T.R.
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STURMIAN THEORY IN IMPULSIVE HYPERBOLIC EQUATIONS

DOCTOR OF PHOLOSOPHY

KÜBRA USLU İŞLER

SUPERVISOR
Prof. Dr. Abdullah ÖZBEKLER

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APPROVAL OF THESIS

STURMIAN THEORY IN IMPULSIVE HYPERBOLIC EQUATIONS submitted by **Kübra USLU İŞLER** and defended before the Examining Committee Members listed below in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Department of **Mathematics, Institute of Graduate Studies of Bolu Abant Izzet Baysal University** in **27/01/2023** by

Examining Committee Members	**Signature**
Supervisor Prof. Dr. Abdullah ÖZBEKLER Atilim University	…………………………
Member Prof. Dr. Zafer ERCAN Bolu Abant İzzet Baysal Üniversitesi	…………………………
Member Prof. Dr. Mehmet TURAN Atilim University	…………………………
Member Prof. Dr. Ayşe Feza GÜVENİLİR Ankara University	…………………………
Member Prof. Dr. Fatma KARAKOÇ Ankara University	…………………………

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To My Lovely Daughter

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KÜBRA USLU İŞLER

ABSTRACT

STURMIAN THEORY IN IMPULSIVE HYPERBOLIC EQUATIONS
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In this thesis, we investigate the Sturmian theory in impulsive hyperbolic equations. We obtained some Sturm-type comparison results for hyperbolic equations on a rectangular region in the plane and on a rectangular prism in 3-space under the impulse effects with fixed moment.

The results obtained in Chapter 2, extend the results of Kreith (1969) and Kreith (1973) for Sturmian comparison results on hyperbolic equations to linear and nonlinear hyperbolic equations under fixed moment of impulse effects on a rectangle.

Chapter 3 deals with new Sturmian comparison results for hyperbolic equations on a rectangular prism under the appropriate initial conditions.

In Chapter 4, we present new Sturmian comparison results for a pair of linear impulsive hyperbolic equations on a rectangular prism having fixed moments of impulse actions. We extend obtained results to nonlinear impulsive hyperbolic equations.

In the last chapter, we give some conclusions and recommendations about exciting problems those would have interesting results in future.

KEY WORDS: Hyperbolic equations, Sturm comparison theorem, Impulsive hyperbolic equations.

ÖZET

İMPALSİF HİPERBOLİK DENKLEMLERDE STURM TEORİSİ
DOKTORA TEZİ
KÜBRA USLU İŞLER
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ
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Bu tezde, impalsif hiperbolik denklemlerde Sturmian teorisini araştırdık. 2 boyuttaki dikdörtgensel alanda ve 3 boyuttaki dikdörtgenler prizması üzerinde sabit zamanlı impalsif hiperbolik denklemlerde Sturmian teorisi için bazı sonuçlar elde ettik.

İkinci bölümde elde edilen sonuçlar, hiperbolik denklemlerdeki Sturmian karşılaştırma sonuçları için Kreith (1969) ve Kreith (1973) sonuçlarını bir dikdörtgen üzerinde impals etkisi sabit zamanlı dogrusal ve doğrusal olmayan hiperbolik denklemlere genişletir.

Üçüncü bölüm, uygun başlangıç koşulları altında bir dikdörtgenler prizması üzerindeki hiperbolik denklemler için yeni Sturmian karşılaştırma sonuçlarını ele alır.

Dördüncü bölümde, dikdörtgen bir prizma üzerindeki impals etkisi sabit zamanlı lineer hiperbolik denklem çifti için yeni Sturmian karşılaştırma sonuçlarını sunduk. Elde edilen sonuçları lineer olmayan impals etkisi sabit zamanlı hiperbolik denklemlere genişlettik.

Son bölümde, ileride ilginç sonuçları olacak problemler hakkında bazı sonuçlar ve öneriler veriyoruz

ANAHTAR KELİMELER: Hiperbolik denklemler, Sturm karşılaştırma teoremi, İmpalsif hiperbolik denklemler.

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1. INTRODUCTION

One of the fundamental goal of the qualitative theory of differential equations is to investigate positive (negative), oscillatory (nonoscillatory) solutions of them. If solution of a differential equation exists, then it is crucial to know about the uniqueness, the number of solutions or separation of the zeros of the solutions.

Let x and y are nontrivial solutions of

$$x'' + \mu_0 x = 0; \qquad \mu_0 \in \mathbb{R},$$
$$y'' + \nu_0 y = 0; \qquad \nu_0 \in \mathbb{R},$$

respectively. Then

(i) $\nu_0 > 0 \Rightarrow y$ has a zero on every interval $[t_0, \infty)$;

(ii) $\mu_0 \leq 0 \Rightarrow x$ has at most one zero on any interval $[t_0, \infty)$;

(iii) $\nu_0 > \mu_0 \Rightarrow$ a zero of y separates each successive zero of x.

for some $t_0 \in \mathbb{R}$. These observations serve as the most basic demonstrations of the oscillation, nonoscillation, and separation of zeros, respectively. Additionally, they contribute as the impetus for a sizable and expanding amount of mathematical literature. The result given by French mathematician J. C. François Sturm who published his ground-breaking research on this topic in Sturm (1836).

The most of the fundamental results in this theory are applicable to self-adjoint Sturm-Liouville equations

$$(k(t)x')' + p(t)x = 0; \qquad k(t) > 0, \tag{1.1}$$
$$(m(t)y')' + q(t)y = 0; \qquad m(t) > 0, \tag{1.2}$$

where p, q, k, and m are real-valued continuous functions on an interval $J \subset \mathbb{R}$.

Theorem 1.0.1 (Sturm comparison theorem, Sturm (1836)). *Let t_1, t_2 be two successive zeros of $x(t)$ of* (1.1) *on J. If $k(t) \equiv m(t) > 0$ and $p(t) \leq q(t)$ for $t \in [t_1, t_2]$, then any solution $y(t)$ of* (1.2) *has a zero in (t_1, t_2).*

Mauro Picone (1909), an Italian mathematician, who was established the Picone identity to provide the Sturm comparison result for self-adjoint equations. Removing the equality requirement on leading coefficients in (1.1) and (1.2), Picone established so called the *"Picone's identity"*:

$$\left[\frac{x}{y}\{k(t)x'y - m(t)xy'\}\right]' = [q(t) - p(t)]x^2 + [k(t) - m(t)](x')^2 + m(t)\left[x' - \frac{xy'}{y}\right]^2$$

which hold for differentiable functions x, y, $k(t)x'$ and $m(t)y'$ with $y \neq 0$. Following generalization of Sturm's theorem based on Picone's identity.

Theorem 1.0.2 (Picone's theorem, Picone (1909))**.** *Let t_1 and t_2 be two consecutive zeros of a nontrivial solution x of* (1.1) *on J, and let $k \geq m > 0$ and $p \leq q$ on* $[t_1, t_2]$. *If either $p(t) - q(t) \not\equiv 0$ or $k(t) - m(t) \not\equiv 0$, then any solution y of* (1.2) *has a zero on (t_1, t_2).*

Since impulsive differential equations are utilized to simulate a wide range of actual phenomena that are found in fields like chemistry, engineering, biology, and physics, theory of the impulsive differential equations has became one of the very important subject of differential equations. Furthermore, impulsive differential equations offer a wider range of applications than the related topic of ordinary differential equations. Much work has been done in this area in last few decades. Numerous scientific fields outside of mathematics have also exploited impulsive differential equation solutions. Examples include problems with the threshold theory in biology, problems with optimum control theory, chemistry, engineering, etc. The first Sturmian comparison result for impulsive equations was published by Bainov et al. (1996) in which the pair of impulsive Hill's equations

$$\begin{aligned} &x'' + p(t)x = 0; && t \neq \tau_k, \\ &\Delta x = 0, \quad \Delta x' + p_k x = 0; && t = \tau_k, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &y'' + q(t)y = 0; && t \neq \tau_k, \\ &\Delta y = 0, \quad \Delta y' + q_k y = 0; && t = \tau_k \end{aligned} \tag{1.4}$$

were considered. By $\Delta z(\tau_k)$, we mean $\Delta z(\tau_k) = z(\tau_k^+) - z(\tau_k^-)$, where

$$z(\tau_k^+) = \lim_{t \to \tau_k^+} z(t) \quad \text{and} \quad z(\tau_k^-) = \lim_{t \to \tau_k^-} z(t).$$

Note that z is continuous at $t = \tau_k$ if and only if $\Delta z(\tau_k) = 0$, $k \in \mathbb{N}$.

Theorem 1.0.3 (Sturmian comparison theorem, Bainov et al. (1996))**.** *Suppose that*

(i) p, $q \in \mathrm{PLC}(I)$;

(ii) *Equation* (1.4) *has a solution* $y(t)$ *such that* $y(t) \neq 0$ *for* $t \in (t_1, t_2)$ *and* $y(t_1) = y(t_2) = 0$;

(iii) $p(t) \geq q(t)$ *for* $t \in (t_1, t_2)$ *and* $p_k \geq q_k$ *for* $t_k \in (t_1, t_2)$ *(*$k \in \mathbb{N}$*) hold;*

(iv) $p - q > 0$ *in some subinterval of* (t_1, t_2) *or* $p_k - q_k > 0$ *for some* $t_k \in (t_1, t_2)$.

Following that every solution x *of equation* (1.3) *has at least one zero in* (t_1, t_2), *where* $\mathrm{PLC}(I)$, $I \subset [t_0, \infty)$, *the set of functions* $z : I \to \mathbb{R}$ *which are continuous for* $t \neq \tau_k$, *continuous from the left with discontinuities of the first kind at* $t = \tau_k$, *i.e,* $z(\tau_k^-) = z(\tau_k)$.

Fix $t_0 \in \mathbb{R}$ and let $\{\tau_k\}$ be a defined sequence in $[t_0, \infty)$, $t_1 > 0$. The space of functions $\mathrm{PLC}^k(I)$ is defined to be the set of functions z such that $z^j \in \mathrm{PLC}(I)$ for $j = 0, 1, \ldots, k$.

Özbekler and Zafer (2005) obtained Sturm-type comparison results for second-order half-linear impulsive equations of the form

$$\begin{aligned} &(k(t)\Phi(x'))' + p(t)\Phi(x) = 0; && t \neq \tau_k, \\ &\Delta x = 0, \qquad \Delta k(t)\Phi(x') + p_k\Phi(x) = 0; && t = \tau_k, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &(m(t)\Phi(y'))' + q(t)\Phi(y) = 0; && t \neq \tau_k, \\ &\Delta y = 0, \qquad \Delta m(t)\Phi(y') + q_k\Phi(y) = 0; && t = \tau_k, \end{aligned} \tag{1.6}$$

where $\Phi(s) = |s|^{\alpha-1}s$, $\alpha > 0$. $\{\tau_k\}$, $\{p_k\}$ and $\{q_k\}$ are sequences of real numbers with $\tau_k < \tau_{k+1}$. The functions p, q, k, m are in $\mathrm{PLC}[t_0, \infty)$ with $k, m > 0$.

Note that (1.5) and (1.6) become linear when $\alpha = 1$.

Theorem 1.0.4 (Picone-type comparison theorem, Özbekler and Zafer (2005))**.** *Let* t_1 *and* t_2 *be two consecutive zeros of a solution* x *of* (1.5) *in* J. *Consider that* $m \leq k$ *and* $p \leq q$ *on* $[t_1, t_2]$, *and that* $p_k \leq q_k$, $k \in \mathbb{N}$, *for which* $t_k \in [t_1, t_2]$. *If either* $p - q \not\equiv 0$ *or* $k - m \not\equiv 0$ *or* $p_k - q_k \not\equiv 0$, *then any solution* y *of* (1.6) *must have a zero in* (t_1, t_2).

Thereafter, Özbekler and Zafer (2006) obtained a Picone-type formula for the second-order non-selfadjoint impulsive differential equations of the form

$$\begin{aligned} &(k_1(t)x')' + r_1(t)x' + p_1(t)x = 0; && t \neq \tau_k, \\ &\Delta x = 0, \qquad \Delta k_1(t)x' + \mu_k x = 0; && t = \tau_k, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &(k_2(t)y')' + r_2(t)y' + p_2(t)y = 0; && t \neq \tau_k, \\ &\Delta y = 0, \qquad \Delta k_2(t)y' + \nu_k y = 0; && t = \tau_k, \end{aligned} \tag{1.8}$$

where $\{\mu_k\}$ and $\{\nu_k\}$ are real sequences, and k_j, r_j, $p_j \in \mathrm{PLC}(I)$ with $k_j(t) > 0$, $j = 1, 2$, for all $t \in I$. They employed the Picone's formula:

$$\begin{aligned} &\frac{x}{y}\Big\{yk_1(t)x' - xk_2(t)y'\Big\}\Big|_{t_1}^{t_2} \\ &\quad = \int_{t_1}^{t_2} \Bigg\{ \Bigg[p_2(t) - p_1(t) - \frac{[r_2(t) - r_1(t)]^2}{4[k_1(t) - k_2(t)]} - \frac{r_2^2(t)}{4k_2(t)}\Bigg] x^2 \\ &\qquad + [k_1(t) - k_2(t)]\Bigg\{x' + \frac{r_2(t) - r_1(t)}{2[k_1(t) - k_2(t)]} r_2(t)\Bigg\}^2 \\ &\qquad + \frac{k_2(t)}{y^2}\Bigg(x'y - xy' - \frac{r_2(t)}{2k_2(t)}xy\Bigg)^2\Bigg\}\mathrm{dt} + \sum_{t_1 \le \tau_k < t_2} (\nu_k - \mu_k)x^2; \qquad y \neq 0. \end{aligned}$$

The recent decades, although the Sturmian theory on partial differential equations have advanced quite quickly, to the our knowledge no work exists with regard to impulsive case in the literature.

Kreith (1969) obtained some remarkable comparison results for hyperbolic equations; he also derived some oscillation criteria for same equations, see also (Kreith, 1973, Theorem 3.8). In view of the Sturm comparison theorem interpretation with the simple harmonic motion, Kreith (1969) considered the pair of hyperbolic equations

$$u_{tt} - u_{xx} + f_1(x,t)u = 0 \tag{1.9}$$

$$v_{tt} - v_{xx} + f_2(x,t)v = 0 \tag{1.10}$$

illustrating how two vibrating strings with the same density and elastic constant move as they oscillate regarding the equilibrium lines $u = 0$ and $v = 0$ under the effect of continuous restorative forces f_1 and f_2, respectively. If $f_2 \geq f_1$, it is reasonable to assume that equation (1.9) should oscillate faster than equation (1.10) in some way. However, experience with enable easy that allow variable separation reveals that this is not the case in the absence of an auxiliary condition. Physically, it is important to analyse finite strings that are elastically bounded at the ends, with the string that is to oscillate more faster being firmly bound. For problems with hyperbolic beginning boundary values, it is feasible to create a mathematical form of the Sturm comparison theorem.

Theorem 1.0.5. *(Kreith, 1973, Theorem 3.8) Let $u > 0$ be a solution of equation* (1.9) *on the rectangle*

$$\mathcal{R} = \{(x,t) : x \in (x_1, x_2),\ t \in (t_1, t_2)\}$$

satisfying

$$\begin{aligned} u(x,t_1) = u(x,t_2) &= 0; & x_1 \le x \le x_2, \\ u_x(x_1,t) - \sigma_1(t)u(x_1,t) &= 0; & t_1 \le t \le t_2, \\ u_x(x_2,t) + \sigma_2(t)u(x_2,t) &= 0; & t_1 \le t \le t_2. \end{aligned}$$

If $f_2(x,t) \ge f_1(x,t)$ on $\mathcal{R}$ and $\tau_j(t) \ge \sigma_j(t)$, $j = 1,2$ for $t \in [t_1,t_2]$, then every solution v of (1.10) *satisfying*

$$\begin{aligned} v_x(x_1,t) - \tau_1(t)v(x_1,t) &= 0; & t_1 \le t \le t_2, \\ v_x(x_2,t) + \tau_2(t)u(x_2,t) &= 0; & t_1 \le t \le t_2 \end{aligned}$$

has a zero in $\bar{\mathcal{R}}$.

In this thesis we research Sturmian theory in impulsive hyperbolic equations. This thesis consists five chapters. In introduction part, a brief history of Sturmian comparison and some Sturm oscillation results are given for some well-known differential equations. In Chapter two, a Sturm comparison criterion is given for a pair of impulsive hyperbolic equations on a rectangular domain in a plane. As some results of them, Sturm oscillation criteria for impulsive hyperbolic equations were given too. Moreover, we investigate how impulse reactions affect oscillatory behavior. In the third Chapter, Sturmian theory for hyperbolic equations on a rectangular prism in three-space and their oscillatory properties are investigated. Chapter four is about a Sturm comparison criterion to a pair of impulsive hyperbolic equations on a rectangular prism in three-space and their oscillatory properties. The last Chapter is the conclusion and recommendation part in which we mentioned a few problems that would have interesting consequences.

2. STURM COMPARISON RESULTS FOR IMPULSIVE HYPERBOLIC EQUATIONS ON A RECTANGLE

In this chapter we give new Sturm-type comparison results for linear and nonlinear impulsive hyperbolic equations under constant moments of impulsive effects. The conclusions obtained expand those in Kreith (1973) for Sturmian comparison theory on linear hyperbolic equations to linear and nonlinear impulsive hyperbolic equations. It has been demonstrated how impulse reactions can significantly alter oscillatory behavior.

