$\begin{aligned}&z^*=l^2\gamma_1^*-l\gamma_2^*+l^2A^*\gamma^*-\gamma^*,\\&w^*=l\gamma_2^*+2\gamma^*,\\&x^*=\gamma^*\end{aligned}$$

$$\gamma_1^* = \frac{z^* + w^* - x^*}{l^2} - A^* x^*,\ \gamma_2^* = \frac{w^* - 2x^*}{l},\ x^* = \gamma^*.$$

As a result from definition of LDM, we get

$$\left(\frac{z^* + w^* - x^*}{l^2} - A^* x^*,\ \frac{w^* - 2x^*}{l} \right) \in F^* \left(x^*; \left(z, \frac{w-z}{l}, \frac{x - (l^2 A - I)z - 2w}{l^2} \right) \right). \tag{31}$$

This means (26) holds. Also since

$x \in Q(z, w, x^*)$ and $\dfrac{x - (l^2 A - I)z - 2Iw}{l^2} \in F\left(z, \dfrac{w-z}{l}; z^* \right)$ then

$$Q^*(x^*; (z, w, x)) \neq \varnothing \text{ and } F^* \left(x^*; \left(z, \frac{w-z}{l}, \frac{x - (l^2 A - I)z - 2Iw}{l^2} \right) \right) \neq \varnothing.$$

By the relation between LDM and Hamiltonian function

$$F^* \left(x^*; (z, w, x) \right) = \partial_{z,w} N(z, w, x^*),\ x \in F(z, w, x^*)$$

holding for convex mapping F, where $\partial_{z,w} N(z, w, x^*) = -\partial_{x,w} \left[-N(z, w, x^*) \right]$, we find that

$$F^* \left(x^*; \left(z, \frac{w-z}{l}, \frac{x - (l^2 A - I)z - 2Iw}{l^2} \right) \right) = \partial_{z,w} N_F \left(z, \frac{w-z}{l}, x^* \right), \text{ where}$$

$$\frac{x - (l^2 A - I)z - 2Iw}{l^2} \in F\left(z, \frac{w-z}{l}; x^* \right).$$

Therefore we may rewrite (26) as

$$\left(\frac{z^* + w^* - x^*}{l^2} - A^* x^*, \frac{w^* - 2x^*}{l} \right) \in \partial_{z,w} N_F \left(z, \frac{w-z}{l}, x^* \right).$$

From the definition of subdifferential, we have

$$N_F\left(z_1,\frac{w_1-z_1}{l},x^*\right)-N_F\left(z,\frac{w-z}{l},x^*\right)$$

$$\leq\left\langle w_1-w,\frac{z^*+w^*-x^*}{l^2}-A^*x^*\right\rangle+\left\langle\frac{w_1-z_1}{l}-\frac{w-z}{l},\frac{w^*-2x^*}{l}\right\rangle.$$

After some necessary arrangements we obtain that

$$\begin{aligned}&\left\langle 2w_1-z_1,x^*\right\rangle+l^2N_F\left(w_1,\frac{w_1-z_1}{l},x^*\right)-\left\langle 2w-z,x^*\right\rangle-l^2N_F\left(z,\frac{w-z}{l},x^*\right)\\&\leq\left\langle w_1-w,w^*\right\rangle+\left\langle z_1-z,z^*\right\rangle.\end{aligned} \tag{32}$$

Using the connection between the Hamiltonian functions N_Q and N_F

$$N_Q(z,w,x^*)=\left\langle 2w-z,x^*\right\rangle+l^2N_F\left(z,\frac{w-z}{l},x^*\right), \tag{33}$$

the inequality (32) is replaced with inequality

$$N_Q(z_1,w_1,x^*)-N_Q(z,w,x^*)\leq\left\langle z_1-z,z^*\right\rangle+\left\langle w_1-w,w^*\right\rangle$$

which implies $\left\langle z^*,w^*\right\rangle\in\partial_{z,w}N_Q(z,w,x^*).$ Since

$Q^*(x^*;(z,w,x))=\partial_{z,w}N_Q(z,w,x^*),\ \ x\in Q(z,w;x^*),$ then we conclude that (27) holds.

