DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON INEQUALITIES INVOLVING THE HADAMARD PRODUCT AND NORMS OF MATRICES

by
Dilek KAYAALP

February, 2015
İZMİR

ON INEQUALITIES INVOLVING THE HADAMARD PRODUCT AND NORMS OF MATRICES

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science
in Mathematics, Mathematics Program

by
Dilek KAYAALP

February, 2015
İZMİR

M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"ON INEQUALITIES INVOLVING THE HADAMARD PRODUCT AND NORMS OF MATRICES"** completed by **DİLEK KAYAALP** under supervision of **ASSOC. PROF. DR. MUSTAFA ÖZEL** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Mustafa ÖZEL

Supervisor

Prof. Dr. Halil ORUÇ

(Jury Member)

Prof Dr Alpay Kırlangıç

(Jury Member)

Prof.Dr. Ayşe OKUR
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Assoc. Prof. Dr. MUSTAFA ÖZEL for his guidance, continual presence, encouragement and endless patience during the course of this research.

Since my sister Necla KAYAALP gave her time and support me frequently for my thesis I would like to thank her specially.

Finally, I am thankful to my family for their confidence in me throughout my life.

Dilek KAYAALP

ON INEQUALITIES INVOLVING THE HADAMARD PRODUCT AND NORMS OF MATRICES

ABSTRACT

Inequalities in Hadamard product has been an active research area in the recent years. In this thesis, the studies about Hadamard products of different type of matrices, spectral radius of those matrices and inequalities on the spectral radius of matrices are investigated. Furthermore, new lower and upper bounds are computed for those inequalities involving spectral radius with the help of the norms of matrices.

Keywords: Spectral radius, Hadamard product, matrix norm.

MATRİSLERİN HADAMARD ÇARPIMLARINI VE NORMLARINI İÇEREN EŞİTSİZLİKLER

ÖZ

Hadamard çarpımlarını içeren eşitsizlikler son yıllarda aktif bir çalışma alanı oluşturmuştur. Bu tezde, farklı matris çeşitlerinin Hadamard çarpımları, bu matrislerin spektral yarıçaplarını içeren eşitsizlikler incelenmiştir. Buna ek olarak, matrislerin normları yardımıyla spektral yarıçapları içeren eşitsizlikler için yeni alt ve üst sınırlar hesaplanmıştır.

Anahtar kelimeler: Spektral yarıçap, Hadamard çarpım, matris normu.

CONTENTS

CHAPTER ONE

INTRODUCTION

Matrices and matrix operations are studied and applied in many areas such as engineering, natural and social sciences. Although they are quite helpful to understand, to classify and to solve the problems in these areas this is not enough to solve all the problems. Different types of matrices and different properties of these matrices are needed in order to overcome such problems. Hence, matrix theory is mainly necessary for classifying matrices, determining the elements of matrices which will be able to generate new type of matrices and new inequalities containing those matrices. Lots of information about matrices can be found in the books of Horn and Johnson (1985, 1991).

There are studies on inequalities involving the spectral radius of a matrix. Ledermann (1950) and Ostrowski (1952) have found an interval for the spectral radius of a positive matrix by using minimum and maximum row sums of this matrix. Moreover, Horn and Johnson (1985) proposed an upper bound for the spectral radius of two nonnegative matrices. A decade later, Audenart (2010) achieved to obtained a better upper bound involving the spectral radius of the Hadamard product of two nonnegative matrices. To prove this theorem, he used the definition of the spectral radius by the help of the trace of a matrix.

Since centrosymmetric matrices have many applications, in recent years a lot of work has been done on centrosymmetric matrices. Li et al. (2011), discussed the bounds of the spectral radius of nonnegative matrices with centrosymmetric structure. In this thesis, we have achieved to obtain and to prove the results for new bounds of spectral radius of the Hadmard and Kronecker products of centrosymmetric matrices.

Now, we will look at brief information about how this thesis has been organized:

Chapter 2 provides main properties, theorems, and definitions which are used within this thesis and it includes three parts. In the first part, important definitions for special products of matrices and related norms of these matrices are given. The special products such as Hadamard, Kronecker and Fan products will be introduced to be able to understand the lemmas, theorems and corollaries in this thesis.In second part, the other basic definitions like spectral radius and centrosymmetric matrix are given.The third part of this chapter will come to the end with some results about trace of a matrix and the Kronecker product of the matrices.

In Chapter 3, we will be interested in inequalities involving spectral radius of matrices.We will analyze the main theorems proved by Ledermann in 1950, Zhao & Dai in 2011 and Ostrowski in 1952. After examining these theorems, we will construct new bounds for the spectral radius of a positive centrosymmetric matrix.In the last part of Chapter 3, we will obtain some new propositions on spectral radius without proving.

In Chapter 4, we will be specially focused on the inequalities involving centrosymmetric matrices. In addition, we will be dealing with special matrix multiplications of two centrosymmetric matrices.We will check whether the the multiplications are centrosymmetric or not. After then, we will try to construct new bounds involving the spectral radius of the centrosymmetric matrices.

CHAPTER TWO
PRELIMINARIES

This chapter provides main properties, theorems and definitions which are used within this thesis.

Let M_{mxn} denote an $m \times n$ matrix and let M_n denote a matrix with size $n \times n$. Also, let $\mathbb{C}_{mxn}$ denote the complex plane containing all $m \times n$ complex matrices.

2.1 Definitions for Products of Matrices and Norms

In order to understand this thesis, it is necessary to give basic whereas important definitions which are special products of matrices and related norms of those matrices:

Definition 2.1.1 (Horn & Johnson, 1985) Let $A = [a_{ij}] \in M_{mxn}$ and $B = [b_{ij}] \in M_{mxn}$ be given as two matrices, then Hadamard product of A and B is

$$A \circ B = \left[a_{ij} b_{ij}\right]. \tag{2.1}$$

Note that Hadamard product is also known as Schur product.

Definition 2.1.2 (Horn & Johnson, 1991) Kronecker product of two matrices A_{mxn} and B_{pxq} is an $mp \times nq$ matrix denoted by $A \otimes B$ and is defined as

$$A \otimes B = [a_{ij} B] \tag{2.2}$$

or

$$A \otimes B = [A b_{ij}]. \tag{2.3}$$

Definition 2.1.3 (Huang, 2008) Let $A_{mxn} = [a_{ij}]$ and $B_{mxn} = [b_{ij}]$ be two matrices in the space of M_{mxn}. Then Fan product of A and B is given as

$$C \equiv A \star B \tag{2.4}$$

where

$$C_{mxn} = [c_{ij}] \tag{2.5}$$

with elements

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j. \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases} \tag{2.6}$$

Definition 2.1.4 (Huang, 2008) Khatri-Rao Product

Let $A \in M_{mxn}$ and $B \in M_{pxq}$ be two matrices. Then

$$A \odot B = \left(A_{ij} \otimes B_{ij}\right)_{ij} \tag{2.7}$$

where

$$A_{ij} \otimes B_{ij} \text{ is of order } m_i p_i \times n_j q_j$$

and

$$A \odot B \text{ is of order } \left(\sum m_i p_i\right) \times \left(\sum n_j q_j\right).$$

In matrix theory, matrix norm is very important and useful tool when determining behavior of a matrix. There are several ways and there exist some formulas in order to describe a norm. Some of them will be given with the help of the vector norms like in the following:

Definition 2.1.5 (Meyer, 2004) *p*-Norms

For $p \geq 1$, the *p*-norm of a vector $x \in \mathbb{C}_p$ is defined as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}. \tag{2.8}$$

In practice, the following three of the *p*-norms are used frequently:

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$, the grid norm.

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$, the Euclidean norm.

$\|x\|_\infty = \lim_{p\to\infty}\|x\|_p = \lim\limits_{p\to\infty} \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} = \max_{1\le i\le n}|x_i|$, the max norm.

Definition 2.1.6 (Meyer, 2004) Induced Matrix Norms

A vector norm that is defined on $\mathbb{C}_p$ for $p = mxn$ induces a matrix norm on $\mathbb{C}_{mxn}$ by setting

$\|A\| = \max_{\|x\|=1}\|Ax\|$ for $A \in \mathbb{C}_{mxn},\ x \in \mathbb{C}_{nx1}$. (2.9)

Definition 2.1.7 (Meyer, 2004) Frobenius Norm

Frobenius norm of $A \in \mathbb{C}_{mxn}$ is defined as follows:

$$\|A\|^2 = \sum_{i,j}\left|a_{ij}\right|^2. \tag{2.10}$$

Note that Frobenius norm can be seen as Hilbert norm or Schur norm in some text books.