2.1 Introduction

For the purpose, fix $x_0, t_0 \in \mathbb{R}$ and let $I \subset [x_0, \infty)$ and $J \subset [t_0, \infty)$ be nondegenerate intervals. It is assumed that the sequence of real numbers $\{\tau_k\}$ is strictly increasing and unbounded, i.e.

$$t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots , \qquad k \in \mathbb{N}.$$

Let $J_{\text{imp}} := \big\{t \in J : t = \tau_k, k \in \mathbb{N}\big\}$, $\mathrm{E} = I \times J$ and

$$\mathrm{E}_{\text{imp}} := I \times J_{\text{imp}} = \big\{(x,t) \in \mathrm{E} : t = \tau_k, k \in \mathbb{N}\big\}.$$

Denote by $\mathrm{C}_{\text{imp}}\left(\bar{E}, \mathbb{R}\right)$ the set of functions $\nu : \bar{E} \to \mathbb{R}$ satisfying

(i) $\nu(x,t) \in \mathrm{C}\left(\bar{E}, \mathbb{R}\right)$ on $(x,t) \in \bar{E} \setminus \bar{E}_{\text{imp}}$;

(ii) There exists limits

$$\lim_{\substack{(x,t)\to(x,\tau_k)\\ t>\tau_k}} \nu(x,t) = \nu(x,\tau_k^+) \quad \text{and} \quad \lim_{\substack{(x,t)\to(x,\tau_k)\\ t<\tau_k}} \nu(x,t) = \nu(x,\tau_k^-);$$

for each $k \in \mathbb{N}$ and $x \in \bar{I}$;

(iii) $\nu(x,t)$ is piecewise left continuous function at each τ_k, $k \in \mathbb{N}$, i.e.

$$\lim_{\substack{(x,t)\to(x,\tau_k)\\ t<\tau_k}} \nu(x,t) = \nu(x,\tau_k)$$

for all $k \in \mathbb{N}$ and $x \in \bar{I}$.

In this chapter we explore the impulsive hyperbolic equations in the form

$$\begin{aligned} z_{tt} - z_{xx} + F(x,t)z &= 0; && (x,t) \in \mathcal{D} \setminus \mathcal{D}_{\text{imp}}, \\ \Delta z_t + F_k(x,t)z &= 0; && (x,t) \in \mathcal{D}_{\text{imp}}, \end{aligned} \tag{2.1}$$

with the initial conditions

$$\begin{aligned} z_x(x_1,t) - a_1(t)z(x_1,t) &= 0; && t \in \bar{J}, \\ z_x(x_2,t) + a_2(t)z(x_2,t) &= 0; && t \in \bar{J} \end{aligned} \tag{2.2}$$

where

(i) $\mathcal{D} = (x_1,x_2) \times J$, $\Delta\nu(x,\tau) = \nu(x,\tau^+) - \nu(x,\tau^-)$, and that $\{\tau_k\}$ is an unbounded sequence of real numbers which is strictly increasing ;

(ii) $a_j(t) : \mathbb{R} \to \mathbb{R}$ are continuous functions for $j = 1, 2$;

(iii) $F, F_k : \mathcal{D} \to \mathbb{R}$, $k \in \mathbb{N}$, are continuous functions.

We remark that $\Delta\nu(x,\tau_k) = 0$, $k \in \mathbb{N}$, if and only if $\nu(x,t)$ is continuous at (x,τ_k).

It is claimed that function z is the solution of (2.1)-(2.2) if

- $z \in \mathrm{C}\left(\bar{\mathcal{D}}, \mathbb{R}\right)$ and $z_t \in \mathrm{C}_{\text{imp}}\left(\bar{\mathcal{D}}, \mathbb{R}\right)$;
- there exist partial derivatives z_{tt}, z_{xx} on $\mathcal{D} \setminus \mathcal{D}_{\text{imp}}$ and z provides the first equation in (2.1) on $\mathcal{D} \setminus \mathcal{D}_{\text{imp}}$;
- z provides the second equation in (2.1) on $\mathcal{D}_{\text{imp}}$ and initial conditions (2.2) on J.

Inspiring by the Sturm comparison theorem's simple harmonic motion interpretation, Kreith (1973) considered the pair of hyperbolic equations

$$\begin{aligned} u_{tt} - u_{xx} + f_1(x,t)u &= 0, \\ v_{tt} - v_{xx} + f_2(x,t)v &= 0. \end{aligned}$$

Motivated by Kreith's results, we provide an extension of his conclusions for the continuous solutions in the pair of hyperbolic impulsive initial value problems in the form (2.1)-(2.2), i.e. the problems

$$\begin{aligned} &u_{tt} - u_{xx} + f(x,t)u = 0; && (x,t) \in \Pi \setminus \Pi_{\text{imp}}, \\ &\Delta u = 0, \qquad \Delta u_t + f_k(x,t)u = 0; && (x,t) \in \Pi_{\text{imp}} \end{aligned} \tag{2.3}$$

satisfying the initial conditions

$$\begin{aligned} &u_x(x_1,t) - r_1(t)u(x_1,t) = 0; \qquad &t \in [t_1,t_2], \\ &u_x(x_2,t) + r_2(t)u(x_2,t) = 0; \qquad &t \in [t_1,t_2], \end{aligned} \tag{2.4}$$

and that

$$\begin{aligned} &v_{tt} - v_{xx} + g(x,t)v = 0; \qquad &(x,t) \in \Pi \setminus \Pi_{\text{imp}}, \\ &\Delta v = 0, \qquad \Delta v_t + g_k(x,t)v = 0; \qquad &(x,t) \in \Pi_{\text{imp}} \end{aligned} \tag{2.5}$$

satisfies the initial conditions

$$\begin{aligned} &v_x(x_1,t) - s_1(t)v(x_1,t) = 0; \qquad &t \in [t_1,t_2], \\ &v_x(x_2,t) + s_2(t)v(x_2,t) = 0; \qquad &t \in [t_1,t_2], \end{aligned} \tag{2.6}$$

where

$$\Pi = \{(x,t) : x \in (x_1,x_2),\ t \in (t_1,t_2)\}, \tag{2.7}$$

and

$$\Pi_{\text{imp}} = \{(x,t) : x \in (x_1,x_2),\ t = \tau_k, k \in \mathbb{N}\}. \tag{2.8}$$

Further it is assumed that $r_j, s_j : J \to \mathbb{R}$, $j = 1,2$, are functions that are continuous defined on $[t_1,t_2] \subset J$, and that $f, g, f_k, g_k : \Pi \to \mathbb{R}$, $k \in \mathbb{N}$, are continuous functions. The sequence $\{\tau_k\}$ is defined as previously.

2.2 Linear Comparison Result

The first main consequence of this chapter is as follows.

Theorem 2.2.1. *Let $u > 0$ be a solution of* (2.3)-(2.4) *on* Π *satisfying* $u(x,t_1) = u(x,t_2) = 0$ *for all* $x \in [x_1,x_2]$.

If the inequalities

$$g(x,t) > f(x,t); \qquad (x,t) \in \Pi, \tag{2.9}$$

$$s_j(t) > r_j(t); \qquad t \in [t_1,t_2] \quad (j = 1,2), \tag{2.10}$$

and that

$$g_k(x,t) > f_k(x,t); \qquad (x,t) \in \Pi_{\text{imp}} \quad (k \in \mathbb{N}) \tag{2.11}$$

hold, then any solution v of (2.5)-(2.6) *has a zero in* Π.

Proof. Contrary assume that v doesn't have a zero in Π. Let's assuming that $v > 0$ in Π. Multiplying first equations of (2.3) and (2.5) by v and u, respectively, then subtract the equations, as may be seen, the identity

$$\big[vu_t - uv_t\big]_t - \big[vu_x - uv_x\big]_x = \big[g(x,t) - f(x,t)\big]uv \tag{2.12}$$

holds for each $(x,t) \in \Pi$. Integration both sides to equation (2.12) over Π, we obtain

$$\iint_{\Pi} \big[g(x,t) - f(x,t)\big]uv\mathrm{d}x\mathrm{d}t = \iint_{\Pi} \Big\{\big[vu_t - uv_t\big]_t - \big[vu_x - uv_x\big]_x\Big\}\mathrm{d}x\mathrm{d}t. \tag{2.13}$$

The functions under integration in (2.13) have jump discontinuities at τ_k, so we have to split the region Π into $(n+1)$-partitions in a way that

$$\Pi_0 := \{(x,t) : x \in (x_1, x_2),\ t \in (t_1, \tau_1]\}, \tag{2.14}$$

$$\Pi_k := \{(x,t) : x \in (x_1, x_2),\ t \in (\tau_k, \tau_{k+1}], k = 1, 2, \dots, n-1\}, \tag{2.15}$$

$$\Pi_n := \{(x,t) : x \in (x_1, x_2),\ t \in (\tau_n, t_2)\} \tag{2.16}$$

so that each integral can be applied to Green's theorem. We also see that each of the preceding partitions is

(a) $\displaystyle\bigcap_{m=0}^{n} \Pi_m = \emptyset$, and

(b) $\displaystyle\Pi = \bigcup_{m=0}^{n} \Pi_m$.

If we apply the Green theorem to the integration of the right hand side on (2.12) on Π_k, then we attain

$$\begin{aligned}\iint_{\Pi_k} \big[g(x,t) - f(x,t)\big]uv\mathrm{d}x\mathrm{d}t &= \iint_{\Pi_k} \Big\{\big[vu_t - uv_t\big]_t - \big[vu_x - uv_x\big]_x\Big\}\mathrm{d}x\mathrm{d}t \\ &= \oint_{\partial\Pi_k} \big[vu_x - uv_x\big]\mathrm{d}t + \big[vu_t - uv_t\big]\mathrm{d}x, \end{aligned}\tag{2.17}$$

where the boundary of each partition $\partial\Pi_k$, $k = 1, 2, \dots, n-1$, is piecewise smooth and positively oriented.

Then the sum of the integrals

$$\begin{aligned}\oint_{\partial\Pi_k} \big[vu_x - uv_x\big]\mathrm{d}t &= \int_{\tau_{k-1}^+}^{\tau_k} \big[v(x_1,t)u_x(x_1,t) - u(x_1,t)v_x(x_1,t)\big]\mathrm{d}t \\ &\quad - \int_{\tau_{k-1}^+}^{\tau_k} \big[v(x_2,t)u_x(x_2,t) - u(x_2,t)v_x(x_2,t)\big]\mathrm{d}t \end{aligned}\tag{2.18}$$

and

$$\oint_{\partial\Pi_k} \big[vu_t - uv_t\big]\mathrm{d}x = \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_k)\mathrm{d}x - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_{k-1}^{+})\mathrm{d}x \tag{2.19}$$

give the last integral in (2.17) for $k = 1, 2, \ldots, n-1$. Due to the initial conditions (2.4) and (2.6) provided by u and v, respectively, (2.18) turns out to be

$$\begin{aligned}\oint_{\partial\Pi_k} \big[vu_x - uv_x\big]\mathrm{d}t = &\int_{\tau_{k-1}^{+}}^{\tau_k} \big[r_1(t) - s_1(t)\big]u(x_1,t)v(x_1,t)\mathrm{d}t \\ &+ \int_{\tau_{k-1}^{+}}^{\tau_k} \big[r_2(t) - s_2(t)\big]u(x_2,t)v(x_2,t)\mathrm{d}t\end{aligned} \tag{2.20}$$

for $k = 1, 2, \ldots, n-1$. In view of (2.19) and (2.20), (2.17) turns to

$$\begin{aligned}&\iint_{\Pi_k} \big[g(x,t) - f(x,t)\big]uv\mathrm{d}x\mathrm{d}t \\ &= \int_{\tau_{k-1}^{+}}^{\tau_k} \big[r_1(t) - s_1(t)\big]u(x_1,t)v(x_1,t)\mathrm{d}t \\ &\quad + \int_{\tau_{k-1}^{+}}^{\tau_k} \big[r_2(t) - s_2(t)\big]u(x_2,t)v(x_2,t)\mathrm{d}t \\ &\quad + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_k)\mathrm{d}x - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_{k-1}^{+})\mathrm{d}x.\end{aligned} \tag{2.21}$$

On the other hand, by using (2.17), and initial conditions (2.4) and (2.6) we obtain

$$\begin{aligned}&\iint_{\Pi_0} \big[g(x,t) - f(x,t)\big]uv\mathrm{d}x\mathrm{d}t \\ &= \oint_{\partial\Pi_0} \big[vu_x - uv_x\big]\mathrm{d}t + \big[vu_t - uv_t\big]\mathrm{d}x \\ &= \int_{t_1}^{\tau_1} \big[v(x_1,t)u_x(x_1,t) - u(x_1,t)v_x(x_1,t)\big]\mathrm{d}t + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_1)\mathrm{d}x \\ &\quad - \int_{t_1}^{\tau_1} \big[v(x_2,t)u_x(x_2,t) - u(x_2,t)v_x(x_2,t)\big]\mathrm{d}t - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,t_1)\mathrm{d}x \\ &= \int_{t_1}^{\tau_1} \big[r_1(t) - s_1(t)\big]u(x_1,t)v(x_1,t)\mathrm{d}t + \int_{t_1}^{\tau_1} \big[r_2(t) - s_2(t)\big]u(x_2,t)v(x_2,t)\mathrm{d}t \\ &\quad + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_1)\mathrm{d}x - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,t_1)\mathrm{d}x\end{aligned} \tag{2.22}$$

for $m = 0$ and

$$\begin{aligned}
&\iint_{\Pi_n} \big[g(x,t) - f(x,t)\big] uv \mathrm{d}x\mathrm{d}t \\
&= \oint_{\partial\Pi_n} \big[vu_x - uv_x\big]\mathrm{d}t + \big[vu_t - uv_t\big]\mathrm{d}x \\
&= \int_{\tau_n^+}^{t_2} \big[v(x_1,t)u_x(x_1,t) - u(x_1,t)v_x(x_1,t)\big]\mathrm{d}t + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,t_2)\mathrm{d}x \\
&\quad - \int_{\tau_n^+}^{t_2} \big[v(x_2,t)u_x(x_2,t) - u(x_2,t)v_x(x_2,t)\big]\mathrm{d}t - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_n^+)\mathrm{d}x \\
&= \int_{\tau_n^+}^{t_2} \big[r_1(t) - s_1(t)\big] u(x_1,t)v(x_1,t)\mathrm{d}t + \int_{\tau_n^+}^{t_2} \big[r_2(t) - s_2(t)\big] u(x_2,t)v(x_2,t)\mathrm{d}t \\
&\quad + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,t_2)\mathrm{d}x - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,\tau_n^+)\mathrm{d}x. \qquad (2.23)
\end{aligned}$$

for $m = n$.

Obviously, as a result of (a) and (b), we have

$$\begin{aligned}
&\iint_{\Pi} \big[g(x,t) - f(x,t)\big] uv \mathrm{d}x\mathrm{d}t \\
&\quad = \iint_{\Pi_0} \big[g(x,t) - f(x,t)\big] uv \mathrm{d}x\mathrm{d}t + \sum_{k=1}^{n-1} \iint_{\Pi_k} \big[g(x,t) - f(x,t)\big] uv \mathrm{d}x\mathrm{d}t \\
&\qquad + \iint_{\Pi_n} \big[g(x,t) - f(x,t)\big] uv \mathrm{d}x\mathrm{d}t. \qquad (2.24)
\end{aligned}$$

Using (2.21), (2.22) and (2.23), (2.24) yields

$$\begin{aligned}
&\iint_{\Pi} \big[g(x,t) - f(x,t)\big] uv \mathrm{d}x\mathrm{d}t \\
&= \int_{t_1}^{t_2} \big[r_1(t) - s_1(t)\big] u(x_1,t)v(x_1,t)\mathrm{d}t + \int_{t_1}^{t_2} \big[r_2(t) - s_2(t)\big] u(x_2,t)v(x_2,t)\mathrm{d}t \\
&\quad + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,t_2)\mathrm{d}x - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x,t_1)\mathrm{d}x \\
&\quad - \int_{x_1}^{x_2} \sum_{k=1}^{n} \Delta\big(vu_t - uv_t\big)(x,\tau_k)\mathrm{d}x. \qquad (2.25)
\end{aligned}$$

Since $\Delta u = \Delta v = 0$ at (x, τ_k), $k \in \mathbb{N}$, the term under sum in (2.25) becomes

$$\sum_{k=1}^{n} \Delta\big(vu_t - uv_t\big)(x,\tau_k) = \sum_{k=1}^{n} \big[v(x,\tau_k)\Delta u_t(x,\tau_k) - u(x,\tau_k)\Delta v_t(x,\tau_k)\big]. \qquad (2.26)$$

The impulse conditions in the second line of (2.3) and (2.5) for u and v, respectively, (2.26) turns out to be

$$\sum_{k=1}^{n} \Delta\big(vu_t - uv_t\big)(x,\tau_k) = \sum_{k=1}^{n} \big[g_k(x,\tau_k) - f_k(x,\tau_k)\big] u(x,\tau_k)v(x,\tau_k). \qquad (2.27)$$

Using (2.27) and $u(x,t_1) = u(x,t_2) = 0$, (2.25) becomes

$$
\begin{aligned}
&\int_{x_1}^{x_2} \big[v(x,t_1)u_t(x,t_1) - v(x,t_2)u_t(x,t_2)\big]\mathrm{d}x \\
&\quad = \iint_{\Pi_k} \big[f(x,t) - g(x,t)\big]uv\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} \big[r_1(t) - s_1(t)\big]u(x_1,t)v(x_1,t)\mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2} \big[r_2(t) - s_2(t)\big]u(x_2,t)v(x_2,t)\mathrm{d}t \\
&\qquad + \int_{x_1}^{x_2} \big[f_k(x,\tau_k) - g_k(x,\tau_k)\big]u(x,\tau_k)v(x,\tau_k)\mathrm{d}x. \qquad (2.28)
\end{aligned}
$$

Conditions (2.9)–(2.11) of Theorem 2.2.1 and (2.28) imply that

$$\int_{x_1}^{x_2} \big[v(x,t_1)u_t(x,t_1) - v(x,t_2)u_t(x,t_2)\big]\mathrm{d}x < 0. \qquad (2.29)$$

However $u_t(x,t_1) \geq 0$, $u_t(x,t_2) \leq 0$, $v(x,t_1) > 0$ and $v(x,t_2) > 0$ in Π, we get

$$0 \leq v(x,t_1)u_t(x,t_1) - v(x,t_2)u_t(x,t_2) \qquad (2.30)$$

for all $x \in [x_1, x_2]$, which contradicts with (2.29). This contradiction proves that v can not be a positive solution of (2.5)-(2.6) on Π. The case $v(x,t) < 0$ in Π can be proved by letting $v = -z$ in Π. Then z becomes a positive solution of (2.5)-(2.6) in Π. Repeating the same proof for z, we obtain that v can not be a negative solution of (2.5)-(2.6) on Π. Therefore v has a zero in Π. Theorem 2.2.1 has been proved. □

Remark 2.2.2. When the impulse effects are removed in equations (2.3) and (2.5), the results obtained in Theorem 2.2.1 cover those given in Kreith (1973) (i.e. Theorem 1.0.5).