Lemma 3.5. [16] Let $T_{lM_2}(\tilde{z}(l)-\tilde{z}(0))$ be the CTD of the set lM_2 at point $\tilde{z}(l)-\tilde{z}(0)$ and $T_{M_2}(\Delta\tilde{z}(0))$ be the CTD of the set M_2 at $\Delta\tilde{z}(0),$ then

$$T_{lM_2}(\tilde{z}(l)-\tilde{z}(0))=T_{M_2}(\Delta\tilde{z}(0)).$$

Moreover the connection between the DCs

$$T^*_{lM_2}(\tilde{z}(l)-\tilde{z}(0))=T^*_{M_2}(\Delta\tilde{z}(0))$$

holds.

Proof [16].

Theorem 3.2. Let DAP be CP. If there exist, not all equal to zero, a number α, taking values 0 or 1 and $\{z^*(t), v^*(t)\}$ that satisfy

$$\left(\Delta^2 z^*(t) + \Delta v^*(t) - A^* z^*(t+2l), v^*(t)\right)$$

$$\in F^*\left(z^*(t+2l); (\tilde{z}(t), \Delta\tilde{z}(t), \Delta^2\tilde{z}(t) - Az(t)\right) - \alpha\,\partial\, g\left(\tilde{z}(t), t\right) \times \{0\} \tag{34}$$

$$t = a+2l, a+3l, ..., b-2l;$$

$$\left(v^*(b-l) + \Delta z^*(b-l), -z^*(b)\right) \in \alpha\,\partial\,\Psi\left(\tilde{z}(b-l), \Delta\tilde{z}(b-l)\right), \tag{35}$$

$$-z^*(a+l) \in T^*_{M_2}(\Delta\tilde{z}(a)),$$

$$v^*(a) + \Delta z^*(a) \in T^*_{M_1}(\tilde{z}(a)) \tag{36}$$

respectively, then the curve $\{\tilde{z}(t)\}$, $t = a, a+l, ..., b,$ is optimal for DAP. There

$$v^*(t) = \frac{\omega^*(t) - 2z^*(t+l)}{l}.$$

Proof. By the Lemma 3.4 the conditions (i) and (ii) for CP gets into the form

$$\left(\frac{z^*(t) - \omega^*(t) + \omega^*(t+l) - z^*(t+2l)}{l^2} - A^* z^*(t+2l), \frac{\omega^*(t+l) - 2z^*(t+2l)}{l}\right)$$

$$\in F^*(z^*(t+2l); (\tilde{z}(t), \Delta\tilde{z}(t), \Delta^2\tilde{z}(t))), \tag{37}$$

$$t = a, a+l, a+2l, a+3l, ..., b-2l;$$

$$\left(\frac{\omega^*(b-l) - z^*(b-l)}{l}, \frac{-z^*(b)}{l}\right) \in \alpha\,\partial\,\overline{\Psi}(\tilde{z}(b-l), \tilde{z}(b)) \tag{38}$$

$$\omega^*(a) \in T^*_{M_1}(\tilde{z}(a)),$$

$$-z^*(a)-z^*(a+l)\in T^*_{M_1}(\tilde{z}(a)), \tag{39}$$

$$-z^*(a+l)\in T^*_{M_2}(\Delta\tilde{z}(a))$$

respectively, just it is considered that LDM is homogeneous on the primary content. (38) is obtained from (19) by signing $l\tilde{z}(t)$ and $l\tilde{\omega}(t)$, and again with $z^*(t)$ and $\omega^*(t)$ individually. The third inclusion in (15) we obtain the third inclusion of (39) in the consequence of Lemma 3.5. Under the steady condition, (37)-(39) are also sufficient for optimality.

Let

$$v^*(t+l)=\frac{\omega^*(t+l)-2z^*(t+2l)}{l} \quad \text{then}$$

$$\frac{z^*(t)-\omega^*(t)+\omega^*(t+l)-z^*(t+2l)}{l^2}-A^*z^*(t+2l)$$

$$=\frac{z^*(t)-lv^*(t)-2z^*(t+l)+lv^*(t+l)+z^*(t+2l)}{l^2}-A^*z^*(t+2l)$$

$$=\frac{z^*(t+2l)-2z^*(t+l)+z^*(t)}{l^2}+\frac{v^*(t+l)-v^*(t)}{l}-A^*z^*(t+2l)$$

$$=\Delta^2z^*(t)+\Delta v^*(t)-A^*z^*(t+2l). \tag{40}$$

So from (37) and (40), (29) holds. However by (38) and Lemma 3.1

$$\left(\frac{\omega^*(b-l)-2z^*(b)+z^*(b)-z^*(b-l)}{l},\frac{-lz^*(b)}{l}\right)\in\alpha\,\partial\,\Psi(\tilde{z}(b-l),\Delta\tilde{z}(b-l))$$

and hence, by the formulae written above this inclusion is (30) or

$$\left(v^*(b-l)+\Delta z^*(b-l),-z^*(b)\right)\in\alpha\,\partial\,\Psi(\tilde{z}(b-l),\Delta\tilde{z}(b-l)).$$

