

Modeling for vibration analysis of thin non-uniform plates using Heaviside Functions

by

Mehmet Murat Gözümler

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Heaviside Functions**

Koc University

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Mehmet Murat Gözüm

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Ipek Başdoğan, Ph.D. (Advisor)

Murat Sözer, Ph.D.

Bekir Bediz, Ph.D.

Çağatay Başdoğan, Ph.D.

Polat Şendur, Ph.D.

Date: _____

ABSTRACT

Plate structures are utilized in many engineering applications where advanced techniques are used to improve their dynamic properties. These techniques may include attaching discrete patches and/or spatially varying the stiffness and mass properties of the plates. For example, piezoelectric transducers can be attached to the laminated plates to perform shunt damping for efficiently reducing the vibrations of the host plate. The vibration analysis of these non-uniform structures requires efficient and reliable mathematical modeling tools. This work focuses on semi-analytical modeling of plate structures having discrete and/or continuous non-uniformities and electroelastic modeling of laminated plates integrated with piezo-patches where it is utilized for analyzing the shunt damping performance of the whole system.

There are many analytical and numerical models developed for the vibration analysis of the plates with attached discrete patches. In parallel, several studies are performed on the vibration analysis of the variable-stiffness laminates where their stiffness and mass properties can be spatially varied. However, there is no analytical model that enables the analysis of plates having discrete(possibly off-axis or rotated) and/or continuous non-uniformities. Considering these needs, the first part of this study aims to address these problems by developing a mathematical model for vibration analysis of plates having different kinds of non-uniformities. A semi-analytical model is developed for the vibration analysis of the plates having discrete (possibly off-axis) and/or continuous non-uniformities where Heaviside step functions are used to represent the discontinuities. Kinematic relations of the plate are based on the Classical Laminated Plate Theory (CLPT) and Rayleigh-Ritz method is used for modal analysis. For modeling rotated discontinuities, a numerical technique is proposed and incorporated into the model. Natural frequencies and mode shapes are calculated for the non-rotated/rotated discrete patch configurations and validated with finite element simulations. It is shown that the rotation of the patch has a significant effect on the natural frequencies and mode shapes which is captured by the developed model. Finally, the modal analysis is performed on the variable-stiffness composite laminates and the natural frequencies obtained by the developed model

are validated with the results from the literature.

In the second part of this thesis, the model is extended such that it can be used for the design and analysis of laminated composite plates with surface-bonded piezo-patches in bimorph configurations. The coupled electromechanical equations are derived to obtain the frequency response functions for displacement and voltage variables under the transverse harmonic point force. The developed model considers mass and stiffness contribution of the piezo-patches and the two-way electromechanical coupling effect. Moreover, an accelerated solution technique is developed for reducing the computation time for solving the coupled differential equations that represent the system dynamics. Several case studies are presented to observe the combined effects of the ply angle and size of the piezo-patches on the natural frequencies and corresponding mode shapes of the host structure. The results are validated by system-level simulations in finite-element software ANSYS. Then, utilizing the developed model, a thorough investigation is performed to determine the system parameters that significantly affect the electromechanical coupling and the shunt damping performance of the overall system. The system parameters such as the host plate ply angle, piezo-patch thickness, size and material, and their combined effects are investigated on the electromechanical coupling coefficient (EMCC). Finally, the shunt damping performance is shown for three configurations of weakly, medium and strongly-coupled systems.

ÖZETÇE

Plaka yapıları, dinamik özelliklerinin geliştirilmesi için gelişmiş tekniklerin kullanıldığı birçok mühendislik uygulamasında kullanılmaktadır. Bu teknikler, ayrık yamalar eklemeyi ve/veya plakaların rijitlik ve kütle özelliklerini düzlemsel olarak değiştirmeyi içerebilir. Örneğin, piezoelektrik transdüktörler, ana plakanın titreşimlerini etkin bir şekilde azaltmak için şant sönümlemesi gerçekleştirmek üzere lamine plakalara bağlanabilir. Bu düzgünsüz yapıların titreşim analizi, verimli ve güvenilir matematiksel modelleme araçları gerektirir. Bu çalışma ayrık ve/veya sürekli düzgünsüzlük bulunduran plakaların yarı-analitik modellemesine ve piezo-yamalar ile entegre edilmiş lamine plakaların elektroelastik modellemesine ve bu modeli kullanarak şant sönümleme performansını analiz etmesine odaklanmaktadır.

Ayrık yamalarla bağlanmış plakaların titreşim analizi için geliştirilmiş birçok analitik ve nümerik model vardır. Paralel olarak, rijitlik ve kütle özelliklerinin düzlemsel olarak değiştirilebildiği değişken-rijitlikli laminatların titreşim analizi üzerinde çeşitli çalışmalar yapılmaktadır. Bununla birlikte, ayrık (muhtemelen eksen dışı veya döndürülmüş) ve/veya sürekli düzgünsüzlüklere sahip olan plakaların analiz edilmesini sağlayan herhangi bir analitik model yoktur. Bu ihtiyaçlar göz önünde bulundurulduğunda, bu çalışmanın ilk kısmı farklı türde düzgünsüz levhaların titreşim analizi için bir matematiksel model geliştirerek bu problemleri ele almayı amaçlamaktadır. Ayrık (muhtemelen eksen dışı) düzgünsüzlüğe sahip olan plakaların titreşim analizi için düzensizliklerin Heaviside birim basamak fonksiyonlarıyla temsil edildiği yarı-analitik bir model geliştirilmiştir. Levhanın kinematik ilişkileri Klasik Laminasyon Teorisine dayanmakta iken modal analiz için Rayleigh-Ritz metodu kullanılmıştır. Döndürülmüş süreksizlikleri modellemek için modele nümerik bir teknik önerilmiş ve dahil edilmiştir. Doğal frekanslar ve mod şekilleri, döndürülmemiş/ döndürülmüş ayrık yama konfigürasyonları için hesaplanmış ve sonlu eleman simülasyonları ile doğrulanmıştır. Yamanın rotasyonunun, geliştirilen model tarafından elde edilen doğal frekanslar ve mod şekilleri üzerinde önemli bir etkisi olduğu gösterilmiştir. Son olarak, değişken-rijitlik kompozit laminatlar üzerinde modal analiz gerçekleştirilmiş ve geliştirilen modelin elde ettiği doğal frekanslar literatürden elde edilen sonuçlar ile doğrulanmıştır.

Bu tezin ikinci bölümünde model, yüzeyinden bimorf konfigürasyonlu piezo-yamalarıyla bağlı olan lamine kompozit plakaların tasarımı ve analizi için kullanılabileceği şekilde genişletilmiştir. Kuple elektromekanik denklemler, enine harmonik nokta kuvveti altında yer değiştirme ve voltaj değişkenleri için frekans tepki fonksiyonlarını elde etmek için türetilmiştir. Geliştirilen model, iki-taraflı elektromekanik kuplaj etkisini ve piezo-yamaların kütle ve rijitlik katkısını dikkate almaktadır. Ayrıca, sistem dinamiklerini temsil eden kuple diferansiyel denklemlerin çözümünün hesaplama zamanını azaltmak için hızlandırılmış bir çözüm tekniği geliştirilmiştir. Ana plakanın tabaka açılarının ve piezo-yamaların boyutlarının, ana yapının doğal frekansları ve mod şekilleri üzerindeki kombine etkilerini gözlemlemek için birkaç vaka çalışması sunulmuştur. Sonuçlar, sonlu elemanlar yazılımı ANSYS'de sistem düzeyinde simülasyonlar tarafından doğrulanmıştır. Daha sonra, geliştirilen modeli kullanarak, elektromekanik kuplajlı ve genel sistemin şant sönümleme performansını önemli ölçüde etkileyen sistem parametrelerini belirlemek için kapsamlı bir inceleme yapılmıştır. Ana plakanın tabaka açısı, piezo-yamanın boyutu, kalınlığı ve malzemesi gibi sistem parametreleri ve bunların kombine etkileri elektromekanik kuplaj katsayısı (EMKM) üzerinde incelenmiştir. Son olarak, zayıf, orta ve güçlü kuple sistemlerin üç konfigürasyonu için şant sönümleme performansı gösterilmiştir.

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Nomenclature



Chapter 1

Introduction

1.1 Mathematical Modeling of Non-Uniform Plates for Vibration Analysis

The plate structures are frequently used in many engineering applications. The mechanical properties of the plates should be improved to be utilized in high-performance applications. Hence, more advanced techniques are required for further enhancement of operational performance. For instance, the dynamic response of the plates can be improved by integrating discrete patches to the surface or by spatially varying in-plane stiffness properties. However, using those superior structures inherently increases the computational complexity of the analysis. Thus, efficient and accurate mathematical models are required for the fast and reliable design of such non-uniform plates.

In the literature, many analytical and numerical models are present for the dynamic analysis of non-uniform plate structures. Some of these models are developed for the analysis of plates integrated with discrete patches. For instance, Banks et al. [Banks et al., 1995] proposed mathematical models using Heaviside functions for the dynamic analysis of beams, plates, and shells with surface-bonded patches. Kung and Singh [Kung and Singh, 1998] developed an analytical model using the Rayleigh-Ritz method for the vibration analysis of the rectangular plates with multiple damping patches. An analytical model using double Fourier series was developed by Alibeigloo and Kari [Alibeigloo and Kari, 2009] for forced vibration analysis of simply-supported laminated composite plates with attached patch mass. Ungbhakorn and Wattanasakulpong [Ungbhakorn and Wattanasakulpong, 2013] developed an analyti-

cal model using Navier solutions for thermo-elastic vibration response of functionally graded plates with distributed patch mass. Recently, Kumar et al. [Kumar et al., 2016] used the finite element method (FEM) for the vibration analysis of an isotropic plate with attached discrete patches and point masses having different thickness variations.

Most of the available models for the analysis of structures involving discrete patches are utilized as piezoelectric sensors and actuators [Sodano et al., 2004, Cook-Chennault, 2008]. For modeling such structures, FEM has been mostly used to deal with the discontinuities. For instance, Saravanos et al. [Saravanos et al., 1997] used FEM for the free vibration analysis of simply-supported composite plates with discrete piezoceramic sensors. Ranjan and Ghosh [Ranjan and Ghosh, 2005] utilized FEM for vibration analysis of the plates with discrete masses and piezoelectric patches. In 2011, de Godoy and Trindade [de Godoy and Trindade, 2011] developed a finite element model for the dynamic analysis of laminated composite plates with embedded active-passive piezoelectric networks. Nevertheless, there is a continuous research work on the development of different analytical and numerical models to improve the ease of implementation and computational efficiency [Shahidi et al., 2013]. In 2000, Correia et al. [Correia et al., 2000] developed an analytical model based on Heaviside functions for dynamic analysis of a plate with surface-bonded piezoelectric actuators and sensors. This model was shown to be computationally more efficient compared to FEM. However, it is only applicable to systems where the piezoelectric patches are aligned with the plate axes all having rectangular geometries. Adali et al. [Adali et al., 2003] proposed an analytical formulation in terms of an integral equation which is based on explicit Green's function for the feedback displacement control of a laminated plate with orthotropic piezoelectric sensors and actuators. Vel et al. [Vel et al., 2004] developed an analytical model based on Fourier series for cylindrical bending of composite plates with distributed or segmented actuators. Their model was shown to perform accurately for the analysis of thick plates. Recently, In 2018 Gozum et al. [Gozum et al., 2018] developed an analytical model using the Rayleigh-Ritz method for vibration analysis of laminated composite plates with rectangular piezoelectric patches aligned with the plate axes. Their work also included an accelerated solution technique called *Separation of Integrals Method* which resulted in significantly lower

computation time compared to FEM.

The orientation of the attached piezoelectric patches also affects the dynamic behavior of flexible structures with attached patches [Qiu et al., 2007]. The orientation effects are more pronounced when the host plate is made of fiber-reinforced composites which might cause off-axis maximum strain directions [Chee et al., 2002], or when the strain constants of the patch along the principal axes are significantly different [Ip et al., 2001]. However, a small number of studies were conducted to analyze the effects of the discrete patch orientation on the dynamic characteristics of the plates [Bruant et al., 2010]. For example, Ip et al. [Ip et al., 2001] used FEM to obtain the optimal position and orientation of a single piezoelectric patch for controlling the first five modes of an isotropic plate. In 2012, Bachmann et al. [Bachmann et al., 2012] developed finite element models of composite turbomachine blade with an attached piezoelectric patch. They obtained optimal values of the patch position, size and rotation angles for the generalized electro-mechanical coupling coefficient and electric charge.

The analysis of continuous non-uniform structures also drew significant attention which results in the development of several numerical and analytical models. On this matter, variable-stiffness laminated plates were extensively investigated, since their spatial change of in-plane stiffness offers a substantial potential to enhance the dynamic performance [Ghiasi et al., 2010, Ribeiro et al., 2014]. For instance, Abdalla et al. [Abdalla et al., 2007] developed a nodal optimization framework coupled with finite element analysis to obtain the maximum fundamental frequency of variable-stiffness composite panels using lamination parameters. Their results show that there is a significant increase in the optimal fundamental frequency compared to constant-stiffness laminated plates. Akhavan and Ribeiro [Akhavan and Ribeiro, 2011], and Ribeiro et al. [Ribeiro et al., 2014] performed vibration analyses to calculate natural frequencies and mode shapes of the variable-stiffness laminates using p-version and h-version FEM, respectively. However, the design optimization of the variable-stiffness laminates using FEM is a computationally expensive procedure as many analyses are required [Setoodeh et al., 2006]. Alternatively, several analytical and numerical models have been proposed. For instance, Honda et al. [Honda et al., 2008] presented an analytical model using Ritz method for vibration analysis of

variable-stiffness laminated plates. Recently, Tan and Nie [Tan and Nie, 2016] developed a numerical model using the method of weighted residuals for analyzing free and forced vibration analysis of variable-stiffness composite annular thin plates with elastically restrained edges.

Among the existing studies for vibration analysis of the plate structures, none of the analytical models enables the analysis of the plates having discrete (possibly off-axis) and/or continuous non-uniformities. First part of this work focuses on the development of a semi-analytical model that enables the analysis of the plates having different kinds of non-uniformities.

1.2 Electroelastic Modeling and Shunt Damping of Laminated Plates with Piezo-Patches

Among the non-uniform plates, the plates integrated with piezoelectric transducers can be utilized for vibration control [Chandrashekhara and Agarwal, 1993, Reddy, 1999], energy harvesting [Arrieta et al., 2010, Rupp et al., 2009] and shunt damping [De Godoy and Trindade, 2001, Saravanos, 1999] applications. Piezoelectric materials are usually preferred in such applications because of their superior properties such as low power consumption, ease of fabrication at different geometric scales, and high-power density [Sodano et al., 2004, Cook-Chennault, 2008]. In addition, it is more advantageous to use laminated composites as plate structures due to their high stiffness-to-weight ratio properties. Stiffness and strength of these structures can be tailored for the specific requirements by altering their fiber orientations, stacking sequence, and layer thickness [Gurdal, 1999, Park et al., 2001]. However, composite laminates can suffer from severe vibrations which result in reduced structural life and mechanical failure [Fleming and Moheimani, 2003]. Hence, it is important to develop damping mechanisms for suppression of unwanted vibrations on laminated composite plates.

As previously stated shunt damping technique can be performed by attaching piezoelectric patches on the laminated composite plate which is an effective approach for vibration reduction [Forward, 1979, Hagood and Flotow, 1991]. Piezoelectric transducers that are connected to an electric shunt circuit (e.g. resistors and induc-

tors) can efficiently damp the vibrations of the host structure without adding extra mass and volumetric occupancy [Chang et al., 2010, Kim and Lee, 2002]. The shunt damping efficiency depends on several parameters including the electromechanical coupling between the piezoelectric material and the host structure, and the shunt circuitry. Therefore, efficient and accurate modeling techniques are required to include the electromechanical coupling effects and dynamics of the shunt circuit.

In the literature, numerous analytical and finite-element models are proposed for vibration analysis of laminated composite plates with integrated piezoelectric elements. In 1990, Lee developed a formulation for the piezoelectric laminates based on the classical lamination theory [Lee, 1990]. Heyliger and Saravanos [Heyliger and Saravanos, 1995] presented the exact solutions for simply supported laminated plates with embedded piezoelectric layers, where they used linear theory of piezoelectricity to solve the coupled three-dimensional (3D) equations of motion and charge equation. Later, Benjeddou et al. [Benjeddou et al., 2002] developed two-dimensional (2D) electromechanically coupled model for simply supported fiber-reinforced laminates with embedded piezoelectric layers and compared the analytical results with the existing uncoupled [Batra and Liang, 1997] and coupled [Heyliger and Saravanos, 1995] Navier-type 3D models and layer-wise finite element solutions [Correia et al., 2000, Saravanos et al., 1997]. Ballhause et al. [Ballhause et al., 2005] proposed that a layer-wise modeling of piezoelectric patches based on the fourth-order expansion in each layer gives the most accurate results when compared to the 3D exact solution in [Heyliger and Saravanos, 1995]. Similarly, a differential quadrature method which is a higher-order interpolation technique was presented for solving the 3D equations of multilayered piezoelectric composite plates [Zhang et al., 2006].

In parallel, finite element formulations were presented for dynamic analysis of the composite laminates with piezoelectric patches used as sensors and/or actuators [Lam et al., 1997, Suleman and Venkaya, 1995]. More recently, Phun-Van et al [Phung-Van et al., 2015] developed a numerical technique using higher-order shear deformation theory and isogeometric analysis for investigating the free vibrations and dynamic control of composite plates with piezoelectric layers.

However, there are relatively few studies that focus on the modeling and optimization of electromechanical coupling coefficient (EMCC) and shunt damping per-

formance in composite laminates with integrated piezoelectric structures. In 1999, Silva and Kikuchi [Silva and Kikuchi, 1999] performed topology optimization on the piezoelectric transducers for maximizing EMCC. However, the optimal distribution of the piezoelectric material they obtained in their study had complex topology which was difficult to be manufactured. In 2001, Chang and Du [Chang and Du, 2001] presented an electro-elastic laminated plate model and calculated an optimum configuration for the material and geometric variables of an asymmetric bimorph disk transducer that consists of a piezoelectric layer with an unequal elastic disk for maximizing EMCC. Nakasone and Silva [Nakasone and Silva, 2010] optimized the distribution of the piezoelectric material on a metallic plate using topology optimization for obtaining maximum EMCC. De Godoy and Trindade [de Godoy and Trindade, 2011] performed a parametric study for obtaining optimal locations and thickness of the piezo-patches on the laminated composite plate for maximizing EMCC. They also optimized the electrical circuit elements for improving the vibration reduction ability. Recently, Silva et al. [Silva et al., 2015] presented a topology optimization algorithm for the distribution of the piezoelectric patches on several host elastic structures including a cantilever beam, an automotive muffler and a free-clamped rectangular plate with the objective of maximizing EMCC. There are also several studies that focus on electrical circuit modeling for maximizing the shunt damping performance of flexible structures such as plates and beams [Ducarne et al., 2010; Lallart et al., 2008; Badel et al., 2006].

The aforementioned analytical models provide an insight into the vibration analysis of laminated composite plates with fully-covered piezoelectric patches whereas many applications such as vibration control and energy harvesting do not require full coverage of the host plate surface. For that reason, accurate analytical models are required for vibration analysis of the laminated composite plates integrated with partially covered piezoelectric patches that consider the electromechanical coupling between the host plate and the piezo-patches. Second part of this thesis focuses on the development of such an electroelastic model that considers mass and stiffness contributions of the piezo-patches as well as the two-way electromechanical coupling. Similarly, among the existing studies about the shunt damping of laminated composite plates, combined effects of the system parameters such as host plate ply angle,

piezo-patch area, thickness and material on the electromechanical coupling and the shunt damping performance were not investigated. Once the electromechanical model is established, the system parameters that significantly affect the electromechanical coupling and the shunt damping performance of the overall system can be easily determined. The second part of this thesis also considers these effects and investigates the system parameters such as the host plate ply angle, piezo-patch thickness, size and material, and their combined effects.

1.3 Dissertation Outline

The goal of this dissertation is to present a novel semi-analytical model for the vibration analysis of non-uniform plates and proposing an electroelastic model for vibration and shunt damping analyses of the laminated composite plates integrated with piezo-patches. Therefore, the outlined chapters are as follows: Chapter 2 introduces the semi-analytical model for the dynamic analysis of non-uniform plate structures, where Heaviside step functions are used to represent the discontinuities. Chapter 3 introduces an electroelastic model of a thin-laminated composite plate with surface-bonded piezo-patches by considering the mass and stiffness contribution of the patches as well as the two-way electromechanical coupling effect. Chapter 4 investigates the system parameters (e.g. geometric and material parameters) and their combined effects on the electromechanical coupling and the shunt damping performance. Chapter 5 presents concluding remarks of this dissertation by summarizing the outcomes and providing future directions in the areas of semi-analytical modeling of non-uniform plates and electroelastic modeling of smart composite plates for vibration and shunt damping analyses.

Chapter 2

A Semi-Analytical Model for Dynamic Analysis of Non-Uniform Plates

2.1 Introduction

The plate structures are commonly used in various engineering applications. Thus, it is a necessity to use advanced techniques to improve mechanical properties of the plates to be utilized in the high-performance applications. For example, dynamic response of the plates can be improved by bonding discrete patches to the surface or by spatially varying inertia and in-plane stiffness properties. However, analysis of these superior structures are difficult due to their complex nature. Therefore, efficient and accurate models are essential for fast and reliable analysis of such non-uniform plates. The analytical and numerical models are developed for the vibration analysis of the plates with discrete patches where mostly piezoelectric transducers are utilized. For modelling such structures with the discontinuities, FEM has been commonly used. In parallel, there has been a continuous work on the development of different analytical and numerical models to improve the ease of application and computational efficiency. It was shown that using Heaviside functions for the dynamic analysis of a plate with surface-bonded piezoelectric sensors/actuators can be computationally more efficient than FEM [Correia et al., 2000]. However, its applicability is restricted to systems where the piezoelectric patches are aligned with the plate axes all having rectangular

geometries. The alignment of the attached piezoelectric patches also affects the dynamic behavior of host structures. Relatively fewer studies have been conducted to analyze the effects of the discrete patch alignment on the dynamic characteristics of the host plate [Ip et al., 2001, Bruant et al., 2010]. At the same time, analytical and numerical models are developed for the analysis of continuous non-uniform structures where they investigated variable-stiffness laminated plates, since the spatial change of in-plane stiffness, offers a substantial potential to enhance the dynamic performance.

Among the existing studies for vibration analysis of the plate structures, there is no analytical model that is capable of analyzing the plates having discrete (possibly off-axis) and/or continuous non-uniformities. In this chapter, a semi-analytical model is presented for the dynamic analysis of non-uniform plate structures where the discontinuities are represented with Heaviside step functions. For the analysis of plates with off-axis discontinuities such as rotated patches, a numerical technique that approximates the non-uniform region is proposed. The governing equations are derived using Hamilton's principle. Modal analysis is performed using the Rayleigh-Ritz method. For test cases, surface-bonded patches and variable-stiffness laminates are used to demonstrate the applicability of the proposed method on discrete and continuous non-uniform structures, respectively. Modal variables such as natural frequencies and mode shapes of the plates with rotated and unrotated discrete patch configurations obtained by the developed semi-analytical model are validated with the results obtained by the finite element analysis (FEA) software ANSYS. It is observed that the rotation of the patch has a significant effect on the natural frequencies and mode shapes of the whole structure which is captured by the developed model. For the variable-stiffness laminates, the natural frequencies obtained by the developed model are validated with the available results in the literature. The results demonstrate that the proposed model can be utilized for accurate and efficient analysis of non-uniform plates.

2.2 Discrete Non-Uniformity

This section presents the semi-analytical model developed for the analysis of plate structures with discrete non-uniformities which are introduced by using surface-bonded

rectangular patches.

2.2.1 Discrete Patches without Rotation

In this subsection, the model for the vibration analysis of the laminated composites integrated with unrotated discrete patches initially developed by Gozum et al. [Gozum et al., 2018] is described as it constitutes the foundation for the present model. Figure 2.1 illustrates the laminated composite plate with a pair of discrete patches. The patches are attached to top and bottom of the plate symmetrically with respect to its midplane. The coordinates of each discrete patch are expressed by (x_1, x_2) and (y_1, y_2) in x and y-axes, respectively. Hence, the length and width of the patches are equal to $a_p = x_2 - x_1$ and $b_p = y_2 - y_1$, respectively. Thickness of each patch is denoted by t_p . Length, width, thickness and layer angles of the base plate are denoted by a_s, b_s, t_s and θ_l , respectively.

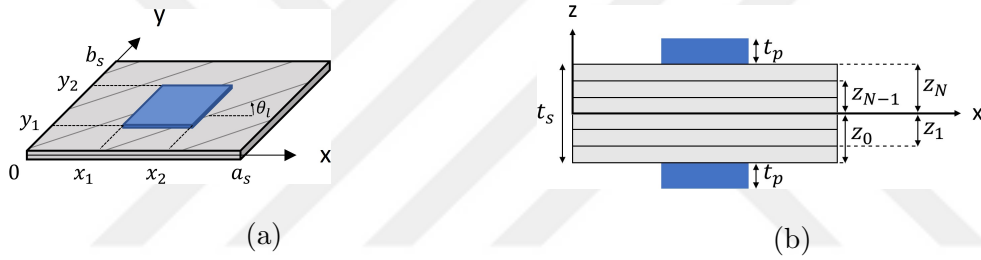


Figure 2.1: Schematics of the laminated composited plate with a pair of unrotated patches: (a) isometric view (b) cross-section view.