Definition 2.1.8 (Meyer, 2004) Matrix 1- Norm

Informally, it is defined as the largest absolute of column sum. Formally, it is defined as

$$\|A\|_1 = \max_{\|x\|_1=1}\|Ax\|_1 = \max_j \sum_j\left|a_{ij}\right|. \tag{2.11}$$

Definition 2.1.9 (Meyer, 2004) Matrix 2- Norm

The matrix norm, induced by the Euclidean vector norm, is

$$\|A\|_2 = \max_{\|x\|_2=1}\|Ax\|_2 = \sqrt{\lambda_{\max}} \tag{2.12}$$

where $\lambda_{\max}$ is the largest eigenvalue λ which satisfies the equation

$$\det(A^*A - \lambda I) = 0.$$

Note that this norm is also known as spectral norm.

Definition 2.1.10 (Meyer, 2004) Matrix ∞- Norm

$$\|A\|_\infty = \max_{\|x\|_\infty=1}\|Ax\|_\infty = \max_i \sum_j |a_{ij}|. \tag{2.13}$$

It can be considered as the largest absolute row sum.

2.2 Definitions for Spectral Radius and Types of Matrices

Definition 2.2.1 (Li & Zhao, 2011) Let $A = [a_{ij}]_{nxn}$ be a matrix with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then

$$\rho(A) = \max\{\, |\lambda_1|, |\lambda_2|, \dots, |\lambda_n| \,\} \tag{2.14}$$

is the spectral radius of *A*.

Definition 2.2.2 (Meyer, 2004) Let $A = [a_{ij}]_{mxn}$, then positive square roots of the nonzero eigenvalues of A^*A (or AA^*) are called nonzero singular values $s_i(A)$.

Note that

$$s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A).$$

Now, let us give other well-known matrix norms.

Definition 2.2.3 (Du, 2010) Spectral Norm

Let $A \in M_n$, then $\|A\|_{(1)} = s_1(A)$.

Definition 2.2.4 (Du, 2010) Trace Norm (nuclear norm)

$$\|A\|_{(n)} = \sum_{j=1}^{n} s_j(A). \tag{2.15}$$

In matrix theory, there are many different types of matrices, but in this thesis we will be frequently dealing with nonnegative, positive, positive definite, positive-semidefinite and centrosymmetric matrices.

Definition 2.2.5 (Li, Zhao, Dai, 2011) Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$. Then

- A is nonnegative if all $a_{ij} \geq 0$ and it is denoted as $A \geq 0$.
- A is positive if all $a_{ij} > 0$ and it is denoted as $A > 0$.
- $A \geq B$ if $a_{ij} \geq b_{ij}$ for all *i,j*.
- $A > B$ if $a_{ij} > b_{ij}$ for all *i,j*.

Definition 2.2.6 (Meyer, 2004) If all eigenvalues of $A \in \mathbb{C}_{nxn}$ are positive then A is called positive definite matrix and it is denoted as $A \succ 0$.

It is obvious that $A \succ B$ means that $A - B \succ 0$.

If all eigenvalues of $A \in \mathbb{C}_{nxn}$ are nonnegative then A is called positive semi-definite matrix and it is shown as $A \succcurlyeq 0$. It is also obvious that $A \succcurlyeq B$ means that $A - B \succcurlyeq 0$.

Definition 2.2.7 (Li, Zhao, Dai, Su, 2011) $A = \left[a_{ij}\right]_{nxn}$ is a centrosymmetric matrix, if

$$a_{ij} = a_{n-i+1,n-j+1} \tag{2.16}$$

where

$$1 \leq i \leq n, \qquad 1 \leq j \leq n$$

or

$$J_n A J_n = A \tag{2.17}$$

where J_n is the flip matrix with ones on the secondary diagonal and zeros elsewhere.

If A is both nonnegative and centrosymmetric then A is called a nonnegative centrosymmetric matrix.

2.3 Basic Lemmas and Theorems on Inequalities Involving Hadamard Product

In this part of this thesis, some necessary lemmas and theorems which will be used in the following chapters are given in order to understand what we have studied.

Theorem 2.3.1 (Horn & Johnson, 1985; Brewer, 1978) Let $A = \left[a_{ij}\right]_{nxn}$ then

$$tr(AB) = tr(BA). \tag{2.18}$$

$$tr(A \otimes B) = tr(A)tr(B). \tag{2.19}$$

Theorem 2.3.2 (Brewer, 1978) Let A, B, C, G be matrices in $\mathbb{C}_{nxn}$ then

$$(A \otimes B)(C \otimes G) = AC \otimes BG. \qquad (2.20)$$

Lemma 2.3.1 (Neudecker, 1990) Let A be a positive semidefinite matrix, then $tr(A) \geq 0$.

Theorem 2.3.3 (Neudecker, 1990) If A and B are positive semidefinite matrices, then $tr(AB) \geq 0$.

Lemma 2.3.2 (Yang, 1988) Let A be a positive definite matrix, then $tr(A) > 0$.

Theorem 2.3.4 (Yang, 1988) If A and B are positive definite matrices, then $tr(AB) > 0$.

CHAPTER THREE
ON SPECTRAL RADIUS OF CENTROSYMMETRIC MATRICES

In this chapter of the thesis, we will be interested in inequalities involving spectral radius of positive centrosymmetric matrices. We will establish some estimations of the spectral radius.

3.1 Inequalities Involving Spectral Radius

Recently, a number of inequalities and result involving the spectral radius of positive centrosymmetric matrices are established by many researchers. In this section, we will analyze some theorems proved by Ledermann (1950), Liu, Zhao, Dai & Su (2011), and Ostrowski (1952).

Theorem 3.1.1 (Ledermann, 1950) Let $A = \left[a_{ij}\right]_{nxn}$ be a positive matrix, then

$$r + a\left(\frac{1}{\sqrt{\delta}} - 1\right) \leq \rho(A) \leq R - a\left(1 - \sqrt{\delta}\right) \tag{3.1}$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \max_{r_i < r_j}\left\{\frac{r_i}{r_j}\right\}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

and

$$a = \min\{a_{ij}\}, \qquad 1 \leq i, j \leq n.$$

H. Li et al. used Thereom 3.1.1 to prove the following theorem.

Theorem 3.1.2 (Liu et al., 2011) Let $A = \left[a_{ij}\right]_{nxn}$ be positive centrosymmetric matrix with the following form

$$A = \begin{bmatrix} A_1 & J_m A_2 J_m \\ A_2 & J_m A_1 J_m \end{bmatrix} \tag{3.2}$$

and

$$D = A_1 + JA_2$$

where $A_1, A_2, \in \mathbb{C}_{mxm}$ and $n = 2m$. Then

$$r + d\left(\frac{1}{\sqrt{\delta_D}} - 1\right) \le \rho(A) \le R - d\left(1 - \sqrt{\delta_D}\right) \qquad (3.3)$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta_D = \max_{r_i < r_j}\left\{\frac{r_i}{r_j}\right\}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

and

$$d = \min\{d_{ij}\}, \qquad 1 \le i, j \le n.$$

Note that this interval is better than Ledermann's interval. Because,

$$r + a\left(\frac{1}{\sqrt{\delta}} - 1\right) \le \boldsymbol{r} + \boldsymbol{d}\left(\frac{\mathbf{1}}{\sqrt{\boldsymbol{\delta_D}}} - \mathbf{1}\right) \le \rho(A) \le \boldsymbol{R} - \boldsymbol{d}\left(\mathbf{1} - \sqrt{\boldsymbol{\delta_D}}\right) \le R - a\left(1 - \sqrt{\delta}\right). \quad (3.4)$$

It is also remarkable that Zhao, Li, & Gong's interval depends only the matrix D with size mxm, not the matrix A with size $2mx2m$.

Theorem 3.1.3 (Ostrowski, 1952) Let $A = [a_{ij}]$ be nonnegative matrix with size nxn. Then

$$r + a\left(\frac{1}{\delta} - 1\right) \le \rho(A) \le R - a(1 - \delta) \qquad (3.5)$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \sqrt{\frac{r-a}{R-a}}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

$$a = \min\{a_{ij}\}, \qquad 1 \le i, j \le n.$$

3.2 A Result for the Bounds of a Spectral Radius

In this section, we will construct and prove a new theorem containing the bounds of a spectral radius. Some new lower and upper bounds of positive centrosymmetric matrices using their minimum and maximum row sums are obtained. Numerical examples are given in order to illustrate the effectiveness of the proposed approach.

