DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INEQUALITIES OF BLOCK MATRICES IN THE CONTEXT OF THE SCHUR COMPLEMENT

by

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İZMİR

INEQUALITIES OF BLOCK MATRICES IN THE CONTEXT OF THE SCHUR COMPLEMENT

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We have read the thesis entitled **"INEQUALITIES OF BLOCK MATRICES IN THE CONTEXT OF THE SCHUR COMPLEMENT"** completed by **AYÇA İLERİ** under supervision of **ASSOC. PROF. DR. MUSTAFA ÖZEL** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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INEQUALITIES OF BLOCK MATRICES IN THE CONTEXT OF THE SCHUR COMPLEMENT

ABSTRACT

Schur complement is a key tool in many areas of matrix analysis and a rich source for matrix inequalities. The use of the Schur complement technique has an algorithmic structure in calculating matrix inverses and linear systems. In this study we examine equalities and inequalities involving the Schur complement of the large sized matrices. For this purpose we use positive definite and positive semi definite block matrices and examine the Schur complement of their products, such as Hadamard product, Kronecker product. Also in this thesis we analyze the block form of Hadamard and Kronecker products. We make an analogy between the inequalities involving Hadarmard, Kronecker products and block form of these products.

Keywords: Schur complement, Hadamard product, Kronecker product, block Hadamard product, block Kronecker product

SCHUR TAMAMLAYICISI BAĞLAMINDA BLOK MATRİS EŞİTSİZLİKLERİ

ÖZ

Schur complement matris analizin birçok alanında önemli bir araçtır ve matris eşitsizlikleri için zengin bir kaynaktır. Schur complement tekniğinin kullanımı lineer sistemler ve matris terslerinin hesabında algoritmik bir yapıya sahiptir. Bu çalışmada büyük boyutlu matrislerin Schur tamamlayıcılarını içeren eşitlik ve eşitsizlikleri inceledik. Bu amaçla pozitif tanımlı ve yarı pozitif tanımlı blok matrisleri kullandık ve bu matrislerin Hadamard, Kronecker çarpımlarının Schur tamamlayıcılarını inceledik. Ayrıca bu tezde Hadamard ve Kronecker çarpımların blok formlarını inceledik. Hadamard ve Kronecker çarpımları içeren eşitsizliklerle bu çarpımların blok formlarını içeren eşitsizlikler arasında analoji kurduk.

Anahtar kelimeler: Schur tamamlayıcısı, Hadamard çarpımı, Kronecker çarpımı, blok Hadamard çarpımı, blok Kronecker çarpımı,

CONTENTS

CHAPTER ONE

INTRODUCTION

Linear algebra has an important place in all other branches of mathematics. And matrices are basic tools in linear algebra. Matrices are used not only in mathematics, but also in many areas including applied mathematics; such as statistics, engineering, economics. Matrix theory is a branch of linear algebra aiming to classify types of matrices, to generate new types of matrices and to determine some useful inequalities involving matrix products. There are a lot of source about matrices and matrix theory. But the books written by Horn and Johnson in 2013 and 1991 are important and main sources in the theory of matrices. In this thesis, we use these books frequently and some other sources written about the matrix analysis and the Schur complement.

Schur complement is a basic tool in the areas of numerical analysis, statistics and matrix analysis. Especially in solving linear systems of equations and finding inverse of matrices the Schur complement is very useful. Schur complement first used by Issai Schur (1875-1941) to prove his lemma (Schur's determinantal formula) in 1917. But the term Schur complement is firstly appeared in a paper written by Emilie Virginia Haynsworth in Basel Mathematical Notes. Since we partition the matrices in blocks, and use block Gaussian elimination for finding Schur complement, probably the history of the Schur complement also goes to Carl Frederick Gauss (1777-1855).

There are a lot of studies about the Schur complement. In 1917 Schur established his determinantal equality. In this formula he proved that the determinant of a matrix is the product of the determinant of its any principal submatrix with that of its Schur complement. Then, in 1937 Banachiewicz stated an inversion formula related with Schur complement. In 1969 Haynsworth proved that the Schur complement of the sum of two positive semidefinite matrices is greater than or equal to the sum of their Schur complements.

Schur complement is also used to characterize the positive (semi) definiteness of a matrix. Horn & Johnson and Fuzhen Zhang have studies about the relation between positive definiteness of a matrix and Schur complement. Zhang and Wang (1997) also proved that the Schur complement of Hadamard product of two positive definite matrices is greater than the Hadamard product of their Schur complements. Moreover, he has stated some inequalities involving the inverse of the Schur complement of matrices and that of their Hadamard products.

In this thesis we especially interested in the block Hadamard and block Kronecker products and Schur complement of block matrices. Now we will look at a brief information about how this thesis has been organized :

In Chapter 2, the definition of types of matrices and the basic elements of a matrix will be given. Also the definitions of special products of matrices such as Hadamard and Kronecker products will be introduced. Since in this thesis we will be commonly using positive (semi) definite matrices, this chapter also includes basic theorems and observations for positive (semi) definite matrices.

In chapter 3, we will give the definition of the Schur complement and some examples of Schur complement of matrices. Then we will try to examine some basic properties of the Schur complement. In addition, we will introduce some basic theorems involving Schur complement and Hadamard product. In the last of this chapter we will give some results that we have obtained concerned with the Schur complement.

In chapter 4, our focus will be on the block matrices. First we will define the block form of the Schur complement Then we will give some examples about the Schur complement in the block form. After that we will introduce the block Hadamard and block Kronecker products and we will give some lemmas and propositions about these products. Then we will construct new lemmas and theorems involving the block Hadamard and block Kronecker products. Finally we will give the definition of centrosymmetric matrices and we will establish some lemmas about the block Hadamard and block Kronecker products of the centrosymmetric matrices.

CHAPTER TWO
PRELIMINARIES

This chapter includes many basic definitions and theorems which are the essential to the development of an understanding of the subject of the thesis and its implications.

Let $\mathbb{C}^{m\times n}$ be the complex space. For $m,n \in Z^{+}$, let $M_{m,n}$ denote the space of $m \times n$ matrices with complex entries and M_n denote the space of $n \times n$ matrices with complex entries.

2.1 Definitions and Some Special Product of Matrices

In this thesis we will mostly use positive definite, positive semidefinite and block matrices. Therefore these definitions about the types of matrices are very useful for us.

Definition 2.1.1 (Meyer, 2000) Let $A = [a_{ij}]$ be a square matrix

- A is said to be a *symmetric matrix* whenever $A = A^T$
- A is said to be a *skew-symmetric matrix* whenever $A = -A^T$
- A is said to be a *Hermitian matrix* whenever $A = A^*$, that is whenever $a_{ij} = \overline{a_{ij}}$
- A is said to be a *skew-Hermitian matrix* whenever $A = -A^*$, that is whenever $a_{ij} = -\overline{a_{ij}}$

Definition 2.1.2 (Horn & Johnson, 2013) An $n \times n$ Hermitian matrix A is said to be *positive definite* if

$$x^*Ax > 0 \text{ for all nonzero } x \in \mathbb{C}^n.$$

Definition 2.1.3 (Horn & Johnson, 2013) An $n \times n$ Hermitian matrix A is said to be *positive semidefinite* if

$$x^*Ax \geq 0 \text{ for all nonzero } x \in \mathbb{C}^n.$$

In the thesis we will use the notation $A > 0$ to mean that A is positive definite and the notation $A \geq 0$.to mean that A is positive semidefinite

Definition 2.1.3 (Horn & Johnson, 1985) Let $A \in M_{m,n}(F)$ (F be any field). For index sets $\alpha \subseteq \{1, \dots, m\}$ and $\beta \subseteq \{1, \dots, n\}$, the submatrix that lies in the rows of A indexed by α and the columns indexed by β is denoted as $A(\alpha, \beta)$.

If $m = n$ and $\alpha = \beta$, then the submatrix $A(\alpha, \alpha)$ is called a *principal submatrix* of A and is denoted by $A(\alpha)$.

If the principal submatrix is on the upper left–hand corner of A, then $A(\alpha)$ is called the *leading principal submatrix* of A .

Definition 2.1.4 (Horn & Johnson, 2013) A matrix $P \in M_n$ is a *permutation matrix* if exactly one entry in each row and column is equal to 1, and all other entries are 0.

In this thesis we will be frequently dealing with some special products of matrices. Here are the definitions:

Definition 2.1.5 (Horn & Johnson, 1985) If $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{m,n}$, then the *Hadamard product* of A and B is the matrix

$$A \circ B = [a_{ij} b_{ij}]. \tag{2.1}$$

Definition 2.1.6 (Horn & Johnson,1991) The *Kronecker product* of an $m \times n$ matrix $A = [a_{ij}]$ and $p \times q$ matrix $B = [b_{ij}]$ is an $mp \times nq$ matrix denoted by $A \otimes B$ and defined to be the block matrix

$$A \otimes B = [a_{ij} B]. \tag{2.2}$$

In general, $A \otimes B \neq B \otimes A$.

2.2 Some Observations and Theorems for Positive Definite Matrices

In this section we will give some observations and theorems on positive-definite and positive-semi definite matrices.

Theorem 2.2.1 (Bhatia, 2007) Let $A \in M_n$ Then, A is positive definite if and only if it is Hermitian and all its eigenvalues are positive.

Theorem 2.2.2 (Bhatia, 2007) Let $A \in M_n$ Then, A is positive semidefinite if and only if it is Hermitian and all its eigenvalues are nonnegative.

Observation 2.2.3 (Horn & Johnson, 2013) Any principal submatrix of a positive definite matrix is positive definite.

Observation 2.2.4 (Horn & Johnson, 2013) Any principal submatrix of a positive semidefinite matrix is positive semidefinite.

Observation 2.2.5 (Horn & Johnson, 1985) The sum of any two positive definite matrices of the same size is positive definite. More generally, any nonnegative linear combination of positive definite matrices is positive definite.

Observation 2.2.6 (Horn & Johnson, 2013) Each eigenvalue of a positive definite matrix is a positive real number.

Corollary 2.2.7 (Horn & Johnson, 2013) The trace, the determinant and all principal minors of a positive definite matrix are positive.

Observation 2.2.8 (Horn & Johnson, 1985) Let $A \in M_n$ be positive definite. If $C \in M_{n,m}$,then C^*AC is positive semidefinite. Furthermore, $rank(C^*AC) = rank(C)$,so that C^*AC is positive definite if and only if C has rank m.

Corollary 2.2.9 (Horn & Johnson, 2013) If $A \in M_n$ is positive semidefinite, then so are all the powers $A^k, k = 1,2,3, \dots$

Theorem 2.2.10 (Zhang & Wang, 1997, Chollet, 1982) If $A > 0$, then

$$A^{-1}(\alpha) - [A(\alpha)]^{-1} \geq 0. \tag{2.3}$$

Theorem 2.2.11 (Horn & Johnson, 1985) If $A \in M_n$ is Hermitian, then A is positive definite if and only if $det\, A_i > 0$ for $i = 1,2, \dots, n$. More generally, the positivity of-any nested sequence of n principal minors of A (not just the leading principal minor) is necessary and sufficient for A to be positive definite.

Corollary 2.2.12 (Horn & Johnson, 1985) If $A, B \in M_n$ are positive definite, then

(a) $A \geq B$ if and only if $B^{-1} \geq A^{-1}$,

(b) If $A \geq B$, then $det\, A \geq detB$ and $trA \geq trB$.

If the matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is positive definite, then $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}^{-1}$ exists and is positive definite. By this, $(A - BC^{-1}B^*)^{-1}$ and $A - BC^{-1}B^*$ are also positive definite. Similarly, $C - B^*A^{-1}B$, A and C are positive definite. Thus, if the partitioned Hermitian matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is positive definite, we have

$$A > 0, \qquad C > 0, \qquad A > BC^{-1}B^* \text{ and } C > B^*A^{-1}B$$

(Horn & Johnson,1985).

CHAPTER THREE

SCHUR COMPLEMENT AND SOME MATRIX INEQUALITIES

This chapter provides definition and some properties of Schur complement. Also this chapter includes some inequalities of matrices involving Schur complement.

3.1 Definition and Some Properties of Schur Complement

Definition 3.1.1 (Crabtree, & Haynsworth, 1969; Zhang, 2005) Let $M \in \mathbb{C}^{m\times n}$ be a matrix partitioned as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A is square and nonsingular. Then

$$M/A = D - CA^{-1}B \quad .(3.1)$$

is called the *Schur complement* of A in M or the Schur complement of A relative to M.

Definition 3.1.2 (Zhang, 2005) Let α and β be given index sets, that is, subsets of $\{1,2,\dots,n\}$. The cardinality of index set is $|\alpha|$ and its complement is $\alpha^c = \{1,2,\dots,n\}\backslash\alpha$. If $|\alpha| = |\beta|$ and if $A(\alpha,\beta)$ is nonsingular, then $A/A(\alpha,\beta)$ is the Schur complement of $A(\alpha,\beta)$ in A and is equal to:

$$A/A(\alpha,\beta) = A(\alpha^c,\beta^c) - A(\alpha^c,\beta)(A(\alpha,\beta))^{-1}A(\alpha,\beta^c). \quad (3.2)$$

For $A/A(\alpha)$ we will mostly use the notation A/α.

