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INTEGRATING FUZZY LOGIC INTO PORTFOLIO OPTIMIZATION: ENHANCING RISK MANAGEMENT AND RETURN MAXIMIZATION

Slim ZOUAOUI

Master's Thesis

Supervisor

Asst. Prof. Dr. Timur İNAN

İstanbul, 2024

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The thesis titled INTEGRATING FUZZY LOGIC INTO PORTFOLIO OPTIMIZATION: ENHANCING RISK MANAGEMENT AND RETURN MAXIMIZATION prepared by SLIM ZOUAOUI and submitted on 13.12.2024 has been **accepted unanimously** for the degree of Master of Science in Data Analytics.

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ABSTRACT

INTEGRATING FUZZY LOGIC INTO PORTFOLIO OPTIMIZATION: ENHANCING RISK MANAGEMENT AND RETURN MAXIMIZATION

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Modern Portfolio Theory (MPT), or mean-variance analysis, is built on principles of risk and return. A core assumption in MPT is investor risk aversion, which significantly influences investment choices. In evidence, highly risk-averse investors often choose safer investments with lower returns over riskier options that may yield higher returns.

This study seeks to optimize investment portfolio utility scores by modelling risk aversion using fuzzy logic. Introduced by Lotfi Zadeh, fuzzy logic addresses human reasoning and decision-making under uncertainty and ambiguity. Thus, this logic is suitable to model investors' diverse utility preferences and to reflect their subjective and varied perceptions of risk.

Traditional utility scoring uses fixed risk aversion coefficients from 1 to 5, with higher values indicating greater risk aversion. Here, we propose a more granular approach, measuring risk aversion on a 0-100 scale, where 0 indicates total risk tolerance and 100 signifies complete risk aversion. We categorize investors as low (5-45), moderate (45-75), or high (75-95) risk-averse, acknowledging that no investor is entirely risk-seeking or risk-averse.

Each risk aversion category is modelled with Trapezoidal and Cauchy distributions using a linear combination of their respective fuzzy membership functions. A sensitivity analysis was conducted, over short, mid and long-term timeframes, on a portfolio consisting of 10 assets which is optimized with fuzzy risk aversion coefficient then compared to the same portfolio optimized with fixed coefficients. Findings showed that fuzzy logic consistently provided superior risk-adjusted returns, particularly in mid-term and long-term scenarios. Although fuzzy-optimized portfolios had higher betas and variances, those risks were offset by enhanced returns. The Treynor ratios were similar across both methods, yet the fuzzy approach delivered higher overall returns, highlighting its effectiveness in managing volatility and risk.

Keywords: Fuzzy Logic, Modern Portfolio Theory, Portfolio Optimization, Return Maximization, Risk Management, Risk Maximization.

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ABBREVIATIONS

CAPM : Capital Asset Pricing Model

MPT : Modern Portfolio Theory

1. INTRODUCTION AND PURPOSE

1.1 BACKGROUND AND MOTIVATION

Maximizing profits while minimizing risks has always been a central concern for financial investors, driving the development of portfolio optimization strategies. Harry Markowitz, the father of modern portfolio theory, introduced the mean-variance model. This model uses estimates of asset standard deviation, variance-covariance, and expected returns to identify optimal asset allocation and determine efficient portfolios [1].

Despite its significance, the mean-variance model has faced criticism for its oversimplifications. It neglects real-world constraints such as taxes, transaction costs, investment strategies, capital allocation for each asset, and the actual number of assets [3]. Moreover, it assumes a normal distribution of returns, failing to account for the impact of extreme economic events, leading to non-normality. The model also treats correlations and variances as static, whereas they fluctuate over time, especially during market crises. Additionally, it does not consider estimation errors. Another limitation is the single-period problem, which assumes that investors make asset allocations only at specific intervals (e.g., annually or monthly) and cannot adjust them until the period ends. This contradicts the reality of changing investor objectives within a period, making the model unsuitable for long-term goals [1].

While linear and quadratic programming have been commonly used exact methods for portfolio optimization, the quadratic programming approach faces challenges in handling real financial constraints, making it a complex and Non-deterministic Polynomial-time hard (NP-hard) problem [2][3].

Optimization is a sophisticated technique leveraging linear and non-linear equations to solve complex problems and evaluate the solutions obtained. Its overarching goal is to achieve optimal outcomes, whether maximizing profits or satisfaction while minimizing costs. This process typically involves the efficient allocation of resources, including financial elements such as money, time, or machinery [4].

In the financial realm, optimization plays a pivotal role, particularly in portfolio management. Investors and portfolio managers aim to optimize the allocation of assets, strategically utilizing available capital to achieve the highest possible returns while concurrently managing risks. This intricate balancing act considers various real-world constraints, ranging from market crises and political or policy changes to investor behaviors and risk appetites. Optimization in portfolio management is a nuanced endeavor that navigates the dynamic landscape of financial markets, seeking to strike an optimal balance amid multifaceted challenges [4].

1.2 MODERN PORTFOLIO THEORY

Modern Portfolio Theory (MPT), also known as mean-variance analysis, is grounded in the fundamental principles of risk and return. Its core premise is that diversification can minimize the inherent risk within a portfolio. Diversification in this context involves holding a variety of assets with low or negative correlations, which mitigates the portfolio's volatility stemming from the individual risks of each asset. MPT posits that the impact of each asset on the overall portfolio is crucial. A key assumption is investors' risk aversion, where the preference is for portfolios that offer the highest return for a given level of risk.

However, investors exhibit varying degrees of risk tolerance. To address this, Harry Markowitz introduced the concept of the efficient frontier [5], which represents the set of optimal portfolios that either maximize expected return for a given level of risk or minimize risk for a predetermined expected return [6]. The efficient frontier graphically illustrates the relationship between return and risk, with each point along the curve signifying the optimal asset weights that yield the highest return for a given level of risk [5]. The below figure from shows the graph of the efficient frontier [5]:

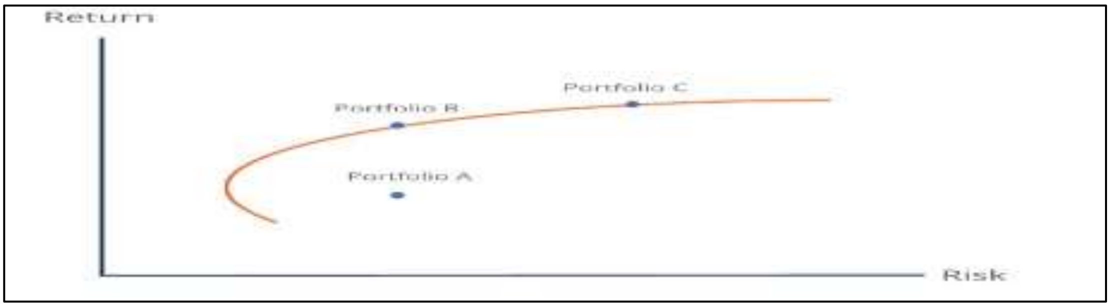

Figure 1.1: The Efficient Frontier.