Theorem 3.2.1 Let $A = [a_{ij}]_{nxn}$ be a positive centrosymmetric matrix with $n = 2m$. Then

$$r + a\left(\frac{1}{\sqrt{\delta}} - 1\right) \leq r + a\left(\frac{1}{\sqrt{\mu}} - 1\right) \tag{3.6}$$

$$R - a(1 - \sqrt{\mu}) \leq R - a\left(1 - \sqrt{\delta}\right) \tag{3.7}$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \max_{r_i < r_j}\left\{\frac{r_i}{r_j}\right\}, \qquad \mu = \min_{r_i < r_j}\left\{\frac{r_i}{r_j}\right\}$$

$$r = \min\{r_i\}, \quad R = \max\{r_i\}, \quad a = \min\{a_{ij}\}, \quad 1 \leq i, j \leq n.$$

Proof

It is clear that $\mu < \delta$. Taking square root of both sides, we obtain

$$\sqrt{\mu} < \sqrt{\delta}.$$

Then,

$$\frac{1}{\sqrt{\delta}} < \frac{1}{\sqrt{\mu}}.$$

Adding -1 to both sides, we obtain the following

$$\frac{1}{\sqrt{\delta}} - 1 < \frac{1}{\sqrt{\mu}} - 1.$$

Multiplying both sides by a gives

$$a\left(\frac{1}{\sqrt{\delta}}-1\right) < a\left(\frac{1}{\sqrt{\mu}}-1\right).$$

Then, adding r to both sides, we get the following inequality:

$$r + a\left(\frac{1}{\sqrt{\delta}}-1\right) < r + a\left(\frac{1}{\sqrt{\mu}}-1\right).$$

Similarly, (3.7) can be proved easily.

It is obvious that $\mu < \delta$. Taking square root of both sides, we obtain

$$\sqrt{\mu} < \sqrt{\delta}.$$

Then,

$$-\sqrt{\delta} < -\sqrt{\mu}\,.$$

Adding 1 to both sides, we obtain the following

$$1-\sqrt{\delta} < 1-\sqrt{\mu}.$$

Multiplying both sides by $-a$ gives

$$-a\left(1-\sqrt{\mu}\right) < -a\left(1-\sqrt{\delta}\right).$$

Then, adding R to both sides, we get the following inequality:

$$R - a\left(1-\sqrt{\mu}\right) < R - a\left(1-\sqrt{\delta}\right).$$

Be aware of that we obtained a bigger lower bound and a smaller upper bound. This shows that our bounds are sharper than Ledermann's.

Example 3.2.1

The following example which is used in the Zhao, Dai, & Su (2011) is chosen.

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 & 2 & 4 \\ 2 & 1 & 1 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 & 3 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 6 & 3 \\ 4 & 2 & 4 \\ 5 & 5 & 5 \end{bmatrix}$$

$\rho(A) = 12{,}6033$

Theorem 3.1.1: $10{,}0742 \leq \rho(A) \leq 14{,}9309$

Theorem 3.1.2: $10{,}1483 \leq \rho(A) \leq 14{,}8619$

Theorem 3.2.1: $10{,}224 \leq \rho(A) \leq \ 14{,}816$

Note that the last interval which we found by using our theorem is the best approach for $\rho(A)$.

Example 3.2.2

$$A = \begin{bmatrix} 1 & 2 & 1 & 6 & 1 & 4 & 2 & 1 \\ 2 & 1 & 3 & 1 & 6 & 3 & 5 & 7 \\ 4 & 2 & 1 & 3 & 1 & 2 & 4 & 3 \\ 1 & 3 & 4 & 2 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 2 & 4 & 3 & 1 \\ 3 & 4 & 2 & 1 & 3 & 1 & 2 & 4 \\ 7 & 5 & 3 & 6 & 1 & 3 & 1 & 2 \\ 1 & 2 & 4 & 1 & 6 & 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 4 & 5 & 7 \\ 9 & 6 & 6 & 7 \\ 7 & 6 & 3 & 4 \\ 3 & 4 & 5 & 5 \end{bmatrix}$$

$\rho(A) = 20{,}5866$

Theorem 3.1.1: $17{,}0289 \leq \rho(A) \leq 27{,}9171$

Theorem 3.1.2: $17{,}1807 \leq \rho(A) \leq 27{,}8342$

Theorem 3.2.1: $17{,}2833 \leq \rho(A) \leq 27{,}779$

Note here that the last interval is the best approach for $\rho(A)$.

The following example is given in order to observe whether there is a difference in the result when the entries of the matrix are increased.

Example 3.2.3

$$A = \begin{bmatrix} 4 & 3 & 9 & 5 & 3 & 11 \\ 3 & 6 & 2 & 8 & 4 & 5 \\ 7 & 5 & 7 & 3 & 2 & 2 \\ 2 & 2 & 3 & 7 & 5 & 7 \\ 5 & 4 & 8 & 2 & 6 & 3 \\ 11 & 3 & 5 & 9 & 3 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 15 & 6 & 14 \\ 8 & 10 & 10 \\ 9 & 7 & 10 \end{bmatrix}$$

$\rho(A) = 29{,}8041$

Theorem 3.1.1: $26{,}0755 \leq \rho(A) \leq 34{,}927$

Theorem 3.1.2: $26{,}227 \quad \leq \rho(A) \leq 34{,}781$

Theorem 3.2.1: $26{,}3204 \leq \rho(A) \leq 34{,}723$

In conclusion, it can be easily seen that the result of the Theorem 3.2.1 is better than the result of the theorems 3.1.1 and 3.1.2.

3.3 Some Propositions on Spectral Radius

While studying on the article of Ostrowski we have found some propositions containing spectral radius. These propositions work in all examples which we have studied. However, in theory we couldn’t prove them. To see effectiveness of those propositions, we will give some numerical examples.

Proposition 3.3.1 Let $A = [a_{ij}]_{nxn}$ be a nonnegative matrix. Then

$$r + a\left(\frac{1}{\delta} - 1\right) \leq \rho(A) \leq R - a(1-\delta) \tag{3.8}$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \sqrt{\frac{r-a}{2R-a}}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

and

$$a = \min\{a_{ij}\}, \qquad 1 \leq i, j \leq n.$$

Proposition 3.3.2 Let $A = [a_{ij}]_{nxn}$ be a nonnegative matrix. Then

$$r + a\left(\frac{1}{\delta} - 1\right) \leq \rho(A) \leq R - a(1 - \delta) \tag{3.9}$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \sqrt{\frac{r - a}{r + R - a}}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

and

$$a = \min\{a_{ij}\}, \qquad 1 \leq i, j \leq n.$$

Proposition 3.3.3 Let $A = [a_{ij}]_{nxn}$ be a nonnegative matrix. Then

$$r + a\left(\frac{1}{\delta} - 1\right) \leq \rho(A) \leq R - a(1 - \delta) \tag{3.10}$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \sqrt{\frac{r - a}{R + 2a}}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

and

$$a = \min\{a_{ij}\}, \qquad 1 \leq i, j \leq n.$$

Proposition 3.3.4 Let $A = [a_{ij}]_{nxn}$ be a nonnegative matrix. Then

$$r + a\left(\frac{1}{\delta} - 1\right) \leq \rho(A) \leq R - a(1 - \delta) \tag{3.11}$$

where

$$r_i = \sum_{k=1}^{n} a_{ik}, \qquad \delta = \sqrt{\frac{r-a}{R+a}}$$

$$r = \min\{r_i\}, \qquad R = \max\{r_i\}$$

and

$$a = \min\{a_{ij}\}, \qquad 1 \leq i,j \leq n.$$

Example 3.3.1

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 & 3 & 1 \\ 1 & 3 & 4 & 1 & 2 & 4 \\ 2 & 1 & 1 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 & 2 & 1 \end{bmatrix}$$

$\rho(A) = 12{,}6033$

Theorem 3.1.3 $\quad :10{,}247 \leq \rho(A) \leq 14{,}801$

Proposition 3.3.1 : $10{,}795 \leq \rho(A) \leq 14{,}557$

Proposition 3.3.2 :$10{,}632 \leq \rho(A) \leq 14{,}612$

Proposition 3.3.3 :$10{,}374 \leq \rho(A) \leq 14{,}727$

Proposition 3.3.4 :$10{,}333 \leq \rho(A) \leq 14{,}75$

Example 3.3.2

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{5} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{5} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