Using first and second inclusion in (38) by the properties of CCs we obtain

$$\frac{\omega^*(a)-2z^*(a+l)+z^*(a+l)-z^*(a)}{l} \in T^*_{M_1}(\tilde{z}(a)).$$

Since $v^*(a)=\dfrac{\omega^*(a)-2z^*(a+l)}{l}$, the last inclusion turns into $v^*(a)+\Delta z^*(a)\in T^*_{M_1}(\tilde{z}(a))$.

Theorem 3.3. Let $g(.,t):\mathbb{R}^n\times[a,b]\to\mathbb{R}$ and $\Psi:\mathbb{R}^n\to\mathbb{R}$ be continuous CFs. Let MF is convex. If there exist absolutely continuous functions $\{z^*(t),v^*(t)\}$, $t\in[a,b]$ that satisfy the continuous E-L second order differential inclusion

$$(i)\left(\frac{d^2z^*(t)}{dt^2}+\frac{dv^*(t)}{dt}-A^*z^*(t),v^*(t)\right),$$

$$\in F^*\left(z^*(t);\left(\tilde{z}(t),\tilde{z}^{'}(t),\tilde{z}^{''}(t)-Az(t)\right)\right)-\alpha\,\partial g\left(\tilde{z}(t),t\right)\times\{0\},t\in[a,b],$$

$$(ii)\left(v^*(b)+\frac{dz^*(b)}{dt},-z^*(b)\right)\in\partial\Psi\left(\tilde{z}(b),\tilde{z}'(b)\right),$$

$$(iii)\ -z^*(a)\in T^*_{M_2}(\tilde{z}'(a)),\ v^*(a)+\frac{dz^*(a)}{dt}\in T^*_{M_1}(\tilde{z}(a)),$$

respectively, and

$$(iv)\ \frac{d^2\tilde{z}(t)}{dt^2}-A\tilde{z}(t)\in F\left(\tilde{z}(t),\tilde{z}'(t),\tilde{z}''(t)\right),\ t\in[a,b],$$

where $F(z,w;x^*)=\{x\in F(z,w)|\langle x,x^*\rangle=N(z,w,x^*)\}$ is the argmaximum set for MMF then $\tilde{z}(t)$, $t\in[a,b]$ is optimal.

There $z^*(t)$, $t\in[a,b]$, its first order derivative and $\dfrac{d^2z^*(.)}{dt^2}\in L_1^n([a,b])$ are absolutely continuous functions. Besides

$v^*(t)$, $t\in[a,b]$ is absolutely continuous and $\dfrac{d^2v^*(.)}{dt}\in L_1^n([a,b])$.

Proof. Using $F^*(x^*;(z,w,x)=\partial_{(z,w)}N(z,w,x^*);\;\; x\in F(z,w;x^*),\;\; -\partial g(.,t)=\partial(-g(.,t))$ and condition (i),

$$\left(\frac{d^2z^*(t)}{dt^2}+\frac{dv^*(t)}{dt}-A^*z^*(t),v^*(t)\right)\in\partial_{(z,w)}\left[N\left(\tilde{z}(t),\tilde{z}'(t),z^*(t)\right)-g(\tilde{z}(t),t)\right]. \tag{41}$$

Using the definition of subdifferential set of the Hamiltonian function, (41) can be replaced by the inequality

$$N_F\left(z(t),z'(t),z^*(t)\right)-N_F\left(\tilde{z}(t),\tilde{z}'(t),z^*(t)\right)-g\left(z(t),t\right)+g\left(\tilde{z}(t),t\right)$$

$$\leq\left\langle z(t)-\tilde{z}(t),\frac{d^2z^*(t)}{dt^2}+\frac{dv^*(t)}{dt}-A^*z^*(t)\right\rangle+\left\langle z'(t)-\tilde{z}'(t),v^*(t)\right\rangle. \tag{42}$$