Portfolios positioned on the efficient frontier are considered more optimal than those below it. For example, Portfolio B is superior to Portfolio A because, for an equivalent level of risk, Portfolio B generates a higher return. However, when two portfolios lie on the same line, it's not accurate to declare one better than the other. The investor's risk appetite plays a crucial role in portfolio selection. For instance, an investor inclined to take substantial risks may opt for Portfolio C, while someone with a lower risk tolerance may prefer Portfolio B. The efficient frontier underscores that higher levels of risk are associated with the potential for greater returns [5].

In a connected context, the condensed mathematical model for the mean-variance approach, formulated by [8], is articulated in [7] as follows:

$$\begin{gathered} Mininimize \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ subject\ to \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\ and\ 0 \leq x_j \leq \mu_j \end{gathered} \tag{1.1}$$

In this formulation, σ_{ij} represents the covariance between assets i and j, x_i and x_j denote the invested amounts in assets i and j, respectively, with μ_j representing the maximum allowable investment in asset j. M_0 stands for the total investment amount, ρ signifies the minimum rate of return mandated by the investor, and r_j denotes the expected rate of return.

1.3 COMPARISON PORTFOLIO OPTIMIZATION TECHNIQUES

In their publication, "A Brief Overview of Portfolio Optimization Techniques" [7], the authors define portfolio optimization as the method of selecting assets in alignment with predefined objectives, such as minimizing risk, maximizing profits, or a combination of both. Each technique in this context has its unique fitness or objective functions. The authors conduct a comprehensive literature review, comparing classical and intelligent approaches, including statistical, evolutionary, and quantum-based techniques. They conclude that evolutionary and quantum-inspired methods exhibit superior performance. Table **1.1**, extracted from [7], succinctly outlines the advantages and disadvantages of various optimization techniques.

The study by [7] advocates for using more adaptable techniques to solve optimization problems. Positioned at the intersection of machine learning and statistical techniques, the study employs sequential least squares programming as an optimization method. Here, the objective function to maximize is referred to as the portfolio's utility score. Additionally, two statistical distributions, Cauchy and trapezoidal, are incorporated to model investors' risk aversion based on fuzzy logic.

Table 1.1: Comparison of Different Techniques Used in Portfolio Optimization.

Method	**Advantage**	**Limitation**
Statistical based approach	Easy to implement. Good for small sets and time periods. Easily interpretable.	Do not consider multiple features into account.
Regression based approach	Simple, Interpretable, easy to implement. Learn iteratively from historical data.	Efficiency is highly dependent on the learning rate
Bayesian based approach	Can be easily extended, unlike other machine learning techniques. Can be employed for reasoning problem.	No universal method to construct on network from data. While training it is not assured that all training patterns are used while training
Neural Networks approach	Better learning model as compared to other traditional techniques. Fault tolerant. Distributed memory and at times can work with inadequate knowledge	Should be used when dealing with large sets. Not self- exploratory and absence of specific rules to determine structure of network
Reinforcement learning based approach	Can solve complex optimization problems. Learning model is similar to learning patterns of human. Can be used to adaptly learn based on changing states.	Should not be used for simple problems or small data sets. Too much reinforcement can adversity affect learning. The cure od dimensionality can limit the learning
Evolutionary Based approach	Conceptually simple. And outperforms classical techniques by nonlinear constraint and nonstationary conditions. Ability to parallelism and robust to dynamic changes.	High computation required. Results at times are not sself explanatory and reproducible
Quantum based approach	Adding quilbits increase storage exponentially and are useful to solve for very complex problem. Faster compared to other methods	Energy required by quantum computer is larger than traditional computers. Still lot of unknows as this is still an ogoing are of research

1.4 OBJECTIVE OF THE STUDY

Risk aversion is a crucial concept for investors. Depending on their level of risk aversion—whether high or low—they will prefer investments that align with their preferred utility levels. In other words, highly risk-averse investors will select investments that guarantee low returns or are risk-free, despite the possibility of higher returns from riskier investments [9].

In his paper "Is There a Need for Fuzzy Logic?" Lotfi Zadeh, the founder of fuzzy logic, asserts that “fuzzy logic is not fuzzy. It is a precise logic of imprecision and approximate reasoning.” He explains that fuzzy logic formalizes human capabilities to converse, reason, and make decisions in environments characterized by incomplete, conflicting, uncertain, and ambiguous information. Additionally, fuzzy logic formalizes the ability to perform mental and physical tasks without needing precise measurements and computations [10].

Risk aversion aligns with the principles of fuzzy logic, as it involves uncertainty and ambiguity derived from the varied utility preferences of investors, each perceiving risks differently. Cognitive biases in risk assessment further complicate risk aversion, influencing investor decision-making. The financial realm is replete with imprecision, risk, uncertainty, and incomplete information. For instance, information asymmetry in financial markets occurs when market participants possess unequal knowledge about transactions. Moreover, the financial world is susceptible to unforeseeable hazardous events such as market turmoil or political crises.

Portfolio optimization revolves around two conflicting objectives—risk and return—which form a trade-off relationship. The conflict lies in maximizing return while minimizing risk, a challenging balance to achieve as the general forms observed are low risk, low return, and high risk, high return [11]. All these factors underscore the suitability of using fuzzy logic to assess financial investors' risk aversion and incorporate it into the portfolio optimization process.

Risk aversion is a key consideration in portfolio optimization and investor utility maximization. Typically, models rely on a fixed risk aversion coefficient, represented by the utility score of the portfolio:

$$Maximize\ U(p) = E(r) - \frac{1}{2} A\, \sigma^2 \tag{1.2}$$

Here, A is a fixed risk aversion coefficient ranging from 1 to 5, where 1 indicates low risk aversion and 5 indicates high risk aversion, and σ^2 is the portfolio variance.

In this study, instead of using a fixed risk aversion coefficient, a fuzzy risk aversion coefficient is proposed. The methodology section will detail how fuzziness is incorporated into the model. The respective expected returns will be derived for both the fuzzy and fixed risk aversion coefficients to compare their performance. This leads to the following hypothesis to be tested:

a. Hypothesis 1: Will employing a fuzzy risk aversion coefficient in the optimization model result in higher expected returns compared to using a fixed risk aversion coefficient?

b. Hypothesis 2: If the expected returns differ when using a fuzzy risk aversion coefficient, does it efficiently handle the multi-objective combination of minimizing risk and maximizing return?

The remainder of the research is organized as follows: The second section, the literature review, explores modern portfolio theory, its limitations, and how these limitations led to the development of behavioral theory. It also discusses the concept of risk aversion and its assessment, and highlights the application of fuzzy logic in portfolio theory. The third section details the methodology, including the optimization function (portfolio utility score), the use of fuzzy logic to model risk aversion, and the rationale behind algorithm selection. The next section presents the model used and evaluates the results of optimizing portfolios using fuzzy risk aversion modeled by Cauchy and trapezoidal membership functions. These results are then compared to the performance metrics of portfolios optimized using fixed risk aversion coefficients ranging from 1 to 5. The last section summarizes the key findings and underscores the advantages of incorporating fuzzy logic into portfolio optimization.