Example 3.1 Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\alpha = \{2,3\}, \qquad \beta = \{1,3\}, \qquad |\alpha| = 2, \text{ and } |\beta| = 2$$

then

$$\alpha^c = \{1,4\}, \quad \text{and} \quad \alpha^c = \{2,4\}$$

Now, let us write the submatrices $A(\alpha,\beta)$, $A(\alpha^c,\beta)$, $A(\alpha,\beta^c)$, and $A(\alpha^c,\beta^c)$ according to the above choices of α and β:

$$A(\alpha,\beta) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad , \qquad A(\alpha^c,\beta^c) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A(\alpha^c,\beta) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad , \qquad A(\alpha,\beta^c) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then the Schur complement of $A(\alpha,\beta)$ is

$$A/A(\alpha,\beta) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5/4 \\ -1 & -1/4 \end{pmatrix}$$

In fact the general form (3.2) of Schur complement is equivalent to the simpler form (3.1). There are permutation matrices (Definition 2.1.4) that put $A(\alpha,\beta)$ into the upper left hand corner of A leaving the rows and columns of $A(\alpha,\beta^c)$ and $A(\alpha^c,\beta)$ in the same increasing order in A. In example (3.1) for putting $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ into the upper left hand corner , we must put the third column in the place of the second column, then the second row in the place of the first row and third row in the place of the second row. By using this process the permutation matrices can be written as follows:

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$UAV = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A(\alpha) & A(\alpha,\beta^c) \\ A(\beta^c,\alpha) & A(\alpha^c,\beta^c) \end{pmatrix}$$

then

$$UAV/A(\alpha) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5/4 \\ -1 & -1/4 \end{pmatrix}$$

is obtained. This example indicates that;

$$A/A(\alpha,\beta) = UAV/A(\alpha).$$

If $\alpha = \beta$ then the two permutations are the same and hence there exists a permutation matrix U such that

$$U^T AU = \begin{pmatrix} A(\alpha) & A(\alpha,\alpha^c) \\ A(\alpha^c,\alpha) & A(\alpha^c) \end{pmatrix}$$

and

$$(U^T AU)/A(\alpha) = A/\alpha.$$

Now we will give properties related with the Schur complement and then we will prove some of them:

Property (I) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a partitioned matrix, where A is nonsingular and $\beta \in F$ for any field F .Then,

$$(\beta M)/A = \beta(M/A). \tag{3.3}$$

Proof:

$$\beta M = \begin{pmatrix} \beta A & \beta B \\ \beta C & \beta D \end{pmatrix}$$

$$\begin{aligned}(\beta M)/A = \beta D - \beta C(\beta A)^{-1}\beta B &= \beta D - \beta C\beta^{-1}A^{-1}\beta B \\ &= \beta(D - CA^{-1}B) \\ &= \beta(M/A)\end{aligned}$$

Property (II) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a partitioned matrix, where A is nonsingular. Then,

$$(M/A)^* = M^*/A. \tag{3.4}$$

Proof: $M/A = D - CA^{-1}B,$

$$(M/A)^* = (D - CA^{-1}B)^* = D^* - B^*(A^*)^{-1}C^* = M^*/A.$$

Property (III) (Zhang & Wang, 1997) Let A> 0 . Then

$$A(\alpha)^c \geq A/\alpha. \tag{3.5}$$

Property (IV) (Zhang, 2005) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an invertible block matrix. If D is invertible, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} (M/D)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}.$$

Property (V) (Zhang, 2005) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an invertible block matrix. If A is invertible, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (M/A)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}.$$

In a similar way we can split into $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as in the following way:

Property (VI) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an invertible block matrix. If B is invertible, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & DB^{-1} \end{pmatrix} \begin{pmatrix} 0 & C - DB^{-1}A \\ B & 0 \end{pmatrix} \begin{pmatrix} B^{-1}A & I \\ I & 0 \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & -B^{-1}A \end{pmatrix} \begin{pmatrix} 0 & B^{-1} \\ (C - DB^{-1}A)^{-1} & 0 \end{pmatrix} \begin{pmatrix} -DB^{-1} & I \\ I & 0 \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & -B^{-1}A \end{pmatrix} \begin{pmatrix} 0 & B^{-1} \\ (M/B)^{-1} & 0 \end{pmatrix} \begin{pmatrix} -DB^{-1} & I \\ I & 0 \end{pmatrix}.$$

Property (VII) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an invertible block matrix. If C is invertible, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} AC^{-1} & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & C \\ B - AC^{-1}D & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & C^{-1}D \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} -C^{-1}D & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & (B - AC^{-1}D)^{-1} \\ C^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & -AC^{-1} \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} -C^{-1}D & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & (M/C)^{-1} \\ C^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & -AC^{-1} \end{pmatrix}.$$

3.2 Basic Lemmas and Theorems on Schur Complement

Theorem 3.2.1 (Zhang, 2005) **(Schur's Formula)** Let M be a square matrix partitioned as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A is nonsingular, then

$$det(M/A) = \frac{detM}{detA} \tag{3.6}$$

Schur's formula may be used to compute characteristic polynomials of block matrices. If A and C commute in M, then

$$det(M - \lambda I) = det(A - \lambda I)\, det[(M - \lambda I)/(A - \lambda I)]$$

$$= det[(A - \lambda I)(D - \lambda I) - CB]\ .$$

Now, we will try to prove this formula:

Proof : By Schur's formula

$$detM = det(M/A)detA$$

$$M - \lambda I = \begin{pmatrix} A - \lambda I & B \\ C & D - \lambda I \end{pmatrix}$$

$$det(\, M - \lambda I) = det \begin{pmatrix} A - \lambda I & B \\ C & D - \lambda I \end{pmatrix}$$

by block Gaussian elimination

$$\begin{pmatrix} A-\lambda I & B \\ C & D-\lambda I \end{pmatrix} \sim \begin{pmatrix} A-\lambda I & B \\ 0 & (D-\lambda I)-C(A-\lambda I)^{-1}B \end{pmatrix}$$

taking the determinant of two equivalent matrices

$$det\begin{pmatrix} A-\lambda I & B \\ C & D-\lambda I \end{pmatrix} = det\begin{pmatrix} A-\lambda I & B \\ 0 & (D-\lambda I)-C(A-\lambda I)^{-1}B \end{pmatrix}$$

$$\begin{aligned} det(M-\lambda I) &= det(A-\lambda I)\, det[(D-\lambda I)-C(A-\lambda I)^{-1}B] \\ &= det[(A-\lambda I)\,((D-\lambda I)-C(A-\lambda I)^{-1}B)] \\ &= det[(A-\lambda I)\,(D-\lambda I)-(A-\lambda I)C(A-\lambda I)^{-1}B] \\ &= det[(A-\lambda I)\,(D-\lambda I)-(CA-\lambda C)(A-\lambda I)^{-1}B] \\ &= det[(A-\lambda I)\,(D-\lambda I)-C(A-\lambda I)(A-\lambda I)^{-1}B] \\ &= det[(A-\lambda I)\,(D-\lambda I)-CB] \end{aligned}$$

is obtained.

Theorem 3.2.2 (Zhang, 2005) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and suppose both M and A are nonsingular. Then M/A is nonsingular and

$$M^{-1} = \begin{pmatrix} A^{-1}+A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{pmatrix}.$$

Thus, the (2,2) block of M^{-1} is $(M/A)^{-1}$:

$$(M^{-1})_{22} = (M/A)^{-1}. \tag{3.7}$$

In a similar way, if both A and D are nonsingular then,

$$M^{-1} = \begin{pmatrix} (M/D)^{-1} & -A^{-1}B(M/A)^{-1} \\ -D^{-1}C(M/D)^{-1} & (M/A)^{-1} \end{pmatrix}$$

and, if $A, B, C,$ and D are all square, invertible and have the same size

$$M^{-1} = \begin{pmatrix} (M/D)^{-1} & (C-DB^{-1}A)^{-1} \\ (B-AC^{-1}D)^{-1} & (M/A)^{-1} \end{pmatrix},$$

$$M^{-1} = \begin{pmatrix} (M/D)^{-1} & (M/B)^{-1} \\ (M/C)^{-1} & (M/A)^{-1} \end{pmatrix}.$$

Theorem 3.2.3 (Zhang, 2005) Let A be a Hermitian matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ {A_{12}}^{*} & A_{22} \end{pmatrix} \tag{3.8}$$

in which A_{11} is square and nonsingular. Then

(a) $A > 0$ if and only if both $A_{11} > 0$ and $A/A_{11} > 0$,
(b) $A \geq 0$ if and only if $A_{11} > 0$ and $A/A_{11} \geq 0$.

This theorem means that if $A \geq 0$ and $A_{11} > 0$, then

$$A/A_{11} = A_{22} - {A_{12}}^{*} {A_{11}}^{-1} A_{12} \geq 0, \text{ so } A_{22} \geq A/A_{11} .$$

By using this, we can conclude that

$$detA_{22} \geq det(A/A_{11}) = \frac{(detA)}{(detA_{11})} \geq 0.$$

Theorem 3.2.4 (Zhang, 2005) **(Fischer's Inequality)** Let A be a positive semidefinite matrix partitioned as in (3.5) . Then

$$detA \leq (detA_{11})(detA_{22}) \tag{3.9}$$

Using the Schur's formula we have

$$det\, A = detA_{11} det(A/A_{11}).$$

If A_{22} is nonsingular, then

$$detA_{11} \geq \frac{(detA)}{(detA_{22})}$$

is obtained.

Combining these results, we obtain a reversed form of the inequality (3.9) :

$$det(A/A_{11}) det(A/A_{22}) \leq detA \tag{3.10}$$

Corollary 3.2.5 (Zhang, 2005) **(Hadamard's Inequality)** Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite matrix. Then

$$detA \leq a_{11} \dots a_{nn}$$

with equality if and only if either A is diagonal or has a zero main diagonal entry (and hence a zero row and column).

Theorem 3.2.6 (Zhang, 2005) Let A be positive definite and α be a given index set. Then

$$A^2/\alpha \leq (A/\alpha)^2, \tag{3.11}$$

$$(A/\alpha)^{1/2} \leq (A^{1/2})/\alpha, \tag{3.12}$$

and

$$A^{-1}/\alpha \leq (A/\alpha)^{-1} \tag{3.13}$$

Theorem 3.2.7 (Zhang & Wang, 1997) If $A > 0$ and $B > 0$ are of the same size and are partitioned conformally, then

$$(A + B)/\alpha \geq A/\alpha + B/\alpha \tag{3.14}$$

Theorem 3.2.8 (Zhang, 2005) If $A, B \in M_n$ are positive semidefinite, then

$$det(A + B) \geq detA + detB \tag{3.15}$$

If A is positive definite, then equality holds if and only if $B = 0$

3.3 Basic Theorems on Hadamard Product and Schur Complement

Theorem 3.3.1 (Zhang, 2005) **(Oppenheim's Inequality)** Let $A = (a_{ij})$ and B be $n \times n$ positive definite matrices. Then

$$det(A \circ B) \geq (a_{11} \dots a_{nn}) detB. \tag{3.16}$$

with equality if and only if B is diagonal.

Theorem 3.3.2 (Horn, 2013, Zhang, 2005) If $A \geq 0$ and $B \geq 0$, then

$$A \circ B \geq 0. \tag{3.17}$$

Theorem 3.3.3 (R. Horn, 2013, F. Zhang, 2005) If $A > 0$ and $B > 0$, then

$$A \circ B > 0. \tag{3.18}$$

Theorem 3.3.4 (Zhang & Wang, 1997) Let *A* and *B* be n-square positive definite matrices, and let *C* and *D* be any matrices of size $m \times n$. Then

$$(C \circ D)(A \circ B)^{-1}(C \circ D)^{*} \leq (CA^{-1}C^{*}) \circ (DB^{-1}D^{*}). \tag{3.19}$$

In particular,

$$(A \circ B)^{-1} \leq A^{-1} \circ B^{-1}. \tag{3.20}$$

and

$$(C \circ D)(C \circ D)^{*} \leq (CC^{*}) \circ (DD^{*}). \tag{3.21}$$

Theorem 3.3.5 (Zhang & Wang, 1997) Let $A > 0$ and $B > 0$ be of the same size. Then

$$(A \circ B)/\alpha \geq A/\alpha \circ B/\alpha. \tag{3.22}$$

Theorem 3.3.6 (Zhang & Wang, 1997) Let $A > 0$ and let *m* be any integer. Then

$$(A^{m}/\alpha)^{1/m} \leq A/\alpha \quad \text{if} \;\; m > 0, \tag{3.23}$$

and

$$(A^{m}/\alpha)^{1/m} \geq A/\alpha \quad \text{if} \;\; m < 0. \tag{3.24}$$

In particular,

$$A^{-1}/\alpha \leq (A/\alpha)^{-1}. \tag{3.25}$$

Corollary 3.3.7 (Zhang & Wang, 1997) Let $A > 0$ and $B > 0$ be of the same size. Then

$$(A \circ B)^{-1}/\alpha \leq [(A \circ B)/\alpha]^{-1} \leq (A/\alpha \circ B/\alpha)^{-1} \leq (A/\alpha)^{-1} \circ (B/\alpha)^{-1} \tag{3.26}$$

and

$$(A\circ B)^{-1}/\alpha \le (A^{-1}/\alpha)\circ(B^{-1}/\alpha) \le (A^{-1}\circ B^{-1})/\alpha \le (A/\alpha)^{-1}\circ(B/\alpha)^{-1} \quad (3.27)$$

Corollary 3.3.8 (Zhang & Wang, 1997) Let $A > 0$ and Let $B > 0$ be of the same size. Then

$$[(A\circ B)/\alpha]^{-1} \le (A^{-1}\circ B^{-1}) \,/\, \alpha. \quad (3.28)$$

3.4 Results on Hadamard Product and Schur Complement of Matrices

Lemma 3.4.1 Let $A > 0$ and $B > 0$ are of the same size. If *m* be an integer such that $m < 0$, then

$$[(A+B)^m/\alpha]^{1/m} \ge (A+B)/\alpha \ge A/\alpha + B/\alpha. \quad (3.29)$$

Proof. Since $A > 0$ and $B > 0$ then so $A + B > 0$ .By using Theorem 3.3.6, inequality (3.24) we have

$$[(A+B)^m/\alpha]^{1/m} \ge (A+B)/\alpha$$

and from the inequality (3.14)

$$(A+B)/\alpha \ge A/\alpha + B/\alpha$$

Hence

$$[(A+B)^m/\alpha]^{1/m} \ge A/\alpha + B/\alpha. \quad \blacksquare$$

Now we will prove the Oppenheim's inequality (3.16) in a different way :