The kinematic relations are based on Classical Lamination Plate Theory (CLPT), where the displacement variables are defined as:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u_0 - z \frac{\partial w}{\partial x} \\ v_0 - z \frac{\partial w}{\partial y} \\ w_0 \end{pmatrix} \quad (2.1)$$

Here, u , v and w are the displacements in x, y and z-directions, respectively; and z is the distance between a specific point and midplane of the plate. u_0 , v_0 and w_0 are the midplane displacements in x, y and z-directions, respectively. Midplane displacements in x and y-directions are neglected based on the assumption of small strains. Based on this assumption and linear elasticity, the strain components can be

determined as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{du}{dx} \\ \frac{dv}{dy} \\ \frac{du}{dy} + \frac{dv}{dx} \end{Bmatrix} = -z \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (2.2)$$

The midplane of the plate is also called as the neutral surface since the strains are zero there. For the k^{th} layer of the laminated composite plate, stress-strain relationship is given by the expression [Reddy, 1997]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^k \quad (2.3)$$

where σ_{ij} and $\bar{Q}_{ij}$ are the in-plane stress terms and transformed reduced stiffness matrix entities for the laminated composite plates. Similarly, stress-strain relationship for each discrete patch can be shown by the expression:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (2.4)$$

where c_{ij} terms are the elastic moduli of each discrete patch attached on the laminated composite plate.

The governing equation of the motions for the laminated composite plate attached with discrete patches is derived using Hamiltons Principle [Meirovitch, 2001]:

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0 \quad (2.5)$$

Here, δ is the variational symbol and $[t_1, t_2]$ is the given time interval. T and U are the kinetic and potential energy terms respectively. The kinetic energy can be expressed as:

$$T = \frac{1}{2} \iint_S m(x, y) \dot{w}^2 dS \quad (2.6)$$

where S and $\dot{w}$ are the total area and the transverse velocity of the laminated composite plate and discrete patches, respectively. $m(x, y)$ is the mass per unit area of

the system which is determined as:

$$m(x, y) = \rho_s t_s + \sum_{i=1}^{N_p} \rho_p t_p P(x, y) \quad (2.7)$$

Here, ρ_s and ρ_p are the densities of the laminated composite plate and discrete patches, respectively. N_p denotes the number of discrete patches. $P(x, y)$ is the indicator function which represents the area covered by each patch in the xy-plane as follows [Yoon et al., 2016]:

$$P(x, y) = [H(x - x_1) - H(x - x_2)] * [H(y - y_1) - H(y - y_2)] \quad (2.8)$$

where H denotes the Heaviside functions in x and y-axes. The total potential energy of the system can be decomposed into two components; the potential energy of the plate (U_s) and the potential energy of the patches (U_p) as follows:

$$U = U_s + U_p = \frac{1}{2} \iiint_{V_s} (\sigma_{xx}\varepsilon_{xx})_s + (\sigma_{yy}\varepsilon_{yy})_s + (\tau_{xy}\gamma_{xy})_s dV_s + \frac{1}{2} \iiint_{V_p} (\sigma_{xx}\varepsilon_{xx})_p + (\sigma_{yy}\varepsilon_{yy})_p + (\tau_{xy}\gamma_{xy})_p dV_p \quad (2.9)$$

where V_s and V_p are the volumes of the plate and the discrete patches, respectively. Using equation (2.2) and equation (2.3), the potential energy of the plate can be expressed as:

$$U_s = \frac{1}{2} \iint_{S_s} D_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12} \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x^2} \right) + D_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{16} \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{26} \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{66} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 dS_s \quad (2.10)$$

where S_s is the surface area of the laminated plate. D_{ij} 's are the bending stiffness terms of the plate with N_l layers, which are calculated by [Shojaee et al., 2012]:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N_l} (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \quad (2.11)$$

The potential energy of the discrete patches can be determined using equation (2.2)

and equation (2.4):

$$U_p = \frac{1}{2} \sum_{i=1}^{N_p} \iint_{S_p} P(x, y) [D_{11}^p \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}^p \left(\frac{\partial^2 w}{\partial y^2}\right) \left(\frac{\partial^2 w}{\partial x^2}\right) + D_{22}^p \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}^p \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2] dS_p \quad (2.12)$$

where S_p is the surface area of the discrete patches. D_{ij}^p 's are the bending stiffness terms of the discrete patches calculated using the following relation:

$$D_{ij}^p = \int_{\frac{t_s}{2}}^{t_p + \frac{t_s}{2}} c_{ij} z^2 dz = \int_{-t_p - \frac{t_s}{2}}^{-\frac{t_s}{2}} c_{ij} z^2 dz = c_{ij} \left(\frac{t_s^3}{3} + \frac{t_s^2 t_p}{4} + \frac{t_s t_p^2}{2} \right) \quad (2.13)$$

After T and U terms are computed, the governing equation of motions can be determined by substituting equations (2.6) and (2.9) into equation (2.5):

$$\begin{aligned} m(x, y) \ddot{w} + [D_{11} + \sum_{i=1}^{N_p} D_{11}^p P(x, y)] \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + [D_{22} + \sum_{i=1}^{N_p} D_{22}^p P(x, y)] \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \\ + [2D_{12} + \sum_{i=1}^{N_p} (2D_{12}^p P(x, y))] \left(\frac{\partial^2 w}{\partial x^2}\right) \left(\frac{\partial^2 w}{\partial y^2}\right) + [4D_{66} + \sum_{i=1}^{N_p} 4D_{66}^p P(x, y)] \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \\ + [4D_{16}] \left(\frac{\partial^2 w}{\partial x^2}\right) \left(\frac{\partial^2 w}{\partial x \partial y}\right) + [4D_{26}] \left(\frac{\partial^2 w}{\partial y^2}\right) \left(\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \end{aligned} \quad (2.14)$$

The modal solution of the governing equation of motions is obtained using Rayleigh-Ritz method where the transverse displacement of a plate structure is approximated as [Meirovitch, 2001]:

$$w(x, y, t) = \sum_{m=1}^M \sum_{n=1}^N \phi_{mn}(x, y) \mu_{mn}(t) \quad (2.15)$$

where ϕ_{mn} , μ_{mn} , M and N are the assumed mode shape, time-dependent modal coordinates and number of modes used in x and y-axes, respectively. The assumed mode shapes can be written as [Bhat, 1985]:

$$\phi_{mn}(x, y) = \sum_{j=1}^M \sum_{i=1}^N A_{mnij} X_i(x) Y_j(y) \quad (2.16)$$

Here $X_i(x)$ and $Y_j(y)$ are the orthogonal polynomials in x and y-axes and A_{mnij} are the arbitrary coefficients. The orthogonal polynomials are generated using Gramm-

Schmidt procedure that satisfies the appropriate boundary conditions [Bhat, 1985]. The arbitrary coefficients, A_{mnij} 's can be obtained by minimizing Rayleigh's quotient which is given by the formula:

$$R[A_{mnij}X_i(x)Y_j(y)] = \frac{U^{max}}{T^{ref}} \quad (2.17)$$

where U^{max} and T^{ref} are the maximum potential energy and reference kinetic energy terms which are shown in equations (2.6) and (2.9). Minimizing Rayleigh's quotient with respect to A_{mnij} yields the following characteristic equation [Meirovitch, 2001]:

$$\frac{\partial R}{\partial A_{mnij}} = \left(\frac{\partial U^{max}}{\partial A_{mnij}} - \omega_{mn}^2 \frac{\partial T^{ref}}{\partial A_{mnij}} \right) = 0 \quad (2.18)$$

Here, ω_{mn} 's are the natural frequencies of the system, and this characteristic equation can be written as an eigenvalue equation:

$$\sum_{m=1}^M \sum_{n=1}^N [K_{mnij} - \omega_{mn}^2 M_{mnij}] A_{mnij} = 0 \quad (2.19)$$

where K_{mnij} and M_{mnij} are the stiffness and mass matrices of the laminated composite plate and the discrete patches. Eigenvalues of the equation (2.19) are the square of the circular natural frequencies ω_{mn} 's, while the eigenvectors of equation (2.19) are the mode shape coefficients A_{mnij} 's.

2.2.2 Discrete Patches with Rotation

In this sub-section, the model is extended for the analysis of plates with off-axis discontinuities. Figure 2.2 shows the schematics of the laminated composite plate with a pair of rotated discrete patches. Same as the previous case, the discrete patches are attached on top and bottom of the plate symmetrically with respect to its midplane. Each discrete patch is rotated around z-axis by θ_r degrees.

Kinematic relations and governing equations are same as the previous section for the laminated composite plate. Due to the rotation, stress-strain relationship for each

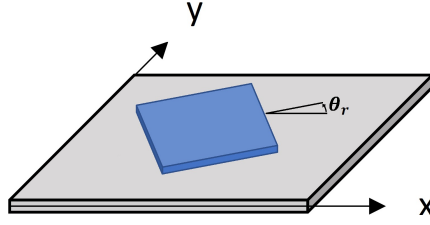


Figure 2.2: Schematics of the laminated composite plate with rotated pair of patches.

discrete patch is modified the way it is transformed in equation (2.3):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{16} \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{26} \\ \bar{c}_{16} & \bar{c}_{26} & \bar{c}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (2.20)$$

where $\bar{c}_{ij}$'s are the transformed reduced stiffness matrix of the rotated discrete patches. As before, the stiffness matrix terms of equation (2.20) can be substituted into equation (2.13) to obtain the bending stiffness matrix of the discrete patches:

$$D^p = \begin{bmatrix} D_{11}^p & D_{12}^p & D_{16}^p \\ D_{12}^p & D_{22}^p & D_{26}^p \\ D_{16}^p & D_{26}^p & D_{66}^p \end{bmatrix} \quad (2.21)$$

D_{16}^p and D_{26}^p are the bending-twisting coupling terms which did not appear in equation (2.12), arising due to the rotation of the patches.

After modifying the indicator function $P(x, y)$ within equation (2.14) to consider rotation effects and adding the bending-twisting coupling terms, the governing equation of motions become:

$$\begin{aligned} m(x, y)\ddot{w} + [D_{11} + \sum_{i=1}^{N_p} (D_{11}^p P^r(x, y))] \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + [D_{22} + \sum_{i=1}^{N_p} (D_{22}^p P^r(x, y))] \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + \\ [2D_{12} + \sum_{i=1}^{N_p} (2D_{12}^p P^r(x, y))] \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) + [4D_{66} + \sum_{i=1}^{N_p} (4D_{66}^p P^r(x, y))] \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + \\ [4D_{16} + \sum_{i=1}^{N_p} (4D_{16}^p P^r(x, y))] \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + [4D_{26} + \sum_{i=1}^{N_p} (4D_{26}^p P^r(x, y))] \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) = 0 \end{aligned} \quad (2.22)$$

where $P^r(x, y)$ is the transformed indicator function obtained using a numerical technique developed for the rotated discrete patches, which is explained in the following

paragraph. Same as the previous section, Rayleigh-Ritz method is used to obtain the modal solution of equation (2.22). Substituting equations (2.15) and (2.16) into equation (2.22) and minimizing it with respect to the mode shape coefficients A_{mnij} 's, an eigenvalue equation similar to equation (2.19) is obtained. Eigenvalues and eigenvectors of this equation provide the square of the natural frequencies and mode shape coefficients, respectively.

As it was mentioned before, one of the main disadvantages of Rayleigh-Ritz model coupled with Heaviside function has been reported to be its inapplicability for the analysis of plates with offaxis discontinuities [Correia et al., 2000]. This limitation can be particularly important when mode shapes are not orthogonal, for example when unbalanced laminates are considered. In such cases, sensors or actuators (e.g.: strain gauges or piezo-electric patches) may need to be installed in rotated configurations, as they function more effectively when they are mounted along maximum strain direction [Ip and Tse, 2001]. As a remedy, a numerical technique is developed to model off-axis discontinuities. The method relies on filling of the indicator function with ones, at the entities whose center point is enclosed by the area of discontinuity. In this way, the sparsity pattern of the resulting matrix approximately coincides with the discontinuity geometry. Consequently, the edges lying along the axes are represented exactly where off-axis edges are approximated by orthogonal components. Therefore, spatial discretization should be fine enough to accurately capture off-axis edges of the discontinuity geometry. In figure 2.3, filling principle of the transformed indicator function is illustrated.

2.3 Continuous Non-Uniformity

In this section, the use of concatenated discontinuities is explained for the analysis of plates with continuous non-uniformities such as variable stiffness laminates. The principle is similar to finite element method, as the plate is initially divided into discrete areas in its xy-plane. Next, each discrete area is assigned different stiffness properties using Heaviside step functions which are used to represent the discrete patches in the previous sections.

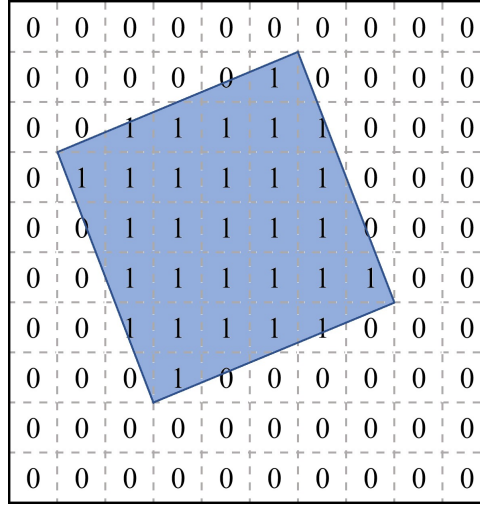


Figure 2.3: Filling of the indicator function with ones, at the entities whose center point is enclosed by the area of off-axis discontinuity.

Similar to equation (2.8), Heaviside unit step functions that represent the concatenated discrete areas can be written as:

$$P(i, j) = [H(x - x_i) - H(x - x_{i+1})] * [H(y - y_j) - H(y - y_{j+1})] \quad (2.23)$$

Here, H) denotes the Heaviside unit step function whereas x_i , x_{i+1} , y_j and y_{j+1} are the vertices of each area in x and y-axes, respectively. The number of divisions used in x and y-axes are N_e^x and N_e^y , respectively; which results in total of $N_e^x \times N_e^y$ concatenated areas in the plate. Figure 2.4 illustrates the discrete areas that are represented by Heaviside step functions.

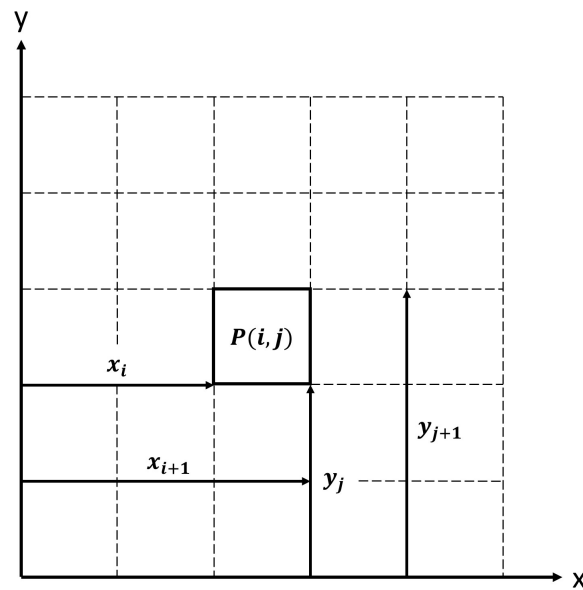


Figure 2.4: Discrete areas that represent the two-dimensional Heaviside unit step functions.

In this chapter, concatenated areas are assumed to be made of same material with the same thickness. Hence, equations (2.3) and (2.7) can be modified for each area which is expressed as:

$$\sigma(i, j) = \bar{Q}(i, j)\varepsilon(i, j) \quad (2.24)$$

$$m(x, y) = \rho_s t_s \sum_{i=1}^{N_e^x} \sum_{j=1}^{N_e^y} P(i, j) \quad (2.25)$$

where $\sigma(i, j)$, $\bar{Q}(i, j)$ and $\varepsilon(i, j)$ are the stress tensor, transformed reduced stiffness matrix and strain tensor of each discrete area, respectively. Potential energy term of the plate is obtained by using equations (2.23) and (2.24) in equation (2.9). Similarly, kinetic energy term can be obtained by substituting equation (2.25) in equation (2.6). By substituting these potential and kinetic energy equations in equation (2.5), the following equation of motion is obtained:

$$\begin{aligned} & \sum_{i=1}^{N_e^x} \sum_{j=1}^{N_e^y} \int_0^{a_s} \int_0^{b_s} P(i, j) [D_{11}(i, j) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + D_{22}(i, j) \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + \\ & 2D_{12}(i, j) \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) + 4D_{16}(i, j) \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + \\ & 4D_{26}(i, j) \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{66}(i, j) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \rho_s t_s \omega^2 w] dx dy = 0 \end{aligned} \quad (2.26)$$

Here, $D(i, j)$ is the bending stiffness matrix for each concatenated area which is calculated using transformed reduced stiffness matrix similar to equation (2.11):

$$D(i, j) = \frac{1}{3} \sum_{k=1}^{N_l} (\bar{Q}(i, j))_k (z_k^3 - z_{k-1}^3) \quad (2.27)$$

Same as the previous section, modal solution of equation (2.26) is obtained using Rayleigh-Ritz method. Equations (2.15) and (2.16) are substituted into equation (2.26) and the obtained result is minimized with respect to the mode shape coefficients, which will yield the eigenvalue equation (2.19). The eigenvalues and eigenvectors of this equation will be the square of the natural frequencies and the mode shape coefficients, respectively.

2.4 Results and Validation

2.4.1 Model Validation for Laminated Composite Plates with Discrete Non-Uniformity

In this section, natural frequencies and mode shapes of the laminated composite plate with two symmetric discrete square patches are obtained using the developed approach. Discrete patches with unrotated and rotated configurations are attached at the center of the laminated composite plate. For the rotated configurations, rotation angle of the discrete patches θ_r is chosen as 45° to maximize the rotation effects on the modal characteristics of the plate. The size of each square patch is selected as 350 mm which is close to the maximum possible value (368 mm) where the patches do not exceed the surface of the composite plate for $\theta_r=45^\circ$. For the composite plate, four layers with the angle configuration $[\theta_l / \theta_l / \theta_l / \theta_l]$ is used where θ_l is chosen as 0° and 45° . All combinations of $0^\circ/45^\circ$ plate and $0^\circ/45^\circ$ patches are investigated to observe the superposed orientation effects on the modal solutions.

The results obtained with the developed model are validated with Finite element analysis (FEA) software ANSYS. The laminated composite plate and the discrete patches are both meshed with 20-node structural solid (SOLID 186). The plate and the discrete patches are bonded to each other by generating constraint equations at their interface. Modal analysis is used to obtain the natural frequencies and mode shapes.

Table 2.1 shows the dimensions and material properties of the base composite plate material made of glass fiber reinforced polymer (T300/976 GFRP). The dimensions and material properties of the discrete patches made of piezoceramic (PZT-5A) are given in Table 2.2 [Gozum et al., 2018]. Two different values are used for the thickness of the symmetric patches which are 0.76 mm and 1.425 mm. These thickness values correspond to thickness ratio of 0.6 and 1.5, respectively (twice the patch thickness divided by the plate thickness). Within the model, the number of modes are selected as 20x20 and 25x25 for the thickness of 0.76 and 1.425 mm, respectively.

Table 2.1: Dimensions and material properties of the composite base plate (T300/976 GFRP).

a_s	mm	580
b_s	mm	540
t_s	mm	1.9
ρ_s	kg/m ³	1600
E_{11}	GPa	150
E_{22}	GPa	9
G_{12}	GPa	7.1
v_{12}^s		0.3

Table 2.2: Dimensions and material properties of the discrete patches (PZT-5A).

a_p	mm	350
b_p	mm	350
t_p	mm	(0.76,1.425)
ρ_p	kg/m ³	7750
c_{11}	GPa	69.48
c_{22}	GPa	24.32
c_{12}	GPa	22.57
v_{12}^p		0.35

Table 2.3 shows the comparison of the natural frequencies obtained using the semi-analytical model (SAM) and FEA for the composite plate with discrete patches that have thickness of 0.76 mm.

Table 2.3: Comparison of the natural frequencies obtained by the semi-analytical model (SAM) and FEA solutions for the composite plate with discrete patches having thickness of 0.76 mm.

θ_l	θ_r	Method	Mode1 (Hz)	Mode2 (Hz)	Mode3 (Hz)	Mode4 (Hz)	Mode5 (Hz)	Mode6 (Hz)	Mode7 (Hz)
0°	0°	ANSYS	43.59	72.07	116.55	140.68	145.42	213.78	239.27
	0°	SAM	43.78	73.49	116.99	142.18	146.65	215.39	240.61
	0°	%difference	0.44	1.97	0.38	1.07	0.84	0.75	0.56
	45°	ANSYS	46.01	91.19	118.37	158.99	172.65	240.25	246.92
	45°	SAM	45.84	90.58	118.41	162.11	169.49	242.93	246.82
	45°	%difference	-0.37	-0.67	0.03	1.96	-1.83	1.11	-0.04
45°	0°	ANSYS	45.53	92.26	107.17	148.08	179.62	204.93	228.05
	0°	SAM	45.57	93.36	108.08	149.78	181.84	207.09	230.68
	0°	%difference	0.09	1.19	0.85	1.15	1.23	1.05	1.15
	45°	ANSYS	45.36	91.31	112.83	150.93	188.95	229.89	258.48
	45°	SAM	45.28	93.81	112.84	156.06	190.98	230.00	260.51
	45°	%difference	0.18	2.74	0.01	3.40	1.07	0.05	0.79

The results obtained with proposed model are in good agreement with FEA simulation results for both unrotated and rotated patches. The difference percentages between the semi-analytical model and ANSYS results are below 2% for all cases except the second and fourth modes of ($\theta_l=45^\circ$, $\theta_r=45^\circ$) configuration. This difference is caused by the combined effects of Rayleigh-Ritz approximation and discontinuity due to rotations. The effect of the patch rotation on the natural frequencies is observed in both semi-analytical model and FEA simulation results.

For the same configurations of the plate and the patches, comparison of the mode shapes obtained by the semi-analytical model and FEA simulation is illustrated in Figure 2.5. Results show that the first five mode shapes are affected by the rotation of the patches (Coloring resolutions of the contours for FEA simulations and the semi-analytical model are not identical). Effect of the rotation on the mode shapes is apparent for both Rayleigh-Ritz and ANSYS results which are in good agreement. The rotation of the patches changes the order of fourth and fifth mode shapes for the laminated plate when the ply angle is 0°.

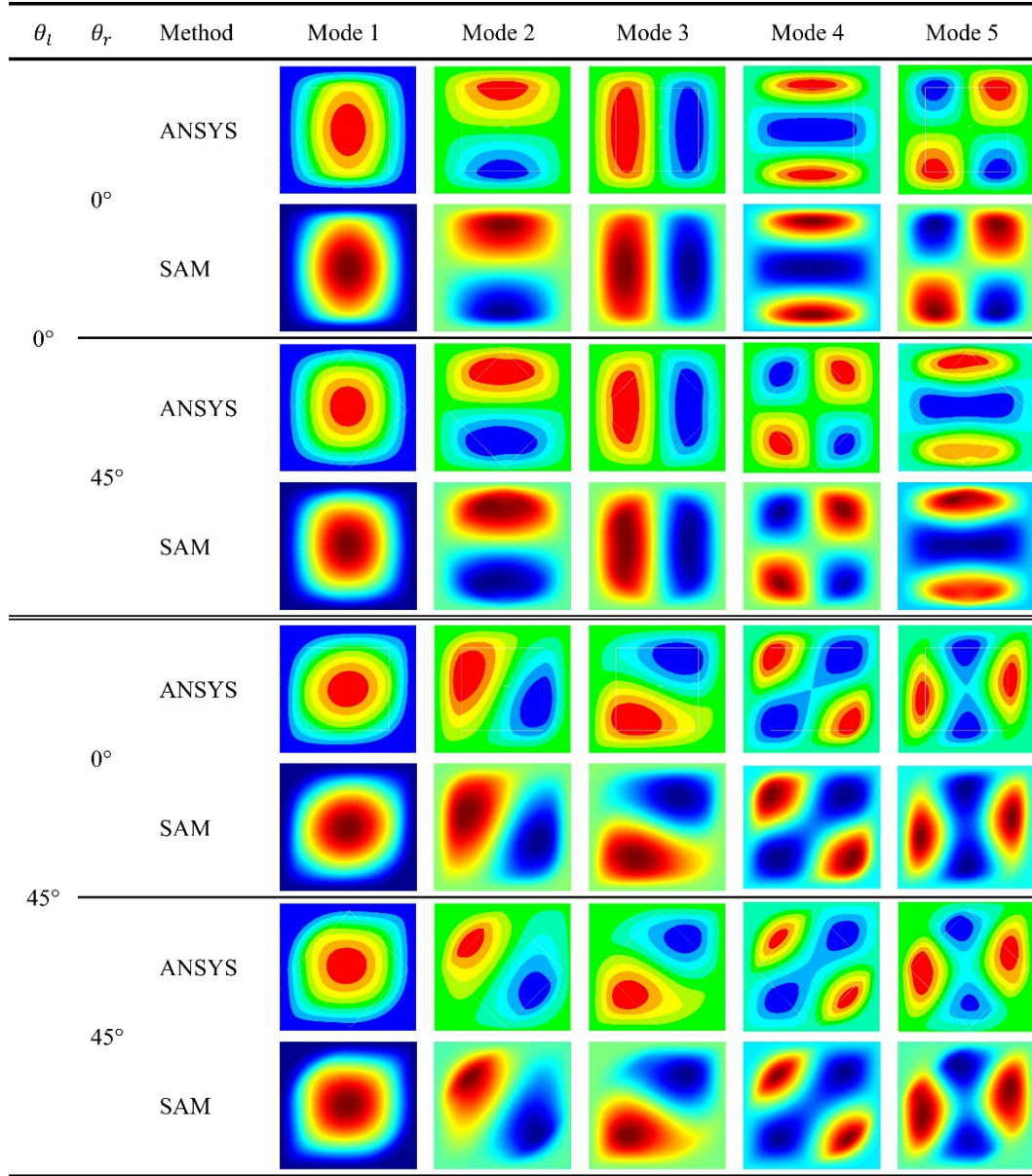


Figure 2.5: First five mode shapes of the laminated composite plate with discrete patches having thickness of 0.76 mm.

Next, the thickness of the patch is increased to 1.425 mm to induce stronger patch rotation effects. Table 2.4 illustrates the comparison of the natural frequencies obtained by the semianalytical model and FEA solutions for the composite plate with discrete patches having thickness of 1.425 mm.

The results obtained by the semi-analytical model and ANSYS agree quite well as the difference percentages are all below 4%. However, since the developed model is based on CLPT which neglects the shear deformation effects, the differences are bigger compared to the cases with thinner patches.