2. GENERAL INFORMATION

2.1 FROM MODERN PORTFOLIO THEORY TO BEHAVIORAL PORTFOLIO THEORY

The foundations of modern portfolio theory (MPT) were established by Harry Markowitz in 1952. MPT encompasses Markowitz's portfolio selection theory and William Sharpe's 1964 contributions to the theory of financial asset pricing, now known as the capital asset pricing model (CAPM). MPT provides a framework for constructing and selecting portfolios that simultaneously aim to maximize profits and minimize risk [12].

The Markowitz model is primarily used to diversify portfolios, while investors use the CAPM model to estimate stock returns and their dynamic behaviors. Specifically, the Markowitz model minimizes risk for a given level of return or maximizes return for a given amount of risk. It assumes that returns are normally distributed and assesses risk as the standard deviation of returns. The portfolio variance is minimized by combining assets that are not positively correlated. Conversely, the CAPM model incorporates the asset's beta (sensitivity to systematic risk), the expected market return, and the return of a theoretical risk-free asset to establish the relationship between investment return and associated risk [13].

Despite the revolutionary contributions of MPT through the CAPM and Markowitz models, these theories have several limitations due to their reliance on unrealistic assumptions that do not always hold in the complex and dynamic financial world. These limitations include:

a. Investor Rationality: MPT assumes investors are rational and aim to minimize risk while maximizing return. However, herding behavior, where investors follow the crowd rather than their analysis, contradicts this assumption. Herding can worsen market volatility, destabilize markets, and intensify financial system fragility, potentially leading to poor investment decisions for all investors [14][15].

b. Perfect Information: The assumption that investors receive timely and complete information is often contradicted by information asymmetry, where one party has more information about an investment than another [12].

c. Efficient Market: Markowitz's theory assumes markets are perfectly efficient. However, asset prices' vulnerability to various factors, such as personal, environmental, strategic, or social dimensions, and market failures like externalities, challenges this assumption. Historical market crises, bubbles, and bursts suggest that markets are far from perfect [12].

d. No Transaction Costs or Taxes: MPT does not account for transaction costs and taxes, yet every investment incurs expenses such as administrative costs or brokerage fees, which can affect optimal portfolio selection [12].

e. Investment Independence: The theory suggests some investments perform independently of others. However, during extreme uncertainty and market stress, supposedly independent investments often exhibit correlation [12].

f. Unlimited Access to Capital: This assumption ignores the reality that investors have credit limits and that only the federal government can borrow at a risk-free rate [12].

g. Higher Risk, Higher Return: The assumption that higher risk always leads to higher returns is contradicted by the use of derivatives or futures to reduce overall risk without necessarily seeking higher returns [12].

Other limitations include the non-existence of a truly risk-free asset (a component of the CAPM model), the inability of historical returns to capture new circumstances, and the neglect of social, personal, and strategic factors by MPT [12].

These limitations and the differences in investors' perceptions of risk, particularly focusing on risk aversion—which is inherently behavioral—suggest a need to view portfolio theory from a behavioral perspective. This research emphasizes understanding risk aversion and addressing behavioral biases and varied utilities in portfolio optimization.

As the Markowitz theory did not account for behavioral aspects of investments and investors—such as risk appetite, investor sentiments, investment objectives, and time horizon—Shefrin and Statman introduced behavioral portfolio theory in 2000 as an alternative. This theory sharply contrasts with Markowitz's approach. Instead of considering return variance as the risk element, behavioral portfolio theory relates risk to the chances of ruin. Investors following this theory set a primary safety constraint to secure wealth under all conditions. The approach also posits that investors do not behave rationally; instead, their behavior is driven by emotions like fear and hope. Furthermore, behavioral portfolio theory incorporates likelihood weights, allowing the coexistence of gambling and lottery-like behavior. It also permits investors to view their portfolios as a collection of sub-portfolios, each optimized for a specific mental account [16][17].

According to [16], the risks associated with behavioral portfolio theory are very high. It suggests that investors with a typical level of risk aversion, who normally employ the Markowitz approach, are unlikely to invest in a behaviorally optimal portfolio. Additionally, [17] demonstrates that risk aversion coefficients induced by behavioral portfolio theory are ten times lower than those for investors using the Markowitz approach. This leads such investors to avoid behaviorally optimal portfolios due to their lower degrees of risk aversion.

These two studies underscore the critical role of risk aversion in shaping investment decisions. This study similarly suggests a comparison between expected returns for investments using fixed risk aversion coefficients (1, 2, 3, 4, etc.) as in Markowitz theory, and expected returns for investments using dynamic risk aversion coefficients modeled with fuzzy logic. This fuzzy logic approach will be discussed in detail in the methodology section.

The findings of [16] and [17] as well as the goal of this research highlight the importance to discuss risk aversion and the rational for modeling it using fuzzy logic in the next sections.

2.2 RISK AVERSION

Economics suggests that more wealth is generally preferred to less wealth, and the concept of marginal utility of wealth refers to the additional satisfaction one derives from a small increase in wealth. Diminishing marginal utility of wealth is considered risk avoidance because, on the utility of wealth curve, a risky investment with the same level of expected return will always have lower utility. When offered a fair game, a risk-averse investor will participate only if their expected utility increases. Conversely, they will avoid the game if their expected utility decreases. However, it should be noted that risk-averse investors may still invest in risky assets if they believe the odds are in their favor.

There are variations in risk behavior among investors:

a. Risk-seeking: These investors make riskier decisions as they believe it increases their utility. They speculate or gamble with the expectation of increased utility.

b. Risk-neutral: These investors are indifferent to risk. Their utility remains unchanged whether they engage in a fair game or not [18].

Although risk aversion is a critical factor in investment decisions, financial advisors often face the challenge of measuring it. In the US, for example, there are no direct surveys to assess risk aversion. Four primary methods are typically used to measure it:

a. Questionnaires about investment choices.

b. Investment and subjective questionnaires.

c. Evaluating behavior.

d. Questionnaires based on hypothetical scenarios [19].

For instance, [20] uses surveys and questions to highlight both objective characteristics and subjective measures of risk preference, as well as principal component analysis to assess risk aversion. Their study concludes that simple, intuitive approaches to illustrating risk preferences are more effective in predicting portfolio allocations than complex measures grounded in economic theory.

This research adopts a similar simple, intuitive approach to measure risk aversion, avoiding the complexities of economic theory such as utility functions or Arrow-Pratt measures. Recognizing that some risk-averse investors are more risk-averse than others, this study uses a dynamic approach. Instead of a fixed risk aversion coefficient (e.g., 1 for low aversion and 5 for high aversion), it introduces fuzzy logic to provide a range of values: low risk aversion, medium risk aversion, and high-risk aversion. This dynamic approach will be detailed further in the methodology section.