$\rho(A) = 1{,}4125$

Theorem 3.1.3 : $1{,}3051 \leq \rho(A) \leq 1{,}5132$

Proposition 3.3.1 : $1{,}4086 \leq \rho(A) \leq 1{,}4559$

Proposition 3.3.2 : $1{,}3941 \leq \rho(A) \leq 1{,}4616$

Proposition 3.3.3 : $1{,}3504 \leq \rho(A) \leq 1{,}4827$

Proposition 3.3.4 : $1{,}3362 \leq \rho(A) \leq 1{,}4911$

Example 3.3.3

$$A = \begin{bmatrix} 1 & 2 & 1 & 6 & 1 & 4 & 2 & 1 \\ 2 & 1 & 3 & 1 & 6 & 3 & 5 & 7 \\ 4 & 2 & 1 & 3 & 1 & 2 & 4 & 3 \\ 1 & 3 & 4 & 2 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 2 & 4 & 3 & 1 \\ 3 & 4 & 2 & 1 & 3 & 1 & 2 & 4 \\ 7 & 5 & 3 & 6 & 1 & 3 & 1 & 2 \\ 1 & 2 & 4 & 1 & 6 & 1 & 2 & 1 \end{bmatrix}$$

$\rho(A) = 20{,}586$

Theorem 3.1.3 : $17{,}299 \leq \rho(A) \leq 27{,}769$

Proposition 3.3.1 : $17{,}850 \leq \rho(A) \leq 27{,}539$

Proposition 3.3.2 : $17{,}658 \leq \rho(A) \leq 27{,}603$

Proposition 3.3.3 : $17{,}369 \leq \rho(A) \leq 27{,}73$

Proposition 3.3.4 : $17{,}346 \leq \rho(A) \leq 27{,}742$

In all examples it is remarkable that the proposed propositions give a better result than Ostrowski's theorem. Moreover, it is seen that the order of importance of $\rho(A)$ is the same. Namely, the best interval is obtained from Proposition 3.3.1 and the worst interval is obtained from the theorem of Ostrowski.

CHAPTER FOUR

INEQUALITIES ON HADAMARD AND KRONECKER PRODUCTS

In this chapter, we will be interested in inequalities involving Hadamard product of matrices. After that we will focus on properties and inequalities of centrosymmetric matrices in detail.

4.1 Spectral Radius and Norm Inequalities Involving Hadamard Product for Matrices

In this section, we will summarize and analyze the literature that is related to our study. Almost all lemmas and theorems which are related to spectral radius and the norm for Hadamard product on different types of matrices such as positive definite, nonnegative and centrosymmetric matrices are represented in details. All inequalities that are mentioned in the literature are compared and investigated in order to achieve the appropriate interval for spectral radius.

Lemma 4.1.1 (Audenaert, 2010)

$$\rho(A^m) = \rho(A)^m \tag{4.1}$$

for all square complex matrix A and all $m = 1, 2, 3, \ldots$

Theorem 4.1.1 (Horn & Johnson, 1985)

Let $A, B \in M_n$, $A \geq 0$, and $B \geq 0$, then

$$\rho\,(A \circ B) \leq \rho\,(A)\rho(B). \tag{4.2}$$

It is obvious that spectral radius is multiplicative with respect to Hadamard product.

Lemma 4.1.2 (Audenaert, 2010) For a given $A \in M_n$, such that $A > 0$,

$$\rho(A) = \lim_{m \to \infty} (trA^m)^{\frac{1}{m}}. \tag{4.3}$$

Corollary 4.1.1 (Zhan, 2009) Zhan's Conjecture

Let A_{nxn}, B_{nxn} be nonnegative matrices. Then

$$\rho\ (A\circ B) < \ \rho\ (AB). \tag{4.4}$$

Theorem 4.1.2 (Audenaert, 2010) For $n \times n$ nonnegative matrices A and B,

$$\rho\ (A \circ B) \leq \rho[(\ A \circ A)(B \circ B\ )]^{1/2}\ \leq\ \rho\ (AB). \tag{4.5}$$

Notice that Audenaert improved Zhan's conjecture using Lemma 4.1.2.

Theorem 4.1.3 (Horn & Zhang, 2010) Let $A, B\ \in M_n$. Suppose that $A\ \geq\ 0$ and $B\ \geq\ 0$. Then

$$\rho\ (A\circ B) \leq\ \rho^{1/2}(\ AB\circ BA\ ) \leq \rho\ (AB). \tag{4.6}$$

Again one can see that (4.6) is an improved version of Zhan's conjecture.

Proposition 4.1.1 (Horn & Johnson, 1991) Let A_{mxn} and B_{mxn} then

$$A\circ B = (A\otimes B)_{[\alpha,\beta]} \tag{4.7}$$

where

$$\alpha = \{1, m+2, 2m+3, \dots, m^2\}$$

and

$$\beta = \{1, n+2, 2n+3, \dots, n^2\}.$$

That is, the Hadamard product is a submatrix of the Kronecker product. In particular, if $m = n$ then $A\circ$B is principal submatrix of $A\otimes B$ which is the submatrix that lies in the rows of A indexed by α and in the columns indexed by β. And it is denoted by $A[\alpha]$ instead of $A[\alpha, \alpha]$.

Lemma 4.1.3 (Horn & Zhang, 2010) Let $A, B\ \in M_n$ be nonnegative and let $\alpha\ \in\ \{1, \dots, n\}$ be given and nonempty.

$$(1)\ If\ A \geq B, then\ \rho(A) \geq \rho(B). \tag{4.8}$$

$$(2)\ \ \rho(\ A[\alpha]\ ) \leq\ \rho(A). \tag{4.9}$$

$$(3)\ A[\alpha]B[\alpha] \leq (AB)[\alpha]. \tag{4.10}$$

$$(4)\ \rho\ (A[\alpha]B[\alpha]) \leq \rho\big((AB)[\alpha]\big) \leq \rho(AB\ ). \tag{4.11}$$

Lemma 4.1.4 (Horn & Zhang, 2010) Let $A = [a_{ij}\ ] \in M_n$ with $n \geq 2$, and suppose that A is nonnegative and irreducible. Let $\alpha \subsetneq \{1, \dots, n\}$ be nonempty. Then

$$\rho\ (A[\alpha])\ <\ \rho\ (A). \tag{4.12}$$

Corollary 4.1.2 (Horn & Zhang, 2010) Suppose that $A > 0$ and $B > 0$ are two matrices in M_n. Then

$$\rho(A \circ \mathrm{B}) < \rho(AB). \tag{4.13}$$

Lemma 4.1.5 (Horn & Zhang, 2010) Let *A*, *B*, and *C* be positive semidefinite matrices in the space M_n and let $\alpha\ \in\ \{1,\dots,n\}$ be given and nonempty. Then

$$(1) If\ A \succcurlyeq B, then\ \rho(A) \geq \rho(B). \tag{4.14}$$

$$(2)\ \rho(\ A[\alpha]\ ) \leq\ \rho(A). \tag{4.15}$$

$$(3)\ \text{If}\ B \succ 0, \text{then}\ \rho\ (A[\alpha]B[\alpha]^{-1}) \leq \rho(AB^{-1}). \tag{4.16}$$

$$(4)\ \text{If}\ A \succcurlyeq B, \text{then}\ \rho(BC) \leq \rho(AC). \tag{4.17}$$

Zhan's conjecture "$\rho(A \circ \mathrm{B}) \leq \rho(AB)$" is proved by K. Audenaert and also by R. Horn and F. Zhang. However, it doesn't work for positive semidefinite matrices. The following theorem offers an alternative idea for positive semidefinite matrices.

Theorem 4.1.4 (Horn & Zhang, 2010) Let A_{nxn} be positive semidefinite matrix and B_{nxn} be positive definite matrix. Then

$$\rho(A \circ B) \leq \beta\rho(AB) \tag{4.18}$$

where

$$\beta = \rho(B^{-1} \circ B) \geq 1.$$

The following theorem is a general form of Zhan's conjecture.

Theorem 4.1.5 (Huang, 2010) Let $A_1, A_2, \dots, A_k$ be nonnegative matrices in the space $\boldsymbol{M_n}$**,** then

$$\rho(A_1 \circ \mathrm{A}_2 \circ \dots \mathrm{A}_\mathrm{k}) \leq \rho(A_1 A_2 \dots A_k). \tag{4.19}$$

It is important to understand the relation between the norms of Hadamard product and spectral radius since this thesis is based on the Hadamard product and norms of matrices.