Furthermore by the definion of N_F , we can rewrite (42) in the form we get

$$\left\langle\frac{d^2z(t)}{dt^2}-Az(t),z^*(t)\right\rangle-\left\langle\frac{d^2\tilde{z}(t)}{dt^2}-A\tilde{z}(t),z^*(t)\right\rangle-g\left(z(t),t\right)+g\left(\tilde{z}(t),t\right)$$

$$\leq\left\langle z(t)-\tilde{z}(t),\frac{d^2z^*(t)}{dt^2}+\frac{dv^*(t)}{dt}-A^*z^*(t)\right\rangle+\left\langle z'(t)-\tilde{z}'(t),v^*(t)\right\rangle$$

arranging the operations

$$\left\langle\frac{d^2z(t)}{dt^2}-\frac{d^2\tilde{z}(t)}{dt^2},z^*(t)\right\rangle-\left\langle Az(t)-A\tilde{z}(t),z^*(t)\right\rangle-g(z(t),t)+g(\tilde{z}'(t),t)$$

$$\leq\left\langle z(t)-\tilde{z}(t),\frac{d^2z^*(t)}{dt^2}\right\rangle+\left\langle z(t)-\tilde{z}(t),\frac{d^2v^*(t)}{dt}\right\rangle-\left\langle z(t)-\tilde{z}(t),A^*z^*(t)\right\rangle+\left\langle z'(t)-\tilde{z}'(t),v^*(t)\right\rangle$$

more specifically we write

$$g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\geq\left\langle\frac{d^2(z(t)-\tilde{z}(t))}{dt^2},z^*(t)\right\rangle-\left\langle A(z(t)-\tilde{z}(t)),z^*(t)\right\rangle-\left\langle z(t)-\tilde{z}(t),\frac{d^2z^*(t)}{dt^2}\right\rangle$$

$$-\frac{d}{dt}\left\langle z(t)-\tilde{z}(t), v^*(t)\right\rangle+\left\langle z(t)-\tilde{z}(t), A^*z^*(t)\right\rangle.$$

By using equality $\left\langle A(z(t)-\tilde{z}(t)), z^*(t)\right\rangle=\left\langle z(t)-\tilde{z}(t), A^*z^*(t)\right\rangle$ the last inequality becomes

$$g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\geq\frac{d}{dt}\left\langle\frac{d(z(t)-\tilde{z}(t))}{dt}, z^*(t)\right\rangle-\frac{d}{dt}\left\langle z(t)-\tilde{z}(t), \frac{dz^*(t)}{dt}\right\rangle$$
$$-\frac{d}{dt}\left\langle z(t)-\tilde{z}(t), v^*(t)\right\rangle. \tag{43}$$

Integrating (43) over the integral $[a,b]$ and considering that $z(.)$, $\tilde{z}(.)$ are feasible we get

$$\int_a^b\left[g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\right]dt\geq\int_a^b\left[\frac{d}{dt}\left\langle\frac{d(z(t)-\tilde{z}(t))}{dt}, z^*(t)\right\rangle-\frac{d}{dt}\left\langle z(t)-\tilde{z}(t), \frac{dz^*(t)}{dt}\right\rangle\right]dt$$
$$-\left\langle z(b)-\tilde{z}(b), v^*(b)\right\rangle+\left\langle z(a)-\tilde{z}(a), v^*(a)\right\rangle.$$

If we calculate the integral in the right side of the above inequality then

$$\int_a^b\left[g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\right]dt\geq\left\langle\frac{d(z(b)-\tilde{z}(b))}{dt}, z^*(b)\right\rangle-\left\langle\frac{d(z(a)-\tilde{z}(a))}{dt}, z^*(a)\right\rangle$$
$$-\left\langle z(b)-\tilde{z}(b), \frac{dz^*(b)}{dt}\right\rangle+\left\langle z(a)-\tilde{z}(a), \frac{dz^*(a)}{dt}\right\rangle-\left\langle z(b)-\tilde{z}(b), v^*(b)\right\rangle \tag{44}$$
$$+\left\langle z(a)-\tilde{z}(a), v^*(a)\right\rangle.$$