2.3 FUZZY LOGIC IN PORTFOLIO THEORY

Introduced by Zadeh in 1965, fuzzy set theory effectively addresses the impreciseness and uncertainty inherent in real-world information. In decision-making contexts, fuzzy logic is particularly advantageous for qualitative assessments and subjective judgments, offering a more suitable alternative to traditional methods that rely on crisp values [21]. By incorporating fuzzy methodology, intuitive and subjective characteristics fundamental to assessing investor preferences can be integrated into portfolio selection [22].

Given the subjective nature of risk aversion and its influence on investor preferences—such as distinguishing between high and low risk aversion—there is a compelling rationale for modeling risk aversion using fuzzy logic.

Several studies have explored the integration of fuzzy logic into modern portfolio theory, particularly concerning the modeling of risk or return. For instance, [23] reformulated the portfolio selection problem as one of fuzzy decision theory, employing fuzzy linear membership functions to account for risk aversion specific to neutral investors while addressing the dual objectives of minimizing risk and maximizing profit. Their findings indicate that the fuzzy portfolio approach outperforms traditional evaluations based on the Sharpe and Treynor ratios when determining optimal portfolio weights.

Similarly, [24] and [22] developed a fuzzy multi-objective portfolio selection model that accounts for the vague aspiration levels of investors. They introduced a nonlinear S-shaped fuzzy membership function to replace the semi-absolute deviation portfolio model, applying their framework to a selection of 20 assets. Their results demonstrated that the fuzzy model yielded effective strategies aligned with investors' aspiration levels, objectives, and satisfaction criteria.

Moreover, [25] employed trapezoidal membership functions to model returns and illustrated an algorithm designed to optimize the utility score of portfolios. This methodology successfully identified portfolios with the highest utility scores as optimal solutions, asserting that these solutions are not merely optimal but also precise.

In a more advanced application of fuzzy logic, [26] developed a system combining fuzzy logic with semantic technologies to recommend investments based on traditional financial metrics and behavioral factors. This "semantic investment portfolio" tool includes a questionnaire that captures investor characteristics, social aspects, and risk preferences. Utilizing predetermined fuzzy rules, the tool assesses the investor's risk profile and identifies financial products described through financial ontology and expert criteria. The system generates tailored portfolios based on these characteristics, which are then evaluated and recommended to investors. In experimental trials, the tool's recommendations closely aligned with those of experienced financial advisors, reflecting high accuracy and effectiveness.

[27] employed triangular fuzzy numbers to identify indicators for portfolio selection and conducted a SWOT analysis of utilizing fuzzy set theory in modern portfolio management.

3. METHODOLOGY

3.1 THE OPTIMIZATION FUNCTION

The primary objective of investors is to maximize portfolio returns while minimizing risks. However, risk perception varies among investors. For instance, a highly risk-averse investor tends to opt for safer investments that, while potentially offering lower returns, provide greater security. In contrast, a low-risk-averse investor is likely to favor riskier investments that promise higher returns. To address this, the equation used for optimization in this research will incorporate a return parameter, reflecting the expected profitability of the portfolio, alongside a risk aversion parameter, which gauges the investor's risk preference. The equation is formulated to emphasize the utility score of the portfolio, representing the level of satisfaction (utility) derived from the portfolio's performance [28][25]:

$$U(p) = E(r) - \frac{1}{2} A \, \sigma^2 \tag{3.1}$$

3.2 ALGORITHM’S CHOICE

The no free lunch theorem posits that all algorithms exhibit similar performance when solving optimization problems. Specifically, the first theorem states that when algorithms’ search results are averaged over all possible search spaces, rather than compared within a specific fitness space, they will yield equal probabilities for the same sample across different algorithms [29]. Furthermore, it has been noted that no single approach serves as a universal remedy for optimization challenges; thus, no search method is effective for all scenarios [30]. However, by incorporating expert knowledge into the system, it is possible to navigate the implications of the no free lunch theorem, allowing for assumptions about the search space that can lead to more effective solutions than generic heuristics.

In this research, the assumption that risk aversion follows a fuzzy distribution is critical, as it introduces expert information into the optimization problem. This inclusion will influence the search space of the optimization.

Given these considerations and in accordance with the no free lunch theorem—which asserts that, on average, all algorithms perform equally, even if they excel in certain problem classes—this study will initially employ the default optimization algorithm available in Python, specifically the Sequential Least Squares Programming (SLSQP) method.

3.3 THE RISK AVERSION MEMBERSHIP FUNCTIONS

3.3.1 Risk Aversion As A Fuzzy Trapezoidal Membership Function

In this research, fuzzy logic is utilized to model risk aversion. Fuzzy logic is particularly suited for this purpose because fuzzy numbers are defined within a bounded range of 0 to 1, paralleling the evaluation of risk aversion as a percentage that also falls within this interval.

A membership function on a set X is any function that maps X to the real interval [0, 1]. Consequently, investor risk aversion can be conceptualized as fuzzy; if we express it as a percentage within the range [0 to 100], it can be reduced to the interval [0, 1]. For example, a risk-seeking investor might accept risks between [60 and 100], while a risk-averse investor may be comfortable with risks ranging from [0 to 40], and a risk-neutral investor could assume a risk level of [50]. This categorization exemplifies a triangular fuzzy number (see Figure 3.1).

However, based on expert knowledge, this research posits that risk aversion follows a trapezoidal distribution (see Figure 2). A trapezoidal membership function is characterized by its outputs forming a trapezoidal shape, making it advantageous for handling parameters with wide peak values, particularly in the context of moderate risk aversion [31]. As per [32] a fuzzy number A is defined as trapezoidal if its membership function is represented as follows:

$$\mu_{\tilde{A}} = \begin{cases} \omega\left(\frac{x-a}{b-a}\right), if\ a \leq x \leq b \\ \omega\left(\frac{d-x}{d-c}\right), if\ c \leq x \leq d \\ 0,\ if\ x \geq d \end{cases} \tag{3.2}$$

where $\omega \in [0,1]$.

In this research, we model risk aversion by categorizing investors based on their levels of risk tolerance. We assume that investors with low risk aversion estimate their risk aversion within the range of [5, 45]. It is presumed that no investor has a risk aversion of 0, as all investors likely prefer some degree of safety over engaging in purely risky investments. Investors exhibiting moderate risk aversion fall within the range of [45, 75], while those with high risk aversion are within [75, 95]. Each of these risk aversion categories—Low, Moderate, and High—is assumed to follow trapezoidal behavior.

Specifically, for Low risk aversion:

a. Low-Low risk aversion ranges from [5, 10],
b. Medium-Low risk aversion from [10, 40],
c. High-Low risk aversion from [40, 45].

For Moderate risk aversion:

d. Low-Moderate risk aversion ranges from [45, 50],
e. Medium-Moderate risk aversion from [50, 70],
f. High-Moderate risk aversion from [70, 75].

Finally, for High risk aversion:

g. Low-High risk aversion is from [75, 80],
h. Medium-High risk aversion from [80, 90],
i. High-High risk aversion from [90, 95].