Corollary 4.1.3 (Horn & Johnson, 1991) Let $A \in M_n$. For any norm

$$\rho(A) \leq \|A\|. \tag{4.20}$$

Corollary 4.1.4 (Huang, 2010) Let $A, B \in M_n$ be nonnegative matrices. Then

$$\|A \circ B\| \leq \rho(A^T B). \tag{4.21}$$

Corollary 4.1.5 (Huang, 2010) Let $A, B \in M_n$ be nonnegative matrices. Then

$$\|A \circ B\| \leq \|A\|\|B\|. \tag{4.22}$$

Another question is that what happens when we have more than two matrices.One can find an answer from the following theorem:

Theorem 4.1.6 (Huang, 2010) Let $A_1, A_2, \dots, A_k \in M_n$ be nonnegative matrices. Then

$$\|A_1 \circ A_2 \circ \dots \circ A_k\| \leq \rho^{\frac{1}{2}}(A_1 A_1^T A_2 A_2^T \dots A_k A_k^T). \tag{4.23}$$

Let A, B be square matrices of the same order. Xingzhi Zhan conjured that

$$\|(A \circ B)(A \circ B)^*\| \leq \|(A \circ \bar{A})(B \circ \bar{B})^*\| \tag{4.24}$$

for any unitarily invariant norm. Kun Du proved this conjecture for spectral norm, trace norm or Frobenius norm.

Theorem 4.1.7 (Horn & Johnson, 1991) For any $A, B \in M_n$ then

$$\|A \circ B\|_{(1)}^2 \leq \|A \circ \bar{A}\|_{(1)} \|(B \circ \bar{B})^*\|_{(1)} \tag{4.25}$$

Theorem 4.1.8 (Du, 2010) For any $A, B \in M_n$

$$\|(A \circ B)(A \circ B)^*\| \leq \|(A \circ \bar{A})(B \circ \bar{B})^T\| \tag{4.26}$$

is satisfied for spectral norm, trace norm or Frobenius norm where $\bar{A}$ is the conjugate of the matrix A.

Lemma 4.1.6 (Horn & Johnson, 1985) Let $A, B \in M_n$. If $|A| \leq B$, then

$$\rho(A) \leq \rho(|A|) \leq \rho(B). \tag{4.27}$$

4.2 A Result on the Norm of the Fan product

In this section, we will construct and prove a new theorem containing the norm of Fan product. The idea of this theorem comes from the Theorem 4.1.7. We will use Fan product instead of Hadamard product.

Theorem 4.2.1

Let A and B be two matrices in M_n. Then

$$\|A \star B\|_{(1)} \leq \|A \star \bar{A}\|_{(1)}. \|B \star \bar{B}\|_{(1)} \tag{4.28}$$

Proof

Let $A = [a_{ij}],\ B = [b_{ij}]$, and $x = [x_j] \in \mathbb{C}_n$ The induced norm of the spectral norm is

$$\|A \star B\|_{(1)} \leq \|(A \star B)x\|_{(1)} \tag{4.29}$$

such that $\|x\|_{(1)} = 1$. Then we have

$$\|A \star B\|_{(1)}^2 = \|(A \star B)x\|_{(1)}^2 = \sum_{i=1}^{n} \left| a_{ii}b_{ii}x_i - \sum_{j \neq i} a_{ij}b_{ij}x_j \right|^2$$

$$= |(a_{11}b_{11}x_1 - a_{12}b_{12}x_2 - \cdots - a_{1n}b_{1n}x_n)^2 +$$

$$+(-a_{21}b_{21}x_1 + a_{22}b_{22}x_2 - \cdots - a_{2n}b_{2n}x_n)^2 +$$

$$\vdots$$

$$+(-a_{n1}b_{n1}x_1 + a_{n2}b_{n2}x_2 - \cdots + a_{nn}b_{nn}x_n)^2|$$

$$\leq \sum_{i=1}^{n}\left[|a_{ii}||x_i|^{\frac{1}{2}}|b_{ii}||x_i|^{\frac{1}{2}} + \sum_{j\neq i}^{n}|-a_{ij}||x_j|^{\frac{1}{2}}|b_{ij}||x_j|^{\frac{1}{2}}\right]^2$$

$$= \left(|a_{11}||x_1|^{\frac{1}{2}}|b_{11}||x_1|^{\frac{1}{2}} + |-a_{12}||x_2|^{\frac{1}{2}}|-b_{12}||x_2|^{\frac{1}{2}} + \cdots + |-a_{1n}||x_n|^{\frac{1}{2}}|-b_{1n}||x_n|^{\frac{1}{2}}\right)^2 +$$

$$\vdots$$

$$+(|a_{nn}||x_n|^{\frac{1}{2}}|b_{nn}||x_n|^{\frac{1}{2}} + \cdots + |-a_{n1}||x_1|^{\frac{1}{2}}|-b_{n1}||x_1|^{\frac{1}{2}}+\ldots+|-a_{n,n-1}||x_{n-1}|^{\frac{1}{2}}|-b_{n,n-1}||x_{n-1}|^{\frac{1}{2}})^2$$

$$= (|a_{11}||x_1|, |-a_{12}||x_2|, \ldots, |-a_{1n}||x_n|)(|b_{11}||x_1|, |-b_{12}||x_2|, \ldots, |-b_{1n}||x_n|) +$$

$$+(|a_{22}||x_2|, |-a_{21}||x_1|, \ldots, |-a_{2n}||x_n|)(|b_{22}||x_2|, |-b_{21}||x_1|, \ldots, |-b_{2n}||x_n|)+$$

$$\vdots$$

$$+\big(|a_{nn}||x_n|, |-a_{n1}||x_1|, \ldots, |-a_{n,n-1}||x_{n-1}|\big)\big(|b_{nn}||x_n|, |-b_{n1}||x_1|, \ldots, |-b_{n,n-1}||x_{n-1}|\big)$$

$$= \sum_i \left\{\left(\left[|a_{ii}|^2|x_i| + \sum_{i\neq j}|-a_{ij}|^2|x_j|\right]^{1/2}\left[|b_{ii}|^2|x_i| + \sum_{i\neq j}|-b_{ij}|^2|x_j|\right]^{1/2}\right)^2\right\}$$

$$\leq \sum_i \left\{\left(\left[|a_{ii}|^2|x_i| + \sum_{i\neq j}|-a_{ij}|^2|x_j|\right]^2\left[|b_{ii}|^2|x_i| + \sum_{i\neq j}|-b_{ij}|^2|x_j|\right]^2\right)^{1/2}\right\}$$

$$= \|(A \star \bar{A})|x|\|_{(1)}\|(B \star \bar{B})|x|\|_{(1)}$$

$$\leq \|(A \star \bar{A})\|_{(1)}\|x\|_{(1)}\|(B \star \bar{B})\|_{(1)}\|x\|_{(1)}$$

$$= \|(A \star \bar{A})\|_{(1)}\|(B \star \bar{B})\|_{(1)}$$

4.3 On Properties of Hadamard and Kronecker Products of Centrosymmetric Matrices

Centrosymmetric matrices have many applications in the fields of numerical analysis, control theory, digital signal and image processing. Therefore, in this section, we will be dealing with centrosymmetric matrices and their properties. At the same time, we will focus on matrix multiplications of two centrosymmetric matrices. We will check whether the multiplications are also centrosymmetric or not.

Proposition 4.3.1 (Weaver, 1985) An nxn matrix P is centrosymmetric iff $JP = PJ$ where J is an nxn matrix with ones along the secondary diagonal and zeros elsewhere.

Theorem 4.3.1 (Weaver, 1985) If P_{nxn} and Q_{nxn} are centrosymmetric matrices then $P + Q$ is also centrosymmetric.

Proposition 4.3.2 (Weaver, 1985) If P_{nxn} is a centrosymmetric matrix, then P^T and P^{-1} is also centrosymmetric.

Chen and Zhong said that matrix multiplication of two centrosymmetric matrices is also centrosymmetric. But they didn't prove it. Let's now prove this.

Theorem 4.3.2

Let A_{nxn} and B_{nxn} be two centrosymmetric matrices. Then AB has also centrosymmetric form.

Proof

Let A_{nxn} and B_{nxn} be two centrosymmetric matrices.From Proposition 4.3.1 we can write $JA = AJ$ and $JB = BJ$.

$JAB = (JA)B = (AJ)B = AJB = A(JB) = A(BJ) = ABJ$.

Hence, this is the end of the proof.