Rearranging (44), then it follows that

$$\int_a^b\left[g\left(\tilde{z}(t),t\right)-g\left(z(t),t\right)\right]dt\leq\left\langle z(b)-\tilde{z}(b), \frac{dz^*(b)}{dt}\right\rangle-\left\langle z(a)-\tilde{z}(a), \frac{dz^*(a)}{dt}\right\rangle$$

$$+\left\langle \frac{dz(a)}{dt}-\frac{d\tilde{z}(a)}{dt},z^*(a)\right\rangle-\left\langle \frac{dz(b)}{dt}-\frac{d\tilde{z}(b)}{dt},z^*(b)\right\rangle$$

$$+\left\langle z(b)-\tilde{z}(b),v^*(b)\right\rangle-\left\langle z(a)-\tilde{z}(a),v^*(a)\right\rangle$$

and hence

$$\int_a^b\left[g\left(\tilde{z}(t),t\right)-g\left(z(t),t\right)\right]dt\le\left\langle z(b)-\tilde{z}(b),\frac{dz^*(b)}{dt}+v^*(b)\right\rangle-\left\langle \frac{dz(b)}{dt}-\frac{d\tilde{z}(b)}{dt},z^*(b)\right\rangle$$
$$+\left\langle \frac{dz(a)}{dt}-\frac{d\tilde{z}(a)}{dt},z^*(a)\right\rangle-\left\langle z(a)-\tilde{z}(a),\frac{dz^*(a)}{dt}+v^*(a)\right\rangle. \tag{45}$$

Applying (ii) and (iii) with all feasible arcs $z(t),\ \ t\in[a,b]$, the relations

$$\left\langle z(b)-\tilde{z}(b),\frac{dz^*(b)}{dt}+v^*(b)\right\rangle-\left\langle \frac{dz(b)}{dt}-\frac{d\tilde{z}(b)}{dt},z^*(b)\right\rangle\le\Psi\left(z(b),z'(b)\right)-\Psi\left(\tilde{z}(b),\tilde{z}'(b)\right) \tag{46}$$

and

$$\left\langle z(a)-\tilde{z}(a),v^*(a)+\frac{dz^*(a)}{dt}\right\rangle\ge 0,$$

$$\left\langle \frac{dz(a)}{dt}-\frac{d\tilde{z}(a)}{dt},-z^*(a)\right\rangle\ge 0$$

holds. Thus (45),(46) and last inequalities imply.

$$\int_a^b\left[g\left(\tilde{z}(t),t\right)-g\left(z(t),t\right)\right]dt+\Psi\left(\tilde{z}(b),\tilde{z}'(b)\right)-\Psi\left(z(b),z'(b)\right)\le 0, \tag{47}$$

$J\left[\tilde{z}(t)\right]\le J\left[z(t)\right]$ which means $\tilde{z}(t),\ t\in[a,b]$ is optimal.

4. DISCUSSION

Now we are going to derive the optimality conditions for the non-convex problem

$$\min\ J\left[z(.)\right]=\int_a^b g\left(z(t),t\right)dt+\Psi\left(z(b),z'(b)\right) \tag{1}$$

$$z''(t)\in Az(t)+F\left(z(t),z'(t)\right),\quad t\in\left[a,b\right] \tag{2}$$

$$z(a)\in M_1,\quad z'(a)\in M_2, \tag{3}$$

where the functions, MF and sets are non-convex, A is $n\times n$ matrix.

Condition 4.1. [14] Let the CTD $T_{gphF}(\tilde{z}_t,\tilde{z}_{t+1},\tilde{z}_{t+2})$ in the problem (1)-(3) be local tents, where $\tilde{z}_t$ are the points of the feasible arc $\{\tilde{z}_t\}_{t=0}^{L}$. Assume in addition that $g(\tilde{z}_{L-1},\tilde{z}_L)$ admits a continuous CUA $h_t(.,\tilde{z}_t)$ at the point $(\tilde{z}_{L-1},\tilde{z}_L)$, which ensures that the subdifferential $\partial g(z_{L-1},z_L)=\partial h_t(0,z_{L-1},z_L)$ is defined.