It is also assumed that no investor is 100% risk averse, as every investor presumably seeks some level of risk to achieve returns based on the trade-off between risk and return. The following figures illustrate the different membership functions representing these distinct fuzzy numbers [33]:

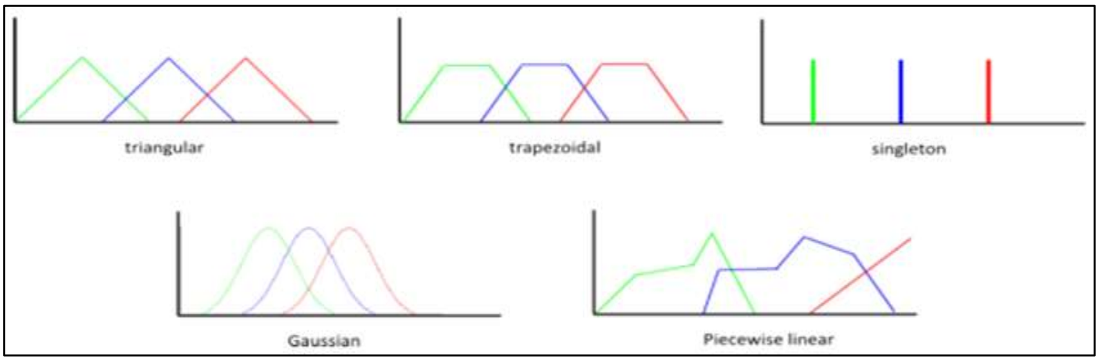

Figure 3.1: Fuzzy Membership Functions.

In this context and for more formal definition, a fuzzy number with left width α, right width β and a tolerance interval [a , b] is trapezoid and denoted A (a, b, α, β) if the membership function takes the following model [22]:

$$A(t) = \begin{cases} 1 - \frac{\alpha - t}{\alpha},\ if\ a - \alpha \le t \le \alpha \\ 1,\ if\ a \le t \le b \\ 1 - \frac{\beta - t}{\beta},\ if\ b - \alpha \le t \le b + \beta \\ 0,\ otherwise \end{cases} \quad (3.3)$$

In this study, risk aversion will be assessed four separate times for comparison and testing purposes. First, a general evaluation will consider risk aversion within the range of [5, 95], as defined earlier. Second, a specific evaluation will focus on the low risk aversion range of [5, 45]. Third, we will analyze the moderate risk aversion preference within the range of [45, 75]. Finally, we will assess high risk aversion, defined by the range of [75, 95]. All risk aversion models will utilize the trapezoidal distributions previously discussed.

One key advantage of trapezoidal distributions is their ability to handle wide peak values. This distribution is a variation of the triangular distribution, allowing for a range of probable values rather than just a single most likely value. This characteristic provides a more realistic representation of potential losses, enabling users to consider a range of values as representative of risk occurrences rather than fixating on a specific value. Trapezoidal distributions combine the benefits of triangular and uniform distributions. To trace the most probable ranges within a trapezoidal distribution, four parameters are established: the minimum, the maximum, the start of the most probable range, and the end of the most probable range [34].

3.3.2 Risk Aversion As A Fuzzy Cauchy Membership Function

In finance, insurance, and engineering, heavy-tailed distributions are commonly used due to the frequent occurrence of extreme events in these fields. Their high kurtosis results in greater probabilities of extreme occurrences compared to distributions with lower kurtosis. A notable example of a heavy-tailed distribution is the Cauchy distribution, which enhances the prediction and management of risks associated with extreme events. One significant advantage of heavy-tailed distributions is that they do not rely on specific assumptions regarding the nature of extreme events [35].

Considering the limitations of modern portfolio theory, along with behavioral biases in portfolio selection, extreme market events, political and geopolitical influences, risk appetite, and the subjective nature of risk aversion—all of which can contribute to extreme outcomes—it is reasonable to use the Cauchy distribution to model risk aversion. Incorporating heavy-tailed distributions into risk modeling improves the capture of tail risks and the accuracy of risk evaluations. For instance, when modeling market crashes or extreme movements, the estimation of probabilities for such outcomes becomes more precise, leading to more informed decision-making and a clearer representation of reality [35].

In addition to trapezoidal distributions, the Cauchy distribution will also be utilized to model risk aversion. As previously mentioned, we assume that risk aversion is measured on a scale from 0 to 100, where 0 indicates no risk aversion and a preference for risk, while 100 represents complete aversion to risk. The Cauchy distribution is unique in that it does not possess a defined mean or variance. For example, [36] argues that someone can make the mean whatever value he this about. Instead, Cauchy distribution is characterized by its location and scale parameters. The median is typically chosen as the location parameter, which is particularly relevant given the subjective nature of risk aversion, as demonstrated in previous literature relying on surveys and questionnaires.

For simplicity, we assume the median value of risk aversion is 50, serving as the location parameter for the Cauchy distribution. The scale parameter can be calculated by taking the difference between the third quartile (75) and the first quartile (25), divided by two, yielding a scale parameter of 25 (Scale Parameter = (75−25)/2 =25).

Although the methods used to determine the location and scale parameters may not be the most efficient, they are straightforward and simple. Thus, in this research, the Cauchy distribution will have a location parameter of 50 and a scale parameter of 25.

With the incorporation of fuzzy logic, the fuzzy membership function representing the Cauchy distribution can be defined as follows [37]:

$$\mu(x) = \frac{1}{1 + \left(\frac{x-p}{q}\right)^2}, x \in \mathbb{R}, q > 0 \tag{3.4}$$

The figure below illustrates the membership function associated with a Cauchy fuzzy number [37]:

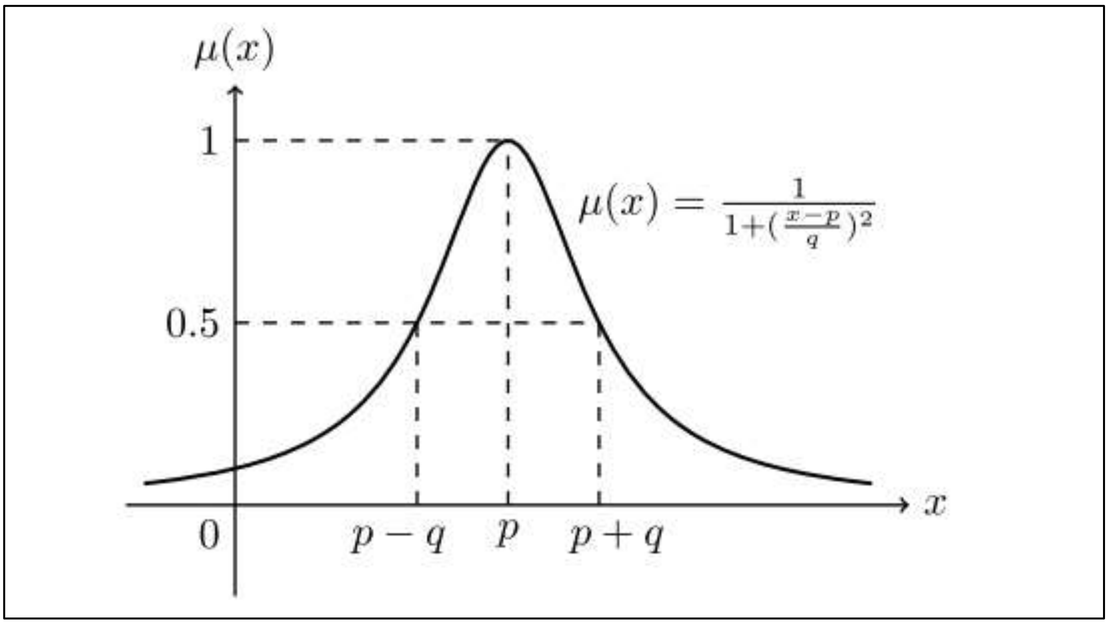

Figure 3.2: Membership Function Of A Cauchy Fuzzy Number.