Lemma 4.3.1 (Li, Zhao, Dai, Su, 2011) Let $A = \left[a_{ij}\right]_{nxn}$ be centrosymmetric matrix with $n = 2m$, then A has the following form

$$A = \begin{bmatrix} A_1 & J_m A_2 C \\ A_2 & J_m A_1 J_m \end{bmatrix} \tag{4.30}$$

and

$$Q^T A Q = \begin{bmatrix} A_1 - JA_2 & 0 \\ 0 & A_1 + JA_2 \end{bmatrix} \tag{4.31}$$

where $A_1, A_2 \in \mathbb{C}_{mxm}$ and

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m & I_m \\ -J_m & J_m \end{bmatrix}. \tag{4.32}$$

In this lemma, the matrix $Q^T AQ$ is similar to the matrix A.

Lemma 4.3.2 (Li, Zhao, Dai, Su, 2011) Let $A = \left[a_{ij}\right]_{2mx2m}$ be a centrosymmetric matrix as in (4.30), then

$$\rho(A) = \rho(\, A_1 + J\, A_2). \tag{4.33}$$

Theorem 4.3.3 (Zhao, Li, Gong, 2012) Let $A = \left[a_{ij}\right]_{nxn}$, $B = \left[b_{ij}\right]_{nxn}$ be two centrosymmetric matrices with the following form

$$A = \begin{pmatrix} A_1 & J_m A_2 J_m \\ A_2 & J_m A_1 J_m \end{pmatrix}, \;\; B = \begin{pmatrix} B_1 & J_m B_2 J_m \\ B_2 & J_m B_1 J_m \end{pmatrix} \tag{4.34}$$

where $A_1, A_2, B_1, B_2 \in \mathbb{C}_{mxm}$ and $n = 2m$.

Then $A \circ B$ is a centrosymmetric matrix.

Proposition 4.3.3 (Mond and Pecaric, 2000) Let A_{nxn} and B_{nxn} be two matrices. Then

$$\mathbb{J}^T (A \otimes B)\mathbb{J} = A \circ B \tag{4.35}$$

where $\mathbb{J}^T$ is a selection matrix in the order of nxn^2.

For example, for *n*, we obtain

$$\mathbb{J}^T_{nxn^2} = \left[\begin{array}{cccc|cccc|c|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 1 \end{array}\right]_{nxn^2}$$

$$= [E_{11} | E_{22} | \ldots | E_{nn}].$$

Proposition 4.3.4 Let J_m and $\mathbb{J}_{m^2xm}$ are the flip and the selection matrices, respectively.

$$\mathbb{J}^T_{mxm^2} J_{m^2} = J_m \mathbb{J}^T_{mxm^2} \tag{4.36}$$

$$J_{m^2}\mathbb{J}_{m^2xm} = \mathbb{J}_{m^2xm}J_m \tag{4.37}$$

Proof

If J_m is an mxm flip matrix and $\mathbb{J}^T_{mxm^2}$is an mxm^2 selection matrix,

$$\begin{aligned}
\mathbb{J}^T_{mxm^2}J_{m^2} &= [E_{11}, E_{22}, \dots, E_{mm}]J_{m^2} \\
&= [E_{11}, E_{22}, \dots, E_{mm}]\begin{bmatrix} 0 & 0 & \cdots & \cdots & J_m \\ 0 & 0 & \cdots & J_m & 0 \\ \vdots & \vdots & & \vdots \; \vdots & \vdots \\ J_m & 0 & \cdots & 0 & 0 \end{bmatrix}_{m^2xm^2} \\
&= \left[E_{mm}J_m, E_{m-1,m-1}J_m, \dots, E_{22}J_m, E_{11}J_m\right] \\
&= \left[J_mE_{11}, J_mE_{22}, \dots, J_mE_{m-1,m-1}, J_mE_{mm}\right] \\
&= J_m[E_{11}, E_{22}, \dots, E_{mm}] \\
&= J_m\mathbb{J}^T_{mxm^2}.
\end{aligned}$$

Thus, the proof of (4.36) is completed. Since $J_m^T = J_m$, (4.37) can be proved easily.

It is proved by Zhao et al. that Hadamard product of two centrosymmetric matrices is also centrosymmetric. They proved that equality by obtaining both sides separately. The following proof will be an alternative proof for the Theorem 4.3.4.

We will prove this theorem from one side to another side. In the proof, we use Thereom 2.3.2, Proposition 4.3.3 and Proposition 4.3.4.

Theorem 4.3.4 Let A_{2mx2m} and B_{2mx2m} be two centrosymmetric matrices as in (4.34). Then $A \circ B$ is also centrosymmetric.

Proof

The Hadamard product of A and B is

$$A \circ B = \begin{bmatrix} A_1 \circ B_1 & (J_mA_2J_m) \circ (J_mB_2J_m) \\ A_2 \circ B_2 & (J_mA_1J_m) \circ (J_mB_1J_m) \end{bmatrix}$$

To prove

$$(J_mA_1J_m) \circ (J_mB_1J_m) = J_m(A_1 \circ B_1)J_m \tag{4.38}$$

$$(J_mA_2J_m)\circ(J_mB_2J_m) = J_m(A_2\circ B_2)J_m \tag{4.39}$$

we shall use Theorem 2.3.2 and Proposition 4.3.3.

$$\begin{aligned}(J_mA_1J_m)\circ(J_mB_1J_m) &= \mathbb{J}_{mxm^2}^T[(J_mA_1J_m)\otimes(J_mB_1J_m)]\mathbb{J}_{m^2xm}\\ &= \mathbb{J}^T[(J_m\otimes J_m)(A_1\otimes B_1)(J_m\otimes J_m)]\mathbb{J}\\ &= \mathbb{J}_{mxm^2}^T[J_{m^2}(A_1\otimes B_1)J_{m^2}]\mathbb{J}_{m^2xm}\\ &= \mathbb{J}_{mxm^2}^TJ_{m^2}(A_1\otimes B_1)J_{m^2}\mathbb{J}_{m^2xm}\end{aligned}$$

is obtained. Using Proposition 4.3.3 and Proposition 4.3.4 we can change the order of multiplication.

$$\begin{aligned}(J_mA_1J_m)\circ(J_mB_1J_m) &= J_m\mathbb{J}_{mxm^2}^T\,(A_1\otimes B_1)\mathbb{J}_{m^2xm}J_{m^2}\\ &= J_m(\mathbb{J}_{mxm^2}^T\,(A_1\otimes B_1)\mathbb{J}_{m^2xm})J_m = J_m(A_1\circ B_1)J_m\end{aligned}$$

which gives (4.3.8).Relation (4.3.9) can be proved in a similar way.

Therefore, the matrix $A\circ B$ is centrosymmetric.

Theorem 4.3.5 Let A_{2mx2m} and B_{2mx2m} be two centrosymmetric matrices as in (4.34). Then $A\otimes B$ is a centrosymmetric matrix.

Proof

Using the definition of Kronecker product, it is known that

$$A\otimes B = [a_{ij}B] = \begin{bmatrix}A_1\otimes B & (J_mA_2J_m)\otimes B\\ A_2\otimes B & (J_mA_1J_m)\otimes B\end{bmatrix}.$$

In order to prove the following inequalities

$$J_{mn}(A_1\otimes B)J_{mn} = (J_mA_1J_m)\otimes B,$$

$$J_{mn}(A_2\otimes B)J_{mn} = (J_mA_2J_m)\otimes B$$

we shall partition the matrix J_{mn} such that

$$J_{mn} = J_m\otimes I_n.$$

Using above partition and Theorem 2.3.2, we obtain

$$\begin{aligned}J_{mn}(A_1\otimes B)J_{mn} &= (J_m\otimes I_n)(A_1\otimes B)(J_m\otimes I_n)\\ &= (J_mA_1J_m)\otimes B\end{aligned}$$

$$J_{mn}(A_2\otimes B)J_{mn} = (J_m\otimes I_n)(A_2\otimes B)(J_m\otimes I_n)$$

$$= (J_m A_2 J_m) \otimes B.$$

Therefore, $A \otimes B$ is a centrosymmetric matrix.

Theorem 4.3.6

Let A_{2mx2m} and B_{2mx2m} be two nonnegative centrosymmetric matrices as in (4.34). Then $A \circ B$ is similar to

$$\widetilde{A \circ B} = Q^T (A \circ B) Q = \begin{bmatrix} \mathfrak{B} - J_{mn}\mathfrak{C} & 0 \\ 0 & \mathfrak{B} + J_{mn}\mathfrak{C} \end{bmatrix} \tag{4.40}$$

where

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix},$$

and

$\mathfrak{B} - J_{mn}\mathfrak{C} = (A_1 \circ B_1) - J(A_2 \circ \mathrm{B}_2)$, $\quad \mathfrak{B} + J_{mn}\mathfrak{C} = (A_1 \circ B_1) + J(A_2 \circ B_2)$.