Theorem 3.4.2 (Oppenheim's Inequality) Let A and B be $n \times n$ positive definite matrices. Then

$$det(A\circ B) \ge (b_{11} \dots\ b_{nn}) detA$$

Proof. By induction, for $n = 1$ is obvious. Suppose that the assertion is true for positive definite matrices of order less than *n*. Partition A and B as follows :

$$A = \begin{pmatrix} A_{11} & \alpha \\ \alpha^* & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & \beta \\ \beta^* & b_{nn} \end{pmatrix}$$

where A_{11} and B_{11} are $(n-1)\times(n-1)$, α and β are $(n-1)\times 1$, a_{nn} and b_{nn} are 1×1 . Then by (3.18)

$$A\circ B = \begin{pmatrix} A_{11}\circ B_{11} & \alpha\circ\beta \\ \alpha^*\circ\beta^* & a_{nn}b_{nn} \end{pmatrix} > 0$$

and by the Schur's formula

$$det(A\circ B) = det(A_{11}\circ B_{11})det(a_{nn}b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta)) \qquad (3.30)$$

Since $A\circ B > 0$ then

$$a_{nn}b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta) > 0$$

Let

$$C = \begin{pmatrix} A_{11} & \alpha \\ \alpha^* & \alpha^* {A_{11}}^{-1}\alpha \end{pmatrix}$$

Then $C \geq 0$, and the Hadamard product of C and B is positive definite. That is

$$C\circ B = \begin{pmatrix} A_{11}\circ B_{11} & \alpha\circ\beta \\ \alpha^*\circ\beta^* & \alpha^* {A_{11}}^{-1}\alpha\, b_{nn} \end{pmatrix} > 0$$

and

$$\alpha^* {A_{11}}^{-1}\alpha\, b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta) > 0.$$

But,

$$a_{nn}b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta) = (a_{nn} - \alpha^* {A_{11}}^{-1}\alpha)b_{nn} + \alpha^* {A_{11}}^{-1}\alpha\, b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta) \qquad (3.31)$$

Since A and B are positive definite, then by the observation 2.2.3 and the theorem 3.2.3

$$\left(a_{nn} - \alpha^* {A_{11}}^{-1}\alpha\right)b_{nn} > 0$$

If we throw the term $\alpha^* {A_{11}}^{-1}\alpha\, b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta)$ in (3.31) then

$$a_{nn}b_{nn} - (\alpha^*\circ\beta^*)(A_{11}\circ B_{11})^{-1}(\alpha\circ\beta) \geq (a_{nn} - \alpha^* {A_{11}}^{-1}\alpha)b_{nn}$$

Taking the determinant of both sides we have

$$det(a_{nn}b_{nn} - (\alpha^* \circ \beta^*)(A_{11} \circ B_{11})^{-1}(\alpha \circ \beta)) \geq det((a_{nn} - \alpha^* {A_{11}}^{-1}\alpha)b_{nn})$$

combining the last inequality and (3.30)

$$\begin{aligned} det(A \circ B) &\geq det(A_{11} \circ B_{11})\, det\left(\left(a_{nn} - \alpha^* {A_{11}}^{-1}\alpha\right)b_{nn}\right) \\ &\geq (\det(A_{11}))b_{11}b_{22} \ldots\ b_{(n-1)(n-1)}(det\left(a_{nn} - \alpha^* {A_{11}}^{-1}\alpha\right))b_{nn} \\ &= b_{11}b_{22} \ldots\ b_{nn} det A_{11} det\left(a_{nn} - \alpha^* {A_{11}}^{-1}\alpha\right) \\ &= (b_{11}b_{22} \ldots\ b_{nn}) det A \quad \blacksquare \end{aligned}$$

CHAPTER FOUR
SCHUR COMPLEMENT OF BLOCK HADAMARD AND BLOCK KRONECKER PRODUCT

In this chapter we will deal with the block Hadamard and block Kronecker products. We will make an analogy between the inequalities involving Hadarmard, Kronecker products and involving the block form of these products.

4.1 Schur Complement in the Block Form

For $m, n, p, q \in Z^+$ let $M_{m,n}$ be the space of $m \times n$ matrices with complex entries and $\boldsymbol{M}_{p,q}(M_{m,n})$ be the space of $p \times q$ block matrices $\boldsymbol{A} = (A_{\alpha,\beta})$, $\alpha = 1, \dots, p$; $\beta = 1, \dots, q$ whose α, β entry belongs to $M_{m,n}$. When m=n then we denote $M_{n,n}$ by M_n and p=q we denote $\boldsymbol{M}_{p,q}$ by $\boldsymbol{M}_p$. Also we denote the $pn \times pn$ identity matrix as $\boldsymbol{I}_p$.

Let $\boldsymbol{\alpha} \subset \{1,2,\dots,p\}$, $\boldsymbol{\beta} \subset \{1,2,\dots,q\}$ and $\boldsymbol{\alpha}^c = \{1,2,\dots,p\} \backslash \boldsymbol{\alpha}$, $\boldsymbol{\beta}^c = \{1,2,\dots,q\} \backslash \boldsymbol{\beta}$ Let $\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the block submatrix of $\boldsymbol{A}$ with the block rows indexed by $\boldsymbol{\alpha}$ and the block columns indexed by $\boldsymbol{\beta}$**.** If $\boldsymbol{\alpha} = \boldsymbol{\beta}$ then we write $\boldsymbol{A}(\boldsymbol{\alpha})$ for $\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\alpha})$.

Definition 4.1.1 Let $\boldsymbol{A} \in \boldsymbol{M}_{p,q}(M_{n,n})$ be a $p \times q$ block matrix such that $\boldsymbol{A} = (A_{\alpha,\beta})$, $\alpha = 1, \dots, p$; $\beta = 1, \dots, q$ in which each block is an $n \times n$ matrix with complex entries. If $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and if $\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is nonsingular then the Schur complement of $\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in $\boldsymbol{A}$ is

$$\boldsymbol{A}/\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{A}(\boldsymbol{\alpha}^c, \boldsymbol{\beta}^c) - \boldsymbol{A}(\boldsymbol{\alpha}^c, \boldsymbol{\beta})(\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}))^{-1}\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}^c). \tag{4.1}$$

We will use the notation $\boldsymbol{A}/\boldsymbol{\alpha}$ for $\boldsymbol{A}/\boldsymbol{A}(\boldsymbol{\alpha})$ **.**

Example 4.1 Let

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \left(\begin{array}{cc:cc:cc} 1 & 0 & 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ \hdashline -1 & 0 & 1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 3 & 2 & 1 \\ \hdashline 0 & 2 & 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 2 & 0 \end{array}\right) \in \boldsymbol{M}_3(M_2).$$

Let

$$\boldsymbol{\alpha} = \{1,2\}, \boldsymbol{\beta} = \{1,3\}\ , \boldsymbol{\alpha}^c = \{3\}, \boldsymbol{\beta}^c = \{2\}, |\boldsymbol{\alpha}| = 2, |\boldsymbol{\beta}| = 2$$

$$\boldsymbol{A}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \left(\begin{array}{cc:cc} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ \hdashline -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{array}\right) \in \boldsymbol{M}_2(M_2)\ , \boldsymbol{A}(\boldsymbol{\alpha^c},\boldsymbol{\beta^c}) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in \boldsymbol{M}_1(M_2)$$

$$\boldsymbol{A}(\boldsymbol{\alpha^c},\boldsymbol{\beta}) = \left(\begin{array}{cc:cc} 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right) \in \boldsymbol{M}_{1,2}(M_2)\ , \boldsymbol{A}(\boldsymbol{\alpha},\boldsymbol{\beta^c}) = \left(\begin{array}{cc} 1 & 2 \\ 1 & 1 \\ \hdashline 1 & 0 \\ 0 & 3 \end{array}\right) \in \boldsymbol{M}_{2,1}(M_2)$$

$$\begin{aligned} \boldsymbol{A}/\boldsymbol{A}(\boldsymbol{\alpha},\boldsymbol{\beta}) &= \boldsymbol{A}/\boldsymbol{A}(2,2) \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Now, let us find the Schur complement of $\boldsymbol{A}$ in the usual way. If we use rows and columns instead of the block rows and columns, then

$$\alpha = \{1,2,3,4\},\ \beta = \{1,2,3,4\}\ , \alpha^c = \{5,6\}\ , \beta^c = \{5,6\},\ |\alpha| = 4, |\beta| = 4$$

$$\boldsymbol{A}(\alpha,\beta) = \left(\begin{array}{cc:cc} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ \hdashline -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{array}\right) \in M_4, \boldsymbol{A}(\alpha,\beta^c) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in M_2$$

$$\boldsymbol{A}(\alpha^{\boldsymbol{c}},\beta) = \left(\begin{array}{cc:cc} 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right) \in M_{2,4}$$

$$\begin{aligned} \boldsymbol{A}/\ \boldsymbol{A}(\alpha,\beta) &= \boldsymbol{A}/\boldsymbol{A}(4,4) \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

As we see above, Schur complement of $\boldsymbol{A}$ in the usual form is equal to the Schur complement of $\boldsymbol{A}$ in the block form.

In the previous example we have used the block square matrix. But we can find the Schur complement of any block matrix, not only block square matrices. Let us look the next example:

Example 4.2 Let

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} = \left(\begin{array}{cc:cc} 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ \hdashline -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ \hdashline 0 & 2 & 2 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \in \boldsymbol{M}_{3,2}(M_2)$$

and

$$\boldsymbol{\alpha} = \{1\}, \boldsymbol{\beta} = \{1\}, \boldsymbol{\alpha}^c = \{2,3\}, \boldsymbol{\beta}^c = \{2\}$$

we can partition the matrix $\boldsymbol{A}$ such that

$$\boldsymbol{A} = \left(\begin{array}{c:c} \boldsymbol{A}_1 & \boldsymbol{A_2} \\ \hdashline \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{array}\right)$$

where

$$\boldsymbol{A}_1 = A_{11}, \boldsymbol{A_2} = A_{12}, \boldsymbol{A}_3 = \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix}, \boldsymbol{A}_4 = \begin{pmatrix} A_{22} \\ A_{32} \end{pmatrix}$$

then the Schur complement of $\boldsymbol{A}_1$ in $\boldsymbol{A}$ is

$$\boldsymbol{A}/\,\boldsymbol{A_1} = \boldsymbol{A_4} - \boldsymbol{A_3}\boldsymbol{A_1}^{-1}\boldsymbol{A_2} \tag{4.2}$$

$$\boldsymbol{A}/\,\boldsymbol{A_1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 1 & 2 \\ 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ -2 & 0 \\ 1 & 0 \\ -1/2 & -3/2 \end{pmatrix}$$

In the block form there are also block permutation matrices that equals the general form (4.1) of the Schur complement of a matrix to the simpler form (4.2) .That is

$$\boldsymbol{U^T A U / A(\alpha) = A/\alpha}. \tag{4.3}$$

where $\boldsymbol{U}$ and $\boldsymbol{U^T}$ are block permutation matrices.

Now we will give an example:

Example 4.3 In this example we will use the matrix $\boldsymbol{A}$ same as in example 4.1. That is

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \left(\begin{array}{cc:cc:cc} 1 & 0 & 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ \hdashline -1 & 0 & 1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 3 & 2 & 1 \\ \hdashline 0 & 2 & 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 2 & 0 \end{array}\right) \in \boldsymbol{M}_3(M_2)$$

and let

$$\boldsymbol{\alpha} = \{1,3\}\,,\ \boldsymbol{\alpha}^c = \{2\}$$

$$\boldsymbol{A(\alpha)} = \left(\begin{array}{cc:cc} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ \hdashline 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right) \in \boldsymbol{M}_2(M_2), \quad \boldsymbol{A(\alpha^c)} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \in \boldsymbol{M}_1(M_2)$$

$$\boldsymbol{A(\alpha^c,\alpha)} = \left(\begin{array}{cc:cc} -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{array}\right) \in \boldsymbol{M}_{1,2}(M_2), \quad \boldsymbol{A(\alpha,\alpha^c)} = \left(\begin{array}{cc} 1 & 2 \\ 1 & 1 \\ \hdashline 2 & 1 \\ 0 & 0 \end{array}\right) \in \boldsymbol{M}_{2,1}(M_2)$$

Then

$$\boldsymbol{A/\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

In this example for putting $\boldsymbol{A(\alpha)}$ into the upper left hand corner, we will use the block permutation matrices

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I} \\ \mathbf{0} & \boldsymbol{I} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{U^T} = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I} \\ \mathbf{0} & \boldsymbol{I} & \mathbf{0} \end{pmatrix}$$

where $\boldsymbol{I} \in \boldsymbol{M}_1(M_2)$ and $\mathbf{0} \in \boldsymbol{M}_1(M_2)$.

Then

$$\boldsymbol{U^TAU} = \begin{pmatrix} \boldsymbol{A(\alpha)} & \boldsymbol{A(\alpha, \alpha^c)} \\ \boldsymbol{A(\alpha^c, \alpha)} & \boldsymbol{A(\alpha, \alpha^c)} \end{pmatrix} = \left(\begin{array}{cc:cc:cc} 1 & 0 & 3 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ \hdashline 0 & 2 & 1 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 & 3 \end{array}\right)$$

$$\boldsymbol{U^TAU/A(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} - \left(\begin{array}{cc:cc} -1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{array}\right) \left(\begin{array}{cc:cc} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ \hdashline 0 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right)^{-1} \left(\begin{array}{cc} 1 & 2 \\ 1 & 1 \\ \hdashline 2 & 1 \\ 0 & 0 \end{array}\right)$$

$$= \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

Hence

$$\boldsymbol{A/\alpha} = \boldsymbol{U^TAU/A(\alpha)} = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

4.2 Block Hadamard and Block Kronecker Product of Matrices

In this section we will be dealing with the block Hadamard and block Kronecker product of matrices. Therefore these definitions, propositions and lemmas will be very useful for us.