Remark 2.2.3. Theorem 2.2.1 can also be proved for a negative solution u of (2.3)-(2.4) in Π, and hence positivity of u can be replaced by "having no zero" in Theorem 2.2.1. This applies to all Sturm comparison theorems in this chapter.

As a result of Remark 2.2.3, we can deduce the oscillation criteria:

Corollary 2.2.4. *If the inequalities*

$$g(x,t) > f(x,t); \qquad (x,t) \in \Pi^*, \qquad (2.31)$$

$$s_j(t) > r_j(t); \qquad t \in [t^*, \infty) \quad (j = 1,2) \qquad (2.32)$$

and that

$$g_k(x,t) > f_k(x,t); \qquad (x,t) \in \Pi^*_{\text{imp}} \quad (k \in \mathbb{N}) \qquad (2.33)$$

hold for all $t^* \geq t_0$, then (2.5)-(2.6) *is oscillatory whenever* (2.3)-(2.4) *is oscillatory, where*

$$\Pi^* = \{(x,t) : x \in (x_1, x_2),\ t \in [t^*, \infty)\} \tag{2.34}$$

and

$$\Pi^*_{\text{imp}} = \{(x,t) : x \in \mathfrak{I},\ t \in [t^*, \infty),\ t = \tau_k, k \in \mathbb{N}\}.$$

Conditions (2.9)–(2.11) of Theorem 2.2.1 can be weakened by the following conclusion.

Corollary 2.2.5. *Let $u(x,t) > 0$ be a solution of* (2.3)-(2.4) *on Π satisfying $u(x,t_1) = u(x,t_2) = 0$ for all $x \in [x_1, x_2]$ and assume that*

$$g(x,t) \geq f(x,t); \qquad (x,t) \in \Pi, \tag{2.35}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_1, t_2] \quad (j = 1, 2), \tag{2.36}$$

and that

$$g_k(x,t) \geq f_k(x,t); \qquad (x,t) \in \Pi_{\text{imp}} \quad (k \in \mathbb{N}) \tag{2.37}$$

are provided. If either

(i) $\operatorname{meas}\{(x,t) \in \Pi : g(x,t) - f(x,t) > 0\} > 0$ *or*

(ii) $\operatorname{meas}\{t \in [t_1, t_2] : s_j(t) - r_j(t) > 0, j = 1, 2\} > 0$*, or that*

(iii) $g_{k_0}(x, \tau_{k_0}) > f_{k_0}(x, \tau_{k_0})$*, $k_0 \in \mathbb{N}$, for which $(x, \tau_{k_0}) \in \Pi_{\text{imp}}$,*

then each solution $v(x,t)$ of (2.5)-(2.6) *has a zero in* Π.

Proof. Contrary assume that v doesn't have a zero in Π. Let' assuming that $v(x,t) > 0$ in Π. Following the similiar methods as in the proof of Theorem 2.2.1, we obtain equation (2.28). All conditions of Corollary 2.2.5 imply that (2.29) and (2.30) hold. The same contradiction establishes that v has a zero in Π. Corollary 2.2.5 has been proved. □

Eventual oscillation results are immediate.

Corollary 2.2.6. *Assuming that the inequalities*

$$g(x,t) \geq f(x,t); \qquad (x,t) \in \Pi^*, \tag{2.38}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t^*,\infty) \quad (j=1,2) \tag{2.39}$$

and that

$$g_k(x,t) \geq f_k(x,t); \qquad (x,t) \in \Pi^*_{\text{imp}} \quad (k \in \mathbb{N}) \tag{2.40}$$

hold for all $t^* \geq t_0$. *If either*

(i) $\operatorname{meas}\{(x,t) \in \Pi^* : (g-f)(x,t) > 0\} > 0$ *or*

(ii) $\operatorname{meas}\{t \in [t^*,\infty) : (s_j - r_j)(t) > 0, j = 1,2\} > 0$*, or that*

(iii) $g_{k_0}(x,\tau_{k_0}) > f_{k_0}(x,\tau_{k_0})$ *for which* $(x,\tau_{k_0}) \in \Pi^*_{\text{imp}}$ *(*$k_0 \in \mathbb{N}$*),*

then (2.5)-(2.6) *is oscillatory whenever* (2.3)-(2.4) *is oscillatory.*

Corollary 2.2.7. *Suppose there exists a domain* $\Pi \subset \Pi^*$ *for a given* $t^* \geq t_0$ *for which either the conditions of Theorem 2.2.1 or Corollary 2.2.5 hold, then* (2.5)-(2.6) *is oscillatory.*

If we replace the strict inequalities (2.9)-(2.11) in Theorem 2.2.1 by the inequalities

$$g(x,t) \geq f(x,t); \qquad (x,t) \in \Pi, \tag{2.41}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_1,t_2] \quad (j=1,2), \tag{2.42}$$

and

$$g_k(x,t) \geq f_k(x,t); \qquad (x,t) \in \Pi_{\text{imp}} \quad (k \in \mathbb{N}), \tag{2.43}$$

then as an alternative of Theorem 2.2.1, we get the following one.

Theorem 2.2.8. *Let* $u > 0$ *be a solution of* (2.3)-(2.4) *on* Π *satisfying* $u(x,t_1) = u(x,t_2) = 0$ *for all* $x \in [x_1,x_2]$. *If* (2.41)–(2.43) *hold, then any solution* v *of* (2.5)-(2.6) *has a zero in* $\bar{\Pi}$.

Proof. We can suppose that $v > 0$ in $\bar{\Pi}$. Utilizing the same techniques of the proof of Theorem 2.2.1, we get equation (2.28). Conditions (2.41)–(2.43) imply

$$\int_{x_1}^{x_2} \big[v(x,t_1)u_t(x,t_1) - v(x,t_2)u_t(x,t_2)\big]\mathrm{d}x \leq 0. \tag{2.44}$$

On the other hand inequality (2.44) together with (2.30) yield

$$v(x,t_1)u_t(x,t_1) - v(x,t_2)u_t(x,t_2) = 0 \tag{2.45}$$

for all $x \in [x_1, x_2]$. Since $u_t(x,t_1) \geq 0$ and $u_t(x,t_2) \leq 0$, (2.45) is possible only when $v(x,t_1) = v(x,t_2) = 0$ for all $x \in [x_1, x_2]$. Otherwise $u_t(x,t_1) = u_t(x,t_2) = 0$ for all $x \in [x_1, x_2]$ which is not possible. This contradiction proves that v can not be a positive solution of (2.5)-(2.6) on $\bar{\Pi}$. The case $v < 0$ can be proved as in the proof of Theorem 2.2.1. Therefore v has a zero on $\bar{\Pi}$. Theorem 2.2.8 has been proved. □

2.3 Nonlinear Comparison Result

In this section we deal with the pair of more generic problems of the form

$$\begin{aligned} &u_{tt} - u_{xx} + F(x,t,u) = 0; && (x,t) \neq (x,\tau_k), \\ &\Delta u = 0, \qquad \Delta u_t + F_k(x,t,u) = 0; && (x,t) = (x,\tau_k), \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} &v_{tt} - v_{xx} + G(x,t,v) = 0; && (x,t) \neq (x,\tau_k), \\ &\Delta v = 0, \qquad \Delta v_t + G_k(x,t,v) = 0; && (x,t) = (x,\tau_k) \end{aligned} \tag{2.47}$$

satisfying initial conditions (2.4) and (2.6) respectively.

We consider that

(i) $F, F_k, G, G_k : \mathbb{R}^3 \to \mathbb{R}$, $k \in \mathbb{N}$, are continuous functions which are odd functions with respect to their third component;

(ii) $p(\delta), q(\delta) \in \mathrm{PLC}(J)$ for which

$$\eta F(\xi,\delta,\eta) \leq p(\delta)\eta^2 \quad \text{and} \quad \eta G(\xi,\delta,\eta) \geq q(\delta)\eta^2; \tag{2.48}$$

(iii) $\{p_k\}$ and $\{q_k\}$ are real-valued sequences for which

$$\eta F_k(\xi,\delta,\eta) \leq p_k\eta^2 \quad \text{and} \quad \eta G_k(\xi,\delta,\eta) \geq q_k\eta^2; \qquad k \in \mathbb{N}. \tag{2.49}$$

The second main consequence of this chapter is as follows.

Theorem 2.3.1. *Let* $u > 0$ *be a solution of* (2.46)-(2.4) *on* Π *satisfying* $u(x, t_1) = u(x, t_2) = 0$ *for all* $x \in [x_1, x_2]$. *Assume that (i)-(iii) hold. If the inequalities*

$$q(t) > p(t); \qquad t \in [t_1, t_2], \tag{2.50}$$

$$s_j(t) > r_j(t); \qquad t \in [t_1, t_2] \quad (j = 1, 2) \tag{2.51}$$

and that

$$q_k > p_k \tag{2.52}$$

hold for each $k \in \mathbb{N}$ *for which* $\tau_k \in [t_1, t_2]$, *then each solution* v *of* (2.47)-(2.6) *has a zero in* Π.

Proof. Consider that $v > 0$ is in Π. Multiplying first equations of (2.46) and (2.47) by v and u, respectively, then subtract the equations, we obtain

$$\big[vu_t - uv_t\big]_t - \big[vu_x - uv_x\big]_x = uG(x, t, v) - vF(x, t, u) \tag{2.53}$$

holds for each $(x, t) \in \Pi$. Then using (2.48), (2.53) becomes

$$\big[vu_t - uv_t\big]_t - \big[vu_x - uv_x\big]_x \geq uq(t)v - vp(t)u = [q(t) - p(t)]uv. \tag{2.54}$$

Integrating both sides of (2.54) over Π we get

$$\iint_\Pi \big[q(t) - p(t)\big]uv\mathrm{d}x\mathrm{d}t \leq \iint_\Pi \Big\{\big[vu_t - uv_t\big]_t - \big[vu_x - uv_x\big]_x\Big\}\mathrm{d}x\mathrm{d}t. \tag{2.55}$$

Since the terms under integration have jump discontinuities at τ_k's, we split the domain Π into $(n + 1)$-partitions in a manner that (2.14)–(2.16) satisfying (a)-(b) so that as in the proof of Theorem 2.2.1, we can apply the Green theorem to each integral on Π_m, $m = 0, \ldots, n$. Using initial conditions (2.4)-(2.6) to each integrals and summing up of them, we obtain

$$\begin{aligned}
&\iint_\Pi \big[q(t) - p(t)\big]uv\mathrm{d}x\mathrm{d}t \\
&\quad \leq \int_{t_1}^{t_2} \big[r_1(t) - s_1(t)\big]u(x_1, t)v(x_1, t)\mathrm{d}t + \int_{t_1}^{t_2} \big[r_2(t) - s_2(t)\big]u(x_2, t)v(x_2, t)\mathrm{d}t \\
&\qquad + \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x, t_2)\mathrm{d}x - \int_{x_1}^{x_2} \big(vu_t - uv_t\big)(x, t_1)\mathrm{d}x \\
&\qquad - \int_{x_1}^{x_2} \sum_{k=1}^{n} \Delta\big(vu_t - uv_t\big)(x, \tau_k)\mathrm{d}x.
\end{aligned} \tag{2.56}$$

Since $\Delta u = \Delta v = 0$ at (x, τ_k), $k \in \mathbb{N}$, the term under the sum above becomes

$$\sum_{k=1}^{n} \Delta\big(vu_t - uv_t\big)(x, \tau_k) = \sum_{k=1}^{n} \big[v(x, \tau_k)\Delta u_t(x, \tau_k) - u(x, \tau_k)\Delta v_t(x, \tau_k)\big]. \tag{2.57}$$

Using the impulse conditions in (2.46) and (2.47) for u and v, respectively, and using (2.49), (2.57) turns out to

$$\begin{aligned}\sum_{k=1}^{n}\Delta\big(vu_t-uv_t\big)(x,\tau_k)&=\sum_{k=1}^{n}\big[u(x,\tau_k)G_k(x,\tau_k,v(x,\tau_k))\\&\quad-v(x,\tau_k)F_k(x,\tau_k,u(x,\tau_k))\big]\\&\geq\sum_{k=1}^{n}[u(x,\tau_k)q_kv(x,\tau_k)-u(x,\tau_k)p_kv(x,\tau_k)]\\&=\sum_{k=1}^{n}[q_k-p_k]u(x,\tau_k)v(x,\tau_k).\end{aligned}\tag{2.58}$$

In view of (2.58), (2.56) becomes

$$\begin{aligned}&\int_{x_1}^{x_2}\big[v(x,t_1)u_t(x,t_1)-v(x,t_2)u_t(x,t_2)\big]\mathrm{d}x\\&\quad\leq\int_{t_1}^{t_2}\big[r_1(t)-s_1(t)\big]u(x_1,t)v(x_1,t)\mathrm{d}t+\int_{t_1}^{t_2}\big[r_2(t)-s_2(t)\big]u(x_2,t)v(x_2,t)\mathrm{d}t\\&\quad+\iint_{\Pi}\Big\{\big[p(t)-q(t)\big]uv\Big\}\mathrm{d}x\mathrm{d}t\\&\quad+\int_{x_1}^{x_2}\sum_{k=1}^{n}\big[p_k-q_k\big]u(x,\tau_k)v(x,\tau_k)\mathrm{d}x.\end{aligned}\tag{2.59}$$

Conditions (2.50)–(2.52) of Theorem 2.3.1 and (2.59) imply that

$$\int_{x_1}^{x_2}\big[v(x,t_1)u_t(x,t_1)-v(x,t_2)u_t(x,t_2)\big]\mathrm{d}x<0.\tag{2.60}$$

However $u_t(x,t_1)\geq 0$, $u_t(x,t_2)\leq 0$, $v(x,t_1)>0$ and $v(x,t_2)>0$ in Π, we have again (2.30). This contradicts with (2.60) which proves that v can not be a positive solution of (2.47)-(2.6) on Π. The case $v<0$ can be done like in the proof of Theorem 2.2.1. Therefore v has a zero in Π. Theorem 2.3.1 has been proved. □

As a result of Theorem 2.3.1, we have the eventual criterion.

Corollary 2.3.2. *Assuming that (i)-(iii) hold. If the inequalities*

$$q(t)>p(t);\qquad t\in[t^*,\infty),\tag{2.61}$$

$$s_j(t)>r_j(t);\qquad t\in[t^*,\infty)\quad(j=1,2)\tag{2.62}$$

hold for each $t^*\geq t_0$, and that

$$q_k>p_k\tag{2.63}$$

for each $k\in\mathbb{N}$ for which $\tau_k\geq t_0$, then (2.47)-(2.6) *is oscillatory whenever* (2.46)-(2.4) *is oscillatory.*

Conditions (2.50)–(2.52) of Theorem 2.3.1 can be weakened by the following conclusion.