3.3.3 Statistical Rationale of Choosing Cauchy And Trapezoidal Membership Functions

As previously mentioned, the Cauchy and trapezoidal functions were selected for their effectiveness in modeling risk. The trapezoidal function allows for a range of probable values rather than a fixed unit, while the Cauchy distribution, with its heavy tails, effectively accounts for extreme values and captures high probabilities of rare events. However, without data reflecting risk aversion, the models' effectiveness in this regard may be questioned.

According to [38], selecting the appropriate distribution is crucial yet challenging, as prior information is often insufficient to determine an exact prior distribution, with various distributions potentially fitting the available data. In this study, any heavy-tailed distribution, such as the Pareto or Student's t-distribution, could model risk aversion. Additionally, the triangular distribution could be utilized instead of the trapezoidal distribution, as the latter is a variant of the former.

[38] also points out that when decision-makers lack the resources and time to identify an exact prior, often the case with available information, they must supplement existing data with subjective inputs to establish the prior distribution. This study exemplifies that approach, as the choice of the Cauchy and trapezoidal distributions was supported by the belief that heavy tails can effectively capture extreme events arising from risk aversion, including behavioral biases, market crashes, and the limitations of modern portfolio theory. Furthermore, the trapezoidal function accommodates a wider range of values rather than fixed ones.

While [38] notes that the selection of a prior distribution can be arbitrary, this may lead to imperfect subsequent inferences. There exists a trade-off between analytical methods and the subjective choice of prior distribution. Thus, it is accurate to state that this research relies on a subjective prior.

As defined by [39], belief is a mental state. When an expert constructs a prior based on their belief and quantifies it, this is termed a subjective prior. [39] emphasizes that quantifying beliefs about the center of a distribution is simpler, often aligning beliefs with distributions centered around the median. In this study, the median of the risk aversion range [0, 100] was assumed to be 50, matching it to the Cauchy distribution, which is symmetrical around the median. The rationale for selecting the Cauchy distribution and trapezoidal function has been discussed earlier. Although [39] outlines additional steps to derive quintiles, this approach was not followed due to the simplicity of the risk aversion scale [0, 100], where the first and third quartiles are easily determined as 25 and 75, respectively.

However, [39] argues that the necessity of selecting a prior for a variable is not universal; subjective beliefs about that variable can vary significantly among experts. It is the expert's responsibility to justify their belief based on current knowledge or existing data. This is evident in the literature, where various measures of risk aversion are employed, ranging from questionnaires and surveys to mathematical models, highlighting the lack of a universal approach to modeling and assessing risk aversion.

Additionally, [39] notes that some priors may be more persuasive than others, suggesting that sensitivity analysis can be utilized to evaluate the model's reliability. In this research, a sensitivity analysis will be conducted to assess the effectiveness of modeling risk aversion using Cauchy and trapezoidal fuzzy membership functions. An illustrative example involving ten stocks will be presented, and results will be compared across different fuzzy risk aversion ranges with a model using a fixed risk aversion parameter on a scale from 1 to 5, which is a common method in portfolio optimization.

Finally, [39] states that if the key aspects of the model remain consistent across various specifications, there is no need for greater precision. Conversely, if the decision regarding main features is sensitive to different specifications, it becomes essential to understand these secondary factors better, which may require further research and alternative methods. If this approach does not resolve the issue, we must acknowledge that current information may not support accurate conclusions. This discussion will occur after deriving the results, where it will be evaluated whether modeling risk aversion through fuzzy logic leads to higher returns and whether this approach enhances the effectiveness of addressing the dual objective of minimizing risk while maximizing returns.

3.4 FUZZY CROSS ENTROPY

Before delving into the optimization problem, the researcher aims to evaluate the cross-entropy between the Cauchy and trapezoidal distributions to assess the dissimilarity in their modeling of risk aversion.

Using entropy as a measure of risk has been previously explored in the literature, drawing from information theory to analyze the loss of information. This approach allows for the measurement of uncertainty associated with specific outcomes. For a risk-averse decision-maker, this criterion applies regardless of the outcome's value, as long as the event is acknowledged. In line with the concept of entropy, risk is associated with future uncertainty, representing a lack of information about potential future states. Formally, entropy can be defined as the degree of uncertainty a system exhibits regarding its possible states, given known probabilities for those states [40]. In information theory, entropy quantifies the surprise or uncertainty linked to a probability distribution or random variable, where high entropy indicates greater uncertainty and low entropy suggests lesser uncertainty. Essentially, it measures the amount of information contained in an outcome: higher entropy corresponds to more surprising outcomes, while lower entropy relates to less surprising ones.

Cross-entropy, on the other hand, compares two distributions and quantifies their dissimilarities, effectively measuring the differences in information content between them [41]. Thus, cross-entropy can be viewed as a measure of model quality [42]. In this research, the goal is to evaluate the information content discrepancies between the Cauchy and trapezoidal distributions.

Given the use of fuzzy logic in this study, the fuzzy cross-entropy will be assessed, as it measures the divergences between two fuzzy variables (specifically, the Cauchy and trapezoidal distributions) [43][44]. The formula for fuzzy cross-entropy is as follows [43][44]:

$$D[\xi\ ,\eta] = \int_{-\infty}^{+\infty} \left(\frac{\mu(x)}{2}\right) \ln\left(\frac{\mu(x)}{v(x)}\right) + \left(1 - \frac{\mu(x)}{2}\right) \ln\left(\frac{2-\mu(x)}{2-v(x)}\right) \tag{3.5}$$

When the integral is evaluated, it yields a value of approximately 1.93. This result encourages the researcher to represent risk aversion as a linear combination of both the fuzzy Cauchy membership function and the trapezoidal one. The value of 1.93 indicates a divergence between the two membership functions in modeling risk aversion. Although this value may be considered low, it still supports the idea that a linear combination is preferable.

4. RESULTS

4.1 THE MODEL

Maximize the utility function:

$$U(p) = E(r) - \frac{1}{2} A\,\sigma^2 \tag{4.1}$$

Subject to:

a. A is defined as the linear combination of trapezoidal and Cauchy distributed risk aversion.
b. σ^2 represents the portfolio variance.
c. $E(r)$ is the expected return, calculated as:

$$E(r) = \sum_{i=1}^{n} r_i\, w_i \tag{4.2}$$

Where w_i is the fraction of the portfolio invested in asset i, r_i is the return of asset i.