Definition 4.2.1 (Günther & Klotz, 2012) Let $m, n, l, p, q, s, t \in Z^+$. A *block Kronecker product* of $A \in M_{m,l}$ and $\boldsymbol{B} = (B_{\mu\vartheta}) \in \boldsymbol{M}_{s,t}(M_{l,n})$ is,

$$A \boxtimes \boldsymbol{B} = AB_{\mu\vartheta}\ , \qquad \mu = 1, \dots, s; \quad \vartheta = 1, \dots, t. \tag{4.4}$$

where $AB_{\mu\vartheta}$ indicates the usual matrix product of A and $B_{\mu\vartheta}$ If $\boldsymbol{A} = (A_{\alpha\beta}) \in \boldsymbol{M}_{p,q}(M_{m,l})$ then the block Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{A} \boxtimes \boldsymbol{B} = A_{\alpha\beta} \boxtimes \boldsymbol{B}, \qquad \alpha = 1, \dots, p; \quad \beta = 1, \dots, q. \tag{4.5}$$

Definition 4.2.2 (Günther & Klotz, 2012) For $= (A_{\alpha\beta})$, $\boldsymbol{B} = (B_{\alpha\beta}) \in \boldsymbol{M}_{p,q}$ a *block Hadamard product* of $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{A} \square \boldsymbol{B} = A_{\alpha\beta} B_{\alpha\beta} \qquad \alpha = 1, \dots, p; \quad \beta = 1, \dots, q. \tag{4.6}$$

Definition 4.2.3 (Günther & Klotz, 2012) The matrices $\boldsymbol{A} \in \boldsymbol{M}_{p,q}$ and $\boldsymbol{B} \in \boldsymbol{M}_{s,t}$ are block commuting if every $n \times n$ block of $\boldsymbol{A}$ commutes with every $n \times n$ block of $\boldsymbol{B}$**.** And it is denoted by $\boldsymbol{A}_{bc}\boldsymbol{B}$

Proposition 4.2.4 (Günther & Klotz, 2012) Let $p, q, s, t \in N$, $\boldsymbol{A} \in \boldsymbol{M}_{p,q}$ $\boldsymbol{B} \in \boldsymbol{M}_{s,t}$, $\boldsymbol{C} \in \boldsymbol{M}_{q,u}$, $\boldsymbol{D} \in \boldsymbol{M}_{t,v}$

(a) If $\boldsymbol{B}_{bc}\boldsymbol{C}$ then $(\boldsymbol{A} \boxtimes \boldsymbol{B})(\boldsymbol{C} \boxtimes \boldsymbol{D}) = \boldsymbol{AC} \boxtimes \boldsymbol{BD}$

(b) $\boldsymbol{A} \boxtimes \boldsymbol{B} = (\boldsymbol{A} \boxtimes \boldsymbol{I_s})(\boldsymbol{I_q} \boxtimes \boldsymbol{B})$

(c) $\boldsymbol{A} \boxtimes \boldsymbol{B} = (\boldsymbol{I_p} \boxtimes \boldsymbol{B})(\boldsymbol{A} \boxtimes \boldsymbol{I_t})$ if and only if $\boldsymbol{A}_{bc}\boldsymbol{B}$.

Lemma 4.2.5 (Günther & Klotz, 2012) Let $\boldsymbol{A} \in \boldsymbol{M}_{p,q}$, $\boldsymbol{B} \in \boldsymbol{M}_{s,t}$

(a) If $\boldsymbol{B}$ is invertible, then $\boldsymbol{A}_{bc}\boldsymbol{B}$ if and only if $\boldsymbol{A}_{bc}\boldsymbol{B}^{-\boldsymbol{1}}$

(b) If $\boldsymbol{A}_{bc}\boldsymbol{B}$ and $\boldsymbol{A}_{bc}\boldsymbol{B}^*$, then $\boldsymbol{A}_{bc}\boldsymbol{B}^+$.

Property 4.2.6 (Günther & Klotz, 2012) $(\boldsymbol{A} \boxtimes \boldsymbol{B})^T = \boldsymbol{A}^T \boxtimes \boldsymbol{B}^{\boldsymbol{T}}$ and $(\boldsymbol{A} \boxtimes \boldsymbol{B})^* = \boldsymbol{A}^* \boxtimes \boldsymbol{B}^*$ if and only if $\boldsymbol{A}_{bc}\boldsymbol{B}$.

Property 4.2.7 (Günther & Klotz, 2012) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitian. Then $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is Hermitian if and only if $\boldsymbol{A}_{bc}\boldsymbol{B}$.

Proposition 4.2.8 (Günther & Klotz, 2012) For $p, s \in N$, there exists a block permutation matrix $\boldsymbol{P} \in \boldsymbol{M}_{p,s}$ such that

$$\boldsymbol{P}^T(\boldsymbol{A} \boxtimes \boldsymbol{B})\boldsymbol{P} = \boldsymbol{B} \boxtimes \boldsymbol{A} \qquad \text{if } \boldsymbol{A} \in \boldsymbol{M}_p , \boldsymbol{B} \in \boldsymbol{M}_s \text{ and } \boldsymbol{A}_{bc}\boldsymbol{B}.$$

In particular,

$$\boldsymbol{P}^T(\boldsymbol{A} \boxtimes \boldsymbol{I}_s)\boldsymbol{P} = \boldsymbol{I}_s \boxtimes \boldsymbol{A} \qquad \text{for any } \boldsymbol{A} \in \boldsymbol{M}_p.$$

Corollary 4.2.9 (Günther & Klotz, 2012) Let $\boldsymbol{A} \in \boldsymbol{M}_p$, $\boldsymbol{B} \in \boldsymbol{M}_s$. Then

(a) $det(\boldsymbol{A} \boxtimes \boldsymbol{B}) = det\boldsymbol{A}^s det\boldsymbol{B}^p$,

(b) $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is invertible if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible,

(c) $(\boldsymbol{A} \boxtimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \boxtimes \boldsymbol{B}^{-1}$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible .

Corollary 4.2.10 (Günther & Klotz, 2012) Let $\boldsymbol{A} \in \boldsymbol{M}_p$, $\boldsymbol{B} \in \boldsymbol{M}_s$

(a) If $\boldsymbol{A}$ is positive semidefinite and $\boldsymbol{B}$ is Hermitian, then $\boldsymbol{A} \boxtimes \boldsymbol{B}$ has only real eigenvalues, the number of positive and negative eigenvalues of $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is not larger than p times the number of positive and negative eigenvalues of $\boldsymbol{B}$, respectively, and the dimension of the null space of $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is not smaller than p times the dimension of the null space of $\boldsymbol{B}$.

(b) If $\boldsymbol{A}$ is positive definite and $\boldsymbol{B}$ is Hermitian, then $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is a diagonalizable matrix, whose number of positive , negative and zero eigenvalues equals p times the number of positive, negative and zero eigenvalues of $\boldsymbol{B}$, respectively.

(c) If $\boldsymbol{A}$ is positive definite and $\boldsymbol{B}$ is such that $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is Hermitian, then $\boldsymbol{B}$ is a diagonalizable matrix that has only real eigenvalues, and the number of positive, negative and zero eigenvalues of $\boldsymbol{A} \boxtimes \boldsymbol{B}$ equals p times the number of positive, negative and zero eigenvalues of $\boldsymbol{B}$, respectively.

Corollary 4.2.11 (Günther & Klotz, 2012) Let $\boldsymbol{A} \in \boldsymbol{M}_p$, $\boldsymbol{B} \in \boldsymbol{M}_s$ be positive semidefinite. Then $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is positive semidefinite if and only if $\boldsymbol{A}_{bc}\boldsymbol{B}$. Furthermore, $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is positive definite if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite and $\boldsymbol{A}_{bc}\boldsymbol{B}$.

Corollary 4.2.12 (Günther & Klotz, 2012) Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p$ and $\boldsymbol{A}_{bc}\boldsymbol{B}$. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both positive semidefinite or both positive definite, then $\boldsymbol{A} \square \boldsymbol{B}$ is positive semidefinite or positive definite, respectively.

Proposition 4.2.13 (Günther &Klotz, 2012) Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p$. If $\boldsymbol{A}_{bc}\boldsymbol{B}$ and $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite, then

$$\boldsymbol{A}^{-1} \square \boldsymbol{B}^{-1} = \boldsymbol{B}^{-1} \square \boldsymbol{A}^{-1} \geq (\boldsymbol{A} \square \boldsymbol{B})^{-1} = (\boldsymbol{B} \square \boldsymbol{A})^{-1}. \tag{4.7}$$

In particular, if $\boldsymbol{A}_{bc}\boldsymbol{A}$ and $\boldsymbol{A}$ is positive definite, then

$$\boldsymbol{A}^{-1} \square \boldsymbol{A}^{-1} \geq (\boldsymbol{A} \square \boldsymbol{A})^{-1}$$

and

$$\boldsymbol{A} \square \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \square \boldsymbol{A} \geq \boldsymbol{I_p} \geq (\boldsymbol{A}^{-1} \square \boldsymbol{A})^{-1} = (\boldsymbol{A} \square \boldsymbol{A}^{-1})^{-1}.$$

4.3 Results on Schur Complement of Matrices Involving Block Hadamard and Block Kronecker Product

In this section we will establish some theorems and lemmas about the Schur complement of block Hadamard and block Kronecker product of matrices. While doing this, the lemmas and theorems in section 3.3 and 4.2 will be used.

Property 4.3.1 Let $\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{D} \in \boldsymbol{M}_{p,q}(M_n)$ and $\boldsymbol{A}_{bc}\boldsymbol{C}$, $\boldsymbol{A}_{bc}\boldsymbol{D}$,, $\boldsymbol{C}_{bc}\boldsymbol{D}$. Then

$$\boldsymbol{A} \square (\boldsymbol{C} + \boldsymbol{D}) = (\boldsymbol{A} \square \boldsymbol{C}) + (\boldsymbol{A} \square \boldsymbol{D}) \tag{4.8}$$

and

$$\boldsymbol{A} \square (\boldsymbol{C} - \boldsymbol{D}) = (\boldsymbol{A} \square \boldsymbol{C}) - (\boldsymbol{A} \square \boldsymbol{D}) \tag{4.9}$$

Proof. Let the block matrices $\boldsymbol{A}, \boldsymbol{C}$, and $\boldsymbol{D}$ be

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A_{11}} & \cdots & \boldsymbol{A_{1q}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{A_{p1}} & \cdots & \boldsymbol{A_{pq}} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} \boldsymbol{C_{11}} & \cdots & \boldsymbol{C_{1q}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{C_{p1}} & \cdots & \boldsymbol{C_{pq}} \end{pmatrix} \text{ and } \boldsymbol{D} = \begin{pmatrix} \boldsymbol{D_{11}} & \cdots & \boldsymbol{D_{1q}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{D_{p1}} & \cdots & \boldsymbol{D_{pq}} \end{pmatrix}$$

then

$$\boldsymbol{C} + \boldsymbol{D} = \begin{pmatrix} \boldsymbol{C_{11}} + \boldsymbol{D_{11}} & \cdots & \boldsymbol{C_{1q}} + \boldsymbol{D_{1q}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{C_{p1}} + \boldsymbol{D_{p1}} & \cdots & \boldsymbol{C_{pq}} + \boldsymbol{D_{pq}} \end{pmatrix}$$

$$\boldsymbol{A}\square(\boldsymbol{C}+\boldsymbol{D}) = \begin{pmatrix} \boldsymbol{A_{11}}(\boldsymbol{C_{11}}+\boldsymbol{D_{11}}) & \cdots & \boldsymbol{A_{1q}}(\boldsymbol{C_{1q}}+\boldsymbol{D_{1q}}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{A_{p1}}(\boldsymbol{C_{p1}}+\boldsymbol{D_{p1}}) & \cdots & \boldsymbol{A_{pq}}(\boldsymbol{C_{pq}}+\boldsymbol{D_{pq}}) \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{A_{11}C_{11}}+\boldsymbol{A_{11}D_{11}} & \cdots & \boldsymbol{A_{1q}C_{1q}}+\boldsymbol{A_{1q}D_{1q}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{A_{p1}C_{p1}}+\boldsymbol{A_{p1}D_{p1}} & \cdots & \boldsymbol{A_{pq}C_{pq}}+\boldsymbol{A_{pq}D_{pq}} \end{pmatrix}$$

$$= (\boldsymbol{A} \square \boldsymbol{C}) + (\boldsymbol{A} \square \boldsymbol{D})$$

the equality $\boldsymbol{A}\square(\boldsymbol{C}-\boldsymbol{D}) = (\boldsymbol{A} \square \boldsymbol{C}) - (\boldsymbol{A} \square \boldsymbol{D})$ can be proved in the same way.