The difference between the natural frequencies of the laminated plates with ro-

Table 2.4: Comparison of the natural frequencies obtained by the semi-analytical model (SAM) and FEA solutions for the composite plate with discrete patches having thickness of 1.425 mm.

θ_l	θ_r	Method	Mode1 (Hz)	Mode2 (Hz)	Mode3 (Hz)	Mode4 (Hz)	Mode5 (Hz)	Mode6 (Hz)	Mode7 (Hz)
0°	0°	ANSYS	41.74	65.85	112.13	146.01	158.39	238.28	241.27
	0°	SAM	42.19	68.08	113.79	148.42	159.64	242.28	243.57
	0°	%difference	1.08	3.38	1.48	1.65	0.79	1.68	0.96
	45°	ANSYS	47.78	96.65	117.13	163.42	195.75	261.58	281.99
	45°	SAM	47.16	93.61	119.06	169.90	189.90	265.91	290.17
	45°	%difference	-1.29	-3.14	1.65	3.96	-2.99	1.66	2.90
45°	0°	ANSYS	45.08	85.26	100.47	147.03	180.53	210.78	249.02
	0°	SAM	45.55	87.86	103.10	150.23	184.28	213.69	252.90
	0°	%difference	1.04	3.05	2.62	2.18	2.08	1.38	1.56
	45°	ANSYS	46.96	92.70	118.05	161.49	202.12	258.86	291.69
	45°	SAM	46.72	95.64	116.83	165.91	207.15	256.52	293.92
	45°	%difference	-0.51	3.17	-1.04	2.74	2.49	-0.90	0.77

tated and unrotated patches is higher compared to the patches with 0.76 mm. This result demonstrates that effect of rotation increases with the increasing patch thickness. For the same configurations of the plate and the patches, comparison of the mode shapes obtained by the semi-analytical model and FEA simulation is presented in Figure 2.6. Similar to the increased changes in the natural frequencies, the effect of patch rotation on the mode shapes is more pronounced for the patches with higher thickness. However, the order of the mode shapes remained the same for all configurations of the laminated composite plate with 1.425 mm patches.

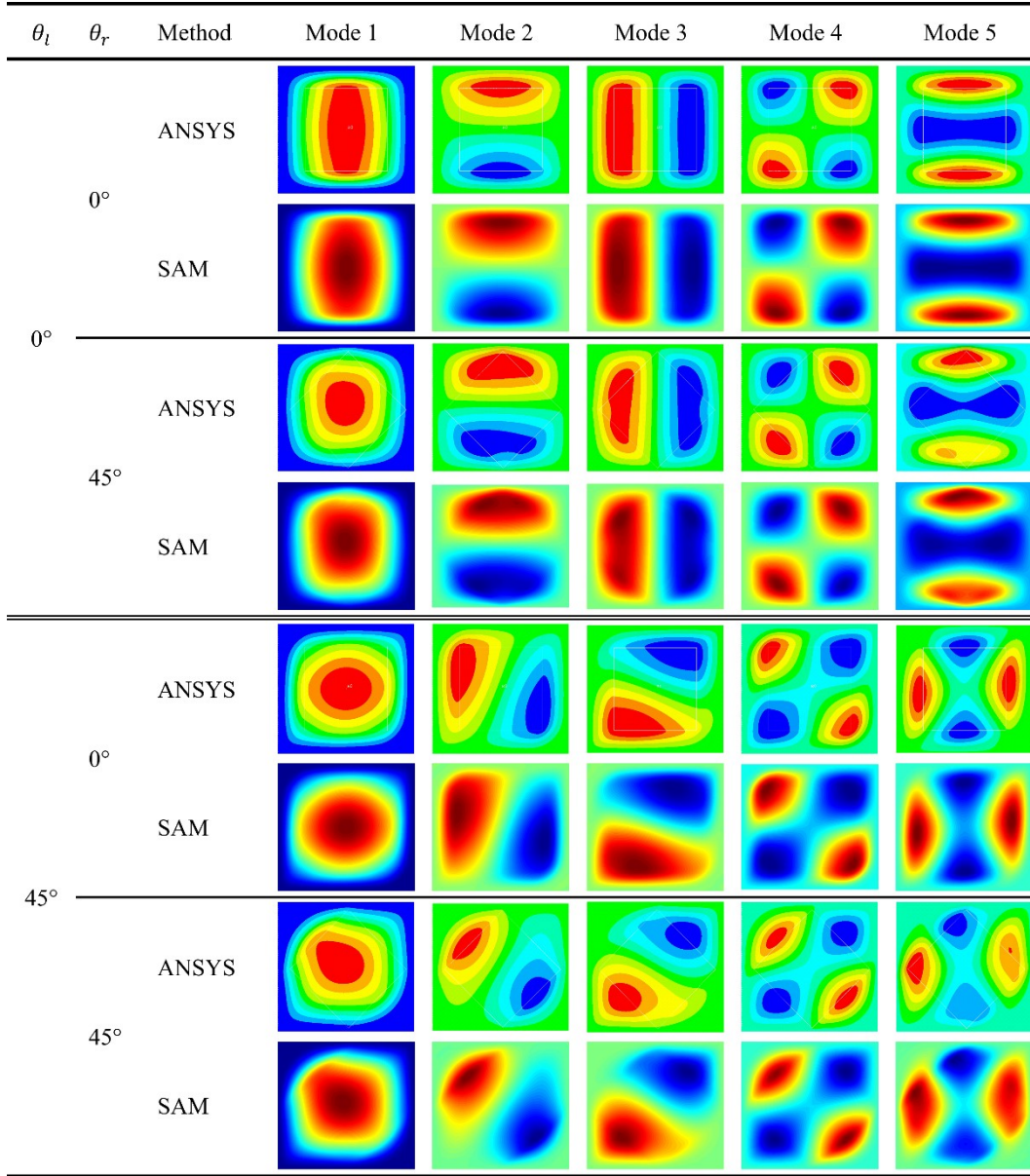


Figure 2.6: First five mode shapes of the laminated composite plate with discrete patches having thickness of 1.425 mm.

2.4.2 Model Validation for Laminated Composite Plates with Continuous Non-Uniformity

In this sub-section, natural frequencies of a variable-stiffness composite plate obtained with proposed model are validated with results from the literature [Akhavan and Riberio, 2011] for different fiber distributions and boundary conditions. After the natural frequencies are validated, the corresponding mode shapes are also presented to show that the developed model can correctly capture the oblique mode shapes of the variable-stiffness designs.

Variable-stiffness composite plate taken from [Akhavan and Riberio, 2011] has a

finite number of plies, and fiber orientation of each ply is defined with the notation $\langle T_0, T_1 \rangle$ where T_0 and T_1 are the path orientations in the center and vertical edges of a rectangular plate (See Figure 2.7).

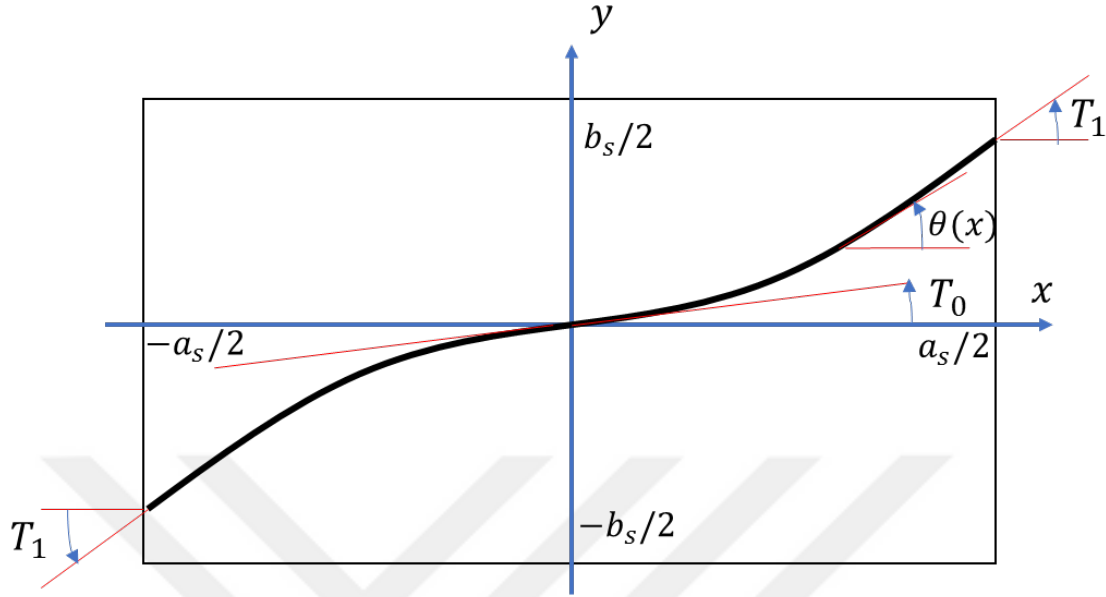


Figure 2.7: Fiber path distribution of the variable-stiffness composite plate.

Here, a_s and b_s are the length and width, respectively. The relation between the path orientations at the center and the vertical edges is given by the equation:

$$\theta(x) = \frac{2(T_1 - T_0)|x|}{a_s} + T_0 \quad (2.28)$$

Material and geometric properties of the plate used in our analyses are given in Table 2.5 [Akhavan and Ribeiro, 2011].

Table 2.5: Material properties of the variable-stiffness composite plate.

a_s	mm	1000
b_s	mm	1000
t_s	mm	10
ρ_s	kg/m ³	1540
E_{11}	GPa	173
E_{22}	GPa	7.2
G_{12}	GPa	3.76
v_{12}^s		0.29

The validation study is performed on the plate with three plies for three different boundary conditions: free (FFFF), simply-supported (SSSS) and clamped (CCCC). For each boundary condition three different fiber distributions are used: $\langle 0^\circ, 45^\circ \rangle$

, $< -45^\circ, -60^\circ >$, $< 0^\circ, 45^\circ >$], [$< 30^\circ, 0^\circ >$, $< 45^\circ, 90^\circ >$, $< 30^\circ, 0^\circ >$] and [$< 90^\circ, 45^\circ >$, $< 60^\circ, 30^\circ >$, $< 90^\circ, 45^\circ >$]. For the calculations of FFFF, SSSS and CCCC number of divisions is selected as 10x10, 13x13 and 15x15 whereas number of modes is kept at 10x10. Tables 2.6, 2.7 and 2.8 show the comparison of the first nine natural frequencies obtained by the developed model and the ones taken from [Akhavan and Ribeiro, 2011] for FFFF, SSSS and CCCC boundary conditions, respectively.

Table 2.6: Comparison of the natural frequencies for FFFF variable-stiffness composite plates.

Mode	Fiber Dist.	[$< 0^\circ, 45^\circ >$, $< -45^\circ, -60^\circ >$, $< 0^\circ, 45^\circ >$]			[$< 30^\circ, 0^\circ >$, $< 45^\circ, 90^\circ >$, $< 30^\circ, 0^\circ >$]			[$< 90^\circ, 45^\circ >$, $< 60^\circ, 30^\circ >$, $< 90^\circ, 45^\circ >$]		
		SAM (rad/s)	Ref. (rad/s)	Diff. (%)	SAM (rad/s)	Ref. (rad/s)	Diff. (%)	SAM (rad/s)	Ref. (rad/s)	Diff. (%)
1		144.1	141.0	2.22	111.7	110.4	1.16	123.4	123.2	0.19
2		171.4	170.2	0.71	178.1	177.5	0.37	152.7	151.4	0.88
3		349.5	344.6	1.43	269.0	266.5	0.91	280.5	277.5	1.05
4		482.3	477.6	1.00	462.9	459.7	0.68	393.8	389.5	1.11
5		596.3	592.5	0.64	473.4	468.7	0.99	409.4	404.5	1.20
6		725.1	716.0	1.28	623.2	618.3	0.79	583.3	577.6	0.98
7		725.5	718.9	0.92	664.4	658.1	0.95	625.3	622.3	0.47
8		887.0	872.2	1.69	780.6	771.6	1.18	704.9	693.6	1.64
9		1021.7	1007.9	1.37	865.1	856.7	0.98	849.7	840.2	1.14

Table 2.7: Comparison of the natural frequencies for SSSS variable-stiffness composite plates.

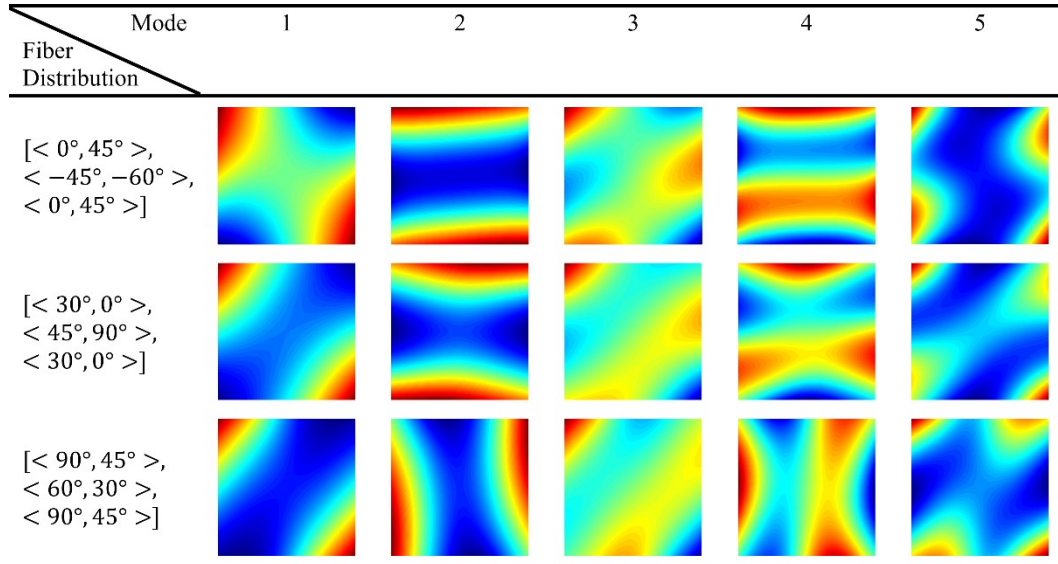
Mode	Fiber Dist.	[$< 0^\circ, 45^\circ >$, $< -45^\circ, -60^\circ >$, $< 0^\circ, 45^\circ >$]			[$< 30^\circ, 0^\circ >$, $< 45^\circ, 90^\circ >$, $< 30^\circ, 0^\circ >$]			[$< 90^\circ, 45^\circ >$, $< 60^\circ, 30^\circ >$, $< 90^\circ, 45^\circ >$]		
		SAM (rad/s)	Ref. (rad/s)	Diff. (%)	SAM (rad/s)	Ref. (rad/s)	Diff. (%)	SAM (rad/s)	Ref. (rad/s)	Diff. (%)
1		357.6	358.5	-0.25	310.2	308.8	0.44	330.3	329.7	0.19
2		590.3	589.9	0.06	505.5	503.8	0.33	539.3	539.4	0.01
3		963.7	960.4	0.34	848.8	845.5	0.39	890.3	886.4	0.44
4		1081.2	1075.2	0.56	1139.5	1131.3	0.72	1099.2	1091.2	0.73
5		1335.1	1327.9	0.54	1285.9	1279.9	0.47	1284.9	1279.9	0.39
6		1473.6	1474.7	-0.07	1316.4	1307.4	0.69	1400.3	1401.9	-0.12
7		1737.7	1726.7	0.64	1709.6	1701.7	0.46	1706.1	1755.5	-2.81
8		2096.6	2137.1	1.90	1763.2	1759.0	0.24	1809.3	1809.8	-0.03
9		2257.4	2262.4	-0.22	2311.9	2342.0	-1.28	2214.0	2216.6	-1.12

The natural frequencies obtained by the proposed model are in good agreement

Table 2.8: Comparison of the natural frequencies for CCCC variable-stiffness composite plates.

Mode	Fiber Dist.	[< 0°,45° >,<-45°,-60° >,< 0°,45° >]			[< 30°,0° >,<45°,90° >,< 30°,0° >]			[< 90°,45° >,<60°,30° >,< 90°,45° >]		
		SAM (rad/s)	Ref. (rad/s)	Diff. (%)	SAM (rad/s)	Ref. (rad/s)	Diff. (%)	SAM (rad/s)	Ref. (rad/s)	Diff. (%)
1		590.6	141.0	2.22	111.7	110.4	1.16	123.4	710.8	0.37
2		832.3	821.5	1.31	867.8	862.9	0.57	911.2	912.2	-0.01
3		1238.6	1225.8	1.05	1241.3	1234.6	0.54	1331.1	1335.5	-0.33
4		1539.5	1493.8	3.06	1719.4	1701.0	1.08	1707.4	1689.7	1.05
5		1775.2	1727.0	2.79	1797.7	1775.6	1.25	1858.9	1836.7	1.21
6		1794.4	1775.2	1.08	1931.6	1902.5	1.53	1990.4	1987.6	0.14
7		2187.5	2135.8	2.42	664.4	658.1	0.95	2270.3	2278.2	-0.35
8		2473.5	2443.5	1.23	2332.2	2310.7	0.93	2478.9	2466.8	0.49
9		2765.4	2706.8	2.17	2914.1	2879.6	1.20	2954.6	2920.4	1.17

with the results from [Akhavan and Ribeiro, 2011] for the specified boundary conditions and fiber distributions. The difference percentage does not exceed 3%, demonstrating the good accuracy of the analytical model. The effects of the fiber distribution and boundary conditions on the mode shapes corresponding to the calculated natural frequencies can be also observed using the proposed model. Figures 2.8, 2.9 and 2.10 show the first five mode shapes obtained by the developed model, for FFFF, SSSS and CCCC boundary conditions, respectively.

**Figure 2.8:** First five mode shapes of the variable-stiffness composite plates with FFFF boundary conditions.

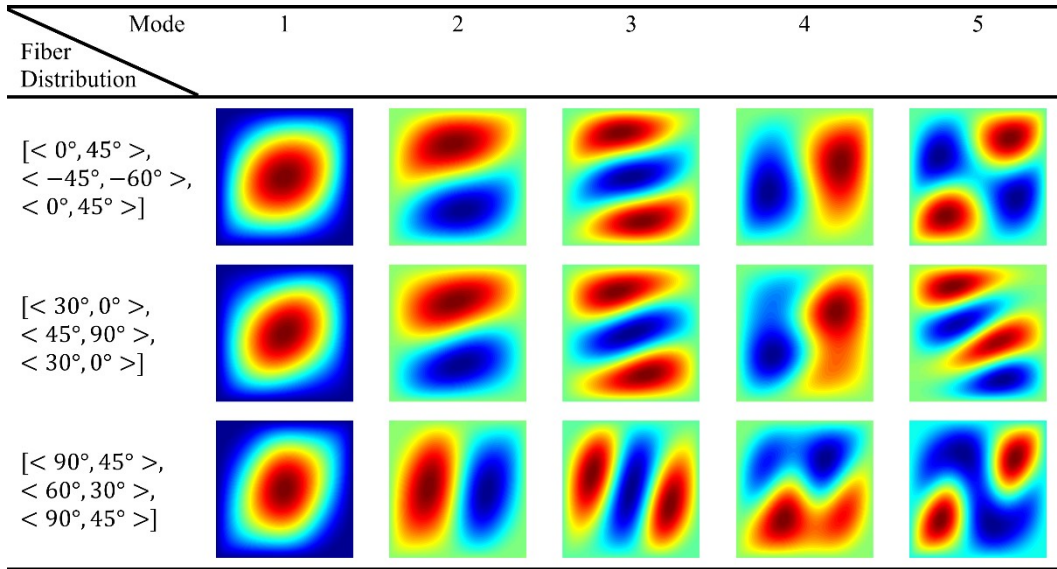


Figure 2.9: First five mode shapes of the variable-stiffness composite plates with SSSS boundary conditions.

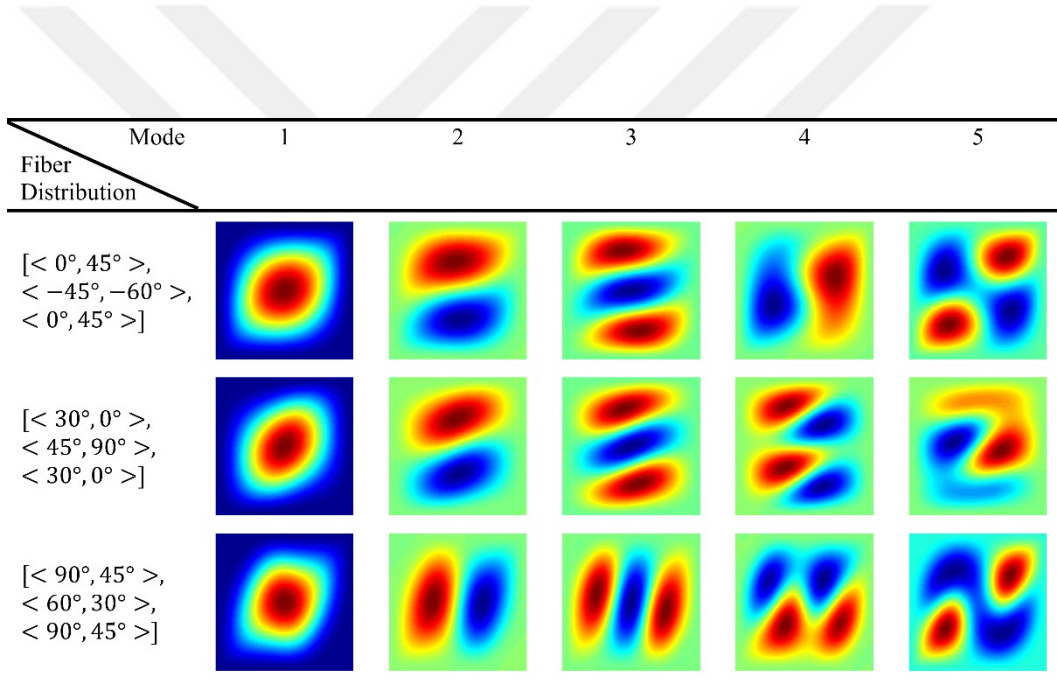


Figure 2.10: First five mode shapes of the variable-stiffness composite plates with CCCC boundary conditions.

The results show that the variation of the fiber distributions and boundary conditions significantly affects the mode shapes which are accurately captured by the proposed model.

2.5 Conclusion

In this chapter, a semi-analytical model for vibration analysis of non-uniform plate structures was presented. Surface-bonded discrete patches and variable-stiffness composite plates were used to demonstrate the applicability of the proposed method on discrete and continuous non-uniformities, respectively. The discontinuities were represented using Heaviside unit step functions. A numerical technique was proposed to model the off-axis discontinuities such as rotated patches. The equations of motion were derived using Hamilton's principle where the kinematic relations were based on CLPT. The natural frequencies and mode shapes were obtained using Rayleigh-Ritz method.

Validation studies were performed on the laminated composite plates with attached unrotated and rotated discrete patches, for two different thickness patches and ply angle orientations. The natural frequencies and mode shapes obtained by the proposed model were validated with FEA software ANSYS results. Patch rotation was observed to have a significant effect on the modal analysis solutions which was more pronounced for the patches with higher thickness. The results obtained with the developed model and FEA were in good agreement for both unrotated and rotated patches. The error percentages for both results were higher for the patches with higher thickness due to the increased shear deformation effects which are not included within the proposed model.

The model was lastly tested for the free vibration analysis of the variable-stiffness laminated plates having different boundary conditions and fiber distributions. The natural frequencies determined by the present model were validated with the results from the literature which showed a good agreement. Finally, the mode shapes obtained by the proposed model were presented for different fiber orientations and boundary conditions.

In conclusion, a semi-analytical model was developed for the accurate and efficient vibration analysis of the plate structures having non-uniform properties. The model provides remarkable advantages of low computational cost and ease of implementation for different analyses where the geometry and/or boundary conditions need to be modified. Besides, time-consuming pre-processing steps of FEA such as remeshing

are avoided. Therefore, the model is particularly suitable for the optimization studies where many different parameters need to be rapidly analyzed.



Chapter 3

Electroelastic Modeling of thin-laminated composite plates with surface-bonded piezo-patches using Rayleigh-Ritz Method

3.1 Introduction

Laminated composite plates have high stiffness-to-weight ratio properties and their stiffness and strength can be tailored by altering their fiber orientations, stacking sequence and layer thickness for the specific design requirements [Gurdal et al., 1999, Park et al., 2001]. These superior structures can be integrated with piezoelectric transducers to be utilized for vibration control [Chandrashekhara and Agarwal, 1993, Reddy, 1999], energy harvesting [Arrieta et al., 2010, Rupp et al., 2009], and shunt damping applications [De Godoy and Trindade., 2011, Saravanos, 1999]. Piezoelectric materials are preferred due to their advanced properties such as low power consumption, ease of fabrication at different geometric scales, and high-power density [Cook-Chennault et al., 2008, Sodano et al., 2004]. Many analytical and finite element models have been developed for vibration analysis of laminated composite plates with piezoelectric layers/patches over the past decades. However, the developed analytical models do not provide a formulation for modeling the piezoelectric patches that partially cover the plate surface. In energy harvesting, vibration control and

actuation/sensing applications, piezoelectric materials are usually in the form of thin square patches which are bonded to specific locations on the surface of the composite plates. Hence, high-fidelity analytical models are required for accurately analyzing the vibration and electrical responses of such structures.

In this chapter, an electroelastic model is presented for the vibration analysis of a thin-laminated composite plate with surface-bonded piezo-patches by considering the mass and stiffness contribution of the patches as well as the two-way electromechanical coupling effect. Heaviside step functions are used to represent the discontinuities of the piezo-patches. Hamilton's principle is used to derive the governing equations of motion. Rayleigh-Ritz method is used for obtaining the natural frequencies and the approximate mode shapes. Using the mode expansion theorem, the governing electromechanical equations in modal coordinates are determined, and closed-form expressions are obtained for the vibration and electrical responses under harmonic point force excitations. In addition, a separation of integral technique is developed in order to reduce the computational time for solving the coupled differential equations. Following the development of the model, free vibration analysis of the whole structure (assuming short-circuit condition) is conducted for various ply angles and piezo-patch sizes, and the results are validated by finite element simulations in ANSYS. The electromechanical frequency response functions (FRFs) are calculated and they converge to the finite element FRF results with increasing the number of modes in the assumed mode shapes. It is also shown that the developed analytical model using separation of integral method is computationally more efficient than the finite element simulations. Finally, the change in the fundamental frequency of the host structure by the increase in the piezo-patch size and thickness is shown to be significant which is captured by the proposed analytical model.