Theorem 4.1. Let (1)-(3) is non-convex. If there are absolutely continuous functions $z^*(t),\ v^*(t),\quad t\in\left[a,b\right]$ ensuring the conditions

$$(i)\ \left(\frac{d^2z^*(t)}{dt^2}+\frac{dv^*(t)}{dt}-A^*z^*(t)+z^*(t),v^*(t)\right)\in F^*\left(z^*(t);(\tilde{z}(t),\tilde{z}'(t),\tilde{z}''(t)-A\tilde{z}(t)\right),\ t\in\left[a,b\right],$$

$$(ii)\ g(z,t)-g\left(\tilde{z}(t),t\right)\geq\left\langle z^*(t),z-\tilde{z}(t)\right\rangle,\ \forall z\in\mathbb{R}^n,$$

$$(iii)\ \Psi(z,v)-\Psi\left(\tilde{z}(b),\tilde{z}'(b)\right)\geq\left\langle v^*(b)+\frac{dz^*(b)}{dt},z-\tilde{z}(b)\right\rangle-\left\langle z^*(b),v-\tilde{z}'(b)\right\rangle,\ \forall(z,v)\in\mathbb{R}^{2n},$$

$$(iv)\ -z^*(a)\in T^*_{M_2}(\tilde{z}'(a)),\quad v^*(a)+\frac{dz^*(a)}{dt}\in T^*_{M_1}(\tilde{z}(a)),$$

$$(v)\ \left\langle \frac{d^2\tilde{z}(t)}{dt} - A\tilde{z}(t), z^*(t) \right\rangle = N_F\left(\tilde{z}(t), \tilde{z}'(t), z^*(t)\right), \quad t \in [a,b]$$

then arc $\tilde{z}(t),\ t \in [a,b]$ is optimal.

Proof. For MF F the LDM on the point $(z,w,x) \in gphF$ is defining as follows;

$$F^*(x^*;(z,w,x)) = \left\{ \begin{array}{r} (z^*,w^*): N(z_1,w_1,x^*) - N(z,w,x^*) \le \left\langle z^*, z_1 - z \right\rangle + \left\langle w^*, w_1 - w \right\rangle \\ , \forall (z_1,w_1,x) \in gphF \end{array} \right\},$$

$$x \in F(z,w;x^*)$$

By condition (i) and definition of LDM in the non-convex case we get

$$N_F\left(z(t), z'(t), z^*(t)\right) - N_F\left(\tilde{z}(t), \tilde{z}'(t), z^*(t)\right)$$

$$\le \left\langle z(t) - \tilde{z}(t), \frac{d^2 z^*(t)}{dt^2} + \frac{dv^*(t)}{dt} - A^* z^*(t) + z^*(t) \right\rangle + \left\langle z'(t) - \tilde{z}'(t), v^* \right\rangle$$

using condition (v) and definition of Hamiltonian function we obtain

$$\left\langle \frac{d^2 z(t)}{dt^2} - Az(t), z^*(t) \right\rangle - \left\langle \frac{d^2\tilde{z}(t)}{dt^2} - A\tilde{z}(t), z^*(t) \right\rangle$$

$$\le \left\langle z(t) - \tilde{z}(t), \frac{d^2 z^*(t)}{dt^2} + \frac{dv^*(t)}{dt} - A^* z^*(t) + z^*(t) \right\rangle + \left\langle z'(t) - \tilde{z}'(t), v^* \right\rangle$$

applying necessary operations

$$0 \ge \left\langle \frac{d^2((z(t) - \tilde{z}(t))}{dt^2}, z^*(t) \right\rangle - \left\langle A(z(t) - \tilde{z}(t)), z^*(t) \right\rangle - \left\langle z(t) - \tilde{z}(t), \frac{d^2 z^*(t)}{dt^2} \right\rangle$$

$$-\frac{d}{dt}\left\langle z(t) - \tilde{z}(t), v^*(t) \right\rangle + \left\langle z(t) - \tilde{z}(t), A^* z^*(t) \right\rangle - \left\langle z(t) - \tilde{z}(t), z^*(t) \right\rangle. \tag{4}$$

From inequality we may simplify this expression since

$\left\langle A(z(t)-\tilde{z}(t)), z^*(t)\right\rangle = \left\langle z(t)-\tilde{z}(t), A^* z^*(t)\right\rangle$ and

also, by conditions (ii) and (iii) the expression (4) becomes

$$g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right) \geq \frac{d}{dt}\left\langle \frac{d(z(t)-\tilde{z}(t))}{dt}, z^*(t)\right\rangle - \frac{d}{dt}\left\langle \frac{dz^*(t)}{dt}, z(t)-\tilde{z}(t)\right\rangle$$

$$-\frac{d}{dt}\left\langle z(t)-\tilde{z}(t), v^*(t)\right\rangle. \qquad (5)$$

Integrate (5) over the interval $[a,b]$ then,

$$\int_a^b \left[g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\right]dt \geq \int_a^b \left[\frac{d}{dt}\left\langle \frac{d(z(t)-\tilde{z}(t))}{dt}, z^*(t)\right\rangle - \frac{d}{dt}\left\langle \frac{dz^*(t)}{dt}, z(t)-\tilde{z}(t)\right\rangle\right]dt$$

$$-\left\langle z(b)-\tilde{z}(b), v^*(b)\right\rangle + \left\langle z(a)-\tilde{z}(a), v^*(a)\right\rangle. \qquad (6)$$