Lemma 4.3.2 Let $\boldsymbol{A} \in \boldsymbol{M}_{p,q}$, $\boldsymbol{B} \in \boldsymbol{M}_{s,t}$ .If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible and $\boldsymbol{A}_{bc}\boldsymbol{B}$ then $\boldsymbol{A^{-1}}_{bc}\boldsymbol{B}$ and $\boldsymbol{A^{-1}}_{bc}\boldsymbol{B^{-1}}$

Proof. By lemma 4.2.5 (a) we know that if $\boldsymbol{B}$ is invertible then, $\boldsymbol{A}_{bc}\boldsymbol{B}$ if and only if $\boldsymbol{A}_{bc}\boldsymbol{B^{-1}}$. And $\boldsymbol{A}_{bc}\boldsymbol{B}$ means that $\boldsymbol{B}_{bc}\boldsymbol{A}$. In this corollary both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible. Hence, since $\boldsymbol{A}$ is invertible, $\boldsymbol{B}_{bc}\boldsymbol{A}$ if and only if $\boldsymbol{B}_{bc}\boldsymbol{A^{-1}}$. And this means that $\boldsymbol{A^{-1}}_{bc}\boldsymbol{B}$. Since $\boldsymbol{B}$ is invertible, $\boldsymbol{A^{-1}}_{bc}\boldsymbol{B}$ if and only if $\boldsymbol{A^{-1}}_{bc}\boldsymbol{B^{-1}}$. ■

Lemma 4.3.3 Let $\boldsymbol{A} \in \boldsymbol{M}_{p,q}$, $\boldsymbol{B} \in \boldsymbol{M}_{s,t}$. If $\boldsymbol{A}_{bc}\boldsymbol{B}$ then $(\boldsymbol{A}/\boldsymbol{\alpha})_{bc}(\boldsymbol{B}/\boldsymbol{\alpha})$

Proof. Since $\boldsymbol{A}_{bc}\boldsymbol{B}$ then all block submatrices of $\boldsymbol{A}$ commutes with all block submatrices of . If we partition $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$\boldsymbol{A} = \left(\begin{array}{c:c} \boldsymbol{A}_1 & \boldsymbol{A_2} \\ \hdashline \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{array}\right), \qquad \boldsymbol{B} = \left(\begin{array}{c:c} \boldsymbol{B}_1 & \boldsymbol{B_2} \\ \hdashline \boldsymbol{B}_3 & \boldsymbol{B}_4 \end{array}\right)$$

where $\boldsymbol{A}_1, \boldsymbol{A_2}, \boldsymbol{A}_3, \boldsymbol{A}_4$ and $\boldsymbol{B}_1, \boldsymbol{B_2}, \boldsymbol{B}_3, \boldsymbol{B}_4$ are block submatrices of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. And if we use $\boldsymbol{A}/\boldsymbol{A_1}$ and $\boldsymbol{B}/\boldsymbol{B_1}$ instead of $\boldsymbol{A}/\boldsymbol{\alpha}$ and $/\boldsymbol{\alpha}$, then

$$\boldsymbol{A}/\boldsymbol{\alpha} = \boldsymbol{A}/\boldsymbol{A_1} = \boldsymbol{A_4} - \boldsymbol{A_3}\boldsymbol{A_1}^{-1}\boldsymbol{A_2} \quad \text{and} \quad \boldsymbol{B}/\boldsymbol{\alpha} = \boldsymbol{B}/\boldsymbol{B_1} = \boldsymbol{B_4} - \boldsymbol{B_3}\boldsymbol{B_1}^{-1}\boldsymbol{B_2}$$

$$(\boldsymbol{A}/\boldsymbol{A_1})(\boldsymbol{B}/\boldsymbol{B_1}) = (\boldsymbol{A_4} - \boldsymbol{A_3}\boldsymbol{A_1}^{-1}\boldsymbol{A_2})(\boldsymbol{B_4} - \boldsymbol{B_3}\boldsymbol{B_1}^{-1}\boldsymbol{B_2})$$

$$= \boldsymbol{A_4B_4} - \boldsymbol{A_4B_3}\boldsymbol{B_1}^{-1}\boldsymbol{B_2} - \boldsymbol{A_3}\boldsymbol{A_1}^{-1}\boldsymbol{A_2B_4} + \boldsymbol{A_3}\boldsymbol{A_1}^{-1}\boldsymbol{A_2B_3}\boldsymbol{B_1}^{-1}\boldsymbol{B_2}$$

By lemma 4.3.2

$$\boldsymbol{A_4B_4 - A_4B_3{B_1}^{-1}B_2 - A_3{A_1}^{-1}A_2B_4 + A_3{A_1}^{-1}A_2B_3{B_1}^{-1}B_2}$$
$$\boldsymbol{= B_4\,A_4 - B_3{B_1}^{-1}B_2A_4 - B_4A_3{A_1}^{-1}A_2 + B_3{B_1}^{-1}B_2A_3{A_1}^{-1}A_2}$$

Hence

$$(\boldsymbol{A}/\boldsymbol{A_1})(\boldsymbol{B}/\boldsymbol{B_1}) = \boldsymbol{B_4\,A_4 - B_3{B_1}^{-1}B_2A_4 - B_4A_3{A_1}^{-1}A_2 + B_3{B_1}^{-1}B_2A_3{A_1}^{-1}A_2}$$

$$= (\boldsymbol{B}/\boldsymbol{B_1})(\boldsymbol{A}/\boldsymbol{A_1})\,. \ \blacksquare$$

Through the following lemma, we will prove the inequality

$$(\boldsymbol{A} \,\square\, \boldsymbol{B})^{-1} \leq \boldsymbol{A^{-1}} \square \boldsymbol{B}^{-1}.$$

Lemma 4.3.4 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_q(M_n)$ and $\boldsymbol{C}, \boldsymbol{D} \in \boldsymbol{M}_{p,q}(M_n)$ be positive definite matrices such that $\boldsymbol{A}_{bc}\boldsymbol{B}$ and $\boldsymbol{C}_{bc}\boldsymbol{D}$ **.** Then

$$(\boldsymbol{C} \,\square\, \boldsymbol{D})(\boldsymbol{A} \,\square\, \boldsymbol{B})^{-1}(\boldsymbol{C} \,\square\, \boldsymbol{D})^{*} \leq (\boldsymbol{CA^{-1}C^{*}}) \square (\boldsymbol{DB^{-1}D^{*}}) \tag{4.11}$$

in particular

$$(\boldsymbol{A} \,\square\, \boldsymbol{B})^{-1} \leq \boldsymbol{A^{-1}} \square \boldsymbol{B}^{-1} \tag{4.12}$$

and

$$(\boldsymbol{C} \,\square\, \boldsymbol{D})(\boldsymbol{C} \,\square\, \boldsymbol{D})^{*} \leq (\boldsymbol{CC^{*}}) \square\, (\boldsymbol{DD^{*}}) \tag{4.13}$$

Proof. Let

$$\widehat{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C}^{*} \\ \boldsymbol{C} & \boldsymbol{CA^{-1}C^{*}} \end{pmatrix}, \qquad \widehat{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{D}^{*} \\ \boldsymbol{D} & \boldsymbol{DB^{-1}D^{*}} \end{pmatrix}$$

Then by the theorem 3.2.3 $\widehat{\boldsymbol{A}}$ and $\widehat{\boldsymbol{B}}$ are positive semidefinite. That is,

$$\widehat{\boldsymbol{A}} \geq 0 \qquad \text{and} \qquad \widehat{\boldsymbol{B}} \geq 0$$

hence

$$\widehat{\boldsymbol{A}} \,\square\, \widehat{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{A} \,\square\, \boldsymbol{B} & \boldsymbol{C}^{*} \,\square\, \boldsymbol{D}^{*} \\ \boldsymbol{C} \,\square\, \boldsymbol{D} & (\boldsymbol{CA^{-1}C^{*}}) \square (\boldsymbol{DB^{-1}D^{*}}) \end{pmatrix} \geq 0$$

Taking the Schur complement of (1,1) block in $\widehat{\boldsymbol{A}} \,\square\, \widehat{\boldsymbol{B}}$

$$(\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{C}^{*})\square(\boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{D}^{*}) - (\boldsymbol{C}\square\boldsymbol{D})(\boldsymbol{A}\square\boldsymbol{B})^{-1}(\boldsymbol{C}^{*}\square\boldsymbol{D}^{*}) \geq 0$$

$$(\boldsymbol{C}\square\boldsymbol{D})(\boldsymbol{A}\square\boldsymbol{B})^{-1}(\boldsymbol{C}^{*}\square\boldsymbol{D}^{*}) \leq (\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{C}^{*})\square(\boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{D}^{*})$$

If we put $\boldsymbol{C} = \boldsymbol{I_q}$ and $\boldsymbol{D} = \boldsymbol{I_q}$ in the inequality (4.11), then we have

$$(\boldsymbol{A}\square\boldsymbol{B})^{-1} \leq \boldsymbol{A}^{-1}\square\boldsymbol{B}^{-1}$$

If $\boldsymbol{A} = \boldsymbol{I_q}$ and $\boldsymbol{B} = \boldsymbol{I_q}$ in (4.11), then

$$(\boldsymbol{C}\square\boldsymbol{D})(\boldsymbol{C}\square\boldsymbol{D})^{*} \leq (\boldsymbol{C}\boldsymbol{C}^{*})\square(\boldsymbol{D}\boldsymbol{D}^{*}). \quad \blacksquare$$

Lemma 4.3.5 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p(M_n)$ and $\boldsymbol{A}_{bc}\boldsymbol{B}$. If $\boldsymbol{A}$ is positive definite and $\boldsymbol{B}$ is positive semidefinite then $\boldsymbol{A}\square\boldsymbol{B}$ is positive semidefinite.

Proof. By corollary 4.2.10 (b) $\boldsymbol{A}\boxtimes\boldsymbol{B}$ is a diagonalizable matrix, whose number of positive , negative and zero eigenvalues equals p times the number of positive, negative and zero eigenvalues of $\boldsymbol{B}$, respectively Since $\boldsymbol{B}$ is positive semidefinite then $\boldsymbol{A}\boxtimes\boldsymbol{B}$ has only positive and zero eigenvalues.

Also since $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitian $\boldsymbol{A}\boxtimes\boldsymbol{B}$ is Hermitian. Therefore $\boldsymbol{A}\boxtimes\boldsymbol{B}$ is positive semidefinite. Since $\boldsymbol{A}\square\boldsymbol{B}$ is a principal submatrix of $\boldsymbol{A}\boxtimes\boldsymbol{B}$, then $\boldsymbol{A}\square\boldsymbol{B}$ is positive semidefinite. $\blacksquare$

Now we will give an inequality which relates the block Hadamard product of Schur complement of two matrices with the Schur complement of their block Hadamard products

Theorem 4.3.6 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p(M_n)$ be positive definite matrices such that $\boldsymbol{A}_{bc}\boldsymbol{B}$. Then

$$(\boldsymbol{A}\square\boldsymbol{B})/\boldsymbol{\alpha} \geq \boldsymbol{A}/\boldsymbol{\alpha}\ \square\ \boldsymbol{B}/\boldsymbol{\alpha}. \qquad (4.14)$$

Proof. If we partition $\boldsymbol{A}$ and $\boldsymbol{B}$ appropriately

$$\boldsymbol{A} = \left(\begin{array}{c|c} \boldsymbol{A_1} & \boldsymbol{A_2} \\ \hline \boldsymbol{A_3} & \boldsymbol{A_4} \end{array}\right) \qquad \text{and} \qquad \boldsymbol{B} = \left(\begin{array}{c|c} \boldsymbol{B_1} & \boldsymbol{B_2} \\ \hline \boldsymbol{B_3} & \boldsymbol{B_4} \end{array}\right)$$

Then

$$\boldsymbol{A} \,\square\, \boldsymbol{B} = \begin{pmatrix} \boldsymbol{A_1} \square\, \boldsymbol{B_1} & \boldsymbol{A_2} \square\, \boldsymbol{B_2} \\ \boldsymbol{A_3} \square\, \boldsymbol{B_3} & \boldsymbol{A_4} \square\, \boldsymbol{B_4} \end{pmatrix}$$

Since $(\boldsymbol{U^T A U})/\boldsymbol{A}(\boldsymbol{\alpha}) = \boldsymbol{A}/\boldsymbol{\alpha}$, then we can use $\boldsymbol{A}/\boldsymbol{A_1}$ and $\boldsymbol{B}/\boldsymbol{B_1}$ instead of $\boldsymbol{A}/\boldsymbol{\alpha}$ and $\boldsymbol{B}/\boldsymbol{\alpha}$.

Using the theorem 3.2.3 we have

$$\boldsymbol{A}_3\boldsymbol{A}_1{}^{-1}\boldsymbol{A}_2 = \boldsymbol{A_4} - \boldsymbol{A}/\boldsymbol{A_1} \qquad \boldsymbol{B}_3\boldsymbol{B}_1{}^{-1}\boldsymbol{B}_2 = \boldsymbol{B_4} - \boldsymbol{B}/\boldsymbol{B_1} \tag{4.15}$$

and

$$\boldsymbol{A_4} - \boldsymbol{A}/\boldsymbol{A_1} \geq 0 \qquad \boldsymbol{B_4} - \boldsymbol{B}/\boldsymbol{B_1} \geq 0 \tag{4.16}$$

Taking the Schur complement of (1,1) block of $\boldsymbol{A} \,\square\, \boldsymbol{B}$ one has

$$(\boldsymbol{A} \,\square\, \boldsymbol{B})/\boldsymbol{\alpha} = (\boldsymbol{A} \,\square\, \boldsymbol{B})/\boldsymbol{A_1} \square\, \boldsymbol{B_1} = \boldsymbol{A_4} \square\, \boldsymbol{B_4} - (\boldsymbol{A_3} \square\, \boldsymbol{B_3})(\boldsymbol{A_1} \square\, \boldsymbol{B_1})^{-\mathbf{1}}(\boldsymbol{A_2} \square\, \boldsymbol{B_2})$$

Using (4.11)

$$\begin{aligned} (\boldsymbol{A} \,\square\, \boldsymbol{B})/\boldsymbol{\alpha} &\geq \boldsymbol{A_4} \square\, \boldsymbol{B_4} - (\boldsymbol{A}_3\boldsymbol{A}_1{}^{-1}\boldsymbol{A}_2) \square (\boldsymbol{B}_3\boldsymbol{B}^{-1}\boldsymbol{B}_2) \\ &= \boldsymbol{A_4} \square\, \boldsymbol{B_4} - (\boldsymbol{A_4} - \boldsymbol{A}/\boldsymbol{A_1}) \square (\boldsymbol{B_4} - \boldsymbol{B}/\boldsymbol{B_1}) \\ &= (\boldsymbol{A}/\boldsymbol{A_1}) \square\, \boldsymbol{B_4} + (\boldsymbol{A_4} - \boldsymbol{A}/\boldsymbol{A_1}) \square (\boldsymbol{B}/\boldsymbol{B_1}) \end{aligned} \tag{4.17}$$

Since $\boldsymbol{A_4} - \boldsymbol{A}/\boldsymbol{A_1} \geq 0$ and $\boldsymbol{B}/\boldsymbol{B_1} > 0$, then by the lemma 4.3.5

$(\boldsymbol{A_4} - \boldsymbol{A}/\boldsymbol{A_1}) \square (\boldsymbol{B}/\boldsymbol{B_1}) \geq 0$. Hence

$$(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha} \geq (\boldsymbol{A}/\boldsymbol{A_1}) \square\, \boldsymbol{B_4} \tag{4.18}$$

$$\geq (\boldsymbol{A}/\boldsymbol{A_1}) \square (\boldsymbol{B}/\boldsymbol{B_1})$$