3.2 Analytical model of a thin laminated composite plate with piezo-patches

Figure 3.1 shows the schematics of the fully clamped composite plate with a pair of piezo-patches attached in bimorph configuration. Plate material is carbon-reinforced composite polymer. Length, width and total thickness of the plate are denoted by

a , b and h_s , respectively. The composite plate is made up of layers which are of the same thickness. Length, width and thickness of the piezo-patches are denoted by l_p, w_p and h_p , respectively. The two piezo-patches are attached to the top and bottom surfaces of the laminated composite plate symmetrically with respect to the mid-plane. They are connected in parallel to a resistive load, which is denoted by R_l . The piezo-patches cover a rectangular region on the composite plate with two corners located at (x_1, y_1) and (x_2, y_2) . As the excitation source for frequency-response analyses, a transverse harmonic point force is applied at (x_0, y_0) coordinates. The laminated composite plate is composed of n layers as shown in Figure 3.1b.

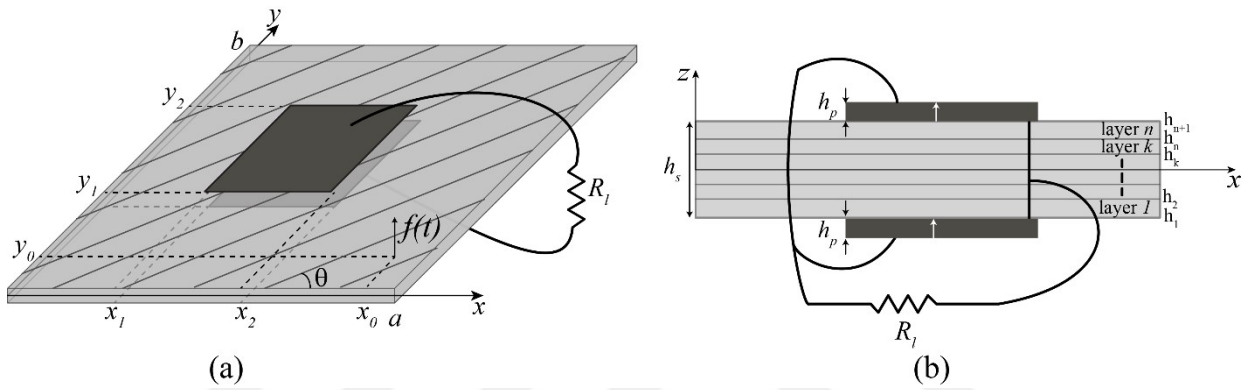


Figure 3.1: Schematics of the laminated composite plate with surface-bonded piezo-patches connected in parallel to a resistive load: (a) isometric view (b) cross-section view

3.2.1 Equations of motion

This section presents the equations of motions for the thin laminated composite plate with surface-bonded piezo-patches. Using Hamiltons principle, the kinetic and potential energy forms of the electromechanical system are written and then the equation of motion is obtained for the case of point force excitation.

The displacement variables of the thin laminated composite plate integrated with the piezo-patches can be written as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u_0(x, y) - z \frac{\partial w}{\partial x} \\ v_0(x, y) - z \frac{\partial w}{\partial y} \\ w_0(x, y) \end{Bmatrix} \quad (3.1)$$

where u_0 , v_0 and w_0 are the mid-plane displacements in x, y and z directions, respectively and z is the vertical distance from the mid-plane to the specified point. Here, u_0 and v_0 are neglected based on assumption of small strains [Yoon et al., 2016]. Strain variables are determined by using linear elastic strain-displacement relations as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \frac{du}{dx} \\ \frac{dv}{dy} \\ \frac{du}{dy} + \frac{dv}{dx} \\ \gamma_{yz} = \gamma_{xz} \end{Bmatrix} = -z \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \\ 0 \end{Bmatrix} \quad (3.2)$$

The stress-strain relationship for the k^{th} layer of the thin composite plate is given by [Phung-Van et al., 2015]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^k \quad (3.3)$$

where $\bar{Q}_{ij}^k$ is the transformed reduced stiffness matrix, σ_{ij}^k and ε_{ij} are in-plane stress and strain terms for the thin laminated composite plate, respectively. Elements of the transformed reduced stiffness matrix are given by [Gohari et al., 2016]:

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}s^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4) \\ \bar{Q}_{22} &= Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4 \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})s^3c - (Q_{22} - Q_{12} - 2Q_{66})c^3s \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(c^4 + s^4) \end{aligned} \quad (3.4)$$

where Q_{ij} 's are the elements of reduced stiffness matrix defined as:

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{E_{22}\nu_{12}}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12} \quad (3.5)$$

c and s represent $\cos(\theta_f)$ and $\sin(\theta_f)$, respectively where θ_f is the ply angle between the fibers and the x-axis. E_{11} , E_{22} and G_{12} are longitudinal, transverse and shear

elastic moduli, respectively. v_{12} is the major Poissons ratio. The minor Poissons ratio can be obtained using the reciprocity relation: $v_{21} = v_{12}E_{22}/E_{11}$.

The two-dimensional constitutive equations for the stress and electric displacement components of each piezo-patch are given by [Aridogan et al., 2014]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ D_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{11} & 0 \\ 0 & 0 & c_{66} \\ \bar{e}_{31} & \bar{e}_{31} & \bar{\epsilon}_{33}^s \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ E_3 \end{Bmatrix} \quad (3.6)$$

where c_{ij} 's are reduced elastic moduli of the piezo-patches computed using equation (3.5), $\bar{e}_{31}$ and $\bar{\epsilon}_{33}^s$ are piezoelectric constant and dielectric permittivity, respectively. The in-plane stress and strain components for the piezo-patches are denoted by σ_{ij} and ε_{ij} , respectively. The electric field E_3 can be expressed in terms of the electrical potential difference $v(t)$ as $E_3 = -v(t)/h_p$.

Hamilton's Principle is used to determine the governing equations of motion [De Godoy and Trindade, 2011]:

$$\delta \int_{t_1}^{t_2} (KE + W_p - PE) dt = 0 \quad (3.7)$$

where δ is the variational symbol, and $[t_1, t_2]$ symbolizes arbitrary time interval. Kinetic energy, potential energy and applied external work are indicated as KE , PE and W_p , respectively. Kinetic energy of the system can be formulated as:

$$KE = \frac{1}{2} \iint_S m(x, y) \dot{w}^2 dS \quad (3.8)$$

where S is the area of the system including the top surfaces of the piezo-patches and the thin composite plate. Here, $m(x, y)$ and $\dot{w}$ are the mass per unit area and the transverse velocity terms. Moreover, mass per unit area can be written as:

$$m(x, y) = \rho_s h_s + \sum_{l=1}^2 \rho_p h_p P(x, y) \quad (3.9)$$

where, ρ_s and ρ_p are the densities of the thin composite plate and the piezoceramic patch, respectively. The Indicator function $P(x, y)$ in equation (3.9) is used to deter-

mine the covered region on the composite plate by the piezo-patches:

$$P(x, y) = [H(x - x_1) - H(x - x_2)] * [H(y - y_1) - H(y - y_2)] \quad (3.10)$$

where x_1, x_2, y_1 and y_2 are the vertices of the area along x and y axes, respectively, and H denotes the Heaviside unit step function.

Potential energy of the system can be written as:

$$PE = PE_s + PE_p = \frac{1}{2} \iiint_{V_s} (\sigma_{xx}\varepsilon_{xx})_s + (\sigma_{yy}\varepsilon_{yy})_s + (\tau_{xy}\gamma_{xy})_s dV_s + \frac{1}{2} \iiint_{V_p} (\sigma_{xx}\varepsilon_{xx})_p + (\sigma_{yy}\varepsilon_{yy})_p + (\tau_{xy}\gamma_{xy})_p dV_p \quad (3.11)$$

where, V_s and V_p are the volumes of the thin composite plate and the piezoceramic patch, respectively. Potential energy stored in the thin composite plate can be written as:

$$PE_s = \frac{1}{2} \iint_{S_s} D_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12} \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x^2} \right) + D_{22} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 4D_{16} \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{26} \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{66} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 dS_s \quad (3.12)$$

Here, S_s is the surface area of the thin composite plate and D_{ij} 's correspond to the elements of the bending stiffness matrix of the host plate with n layers, which can be determined as follows [Shojaee et al.,2012]:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^n (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \quad \text{for } i = j = 1, 2, 6 \quad (3.13)$$

The potential energy stored in the piezo-patches can be written as:

$$PE_p = \frac{1}{2} \sum_{l=1}^2 \iint_{S_p} P(x, y) [D_{11}^p \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12}^p \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x^2} \right) + D_{11}^p \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{16}^p \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{26}^p \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{66}^p \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \bar{e}_{31} v(t) \left(\frac{h_s + h_p}{2} \right) \left[\left(\frac{\partial^2 w}{\partial x^2} \right) + \left(\frac{\partial^2 w}{\partial y^2} \right) \right]] dS_p \quad (3.14)$$

where S_p is the surface area of the piezo-patches, $v(t)$ is the voltage and D_{ij}^p is the bending stiffness matrix of the patches which is identical to equation (2.13).

The applied external work of the point force can be written as:

$$W_p = \iint_S f(t)\delta(x - x_0)\delta(y - y_0) dS \quad (3.15)$$

where $f(t)$ is the force amplitude, and $\delta(x)$ and $\delta(y)$ are the Dirac delta functions along the x and y axes. The governing equation of motion for the whole electromechanical system thus can be obtained by substituting equations (3.8), (3.11), and (3.15) into equation (3.7):

$$\begin{aligned} m(x, y)\ddot{w} + [D_{11} + \sum_{l=1}^2 D^p_{11}P(x, y)](\frac{\partial^2 w}{\partial x^2})^2 \\ + [D_{22} + \sum_{l=1}^2 D^p_{22}P(x, y)](\frac{\partial^2 w}{\partial y^2})^2 \\ + [2D_{12} + \sum_{l=1}^2 2D^p_{12}P(x, y)](\frac{\partial^2 w}{\partial x^2})(\frac{\partial^2 w}{\partial y^2}) \\ + [4D_{66} + \sum_{l=1}^2 4D^p_{66}P(x, y)](\frac{\partial^2 w}{\partial x \partial y})^2 \\ + [4D_{16}] (\frac{\partial^2 w}{\partial x^2})(\frac{\partial^2 w}{\partial x \partial y}) + [4D_{26}] (\frac{\partial^2 w}{\partial y^2})(\frac{\partial^2 w}{\partial x \partial y}) \\ - \sum_{l=1}^2 \bar{e}_{31}v(t)P(x, y)(\frac{h_s + h_p}{2})[(\frac{\partial^2 w}{\partial x^2}) + (\frac{\partial^2 w}{\partial y^2})] \\ = f(t)\delta(x - x_0)\delta(y - y_0) \end{aligned} \quad (3.16)$$

3.2.2 Modal Analysis Using Rayleigh-Ritz Method

This section presents the modal analysis solution of the laminated composite plate with structurally integrated piezo-patches using Rayleigh-Ritz method. The natural frequencies and corresponding mode shapes are calculated for the fully clamped (CCCC) boundary conditions. The displacement of the system can be defined in terms of the assumed modes [Yoon et al, 2016]:

$$w(x, y, t) = \sum_{i=1}^N \sum_{j=1}^N U_{ij}W_{ij}(x, y)\mu_{ij}(t) \quad (3.17)$$

where $\mu_{ij}(t)$'s are the generalized modal coordinates, N is the total number of vibration modes in x and y coordinates. Assumed modes are indicated by $U_{ij}W_{ij}(x, y)$, where $W_{ij}(x, y)$'s are the trial functions satisfying the boundary conditions, and U_{ij}

's are the corresponding coefficients. The assumed mode shapes of the structure can be written in terms of the orthogonal polynomials in the x and y directions as [Bhat, 1985]:

$$W_{ij}(x, y) = \phi_i(x) \cdot \varphi_j(y) \quad (3.18)$$

The orthogonal polynomials are generated by using Gramm-Schmidt procedure for the interval $0 \leq x < a$ [Bhat, 1985]:

$$\phi_2(x) = (x - B_2)\phi_1(x), \quad \phi_r(x) = (x - B_r)\phi_{r-1}(x) - C_r\phi_{r-2}(x) \quad (3.19)$$

where

$$B_r = \frac{\int_0^a x k(x) \phi_{r-1}^2(x) dx}{\int_0^a k(x) \phi_{r-1}^2(x) dx} \quad (3.20)$$

$$C_r = \frac{\int_0^a x k(x) \phi_{r-1}(x) \phi_{r-2}(x) dx}{\int_0^a k(x) \phi_{r-2}^2(x) dx} \quad (3.21)$$

here, $k(x)$ is the weighting function which is chosen as unity. Coefficients of the orthogonal polynomials are determined to satisfy the orthogonality of the polynomials [Bhat, 1985]:

$$\int_0^a \phi_r(x) \phi_f(x) dx = \begin{cases} 0, & \text{if } r \neq f \\ 1, & \text{if } r = f \end{cases} \quad (3.22)$$

The first term $\phi_1(x)$ of the orthogonal polynomials is for the fixed-fixed boundary condition of a rectangular beam with length a is given by [Yoon et al.,2016]:

$$\phi_1(x) = a^2 x^2 - 2ax^3 + x^4, \quad (0 \leq x \leq a) \quad (3.23)$$

The same procedure can be repeated for generating orthogonal polynomials $\varphi_j(y)$ in y direction for the interval $0 \leq y < b$.

Here, depending on other homogeneous boundary conditions (e.g., free, clamped and simply-supported) of each edge, different polynomials can be generated using the

same procedure. Moreover, for more complex boundary conditions, instead of using orthogonal polynomials generated by Gramm-Schmidt process, the transverse displacement in z-direction can be expressed in terms of simple trigonometric series and the elastically restrained boundary condition of each edge which can be described in terms of fictitious springs having arbitrary stiffness [Jiang et al, 2016]. This way, each edge can have any boundary conditions including the homogenous boundary conditions. Furthermore, the terms with fictitious springs can be added to the potential energy terms such that a modal analysis can be performed with complex boundary conditions.

U_{ij} coefficients in equation (3.17) can be determined by minimizing Rayleigh's quotient with respect to the corresponding U_{ij} 's. Rayleighs quotient is given by [Meirovitch, 2001]:

$$R[U_{ij}W_{ij}(x, y)] = \frac{PE_{max}}{KE_{ref}} \quad (3.24)$$

For the minimization of Rayleighs quotient with respect to U_{ij} , the following procedure can be used [Meirovitch,2001]:

$$\frac{\partial R}{\partial U_{ij}} = \left(\frac{\partial PE_{max}}{\partial U_{ij}} - \omega_{ij}^2 \frac{\partial KE_{ref}}{\partial U_{ij}} \right) = 0 \quad (3.25)$$

where PE_{max} and KE_{ref} are the maximum potential energy and reference kinetic energy, respectively and can be written as following:

$$PE_{max} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N K_{ij,kl} U_{ij} U_{kl} \quad (3.26)$$

$$KE_{ref} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N M_{ij,kl} U_{ij} U_{kl} \quad (3.27)$$

where $K_{ij,kl}$ is the stiffness term, and $M_{ij,kl}$ is the mass term, both of which can be obtained as:

$$\begin{aligned}
 K_{ij,kl} = & \iint_S [D_{11} + \sum_{l=1}^2 D^p_{11} P(x, y)] [\frac{\partial^2 W_{ij}}{\partial x^2} \frac{\partial^2 W_{kl}}{\partial x^2}] + \\
 & [D_{22} + \sum_{l=1}^2 D^p_{11} P(x, y)] [\frac{\partial^2 W_{ij}}{\partial y^2} \frac{\partial^2 W_{kl}}{\partial y^2}] + \\
 & [D_{12} + \sum_{l=1}^2 D^p_{12} P(x, y)] [\frac{\partial^2 W_{ij}}{\partial x^2} \frac{\partial^2 W_{kl}}{\partial y^2} + \frac{\partial^2 W_{ij}}{\partial y^2} \frac{\partial^2 W_{kl}}{\partial x^2}] \\
 & + [4D_{66} + \sum_{l=1}^2 4D^p_{66} P(x, y)] [\frac{\partial^2 W_{ij}}{\partial x \partial y} \frac{\partial^2 W_{kl}}{\partial x \partial y}] \\
 & + [2D_{16}] [\frac{\partial^2 W_{ij}}{\partial x^2} \frac{\partial^2 W_{kl}}{\partial x \partial y} + \frac{\partial^2 W_{ij}}{\partial x \partial y} \frac{\partial^2 W_{kl}}{\partial x^2}] + \\
 & [2D_{26}] [\frac{\partial^2 W_{ij}}{\partial y^2} \frac{\partial^2 W_{kl}}{\partial x \partial y} + \frac{\partial^2 W_{ij}}{\partial x \partial y} \frac{\partial^2 W_{kl}}{\partial y^2}] dS
 \end{aligned} \tag{3.28}$$

$$M_{ij,kl} = \iint_S W_{ij}(x, y) m(x, y) W_{kl}(x, y) dS \tag{3.29}$$

After calculating the mass and stiffness parameters for corresponding approximation indices, the eigenvalue equation can be written as:

$$[K_{ij,kl} - \omega_{ij}^2 M_{ij,kl}] [U_{ij}] = 0 \tag{3.30}$$

Here, assumed mode shape coefficients U_{ij} 's are the eigenvectors and natural frequencies ω_{ij} 's are square root of the eigenvalues of equation (3.30).

3.2.3 Electrical Circuit Equation

The equivalent circuit representation of the two piezo-patches connected to a resistive load is shown in Figure 3.2. Its worth noting that, if nonlinear circuits are connected to the piezo-patches, an equivalent impedance method can be used [Aghakhani and Basdogan, 2017].

Using Kirchhoffs current law, the circuit equation (Figure 3.2) can be written as

[Aridogan et al., 2016]:

$$\frac{dv(t)}{d(t)} \sum_{l=1}^2 (C_p)_l + \frac{v(t)}{R_l} = \sum_{l=1}^2 i_l^p(t) \quad (3.31)$$

where,

$$i_l^p(t) = -\theta_l \frac{\partial}{\partial t} \iint_{S_p} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) dS_p \quad (3.32)$$

$$(C_p)_l = (\bar{\varepsilon}_{33}^s) \frac{(l_p)(w_p)}{h_p} \quad (3.33)$$

$$\theta_l = \bar{e}_{31} \frac{(h_s + h_p)}{2} \quad (3.34)$$

Here, $i_l^p(t)$ is the velocity dependent current source, $(C_p)_l$ is the equivalent capacitance and θ_l is the electromechanical coupling term for the l th piezo-patch.

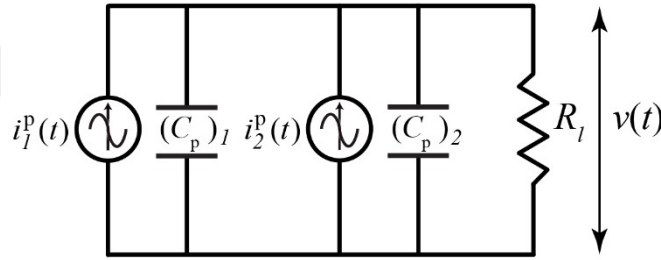


Figure 3.2: Equivalent circuit representation of the electromechanical system .

3.2.4 Steady-state Response to a Harmonic Point Force Excitation

The governing equations of the system in physical coordinates given in equation (3.16) can be transformed to modal coordinates by modal coordinate transformation. Orthogonality of the assumed mode shape coefficients U_{ij} with respect to mass and stiffness matrices results in uncoupled equations of motion after modal coordinate transformation [Yoon et al., 2016]:

$$M'_{ij,kl} = U_{ij}^T M_{ij,kl} U_{kl} = \begin{cases} 0, & \text{if } ij \neq kl \\ 1, & \text{if } ij = kl \end{cases} \quad (3.35)$$

$$K'_{ij,kl} = U_{ij}^T K_{ij,kl} U_{kl} = \begin{cases} 0, & \text{if } ij \neq kl \\ \omega_{ij}^2, & \text{if } ij = kl \end{cases} \quad (3.36)$$

where $M'_{ij,kl}$ and $K'_{ij,kl}$ are the modal mass and stiffness matrices, respectively. Applied point force in modal coordinates can be obtained as:

$$f_{rs} = \iint_S f(t) \delta(x - x_0) \delta(y - y_0) U_{rs} W_{rs}(x, y) dS = f(t) U_{rs} W_{rs}(x_0, y_0) \quad (3.37)$$

$$r=1,2,3,\dots,N \quad s=1,2,3,\dots,N.$$

The current output in modal coordinates can be obtained by substituting equation (3.17) into equation (3.32):

$$i_l^p(t) = - \sum_{r=1}^N \sum_{s=1}^N \frac{d\mu_{rs}(t)}{dt} (\tilde{\theta}_{rs})_l \quad (3.38)$$

where the electromechanical coupling term $(\tilde{\theta}_{rs})_l$ for the l th piezo-patch in modal coordinates can be written as:

$$(\tilde{\theta}_{rs})_l = \theta_l \iint_{S_p} U_{rs} P(x, y) \left(\frac{\partial^2 W_{rs}}{\partial x^2} + \frac{\partial^2 W_{rs}}{\partial y^2} \right) dS_p \quad (3.39)$$

The electromechanically coupled equations of the system in modal coordinates, thus, can be expressed as [Gohari et al., 2016]:

$$\frac{\partial^2 \mu_{rs}}{\partial t^2} + 2\omega_{rs} \zeta_{rs} \frac{\partial \mu_{rs}}{\partial t} + \omega_{rs}^2 \mu_{rs}(t) - \sum_{l=1}^2 (\tilde{\theta}_{rs})_l v(t) = f_{rs}(t) \quad (3.40)$$

$$\frac{dv(t)}{dt} \sum_{l=1}^2 (C_p)_l + \frac{v(t)}{R_l} + \sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{d\mu_{rs}(t)}{dt} (\tilde{\theta}_{rs})_l = 0 \quad (3.41)$$

Based on linear system assumption, for the harmonic force input of the form $f(t) = F_0 e^{j\omega t}$, the steady state voltage output and mechanical modal response can be

written as

$$v(t) = V e^{j\omega t} \quad (3.42)$$

$$\mu_{rs}(t) = H_{rs} e^{j\omega t} \quad (3.43)$$

where ω_{rs} , ζ_{rs} in equation (3.40) are the corresponding natural frequency and modal damping ratio. Also, V and H_{rs} are the complex amplitudes of the voltage output and the modal response, respectively. Having substituted $\mu_{rs}(t)$, $v(t)$ and f_{rs} into equation (3.40), the complex modal amplitude H_{rs} can be extracted as [Aridogan et al., 2014]:

$$H_{rs} = \frac{F_0 U_{rs} W_{rs}(x_0, y_0) + V \sum_{l=1}^2 (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega} \quad (3.44)$$

The complex voltage amplitude V can be obtained similarly by substituting equation (3.44) into equation (3.41):

$$V = \frac{-\sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{j\omega F_0 U_{rs} W_{rs}(x_0, y_0) (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega}}{j\omega \sum_{l=1}^2 (C_p)_l + \frac{1}{R_l} + \sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{j\omega (\tilde{\theta}_{rs})_l \sum_{l=1}^2 (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega}} \quad (3.45)$$

Similarly, displacement of the system in modal coordinates can be obtained by substituting equation (3.44) into equation (3.17) as:

$$w(x, y, t) = \sum_{r=1}^N \sum_{s=1}^N \frac{U_{rs} W_{rs}(x, y) [F_0 U_{rs} W_{rs}(x_0, y_0) + V \sum_{l=1}^2 (\tilde{\theta}_{rs})_l] e^{j\omega t}}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega} \quad (3.46)$$

The voltage frequency response function, which relates the voltage output to input force, can be obtained using equation (3.45) and the harmonic force input:

$$\alpha(\omega) = \frac{v(t)}{F_0 e^{j\omega t}} = \frac{-\sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{j\omega U_{rs} W_{rs}(x_0, y_0) (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega}}{j\omega \sum_{l=1}^2 (C_p)_l + \frac{1}{R_l} + \sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{j\omega (\tilde{\theta}_{rs})_l \sum_{l=1}^2 (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega}} \quad (3.47)$$

Furthermore, the displacement frequency response function, which relates the displacement output to input point force, can be obtained dividing equation (3.46) by

the harmonic force input as:

$$\beta(x, y, \omega) = \frac{w(x, y, t)}{F_0 e^{j\omega t}} = \sum_{r=1}^N \sum_{s=1}^N \frac{U_{rs} W_{rs}(x, y) [U_{rs} W_{rs}(x_0, y_0) + V \sum_{l=1}^2 (\tilde{\theta}_{rs})_l] e^{j\omega t}}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega} \quad (3.48)$$

3.2.5 Separation of Integrals Method

In the previous section, modal analysis solution of the composite plate with the piezo-patches using Rayleigh-Ritz method was presented, which required the surface integral operation. However, this operation costs significantly long computational time, especially for higher number of modes. In this section, an alternative method is proposed which relies on the separation of integrals (i.e., splitting the surface integral into two single integrals). This method drastically decreases the computation time.