Corollary 2.3.3. *Let* $u > 0$ *be a solution of* (2.46)-(2.4) *on* Π *satisfying* $u(x,t_1) = u(x,t_2) = 0$ *for all* $x \in [x_1, x_2]$*. Assume that (i)-(iii) hold. If the inequalities*

$$q(t) \geq p(t); \qquad t \in [t_1, t_2], \tag{2.64}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_1, t_2] \quad (j = 1, 2), \tag{2.65}$$

and that

$$q_k \geq p_k \tag{2.66}$$

hold for each $k \in \mathbb{N}$ *for which* $\tau_k \in [t_1, t_2]$*. If either*

(i) $\operatorname{meas}\{t \in [t_1, t_2] : q(t) - p(t) > 0\} > 0$ *or*

(ii) $\operatorname{meas}\{t \in [t_1, t_2] : s_j(t) - r_j(t) > 0, j = 1, 2\} > 0$*, or that*

(iii) $q_{k_0} > p_{k_0}$ *for some* $k_0 \in \mathbb{N}$*,*

then any solution v *of* (2.47)-(2.6) *has a zero in* Π*.*

Proof. Contrarily assume that $v > 0$ in Π. Equation (2.59) is obtained by following the analogous methods as in the proving of Theorem 2.3.1. Conditions of Corollary 2.3.3 imply that (2.60) and (2.30) hold. Same contradiction illustrates that v has a zero in Π. Corollary 2.3.3 has been proved. □

The oscillation criterion is as follows.

Corollary 2.3.4. *Assuming that for a given* $t_* > 0$*, there exists the interval* $(t_1, t_2) \subset [t_*, \infty)$ *for which either the conditions of Theorem 2.3.1 or Corollary 2.3.3 hold, then* (2.47)-(2.6) *is oscillatory.*

If strict inequalities (2.50)–(2.52) are ruled out in Theorem 2.3.1, namely,

$$q(t) \geq p(t); \qquad t \in [t_1, t_2], \tag{2.67}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_1, t_2] \quad (j = 1, 2) \tag{2.68}$$

and that

$$q_k \geq p_k \quad \text{for each} \quad k \in \mathbb{N} \quad \text{for which} \quad \tau_k \in [t_1, t_2], \tag{2.69}$$

then as a alternative result for Theorem 2.3.1, we get the following one.

Theorem 2.3.5. *Let $u > 0$ be a solution of* (2.46)-(2.4) *on* Π *satisfying* $u(x,t_1) = u(x,t_2) = 0$ *for all* $x \in [x_1, x_2]$. *Assume that (i)-(iii) hold. If conditions* (2.67)–(2.69) *hold, then any solution* v *of* (2.47)-(2.6) *has a zero in* $\bar{\Pi}$.

Proof. Contrary assume that $v > 0$ in $\bar{\Pi}$. Using the same techniques of the proof of Theorem 2.2.1, we get equation (2.59). Conditions (2.67)–(2.69) imply that

$$\int_{x_1}^{x_2} \big[v(x,t_1)u_t(x,t_1) - v(x,t_2)u_t(x,t_2)\big]\mathrm{d}x \le 0. \tag{2.70}$$

On the other hand inequality (2.70) together with (2.30) yield (2.45) for all $x \in [x_1, x_2]$. Since $u_t(x,t_1) \ge 0$ and $u_t(x,t_2) \le 0$, (2.45) is possible only when $v(x,t_1) = v(x,t_2) = 0$ for all $x \in [x_1,x_2]$. Otherwise $u_t(x,t_1) = u_t(x,t_2) = 0$ for all $x \in [x_1,x_2]$ which is not possible. This contradiction proves that v can not be a positive solution of (2.47)-(2.6) on $\bar{\Pi}$. The case $v < 0$ is as similar just like in the proof of Theorem 2.2.1, so it can be omitted. Therefore v has a zero in $\bar{\Pi}$. Theorem 2.3.5 has been proved. □

Corollary 2.3.6. *Assume that (i)-(iii) hold. If the inequalities*

$$q(t) \ge p(t); \qquad t \in [t^*, \infty), \tag{2.71}$$

$$s_j(t) \ge r_j(t); \qquad t \in [t^*, \infty) \quad (j = 1, 2) \tag{2.72}$$

hold for all $t^* \ge t_0$, *and that*

$$q_k \ge p_k \tag{2.73}$$

for each $k \in \mathbb{N}$ *for which* $\tau_k \ge t^*$, *then* (2.47)-(2.6) *is oscillatory whenever* (2.46)-(2.4) *is oscillatory.*

3. STURM COMPARISON RESULTS FOR HYPERBOLIC EQUATIONS ON A RECTANGULAR PRISM

In this chapter, new Sturmian comparison results are obtained for linear and nonlinear hyperbolic equations in a rectangular prism in three-space. The results we obtained are extension of the Sturmian comparison theorem for linear hyperbolic equations on a rectangular region in the plane made by Kreith (1969). An application is also presented via an eigenvalue problem.

3.1 Introduction

In our case, we fix $x_0, y_0, t_0 \in \mathbb{R}$. Let $\mathcal{I} = (x_1, x_2) \subset [x_0, \infty)$, $\mathcal{J} = (y_1, y_2) \subset [y_0, \infty)$, and $\mathcal{K} = (t_1, t_2) \subset [t_0, \infty)$ be three non-degenerate intervals and define the domain (*a rectangular prism*) as

$$\Omega = \mathcal{I} \times \mathcal{J} \times \mathcal{K}. \tag{3.1}$$

In this chapter, we provide some comparison results to the continuous solutions of the pair of hyperbolic equations

$$u_{tt} - \Delta u + f(x, y, t)u = 0 \tag{3.2}$$

and

$$v_{tt} - \Delta v + g(x, y, t)v = 0 \tag{3.3}$$

for $(x, y, t) \in \Omega$, satisfying the initial conditions

$$\begin{aligned} u_x(x_k, y, t) + (-1)^k r_k(t)u(x_k, y, t) = 0; \qquad & (y, t) \in \bar{\mathcal{J}} \times \bar{\mathcal{K}}, \\ u_y(x, y_k, t) + (-1)^k r_{k+2}(t)u(x, y_k, t) = 0; \qquad & (x, t) \in \bar{\mathcal{I}} \times \bar{\mathcal{K}}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} v_x(x_k, y, t) + (-1)^k s_k(t)v(x_k, y, t) = 0; \qquad & (y, t) \in \bar{\mathcal{J}} \times \bar{\mathcal{K}}, \\ v_y(x, y_k, t) + (-1)^k s_{k+2}(t)v(x, y_k, t) = 0; \qquad & (x, t) \in \bar{\mathcal{I}} \times \bar{\mathcal{K}} \end{aligned} \tag{3.5}$$

for $k = 1, 2$, respectively, where $f, g \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$, $r_k, s_k \in \mathrm{C}\left(\bar{\mathcal{K}}, \mathbb{R}\right)$, $(k = 1, 2, 3, 4)$, and that Δ is the usual *Laplace operator* in $\mathbb{R}^2$, i.e.

$$\Delta = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}.$$

A nontrivial function $z(x, y, t)$ is said to be a solution of (3.2)-(3.4) if

- $z : \bar{\Omega} \to \mathbb{R}$ is a continuous function;
- z has second-order partial derivatives z_{xx}, z_{yy} and z_{tt} for each $(x, y, t) \in \Omega$, and that z satisfies equation (3.2);
- z satisfies the initial conditions (3.4) on $\bar{\Omega}$.

A solution z of (3.2)-(3.4) is said to be *oscillatory* if a sequence exists $\{\zeta_n\}_{n=1}^{\infty}$ of real numbers such that $z(x, y, \zeta_n) = 0$ with

$$\lim_{n\to\infty} \zeta_n = 0.$$

Otherwise $z(x, y, t)$ is said to be *nonoscillatory*. Moreover, if all solutions are oscillatory, then (3.2)-(3.4) is said to be *oscillatory* . The similar definitions given above are also valid for (3.3)-(3.5). Motivated by Kreith (1969); a remarkable comparison result obtained on the rectangular domain in the plane, we provide some analogous conclusions for continuous solutions of a pair of linear hyperbolic initial value problems (3.2)-(3.4) and (3.3)-(3.5) on a *rectangular prism* in Section 3.2. The obtained results are extended to the nonlinear problems in Section 3.3.

3.2 Linear Comparison Results

In this section, we offer Sturm comparison results for linear hyperbolic initial value problems.

The first main consequence of this chapter is as follows.

Theorem 3.2.1. *Let $u > 0$ be a solution of* (3.2)-(3.4) *satisfying the boundary conditions*

$$u(x, y, t_1) = u(x, y, t_2) = 0; \qquad (x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}, \tag{3.6}$$

which is positive on $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathcal{K}$. If

$$g(x, y, t) > f(x, y, t); \qquad (x, y, t) \in \Omega, \tag{3.7}$$

and

$$s_k(t) > r_k(t); \qquad t \in \bar{\mathcal{K}} \quad (k = 1, 2, 3, 4) \tag{3.8}$$

are satisfied, then each solution v of (3.3)-(3.5) *has a zero in Ω.*

Proof. Contrarily assume that the solution v of (3.3)-(3.5) does not have a zero in Ω. We can suppose that $v > 0$ in Ω. Multiply equations (3.2) and (3.3) by v and u, respectively, then subtract the equations, we can observe that the identity

$$\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t = \big[g(x,y,t) - f(x,y,t)\big]uv \quad (3.9)$$

holds for each $(x, y, t) \in \Omega$. Integration both sides of equation (3.9) over Ω, we attain

$$\begin{aligned}&\iiint_\Omega \big[g(x,y,t) - f(x,y,t)\big]uv\mathrm{d}V\\ &\quad= \iiint_\Omega \Big\{\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t\Big\}\mathrm{d}V,\end{aligned} \quad (3.10)$$

where $\mathrm{d}V$ is the volume element. Note that Ω is a simple solid region with the piecewise smooth boundary $\mathcal{S}$, so using divergence theorem to the smooth vector field $\mathbf{F}$ on Ω defined by

$$\mathbf{F}(x,y,t) := (uv_x - vu_x)\mathbf{i} + (uv_y - vu_y)\mathbf{j} + (vu_t - uv_t)\mathbf{k}, \quad (3.11)$$

the right hand side of (3.10) turns out to be

$$\begin{aligned}&\iiint_\Omega \Big\{\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t\Big\}\mathrm{d}V\\ &\quad= \iiint_\Omega \nabla \bullet \mathbf{F}\mathrm{d}V \quad \Big(= \iiint_\Omega \operatorname{div}\mathbf{F}\mathrm{d}V\Big)\\ &\quad= \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\mathrm{d}S,\end{aligned} \quad (3.12)$$

where $\hat{\mathbf{N}}$ is the unit outward normal on the surface $\mathcal{S}$ $(= \partial\Omega)$, $\mathrm{d}S$ is surface area element and the ∇ is the usual nabla (gradient) operator defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial t}. \quad (3.13)$$

Note that $\mathcal{S}$ is the union of six rectangular regions, it can be expressed as

$$\mathcal{S} = \bigcup_{j=1}^{6} \mathcal{S}_j,$$

where each $\mathcal{S}_j$ are disjoint, oriented, closed surfaces and defined by

$$\mathcal{S}_1 = \{(x,y,t) : x = x_1, (y,t) \in \bar{\mathcal{J}} \times \bar{\mathcal{K}}\}, \quad (3.14)$$

$$\mathcal{S}_2 = \{(x,y,t) : x = x_2, (y,t) \in \bar{\mathcal{J}} \times \bar{\mathcal{K}}\}, \quad (3.15)$$

$$\mathcal{S}_3 = \{(x,y,t) : y = y_1, (x,t) \in \bar{\mathcal{I}} \times \bar{\mathcal{K}}\}, \quad (3.16)$$

$$\mathcal{S}_4 = \{(x,y,t) : y = y_2, (x,t) \in \bar{\mathcal{I}} \times \bar{\mathcal{K}}\}, \quad (3.17)$$

$$\mathcal{S}_5 = \{(x,y,t) : t = t_1, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\} \quad (3.18)$$

and

$$\mathcal{S}_6 = \{(x, y, t) : t = t_2, (x, y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\}. \tag{3.19}$$

Then the (surface) integral in (3.12) can be represented as

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{j=1}^{6} \iint_{\mathcal{S}_j} \mathbf{F} \bullet \hat{\mathbf{N}}_j \mathrm{d}S, \tag{3.20}$$

where the vectors $\hat{\mathbf{N}}_j$ are the unit outward normal vectors for the surfaces $\mathcal{S}_j$, $j = 1, \dots, 6$, and hence we have

$$\begin{aligned}\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = &\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1 \mathrm{d}S + \iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2 \mathrm{d}S + \iint_{\mathcal{S}_3} \mathbf{F} \bullet \hat{\mathbf{N}}_3 \mathrm{d}S \\ &+ \iint_{\mathcal{S}_4} \mathbf{F} \bullet \hat{\mathbf{N}}_4 \mathrm{d}S + \iint_{\mathcal{S}_5} \mathbf{F} \bullet \hat{\mathbf{N}}_5 \mathrm{d}S + \iint_{\mathcal{S}_6} \mathbf{F} \bullet \hat{\mathbf{N}}_6 \mathrm{d}S. \end{aligned} \tag{3.21}$$

Since $\hat{\mathbf{N}}_1 = -\mathbf{i}$, $\hat{\mathbf{N}}_2 = \mathbf{i}$, $\hat{\mathbf{N}}_3 = -\mathbf{j}$, $\hat{\mathbf{N}}_4 = \mathbf{j}$, $\hat{\mathbf{N}}_5 = -\mathbf{k}$ and $\hat{\mathbf{N}}_6 = \mathbf{k}$, the integrals on the right hand side of (3.21) become

$$\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1 \mathrm{d}S = -\iint_{\mathcal{S}_1} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S = -\int_{t_1}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t, \tag{3.22}$$

$$\iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2 \mathrm{d}S = \iint_{\mathcal{S}_2} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S = \int_{t_1}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t, \tag{3.23}$$

$$\iint_{\mathcal{S}_3} \mathbf{F} \bullet \hat{\mathbf{N}}_3 \mathrm{d}S = -\iint_{\mathcal{S}_3} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S = -\int_{t_1}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t, \tag{3.24}$$

$$\iint_{\mathcal{S}_4} \mathbf{F} \bullet \hat{\mathbf{N}}_4 \mathrm{d}S = \iint_{\mathcal{S}_4} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t, \tag{3.25}$$

$$\iint_{\mathcal{S}_5} \mathbf{F} \bullet \hat{\mathbf{N}}_5 \mathrm{d}S = -\iint_{\mathcal{S}_5} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_1) \mathrm{d}x \mathrm{d}y, \tag{3.26}$$

and

$$\iint_{\mathcal{S}_6} \mathbf{F} \bullet \hat{\mathbf{N}}_6 \mathrm{d}S = \iint_{\mathcal{S}_6} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_2) \mathrm{d}x \mathrm{d}y. \tag{3.27}$$

Imposing initial conditions (3.4) and (3.5) in the integration on the right-hand side of (3.22)–(3.25), we get that

$$\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1 \mathrm{d}S = \int_{t_1}^{t_2} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t, \tag{3.28}$$

$$\iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2 \mathrm{d}S = \int_{t_1}^{t_2} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t, \tag{3.29}$$

$$\iint_{\mathcal{S}_3} \mathbf{F} \bullet \hat{\mathbf{N}}_3 \mathrm{d}S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t \tag{3.30}$$

and

$$\iint_{\mathbb{S}_4} \mathbf{F} \bullet \hat{\mathbf{N}}_4 \mathrm{d}S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t. \tag{3.31}$$

On the other hand, boundary conditions (3.6) imply that (3.26) and (3.27) reduce to

$$\iint_{\mathbb{S}_5} \mathbf{F} \bullet \hat{\mathbf{N}}_5 \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} v(x, y, t_1) u_t(x, y, t_1) \mathrm{d}x \mathrm{d}y, \tag{3.32}$$

and

$$\iint_{\mathbb{S}_6} \mathbf{F} \bullet \hat{\mathbf{N}}_6 \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} v(x, y, t_2) u_t(x, y, t_2) \mathrm{d}x \mathrm{d}y. \tag{3.33}$$

Summing up equations (3.28)–(3.33), (3.20) can be expressed as

$$\begin{aligned}
&\int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ v(x, y, t_1) u_t(x, y, t_1) - v(x, y, t_2) u_t(x, y, t_2) \Big\} \mathrm{d}x \mathrm{d}y \\
&= \int_{t_1}^{t_2} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t \\
&\quad + \iiint_{\Omega} \big[f(x, y, t) - g(x, y, t) \big] uv \mathrm{d}V.
\end{aligned} \tag{3.34}$$

Conditions (3.7)-(3.8) of Theorem 3.2.1 and (3.34) imply that

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \big[v(x, y, t_1) u_t(x, y, t_1) - v(x, y, t_2) u_t(x, y, t_2) \big] \mathrm{d}x \mathrm{d}y < 0. \tag{3.35}$$

However $u_t(x, y, t_1) \geq 0$, $u_t(x, y, t_2) \leq 0$, $v(x, y, t_1) > 0$ and $v(x, y, t_2) > 0$ in Ω, we have

$$0 \leq v(x, y, t_1) u_t(x, y, t_1) - v(x, y, t_2) u_t(x, y, t_2) \tag{3.36}$$

for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$, which contradicts with (3.35). This contradiction proves that v can not be a positive solution of (3.3)-(3.5) on Ω. For the proof of the case $v < 0$ in Ω, we let $v = -z$ in Ω. Then z becomes a positive solution of (3.3)-(3.5) in Ω. Repeating the same proof for z, we obtain that v can not be a negative solution of (3.3)-(3.5) in Ω. Therefore v has a zero in Ω. Theorem 3.2.1 has been proved. $\square$

Remark 3.2.2. Theorem 3.2.1 can also be proved for a negative solution u of (3.2)-(3.4) in Ω, and hence positivity of u can be replaced by "having no zero" in Theorem 3.2.1. This applies to all Sturm comparison theorems in this chapter.