Therefore

$$(\boldsymbol{A} \,\square\, \boldsymbol{B})/\boldsymbol{\alpha} \geq \boldsymbol{A}/\boldsymbol{\alpha} \;\square\, \boldsymbol{B}/\boldsymbol{\alpha} . \quad \blacksquare$$

Example 4.4 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_2(M_2)$ be positive definite matrices such that

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \left(\begin{array}{cc:cc} 5 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ \hdashline 1 & 0 & 5 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right) \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc:cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hdashline 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\boldsymbol{\alpha} = \{1\}\ ,\ \boldsymbol{\beta} = \{1\}\ ,\ \boldsymbol{\alpha}^c = \{2\}\ ,\ \boldsymbol{\beta}^c = \{2\}$$

$$\boldsymbol{A}/\boldsymbol{\alpha} = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 52/11 & 24/11 \\ 24/11 & 28/11 \end{pmatrix}$$

$$\boldsymbol{B}/\boldsymbol{\alpha} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 13/5 & 6/5 \\ 6/5 & 7/5 \end{pmatrix}$$

$$\boldsymbol{A}/\boldsymbol{\alpha}\ \Box\ \boldsymbol{B}/\boldsymbol{\alpha} = \begin{pmatrix} 164/11 & 96/11 \\ 96/11 & 68/11 \end{pmatrix}$$

$$\boldsymbol{A} \Box \boldsymbol{B} = \left(\begin{array}{cc:cc} 17 & 9 & 1 & 0 \\ 9 & 8 & 0 & 1 \\ \hdashline 1 & 0 & 17 & 9 \\ 0 & 1 & 9 & 8 \end{array}\right)$$

$$(\boldsymbol{A} \Box \boldsymbol{B})/\boldsymbol{\alpha} = \begin{pmatrix} 17 & 9 \\ 9 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 17 & 9 \\ 9 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 927/55 & 504/55 \\ 504/55 & 423/55 \end{pmatrix}$$

Now we look for if $(\boldsymbol{A} \Box \boldsymbol{B})/\boldsymbol{\alpha}$ is greater than or equal to $\boldsymbol{A}/\boldsymbol{\alpha}\ \Box\ \boldsymbol{B}/\boldsymbol{\alpha}$, that is

$$\begin{pmatrix} 927/55 & 504/55 \\ 504/55 & 423/55 \end{pmatrix} \geq \begin{pmatrix} 164/11 & 96/11 \\ 96/11 & 68/11 \end{pmatrix}.$$

For doing this we must look for

$$\begin{pmatrix} 927/55 & 504/55 \\ 504/55 & 423/55 \end{pmatrix} - \begin{pmatrix} 164/11 & 96/11 \\ 96/11 & 68/11 \end{pmatrix} \geq 0.$$

Since $\begin{pmatrix} 927/55 & 504/55 \\ 504/55 & 423/55 \end{pmatrix} - \begin{pmatrix} 164/11 & 96/11 \\ 96/11 & 68/11 \end{pmatrix} = \begin{pmatrix} 107/55 & 24/55 \\ 24/55 & 83/55 \end{pmatrix}$

and

$\begin{pmatrix} 107/55 & 24/55 \\ 24/55 & 83/55 \end{pmatrix}$ is a Hermitian matrix that has eigenvalues $\lambda_1 = 1.2394$, $\lambda_2 = 2.2151$.

then $\begin{pmatrix} 107/55 & 24/55 \\ 24/55 & 83/55 \end{pmatrix}$ is a positive semidefinite matrix.

Therefore

$$(\boldsymbol{A} \Box \boldsymbol{B})/\boldsymbol{\alpha} \geq \boldsymbol{A}/\boldsymbol{\alpha}\ \Box\ \boldsymbol{B}/\boldsymbol{\alpha}\ .$$

Now we will state a corollary of theorem 4.3.6. While doing this, we will use the previous lemmas and theorems about the Schur complement.

Corollary 4.3.7 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p(M_n)$ be positive definite matrices such that $\boldsymbol{A}_{bc}\boldsymbol{B}$. Then

(a) $(\boldsymbol{A} \square \boldsymbol{B})^{-1}/\boldsymbol{\alpha} \leq [(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} \leq (\boldsymbol{A}/\boldsymbol{\alpha} \ \square \boldsymbol{B}/\boldsymbol{\alpha})^{-1} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}$,

(b) $(\boldsymbol{A} \square \boldsymbol{B})^{-1}/\boldsymbol{\alpha} \leq (\boldsymbol{A}^{-1}/\boldsymbol{\alpha}) \square (\boldsymbol{B}^{-1}/\boldsymbol{\alpha}) \leq (\boldsymbol{A}^{-1} \square \boldsymbol{B}^{-1})/\boldsymbol{\alpha} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}$

Proof. For (a) : Since $\boldsymbol{A}^{-1}/\boldsymbol{\alpha} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1}$, then

$$(\boldsymbol{A} \square \boldsymbol{B})^{-1}/\boldsymbol{\alpha} \leq [(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1}$$

by theorem 4.3.6

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} \leq (\boldsymbol{A}/\boldsymbol{\alpha} \ \square \boldsymbol{B}/\boldsymbol{\alpha})^{-1}$$

by (4.12)

$$(\boldsymbol{A}/\boldsymbol{\alpha} \ \square \boldsymbol{B}/\boldsymbol{\alpha})^{-1} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}.$$

For (b), if we partition $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\boldsymbol{A} = \left(\begin{array}{c:c} \boldsymbol{A_1} & \boldsymbol{A_2} \\ \hdashline \boldsymbol{A_3} & \boldsymbol{A_4} \end{array}\right) \qquad \boldsymbol{B} = \left(\begin{array}{c:c} \boldsymbol{B_1} & \boldsymbol{B_2} \\ \hdashline \boldsymbol{B_3} & \boldsymbol{B_4} \end{array}\right)$$

then we have

$$\boldsymbol{A} \square \boldsymbol{B} = \begin{pmatrix} \boldsymbol{A_1} \square \boldsymbol{B_1} & \boldsymbol{A_2} \square \boldsymbol{B_2} \\ \boldsymbol{A_3} \square \boldsymbol{B_3} & \boldsymbol{A_4} \square \boldsymbol{B_4} \end{pmatrix}.$$

We will use the notation $\widetilde{\boldsymbol{A_1} \square \boldsymbol{B_1}}$ in place of $(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}$, that is, the Schur complement of (1,1) block of $\boldsymbol{A} \square \boldsymbol{B}$. Similarly, $\widetilde{\boldsymbol{A}_2}$, $\widetilde{\boldsymbol{B}_2}$ and $\widetilde{\boldsymbol{A_2} \square \boldsymbol{B_2}}$ mean that the Schur complement of (2,2) block of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{A} \square \boldsymbol{B}$ respectively. By (3.7)

$$(\boldsymbol{A} \square \boldsymbol{B})^{-1} = \begin{pmatrix} (\widetilde{\boldsymbol{A}_4 \square \boldsymbol{B}_4})^{-1} & * \\ * & (\widetilde{\boldsymbol{A_1} \square \boldsymbol{B_1}})^{-1} \end{pmatrix}, \qquad \boldsymbol{A}^{-1} = \begin{pmatrix} \widetilde{\boldsymbol{A}_4}^{-1} & * \\ * & \widetilde{\boldsymbol{A}_1}^{-1} \end{pmatrix}$$

and

$$\boldsymbol{B^{-1}} = \begin{pmatrix} \widetilde{\boldsymbol{B}_4}^{-1} & * \\ * & \widetilde{\boldsymbol{B}_1}^{-1} \end{pmatrix}$$

by this

$$(\widetilde{\boldsymbol{A}_4 \square \boldsymbol{B}_4})^{-1} = [(\boldsymbol{A} \square \boldsymbol{B})^{-1}]_1, \quad \widetilde{\boldsymbol{A}_4}^{-1} = (\boldsymbol{A}^{-1})_1, \quad \text{and} \quad \widetilde{\boldsymbol{B}_4}^{-1} = (\boldsymbol{B}^{-1})_1.$$

If we take the Schur complement of these expressions, then

$$\widetilde{(\widetilde{\boldsymbol{A}_4 \square \boldsymbol{B}_4})^{-1}}^{-1} = \boldsymbol{A}_4 \square \boldsymbol{B}_4, \quad \widetilde{\widetilde{\boldsymbol{A}_4}^{-1}}^{-1} = \boldsymbol{A}_4, \qquad \widetilde{\widetilde{\boldsymbol{A}_1}^{-1}}^{-1} = \boldsymbol{A}_1$$

hence

$$\begin{aligned} (\widetilde{\widetilde{\boldsymbol{A}_4 \square \boldsymbol{B}_4}})^{-1} &= (\boldsymbol{A}_4 \square \boldsymbol{B}_4)^{-1} \le \boldsymbol{A}_4{}^{-1} \square \boldsymbol{B}_4{}^{-1} \\ &= \widetilde{\widetilde{\boldsymbol{A}_4}^{-1}} \square \widetilde{\widetilde{\boldsymbol{B}_4}^{-1}} \\ &= (\widetilde{\boldsymbol{A}^{-1}})_1 \square (\widetilde{\boldsymbol{B}^{-1}})_1 \\ &= (\boldsymbol{A^{-1}}/\boldsymbol{\alpha}) \square (\boldsymbol{B^{-1}}/\boldsymbol{\alpha}) \end{aligned}$$

by theorem (4.3.6)

$$(\boldsymbol{A^{-1}}/\boldsymbol{\alpha}) \square (\boldsymbol{B^{-1}}/\boldsymbol{\alpha}) \le (\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}})/\boldsymbol{\alpha}$$

For $(\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}})/\boldsymbol{\alpha} \le (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}$ we partition $\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}}$ as follows

$$\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{N} \\ \boldsymbol{N}^* & \widetilde{\boldsymbol{A}_1}^{-1} \square \widetilde{\boldsymbol{B}_1}^{-1} \end{pmatrix}$$

M and **N** any matrices. Then the Schur complement of (1,1) block of $\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}}$ is

$$(\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}})/\boldsymbol{\alpha} = (\widetilde{\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}}})_1 = \widetilde{\boldsymbol{A}_1}^{-1} \square \widetilde{\boldsymbol{B}_1}^{-1} - \boldsymbol{N}^* \boldsymbol{M^{-1}} \boldsymbol{N} \le \widetilde{\boldsymbol{A}_1}^{-1} \square \widetilde{\boldsymbol{B}_1}^{-1}$$

by the observation 2.2.8

$$\boldsymbol{N}^* \boldsymbol{M^{-1}} \boldsymbol{N} \ge 0$$

hence

$$\begin{aligned} \widetilde{\boldsymbol{A}_1}^{-1} \square \widetilde{\boldsymbol{B}_1}^{-1} - \boldsymbol{N}^* \boldsymbol{M^{-1}} \boldsymbol{N} &\le \widetilde{\boldsymbol{A}_1}^{-1} \square \widetilde{\boldsymbol{B}_1}^{-1} \\ &= (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1} \end{aligned}$$

by this

$$(\boldsymbol{A}^{-\mathbf{1}} \square \boldsymbol{B}^{-\mathbf{1}})/\boldsymbol{\alpha} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}. \ \blacksquare$$

Example 4.5 Let us apply the corollary 4.3.7, using the matrices in the example 4.4

$$\boldsymbol{A} = \left(\begin{array}{cc:cc} 5 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ \hdashline 1 & 0 & 5 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right), \qquad \boldsymbol{B} = \left(\begin{array}{cc:cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hdashline 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

then

$$\boldsymbol{A} \square \boldsymbol{B} = \left(\begin{array}{cc:cc} 17 & 9 & 1 & 0 \\ 9 & 8 & 0 & 1 \\ \hdashline 1 & 0 & 17 & 9 \\ 0 & 1 & 9 & 8 \end{array}\right)$$

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} = \begin{pmatrix} 47/279 & -56/279 \\ -56/279 & 103/279 \end{pmatrix}$$

$$(\boldsymbol{A} \square \boldsymbol{B})^{-1} = \begin{pmatrix} 47/279 & -56/279 & -16/279 & 25/279 \\ -56/279 & 103/279 & 25/279 & -41/279 \\ -16/279 & 25/279 & 47/279 & -56/279 \\ 25/279 & -41/279 & -56/279 & 103/279 \end{pmatrix}$$

$$(\boldsymbol{A} \square \boldsymbol{B})^{-\mathbf{1}} \ / \ \boldsymbol{\alpha} = \begin{pmatrix} 8/55 & -9/55 \\ -9/55 & 17/55 \end{pmatrix}$$

$$(\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1} = \begin{pmatrix} 17/44 & -6/11 \\ -6/11 & 41/44 \end{pmatrix}$$

$$(\boldsymbol{A}/\boldsymbol{\alpha} \ \square \boldsymbol{B}/\boldsymbol{\alpha})^{-1} = \begin{pmatrix} 17/44 & -6/11 \\ -6/11 & 41/44 \end{pmatrix}$$

$$(\boldsymbol{A}^{-\mathbf{1}}/\boldsymbol{\alpha}) \square (\boldsymbol{B}^{-\mathbf{1}}/\boldsymbol{\alpha}) = \begin{pmatrix} 8/55 & -9/55 \\ -9/55 & 17/55 \end{pmatrix}$$

$$(\boldsymbol{A}^{-\mathbf{1}} \square \boldsymbol{B}^{-\mathbf{1}}) \ / \ \boldsymbol{\alpha} = \begin{pmatrix} 18/55 & -9/20 \\ -9/20 & 171/220 \end{pmatrix}$$

For (a)

$$(\boldsymbol{A} \square \boldsymbol{B})^{-\mathbf{1}} \,/\, \boldsymbol{\alpha} \leq [(\boldsymbol{A} \square \boldsymbol{B}) \,/\, \boldsymbol{\alpha}]^{-1}$$

$$\begin{pmatrix} 8/55 & -9/55 \\ -9/55 & 17/55 \end{pmatrix} \leq \begin{pmatrix} 47/279 & -56/279 \\ -56/279 & 103/279 \end{pmatrix}$$