Let ε and η be the normalized variables in x and y coordinates as:

$$\varepsilon = x/a \quad \eta = y/b \quad (3.49)$$

The displacement of the system in physical coordinates can be interpreted in terms of the normalized variables [Bhat, 1985]:

$$w(\varepsilon, \eta, t) = \sum_{m=1}^N \sum_{n=1}^N U_{mn} X_m(\varepsilon) Y_n(\eta) \mu_{mn}(t) \quad (3.50)$$

where $X_m(\varepsilon)$, $Y_n(\eta)$ are the orthogonal polynomials in the x and y coordinates with normalized variables. U_{mn} and $\mu_{mn}(t)$ are the corresponding coefficient and generalized modal coordinates, respectively. The indicator function $P(\varepsilon, \eta)$ can also be split into two parts:

$$P(\varepsilon, \eta) = Px(\varepsilon).Py(\eta) \Rightarrow \begin{cases} Px(\varepsilon) = [H(\varepsilon - x_1/a) - H(\varepsilon - x_2/a)] \\ Py(\eta) = [H(\eta - y_1/b) - H(\eta - y_2/b)] \end{cases} \quad (3.51)$$

Here, $Px(\varepsilon)$ and $Py(\eta)$ are the two separate Heaviside-step functions in x and y-

directions, respectively. ε and η are the normalized variables. Substituting equation (3.50) into equations. (3.26), (3.27) and (3.25), an eigenvalue equation similar to the one in equation (3.30) can be written as:

$$\sum_{m=1}^N \sum_{n=1}^N [C_{mnij} - \lambda M_{mnij}] U_{mn} = 0 \quad (3.52)$$

where $\lambda = \rho_s h_s \omega_m^2 n a^4 / D_{11}$ are the normalized natural frequencies and C_{mnij} is stiffness term of the host structure and the piezo-patches and is defined as:

$$C_{mnij} = C s_{mnij} + 2C p_{mnij} \quad (3.53)$$

where stiffness term of the host structure $C s_{mnij}$ and the piezo-patches $C p_{mnij}$ can be expressed as:

$$\begin{aligned} C s_{mnij} = & E_{mi}^{(2,2)} F_{nj}^{(0,0)} + \alpha^4 E_{mi}^{(0,0)} F_{nj}^{(2,2)} \left(\frac{D_{22}}{D_{11}} \right) \\ & + \alpha^2 [E_{mi}^{(0,2)} F_{nj}^{(2,0)} + E_{mi}^{(2,0)} F_{nj}^{(0,2)}] \left(\frac{D_{12}}{D_{11}} \right) \\ & + 2\alpha [E_{mi}^{(2,1)} F_{nj}^{(0,1)} + E_{mi}^{(1,2)} F_{nj}^{(1,0)}] \left(\frac{D_{16}}{D_{11}} \right) \\ & + 2\alpha^3 [E_{mi}^{(0,1)} F_{nj}^{(2,1)} + E_{mi}^{(1,0)} F_{nj}^{(1,2)}] \left(\frac{D_{26}}{D_{11}} \right) + 4\alpha^2 E_{mi}^{(1,1)} F_{nj}^{(1,1)} \left(\frac{D_{66}}{D_{11}} \right) \end{aligned} \quad (3.54)$$

$$\begin{aligned} C p_{mnij} = & E p_{mi}^{(2,2)} F p_{nj}^{(0,0)} \left(\frac{D_{11}^p}{D_{11}} \right) + \alpha^4 E p_{mi}^{(0,0)} F p_{nj}^{(2,2)} \left(\frac{D_{22}^p}{D_{11}} \right) \\ & + \alpha^2 [E p_{mi}^{(0,2)} F p_{nj}^{(2,0)} + E p_{mi}^{(2,0)} F p_{nj}^{(0,2)}] \left(\frac{D_{12}^p}{D_{11}} \right) \\ & + 4\alpha^2 E p_{mi}^{(1,1)} F p_{nj}^{(1,1)} \left(\frac{D_{66}^p}{D_{11}} \right) \end{aligned} \quad (3.55)$$

Here, $\alpha = a/b$ is the aspect ratio of the host structure. While M_{mnij} is the mass term of the host structure and the piezo-patches,

$$M_{mnij} = M s_{mnij} + 2M p_{mnij} \rho_p h_p / \rho_s h_s \quad (3.56)$$

where mass term of the host structure $M s_{mnij}$ and piezo-patch $M p_{mnij}$ can be written as:

$$M s_{mnij} = E_{mi}^{(0,0)} F_{nj}^{(0,0)} \quad (3.57)$$

$$Mp_{mnij} = Ep_{mi}^{(0,0)} Fp_{nj}^{(0,0)} \quad (3.58)$$

where, the terms for the host structure $E_{mi}^{(r,s)}$, $F_{nj}^{(r,s)}$ and the terms for the piezo-patches $Ep_{mi}^{(r,s)}$, $Fp_{nj}^{(r,s)}$ are defined as:

$$E_{mi}^{(r,s)} = \int_0^1 \frac{\partial^r X_m(\varepsilon)}{\partial \varepsilon^r} \frac{\partial^s X_i(\varepsilon)}{\partial \varepsilon^s} d\varepsilon \quad (3.59)$$

$$F_{nj}^{(r,s)} = \int_0^1 \frac{\partial^r Y_n(\eta)}{\partial \eta^r} \frac{\partial^s Y_j(\eta)}{\partial \eta^s} d\eta \quad (3.60)$$

$$Ep_{mi}^{(r,s)} = \int_0^1 Px(\varepsilon) \frac{\partial^r X_m(\varepsilon)}{\partial \varepsilon^r} \frac{\partial^s X_i(\varepsilon)}{\partial \varepsilon^s} d\varepsilon \quad (3.61)$$

$$Fp_{nj}^{(r,s)} = \int_0^1 Py(\eta) \frac{\partial^r Y_n(\eta)}{\partial \eta^r} \frac{\partial^s Y_j(\eta)}{\partial \eta^s} d\eta \quad (3.62)$$

where $m, i = 1, 2, 3, \dots, N$ $n, j = 1, 2, 3, \dots, N$ $r, s = 0, 1, 2$.

Eigenvalues and eigenvectors of the equation (3.52) are λ and U_{mn} , respectively. Natural frequencies of the system ω_{mn} can then be determined from equation (3.52). Note that, for harmonic response, modal mass and stiffness matrices can be obtained by substituting equations (3.53), (3.54) and (3.55) into equations (3.35) and (3.36). Applied force and electromechanical coupling term in modal coordinates can be obtained by replacing $U_{mn}X_m(\varepsilon)Y_n(\eta)$ with $U_{rs}W_{rs}(x, y)$ in equations (3.37), (3.38) and using separation of integrals. Hence, governing equations of motion can be obtained in modal coordinates. Same procedure in the previous section can be used to obtain displacement, voltage outputs of the system and their frequency response functions in modal coordinates.

3.3 Finite Element Method

Commercial finite element analysis (FEA) software ANSYS is used for validating the modal and harmonic response solutions obtained by using the analytical model. The laminated composite plate is meshed with 20-node structural solid elements (SOLID 186). The piezo-patches are meshed with 20-node coupled field solid elements (SOLID 226). The host structure and the piezo-patches are bonded by generating constraint equations at the interface [Bayik et al., 2016]. External resistive load connected to piezo-patches is modeled with piezoelectric circuit element (CIRCU 94). Natural

frequencies and mode shapes are obtained by modal analysis. Then, voltage and displacement FRFs are obtained using mode superposition method, considering the first thirty vibration modes.

3.4 Model Validation

In this section, several results are obtained by using the developed analytical model for laminated composite plates integrated with symmetric piezo-patches. Initially, validation studies are performed on the analytical model to validate its accuracy. Natural frequencies, corresponding mode shapes, displacement and voltage frequency response functions of the structure are compared with ANSYS results. Finally, change of fundamental frequency with respect to piezo-patch size is obtained by the analytical model to observe effect of inertia and stiffness properties of the piezo-patch. Material properties of the composite plate and piezo-patches are given in Table 3.1 [Gohari et al., 2016; Liu et al., 1999].

Table 3.1: Geometric and material properties.

Properties	Composite Plate (T300/976 GFRP)	Piezoceramic (PZT-5A)
Length(mm)	580	72.4
Width(mm)	540	72.4
Thickness(mm)	1.9	0.267
Mass density(kg/mm ³)	1600	7750
Piezoelectric constant $\bar{e}_{31}$ (C/m ²)	-	-16.04
Permittivity constant $\bar{\epsilon}_{33}^s$ (nF/m)	-	9.56
Longitudinal elastic modulus (GPa)	$E_{11} = 150$	-
Transverse elastic mod- ulus (GPa)	$E_{22} = 9$	-
Shear elastic modulus (GPa)	$G_{12} = 7.1$	-
Reduced elastic moduli (GPa)	-	$C_{11} = 69.48$
	-	$C_{12} = 24.32$
	-	$C_{66} = 22.57$
Poisson's ratio (v_{11}, v_{12})	0.3	0.35

3.4.1 Natural Frequencies and Mode Shapes

The modal analysis solution of the surface-bonded piezo-patches integrated to the composite plate are obtained using Rayleigh-Ritz method and finite-element simulations in ANSYS. The first four resonance frequencies of the system for combinations of three piezo-patch sizes and three ply angles of the laminated plate are presented in Table 3.2. Note that, for all the cases, piezo-patches are located at the center of plate. The analytical results for the first four vibration modes agree well with the FE simulation results. For all the ply angles in the first mode as the patch area is increased from 72.4x72.4 mm² to 300x300 mm², the resonance frequency decreases. However, the same trend is not observed in the other modes and the frequency values change depending on the ply angle. Table 3.4 presents the normalized mode shapes for the first four vibration modes of the structures. The results are given for three different piezo-patch sizes and three different ply angles. As it can be observed from the graphs, the deformation shapes are affected both by the ply angle and piezo-patch size. The changes in the natural frequencies are also very significant in some of the modes depending on the patch size and the ply angle (Table 3.2).

Table 3.2: Rayleigh-Ritz and ANSYS simulation results for the first four natural frequencies (Hz) of the composite plate with two piezo-patches.

Vibration mode	Size(mm ²)	Rayleigh-Ritz			ANSYS		
		Ply angle					
		0°	45°	90°	0°	45°	90°
First mode	72.4x72.4	57.7	51.7	64.9	58.0	51.8	65.1
	150x150	53.4	49.5	59.6	53.4	49.4	59.7
	300x300	48.8	47.1	53.5	48.9	47.3	53.6
Second mode	72.4x72.4	76.9	87.5	80.4	77.4	88.0	81.0
	150x150	76.6	88.5	79.5	76.0	88.1	79.3
	300x300	78.3	93.0	77.9	76.1	92.2	76.1
Third mode	72.4x72.4	115.9	124.4	110.9	115.1	125.2	110.1
	150x150	128.3	122.7	120.8	127.1	122.8	119.4
	300x300	132.1	114.4	125.6	131.1	114.2	124.6
Fourth mode	72.4x72.4	157.3	133.3	152.5	158.4	132.7	151.1
	150x150	152.0	145.0	163.1	152.9	144.5	157.0
	300x300	133.5	147.1	148.7	132.4	145.4	149.2

3.4.2 Electromechanical FRF's

For obtaining the displacement-to-force input FRF, transverse point force with magnitude of 1N is applied at the top of the host structure with position coordinates (145, 135) mm, and the vibration output is measured at the coordinates (435, 405) mm. Figure 3.3 shows the analytical predictions for displacement FRFs with three different number of modes used in the solution. As the number of modes increase from $N = 3$ to $N = 13$, the predictions get more accurate and agree well with the finite-element simulations. Note that the number of modes correspond to both x and y directions, therefore a total of N^2 vibration modes are included in the predictions for the two-dimensional composite plate. Here, for the frequency range up to 210 Hz, increasing the number of modes to more than $N = 13$ does not improve the vibration response significantly. However, for prediction of the higher vibration modes, one can increase the number of modes until a convergence is obtained for the rest of the vibration modes.

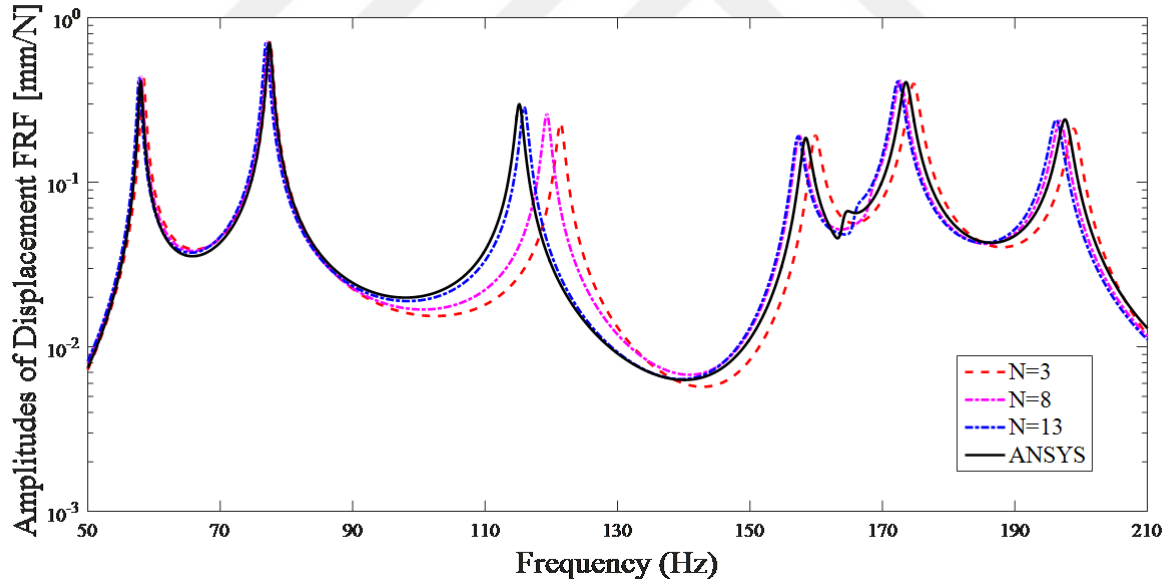


Figure 3.3: Comparison of displacement FRFs against ANSYS simulation results with variations in number of modes (N) used in the analytical model. The piezo-patch area is $A_p=72.4 \times 72.4 \text{ mm}^2$, and the ply angle is $\theta_f=0$ degrees.

For obtaining the voltage-to-force input FRF, the voltage output is measured across the resistive load. For both analytical and finite element models, the external resistance of $1\text{M}\Omega$ is used which is close to the open-circuit condition. Figure 3.4 shows the analytical predictions for voltage FRFs with three different number of

modes used in the solution. Again, as the number of modes increase from $N = 3$ to $N = 13$, the predictions converge and match well with the finite-element simulations.

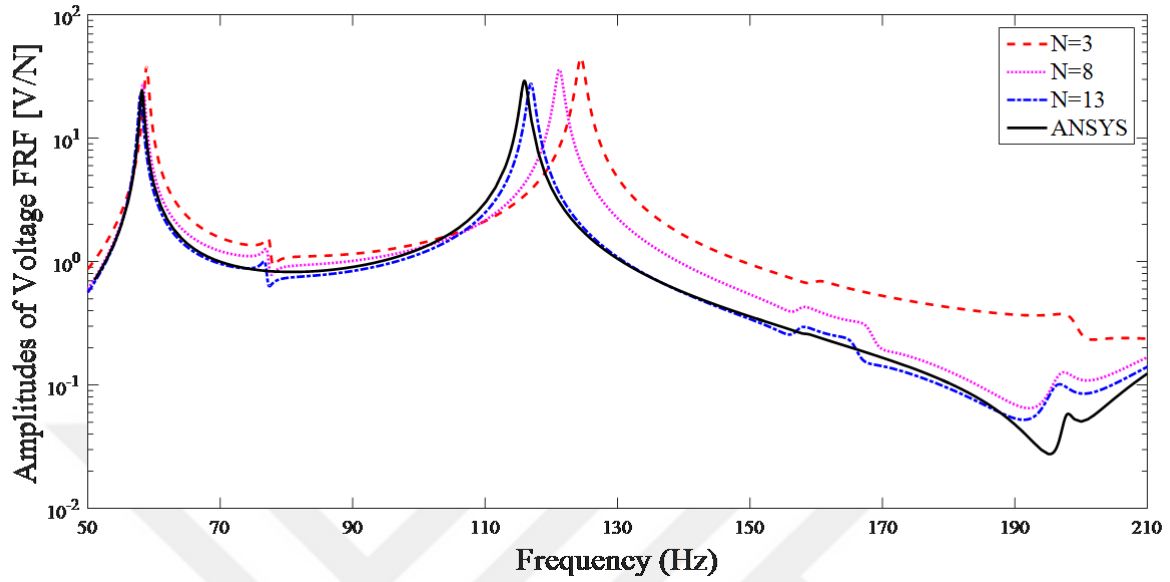


Figure 3.4: Comparison of voltage FRFs against ANSYS simulation results with variations in number of modes (N) used in the analytical model. The piezo-patch area is $A_p = 72.4 \times 72.4$ mm², and the ply angle is $\theta_f = 0$ degrees.

Note that, based on the locations of the point force input and displacement output on the plate surface, all the resonance modes are displayed in the displacement FRFs. However, for the case of voltage FRFs, the piezo-patches are attached to the center of plate and since the electromechanical coupling term (given in equation (3.39)) is directly proportional to the integral of the mode shapes over the area that is covered by the piezo-patches, there would be no peak voltage output at some modes (second, fourth and fifth modes) . The small peaks around 150-190 Hz in the analytical model, however, are due to the discrepancies in the numerical integration which results in the nonzero electromechanical coupling and thus small voltage generation at those modes.

3.4.3 Computational Cost Comparison

In this section, the computational times of original method , separation of integrals method, and ANSYS simulations are compared (See Table 3.3). Comparisons are based on different number of modes used in the harmonic analysis procedure.

Table 3.3: Comparison of computational times (seconds) for the developed methods and ANSYS.

Analysis method	Number of modes				
	3x3	5x5	8x8	10x10	13x13
ANSYS	142.3	169.1	264.1	427.1	642.2
Original method	7.9	35.3	231.9	551.4	1607.1
Separation of integrals method	3.4	6.8	19.3	35.3	81.4

From Table 3.3, the computational efficiency of separation of integrals method can be observed, especially when higher number of modes are used in the analysis. The original method has lower computational cost than ANSYS until the number of modes reaches 100, after that value, ANSYS is more efficient. However, the separation of integrals method is always faster than the other two methods. For the highest number of modes used in this study, it is 6 times faster than ANSYS and 20 times faster than the original method. Compared to the original method, with increasing number of modes, it becomes faster and more efficient. The main limitation of the separational of integrals method is its inapplicability for the analysis of composite plates with rotated surface-bonded piezo-patches, since orthogonality of the polynomials in x and y coordinates is disrupted for those cases. Therefore, the original method should

be used for the cases involving off-axis piezo-patches attached to composite plates.

3.4.4 Mass and Stiffness Contribution of Piezo-Patches

Figure 3.5a shows the effect of the piezo patch size and thickness on the fundamental natural frequency of the laminated plate with surface-bonded piezo-patches. In Figure 3.5a, x-axis shows the normalized piezo-patch size (0 is the case when there is no piezo-patch on the host structure whereas 1 is the case when the plate is fully covered by the piezo-patch); and y-axis shows the thickness ratio of the piezo-patches to the plate (total thickness of the piezo patches is used here which is twice the thickness of each patch). The length of each piezo-patch varies between 72.4 and 580 mm whereas its width varies between 72.4 and 540 mm. The thickness of each piezo-patch varies between 0.016 and 0.95 mm (half of the plate thickness). Hence, both the normalized piezo-patch size and thickness ratio are between 0.017 and 1. The ply angles of the laminated plate are kept at 0 degrees. Fundamental frequency obtained by the distributed parameter model [Xing and Liu, 2009], by neglecting the mass and stiffness contribution of the piezo-patches, is also demonstrated in Figure 3.5a. Since the distributed parameter model does not consider the stiffness and mass effects of the piezo-patches, the fundamental frequency remains constant as the piezo size and thickness are increased. However, the fundamental frequency obtained by the Rayleigh-Ritz model drops while the normalized piezo-patch size and thickness ratios are increased until the normalized piezo-patch size reaches 0.4. For the values of the normalized piezo-patch size piezo-patch size between 0.4 and 0.93, the fundamental frequency increases by increasing the piezo-patch thickness ratio.

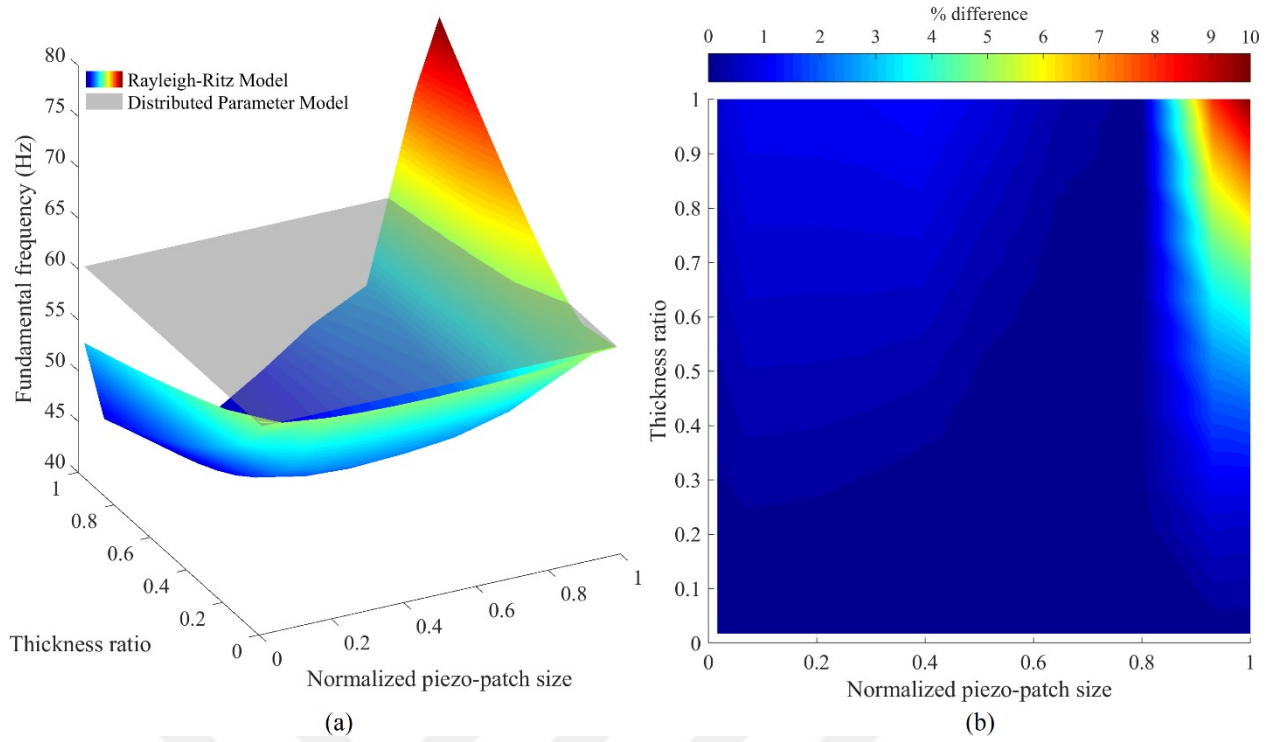


Figure 3.5: a) Fundamental frequency of the laminated plate with surface-bonded piezo-patch, as a function of normalized patch size and thickness ratio. b) Fundamental frequency difference percentage between Rayleigh-Ritz method and FEA.

Figure 3.5b illustrates the percentage difference between the fundamental frequency results obtained by Rayleigh-Ritz method and ANSYS. The fundamental frequencies predicted by the analytical model and ANSYS are in good agreement. Larger differences occur for the normalized piezo-patch sizes between 0.8 and 1 and thickness ratios between 0.6 and 1, which are due to the absence of shear deformation components in CLPT method.

The alternative methods for vibration analysis of moderately thick plates and surface-bonded patches include First Order Shear Deformation Theory (FSDT), whereas for very thick plates and piezo-patches, Higher Order Shear Deformation Theory (HSDT) can be used. Its also worth to note that, neglecting the shear deformation effects would cause CLPT method to yield inaccurate results in high frequency ranges (roughly above 500 Hz) which is not in the scope of the current study.

3.5 Conclusion

In this study, an analytical method for the design and analysis of laminated composite plates with surface-bonded piezo-patches was developed. Piezo-patches were

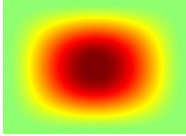
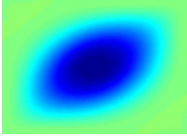
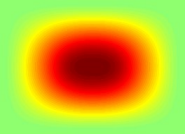
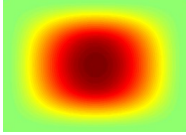
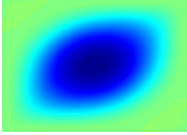
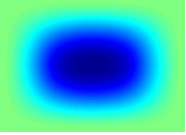
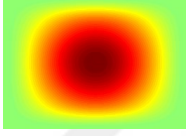
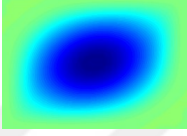
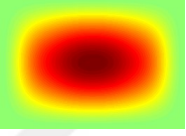
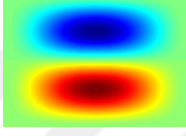
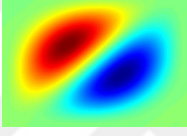
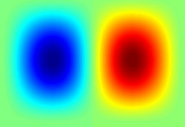
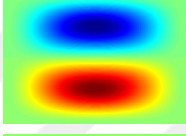
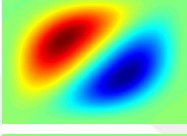
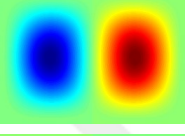
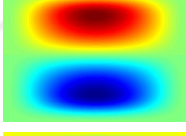
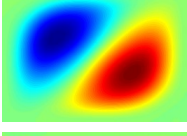
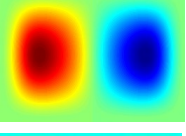
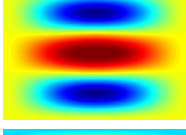
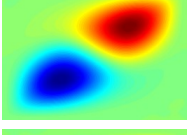
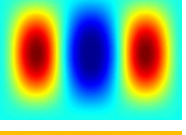
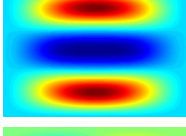
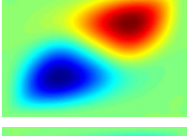
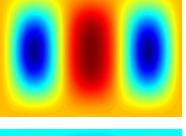
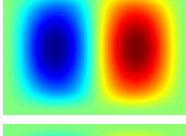
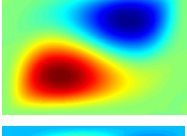
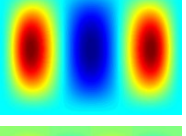
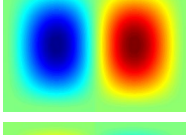
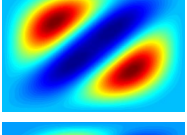
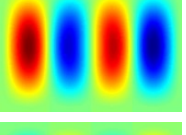
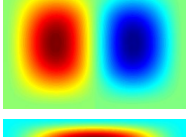
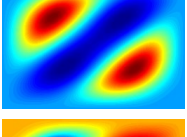
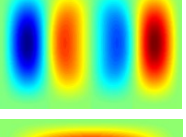
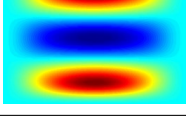
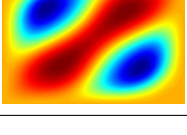
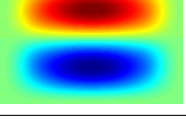
symmetrically attached to the top and bottom of the plate. Kinematic relations of the composite plate are based on Classical Laminated Plate Theory (CLPT). Hamilton's principle was used for the derivation of equations of motion whereas Rayleigh-Ritz method is used for the modal analysis and frequency response function (FRF) calculations. The developed modeling framework considers mass and stiffness properties of the piezo-patches and also the two-way electromechanical coupling between the composite plate and the patches. In addition, an accelerated solution technique was developed for reducing the computational cost.