Modeling Returns:

We will use historical average returns for r_i and model them using the Capital Asset Pricing Model (CAPM).

Risk Aversion Parameter A:

Assuming a linear combination of trapezoidal and Cauchy membership functions, we will use the previously established bounds. For the Cauchy distribution, we will set the location parameter to 50 and the scale parameter to 25. The linear combination is expressed as:

$$A = \alpha \cdot \mathit{Cauchy\ distribution} + \beta \cdot \mathit{trapezoidal\ distribution} \tag{4.3}$$

where $\alpha = 0.5$ and $\beta = 0.5$.

In this research, we model returns using the Capital Asset Pricing Model (CAPM). The returns can be expressed as follows [44]:

$$ER_i = R_f + B_i\left(ER_m - R_f\right) \tag{4.4}$$

Where ER_i is the expected return of investment, R_f the risk-free rate, B_i represents the beta of the investment and (ER_m – R_f) the market risk premium.

In addition to being a key component of the CAPM model and evaluating the individual beta of each stock, this research also assesses the beta of the entire optimized portfolio to measure its volatility in relation to the systemic risk associated with the market.

The beta coefficient can be modeled as follows [45]:

$$\beta = \frac{Covariance(R_e, R_m)}{Variance(R_m)} \tag{4.5}$$

where:

a. R_e = return on an individual stock
b. R_m = return on the overall market

To evaluate the performance of the optimized portfolio using fuzzy logic, the Trine Ratio is employed as a performance metric. This ratio helps to quantify the excess return generated per unit of risk taken in the portfolio. It can be modeled as follows [46]:

$$\text{Treynor Ratio } = \frac{(\text{Rp}-\text{Rf})}{\beta\text{p}} \tag{4.6}$$

Where R_p is the portfolio return, r_f the risk-free rate and β_p the beta of the portfolio.

4.2 RESULTS AND DISCUSSION

In this research, the portfolio evaluation is conducted over three timelines: long-term (10 years), mid-term (5 years), and short-term (1 year). The year 2024 will be excluded as it is not yet complete.

For illustrative purposes, the portfolio will consist of 10 stocks: Amazon, Apple, JP Morgan, Microsoft, Tesla, Netflix, Visa, Boeing, Meta, and Google. The chosen benchmark for comparison is the S&P 500.

4.2.1 Descriptive Statistics

Before conducting the optimization, a statistical analysis of the portfolio is performed to identify the mean returns, standard deviations, and minimum and maximum values for each stock. Additionally, each stock's beta is evaluated, along with the correlation matrix among the stocks, to understand their sensitivities to one another.

Long-term; 10 years from (01 – 01 – 2014 to 31 – 12 – 2023)

Table 4.1: Descriptive Statistics of Daily Historical Returns.

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
Minimum	-0.138	-0.124	-0.159	-0.151	-0.433	-0.146	-0.272	-0.162	-0.237	-0.306
Maximum	0.113	0.150	0.133	0.132	0.174	0.130	0.128	0.166	0.181	0.209
Mean	0.00096	0.00064	0.00098	0.00081	0.00089	0.00064	0.00032	0.00053	0.00128	0.00074
Std	0.01789	0.01754	0.01704	0.02086	0.028485	0.01558	0.02479	0.01694	0.03501	0.02386

Table 4.2: Stock Beta's.

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
Beta	1.19	1.15	1.20	1.14	1.17	1.08	1.39	1.12	1.48	1.25

The betas of the stocks indicate that they all exhibit greater volatility than the market, as all values are above 1. Tesla is the most volatile, with a volatility of 48%, while Visa is the least volatile at 8%. This observation is further supported by the descriptive statistics, where Tesla shows the highest standard deviation at 3.5%, whereas Visa has the lowest standard deviation at 1.5%.

Table 4.3: Correlation Matrix of Stocks (Significant at p = 0.05 and p = 0.01).

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
AAPL	1.00	0.61	0.67	0.54	0.40	0.56	0.41	0.44	0.42	0.52
GOOGL	0.61	1.00	0.72	0.64	0.46	0.58	0.40	0.43	0.39	0.64
MSFT	0.67	0.72	1.00	0.62	0.45	0.63	0.40	0.46	0.41	0.56
AMZN	0.54	0.64	0.62	1.00	0.50	0.48	0.30	0.30	0.40	0.58
NFLX	0.40	0.46	0.45	0.50	1.00	0.35	0.24	0.24	0.35	0.45
V	0.56	0.58	0.63	0.48	0.35	1.00	0.49	0.59	0.35	0.47
BA	0.41	0.40	0.40	0.30	0.24	0.49	1.00	0.58	0.31	0.31
JPM	0.44	0.43	0.46	0.30	0.24	0.59	0.58	1.00	0.27	0.34
TSLA	0.42	0.39	0.41	0.40	0.35	0.35	0.31	0.27	1.00	0.35
META	0.52	0.64	0.56	0.58	0.45	0.47	0.31	0.34	0.35	1.00

Table 4.3 shows that most stocks are correlated, with the majority of correlation coefficients exceeding 0.5. This is expected, as companies like Meta, Apple, Amazon, Microsoft, and Google are all part of the big five American tech firms. Notably, the correlation coefficients among these companies are higher than 0.5. The prevalence of high to moderate positive correlations among these stocks supports the observation that all betas are above 1, indicating that they are more volatile than the market.

Midterm; 5 years from 01 – 01 – 2019 to 31 – 12 – 2023:

Table 4.4: Descriptive Statistics of Daily Historical Returns.

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
Minimum	-0.138	-0.124	-0.159	-0.151	-0.433	-0.146	-0.272	-0.162	-0.237	-0.306
Maximum	0.113	0.092	0.133	0.127	0.156	0.130	0.218	0.166	0.181	0.209
Mean	0.00129	0.00078	0.00109	0.00054	0.00048	0.00056	-0.00015	0.00055	0.00198	0.00076
Std	0.02032	0.02001	0.01921	0.02218	0.02996	0.01764	0.03184	0.02006	0.04077	0.02788

Table 4.4 indicates that the minimum returns for the periods of 2019-2023 and 2014-2023 are the same. This is likely due to the impact of COVID-19, which caused many stocks to reach their lowest points. Additionally, Boeing recorded a negative return, reflecting the adverse effects of COVID-19 on the airline industry and manufacturing as a whole.

Table 4.5: Stock Beta’s.

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
Beta	1.21	1.13	1.18	1.06	1.05	1.04	1.49	1.1	1.52	1.29

The beta values for stocks in the period from 2019 to 2023 demonstrated similar trends to those observed from 2014 to 2023, with all stock betas exceeding 1. Tesla was the most volatile, showing 52% volatility compared to the market, while Visa had the least volatility at 4%. Consistent with the earlier period, Tesla remained the most volatile and Visa the least. This observation is further supported by the descriptive statistics, where Tesla displays the highest standard deviation at 4.07%, whereas Visa has the lowest at 1.7%.

Table 4.6: Correlation Matrix of Stocks (Significant at p =0.01 and 0.05).