The eigenvalues of $\begin{pmatrix} 47/279 & -56/279 \\ -56/279 & 103/279 \end{pmatrix} - \begin{pmatrix} 8/55 & -9/55 \\ -9/55 & 17/55 \end{pmatrix}$ are

$$\lambda_1 = 0{,}000087 \quad \lambda_2 = 0{,}083002.$$

If we do the same process the following inequalities, then we will see that the inequalities are true

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} \leq (\boldsymbol{A}/\boldsymbol{\alpha} \ \square\, \boldsymbol{B}/\boldsymbol{\alpha})^{-1}$$

$$\begin{pmatrix} 47/279 & -56/279 \\ -56/279 & 103/279 \end{pmatrix} \leq \begin{pmatrix} 17/44 & -6/11 \\ -6/11 & 41/44 \end{pmatrix}$$

$$(\boldsymbol{A}/\boldsymbol{\alpha} \ \square\, \boldsymbol{B}/\boldsymbol{\alpha})^{-1} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}$$

$$\begin{pmatrix} 17/44 & -6/11 \\ -6/11 & 41/44 \end{pmatrix} \leq \begin{pmatrix} 17/44 & -6/11 \\ -6/11 & 41/44 \end{pmatrix}$$

For (b)

$$(\boldsymbol{A} \square \boldsymbol{B})^{-\mathbf{1}}/\boldsymbol{\alpha} \leq (\boldsymbol{A}^{-\mathbf{1}}/\boldsymbol{\alpha}) \square (\boldsymbol{B}^{-\mathbf{1}}/\boldsymbol{\alpha})$$

$$\begin{pmatrix} 8/55 & -9/55 \\ -9/55 & 17/55 \end{pmatrix} \leq \begin{pmatrix} 8/55 & -9/55 \\ -9/55 & 17/55 \end{pmatrix},$$

$$(\boldsymbol{A}^{-\mathbf{1}}/\boldsymbol{\alpha}) \square (\boldsymbol{B}^{-\mathbf{1}}/\boldsymbol{\alpha}) \leq (\boldsymbol{A}^{-\mathbf{1}} \square \boldsymbol{B}^{-\mathbf{1}})/\boldsymbol{\alpha}$$

$$\begin{pmatrix} 0.145 & -0.164 \\ -0.164 & 0{,}309 \end{pmatrix} \leq \begin{pmatrix} 0.327 & -0.449 \\ -0.449 & 0.777 \end{pmatrix}$$

and

$$(\boldsymbol{A}^{-\mathbf{1}} \square \boldsymbol{B}^{-\mathbf{1}})/\boldsymbol{\alpha} \leq (\boldsymbol{A}/\boldsymbol{\alpha})^{-1} \square (\boldsymbol{B}/\boldsymbol{\alpha})^{-1}$$

$$\begin{pmatrix} 0.327 & -0.449 \\ -0.449 & 0.777 \end{pmatrix} \leq \begin{pmatrix} 0.386 & -0.545 \\ -0.545 & 0.932 \end{pmatrix}.$$

Corollary 4.3.8 Let $\boldsymbol{A} > 0$, $\boldsymbol{B} > 0 \in \boldsymbol{M}_p(M_n)$ and $\boldsymbol{A}_{bc}\boldsymbol{B}$. Then

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} \leq (\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}})/\boldsymbol{\alpha}$$

Proof. By the inequality (4.18)

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{A_1} \square \boldsymbol{B_1}] \geq (\boldsymbol{A}/\boldsymbol{A_1}) \square \boldsymbol{B_4}$$

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{A_1} \square \boldsymbol{B_1}]^{-1} \leq [(\boldsymbol{A}/\boldsymbol{A_1}) \square \boldsymbol{B_4}]^{-1}$$

$$= [(\boldsymbol{A}/\boldsymbol{A_1}) \square (\boldsymbol{B^{-1}}/\boldsymbol{B_1})^{-1}]^{\boldsymbol{-1}}$$

$$\leq (\boldsymbol{A}/\boldsymbol{A_1})^{\boldsymbol{-1}} \square (\boldsymbol{B^{-1}}/\boldsymbol{B_1})$$

$$= \boldsymbol{A}_4 \square (\boldsymbol{B^{-1}}/\boldsymbol{B_1})$$

$$\leq \boldsymbol{A}_4 \square (\boldsymbol{B^{-1}}/\boldsymbol{B_1}) + \boldsymbol{A}_4 \square [(\boldsymbol{B}/\boldsymbol{B_1})^{-1} - \boldsymbol{B^{-1}}/\boldsymbol{B_1}]$$

Since $\boldsymbol{A}_{bc}\boldsymbol{B}$, then

$$\boldsymbol{A}_4 \square (\boldsymbol{B^{-1}}/\boldsymbol{B_1}) + \boldsymbol{A}_4 \square [(\boldsymbol{B}/\boldsymbol{B_1})^{-1} - \boldsymbol{B^{-1}}/\boldsymbol{B_1}]$$
$$= (\boldsymbol{B^{-1}}/\boldsymbol{B_1}) \square \boldsymbol{A}_4 + [(\boldsymbol{B}/\boldsymbol{B_1})^{-1} - \boldsymbol{B^{-1}}/\boldsymbol{B_1}] \square \boldsymbol{A}_4$$

by (4.17)

$$(\boldsymbol{B^{-1}}/\boldsymbol{B_1}) \square \boldsymbol{A}_4 + [(\boldsymbol{B}/\boldsymbol{B_1})^{-1} - \boldsymbol{B^{-1}}/\boldsymbol{B_1}] \square \boldsymbol{A}_4 \leq (\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}})/\boldsymbol{A_1} \square \boldsymbol{B_1}$$

hence

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{A_1} \square \boldsymbol{B_1}]^{-1} \leq (\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}}) \,/\, \boldsymbol{A_1} \square \boldsymbol{B_1}$$

Since $\boldsymbol{U^T A U}/\boldsymbol{A}(\boldsymbol{\alpha}) = \boldsymbol{A}/\boldsymbol{\alpha}$, then $(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{A_1} \square \boldsymbol{B_1} = (\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}$. Hence

$$[(\boldsymbol{A} \square \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} \leq (\boldsymbol{A^{-1}} \square \boldsymbol{B^{-1}})/\boldsymbol{\alpha}. \quad \blacksquare$$

Example 4.6 We will use the same matrices as in the example 4.5

$$\boldsymbol{A} = \left(\begin{array}{cc:cc} 5 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ \hdashline 1 & 0 & 5 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right) \quad \text{and} \quad \boldsymbol{B} = \left(\begin{array}{cc:cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hdashline 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

Then

$$[(\boldsymbol{A} \,\square\, \boldsymbol{B})/\boldsymbol{\alpha}]^{-1} = \begin{pmatrix} 47/279 & -56/279 \\ -56/279 & 103/279 \end{pmatrix}$$

$$(\boldsymbol{A}^{-1} \,\square\, \boldsymbol{B}^{-1})/\boldsymbol{\alpha} = \begin{pmatrix} 18/55 & -9/20 \\ -9/20 & 171/220 \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} 47/279 & -56/279 \\ -56/279 & 103/279 \end{pmatrix} - \begin{pmatrix} 18/55 & -9/20 \\ -9/20 & 171/220 \end{pmatrix}$ are

$$\lambda_1 = 0.0047 \quad \lambda_2 = 0.5621.$$

In the theorem 3.2.4 we stated an inequality between the determinants of a matrix and its Schur complement. Now we will find a same relation between the block Hadamard product of two matrices and their Schur complements.

Lemma 4.3.9 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 2×2 positive semidefinite block matrices such that

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

and $\boldsymbol{A}_{bc}\boldsymbol{B}$**.** Then

$$det[(\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11} + \boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11} - A_{22}B_{22}) +$$

$$(A_{22}B_{22} - \boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \,\square\, \boldsymbol{B}/A_{22}B_{22})(A_{11}B_{11})^{-1}] \leq \det(\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11})$$

Proof. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite then

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} \\ {A_{12}}^* & A_{22} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} B_{11} & B_{12} \\ {B_{12}}^* & B_{22} \end{pmatrix}$$

$$\boldsymbol{A} \,\square\, \boldsymbol{B} = \begin{pmatrix} A_{11}B_{11} & A_{12}B_{12} \\ {A_{12}}^*{B_{12}}^* & A_{22}B_{22} \end{pmatrix}$$

$$\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11} = A_{22}B_{22} - {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12}$$

$$\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{22}B_{22} = A_{11}B_{11} - A_{12}B_{12}(A_{22}B_{22})^{-1}{A_{12}}^*{B_{12}}^*$$

And we know by the theorem (3.2.4) that

$$det(\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11})\, det(\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{22}B_{22}) \leq det(\boldsymbol{A} \,\square\, \boldsymbol{B})$$

Now let us begin the proof :

$$det(\boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11})\, det(\boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22}) = det[(\boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(\boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})]$$

$$= det[(A_{22}B_{22} - {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12})][(A_{11}B_{11} - A_{12}B_{12}(A_{22}B_{22})^{-1}{A_{12}}^*{B_{12}}^*)]$$

$$= det[(A_{22}B_{22}A_{11}B_{11} - A_{12}B_{12}{A_{12}}^*{B_{12}}^* - {A_{12}}^*{B_{12}}^*A_{12}B_{12} + {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12}A_{12}B_{12}(A_{22}B_{22})^{-1}{A_{12}}^*{B_{12}}^*)]$$

$$= det[(A_{22}B_{22}A_{11}B_{11} - {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12}(A_{11}B_{11}) - {A_{12}}^*{B_{12}}^*A_{12}B_{12}) + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})]$$

$$= det[(A_{22}B_{22} - {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12})\,(A_{11}B_{11}) - {A_{12}}^*{B_{12}}^*A_{12}B_{12} + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})]$$

$$= det[(\boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11})(A_{11}B_{11}) - {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12}(A_{11}B_{11}) + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})]$$

$$= det[((\boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11}) - {A_{12}}^*{B_{12}}^*(A_{11}B_{11})^{-1}A_{12}B_{12}))(A_{11}B_{11}) + (A_{22}B_{22} - (\boldsymbol{A} \square \boldsymbol{B})/A_{11}B_{11})(A_{11}B_{11} - (\boldsymbol{A} \square \boldsymbol{B})/ A_{22}B_{22})]$$

$$= det[(\boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11} + \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11} - A_{22}B_{22})(A_{11}B_{11}) + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})]$$

$$= det[(\boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11} + \boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11} - A_{22}B_{22})(A_{11}B_{11}) + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})(A_{11}B_{11})(A_{11}B_{11})^{-1}]$$

$$= det[((\boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11} + \boldsymbol{A} \square \boldsymbol{B}/ A_{11}B_{11} - A_{22}B_{22}) + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})(A_{11}B_{11})^{-1})(A_{11}B_{11})]$$

$$= det[(\boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11} + \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11} - A_{22}B_{22}) + (A_{22}B_{22} - \boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \square \boldsymbol{B}/A_{22}B_{22})(A_{11}B_{11})^{-1}]det(A_{11}B_{11}) \leq det(\boldsymbol{A} \square \boldsymbol{B}).$$

Since

$$det(\boldsymbol{A} \square \boldsymbol{B}) = det(\boldsymbol{A} \square \boldsymbol{B}/A_{11}B_{11})det(A_{11}B_{11})$$

then

$$det[(\boldsymbol{A} \,\square\, \boldsymbol{B}/\, A_{11}B_{11} + \boldsymbol{A} \,\square\, \boldsymbol{B}/\, A_{11}B_{11} - A_{22}B_{22}) + (A_{22}B_{22} - \boldsymbol{A} \,\square\, \boldsymbol{B}/\, A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \,\square\, \boldsymbol{B}/A_{22}B_{22})(A_{11}B_{11})^{-1}\,]det(A_{11}B_{11}) \leq det(\boldsymbol{A} \,\square\, \boldsymbol{B}/\, A_{11}B_{11})det(A_{11}B_{11})$$

hence

$$det[(\boldsymbol{A} \,\square\, \boldsymbol{B}/\, A_{11}B_{11} + \boldsymbol{A} \,\square\, \boldsymbol{B}/\, A_{11}B_{11} - A_{22}B_{22}) + (A_{22}B_{22} - \boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11})(A_{11}B_{11} - \boldsymbol{A} \,\square\, \boldsymbol{B}/A_{22}B_{22})(A_{11}B_{11})^{-1} \leq det(\boldsymbol{A} \,\square\, \boldsymbol{B}/A_{11}B_{11}) \quad \blacksquare$$

In theorem 3.3.1 we stated the Oppenheim's inequality for Hadamard product of positive difenite matrices. But this thereom also valid for block Hadamard product. Günther & Klotz proved Oppenheim's inequality for block Hadamard product. Now we try to prove the inequality for positive definite block matrices by using the method as in the proof of theorem 1.17 in (F. Zhang, 2005) :

Theorem 4.3.10 Let, $\boldsymbol{B} \in \boldsymbol{M}_p(M_n)$ and $\boldsymbol{A}_{bc}\boldsymbol{B}$**.** If $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite, then

$$det(\boldsymbol{A} \,\square\, \boldsymbol{B}) = det(\boldsymbol{B} \,\square\, \boldsymbol{A}) \geq det\boldsymbol{B} \prod_{i=1}^{p} A_{ii} \tag{4.19}$$

Proof. We will prove by using induction. For $p = 1$

$$\boldsymbol{A} \,\square\, \boldsymbol{B} = A_{11}\boldsymbol{B}$$

if we take the determinant of $\boldsymbol{A} \,\square\, \boldsymbol{B}$, then

$$det(\boldsymbol{A} \,\square\, \boldsymbol{B}) = det(A_{11}\boldsymbol{B}) = detA_{11}det\boldsymbol{B}.$$