Case studies included combined effects of the ply angle and piezo-patch size on the natural frequencies and corresponding mode shapes of the structure. It was observed that mode shapes were significantly affected by the ply angle and the piezo-patch size. In addition, displacement and voltage FRFs under transverse harmonic point force were obtained. The natural frequencies, mode shapes and the forced response of the composite plate integrated with piezo-patches were compared and validated by ANSYS results. It was shown that the developed model accurately predicted the dynamics of the composite plate with surface-bonded bi-morph piezo-patches.

Finally, the change in the natural frequency of the first mode is calculated with respect to the normalized piezo-patch size and thickness. The results of the three different approaches including the developed analytical model, ANSYS and the distributed parameter model were compared. For the first two models, as the piezo-patch size is increased, the natural frequency of the coupled system decreased until a certain value. After that value, the stiffening effect of the piezo-patch dominated the mass effect, and thus the natural frequency of the coupled system increased. Distributed parameter model fails to capture this trend which demonstrates the advantage of the developed model where the piezo mass and stiffness contributions are included.

As a result, the modeling framework developed in this study can be utilized for accurate and efficient analysis of laminated composite plates with surface-bonded piezo-patches. This model can be utilized in many other applications such as vibration control and energy harvesting where piezo patches are attached to composite laminates.

Table 3.4: The first four mode shapes of the fully clamped laminated plates with surface-bonded piezo-patches, for different ply angles (θ_f) and different patch sizes (A_p).

Vib. Mode	A_p	θ_f		
		0°	45°	90°
First mode	$72.4 \times 72.4 \text{ mm}^2$			
	$150 \times 150 \text{ mm}^2$			
	$300 \times 300 \text{ mm}^2$			
Second mode	$72.4 \times 72.4 \text{ mm}^2$			
	$150 \times 150 \text{ mm}^2$			
	$300 \times 300 \text{ mm}^2$			
Third mode	$72.4 \times 72.4 \text{ mm}^2$			
	$150 \times 150 \text{ mm}^2$			
	$300 \times 300 \text{ mm}^2$			
Fourth mode	$72.4 \times 72.4 \text{ mm}^2$			
	$150 \times 150 \text{ mm}^2$			
	$300 \times 300 \text{ mm}^2$			

Chapter 4

Investigating the Electromechanical Coupling and Shunt Damping Performance of Composite Plates with Integrated Piezo-Patches

4.1 Introduction

The use of laminated composite plates is increasing in various engineering applications due to their superior structural properties. However, such structures can be exposed to from severe vibrations which result in reduced structural life and mechanical failure. Hence, it is significant to develop damping mechanisms for suppression of unwanted vibrations on laminated composite plates. Shunt damping technique can be performed for the vibration reduction of the laminated composites by attaching or embedding piezoelectric elements. Piezoelectric layers/patches that are connected to an electrical shunt circuit (e.g. resistors and inductors) can efficiently damp the vibrations of the host structure without adding extra mass and volumetric occupancy. The shunt damping efficiency is dependent on several parameters such as the electrical coupling between the piezoelectric material and the host structure, and the parameters of the shunt circuit. Hence, efficient and accurate modeling tools are required

to include the electromechanical coupling effect and dynamics of the shunt circuit. Many analytical and finite element models are developed for vibration analysis of the laminated composite plates with integrated piezoelectric elements [Heyliger and Saravanas, 1995, Benjeddou et al., 2002, Ballhause et al., 2005]. However, there are a few studies that focus on the modeling and optimization of electromechanical coupling coefficient (EMCC) and shunt damping performance in composite laminates with integrated piezoelectric structures. In addition, there are several studies that focused on electrical circuit modeling for maximizing the shunt damping performance [Badel et al., 2006, Lallart et al., 2008, Ducarne et al., 2012]. The existing studies about the shunt damping of the smart composite plates in the literature mostly focus on either topology optimization and/or employing electrical shunt circuits [Sun et al., 2009, Han et al., 2013]. However, prior to applying these techniques, the most critical system parameters and their combined effects on the dynamics of smart composite plate must be correctly analyzed to achieve the best shunt damping performance.

This chapter focuses on determining the most significant system parameters and their combined effects on the system-level EMCC and consequently maximizing the shunt damping performance of the composite laminated plate with integrated piezoelectric patches. The most significant system parameters are selected to be ply angles of the host laminated plate, piezo-patch thickness, area, and material. The combined effects of these system parameters are usually disregarded in the existing studies. In this chapter, modal solution is obtained with the Rayleigh-Ritz method which considers the mass and stiffness contribution of the piezo-patches as well as the two-way electromechanical coupling effect [Gozum et al., 2018]. To demonstrate the accuracy of the analytical model, the first four analytical natural frequencies are validated against the finite-element simulations for a variety of piezo-patch area and thickness values. The design of experiments (DOE) method is implemented to determine the critical system parameters that significantly affect the system-level EMCC which is calculated at the first vibration mode. A full factorial design [Montgomery, 2008] with three factors is performed such that the design parameters and their combined effects can be investigated. Furthermore, the direct effect of the EMCC on the shunt damping performance is demonstrated for the weak, medium, and strongly coupled configurations. Finally, the displacement FRFs for four different piezoelectric mate-

materials with optimum resistive shunt circuits are calculated, and it is observed that the piezoelectric single crystal (PMN-PT (33%PT)) has the largest shunt damping effect on the system compared to the other piezo materials considered in this chapter.

4.2 Analytical Model of a composite plate with piezo-patches

Figure 4.1 illustrates the fully clamped composite plate with a pair of surface-bonded piezo-patches in bimorph configuration. The plate is made of glass fiber reinforced polymer (GFRP) composite material. Length, width, and thickness of the plate are denoted by a, b and t_s , respectively. Layers of the plate have the same thickness. Length, width and thickness of the piezo-patch are denoted by l_p, w_p and t_p , respectively. The pair of piezo-patches are bonded to the top and bottom surfaces of the plate which are symmetrical with respect to the mid-plane of the composite plate. They are connected in parallel to a resistive load, R_l as shown in figure 4.2(b). The equivalent circuit representation of the piezo-patches on the host plate is shown in figure 4.2, where the patches are connected in parallel to a resistive load. The vertices of the piezoelectric patches in x-y plane are denoted by (x_1, y_1) and (x_2, y_2) . A transverse harmonic point force is applied at (x_0, y_0) coordinates.

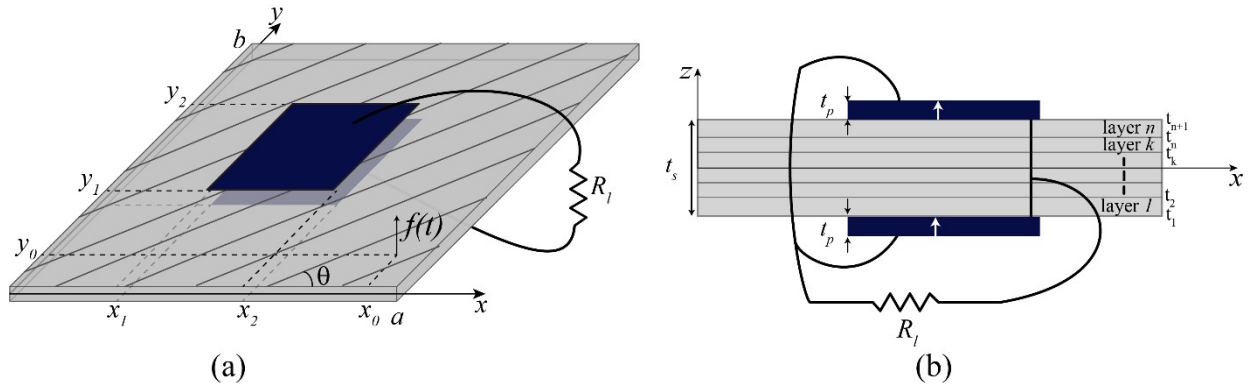


Figure 4.1: Schematics of the composite plate with integrated piezo-patches.

4.2.1 Governing equations

Governing equations of motion for the laminated composite plate with piezo-patches under harmonic point force excitation can be derived using Hamilton's principle [Gozum

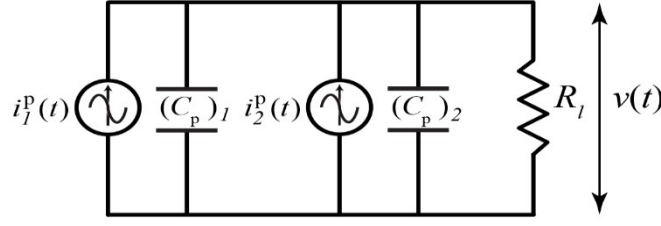


Figure 4.2: Equivalent circuit representation of the electromechanical system connected to a resistive load.

et al., 2018]:

$$\begin{aligned}
 & m(x, y) \ddot{w} + [D_{11} + 2D_{11}^p H_x(x) H_y(y)] \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \\
 & + [D_{22} + 2D_{22}^p H_x(x) H_y(y)] \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \\
 & + [2D_{12} + 4D_{12}^p H_x(x) H_y(y)] \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) \\
 & + [4D_{66} + 8D_{66}^p H_x(x) H_y(y)] \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \\
 & + [4D_{16}] \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + [4D_{26}] \left(\frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) - \\
 & 2\bar{e}_{31} v(t) H_x(x) H_y(y) \left(\frac{t_s + t_p}{2} \right) \left[\left(\frac{\partial^2 w}{\partial x^2} \right) + \left(\frac{\partial^2 w}{\partial y^2} \right) \right] \\
 & = f(t) \delta(x - x_0) \delta(y - y_0)
 \end{aligned} \tag{4.1}$$

Here, w is the displacement in the z-direction, whereas, D_{ij} 's and D_{ij}^p 's are the bending stiffness matrix elements of the laminated composite plate and piezo-patches, respectively. $H_x(x)$ and $H_y(y)$ are the Heaviside functions that are used to determine the region covered by piezo-patches in x- and y-axes, respectively. The terms $\bar{e}_{31}$ and $v(t)$ are the piezoelectric constant and the voltage, respectively; $f(t)$ is the amplitude of the point force, $\delta(x)$ and $\delta(y)$ are the Dirac delta functions along the x- and y-axes. Further, the total mass of the system $m(x, y)$ can be written as:

$$m(x, y) = \rho_s h_s + \sum_{l=1}^2 \rho_p h_p H_x(x) H_y(y) \tag{4.2}$$

where ρ_s and ρ_p are the densities of the laminated composite plate and the piezo-patches, respectively.

The electrical circuit equation (the circuit is shown in figure 4.2) is obtained by

using Kirchhoff's current law [Aridogan et al., 2016]:

$$\frac{dv(t)}{dt} \sum_{l=1}^2 (C_p)_l + v(t)Y = \sum_{l=1}^2 i_l^p(t) \quad (4.3)$$

where

$$i_l^p(t) = -\theta_l \frac{\partial}{\partial t} \iint_{S_p} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) dS_p \quad (4.4)$$

$$(C_p)_l = (\bar{\epsilon}_{33}^s) \frac{(l_p)(w_p)}{t_p} \quad (4.5)$$

$$\theta_l = \bar{e}_{31} \frac{(t_s + t_p)}{2} \quad (4.6)$$

Here, $i_l^p(t)$ is the current source, $(C_p)_l$ and θ_l are the equivalent capacitance and the electromechanical coupling term for the l th piezo-patch, $\bar{\epsilon}_{33}^s$ and S_p are the dielectric permittivity and surface area of the piezo-patches, respectively. The frequency dependent admittance of the shunt circuit composed of resistive load is given by:

$$Y = \frac{1}{R_l} \quad (4.7)$$

4.2.2 Modal Analysis by Rayleigh-Ritz Method

Rayleigh-Ritz method is used to obtain the modal analysis solution of the laminated composite plate with surface-bonded piezo-patches. The natural frequencies are calculated and presented for the fully clamped (CCCC) boundary conditions. The displacement of the system can be expressed as:

$$w(x, y, t) = \sum_{m=1}^N \sum_{n=1}^N \phi_{mn}(x, y) \mu_{mn}(t) \quad (4.8)$$

where $\phi_{mn}(x, y)$ is the assumed mode shape, $\mu_{mn}(t)$ is the generalized modal coordinates and N is the number of vibration modes in x and y coordinates. The assumed mode shapes can be expressed as [Meirovitch, 2001]:

$$\phi_{mn}(x, y) = \sum_{i=1}^N \sum_{j=1}^N U_{mni} X_i(x) Y_j(y) \quad (4.9)$$

where $X_i(x)$ and $Y_j(y)$ are the orthogonal polynomials in the x and y coordinates and U_{mnij} are the arbitrary coefficients. The orthogonal polynomials are generated using Gram-Schmidt procedure [Bhat,1985]. If the forcing and voltage terms in equation (4.1) are assumed zero, minimization of equation (4.1) with respect to U_{mnij} 's gives the eigenvalue equation [Bhat,1985]:

$$\sum_{m=1}^N \sum_{n=1}^N [C_{mnij} - \lambda M_{mnij}] U_{mnij} = 0 \quad (4.10)$$

Here, C_{mnij} , M_{mnij} are the stiffness and mass terms of the laminated composite plate and the piezo-patches. λ is the square of ω_{mn} 's, the natural frequencies, which are the eigenvalues of equation (4.10), and U_{mnij} 's are the mode shape coefficients which are the eigenvectors of equation (4.10). For the detailed derivation of equation (4.10), the reader is referred to [Gozum et al., 2018].

4.2.3 Forced Vibration Response to a Harmonic Point Force

The governing equations of motion, equation (4.1) and electrical circuit equation, equation (4.3) in physical coordinates can be transformed to modal coordinates, μ_{rs} , using the orthogonality property of the assumed mode shapes [Aridogan et al., 2014; Aghakhani and Basdogan, 2017] as:

$$\frac{\partial^2 \mu_{rs}}{\partial t^2} + 2\omega_{rs}\zeta_{rs} \frac{\partial \mu_{rs}}{\partial t} + \omega_{rs}^2 \mu_{rs}(t) - \sum_{l=1}^2 (\tilde{\theta}_{rs})_l v(t) = f_{rs}(t) \quad (4.11)$$

$$\frac{dv(t)}{dt} \sum_{l=1}^2 (C_p)_l + v(t)Y + \sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{d\mu_{rs}(t)}{dt} (\tilde{\theta}_{rs})_l = 0 \quad (4.12)$$

where the applied point force in modal coordinates $f_{rs}(t)$ is given by

$$f_{rs} = f(t)\phi_{rs}(x_0, y_0) \quad (4.13)$$

The electromechanical coupling term for the l th piezo-patch in modal coordinates

$(\tilde{\theta}_{rs})_l$ is given by:

$$(\tilde{\theta}_{rs})_l = \theta_l \int_{-\frac{l_p}{2}}^{\frac{l_p}{2}} \int_{-\frac{w_p}{2}}^{\frac{w_p}{2}} \left(\frac{\partial^2 \phi_{rs}}{\partial x^2} + \frac{\partial^2 \phi_{rs}}{\partial y^2} \right) dx dy \quad (4.14)$$

For the harmonic force input, $F_0 e^{j\omega t}$, the voltage and modal response can be written similarly as

$$v(t) = V e^{j\omega t} \quad (4.15)$$

$$\mu_{rs}(t) = H_{rs} e^{j\omega t} \quad (4.16)$$

In equation (4.11), ω_{rs} , ζ_{rs} are natural frequency and modal damping ratio, respectively. V in equation (4.15) and H_{rs} in equation (4.16) are the complex amplitudes of voltage and modal response, respectively. The complex mechanical coordinate amplitude H_{rs} can be determined by substituting equations (4.15), (4.16), (4.13) and (4.14) into equation (4.11)

$$H_{rs} = \frac{F_0 \phi_{rs}(x_0, y_0) + V \sum_{l=1}^2 (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega} \quad (4.17)$$

Then, the complex voltage amplitude V can be determined by substituting equation (4.17) into equation (4.12):

$$V = \frac{-\sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{j\omega F_0 \phi_{rs}(x_0, y_0) (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega}}{j\omega \sum_{l=1}^2 (C_p)_l + Y + \sum_{l=1}^2 \sum_{r=1}^N \sum_{s=1}^N \frac{j\omega (\tilde{\theta}_{rs})_l \sum_{l=1}^2 (\tilde{\theta}_{rs})_l}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega}} \quad (4.18)$$

Furthermore, displacement of the system in modal coordinates can be obtained by substituting equation (4.17) into equation (4.8)

$$w(x, y, t) = \sum_{r=1}^N \sum_{s=1}^N \frac{\phi_{rs}(x, y) [F_0 \phi_{rs}(x_0, y_0) + V \sum_{l=1}^2 (\tilde{\theta}_{rs})_l] e^{j\omega t}}{\omega_{rs}^2 - \omega^2 + 2j\zeta_{rs}\omega_{rs}\omega} \quad (4.19)$$

4.3 Vibration Analysis Using Analytical and Finite Element Models

In this section, a modal analysis solution is obtained using the Rayleigh-Ritz method for the laminated composite plate with piezo-patches for different patch sizes and thicknesses. The first four natural frequencies in short-circuit condition are obtained and validated against the finite element method (FEM) results. Accurate prediction of the natural frequencies is critical for the EMCC which will be discussed in the next section (Section 4). In ANSYS, the laminated composite plate is meshed with 20-node structural solid elements (SOLID 186), whereas the piezo-patches are meshed with 20-node coupled field solid elements (SOLID 226). CEINTF command which generates constraint equations at the interface is used for bonding the host structure and the piezo-patches [Bayik et al., 2016].

Material and geometric properties of the composite plate and piezo-patches are given in Table 3.1 from Chapter 3. Note that table 3.1 lists the nominal size and thickness values for the piezo-patches which are used in this chapter.

Natural frequencies of the host plate integrated with piezo patches are calculated using equation (4.10) and compared with ANSYS results, for the varying area and thickness ratios of the piezo-patches to the host structure. The area ratio is defined to be the ratio of surface area of one piezo-patch to the surface area of the host structure whereas, thickness ratio is defined to be the total thickness of the piezo-patches (twice the thickness of each piezo-patch) divided by the thickness of the host structure. The minimum size of the patches is chosen to be the nominal size ($72.4 \times 72.4 \text{ mm}^2$) while the maximum size is equal to $580 \times 540 \text{ mm}^2$ which is the case when the plate is fully covered by the piezo-patch. The minimum total thickness of the piezo-patches is chosen to be 0.4 mm of the piezo-patch which is smaller than the nominal thickness and the maximum total thickness is selected as 1.448 mm. The value of the maximum total thickness complies with the classical laminated plate theory (CLPT) assumption for the thin plates with a thickness to length ratio limit of 0.01 when the minimum size of piezo-patch is used (nominal size). Note that, for all the cases, the piezo patches are located at the center of the host plate since the shunt performance is evaluated at the first mode of the composite plate.

The first column in figure 4.3 illustrates the change of first four natural frequencies obtained by the analytical model while the area and thickness ratios of the piezo-patches are varied. The second column shows the natural frequency differences in percentage between the analytical model and the FEM results.



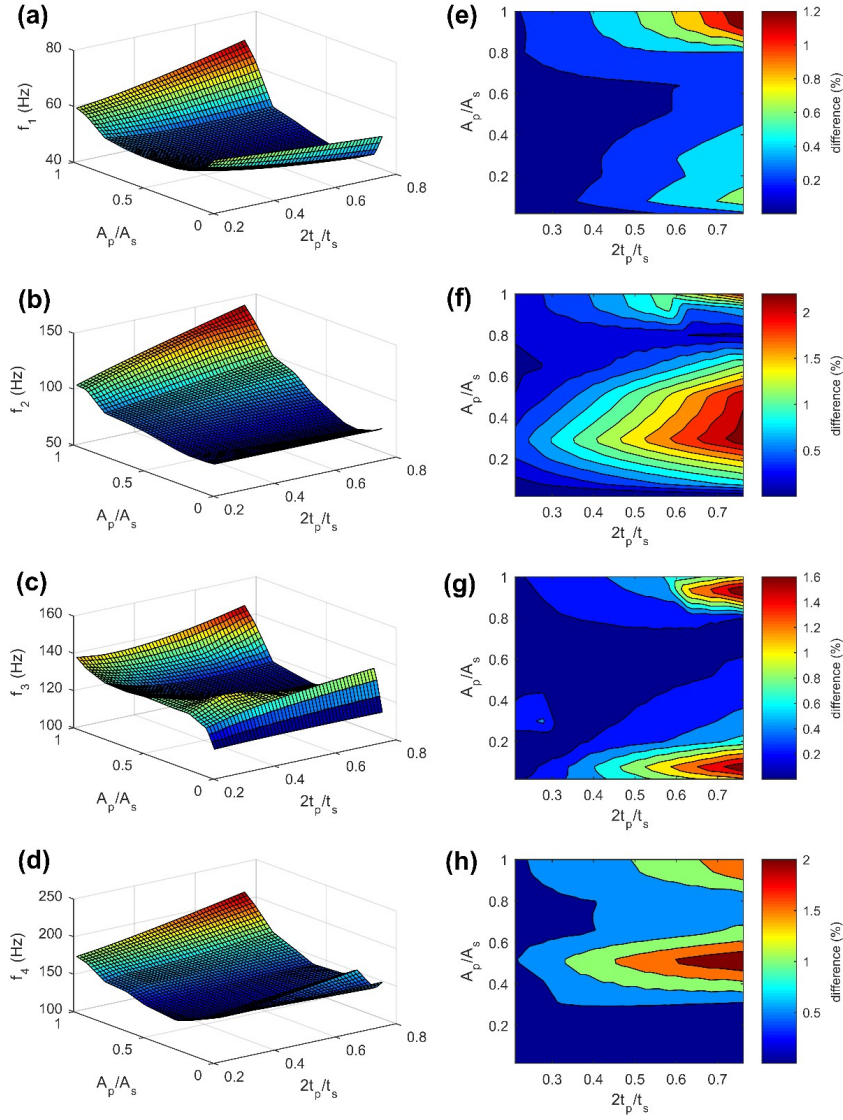


Figure 4.3: The first four natural frequencies ((a)-(d) in the left column) of the composite plate with the surface bonded piezo-patches for a set of normalized area and thicknesses. For each natural frequency, the difference percentages between the analytical and FEM results are shown in the right column (e)-(h).

The first four natural frequencies of the laminated composite plate with piezo-patches having the nominal size and thickness are 57.88, 77.09, 117.2, and 157.2 Hz. From the results presented in figure 4.3, it can be concluded that the first four natural frequencies are significantly affected by the area and the thickness ratios of the piezo-patches. Results also show that both analytical and ANSYS models are in good agreement where the maximum difference for the first four modes remains below 2.5 %. The differences between the models could be due to the discontinuities in the analytical model, caused by the piezoelectrically induced moments at the edges

of the patches, which are reduced by increasing the number of expansion modes in the simulations.

4.4 Investigation of critical system parameters and their combined effects on the Electromechanical Coupling Coefficient

4.4.1 Electromechanical Coupling

In this section, the system-level electromechanical coupling coefficient, a metric for measuring how strong the piezo-patches are coupled to the host laminated plate, is calculated. The system-level EMCC which quantifies the energy conversion between electrical and mechanical domains can be obtained using the following expression [Trindade and Benjeddou,2009]:

$$k_e^2 = 1 - \frac{\omega_{sc}^2}{\omega_{oc}^2} \quad (4.20)$$

Here, ω_{sc} is the short-circuit natural frequency where the load resistance is low ($R_l \rightarrow 0$), and ω_{oc} is the open-circuit natural frequency where the load resistance is very high ($R_l \rightarrow \infty$) [Aridogan et al.,2014] and k_e^2 is the square of EMCC. In the model proposed by [Shu and Lien,2006], the range of electromechanical coupling is categorized into the weakly-, medium- and strongly-coupled cases based on the value of k_e^2 divided by the modal damping ratio.

The square of piezoelectric materials EMCC is defined as [Lesieutre and Davis,1997]:

$$k_{31}^2 = \frac{d_{31}^2}{s_{11}^E \varepsilon_{33}^T} \quad (4.21)$$

where d_{31} and s_{11}^E are the piezoelectric strain coefficient and compliance terms, and ε_{33}^T is the dielectric permittivity at constant stress. In this study, we characterize the level of electromechanical coupling by dividing the square of system-level EMCC to the square of piezoelectric materials EMCC. For different designs of the lightly

damped composite electromechanical system, value of the normalized electromechanical coupling (i.e., $(k_e^2)/(k_{31}^2)$), shows whether the electromechanical coupling of the device could reach to or even be higher than that of the piezoelectric material. This could be an effective way of measuring the level of electromechanical coupling and thus categorizing the weakly, medium, and strongly coupled systems.