As a result of the Remark above, the oscillation conclusion is as follows.

Corollary 3.2.3. *If*

$$g(x,y,t) > f(x,y,t); \qquad (x,y,t) \in \Omega^*, \tag{3.37}$$

and

$$s_k(t) > r_k(t); \qquad t \in [t^*, \infty) \quad (k = 1,2,3,4) \tag{3.38}$$

are satisfied for each $t^* \geq t_0$, then (3.3)-(3.5) *is oscillatory whenever* (3.2)-(3.4) *is oscillatory, where*

$$\Omega^* = \{(x,y,t) : (x,y) \in \mathfrak{I} \times \mathfrak{J}, t \in [t^*, \infty)\}. \tag{3.39}$$

Conditions (3.7)-(3.8) of Theorem 3.2.1 can be weakened by the following consequence.

Corollary 3.2.4. *Let u be a solution of* (3.2)-(3.4) *providing the boundary conditions* (3.6)*, which is positive on $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathfrak{K}$, and assume that*

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega, \tag{3.40}$$

and

$$s_k(t) \geq r_k(t); \qquad t \in \bar{\mathfrak{K}} \quad (k = 1,2,3,4) \tag{3.41}$$

are provided. If either

(i) $\text{meas}\{(x,y,t) \in \Omega : g(x,y,t) - f(x,y,t) > 0\} > 0$ *or*

(ii) $\text{meas}\{t \in \bar{\mathfrak{K}} : s_k(t) - r_k(t) > 0, k = 1,2,3,4\} > 0$,

then each solution v of (3.3)-(3.5) *has a zero in* Ω*.*

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ in Ω. Following is the similiar steps as in the proof of Theorem 3.2.1, we get equation (3.34). All conditions of Corollary 3.2.4 imply that (3.35) and (3.36) hold. The similar contradiction shows that v has a zero in Ω. Corollary 3.2.4 has been proved. □

Following oscillation results are immediate.

Corollary 3.2.5. *Assume that*

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega^*, \tag{3.42}$$

$$s_k(t) \geq r_k(t); \qquad t \in [t^*,\infty) \quad (k=1,2,3,4) \tag{3.43}$$

are provided for all $t^* \geq t_0$, where Ω^* is defined in (3.39). *If either*

(i) $\text{meas}\,\{(x,y,t) \in \Omega^* : g(x,y,t) - f(x,y,t) > 0\} > 0$ *or*

(ii) $\text{meas}\,\{t \in [t^*,\infty) : s_k(t) - r_k(t) > 0, k=1,2,3,4\} > 0$,

then (3.3)-(3.5) *is oscillatory whenever* (3.2)-(3.4) *is oscillatory.*

Corollary 3.2.6. *Assume for a given $t^* \geq t_0$, there exists a region $\Omega \subset \Omega^*$, for which either the conditions of Theorem 3.2.1 or Corollary 3.2.4 are provided, then* (3.3)-(3.5) *is oscillatory.*

If we replace the strict inequalities (3.7)-(3.8) in Theorem 3.2.1 by the inequalities

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega, \tag{3.44}$$

$$s_k(t) \geq r_k(t); \qquad t \in [t_1,t_2] \quad (k=1,2,3,4) \tag{3.45}$$

as an alternative to Theorem 3.2.1, we acquire the following conclusion.

Theorem 3.2.7. *Let $u > 0$ be a solution of* (3.2)-(3.4) *on Ω satisfying $u(x,y,t_1) = u(x,y,t_2) = 0$ for all $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. If* (3.44)-(3.45) *are satisfied, then each solution v of* (3.3)-(3.5) *has a zero in $\bar{\Omega}$.*

Proof. Contrary assume that v doesn't have a zero in $\bar{\Omega}$. We can say that $v > 0$ in $\bar{\Omega}$. Using the exact same methods as the proof of Theorem 3.2.1, we get equation (3.34). Conditions (3.44)-(3.45) of Theorem 3.2.7 imply that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} \big[v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\big]\mathrm{d}x\mathrm{d}y \leq 0. \tag{3.46}$$

On the other hand inequality (3.46) together with (3.36) yield

$$v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2) = 0 \tag{3.47}$$

for each $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Since $u_t(x,y,t_1) \geq 0$ and $u_t(x,y,t_2) \leq 0$, (3.47) is possible only when $v(x,y,t_1) = v(x,y,t_2) = 0$ for all $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Otherwise $u_t(x,y,t_1) = u_t(x,y,t_2) = 0$ for all $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$ which is not possible. This contradiction proves that v can not be a positive solution of (3.3)-(3.5) on $\bar{\Omega}$. The proof of the situation $v < 0$ is explained as in the proof of Theorem 3.2.1, hence it is omitted. Therefore v has a zero on $\bar{\Omega}$. Theorem 3.2.7 has been proved. □

3.3 Nonlinear Comparison Results

The results obtained for linear equations in prior section can be expanded to the pair of nonlinear hyperbolic equations in the form

$$u_{tt} - \Delta u + \mathcal{F}(u, x, y, t) = 0; \qquad (x, y, t) \in \Omega, \tag{3.48}$$

and

$$v_{tt} - \Delta v + \mathcal{G}(v, x, y, t) = 0; \qquad (x, y, t) \in \Omega \tag{3.49}$$

providing the initial conditions (3.4) and (3.5), respectively.

We consider that

(i) $F(\eta, \xi, \delta, \beta)$ and $G(\eta, \xi, \delta, \beta)$, $k \in \mathbb{N}$, are real-valued continuous functions on $\mathbb{R}^4$ which are odd functions with respect to their first component;

(ii) $p(\beta), q(\beta) : \bar{\mathcal{K}} \to \mathbb{R}$ are continuous functions for which

$$\eta\mathcal{F}(\eta, \xi, \delta, \beta) \leq p(\beta)\eta^2; \qquad (\eta, \xi, \delta, \beta) \in \mathbb{R} \times \bar{\Omega}, \tag{3.50}$$

$$\eta\mathcal{G}(\eta, \xi, \delta, \beta) \geq q(\beta)\eta^2; \qquad (\eta, \xi, \delta, \beta) \in \mathbb{R} \times \bar{\Omega}. \tag{3.51}$$

The second primary conclusion of this chapter is as follows.

Theorem 3.3.1. *Let $u(x, y, t)$ be a solution of* (3.48)-(3.4) *satisfying the boundary conditions* (3.6)*, which is positive on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. Assume that (i)-(iii) hold. If*

$$q(t) > p(t) \quad and \quad s_k(t) > r_k(t) \quad (k = 1, 2, 3, 4) \tag{3.52}$$

hold for $t \in \bar{\mathcal{K}}$, then each solution v of (3.49)-(3.5) *has a zero in Ω.*

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ is in Ω. Multiplying equations (3.48) and (3.49) by v and u, respectively, then subtract the equations, as may be seen, the identity

$$\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t = u\mathcal{G}(v, x, y, t) - v\mathcal{F}(u, x, y, t) \tag{3.53}$$

holds for all $(x,y,t) \in \Omega$, $u \in \mathrm{C}\left(\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathcal{K}, \mathbb{R}\right)$ and $v \in \mathrm{C}(\Omega, \mathbb{R})$. Then using (3.50) and (3.51) in the right-hand side of (3.53), it turns out the inequality

$$\begin{aligned}\left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t &\geq uq(t)v - vp(t)u \\ &= [q(t) - p(t)]uv. \qquad (3.54)\end{aligned}$$

Integrating both sides of (3.54) over Ω we get

$$\begin{aligned}&\iiint_{\Omega} [q(t) - p(t)]u(x,y,t)v(x,y,t)\mathrm{d}V \\ &\quad \leq \iiint_{\Omega} \Big\{ \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t \Big\} \mathrm{d}V, \qquad (3.55)\end{aligned}$$

where $\mathrm{d}V$ is the volume element. Repeating same steps as in the proof of Theorem 3.2.1, we attain

$$\begin{aligned}&\int_{y_1}^{y_2}\int_{x_1}^{x_2} \Big\{ v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2) \Big\} \mathrm{d}x\mathrm{d}y \\ &\quad \leq \int_{t_1}^{t_2}\int_{y_1}^{y_2} [r_1(t) - s_1(t)]u(x_1,y,t)v(x_1,y,t)\mathrm{d}y\mathrm{d}t \\ &\qquad + \int_{t_1}^{t_2}\int_{y_1}^{y_2} [r_2(t) - s_2(t)]u(x_2,y,t)v(x_2,y,t)\mathrm{d}y\mathrm{d}t \\ &\qquad + \int_{t_1}^{t_2}\int_{x_1}^{x_2} [r_3(t) - s_3(t)]u(x,y_1,t)v(x,y_1,t)\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{t_1}^{t_2}\int_{x_1}^{x_2} [r_4(t) - s_4(t)]u(x,y_2,t)v(x,y_2,t)\mathrm{d}x\mathrm{d}t \\ &\qquad + \iiint_{\Omega} [p(t) - q(t)]uv\mathrm{d}V. \qquad (3.56)\end{aligned}$$

Conditions (3.52) of Theorem 3.3.1 imply that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} \left[v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\right]\mathrm{d}x\mathrm{d}y < 0. \qquad (3.57)$$

However $u_t(x,y,t_1) \geq 0$, $u_t(x,y,t_2) \leq 0$, $v(x,y,t_1) > 0$ and $v(x,y,t_2) > 0$ in Ω, we have again (3.36). This contradicts with (3.57) and which proves that v can not be a positive solution of (3.49)-(3.5) on Ω. The proof of the situation $v < 0$ is explained as in the proof of the Theorem 2.2.1, hence it is omited. Therefore v has a zero in Ω. Theorem 3.3.1 has been proved. □

The oscillation criteria are as follows as a conclusion of Theorem 3.3.1.

Corollary 3.3.2. *Assuming that (i)-(iii) hold. If*

$$q(t) > p(t); \qquad t \in [t^*, \infty), \qquad (3.58)$$

$$s_k(t) > r_k(t); \qquad t \in [t^*, \infty) \quad (k = 1,2,3,4) \qquad (3.59)$$

are provided for all $t^* \geq t_0$, then (3.49)-(3.5) *is oscillatory whenever* (3.48)-(3.4) *is oscillatory.*

Conditions (3.52) of Theorem 3.3.1 can be weakened by the following conclusion.

Corollary 3.3.3. *Let u be a solution of* (3.48)-(3.4) *providing the boundary conditions* (3.6)*, which is positive on $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathcal{K}$. Assume that (i)-(iii) hold. If*

$$q(t) \geq p(t) \quad \textit{and} \quad s_k(t) \geq r_k(t) \quad (k = 1, 2, 3, 4) \tag{3.60}$$

hold for $t \in \bar{\mathcal{K}}$. If either

(i) $\text{meas}\big\{t \in \bar{\mathcal{K}} : q(t) - p(t) > 0\big\} > 0$ *or*

(i) $\text{meas}\big\{t \in \bar{\mathcal{K}} : s_k(t) - r_k(t) > 0, k = 1, 2, 3, 4\big\} > 0,$

then each solution v of (3.49)-(3.5) *has a zero in Ω.*

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ in Ω. Following the similiar steps as in the proof of Theorem 3.3.1, we obtain equation (3.56). Conditions of Corollary 3.3.3 imply that (3.57) and (3.36) hold. Similar contradiction illustrates that v has a zero in Ω. Corollary 3.3.3 has been proved. □

The following oscillation criterion is immediate.

Corollary 3.3.4. *Assume for a given $t^* > 0$ there exists an interval $(t_1, t_2) \subset [t^*, \infty)$ for which either the conditions of Theorem 3.3.1 or Corollary 3.3.3 are provided, then* (3.49)-(3.5) *is oscillatory.*

If strict inequalities (3.52) are ruled out in Theorem 3.3.1, namely,

$$q(t) \geq p(t); \qquad t \in \bar{\mathcal{K}} \tag{3.61}$$

$$s_k(t) \geq r_k(t); \qquad t \in \bar{\mathcal{K}} \quad (k = 1, 2, 3, 4) \tag{3.62}$$

then we have the pursuing alternative result for Theorem 3.3.1.

Theorem 3.3.5. *Let u be a solution of* (3.48)-(3.4) *providing the boundary conditions* (3.6)*, which is positive on $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathcal{K}$. Assume that (i)-(iii) hold. If conditions* (3.61) *and* (3.62) *are satisfying, then each solution v of* (3.49)-(3.5) *has a zero in $\bar{\Omega}$.*

Proof. Contrary assume that v doesn't have a zero on $\bar{\Omega}$. We can say that $v > 0$ in $\bar{\Omega}$. Using the same techniques of proof of Theorem 3.3.1, we get equation (3.56). Conditions (3.61)-(3.62) of Theorem 3.3.5 imply that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\big[v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\big]\mathrm{d}x\mathrm{d}y \leq 0. \qquad (3.63)$$

On the other hand inequality (3.63) together with (3.36) yield (3.47) for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Since $u_t(x, y, t_1) \geq 0$, $u_t(x, y, t_2) \leq 0$, (3.47) is possible only when $v(x, y, t_1) = v(x, y, t_2) = 0$ for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Otherwise $u_t(x, y, t_1) = u_t(x, y, t_2) = 0$ for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. This contradiction proves that v can not be a positive solution of (3.49)-(3.5) on $\bar{\Omega}$. The proof of the situation $v < 0$ is explained as in the proof of Theorem 3.2.1, and hence it is omitted. Therefore v has a zero on $\bar{\Omega}$. Theorem 3.3.5 has been proved. □

Corollary 3.3.6. *Assume that (i)-(iii) hold. If*

$$q(t) \geq p(t); \qquad t \in [t^*, \infty), \qquad (3.64)$$

$$s_k(t) \geq r_k(t); \qquad t \in [t^*, \infty) \quad (k = 1, 2, 3, 4) \qquad (3.65)$$

are provided for all $t^* \geq t_0$, then (3.49)-(3.5) *is oscillatory whenever* (3.48)-(3.4) *is oscillatory.*

3.4 An Application

Consider the hyperbolic equation (3.2) with only the time dependent potential $\hat{f}(t)$, i.e.,

$$w_{tt} - \Delta w + \hat{f}(t)w = 0; \qquad (x, y, t) \in \mathfrak{I} \times \mathfrak{J} \times (\hat{t}, \infty) \qquad (3.66)$$

satisfying the initial conditions

$$\begin{aligned} w_x(x_k, y, t) + (-1)^k\alpha_k w(x_k, y, t) &= 0; && (y, t) \in \bar{\mathfrak{J}} \times [\hat{t}, \infty), \\ w_y(x, y_k, t) + (-1)^k\alpha_{k+2} w(x, y_k, t) &= 0; && (x, t) \in \bar{\mathfrak{I}} \times [\hat{t}, \infty), \end{aligned} \qquad (3.67)$$

where α_1, α_2 are real constants and $\hat{t} \geq t_0$. Now, equation (3.66) allows a separation of variables.

Setting $w(x, y, t) = H(x, y)T(t)$, we can solve the eigenvalue problem (3.66)

$$\Delta H = \lambda H; \qquad (x, y) \in \mathfrak{I} \times \mathfrak{J}, \qquad (3.68)$$

with the initial conditions

$$\begin{aligned} H_x(x_k, y) + (-1)^k \alpha_k H(x_k, y) &= 0; \qquad & y \in \bar{\mathfrak{J}}, \\ H_y(x, y_k) + (-1)^k \alpha_{k+2} H(x, y_k) &= 0; \qquad & x \in \bar{\mathfrak{I}} \end{aligned} \tag{3.69}$$

for $k = 1, 2$ and

$$T'' + \hat{f}(t)T = \lambda T; \qquad t \in [\hat{t}, \infty).$$

Applying Corollary 3.2.3, we derive an interesting oscillation criteria to a class of hyperbolic equations of the form (3.66)-(3.67).

Theorem 3.4.1. *Let w be a nontrivial solution of* (3.66)-(3.67)*, and assume that equation*

$$T'' + [\hat{f}(t) - \lambda_0]T = 0 \tag{3.70}$$

is oscillatory, where λ_0 is the first eigenvalue of (3.68)-(3.69).

If the inequalities

$$g(x, y, t) \geq \hat{f}(t); \qquad (x, y, t) \in \mathfrak{I} \times \mathfrak{J} \times [\tilde{t}, \infty), \tag{3.71}$$

and

$$s_k(t) \geq \alpha_k; \qquad t \in [\tilde{t}, \infty) \quad (k = 1, 2, 3, 4) \tag{3.72}$$

hold for every $\tilde{t} \geq \hat{t}$, then each solution v of (3.3)-(3.5) *has a zero in $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times [\tilde{t}, \infty)$.*

For the elliptic case, analogous results with Theorem 3.4.1 can be found in Kreith (1964).

Remark 3.4.2. When the potential doesn't depend on time variable; the method of Theorem 3.4.1 also applies to equation

$$\nu_{tt} - \Delta\nu + Q(x, y)\nu = 0; \qquad (x, y, t) \in \mathfrak{I} \times \mathfrak{J} \times (\hat{t}, \infty),$$

under the analogous boundary conditions with (3.67).

4. STURM COMPARISON RESULTS FOR IMPULSIVE HYPERBOLIC EQUATIONS ON A RECTANGULAR PRISM

In this chapter, new Sturmian comparison results are obtained for linear and nonlinear impulsive hyperbolic equations on a rectangular prism under fixed moment of impulse effects. Even the impulse effects are dropped, the obtained results are still new and were presented in Chapter three.