If we calculate the integral in the right side of the above inequality then

$$\int_a^b \left[g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\right]dt \leq \left\langle z(b)-\tilde{z}(b), \frac{dz^*(b)}{dt}\right\rangle - \left\langle z(a)-\tilde{z}(a), \frac{dz^*(a)}{dt}\right\rangle$$

$$+\left\langle \frac{dz(a)}{dt}-\frac{d\tilde{z}(a)}{dt}, z^*(a)\right\rangle - \left\langle \frac{dz(b)}{dt}-\frac{d\tilde{z}(b)}{dt}, z^*(b)\right\rangle$$

$$+\left\langle z(b)-\tilde{z}(b), v^*(b)\right\rangle - \left\langle z(a)-\tilde{z}(a), v^*(a)\right\rangle$$

and hence

$$\int_a^b \left[g(z(t),t)-g(\tilde{z}(t),t)\right]dt \leq \left\langle z(b)-\tilde{z}(b), \frac{dz^*(b)}{dt}+v^*(b)\right\rangle - \left\langle \frac{dz(b)}{dt}-\frac{d\tilde{z}(b)}{dt}, z^*(b)\right\rangle$$

$$+\left\langle \frac{dz(a)}{dt}-\frac{d\tilde{z}(a)}{dt}, z^*(a)\right\rangle - \left\langle z(a)-\tilde{z}(a), \frac{dz^*(a)}{dt}+v^*(a)\right\rangle. \qquad (7)$$

Using transvesality condition (iii), for all feasible arcs $x(t)$, $t\in[a,b]$ the relations

$$\Psi(z(b))-\Psi(\tilde{z}(b))\geq\left\langle z(b)-\tilde{z}(b),v^{*}(b)+\frac{dz^{*}(b)}{dt}\right\rangle-\left\langle z^{*}(b),v-\tilde{z}'(b)\right\rangle \quad (8)$$

Thus (7), (8) and last inequalities imply

$$\int_a^b\left[g\left(z(t),t\right)-g\left(\tilde{z}(t),t\right)\right]dt+\Psi\left(z(b),z'(b)\right)-\Psi\left(\tilde{z}(b),\tilde{z}'(b)\right)\geq 0$$

then we obtain $J\left[\tilde{z}(t)\right]\leq J\left[z(t)\right]$ so $\tilde{z}(t),\ t\in\left[a,b\right]$ is optimal.

5. CONCLUSION AND RECOMMENDATIONS

Through the thesis, it has been studied over multivalued functions, local dual functions, discrete and differential inclusion problems in convex analysis by assigning fundemantel concepts. Convex functions are analyzied by supplying features of convex sets and convex cones. Optimization problems with second order semilinear differential inclusion are studied. First of all, basic description and theorems concerning convex analysis are given. Later on introduction to multivalued functions, dual functions, discrete and differential inclusion optimization problems are given. In the discussion section, in the light of Mahmudov's studies, second order semilinear differential inclusion of convex and nonconvex problems are discussed. When we change the boundary conditions, optimization problems gets formed as a totally different optimization problems.

My intent is to improve the outcomes required regarding the optimization out of this thesis in the mentioned ascept in my following studies.

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CURRICULUM VITAE

Personal Information	
Name Surname	Şükran KESKİN
Place of Birth	Suşehri
Date of Birth	14.07 1987
Nationality	❒ T.C. ❒ Other:
Phone Number	05467645168
Email	sukran-mat@hotmail.com

Educational Information	
B. Sc.	
University	Uludağ University
Faculty	Faculty of Science and Arts
Department	Departmant of Mathematics
Graduation Year	08.06.2009

M. Sc.	
University	Istanbul University
Institute	Graduate School of Science and Engineering
Department	Departmant of Mathematics
Programme	Mathematics Programme