Once the harmonic analysis solution is obtained, the displacement FRFs are calculated to obtain the normalized electromechanical coupling for the first mode. The displacement FRF is defined as the ratio of displacement and the applied transverse point force which can be obtained via dividing equation (4.19) by $f(t)$:

$$\beta(x, y, \omega) = w(x, y, t)/f(t) \quad (4.22)$$

Similarly, voltage FRF can be obtained by:

$$\alpha(\omega) = V(t)/f(t) \quad (4.23)$$

The displacement FRFs are calculated with a transverse point force with an amplitude of 1 N, which is applied at the top of the host structure with position coordinates (145,135) mm and the displacement is measured at the coordinates (435,405) mm. The modal damping ratio of 0.005 is used which represents the lightly-damped case for the composite plate structures. Location of the center for the piezo-patches in the x-y plane is chosen to be at the middle of the laminated composite plate which is the optimum location for exciting the first mode of the fully clamped plate [Aridogan et al.,2014].

4.4.2 Effects of Piezo-Patch and Host Plate thicknesses on the Fundamental Frequency and Normalized Electromechanical Coupling

In order to provide an insight for the upcoming design studies that uses the piezo to plate thicknesses ratio ($2t_p/t_s$), and also investigate the thickness effects of the laminated plate and piezo-patches on the dynamics of the electromechanical system, the fundamental frequency, and normalized electromechanical coupling are calculated using the material properties in table 3.1 (Chapter 3). The analysis is repeated for three different piezo-patch areas which are $72.4 \times 72.4 \text{ mm}^2$, $150 \times 150 \text{ mm}^2$ and 400×400

mm^2 and the results are shown in figure 4.4. Ten different values are used for thickness values of the piezo-patches ($2t_p$) and for the laminated plate (t_s) between 0.5 mm and 3.5 mm.

Its clear that the host plate thickness is a dominant factor in changing the natural frequency compared to the piezo-patch thickness (see figures 4.4(a)-(c)). However, the normalized electromechanical coupling contours in figures 4.4(d)-(f) (calculated using equations (4.20) and (4.21)) show that both the thicknesses of piezo-patches and plate have important roles. In other words, the normalized electromechanical coupling values seem to remain constant along the equal piezo-to-plate ratios. For example, by focusing on the dashed black line which indicates the piezo-to-plate ratio of one ($2t_p/t_s = 1$), it can be seen that the normalized electromechanical coupling remains the same for all three-different piezo-patch areas (see figures 4.4 (d)-(f)). Moreover, for the case with smallest piezo-patch area (figure 4.4(d)), the maximum value of the normalized electromechanical coupling lies on the regions close to the dashed line; for the case with medium piezo-patch area (figure 4.4(e)), it remains below the dashed line; and for the case with the largest piezo-patch area (figure 4.4(f)), it remains above the dashed line.

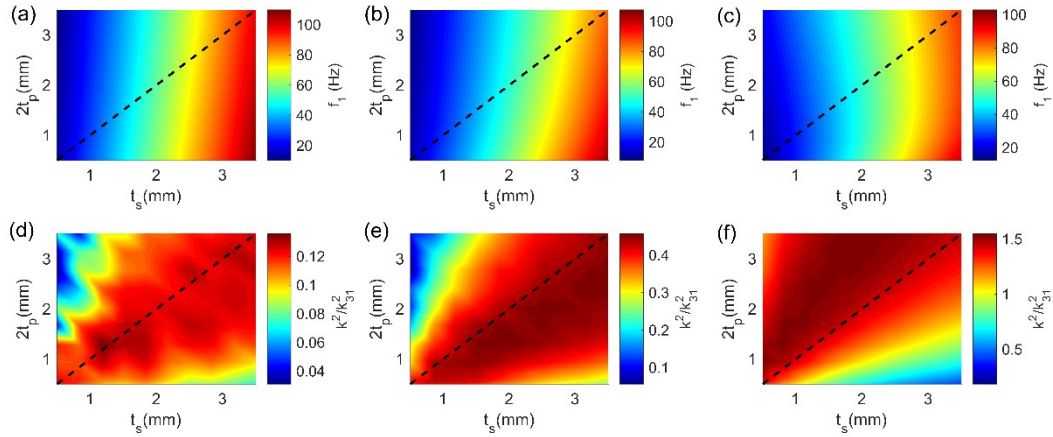


Figure 4.4: The fundamental natural frequency (the first row) and the normalized electromechanical coupling (the second row) contours of the composite electromechanical system calculated for different thickness values of piezo-patch and laminated plate, and for three different piezo-patch areas of (a),(d) $A_p=72.4 \times 72.4 \text{ mm}^2$; (b),(e) $A_p=150 \times 150 \text{ mm}^2$; (c),(f) $A_p=400 \times 400 \text{ mm}^2$.

4.4.3 Investigation of the Most Critical System Parameters

In this section, the critical system parameters and their combined effects are investigated to maximize the system-level EMCC and consequently the shunt damping performance. The system parameters are the geometric and material properties of the plate and the piezo-patches. The geometric properties (e.g., the size and thickness) of the host laminated plate is kept constant (please refer to the Table 3.1 (Chapter 3) for the dimensions), whereas its ply angle is included in the analysis as a design parameter. For the piezo-patches, the normalized thickness and area ratios and material types are used as the design parameters. In order to understand the effects of the material type of the patch, four different materials (please see Table 4.1 for different materials) are investigated.

Table 4.1: Different piezoelectric ceramic and single crystal material properties [Erturk and Inman, 2011 ; Cao et al.,2004].

Piezoelectric material	ρ_p (kg.m ⁻³)	C_{11} (GPa)	C_{12} (GPa)	C_{66} (GPa)	e_{31}^- (C/m ²)	$\bar{\epsilon}_{33}^-$ (nF.m ⁻¹)	k_{31}^2
PZT-5A	7750	69.4	24.3	22.6	-16.0	9.6	0.118
PZT-5H	7500	66.1	19.2	23.5	-23.4	17.3	0.151
PMN-PT (30%PT)	8040	22.1	8.1	65.8	-27.8	17.8	0.236
PMN-PT (33%PT)	8060	14.9	2.4	65.8	-23.0	11.5	0.353

Table 4.2: First four resonant frequencies of the composite plate with different piezoelectric materials.

Natural Frequencies (Hz)	PZT-5A	PZT-5H	PMN-PT (30%PT)	PMN-PT (33%PT)
1 st	57.88	57.91	56.99	56.72
2 nd	77.09	77.08	76.89	76.82
3 rd	117.2	117.0	111.7	110.3
4 th	157.2	157.2	157.1	157.0

Table 4.2 shows the change in the first four natural frequencies of the system when different materials of piezo-patches having nominal size and thickness are used. As mentioned before, the analytical model used in this study includes the mass and stiffness effects of the piezo such that the shifts in the natural frequencies can be predicted when different piezo materials are used.

The parameters that maximize the normalized electromechanical coupling are de-

terminated using the full factorial design with three factors which are the normalized piezo-patch area and thickness ratios and the ply angle configuration of the host plate. For the host plate, total of eight layers with symmetric and balanced stacking sequence is assumed: $[\theta/\theta/\theta/\theta]_s$ where θ is the ply angle. Total of eleven different θ values, which range from 0 to 90 degrees with an increment of 9 degrees, are used. For the piezo-patches, total of nine and eleven different values are used for the area and thickness ratios, respectively (refer to Section 4.3 for the minimum and maximum values). Hence, total of 1089 design points are used to determine the values of the parameters that maximize the normalized electromechanical coupling and the full factorial design is repeated for each material. The values of the parameters and their corresponding normalized electromechanical coupling and system-level EMCC values for each material are summarized in Table 4.3. Based on the full factorial design study, 3D plots of the normalized electromechanical coupling with three system parameters of piezoelectric thickness and area ratio, and the host plate ply-angle for all materials are obtained and shown in figures A.1-A.4 in the Appendix. Note that, the peak values of the normalized electromechanical coupling cannot be demonstrated clearly by the 3D plots. For that reason, only the contour lines of the full factorial design study will be presented in the following sections to demonstrate the combined effects of the design parameters.

Table 4.3: The ply angle and piezo-patch area and thickness ratios values that maximize the normalized electromechanical coupling and system-level EMCC for each material.

Materials	θ°	A_p/A_s	$2t_p/t_s$	Maximum System-Level EMCC (k_e^2)	Maximum Normalized Electromechanical Coupling (k_e^2/k_{31}^2)
PZT-5A	45	0.511	0.762	0.182	1.54
PZT-5H	45	0.511	0.762	0.214	1.42
PMN-PT(30%PT)	45	0.298	0.762	0.356	1.52
PMN-PT(33%PT)	45	0.298	0.762	0.413	1.17

4.4.4 Combined Effects of Piezo-Patch Area Ratio and Composite Plate Ply Angle on Normalized Electromechanical Coupling

In this section, the combined effects of piezo-patch area and ply angle at a constant thickness of piezo-patches are investigated on the normalized electromechanical coupling. The piezo-patches with thickness ratio of 0.762 (1.448 mm) are used which is the piezo-patch thickness ratio that maximizes the normalized electromechanical coupling for PZT-5A. Figure 4.5 illustrates the normalized electromechanical coupling with respect to piezo-patch area and ply angle when the total thickness of piezo-patches is 1.448 mm.

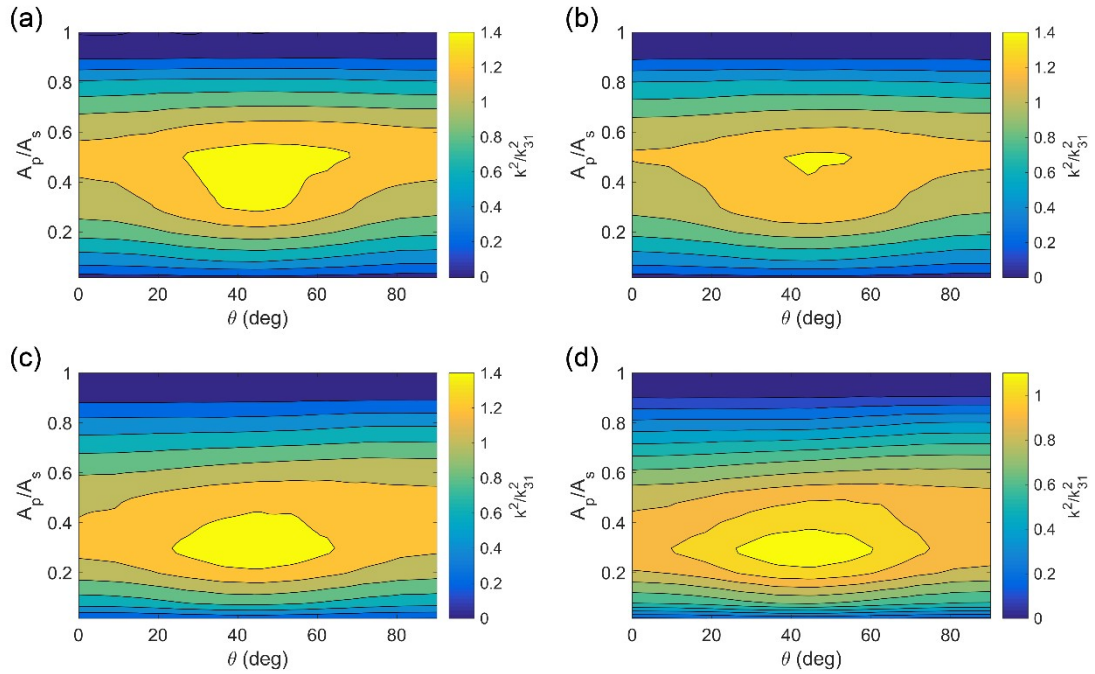


Figure 4.5: Contour plots of the normalized electromechanical coupling $(k_e^2)/(k_{31}^2)$ with respect to piezo-patch area and the plate ply angle for: (a) PZT-5A (b) PZT-5H (c) PMN-PT (30%PT) (d) PMN-PT (33%PT). The piezo-patches have the total thickness of 1.448 mm.

The maximum normalized electromechanical coupling can be achieved for the same area ratio for the piezo ceramic materials (PZT-5A and PZT-5H) and single crystal materials (PMN-PT (30%PT) and PMN-PT (33%PT)). The value of the ply angle that maximizes the normalized electromechanical coupling is 45 degrees for all materials. For the fully clamped plate, the ply angle of 45 degrees yields the most flexible (or the least stiff) composite plate [Serhat and Basdogan,2018] and thus the

piezo-patches have higher electromechanical coupling with the host structure. The average magnitude of the normalized electromechanical coupling for PMN-PT (33%) is the lowest due to its high piezoelectric material EMCC compared to the other materials (see Table 4.1 for the k_{31}^2 values).

4.4.5 Combined Effects of Piezo-Patch Thickness Ratio and Composite Plate Ply Angle on Normalized Electromechanical Coupling

In this section, the combined effects of the piezo-patch thickness and ply angle of the laminated plate at constant surface area of the piezo-patches are investigated on the electromechanical coupling for each material. The length of 400 mm is used for the length and width of the piezo-patches which corresponds to the piezo-patch area ratio of 0.511 that maximizes the normalized electromechanical coupling for PZT-5A. Figure 4.6 illustrates the normalized electromechanical coupling contour with respect to the piezo-patch area and ply angle when length and width of piezo-patch is fixed at 400 mm each.

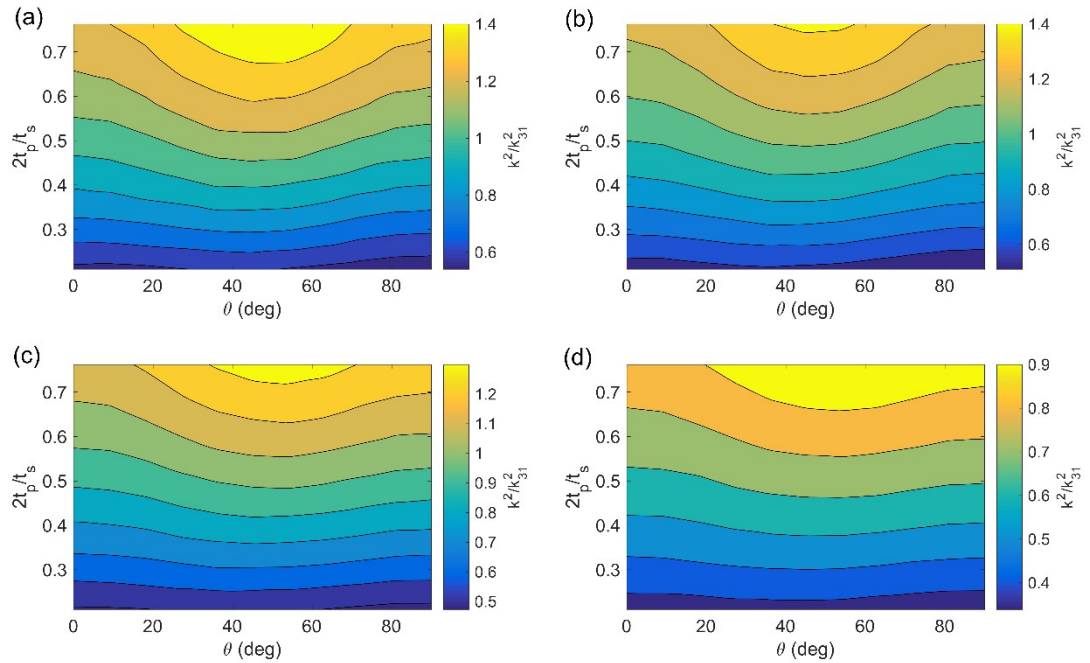


Figure 4.6: Contour plots of the normalized electromechanical coupling $(k_e^2)/(k_{31}^2)$ at the piezo-patch size of 400 mm with respect to piezo-patch thickness ratio and the plate ply angle for: (a) PZT-5A (b) PZT-5H (c) PMN-PT (30%PT) (d) PMN-PT (33%PT).

It is observed that when the thickness ratio increases, the normalized electrome-

chanical coupling increases for all the materials. This can be explained by equation (4.6) which relates the electromechanical coupling term with the reference thickness which is half of the sum of the host structure and piezo-patch thickness. The ply angle which maximizes the normalized electromechanical coupling is determined to be 45 degrees as it was in the previous section

4.4.6 Combined effects of piezo-patches thickness and area ratio on Normalized Electromechanical Coupling

In this section, the combined effects of piezo-patches area and thickness ratios are investigated on the electromechanical coupling for each piezoelectric material. The ply angle of the composite plate is fixed at 45 degrees which is the ply angle that maximizes the normalized electromechanical coupling for PZT-5A. Figure 4.7 depicts the normalized electromechanical coupling contour with respect to piezo-patches area and thickness ratios when ply angle is 45 for four different materials.

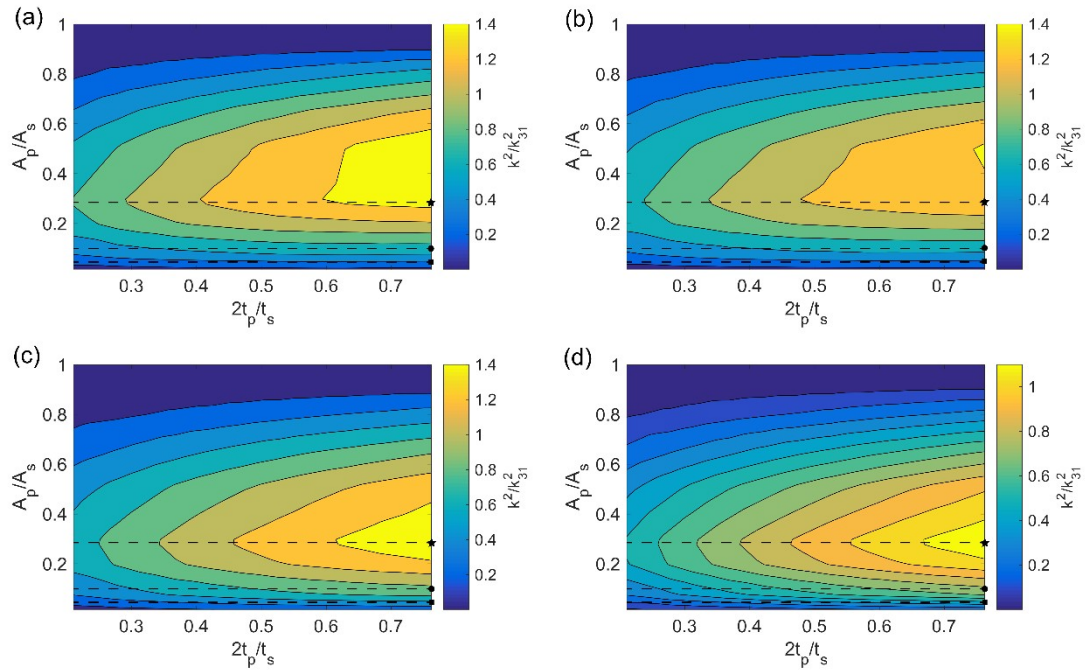


Figure 4.7: Contour plots of the normalized electromechanical coupling $(k_e^2)/(k_{31}^2)$ at ply angle of 45 degrees with respect to piezo-patch area and thickness ratios for: (a) PZT-5A (b) PZT-5H (c) PMN-PT (30%PT) (d) PMN-PT (33%PT).

The level of the electromechanical coupling is grouped into weak, medium and strong cases based on the magnitude of the normalized electromechanical coupling

obtained with PZT-5A. In this study, the weak case corresponds to the normalized electromechanical coupling values between 0 (minimum value for PZT-5A) and 0.5, the medium case corresponds to the normalized electromechanical coupling values between 0.5 and 1.0 and the strong case corresponds to the normalized electromechanical coupling values between 1.0 and 1.54 (maximum value for PZT-5A). Based on this classification, three points that correspond to the weakly, medium and strongly-coupled cases for PZT-5A are selected (see figures 4.7(a)-(d)). From figures 4.7(a) - (d) it can be observed that the prescribed points correspond to the normalized electromechanical coupling with different magnitudes for each material. Table 16 shows the prescribed points and their corresponding normalized electromechanical coupling and system-level EMCC magnitudes for each material.

Table 4.4: The selected points that correspond to the weakly, medium and strongly-coupled cases for PZT -5A and their corresponding normalized electromechanical coupling and system-level EMCC magnitudes for all materials.

Cases	Materials	(A_p/A_s)	$(2t_p/t_s)$	System-level EMCC (k_e^2)	Norm. electromechanical coupling ($k_e^2/(k_{31}^2)$)
Weak	PZT-5A	0.017	0.706	0.019	0.158
	PZT-5H	0.017	0.706	0.023	0.152
	PMN-PT(30%PT)	0.017	0.706	0.062	0.261
	PMN-PT(33%PT)	0.017	0.706	0.078	0.221
Medium	PZT-5A	0.077	0.706	0.072	0.606
	PZT-5H	0.077	0.706	0.086	0.571
	PMN-PT(30%PT)	0.077	0.706	0.197	0.836
	PMN-PT(33%PT)	0.077	0.706	0.242	0.685
Strong	PZT-5A	0.298	0.706	0.172	1.464
	PZT-5H	0.298	0.706	0.205	1.359
	PMN-PT(30%PT)	0.298	0.706	0.352	1.491
	PMN-PT(33%PT)	0.298	0.706	0.398	1.129

4.5 Piezoelectric Shunt Damping

In this section, two case studies are conducted for demonstrating effects of the normalized electromechanical coupling on piezoelectric shunt damping performance. First, the prescribed points in figure 4.7(a) which correspond to three design cases having a weak, medium and strong coupling values are analyzed and their displacement FRFs are plotted for various resistive loads (see also Table 4.4 for corresponding normalized coupling values). Then the effects of the piezo material types are investigated on the

shunt damping performance. Then, the piezo material effect on the shunt damping performance is investigated using different piezo types.

4.5.1 Effects of Normalized Electromechanical Coupling on Piezoelectric Shunt Damping

Displacement FRFs of three design configurations (figure 4.7(a)) for PZT-5A material corresponding to weak, medium and strongly- coupled cases are compared to demonstrate the effect of normalized electromechanical coupling on the shunt damping performance. The design configurations correspond to the normalized electromechanical coupling values of 0.158, 0.606 and 1.464 (Table 4.4) in ascending order. The material and geometric properties in Table 3.1 (Chapter 3) are used for the composite plate.

The location of the transverse force, its magnitude, and the modal damping ratio and measurement point of the displacement are the same as the previous section. The frequency range is selected as 45-60 Hz to focus on the shunt damping effect on the first mode. The FRF results are plotted for a set of resistive loads 100Ω - $1M\Omega$ for each configuration. Figure 4.8 illustrates the displacement FRFs for all the selected design configurations. From figure 4.8, it can be observed that for each design configuration, the lowest magnitude of the displacement FRF at the first mode is obtained at an optimum resistance value. Table 4.5 summarizes the shunt damping performance of each design configuration with their corresponding optimal resistance values.

Table 4.5: Shunt Damping performance of each design configuration with their corresponding optimal resistance values.

Cases	$f_{sc}(Hz)$	$f_{oc}(Hz)$	Freq. Shift (Hz)	Opt. Resistance (Ω)	Mimimum Displacement (dB)
Weak	56.52	57.08	0.56	10000	-70.65
Medium	49.87	51.78	1.91	2683	-79.30
Strong	46.95	51.63	4.68	1390	-90.78

Frequency shift which is the difference between f_{sc} (short-circuit natural frequency in Hz) and f_{oc} (open-circuit natural frequency in Hz) increases, and the shunt damping performance at 1st mode shows improvement with the level of electromechanical coupling. Especially, between the weak and strong cases the frequency shift was ob-

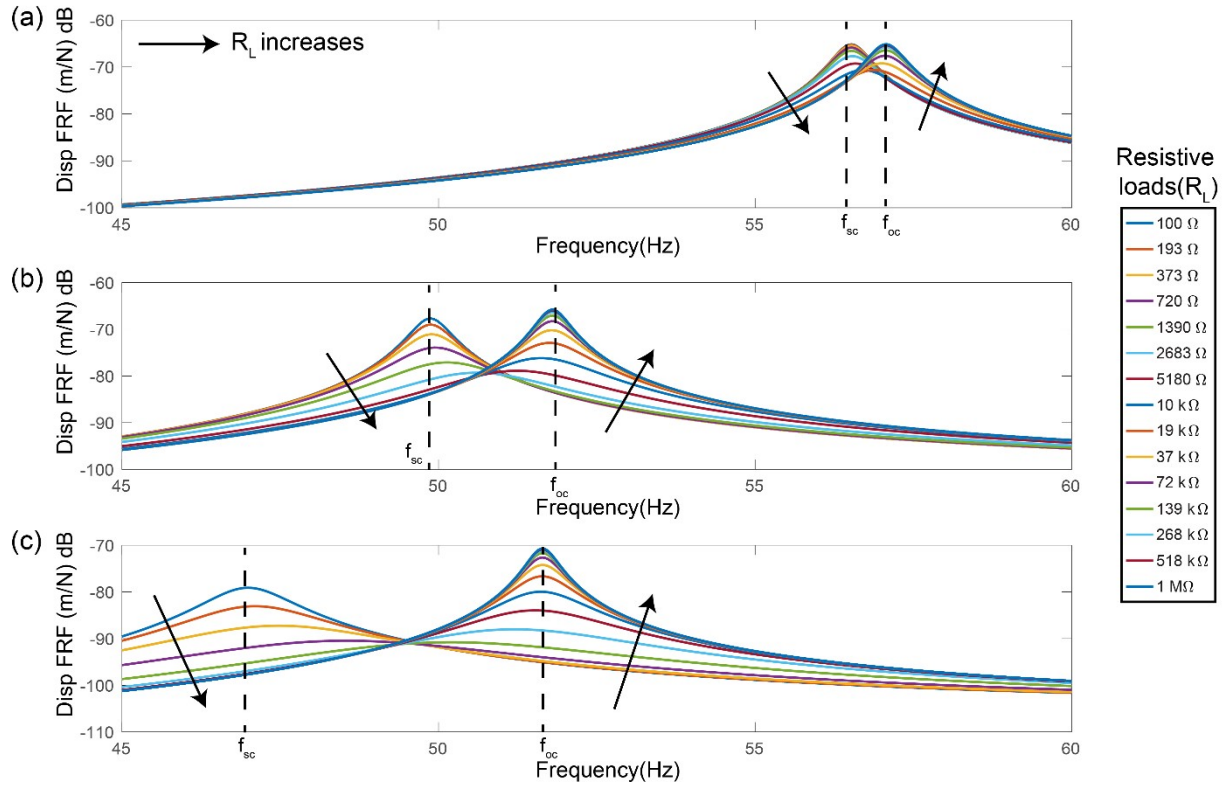


Figure 4.8: Displacement FRFs for resistive loads with the design configurations providing the normalized electromechanical coupling (k_e^2/k_{31}^2) values of: (a) 0.158 (Weak) (b) 0.606 (Medium) (c) 1.464 (Strong).

served to be around 4 Hz whereas the displacement reduction was more than 20 dB. Hence, as expected, the shunt damping gets more effective and the frequency shift increases with the increasing magnitude of normalized electromechanical coupling.