4.1 Introduction

Kreith obtained a remarkable conclusion of the Sturm comparison theorem between the pair of hyperbolic initial boundary value problems of the form (1.9) and (1.10) which was published in a book by Kreith (1969), see also (Kreith, 1973, pp. 24–26). Thereafter we gave some Sturm-type comparison criteria for impulsive hyperbolic equations on a rectangular domain in Özbekler and Uslu İşler (2020). In this chapter we obtain some similar comparison results for a pair of impulsive hyperbolic equations on a rectangular prism.

Fix $x_0, y_0, t_0 \in \mathbb{R}$. Let $\mathcal{I} = (x_1, x_2) \subset [x_0, \infty)$, $\mathcal{J} = (y_1, y_2) \subset [y_0, \infty)$ and $\mathcal{K} = (t_1, t_2) \subset [t_0, \infty)$ be three non-degenerate intervals and define the domain, a rectangular prism, as

$$\Omega = \mathcal{I} \times \mathcal{J} \times \mathcal{K}.$$

Furthermore, define the domains

$$\mathcal{K}_{\text{imp}} := \{t \in \mathcal{K} : t = \tau_k, k \in \mathbb{N}\},$$
$$\Omega_{\text{imp}} := \mathcal{I} \times \mathcal{J} \times \mathcal{K}_{\text{imp}},$$

where $\{\tau_k\}$ is a sequence of real numbers which is unbounded and strictly increasing, i.e.

$$t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots , \qquad (k \in \mathbb{N}).$$

Defined by $\mathrm{C}_{\text{imp}}\left(\bar{\Omega}, \mathbb{R}\right)$ the class of all functions $\nu : \bar{\Omega} \to \mathbb{R}$ such that:

(i) $\nu(x,y,t)$ is a continuous function for $(x,y,t) \in \bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$;

(ii) There exist limits

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t>\tau_k}} \nu(x,y,t) = \nu(x,y,\tau_k^+)$$

and

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t<\tau_k}} \nu(x,y,t) = \nu(x,y,\tau_k^-)$$

for each $k \in \mathbb{N}$ and $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$;

(iii) $\nu(x,y,t)$ is piecewise left continuous function at each τ_k, $k \in \mathbb{N}$, i.e.

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t<\tau_k}} \nu(x,y,t) = \nu(x,y,\tau_k)$$

for each $k \in \mathbb{N}$ and $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$.

In this chapter, we give some Sturm-type comparison consequences for the continuous solutions of the pair of impulsive hyperbolic initial value problems of the form

$$\begin{aligned} u_{tt} - \triangle u + f(x,y,t)u = 0; \qquad & (x,y,t) \in \Omega \setminus \Omega_{\text{imp}}, \\ \Delta u_t + f_k(x,y,t)u = 0; \qquad & (x,y,t) \in \Omega_{\text{imp}}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} v_{tt} - \triangle v + g(x,y,t)v = 0; \qquad & (x,y,t) \in \Omega \setminus \Omega_{\text{imp}}, \\ \Delta v_t + g_k(x,y,t)v = 0; \qquad & (x,y,t) \in \Omega_{\text{imp}} \end{aligned} \tag{4.2}$$

satisfying the initial conditions

$$\begin{aligned} u_x(x_j,y,t) + (-1)^j r_j(t) u(x_j,y,t) = 0; \qquad & (y,t) \in \bar{\mathfrak{J}} \times \bar{\mathfrak{K}}, \\ u_y(x,y_j,t) + (-1)^j r_{j+2}(t) u(x,y_j,t) = 0; \qquad & (x,t) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{K}}, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} v_x(x_j,y,t) + (-1)^j s_j(t) v(x_j,y,t) = 0; \qquad & (y,t) \in \bar{\mathfrak{J}} \times \bar{\mathfrak{K}}, \\ v_y(x,y_j,t) + (-1)^j s_{j+2}(t) v(x,y_j,t) = 0; \qquad & (x,t) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{K}} \end{aligned} \tag{4.4}$$

for $j = 1,2$, respectively, where $f, g \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$, $r_\ell, s_\ell \in \mathrm{C}\left(\bar{\mathfrak{K}}, \mathbb{R}\right)$, $\ell = 1,2,3,4$,

$$\Delta\nu(x,y,t) = \nu(x,y,t^+) - \nu(x,y,t^-),$$

and that $\triangle$ is two dimensional *Laplace operator*:

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A nontrivial function z is said to be a solution of (4.1)-(4.3) if

- $z \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$ (i.e., $\Delta z(x, y, \tau_k) = 0$ for all $k \in \mathbb{N}$) and $z_t \in \mathrm{C}_{\mathrm{imp}}\left(\bar{\Omega}, \mathbb{R}\right)$;
- there exist second-order partial derivatives z_{tt}, z_{xx} and z_{yy} satisfying the first line of equation (4.1) for each $(x, y, t) \in \Omega \setminus \Omega_{\mathrm{imp}}$;
- z satisfies the second line of equation (4.1) in Ω_{imp} and the initial conditions in (4.3).

4.2 Linear Comparison Results

In this section we give some analogous conclusions for solutions of the pair of linear impulsive hyperbolic initial value problems (4.1)-(4.3) and (4.2)-(4.4) on a *rectangular prism* in three-space.

The primary conclusion of this chapter is as follows.

Theorem 4.2.1. *Let $u > 0$ be a solution of* (4.1)-(4.3) *on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$, satisfying the boundary conditions*

$$u(x, y, t_1) = u(x, y, t_2) = 0; \qquad (x, y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}. \tag{4.5}$$

If

$$g(x, y, t) > f(x, y, t); \qquad (x, y, t) \in \Omega, \tag{4.6}$$

$$s_j(t) > r_j(t); \qquad t \in \bar{\mathcal{K}} \quad (j = 1, 2, 3, 4) \tag{4.7}$$

and that

$$g_k(x, y, t) > f_k(x, y, t); \qquad (x, y, t) \in \Omega_{\mathrm{imp}} \quad (k \in \mathbb{N}) \tag{4.8}$$

are satisfied, then each solution v of (4.2)-(4.4) *has a zero in* Ω.

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ in Ω. Multiplying first equations of (4.1) and (4.2) by v and u respectively, then subtract the equations, as may be seen, the identity

$$\left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t = \left[g(x, y, t) - f(x, y, t)\right]uv \tag{4.9}$$

holds for all $(x, y, t) \in \Omega$. Integrating both sides of equation (4.9) over Ω, we obtain

$$\begin{aligned} &\iiint_\Omega \left[g(x, y, t) - f(x, y, t)\right]uv\mathrm{d}V \\ &\quad = \iiint_\Omega \left\{\left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t\right\}\mathrm{d}V, \end{aligned} \tag{4.10}$$

where $\mathrm{d}V$ is the volume element. Due to the functions with integration have jump discontinuities at the impulse points τ_k, we split the domain Ω into $(n+1)$-partitions in the following manner:

$$\Omega_0 := \{(x,y,t) : (x,y) \in \mathcal{I} \times \mathcal{J}, t \in (t_1, \tau_1]\}, \tag{4.11}$$

$$\Omega_k := \{(x,y,t) : (x,y) \in \mathcal{I} \times \mathcal{J}, t \in (\tau_k, \tau_{k+1}], k = 1,2,\ldots,n-1\}, \tag{4.12}$$

$$\Omega_n := \{(x,y,t) : (x,y) \in \mathcal{I} \times \mathcal{J}, t \in (\tau_{k+1}, t_2)\} \tag{4.13}$$

such that each triple integral can be applied to the divergence theorem

$$\iiint_{\Omega_m} \left\{ \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t \right\} \mathrm{d}V \tag{4.14}$$

for $m = 0, 1, \ldots, n$. We also see that each of the above partitions is

(a) $\displaystyle\bigcap_{\ell=0}^{n} \Omega_\ell = \emptyset$,

(b) $\displaystyle\Omega = \bigcup_{\ell=0}^{n} \Omega_\ell$.

Obviously, from (a) and (b), we have

$$\begin{aligned}
&\iiint_{\Omega} \left[g(x,y,t) - f(x,y,t)\right] uv\mathrm{d}V \\
&\quad = \iiint_{\Omega_0} \left[g(x,y,t) - f(x,y,t)\right] uv\mathrm{d}V + \sum_{k=1}^{n-1} \iiint_{\Omega_k} \left[g(x,y,t) - f(x,y,t)\right] uv\mathrm{d}V \\
&\qquad + \iiint_{\Omega_n} \left[g(x,y,t) - f(x,y,t)\right] uv\mathrm{d}V.
\end{aligned} \tag{4.15}$$

Note that each Ω_m, $m = 0, 1, \ldots, n$, is a simple solid region with the piecewise smooth boundary $\mathcal{S}_m$, so by using divergence theorem to the smooth vector field

$$\mathbf{F}(x,y,t) := (uv_x - vu_x)\mathbf{i} + (uv_y - vu_y)\mathbf{j} + (vu_t - uv_t)\mathbf{k}, \tag{4.16}$$

on Ω_m, $m = 0, 1, \ldots, n$, the integral given in (4.14) turns out to be

$$\begin{aligned}
&\iiint_{\Omega_m} \left\{ \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t \right\} \mathrm{d}V \\
&\quad = \iiint_{\Omega_m} \operatorname{div} \mathbf{F} \mathrm{d}V \qquad \left(= \iiint_{\Omega_m} \nabla \bullet \mathbf{F} \mathrm{d}V \right) \\
&\quad = \oiint_{\mathcal{S}_m} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S \qquad (m = 0, 1, \ldots, n),
\end{aligned} \tag{4.17}$$

where $\hat{\mathbf{N}}$ is the unit outward normal to the surface $\mathcal{S}_m$, $\mathrm{d}S$ is the surface area element, and the ∇ is the usual nabla (gradient) operator defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial t}. \tag{4.18}$$

Since $\mathcal{S}_m$ $(= \partial\Omega_m)$, $m = 0, 1, \ldots, n$, is the union of six regions, it can be expressed as

$$\mathcal{S}_m = \bigcup_{\mu=1}^{6} \mathcal{S}_{m\mu} \tag{4.19}$$

where each $\mathcal{S}_{m\mu}$, $\mu = 1, \ldots, 6$, are disjoint, rectangular, oriented, closed surfaces. Because of the fact that (4.19), the integration in the right hand side of (4.17) can be expressed as

$$\begin{aligned} &\iiint_{\Omega_m} \Big\{ \big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t \Big\} \, \mathrm{d}V \\ &\quad = \sum_{\mu=1}^{6} \iint_{\mathcal{S}_{m\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{m\mu} \mathrm{d}S \qquad (m = 0, 1, \ldots, n), \end{aligned} \tag{4.20}$$

where each $\hat{\mathbf{N}}_{m\mu}$ are the unit outward normal vectors to each surface $\mathcal{S}_{m\mu}$ and defined by

$$\begin{aligned} &\hat{\mathbf{N}}_{m1} = -\mathbf{i}, \quad \hat{\mathbf{N}}_{m2} = \mathbf{i}, \qquad \hat{\mathbf{N}}_{m3} = -\mathbf{j}, \\ &\hat{\mathbf{N}}_{m4} = \mathbf{j}, \qquad \hat{\mathbf{N}}_{m5} = -\mathbf{k}, \quad \hat{\mathbf{N}}_{m6} = \mathbf{k} \end{aligned} \tag{4.21}$$

for $m = 0, 1, \ldots, n$, and $\mathbf{F}$ is defined in (4.16). Now, we start with the first integral in the right hand side on (4.15). Taking $m = 0$ in (4.20) and using (4.9) and (4.21), it can be displayed as

$$\iiint_{\Omega_0} \big[g(x,y,t) - f(x,y,t)\big] uv \mathrm{d}V = \oiint_{\mathcal{S}_0} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{\mu=1}^{6} \iint_{\mathcal{S}_{0\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{0\mu} \mathrm{d}S, \tag{4.22}$$

where each surfaces $\mathcal{S}_{0\mu}$ are defined by

$$\begin{aligned} \mathcal{S}_{01} &= \{(x,y,t) : x = x_1, y \in \bar{\mathcal{J}}, t \in [t_1, \tau_1]\}, \\ \mathcal{S}_{02} &= \{(x,y,t) : x = x_2, y \in \bar{\mathcal{J}}, t \in [t_1, \tau_1]\}, \\ \mathcal{S}_{03} &= \{(x,y,t) : y = y_1, x \in \bar{\mathcal{I}}, t \in [t_1, \tau_1]\}, \\ \mathcal{S}_{04} &= \{(x,y,t) : y = y_2, x \in \bar{\mathcal{I}}, t \in [t_1, \tau_1]\}, \\ \mathcal{S}_{05} &= \{(x,y,t) : t = t_1, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\} \end{aligned}$$

and

$$\mathcal{S}_{06} = \{(x,y,t) : t = \tau_1, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\}.$$

Then by using the initial conditions (4.3) and (4.4), each integral on the right hand side of (4.22) turn out to be

$$\begin{aligned}\iint_{\mathbb{S}_{01}} \mathbf{F} \bullet \hat{\mathbf{N}}_{01} \mathrm{d}S &= -\iint_{\mathbb{S}_{01}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\ &= -\int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t, \end{aligned} \tag{4.23}$$

$$\begin{aligned}\iint_{\mathbb{S}_{02}} \mathbf{F} \bullet \hat{\mathbf{N}}_{02} \mathrm{d}S &= \iint_{\mathbb{S}_{02}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\ &= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t, \end{aligned} \tag{4.24}$$

$$\begin{aligned}\iint_{\mathbb{S}_{03}} \mathbf{F} \bullet \hat{\mathbf{N}}_{03} \mathrm{d}S &= -\iint_{\mathbb{S}_{03}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= -\int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t, \end{aligned} \tag{4.25}$$

$$\begin{aligned}\iint_{\mathbb{S}_{04}} \mathbf{F} \bullet \hat{\mathbf{N}}_{04} \mathrm{d}S &= \iint_{\mathbb{S}_{04}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t, \end{aligned} \tag{4.26}$$

$$\iint_{\mathbb{S}_{05}} \mathbf{F} \bullet \hat{\mathbf{N}}_{05} \mathrm{d}S = -\iint_{\mathbb{S}_{05}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_1) \mathrm{d}x \mathrm{d}y \tag{4.27}$$

and

$$\iint_{\mathbb{S}_{06}} \mathbf{F} \bullet \hat{\mathbf{N}}_{06} \mathrm{d}S = \iint_{\mathbb{S}_{06}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_1) \mathrm{d}x \mathrm{d}y. \tag{4.28}$$

Summing up equations (4.23)–(4.28), equation (4.22) turns out to be

$$\begin{aligned}
&\iiint_{\Omega_0} \big[g(x,y,t) - f(x,y,t)\big] uv \mathrm{d}V \\
&\quad = \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1,y,t) v(x_1,y,t) \\
&\qquad + [r_2(t) - s_2(t)] u(x_2,y,t) v(x_2,y,t) \Big\} \mathrm{d}y \mathrm{d}t \\
&\qquad + \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x,y_1,t) v(x,y_1,t) \\
&\qquad + [r_4(t) - s_4(t)] u(x,y_2,t) v(x,y_2,t) \Big\} \mathrm{d}x \mathrm{d}t \\
&\qquad + \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ [vu_t - uv_t](x,y,\tau_1) - [vu_t - uv_t](x,y,t_1) \Big\} \mathrm{d}x \mathrm{d}y. \qquad (4.29)
\end{aligned}$$

Similarly, by taking $m = n$ in (4.20) and using (4.9) and (4.21), the last integral in the right hand side on (4.15) can be displayed as

$$\iiint_{\Omega_n} \big[g(x,y,t) - f(x,y,t)\big] uv \mathrm{d}V = \oiint_{\mathcal{S}_n} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{\mu=1}^{6} \iint_{\mathcal{S}_{n\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n\mu} \mathrm{d}S, \qquad (4.30)$$

where

$$\begin{aligned}
\mathcal{S}_{n1} &= \{(x,y,t) : x = x_1, y \in \bar{\mathcal{J}}, t \in [\tau_n, t_2]\}, \\
\mathcal{S}_{n2} &= \{(x,y,t) : x = x_2, y \in \bar{\mathcal{J}}, t \in [\tau_n, t_2]\}, \\
\mathcal{S}_{n3} &= \{(x,y,t) : y = y_1, x \in \bar{\mathcal{I}}, t \in [\tau_n, t_2]\}, \\
\mathcal{S}_{n4} &= \{(x,y,t) : y = y_2, x \in \bar{\mathcal{I}}, t \in [\tau_n, t_2]\}, \\
\mathcal{S}_{n5} &= \{(x,y,t) : t = \tau_n, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\}
\end{aligned}$$

and

$$\mathcal{S}_{n6} = \{(x,y,t) : t = t_2, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\}.$$