Proof

$$Q^T (A \circ B) \mathrm{Q} = \frac{1}{2} \begin{bmatrix} I_{mn} & -J_{mn} \\ I_{mn} & J_{mn} \end{bmatrix} \begin{bmatrix} A_1 \circ B_1 & J(A_2 \circ B_2)J \\ A_2 \circ B_2 & J(A_1 \circ B_1)J \end{bmatrix} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} A_1 \circ B_1 - J_{mn}(A_2 \circ B_2) & J(A_2 \circ B_2)J - J_{mn}J(A_1 \circ B_1)J \\ A_1 \circ B_1 + J_{mn}(A_2 \circ B_2) & J(A_2 \circ B_2)J + J_{mn}J(A_1 \circ B_1)J \end{bmatrix} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} (A_1 \circ B_1)I_{mn} - J_{mn}(A_2 \circ B_2)I_{mn} & (A_1 \circ B_1)I_{mn} - J_{mn}(A_2 \circ B_2)I_{mn} \\ -J(A_2 \circ B_2)J^2 + I_{mn}(A_1 \circ B_1)J^2 & +J(A_2 \circ B_2)J^2 - J_{mn}J(A_1 \circ B_1)J^2 \\ \\ (A_1 \circ B_1)I + J(A_2 \circ B_2)I & (A_1 \circ B_1)I + J(A_2 \circ B_2)I \\ -J(A_2 \circ B_2)J^2 - J_{mn}J(A_1 \circ B_1)J^2 & +J(A_2 \circ B_2)J^2 + J_{mn}J(A_1 \circ B_1)J^2 \end{bmatrix}$$

$J^2 = I$ gives;

$$= \frac{1}{2} \begin{bmatrix} 2(A_1 \circ B_1) - 2J(A_2 \circ B_2) & 0 \\ 0 & 2(A_1 \circ B_1) + 2J(A_2 \circ B_2) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (A_1 \circ B_1) - J(A_2 \circ \mathrm{B}_2) & 0 \\ 0 & (A_1 \circ B_1) + J(A_2 \circ B_2) \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{B} - J_{mn}\mathfrak{C} & 0 \\ 0 & \mathfrak{B} + J_{mn}\mathfrak{C} \end{bmatrix}.$$

From Lemma 4.3.2, we know that spectral radius of a centrosymmetric matrix can be shown in terms of the partition matrices. The following lemma inspired by Lemma 4.3.2.

Lemma 4.3.3 Let A_{2mx2m} and B_{2mx2m} be nonnegative centrosymmetric matrices as in (4.34). Then

$$\rho(A \circ B) = \rho[A_1 \circ B_1 + J(A_2 \circ B_2)]. \tag{4.41}$$

Proof

It is known that

$$Q^T(A \circ B)Q = \begin{bmatrix} A_1 \circ B_1 - J(A_2 \circ B_2) & 0 \\ 0 & A_1 \circ B_1 + J(A_2 \circ B_2) \end{bmatrix}$$

by Theorem 4.3.6. It is clear that all A_1, B_1, A_2, B_2 are also nonnegative because of nonnegativity of the matrices. This implies that $A_1 \circ B_1$ and $A_2 \circ B_2$ are nonnegative too. Hence,

$$-(A_1 \circ B_1) - J(A_2 \circ B_2) \leq (A_1 \circ B_1) - J(A_2 \circ B_2) \leq (A_1 \circ B_1) + J(A_2 \circ B_2).$$

Taking the absolute value of entries, the following inequality

$$A_1 \circ B_1 - J(A_2 \circ B_2) \leq |A_1 \circ B_1 - J(A_2 \circ B_2)| \leq A_1 \circ B_1 + J(A_2 \circ B_2)$$

is obtained. By Lemma 4.1.6

$$\rho[(A_1 \circ B_1 - J(A_2 \circ B_2)] \leq \rho[|A_1 \circ B_1 - J(A_2 \circ B_2)|] \leq \rho[A_1 \circ B_1 + J(A_2 \circ B_2)]$$

Thus,

$$\rho[A_1 \circ B_1 - J(A_2 \circ B_2)] \leq \rho[A_1 \circ B_1 - J(A_2 \circ B_2)]$$

By lemma 4.3.2, we have the following equality.

$$\rho(A \circ B) = \rho[Q^T(A \circ B)Q]$$

$$= max\{\rho[A_1 \circ B_1 - J(A_2 \circ B_2)], \rho[A_1 \circ B_1 + J(A_2 \circ B_2)]\}$$

$$= \rho[A_1 \circ B_1 + J(A_2 \circ B_2)].$$

The following lemma is an improved version of the Lemma 4.3.3. Using the relation between norm and spectral radius, we will find an upper bound for the spectral radius.

Lemma 4.3.4 Let A_{2mx2m} and B_{2mx2m} be nonnegative centrosymmetric matrices as in (4.3.4).Then

$$\rho(A \circ B) \leq \rho(A_1^T B_1) + \rho(A_2^T B_2). \tag{4.42}$$

Proof

By Lemma 4.3.3 and Corollary 4.1.3 we have following

$$\begin{aligned} \rho(A \circ B) &= \rho\big((A_1 \circ B_1) + J(A_2 \circ B_2)\big) \\ &\leq \|A_1 \circ B_1 + J(A_2 \circ B_2)\| \\ &\leq \|A_1 \circ B_1\| + \|JA_2 \circ B_2\| \\ &\leq \|A_1 \circ B_1\| + \|J\| \|(A_2 \circ B_2)\|. \end{aligned}$$

Since $\|A \circ B\| \leq \rho(A^T B)$ for any nonnegative matrices (Huang, 2011);

$$\rho(A \circ B) \leq \rho(A_1^T B_1) + \|J\| \rho(A_2^T B_2) \tag{4.43}$$

is obtained.

To complete the proof, we must choose $\|J\| = 1$. This is possible for matrix norms $\| \ \|_1$, $\| \ \|_2$, and $\| \ \|_\infty$. So (4.42) holds. This completes the proof.

Notice that, we can't write equality (4.42) for the Frobenius norm of the matrix J_m.

The idea of the following lemma comes from Lemma 4.3.4. We will try to find an upper bound for the spectral radius of the m-th power of $A \circ B$.

Lemma 4.3.5

Let A and B be nonnegative centrosymmetric matrices as in (4.34).

Then

$$\rho[(A \circ B)^m] \leq \rho^m(A_1 + JA_2)\rho^m(B_1 + JB_2). \tag{4.44}$$

Proof

Since A and B are centrosymmetric matrices, using Lemma 4.3.1, the following expressions

$$\tilde{A} = Q^T A Q = \begin{bmatrix} A_1 - JA_2 & 0 \\ 0 & A_1 + JA_2 \end{bmatrix}$$

$$\tilde{B} = Q^T B Q = \begin{bmatrix} B_1 - JB_2 & 0 \\ 0 & B_1 + JB_2 \end{bmatrix}$$

can be written and using Lemma 4.1.1,

$$\rho[(A \circ B)^m] = \rho[(A \circ B)]^m = \rho^m(A \circ B)$$
$$= \rho(A \circ B)\rho(A \circ B) \dots \rho(A \circ B)$$

Theorem 4.1.1 gives the following inequality

$$\rho[(A \circ B)^m] \leq \rho(A)\rho(B)\rho(A)\rho(B) \dots \rho(A)\rho(B)$$
$$= \rho^m(A)\rho^m(B).$$

And then using Lemma 4.3.2, we obtain

$$\rho[(A \circ B)^m] \leq \rho^m(A_1 + JA_2)\rho^m(B_1 + JB_2).$$

The idea of the following theorem comes from Theorem 4.3.6. Similar matrix of $A \circ B$ is found in the Theorem 4.3.6. In Theorem 4.3.7 we will look for similar matrix of $A \otimes B$ too.