Assume that (4.19) is true for all positive definite block matrices that have size less than p. If we partition $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & \boldsymbol{a} \\ \boldsymbol{a}^* & \boldsymbol{A_{p-1}} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} B_{11} & \boldsymbol{b} \\ \boldsymbol{b}^* & \boldsymbol{B_{p-1}} \end{pmatrix}$$

where

$$\boldsymbol{a} = (A_{12} \quad \cdots \quad A_{1p}), \quad \boldsymbol{a}^* = \begin{pmatrix} {A_{12}}^* \\ \vdots \\ {A_{1p}}^* \end{pmatrix}, \quad \boldsymbol{A_{p-1}} = \begin{pmatrix} A_{22} & & & \\ & A_{33} & & \\ & & \ddots & \\ & & & A_{pp} \end{pmatrix}$$

and

$$\boldsymbol{b} = (B_{12} \quad \cdots \quad B_{1p}), \quad \boldsymbol{b}^* = \begin{pmatrix} {B_{12}}^* \\ \vdots \\ {B_{1p}}^* \end{pmatrix}, \quad \boldsymbol{B_{p-1}} = \begin{pmatrix} B_{22} & & & \\ & B_{33} & & \\ & & \ddots & \\ & & & B_{pp} \end{pmatrix}.$$

And if we define the matrices

$$\widehat{\boldsymbol{A}} = \boldsymbol{a}^* {A_{11}}^{-1} \boldsymbol{a} = \begin{pmatrix} {A_{12}}^* \\ \vdots \\ {A_{1p}}^* \end{pmatrix} {A_{11}}^{-1} (A_{12} \quad \cdots \quad A_{1p})$$

$$\widehat{\boldsymbol{B}} = \boldsymbol{b}^* {B_{11}}^{-1} \boldsymbol{b} = \begin{pmatrix} {B_{12}}^* \\ \vdots \\ {B_{1p}}^* \end{pmatrix} {B_{11}}^{-1} (B_{12} \quad \cdots \quad B_{1p})$$

then

$$\boldsymbol{A}/A_{11} = \boldsymbol{A_{p-1}} - \boldsymbol{a}^* {A_{11}}^{-1} \boldsymbol{a} = \boldsymbol{A_{p-1}} - \widehat{\boldsymbol{A}}$$

$$\boldsymbol{B}/B_{11} = \boldsymbol{B_{p-1}} - \boldsymbol{b}^* {B_{11}}^{-1} \boldsymbol{b} = \boldsymbol{B_{p-1}} - \widehat{\boldsymbol{B}}$$

And $\boldsymbol{A}/A_{11}$, $\boldsymbol{B}/B_{11}$ are positive definite. Now let us look at this computation:

$$\boldsymbol{A_{p-1}} \square (\boldsymbol{B_{p-1}} - \widehat{\boldsymbol{B}}) + \left(\boldsymbol{A_{p-1}} - \widehat{\boldsymbol{A}}\right) \square \widehat{\boldsymbol{B}} = \boldsymbol{A_{p-1}} \square \boldsymbol{B_{p-1}} - \widehat{\boldsymbol{A}} \square \widehat{\boldsymbol{B}}$$

$$= \boldsymbol{A} \square \boldsymbol{B} / A_{11} B_{11}$$

$$det(\boldsymbol{A} \square \boldsymbol{B}) = det(A_{11} B_{11}) det(\boldsymbol{A} \square \boldsymbol{B} / A_{11} B_{11})$$

$$= det(A_{11} B_{11}) det[\boldsymbol{A_{p-1}} \square (\boldsymbol{B}/B_{11}) + (\boldsymbol{A}/A_{11}) \square \widehat{\boldsymbol{B}}].$$

Since $\boldsymbol{A_{p-1}} \square (\boldsymbol{B}/B_{11})$ is positive definite and $(\boldsymbol{A}/A_{11}) \square \widehat{\boldsymbol{B}}$ is positive semidefinite. By the theorem 3.2.8,

$$det(\boldsymbol{A} \square \boldsymbol{B}) \geq det(A_{11} B_{11}) \left[det[\boldsymbol{A_{p-1}} \square (\boldsymbol{B}/B_{11})] + det[(\boldsymbol{A}/A_{11}) \square \widehat{\boldsymbol{B}}] \right]$$

$$= det(A_{11} B_{11}) det\left[\boldsymbol{A_{p-1}} \square (\boldsymbol{B}/B_{11})\right] +$$

$$det(A_{11} B_{11}) det\left[(\boldsymbol{A}/A_{11}) \square \widehat{\boldsymbol{B}}\right]$$

$$\geq det(A_{11} B_{11}) det\left[\boldsymbol{A_{p-1}} \square (\boldsymbol{B}/B_{11})\right]$$

$$\geq det(A_{11}B_{11})detA_{22}detA_{33} \dots detA_{pp}det(\boldsymbol{B}/B_{11})$$

$$= det\,A_{11}detB_{11}detA_{22}detA_{33} \dots detA_{pp}det(\boldsymbol{B}/B_{11})$$

$$= detB_{11}det(\boldsymbol{B}/B_{11})det\,A_{11}detA_{22}detA_{33} \dots detA_{pp}$$

$$= det\boldsymbol{B}\prod_{i=1}^{p} A_{ii} \quad \blacksquare$$

From now on we will use some special matrices such as centrosymmetric. So that we must give the definition of these matrices:

Definition 4.3.11 (Li, Zhao, Dai, Su, 2011) $A = [a_{ij}]_{nxn}$ is a *centrosymmetric matrix*, if

$$a_{ij} = a_{n-i+1,n-j+1} \tag{4.20}$$

$$1 \leq i \leq n, \qquad 1 \leq j \leq n$$

or

$$J_nAJ_n = A \tag{4.21}$$

where J_n is the flip matrix with ones on the secondary diagonal and zeros elsewhere.

Lemma 4.3.12 Let $A = [A_{ij}]_{nxn}$, $B = [B_{ij}]_{nxn}$ $(n = 2m)$ be two centrosymmetric matrices with the following form:

$$A = \begin{pmatrix} A_1 & J_mA_2J_m \\ A_2 & J_mA_1J_m \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & J_mB_2J_m \\ B_2 & J_mB_1J_m \end{pmatrix}$$

$A_1, A_2, B_1, B_2 \in \mathbb{C}^{m\times m}$. Then, $A \,\square\, B$ is centrosymmetric.

Proof.

$$A \,\square\, B = \begin{pmatrix} A_1B_1 & J_mA_2J_mJ_mB_2J_m \\ A_2B_2 & J_mA_1J_mJ_mB_1J_m \end{pmatrix}$$

since $J_mJ_m = I_m$ then

$$A \,\square\, B = \begin{pmatrix} A_1B_1 & J_mA_2B_2J_m \\ A_2B_2 & J_mA_1B_1J_m \end{pmatrix}$$

Therefore $A \square B$ is also centrosymmetric. ■

Example 4.7 Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 2 & 3 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 3 \end{pmatrix}$$

then

$$A \square B = \begin{pmatrix} 7 & 4 & 3 & 12 & 9 & 9 \\ 3 & 1 & 2 & 4 & 1 & 1 \\ 7 & 5 & 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 & 5 & 7 \\ 1 & 1 & 4 & 2 & 1 & 3 \\ 9 & 9 & 12 & 3 & 4 & 7 \end{pmatrix}$$

which is centrosymmetric.

Observation 4.3.13 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p(M_n)$ be two centrosymmetric matrices with the following form:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A_1} & \boldsymbol{J_m A_2 J_m} \\ \boldsymbol{A_2} & \boldsymbol{J_m A_1 J_m} \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{J_m B_2 J_m} \\ \boldsymbol{B_2} & \boldsymbol{J_m B_1 J_m} \end{pmatrix}$$

where $p = 2m$ and $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{B_1}, \boldsymbol{B_2}, \boldsymbol{J_m} \in \boldsymbol{M}_m(M_n)$ and where $\boldsymbol{J_m}$ is a block flip matrix . Then $\boldsymbol{A} \square \boldsymbol{B}$ is also centrosymmetric.

Example 4.8 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_4(M_2)$

$$\boldsymbol{A} = \left(\begin{array}{cc|cc|cc|cc} 1 & 3 & 1 & 2 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 & 3 & 0 \\ \hline 2 & 0 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 2 & 0 & 3 \\ \hline 3 & 0 & 2 & 0 & 3 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 & 2 \\ \hline 0 & 3 & 1 & 2 & 0 & 1 & 2 & 0 \\ 2 & 2 & 1 & 0 & 2 & 1 & 3 & 1 \end{array}\right) \quad \text{and} \quad \boldsymbol{B} = \left(\begin{array}{cc|cc|cc|cc} 1 & 2 & 1 & 3 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 & 2 & 1 & 2 & 3 \\ \hline 3 & 2 & 1 & 2 & 1 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 1 & 3 & 1 & 2 & 1 \end{array}\right)$$

then

$$A \,\square\, B = \left(\begin{array}{cc:cc:cc:cc} 4 & 2 & 3 & 3 & 1 & 0 & 8 & 2 \\ 2 & 0 & 1 & 3 & 3 & 2 & 9 & 3 \\ \hdashline 0 & 2 & 3 & 1 & 4 & 3 & 2 & 4 \\ 2 & 3 & 0 & 3 & 4 & 2 & 6 & 9 \\ \hdashline 9 & 6 & 2 & 4 & 3 & 0 & 3 & 2 \\ 4 & 2 & 3 & 4 & 1 & 3 & 2 & 0 \\ \hdashline 3 & 9 & 2 & 3 & 3 & 1 & 0 & 2 \\ 2 & 8 & 0 & 1 & 3 & 3 & 2 & 4 \end{array}\right)$$

$\boldsymbol{A} \,\square\, \boldsymbol{B}$ is centrosymmetric.

Lemma 4.3.14 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_p\,(M_n\,)$ be two centrosymmetric matrices with the following form:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A_1} & \boldsymbol{J_m A_2 J_m} \\ \boldsymbol{A_2} & \boldsymbol{J_m A_1 J_m} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{J_m B_2 J_m} \\ \boldsymbol{B_2} & \boldsymbol{J_m B_1 J_m} \end{pmatrix}$$

where, $p = 2m$ $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{B_1}, \boldsymbol{B_2}, \boldsymbol{J_m} \in \boldsymbol{M}_p(M_n)$ and where $\boldsymbol{J_m}$ is a block flip matrix. Then $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is also centrosymmetric.

Proof. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are centrosymmetric then

$$\boldsymbol{A} = \boldsymbol{J_{2m} A J_{2m}} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{J_{2m} B J_{2m}}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{J_{2m}} \in \boldsymbol{M}_p(M_n)$ and $\boldsymbol{J_{2m}}$ is block flip matrix. Then

$$\begin{aligned} \boldsymbol{A} \boxtimes \boldsymbol{B} &= (\boldsymbol{J_{2m} A J_{2m}}) \boxtimes (\boldsymbol{J_{2m} B J_{2m}}) \\ &= [(\boldsymbol{J_{2m} A}) \boxtimes (\boldsymbol{J_{2m} B})](\boldsymbol{J_{2m}} \boxtimes \boldsymbol{J_{2m}}) \quad \text{by proposition 4.2.4(a)} \\ &= (\boldsymbol{J_{2m}} \boxtimes \boldsymbol{J_{2m}})(\boldsymbol{A} \boxtimes \boldsymbol{B})\boldsymbol{J_{4m^2}} \quad \text{by proposition 4.2.4 (a)} \\ &= \boldsymbol{J_{4m^2}}(\boldsymbol{A} \boxtimes \boldsymbol{B})\boldsymbol{J_{4m^2}} \end{aligned}$$

Hence, $\boldsymbol{A} \boxtimes \boldsymbol{B}$ is also centrosymmetric. ■

Example 4.9 Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_2(M_2)$. So

$p = 2, m = 1 \quad J_m = \boldsymbol{J}_1 \quad \boldsymbol{A} = \boldsymbol{J_2 A J_2} \quad \boldsymbol{B} = \boldsymbol{J_2 B J_2}$

$$\boldsymbol{A}=\left(\begin{array}{cc:cc}1&3&1&1\\0&2&0&2\\\hdashline 2&0&2&0\\1&1&3&1\end{array}\right)\qquad\text{and}\qquad\boldsymbol{B}=\left(\begin{array}{cc:cc}2&1&2&3\\1&1&0&0\\\hdashline 0&0&1&1\\3&2&1&2\end{array}\right)$$

Then

$$\boldsymbol{A}\boxtimes\boldsymbol{B}=\left(\begin{array}{cc:cc:cc:cc}5&4&2&3&3&2&2&3\\2&2&0&0&2&2&0&0\\\hdashline 9&2&4&7&3&2&2&3\\6&4&2&4&6&4&2&4\\\hdashline 4&2&4&6&4&2&4&6\\3&2&2&3&7&4&6&9\\\hdashline 0&0&2&2&0&0&2&2\\3&2&2&3&3&2&4&5\end{array}\right)$$

That is $\quad\boldsymbol{A}\boxtimes\boldsymbol{B}=\boldsymbol{J_4}(\boldsymbol{A}\boxtimes\boldsymbol{B})\boldsymbol{J_4}$.

CHAPTER FIVE
CONCLUSION

In this thesis we have worked out the Schur complement of matrices. For this purpose, we have studied the articles on inequalities involving Schur complement and Hadamard product. Then we have obtained some results concerning Schur complement and Hadamard product.

After examining the usual Hadamard and Kronecker products of matrices, then we studied the block Hadamard and block Kronecker products. By using these products, we focused on the Schur complement of block matrices. Firstly, we have established the definition of the Schur complement in the block form and we gave some examples about this. After that, we have analyzed the block Hadamard and block Kronecker products. We achieved some new results and constructed new lemmas and theorems on the Schur complement of matrices involving block Hadamard and block Kronecker products. Also we found an analogy between the inequalities involving Hadarmard, Kronecker products and block form of these products. After this, we gave some examples that providing the lemmas and theorems that we construct .

Finally, we studied the centrosymmetric matrices. We analyzed the block Hadamard and block Kronecker products of centrosymmetric matrices. Then we established some new lemmas and give some examples that providing the lemmas about the centrosymmetric matrices.

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