Initial conditions (4.3) and (4.4) imply that each integral of the right hand side on (4.30) turn out to be

$$\begin{aligned}
\iint_{\mathcal{S}_{n1}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n1} \mathrm{d}S &= -\iint_{\mathcal{S}_{n1}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\
&= -\int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_1,y,t) \mathrm{d}y \mathrm{d}t \\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1,y,t) v(x_1,y,t) \mathrm{d}y \mathrm{d}t, \qquad (4.31)
\end{aligned}$$

$$\begin{aligned}\iint_{\mathcal{S}_{n2}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n2} \mathrm{d}S &= \iint_{\mathcal{S}_{n2}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\ &= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t \\ &= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t, \end{aligned} \quad (4.32)$$

$$\begin{aligned}\iint_{\mathcal{S}_{n3}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n3} \mathrm{d}S &= -\iint_{\mathcal{S}_{n3}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= -\int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t, \end{aligned} \quad (4.33)$$

$$\begin{aligned}\iint_{\mathcal{S}_{n4}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n4} \mathrm{d}S &= \iint_{\mathcal{S}_{n4}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t, \end{aligned} \quad (4.34)$$

$$\iint_{\mathcal{S}_{n5}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n5} \mathrm{d}S = -\iint_{\mathcal{S}_{n5}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_n^+) \mathrm{d}x \mathrm{d}y \quad (4.35)$$

and

$$\iint_{\mathcal{S}_{n6}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n6} \mathrm{d}S = \iint_{\mathcal{S}_{n6}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_2) \mathrm{d}x \mathrm{d}y. \quad (4.36)$$

By addition of integrals (4.31)–(4.36), equation (4.30) can be displayed as

$$\begin{aligned}&\iiint_{\Omega_n} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V \\ &= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \\ &\quad + [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \Big\} \mathrm{d}y \mathrm{d}t \\ &\quad + \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \\ &\quad + [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \Big\} \mathrm{d}x \mathrm{d}t \\ &\quad + \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ [vu_t - uv_t](x, y, t_2) - [vu_t - uv_t](x, y, \tau_n^+) \Big\} \mathrm{d}x \mathrm{d}y. \end{aligned} \quad (4.37)$$

Finally, we will examine the integrals in the mid part of (4.15), i.e.

$$\iiint_{\Omega_k} [g(x,y,t) - f(x,y,t)]uv\mathrm{d}V = \oiint_{\mathcal{S}_k} \mathbf{F} \bullet \hat{\mathbf{N}}\mathrm{d}S = \sum_{\mu=1}^{6} \iint_{\mathcal{S}_{k\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k\mu}\mathrm{d}S, \tag{4.38}$$

where

$$\begin{aligned}
\mathcal{S}_{k1} &= \{(x,y,t) : x = x_1, y \in \bar{\mathcal{J}}, t \in [\tau_k, \tau_{k+1}]\},\\
\mathcal{S}_{k2} &= \{(x,y,t) : x = x_2, y \in \bar{\mathcal{J}}, t \in [\tau_k, \tau_{k+1}]\},\\
\mathcal{S}_{k3} &= \{(x,y,t) : y = y_1, x \in \bar{\mathcal{I}}, t \in [\tau_k, \tau_{k+1}]\},\\
\mathcal{S}_{k4} &= \{(x,y,t) : y = y_2, x \in \bar{\mathcal{I}}, t \in [\tau_k, \tau_{k+1}]\},\\
\mathcal{S}_{k5} &= \{(x,y,t) : t = \tau_k, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\}
\end{aligned}$$

and

$$\mathcal{S}_{k6} = \{(x,y,t) : t = \tau_{k+1}, (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}\}$$

for $k = 1, 2, \ldots, n-1$. Then integration on the right hand side of (4.38) become

$$\begin{aligned}
\iint_{\mathcal{S}_{k1}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k1}\mathrm{d}S &= -\iint_{\mathcal{S}_{k1}} \mathbf{F} \bullet \mathbf{i}\mathrm{d}S\\
&= -\int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t)\mathrm{d}y\mathrm{d}t\\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [r_1(t) - s_1(t)]u(x_1, y, t)v(x_1, y, t)\mathrm{d}y\mathrm{d}t,
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\iint_{\mathcal{S}_{k2}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k2}\mathrm{d}S &= \iint_{\mathcal{S}_{k2}} \mathbf{F} \bullet \mathbf{i}\mathrm{d}S\\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t)\mathrm{d}y\mathrm{d}t\\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [r_2(t) - s_2(t)]u(x_2, y, t)v(x_2, y, t)\mathrm{d}y\mathrm{d}t,
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
\iint_{\mathcal{S}_{k3}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k3}\mathrm{d}S &= -\iint_{\mathcal{S}_{k3}} \mathbf{F} \bullet \mathbf{j}\mathrm{d}S\\
&= -\int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t)\mathrm{d}x\mathrm{d}t\\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [r_3(t) - s_3(t)]u(x, y_1, t)v(x, y_1, t)\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.41}$$

$$\begin{aligned}\iint_{\mathbb{S}_{k4}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k4} \mathrm{d}S &= \iint_{\mathbb{S}_{k4}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t, \end{aligned} \tag{4.42}$$

$$\iint_{\mathbb{S}_{k5}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k5} \mathrm{d}S = -\iint_{\mathbb{S}_{k5}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_k^+) \mathrm{d}x \mathrm{d}y \tag{4.43}$$

and

$$\iint_{\mathbb{S}_{k6}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k6} \mathrm{d}S = \iint_{\mathbb{S}_{k6}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_{k+1}) \mathrm{d}x \mathrm{d}y \tag{4.44}$$

for $k = 1, 2, \ldots, n-1$. Integrals (4.39)–(4.44) yield

$$\begin{aligned}&\iiint_{\Omega_k} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V \\ &\quad = \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \\ &\qquad + [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \Big\} \mathrm{d}y \mathrm{d}t \\ &\qquad + \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \\ &\qquad + [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \Big\} \mathrm{d}x \mathrm{d}t \\ &\qquad + \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ -[vu_t - uv_t](x, y, \tau_k^+) + [vu_t - uv_t](x, y, \tau_{k+1}) \Big\} \mathrm{d}x \mathrm{d}y \end{aligned} \tag{4.45}$$

for $k = 1, 2, \ldots, n-1$.

Finally we add integrals (4.29), (4.37) and (4.45) to obtain integral (4.15) as

$$
\begin{aligned}
&\iiint_{\Omega} \big[g(x,y,t) - f(x,y,t)\big] uv \mathrm{d}V \\
&= \left\{ \int_{t_1}^{\tau_1} + \int_{\tau_1^+}^{\tau_2} + \cdots + \int_{\tau_{n-1}^+}^{\tau_n} + \int_{\tau_n^+}^{t_2} \right\} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1,y,t) v(x_1,y,t) \\
&+ [r_2(t) - s_2(t)] u(x_2,y,t) v(x_2,y,t) \Big\} \mathrm{d}y \mathrm{d}t \\
&+ \left\{ \int_{t_1}^{\tau_1} + \int_{\tau_1^+}^{\tau_2} + \cdots + \int_{\tau_{n-1}^+}^{\tau_n} + \int_{\tau_n^+}^{t_2} \right\} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x,y_1,t) v(x,y_1,t) \\
&+ [r_4(t) - s_4(t)] u(x,y_2,t) v(x,y_2,t) \Big\} \mathrm{d}x \mathrm{d}t \\
&+ \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ [vu_t - uv_t](x,y,t_2) - [vu_t - uv_t](x,y,t_1) + [vu_t - uv_t](x,y,\tau_1) \\
&+ \sum_{k=1}^{n-1} \Big\{ - [vu_t - uv_t](x,y,\tau_k^+) + [vu_t - uv_t](x,y,\tau_{k+1}) \Big\} \\
&- [vu_t - uv_t](x,y,\tau_n^+) \Big\} \mathrm{d}x \mathrm{d}y. \qquad (4.46)
\end{aligned}
$$

Using $\Delta u(x,y,\tau_k) = \Delta v(x,y,\tau_k) = 0$, $k \in \mathbb{N}$, and the impulse conditions to the functions u_t and v_t in the second lines of (4.1) and (4.2), respectively, (4.46) turns into

$$
\begin{aligned}
&- [vu_t - uv_t](x,y,\tau_1) + \sum_{k=1}^{n-1} \Big[[vu_t - uv_t](x,y,\tau_k^+) - [vu_t - uv_t](x,y,\tau_{k+1}) \Big] \\
&+ [vu_t - uv_t](x,y,\tau_n^+) \\
&\quad = \sum_{k=1}^{n} \Delta [vu_t - uv_t](x,y,\tau_k) \\
&\quad = \sum_{t_1 \le \tau_k < t_2} \Delta [vu_t - uv_t](x,y,\tau_k) \\
&\quad = \sum_{t_1 \le \tau_k < t_2} \Big[v(x,y,\tau_k) \Delta u_t(x,y,\tau_k) - u(x,y,\tau_k) \Delta v_t(x,y,\tau_k) \Big] \\
&\quad = \sum_{t_1 \le \tau_k < t_2} \big[g_k(x,y,\tau_k) - f_k(x,y,\tau_k) \big] u(x,y,\tau_k) v(x,y,\tau_k). \qquad (4.47)
\end{aligned}
$$

Using boundary conditions (4.5) and imposing (4.47) into (4.46), we obtain the identity

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{(vu_t)(x,y,t_1)-(vu_t)(x,y,t_2)\Big\}\mathrm{d}x\mathrm{d}y$$
$$=\int_{t_1}^{t_2}\int_{y_1}^{y_2}\Big\{[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t)$$
$$+[r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\Big\}\mathrm{d}y\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_{x_1}^{x_2}\Big\{[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t)$$
$$+[r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\Big\}\mathrm{d}x\mathrm{d}t$$
$$+\iiint_\Omega\big[f(x,y,t)-g(x,y,t)\big]uv\mathrm{d}V$$
$$+\sum_{t_1\le\tau_k<t_2}\big[f_k(x,y,\tau_k)-g_k(x,y,\tau_k)\big]uv. \tag{4.48}$$

Conditions (4.6)–(4.8) of Theorem 4.2.1 and (4.48) imply that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\big[v(x,y,t_1)u_t(x,y,t_1)-v(x,y,t_2)u_t(x,y,t_2)\big]\mathrm{d}x\mathrm{d}y<0. \tag{4.49}$$

However $u_t(x,y,t_1)\ge 0$, $u_t(x,y,t_2)\le 0$, $v(x,y,t_1)>0$ and $v(x,y,t_2)>0$ in Ω, we have

$$0\le v(x,y,t_1)u_t(x,y,t_1)-v(x,y,t_2)u_t(x,y,t_2) \tag{4.50}$$

for all $(x,y)\in\bar{\mathfrak{I}}\times\bar{\mathfrak{J}}$, which contradicts with (4.49). This contradiction proves that v can not be a positive solution of (4.1)-(4.3) on Ω. For the proof of the case $v<0$ in Ω, we let $v=-z$ in Ω. Then z becomes a positive solution of (4.1)-(4.3) in Ω. Repeating the same proof for z, we obtain that v can not be a negative solution of (4.1)-(4.3) on Ω. Therefore v has a zero in Ω. Theorem 4.2.1 has been proved. □

Remark 4.2.2. Theorem 4.2.1 can also be proved for a negative solution u of (4.1)-(4.3) in Ω, and hence positivity of u can be replaced by "having no zero" in Theorem 4.2.1. This applies to all Sturm comparison theorems in this chapter.

As a result of Remark 4.2.2, we have the eventual oscillation consequence.

Corollary 4.2.3. *If*

$$g(x,y,t)>f(x,y,t);\qquad (x,y,t)\in\Omega^*, \tag{4.51}$$
$$s_j(t)>r_j(t);\qquad t\in[t^*,\infty)\quad (j=1,2,3,4) \tag{4.52}$$

and that

$$g_k(x,y,t)>f_k(x,y,t);\qquad (x,y,t)\in\Omega^*_{\mathrm{imp}}\quad (k\in\mathbb{N}) \tag{4.53}$$

are provided for all $t^* \geq t_0$, then (4.2)-(4.4) *is oscillatory whenever* (4.1)-(4.3) *is oscillatory where*

$$\Omega^* = \{(x, y, t) : x \in \mathfrak{I}, y \in \mathfrak{J},\ t \in [t^*, \infty)\} \tag{4.54}$$

and

$$\Omega^*_{\text{imp}} = \{(x, y, t) : x \in \mathfrak{I}, y \in \mathfrak{J},\ t \in [t^*, \infty),\ t = \tau_k, k \in \mathbb{N}\}. \tag{4.55}$$

Conditions (4.6)-(4.8) of Theorem 4.2.1 can be weakened by the following consequence.

Corollary 4.2.4. *Let $u > 0$ be a solution of* (4.1)-(4.3) *on $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathfrak{K}$, satisfying the boundary conditions* (4.5)*, which is positive on $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathfrak{K}$, and assume that*

$$g(x, y, t) \geq f(x, y, t); \qquad (x, y, t) \in \Omega, \tag{4.56}$$

$$s_j(t) \geq r_j(t); \qquad t \in \bar{\mathfrak{K}} \quad (j = 1, 2, 3, 4) \tag{4.57}$$

and that

$$g_k(x, y, t) \geq f_k(x, y, t); \qquad (x, y, t) \in \Omega_{\text{imp}} \quad (k \in \mathbb{N}) \tag{4.58}$$

are provided. If either

(i) $\text{meas}\{(x, y, t) \in \Omega : g(x, y, t) - f(x, y, t) > 0\} > 0$ *or*

(ii) $\text{meas}\{t \in \bar{\mathfrak{K}} : s_k(t) - r_k(t) > 0, k = 1, 2, 3, 4\} > 0$ *or that*

(iii) $g_{k_0}(x, y, \tau_{k_0}) > f_{k_0}(x, y, \tau_{k_0})$, $k_0 \in \mathbb{N}$*, for which* $(x, y, \tau_{k_0}) \in \Omega_{\text{imp}}$,

then each solution v of (4.1)-(4.3) *has a zero in Ω.*

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ in Ω. Following the similar steps as in the proof of Theorem 4.2.1, we obtain equation (4.48). All conditions of Corollary 4.2.4 imply that (4.49) and (4.50) hold. Similar contradiction shows that v has a zero in Ω. Corollary 4.2.4 has been proved. □

Following oscillation results are immediate.

Corollary 4.2.5. *Assume that*

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega^*, \tag{4.59}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t^*, \infty) \quad (j = 1,2,3,4) \tag{4.60}$$

and that

$$g_k(x,y,t) \geq f_k(x,y,t); \qquad (x,y,t) \in \Omega^*_{\text{imp}} \quad (k \in \mathbb{N}) \tag{4.61}$$

are provided for all $t^* \geq t_0$*, if either*

(i) $\operatorname{meas}\{(x,y,t) \in \Omega^* : g(x,y,t) - f(x,y,t) > 0\} > 0$ *or*

(ii) $\operatorname{meas}\{t \in [t^*, \infty) : s_j(t) - r_j(t) > 0, j = 1,2,3,4\} > 0$*, or that*

(iii) $g_{k_0}(x,y,\tau_{k_0}) > f_{k_0}(x,y,\tau_{k_0})$, $k_0 \in \mathbb{N}$*, for which* $(x,y,\tau_{k_0}) \in \Omega^*_{\text{imp}}$,

then (4.2)-(4.4) *is oscillatory whenever* (4.1)-(4.3) *is oscillatory.*

Corollary 4.2.6. *Assuming that for a given* $t^* \geq t_0$*, there exists a region* $\Omega \subset \Omega^*$ *for which either the conditions of Theorem 4.2.1 or Corollary 4.2.4 are provided, then* (4.2)-(4.4) *is oscillatory.*

If we replace the strict inequalities (4.6)-(4.8) in Theorem 4.2.1 by the inequalities

$$q(t) \geq p(t); \qquad (x,y,t) \in \Omega, \tag{4.62}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_1, t_2] \quad (j = 1,2,3,4) \tag{4.63}$$

and that

$$g_k(x,y,t) \geq f_k(x,y,t); \qquad (x,y,t) \in \Omega_{\text{imp}} \quad (k \in \mathbb{N}) \tag{4.64}$$

as a result, Theorem 4.2.1 has the following alternate result.

Theorem 4.2.7. *Let* u *be a solution of* (4.1)-(4.3) *satisfying the boundary conditions* (4.5)*, which is positive on* $\bar{\mathfrak{I}} \times \bar{\mathfrak{J}} \times \mathcal{K}$*. If* (4.62)–(4.64) *are satisfied, then each solution* v *of* (4.2)-(4.4) *has a zero in* $\bar{\Omega}$*.*

Proof. Assuming the contrary that v does not have a zero in $\bar{\Omega}$. We can say that $v > 0$ in $\bar{\Omega}$. Using the exact same techniques as the proof of Theorem 4.2.1, we get equation (4.48). Conditions (4.62)–(4.64) of Theorem 4.2.7 imply that

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \big[v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\big] \mathrm{d}x\mathrm{d}y \leq 0. \tag{4.65}$$

On the other hand inequality (4.65) yield

$$v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2) = 0 \tag{4.66}$$

for all $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Since $u_t(x,y,t_1) \geq 0$ and $u_t(x,y,t_2) \leq 0$, (4.66) is possible only when $v(x,y,t_1) = v(x,y,t_2) = 0$ for all $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Otherwise $u_t(x,y,t_1) = u_t(x,y,t_2) = 0$ for all $(x,y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$ which is not possible. This contradiction proves that v can not be a positive solution of (4.2)-(4.4) on $\bar{\Omega}$. The proof of the case $v < 0$ is explained as in the proof of Theorem 4.2.1, hence it is omitted. Therefore v has a zero on $\bar{\Omega}$. Theorem 4.2.7 has been proved. □

4.3 Nonlinear Comparison Results

The results in previous section obtained for linear equations can be expanded to the nonlinear hyperbolic equations of the form

$$\begin{aligned} &u_{tt} - \triangle u + \mathcal{F}(u,x,y,t) = 0; && (x,y,t) \in \Omega \setminus \Omega_{\text{imp}}, \\ &\Delta u = 0, \qquad \Delta u_t + \mathcal{F}_k(u,x,y,t) = 0; && (x,y,t) \in \Omega_{\text{imp}}, \end{aligned} \tag{4.67}$$

and

$$\begin{aligned} &v_{tt} - \triangle u + \mathcal{G}(v,x,y,t) = 0; && (x,y,t) \in \Omega \setminus \Omega_{\text{imp}}, \\ &\Delta v = 0, \qquad \Delta v_t + \mathcal{G}_k(v,x,y,t) = 0; && (x,y,t) \in \Omega_{\text{imp}} \end{aligned} \tag{4.68}$$

satisfying the initial conditions (4.3) and (4.4), respectively.