Theorem 4.3.7 Let A_{2mx2m} and B_{2mx2m} be two centrosymmetric matrices as in (4.3.4). Then $A \otimes B$ is similar to

$$\widetilde{A \otimes B} = Q^T (A \otimes B) Q = \begin{bmatrix} \mathfrak{B} - J\mathfrak{C} & 0 \\ 0 & \mathfrak{B} + J\mathfrak{C} \end{bmatrix} \quad (4.45)$$

where

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix}.$$

Proof

By using definition of Kronecker product, we write

$$\widetilde{A \otimes B} = Q^T (A \otimes B) Q$$
$$= \frac{1}{2} \begin{bmatrix} I_{mn} & -J_{mn} \\ I_{mn} & J_{mn} \end{bmatrix} \begin{bmatrix} A_1 \otimes B & (J_m A_2 J_m) \otimes B \\ A_2 \otimes B & (J_m A_1 J_m) \otimes B \end{bmatrix} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} A_1 \otimes B - J_{mn}(A_2 \otimes B) & (J_m A_2 J_m) \otimes B - J_{mn}[(J_m A_1 J_m) \otimes B] \\ A_1 \otimes B + J_{mn}(A_2 \otimes B) & (J_m A_2 J_m) \otimes B + J_{mn}[(J_m A_1 J_m) \otimes B] \end{bmatrix} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix}$$

By using Theorem 2.3.2 we have the following equality:

$$J_{mn}(A_2 \otimes B) = (J_m \otimes I_n)(A_2 \otimes B) = J_m A_2 \otimes B.$$

Hence;

$\widetilde{A \otimes B} = Q^T(A \otimes B)Q$

$$= \frac{1}{2}\begin{bmatrix} A_1 \otimes B - (J_m \otimes I_n)(A_2 \otimes B) & (J_m A_2 J_m) \otimes B - (J_m \otimes I_n)[(J_m A_1 J_m) \otimes B] \\ A_1 \otimes B + (J_m \otimes I_n)(A_2 \otimes B) & (J_m A_2 J_m) \otimes B + (J_m \otimes I_n)[(J_m A_1 J_m) \otimes B] \end{bmatrix} \begin{bmatrix} I_{mn} & I_{mn} \\ -J_{mn} & J_{mn} \end{bmatrix}$$

By multiplying the above two matrices and using Theorem 2.3.2

$$\begin{aligned} Q^T(A \otimes B)Q &= \frac{1}{2}\begin{bmatrix} 2[A_1 \otimes B - J_m(A_2 \otimes B)] & 0 \\ 0 & 2[A_1 \otimes B + J_m(A_2 \otimes B] \end{bmatrix} \\ &= \begin{bmatrix} A_1 \otimes B - J_{mn}(A_2 \otimes B) & 0 \\ 0 & A_1 \otimes B + J_{mn}(A_2 \otimes B) \end{bmatrix} \end{aligned}$$

is obtained. For the matrices A_1, A_2, and B set

$$\mathfrak{B} = A_1 \otimes B \quad \text{and} \quad \mathfrak{C} = A_2 \otimes B$$

and the proof is completed.

Since we have found the similar matrix of $A \otimes B$, it is easy to find spectral radius of $A \otimes B$. In the proof we will use Theorem 4.3.7, Lemma 4.1.6, and Lemma 4.3.2.

Lemma 4.3.6 Let A_{2mx2m} and B_{2mx2m} be nonnegative centrosymmetric matrices as in (4.34). Then

$$\rho(A \otimes B) = \rho[A_1 \otimes B + J(A_2 \otimes B)] \tag{4.46}$$

Proof

Using Theorem 4.3.7,

$$Q^T(A \otimes B)Q = \begin{bmatrix} A_1 \otimes B - J_{mn}(A_2 \otimes B) & 0 \\ 0 & A_1 \otimes B + J_{mn}(A_2 \otimes B) \end{bmatrix}$$

can be written. Since A and B are nonnegative matrices, then A_1, A_1, B_1, B_2 are nonnegative, too. This implies that $A_1 \otimes B$ and $A_2 \otimes B$ are also nonnegative.

Thus,

$$-(A_1 \otimes B) - J(A_2 \otimes B) \leq (A_1 \otimes B) - J(A_2 \otimes B) \leq (A_1 \otimes B) + J(A_2 \otimes B)$$

By Lemma 4.1.6,

$$\begin{aligned}\rho((A_1 \otimes B) - J(A_2 \otimes B)) &\leq \rho(|(A_1 \otimes B) - J(A_2 \otimes B)|) \\ &\leq \rho(A_1 \otimes B + J(A_2 \otimes B)).\end{aligned}$$

Thus we obtained that,

$$\rho(A_1 \otimes B - J(A_2 \otimes B)) \leq \rho\big(A_1 \otimes B + J(A_2 \otimes B)\big)$$

and using Lemma 4.3.2, we have the following equality.

$$\begin{aligned}\rho(A \otimes B) &= \rho(Q^T (A \otimes B) Q) \\ &= max\big\{\rho(A_1 \otimes B - J(A_2 \otimes B), \rho\big(A_1 \otimes B + J(A_2 \otimes B)\big)\big\} \\ &= \rho\big(A_1 \otimes B + J(A_2 \otimes B)\big).\end{aligned}$$

CHAPTER FIVE

CONCLUSION

In this thesis, we have studied the articles on inequalities involving the Hadamard product and norm of matrices. We mainly focused on centrosymmetric matrices. Firstly, we have tried to find both lower and upper bounds for the spectral radius of these centrosymmetric matrices. After then, we have obtained some new results on Hadamard and Kronecker products of the centrosymmetric matrices. In order to obtain a better approach for spectral radius of cenrosymmetric matrices we have also analyzed equivalent form of these matrices. After examining the results on the equivalent forms, we have achieved to produce the spectral radius of the Hadamard product of the centrosymmetric matrices $A \circ B$ and $(A \circ B)^m$. Finally, we have obtained an upper bound for the spectral radius of $A \circ B$ and $(A \circ B)^m$.

REFERENCES

Audenaert, K. M. R. (2010). Spectral radius of Hadamard product versus conventional product for nonnegative matrices. *Linear Algebra and Its Applications*, *432*, 366-368.

Brewer, J. W. (1978). Kronecker Products and matrix calculus in system theory. *Institue of Electrical and Electronics Engineers*, *25* (9).

Chen, W., Wong, X & Zhong, T. (1996). The structure of weighting coefficient matrices of harmanic differential quadrature and its applications. *Communications in Numerical Methods in Engineering*, (12), 455-459.

Du, K. (2010). Norm inequalities involving Hadamard products. *Linear and Multilinear Algebra*, *58* (2), 235-238.

Horn, R. A., & Johnson, C. R. (1985). *Matrix analysis*. New York: Cambridge University Press.

Horn, R. A., & Johnson, C. R. (1991). *Topics in matrix analysis*. New York: Cambridge University Press.

Horn, R. A., & Zhang, F. (2010). Bounds on the spectral radius of a Hadamard product of nonnegative or positive semidefinite matrices. *Electronic Journal of Linear Algebra*, *20*, 90-94.

Huang, R. (2008). Some inequalities for the Hadamard product and the Fan product of matrices. *Linear Algebra and its Applications*, *428* (7), 1551-1559.

Huang, Z. (2011). On the spectral radius and the spectral norm of Hadamard products of nonnegative matrices. *Linear Algebra and its Applications*, *434* (2), 457-462.

Ledermann, W. (1950). Bounds for the latent rooot of a positive matrix. *Journal London Mathematical Society*, *25*, 265-268.

Liu, H., Zhao, D., Dai, F. & Su, D. (2011). On the spectral radius of a nonnegative centrosymmetric matrices. *Applied Mathematics and Computation*, *218*, 4962-4966.

Liu, S., & Trenkler, G. (2008). Hadamard, Khatri-Rao, Kronecker and other matrix products. *International Journal of Information and Systems Sciences*, *4* (1), 160-177.

Meyer, C. D. (2004). *Matrix analysis and applied linear algebra*. Philadelphia: Society for Industrial and Applied Mathematics.

Mond, B., & Pecaric, J. (2000). On inequalities involving the Hadamard product of matrices. *Electronic Journal of Linear Algebra*, *6*, 56-61.

Neudecker, H. (1990). A matrix trace inequality. *Journal of Mathematical Analysis and Applications*, *166*, 302-303.

Ostrowski, A. (1952). Bounds for the greatest latent root of a positive matrices. *Journal London .Mathematical .Society*, *27*, 253-256.

Ulukök, Z., & Türkmen, R. (2010). On some matrix trace inequalities. *Journal of Inequalities and Applications*, 1-8.

Weaver, J. R. (1985). Centrosymmetric matrices,their basic properties,eigenvalues and eigenvectors. *Mathematical Association of America*, *92* (10), 711-717.

Yang, Y. (1988). A matrix trace inequality. *Journal of Mathematical Analysis and Applications*, *133* (2), 573-574.

Zhao, D., Li, H. & Gong, Z. (2012). Trace inequalities for positive semidefinite matrices with centrosymmetric structure. *Journal of Inequalities and Applications*, *62*, 1-6.