4.5.2 Effects of Material Type of Piezo-Patches on Piezoelectric Shunt Damping

In this part, material types of piezo-patches are investigated on the piezoelectric shunt damping performance. The material and geometric properties in Table 3.1 are used for the composite plate. Four different types of materials are used which are PZT-5A, PZT-5H, PMN-PT (30%PT) and PMN-PT (33%PT), respectively. The first two material types are piezoceramic materials while the last two material types are single crystal materials. Three points prescribed in figures 4.7(a)-(d) is selected for each material. All the points correspond to the weak, medium and strongly-coupled cases

for PZT-5A. For each design case, displacement FRFs are obtained using resistive circuit elements in the frequency range of 40-60 Hz to focus on the piezoelectric shunt damping performance in the first mode. The location of the transverse force, its magnitude, and the modal damping ratio and measurement point of the displacement are the same as the previous section. Set of resistive loads between 100Ω and $1M\Omega$ are used and the optimum resistance that minimizes the displacement FRF magnitude for the first mode is determined for each design case.

Figures 4.9 illustrates the displacement FRFs of each material for weak, medium and strong cases, respectively. For each material, the FRFs with the open-circuit (OC) and the optimum resistive circuit designs are compared.

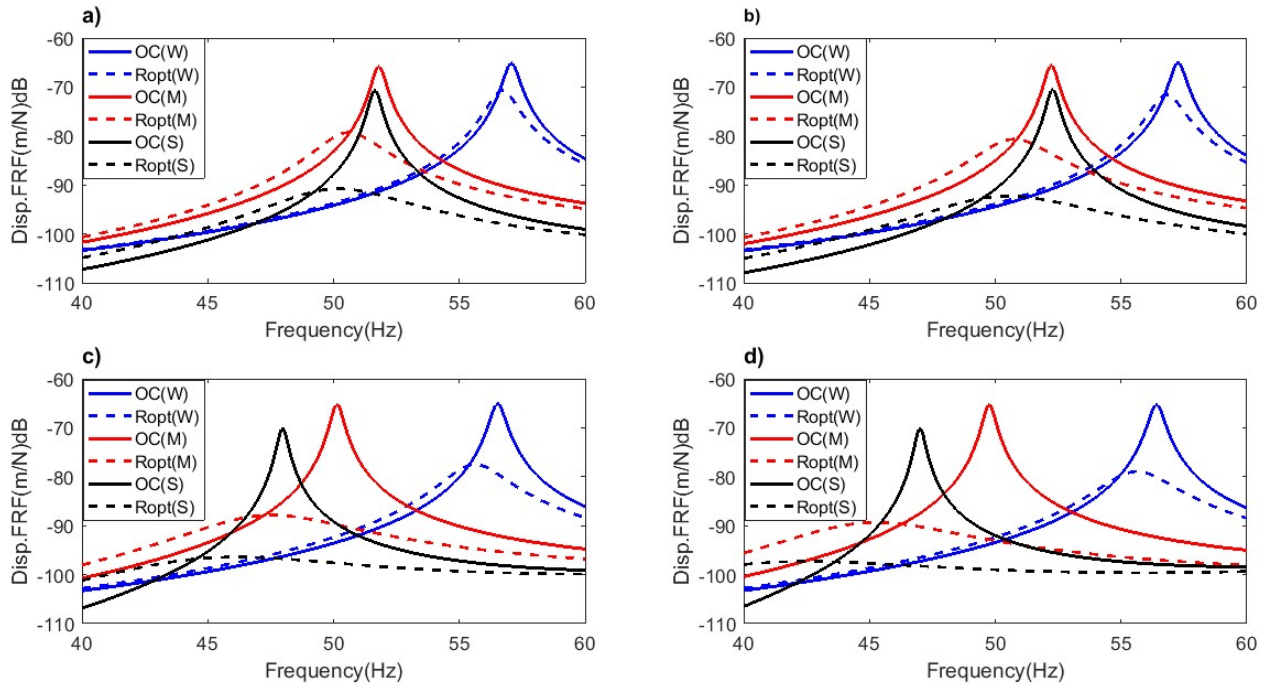


Figure 4.9: Displacement FRFs of open-circuit and optimum resistive circuit designs for weakly (W), medium (M) and strongly (S)-coupled cases (piezo-patch area and thickness ratios of 0.017, 0.077, 0.298 and 0.706, 0.706, 0.706). a) PZT-5A b) PZT-5H c) PMN-PT (30%PT) d) PMN-PT (33%PT).

From figure 4.9, it is observed that the vibration reduction performance of the optimum resistive circuit is affected by the material type of the patch and level of the coupling. The amount of displacement amplitude reduction of the optimum resistive circuit design from the open-circuit design for all materials and cases are summarized in Table 4.6.

Table 4.6: Displacement amplitude reduction (dB) of the optimum resistive circuit design from the open-circuit design for the weak, medium and strong cases using different materials.

Materials	Weak Case	Medium Case	Strong Case
PZT-5A	5.52	13.51	20.09
PZT-5H	6.50	15.08	21.76
PMN-PT(30%PT)	12.43	22.63	26.33
PMN-PT(33 %PT)	13.66	24.09	27.17

The results show that the damping performance gets better with the level of the electromechanical coupling. The vibration amplitude reduction of the single crystal materials is twice the value obtained by the piezoceramic materials for the weak case. For all cases, the amplitude reduction of the single crystal materials is larger than the piezoceramic materials. In addition, the frequency shift between the open-circuit and optimum resistive circuit designs of single crystal materials is much higher than the piezoceramic materials due to their high level system-level (EMCCC) (refer to Table 4.4) compared to the piezoceramic materials. Among the materials used, the best damping performance is obtained with PMN-PT (33%PT) for all the designs. This result is evident from the high system-level EMCC magnitudes obtained with PMN-PT (33%PT) compared to the other materials (refer to Table 4.4). Although the normalized electromechanical coupling magnitude obtained for all cases (weak, medium and strongly-coupled) is lower than the other materials, the design with PMN-PT (33%PT) has the best shunting performance due to its high piezoelectric material EMCC compared to other materials.

These results show that the normalized electromechanical coupling value can be used for identifying the coupling state (i.e., weak, medium, and strong) in such smart composite plates; whereas, the system-level EMCC is shown to be an important metric for addressing the effect of different design variables on shunt damping performance.

4.6 Conclusion

Undesired vibrations in composite laminated plates can be reduced by integrating piezoelectric patches with electrically resistive components and hence utilizing the shunt damping technique. This method enables a compact and light-weight damping mechanism as compared to the bulky mass-and-spring vibration absorber systems. In order to achieve the maximum shunt damping on the composite structure, a large system-level electromechanical coupling coefficient (EMCC) must be obtained. Therefore, in this study, the effects of various geometrical and material properties of the surface-bonded piezo-patches and the effects of geometric properties of the host plate on the system-level EMCC were investigated. Based on the previously developed electro-elastic model [Gozum et al.,2018], the numerical simulations were performed to obtain an understanding of system-level EMCC for a range of geometrical properties (piezo-patches thickness and area ratio, and the host plate ply-angle) and for four different piezoelectric materials (PZT-5A, PZT-5H, PMN-PT(30%PT), and PMN-PT(33%PT)). A full factorial design was performed to identify the critical system parameters such as the piezo-patch area, thickness and the laminated plate ply angle which have a significant effect on the electromechanical coupling and their combined effects on the electromechanical coupling. The maximum electromechanical coupling was achieved when the ply angle of the host plate was 45 degrees for all the piezoelectric materials. Three cases of weakly-, medium- and strongly-coupled cases were selected based on the values of normalized electromechanical coupling which was defined as a ratio of system-level EMCC to piezoelectric EMCC. Then, for these three cases, the shunt damping performance was demonstrated for all four piezoelectric materials by the displacement FRF curves at an optimum resistive value. It was concluded that among the piezoelectric materials used, the best damping performance was obtained by PMN-PT (33%PT) piezo-patches.

Chapter 5

Conclusion and Future Work

5.1 Mathematical Modeling of Non-Uniform Plates for Vibration Analysis

Dynamic performance of thin plates can be improved by introducing different types of discontinuities such as attaching surface-bonded discrete patches or spatially varying the stiffness and inertia properties of the plate structures. The design and analysis of these complex structures require advanced techniques. In the literature, various analytical and numerical models are developed for vibration analysis of these non-uniform plate structures. However, among these studies there is no analytical model that enables the analysis of plates having discrete (possibly off-axis or rotation) and/or continuous non-uniformities. The second chapter of this thesis addressed those problems by developing and presenting a novel semi-analytical model for dynamic analysis of non-uniform plate structures where the discontinuities were represented by Heaviside step functions. The major contributions of Chapter 2 were summarized below:

- A semi-analytical model was developed for dynamic analysis of plates having discrete (possibly off-axis) and/or continuous non-uniformities. The governing equations of motion were derived based on Hamilton's principle, the kinematic relations were based on Classical Laminated Plate Theory (CLPT) and modal analysis was performed using Rayleigh-Ritz Method;

- Discrete and continuous non-uniformities were modeled using Heaviside unit step functions. For modeling off-axis discontinuities (rotated discrete patches), a numerical technique was developed;

- The natural frequencies and mode shapes of the laminated composite plates with attached unrotated and rotated discrete patches were presented for two different thickness patches and ply angle orientations. The results were validated with FEA software ANSYS;

- Effect of the patch rotation on the natural frequencies and the mode shapes was illustrated and it was more pronounced for the patches with higher thickness;

- The natural frequencies and mode shapes of the variable-stiffness composite laminates having different boundary conditions and fiber distributions were presented. The natural frequencies obtained with the proposed model were validated with the results from the literature [Akhavan and Ribeiro, 2011];

- Providing advantages such as low computational cost and ease of implementation for various analyses that require modifications of the geometry and/or boundary conditions. In addition, compared to FEA, this model does not require time-consuming pre-processing steps such as remeshing.

For future work, the developed model can be implemented to the forced vibration analysis by considering two-way electromechanical coupling similar to the electroelastic model developed in Chapter 3. Hence, the rotation of the patch and/or fiber distributions of the variable-stiffness laminated plate on the electromechanical coupling, shunt damping and energy harvesting performances can be investigated. Furthermore, the developed model can be extended to thick plate structures by using Mindlin or first-shear deformation theory (FSDT) instead of CLPT. The developed model can also be utilized for optimization studies where many different parameters such as piezo-patch size, thickness, rotation and host plate size, thickness and fiber distributions can be rapidly modified.

5.2 Electroelastic Modeling and Shunt Damping of Thin Plates with Piezo-Patches

Piezoelectric patches/layers can be integrated into laminated composite plate structures for energy harvesting, vibration control and shunt damping applications. Hence, accurate and fast modeling tools are required for the dynamic analysis of these electromechanical systems which properly consider the coupling between the piezoelectric transducers and the host plate. In the literature, many analytical models were developed for the vibration analysis of the laminated composite plates with fully covered piezoelectric layers. However, these models cannot be utilized for modeling the piezoelectric patches that partially cover the plate surface which is significant for energy harvesting, shunt damping and vibration control applications where electromechanical composite plates with patch-based configurations are required. Considering these needs, Chapter 3 of this thesis presents an electroelastic model of a thin-laminated composite plate with surface-bonded piezo-patches that considers the mass and stiffness contribution of the patches as well as the two-way electromechanical coupling effect. In Chapter 4, the developed electroelastic model was utilized to analyze shunt damping performance of the laminated composite plate. Then, the critical system parameters (e.g. ply angle of the host plate and piezo-patch size, thickness and material type) and their combined effects on the electromechanical coupling and shunt damping performance were determined. The major contributions of these chapters were summarized below:

- An electroelastic model was developed for vibration analysis of a thin-laminated composite plate integrated with piezo-patches which considers the mass and stiffness contribution of the patches as well as the two-way electromechanical coupling effect. The governing equations of motion were derived based on Hamilton's principle, the kinematic relations were based on Classical Laminated Plate Theory (CLPT) and modal analysis was performed using Rayleigh-Ritz Method (Chapter 3);
- Closed-form expressions were obtained for the vibration and electrical frequency responses under harmonic point force excitations (Chapter 3);

- The modal analysis results and electromechanical frequency response functions (FRF's) were presented for the laminated plate with piezo patches having different ply angles and piezo-patch sizes. The results were validated with FEA Software ANSYS (Chapter 3);
- A separation of integral technique was developed for reducing the computational time of the modal and harmonic solutions. It was shown that the accelerated technique is faster than the finite element simulations (Chapter 3);
- Effects of the piezo-patch size and thickness on the fundamental frequency of the laminated composite plate were illustrated. The trend of the natural frequency with respect to the piezo-patch size and thickness was captured by the developed model which demonstrated the advantage of the model where the piezo inertia and stiffness contributions were included (Chapter 3);
- The effects of various geometrical and material properties of the surface-bonded patches and the effects of the geometric properties of the host plate on the system-level electromechanical coupling coefficient (EMCC) were investigated. The system-level EMCC was obtained utilizing the developed model (Chapter 3) for a range of geometrical properties (piezo-patches thickness and area, and the host plate ply-angle) and for four different piezoelectric materials (PZT-5A, PZT-5H, PMN-PT(30%PT), and PMN-PT(33%PT)) (Chapter 4);
- A full factorial design was performed to identify the critical system parameters such as the piezo-patch area, thickness and the laminated ply angle and their combined effects on the electromechanical coupling (Chapter 4);
- The shunt damping performance for three different cases was demonstrated depending on the level of electromechanical coupling (weakly, medium and strongly-coupled) and four different piezoelectric materials by the displacement FRF curves at optimum resistive values. The best shunt damping performance was obtained by

PMN-PT(33%PT) piezo-patches (Chapter 4).

For future work, experimental verification of the electroelastic model for the laminated plate with piezoelectric patches can be conducted. Furthermore, the electroelastic model can be extended to shell-type structures to observe the effects of the curvature on the vibration and electric responses. Another different approach can be implementing this model with multiple-piezo patches to control multiple vibration modes. Hence, the shunt damping performance of the laminated plate for multiple modes can be analyzed by using the full factorial design in Chapter 4. Moreover, the damping performance of the laminated composite plate can be improved by using nonlinear switch-based interfaces such as synchronized switch harvesting on inductor (SSHI) and synchronized switch damping (SSD) techniques. Similar to shunt damping, this model can be implemented for vibration control, structural health monitoring and energy harvesting applications.

Appendix

3D plots are obtained using the full factorial design for the normalized electromechanical coupling with system parameters (the plate ply angle, the piezo-patch area and thickness ratios) for all materials. Figures A.1-A.4 demonstrate the plots for PZT-5A, PZT-5H, PMN-PT (30%PT) and PMN-PT(33%PT), respectively.

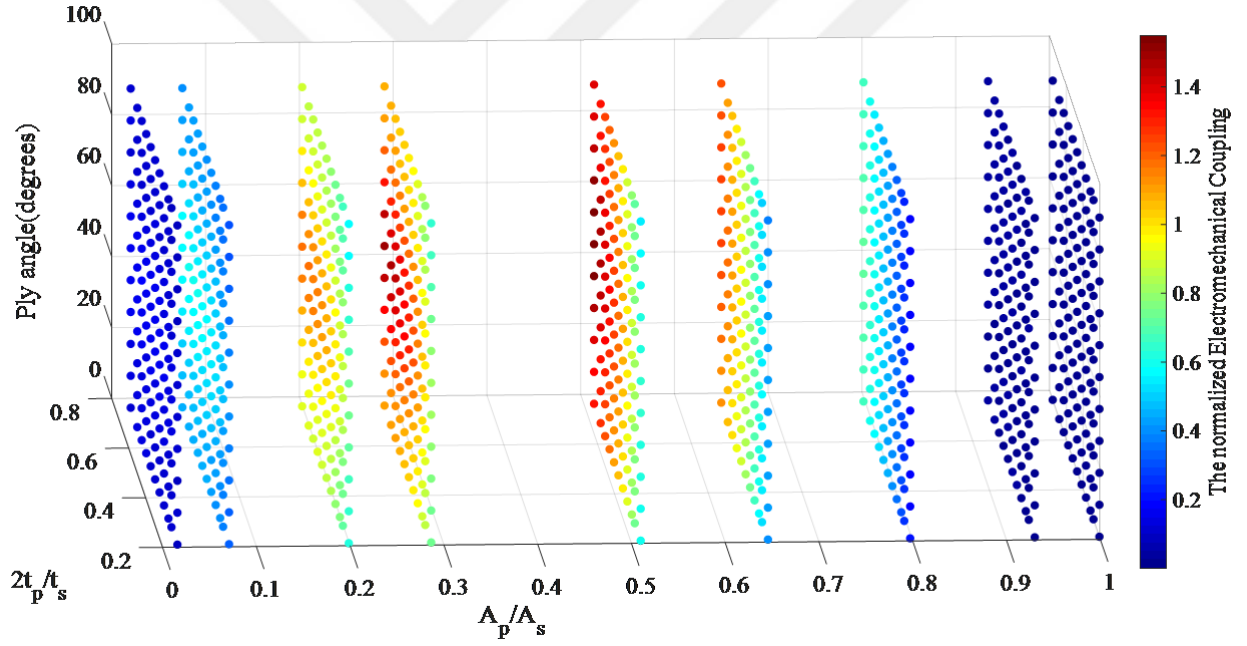


Figure A.1: The normalized electromechanical coupling with respect to the piezo-patch area and thickness ratios and the plate ply angle for PZT-5A.

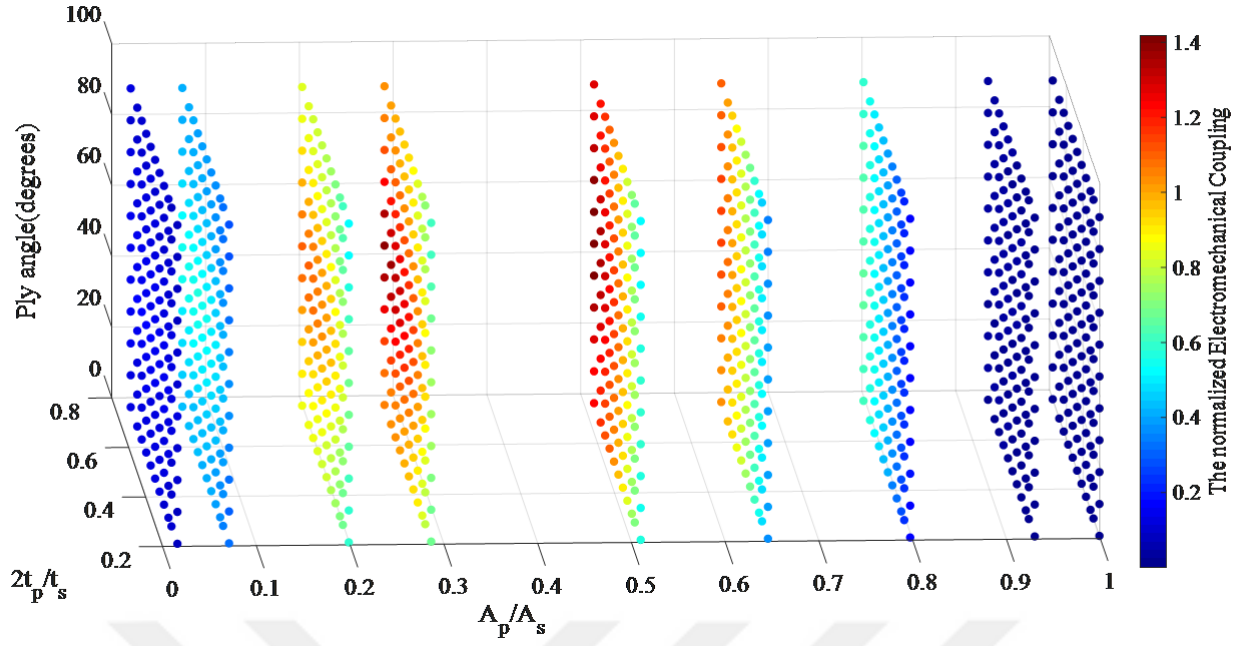


Figure A.2: The normalized electromechanical coupling with respect to the piezo-patch area and thickness ratios and the plate ply angle for PZT-5H.

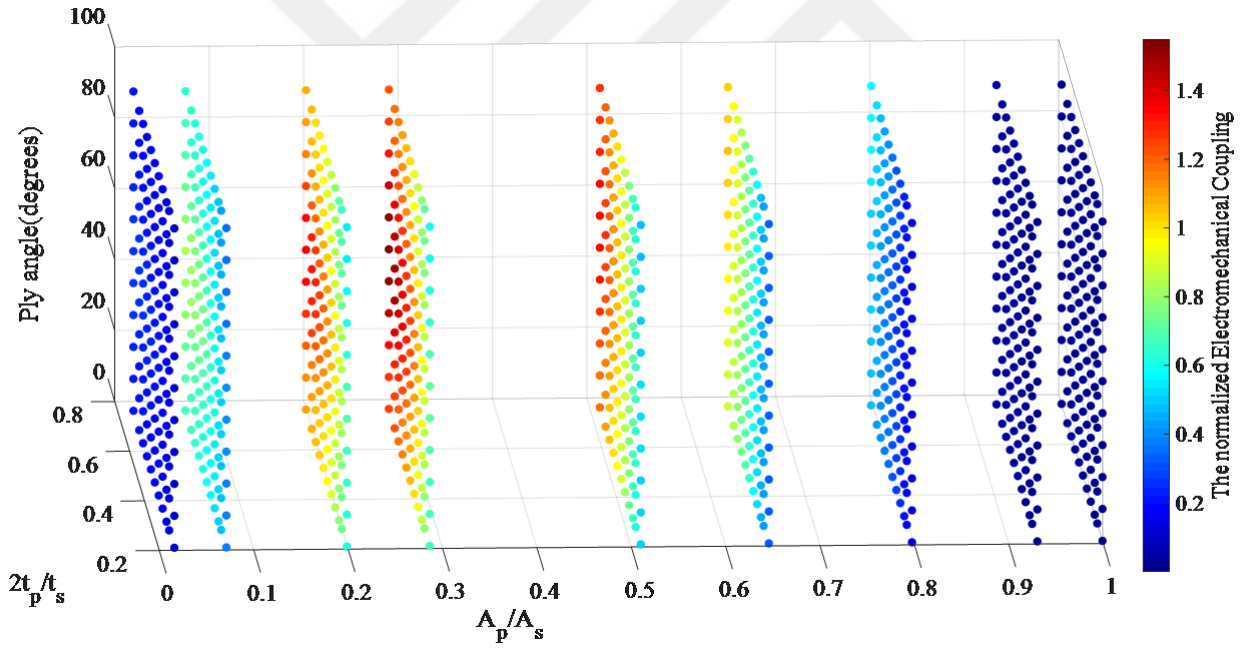


Figure A.3: The normalized electromechanical coupling with respect to the piezo-patch area and thickness ratios and the plate ply angle for PMN-PT(30%PT).

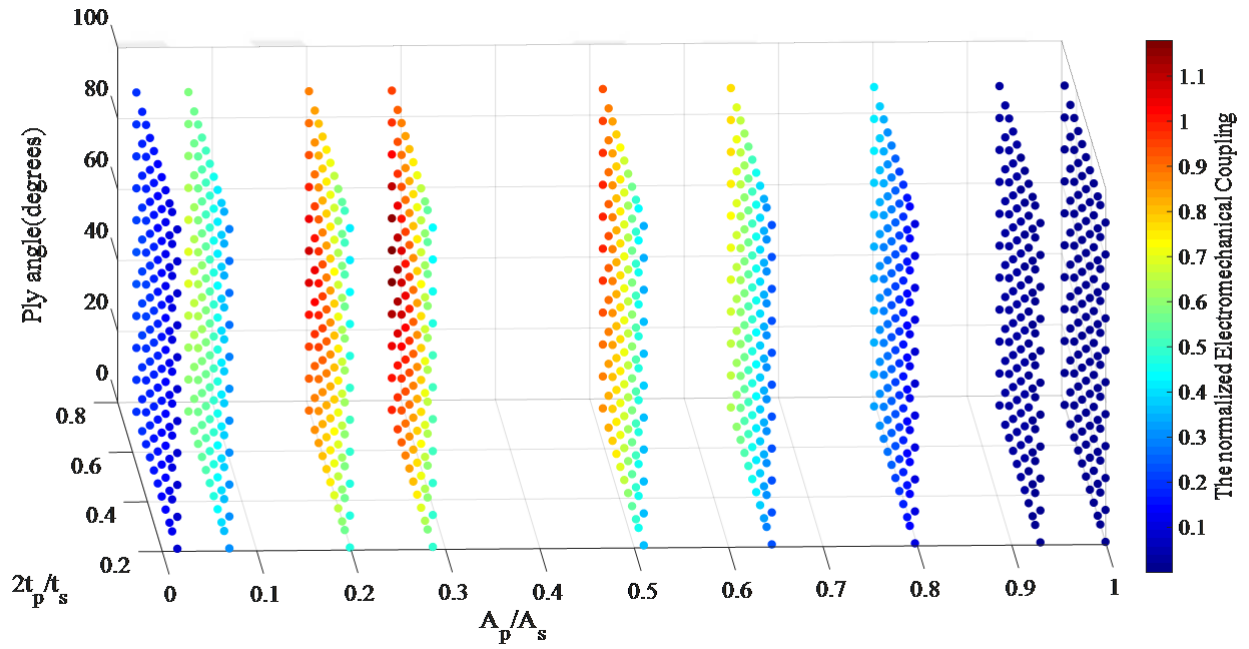


Figure A.4: The normalized electromechanical coupling with respect to the piezo-patch area and thickness ratios and the plate ply angle for PMN-PT(33%PT).

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