We suppose without further mention that

(i) $\mathcal{F}(u,x,y,t)$, $\mathcal{F}_k(u,x,y,t)$, $\mathcal{G}(v,x,y,t)$ and $\mathcal{G}_k(v,x,y,t)$, $k \in \mathbb{N}$ are real-valued continuous functions on $\mathbb{R}^4$, which are odd functions with respect to their first component;

(ii) $p(\beta)$, $q(\beta) : \bar{\mathcal{K}} \to \mathbb{R}$ and $(\eta,\xi,\delta,\beta) \in \mathbb{R} \times \bar{\Omega}$ are continuous functions for which

$$\eta\mathcal{F}(\eta,\xi,\delta,\beta) \leq p(\beta)\eta^2 \quad \text{and} \quad \eta\mathcal{G}(\eta,\xi,\delta,\beta) \geq q(\beta)\eta^2; \tag{4.69}$$

(iii) $\{p_k\}$ and $\{q_k\}$ are sequences of real numbers for which

$$\eta\mathcal{F}_k(\eta,\xi,\delta,\beta) \leq p_k\eta^2 \quad \text{and} \quad \eta\mathcal{G}_k(\eta,\xi,\delta,\beta) \geq q_k\eta^2. \tag{4.70}$$

The second primary conclusion of this chapter is as follows.

Theorem 4.3.1. *Let u be a solution of* (4.67)-(4.3) *satisfying the boundary conditions* (4.5)*, which is positive on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. Assume that (i)-(iii) hold. If*

$$q(t) > p(t) \quad and \quad s_j(t) > r_j(t) \quad (j = 1, 2, 3, 4) \tag{4.71}$$

hold for $t \in \bar{\mathcal{K}}$, and that

$$q_k > p_k \quad for\ all\ k \in \mathbb{N} \quad for\ which\ \tau_k \in \bar{\mathcal{K}}, \tag{4.72}$$

then every solution v of (4.68)-(4.4) *has a zero in Ω.*

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ is in Ω. Multiply first equations of (4.67) and (4.68) by v and u, respectively, then subtract the equations, as may be seen, the identity

$$\left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t = u\mathcal{G}(v, x, y, t) - v\mathcal{F}(u, x, y, t) \tag{4.73}$$

holds for all $(x, y, t) \in \Omega$, $u \in \mathrm{C}\left(\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}, \mathbb{R}\right)$ and $v \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$. Then using (4.69) in the right hand side of (4.73), it turns out the inequality

$$\begin{aligned} \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t &\geq uq(t)v - vp(t)u \\ &= [q(t) - p(t)]uv. \end{aligned} \tag{4.74}$$

Integrating both sides of (4.74) over Ω we get

$$\begin{aligned} &\iiint_\Omega [q(t) - p(t)]u(x, y, t)v(x, y, t)\mathrm{d}V \\ &\quad \leq \iiint_\Omega \left\{\left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t\right\}\mathrm{d}V, \end{aligned} \tag{4.75}$$

where $\mathrm{d}V$ is the volume element. Since the functions with integration have jump discontinuities at the impulse points τ_k, we split the domain Ω into $(n+1)$-partitions in a manner that (4.11)-(4.13) satisfying (a)-(b) so that divergence theorem can be applied to each integral on Ω_m, $m = 0, \ldots, n$, as in the proof of Theorem 4.2.1. Using initial conditions (4.3)-(4.4) to each integrals and summing up of them, we obtain

$$
\begin{aligned}
&\iiint_{\Omega}[q(t)-p(t)]u(x,y,t)v(x,y,t)\mathrm{d}V \\
&\quad\leq \left\{\int_{t_1}^{\tau_1}+\int_{\tau_1^+}^{\tau_2}+\cdots+\int_{\tau_{n-1}^+}^{\tau_n}+\int_{\tau_n^+}^{t_2}\right\}\int_{y_1}^{y_2}\Big\{[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t) \\
&\quad+[r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\Big\}\mathrm{d}y\mathrm{d}t \\
&\quad+\left\{\int_{t_1}^{\tau_1}+\int_{\tau_1^+}^{\tau_2}+\cdots+\int_{\tau_{n-1}^+}^{\tau_n}+\int_{\tau_n^+}^{t_2}\right\}\int_{x_1}^{x_2}\Big\{[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t) \\
&\quad+[r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\Big\}\mathrm{d}x\mathrm{d}t \\
&\quad+\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{[vu_t-uv_t](x,y,t_2)-[vu_t-uv_t](x,y,t_1)+[vu_t-uv_t](x,y,\tau_1) \\
&\quad+\sum_{k=1}^{n-1}\Big\{-[vu_t-uv_t](x,y,\tau_k^+)+[vu_t-uv_t](x,y,\tau_{k+1})\Big\} \\
&\quad-[vu_t-uv_t](x,y,\tau_n^+)\Big\}\mathrm{d}x\mathrm{d}y. \qquad (4.76)
\end{aligned}
$$

Using $\Delta u(x,y,\tau_k)=\Delta v(x,y,\tau_k)=0$, $k\in\mathbb{N}$, and the impulse conditions for the functions u_t and v_t in the second lines of (4.67) and (4.68), respectively, (4.76) turns into

$$
\begin{aligned}
&-[vu_t-uv_t](x,y,\tau_1)+\sum_{k=1}^{n-1}[vu_t-uv_t](x,y,\tau_k^+)-[vu_t-uv_t](x,y,\tau_{k+1}) \\
&+[vu_t-uv_t](x,y,\tau_n^+) \\
&\quad=\sum_{k=1}^{n}\Delta[vu_t-uv_t](x,y,\tau_k) \\
&\quad=\sum_{t_1\leq\tau_k<t_2}\Delta[vu_t-uv_t](x,y,\tau_k) \\
&\quad=\sum_{t_1\leq\tau_k<t_2}v(x,y,\tau_k)\Delta u_t(x,y,\tau_k)-u(x,y,\tau_k)\Delta v_t(x,y,\tau_k) \\
&\quad=\sum_{t_1\leq\tau_k<t_2}\big[u(x,y,\tau_k)\mathcal{G}_k(v,x,y,\tau_k)-v(x,y,\tau_k)\mathcal{F}_k(u,x,y,\tau_k)\big] \\
&\quad\geq\sum_{t_1\leq\tau_k<t_2}\big[u(x,y,\tau_k)q_kv(x,y,\tau_k)-v(x,y,\tau_k)p_ku(x,y,\tau_k)\big] \\
&\quad=\sum_{t_1\leq\tau_k<t_2}\big[q_k-p_k\big]u(x,y,\tau_k)v(x,y,\tau_k). \qquad (4.77)
\end{aligned}
$$

In view of (4.77), the analogous methods as in the proof of Theorem 4.2.1,

$$\begin{aligned}
&\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{v(x,y,t_1)u_t(x,y,t_1)-v(x,y,t_2)u_t(x,y,t_2)\Big\}\mathrm{d}x\mathrm{d}y\\
&\quad\le \iiint_\Omega \big[p(t)-q(t)\big]u(x,y,t)v(x,y,t)\mathrm{d}V\\
&\qquad+\int_{t_1}^{t_2}\int_{y_1}^{y_2}[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t)\mathrm{d}y\mathrm{d}t\\
&\qquad+\int_{t_1}^{t_2}\int_{y_1}^{y_2}[r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\mathrm{d}y\mathrm{d}t\\
&\qquad+\int_{t_1}^{t_2}\int_{x_1}^{x_2}[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t)\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{t_1}^{t_2}\int_{x_1}^{x_2}[r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{y_1}^{y_2}\int_{x_1}^{x_2}[p_k-q_k]u(x,y,\tau_k)v(x,y,\tau_k)\mathrm{d}x\mathrm{d}y. \qquad (4.78)
\end{aligned}$$

Conditions (4.71)-(4.72) of Theorem 4.3.1 and (4.78) imply that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\big[v(x,y,t_1)u_t(x,y,t_1)-v(x,y,t_2)u_t(x,y,t_2)\big]\mathrm{d}x\mathrm{d}y<0. \qquad (4.79)$$

However $u_t(x,y,t_1)\ge 0$, $u_t(x,y,t_2)\le 0$, $v(x,y,t_1)>0$ and $v(x,y,t_2)>0$ in Ω, we have again (4.50). This contradicts with (4.79) which proves that v can not be a positive solution of (4.67)-(4.3) on Ω. The proof of the situation $v<0$ is explained as in the proof of Theorem 4.2.1, hence it is omitted. Therefore v has a zero in Ω. Theorem 4.3.1 has been proved. □

The oscillation criterion is as follows.

Corollary 4.3.2. *Assuming that (i)-(iii) hold. If*

$$q(t)>p(t); \qquad t\in[t^*,\infty), \qquad (4.80)$$

$$s_j(t)>r_j(t); \qquad t\in[t^*,\infty) \quad (j=1,2,3,4) \qquad (4.81)$$

and that

$$q_k>p_k; \qquad (x,t)\in\Omega^*_{\mathrm{imp}} \quad (k\in\mathbb{N}) \qquad (4.82)$$

are provided for all $t^*\ge t_0$*, then* (4.68)-(4.4) *is oscillatory whenever* (4.67)-(4.3) *is oscillatory.*

Conditions (4.71)-(4.72) of Theorem 4.3.1 can be weakened by the following conclusion.

Corollary 4.3.3. *Let u be a solution of* (4.67)-(4.3) *satisfying the boundary conditions* (4.5)*, which is positive on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. Assume that (i)-(iii) hold. If*

$$q(t) \geq p(t) \quad \textit{and} \quad s_j(t) \geq r_j(t) \quad (j = 1, 2, 3, 4) \tag{4.83}$$

for $t \in \bar{\mathcal{K}}$, and that

$$q_k \geq p_k \quad \textit{for all} \;\; k \in \mathbb{N} \quad \textit{for which} \;\; \tau_k \in \bar{\mathcal{K}}. \tag{4.84}$$

If either

(i) $\operatorname{meas}\big\{t \in \bar{\mathcal{K}} : q(t) - p(t) > 0\big\} > 0$ *or*

(ii) $\operatorname{meas}\big\{t \in \bar{\mathcal{K}} : s_j(t) - r_j(t) > 0, j = 1, 2, 3, 4\big\} > 0$, *or that*

(iii) $q_{k_0} > p_{k_0}$ *for some* $k_0 \in \mathbb{N}$,

then each solution v of (4.68)-(4.4) *has a zero in Ω.*

Proof. Contrary assume that v doesn't have a zero in Ω. We can say that $v > 0$ in Ω. Following the similiar steps as in the proof of Theorem 4.3.1, we obtain equation (4.78). Conditions of Corollary 4.3.3 imply that (4.79) and (4.50) hold. A similar contradiction shows that v has a zero in Ω. Corollary 4.3.3 has been proved. □

The following oscillation criterion is immediate.

Corollary 4.3.4. *Assuming that for a given $t^* > 0$ there exists an interval $(t_1, t_2) \subset [t^*, \infty)$ for which either the conditions of Theorem 4.3.1 or Corollary 4.3.3 are provided, then* (4.68)-(4.4) *is oscillatory.*

If strict inequalities (4.71) and (4.72) are ruled out in Theorem 4.3.1, namely,

$$q(t) \geq p(t) \quad \text{and} \quad s_j(t) \geq r_j(t) \quad (j = 1, 2, 3, 4) \tag{4.85}$$

hold for $t \in \bar{\mathcal{K}}$, and that

$$q_k \geq p_k \quad \text{for all} \;\; k \in \mathbb{N} \quad \text{for which} \;\; \tau_k \in \bar{\mathcal{K}} \tag{4.86}$$

then we have the eventual alternative result for Theorem 4.3.1.

Theorem 4.3.5. *Let u be a solution of* (4.67)-(4.3) *satisfying the boundary conditions* (4.5)*, which is positive on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$. Assume that (i)-(iii) hold. If conditions* (4.85)-(4.86) *hold, then each solution v of* (4.68)-(4.4) *has a zero in $\bar{\Omega}$.*

Proof. Contrary assume that v doesn't have a zero in $\bar{\Omega}$. We can say that $v > 0$ in $\bar{\Omega}$. Using the same techniques of the proof of Theorem 4.3.1, we get equation (4.78). Conditions (4.85)-(4.86) of Theorem 4.3.5 imply that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\big[v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\big]\mathrm{d}x\mathrm{d}y \le 0. \tag{4.87}$$

On the other hand inequality (4.87) together with (4.50) yield (4.66) for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Since $u_t(x, y, t_1) \ge 0$ and $u_t(x, y, t_2) \le 0$, (4.66) is possible only when $v(x, y, t_1) = v(x, y, t_2) = 0$ for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$. Otherwise $u_t(x, y, t_1) = u_t(x, y, t_2) = 0$ for all $(x, y) \in \bar{\mathfrak{I}} \times \bar{\mathfrak{J}}$ which is not possible. This contradiction proves that v can not be a positive solution of (4.68)-(4.4) on $\bar{\Omega}$. The proof of the case $v < 0$ is explained as in the proof of Theorem 4.3.1, and hence it is omitted. Therefore v has a zero in $\bar{\Omega}$. Theorem 4.3.5 has been proved. □

Corollary 4.3.6. *Assume that (i)-(iii) hold. If*

$$q(t) \ge p(t); \qquad t \in [t^*, \infty), \tag{4.88}$$

$$s_j(t) \ge r_j(t); \qquad t \in [t^*, \infty) \quad (j = 1, 2, 3, 4) \tag{4.89}$$

are provided for all $t^* \ge t_0$, and that

$$q_k \ge p_k \tag{4.90}$$

for each $k \in \mathbb{N}$ for which $\tau_k \ge t^*$, then (4.68)-(4.4) *is oscillatory whenever* (4.67)-(4.3) *is oscillatory.*

5. CONCLUSIONS AND RECOMMENDATIONS

The achievement of analogous comparison conclusions to those in Chapter 3 will be of interest, on the $(n+1)$-orthotope (*hyperrectangle*) for the hyperbolic equations of the form

$$u_{tt} - \Delta u + \mathcal{H}(\mathbf{x}, t)u = 0; \quad (\mathbf{x}, t) \in \Gamma \subset \mathbb{R}^n \times (t_0, \infty)$$

under the proper boundary conditions, where $\mathbf{x} = (x_1, \ldots, x_n)$ and Δ is the usual *Laplace operator* in $\mathbb{R}^n$, i.e.,

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j}.$$

Here Γ is the hyperrectangle defined by

$$\Gamma := \big((x_1)_1, (x_1)_2\big) \times \big((x_2)_1, (x_2)_2\big) \times \cdots \times \big((x_n)_1, (x_n)_2\big) \times (t_1, t_2)$$

for points $(x_j)_1, (x_j)_2 \in \mathbb{R}$ on the x_j-axis, $j = 1, 2, \ldots, n$.

Our future plan is to provide some Sturm comparison results for a pair of hyperbolic impulsive initial value problems of the form

$$\begin{aligned} u_{xt} + f(x,t)u &= 0; & (x,t) &\in \Pi \setminus \Pi_{\text{imp}}, \\ \Delta u_t + f_k(x,t)u &= 0; & (x,t) &\in \Pi_{\text{imp}} \end{aligned} \tag{5.1}$$

providing the initial conditions

$$\begin{aligned} u_x(x_1,t) - r_1(t)u(x_1,t) &= 0; & t &\in [t_1, t_2], \\ u_x(x_2,t) + r_2(t)u(x_2,t) &= 0; & t &\in [t_1, t_2], \end{aligned} \tag{5.2}$$

and that

$$\begin{aligned} v_{xt} + g(x,t)v &= 0; & (x,t) &\in \Pi \setminus \Pi_{\text{imp}}, \\ \Delta v_t + g_k(x,t)v &= 0; & (x,t) &\in \Pi_{\text{imp}} \end{aligned} \tag{5.3}$$

providing the initial conditions

$$\begin{aligned} v_t(x_1,t) - s_1(t)v(x_1,t) &= 0; & t &\in [t_1, t_2], \\ v_t(x_2,t) + s_2(t)v(x_2,t) &= 0; & t &\in [t_1, t_2] \end{aligned} \tag{5.4}$$

under some appropriate suppositions on the functions f, g, f_k and g_k, $k \in \mathbb{N}$, and that r_j, s_j, $j = 1, 2$, where Π and Π_{imp} are defined in (2.7) and (2.8), respectively.

6. REFERENCES

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