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
AAPL	1.00	0.68	0.76	0.62	0.45	0.62	0.43	0.46	0.5	0.59
GOOGL	0.68	1.00	0.76	0.66	0.46	0.58	0.41	0.44	0.41	0.66
MSFT	0.76	0.76	1.00	0.68	0.48	0.64	0.40	0.45	0.46	0.61
AMZN	0.62	0.66	0.68	1.00	0.54	0.44	0.29	0.28	0.44	0.61
NFLX	0.45	0.46	0.48	0.54	1.00	0.32	0.24	0.21	0.37	0.5
V	0.62	0.58	0.64	0.44	0.32	1.00	0.51	0.63	0.37	0.46
BA	0.43	0.41	0.40	0.29	0.24	0.51	1.00	0.61	0.33	0.32
JPM	0.46	0.44	0.45	0.28	0.21	0.63	0.61	1.00	0.29	0.33
TSLA	0.5	0.41	0.46	0.44	0.37	0.37	0.33	0.29	1.00	0.35
META	0.59	0.66	0.61	0.61	0.5	0.46	0.32	0.33	0.35	1.00

Table 4.6 shows that most stocks are correlated, with the majority of correlation coefficients exceeding 0.5. This trend is consistent with the period from 2014 to 2023, although correlations in the period from 2019 to 2023 are slightly higher. As mentioned earlier, this is expected since many of the stocks belong to the same industry. The presence of high to moderate positive correlations among these companies further supports the finding that all betas are above 1, indicating that they are all more volatile than the market.

Short-term; 1 year from (01 – 01 – 2023 to 31 – 12 – 2023):

Table 4.7: Descriptive Statistics of Daily Historical Returns.

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
Minimum	-0.049	-0.0999	-0.045	-0.088	-0.087	-0.029	-0.057	-0.056	-0.103	-0.047
Maximum	0.045	0.0703	0.0699	0.079	0.149	0.031	0.084	0.073	0.104	0.209
Mean	0.00176	0.00181	0.00185	0.00229	0.00201	0.00095	0.00116	0.00104	0.00334	0.00419
Std	0.01255	0.01918	0.01575	0.02074	0.02325	0.0098	0.01705	0.01308	0.0330	0.02402

Table 4.7 indicates that the mean returns for the stocks in year 2023 are higher compared to the periods of 2014 to 2023 and 2019 to 2023.

Table 4.8: Stock Beta's.

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
Beta	1.10	1.39	1.18	1.54	1.32	0.76	1.02	0.88	2.21	1.74

The beta results for stocks in the period from 2022 to 2023 showed patterns similar to those observed in the periods from 2014 to 2023 and 2019 to 2023, with individual stock betas exceeding 1. Tesla continued to be the most volatile, exhibiting volatility nearly twice that of the market (S&P 500), while Visa remained the least volatile, with a beta of less than 1.

Table 4.9: Correlation Matrix of Stocks (Significant at p=0.05 and p=0.01).

TICKER	AAPL	GOOGL	MSFT	AMZN	NFLX	V	BA	JPM	TSLA	META
AAPL	1.00	0.53	0.55	0.44	0.43	0.48	0.29	0.30	0.44	0.55
GOOGL	0.53	1.00	0.51	0.60	0.32	0.31	0.18	0.16	0.34	0.61
MSFT	0.55	0.50	1.00	0.58	0.39	0.33	0.14	0.09	0.32	0.53
AMZN	0.44	0.60	0.58	1.00	0.38	0.31	0.19	0.13	0.37	0.59
NFLX	0.43	0.32	0.39	0.38	1.00	0.30	0.19	0.15	0.28	0.34
V	0.48	0.31	0.33	0.31	0.31	1.00	0.29	0.43	0.33	0.30
BA	0.29	0.18	0.14	0.19	0.19	0.29	1.00	0.32	0.24	0.19
JPM	0.30	0.16	0.09	0.13	0.16	0.43	0.32	1.00	0.30	0.20
TSLA	0.44	0.34	0.32	0.37	0.28	0.33	0.24	0.30	1.00	0.39
META	0.55	0.61	0.53	0.59	0.34	0.30	0.19	0.20	0.39	1.00

4.2.2 Portfolio Optimization Results

In optimizing the quadratic utility of a portfolio (the utility score), risk aversion is typically assigned a coefficient ranging from 1 to 4 or 1 to 5, depending on the investor's level of risk aversion. We will present the results of the portfolio optimization using risk aversion coefficients of 1, 2, 3, 4, and 5. Subsequently, we will compare these results with those obtained when risk aversion is modeled using fuzzy logic rather than a fixed coefficient.

Initially, each of the 10 stocks will be assigned a weight of 10%. The utility will then be optimized using the Sequential Least Squares Programming algorithm, which is the default method in the minimize function of the optimization package in Python. After optimization, we will derive and compare the respective expected returns (corresponding to the highest utility score), portfolio betas, weight allocations, and the Trine ratios of both portfolios (fixed risk aversion vs. fuzzy variable risk aversion).

Portfolio optimization using fixed risk aversion from 1 to 5

In this context, the following objective function (utility score of the portfolio) will be evaluated using risk aversion coefficients (A) of 1 to 5:

$$Maximize\ U(p) = E(r) - \frac{1}{2} A\, \sigma^2 \qquad (4.7)$$

The optimization results presented in the table 4.10 using fixed risk aversion coefficients show that higher risk aversion correlates with lower returns, consistent with existing literature. However, as the time horizon increases, portfolio returns decrease across all risk aversion coefficients from 1 to 5.

Table 4.10: Fixed Portfolio Optimization Results

Risk aversion		A =1	A =2	A = 3	A = 4	A=5
Short-term from 01-01-2023 to 31-12-2023	**Optimal weights**	AMZN: 0.0092, TSLA: 0.5764, META: 0.4145	GOOGL: 0.0535, AMZN: 0.2189, NFLX: 0.0570, BA: 0.0507, TSLA: 0.3263, META: 0.2936	AAPL: 0.0495, GOOGL: 0.0828, MSFT: 0.0468, AMZN: 0.1710, NFLX: 0.0592, BA: 0.1285, JPM: 0.0790, TSLA: 0.2049, META: 0.1783	AAPL: 0.1124, GOOGL: 0.0756, MSFT: 0.0905, AMZN: 0.1375, NFLX: 0.0464, BA: 0.1292, JPM: 0.1603, TSLA: 0.1375, META: 0.1106	AAPL: 0.1508, GOOGL: 0.0702, MSFT: 0.1122, AMZN: 0.1166, NFLX: 0.0381, V: 0.0101, BA: 0.1303, JPM: 0.2033, TSLA: 0.0967, META: 0.0717
	Portfolio return	0.407	0.364	0.3202	0.292	0.275
	Variance	0.154	0.094	0.059	0.042	0.034
	Portfolio beta	2.012	1.772	1.53	1.38	1.28
	Treynor ratio	0.182	0.182	0.182	0.182	0.182
Mid-term from 01-01-2019 to 31-12-2023	**Optimal weights**	AAPL: 0.2081, GOOGL: 0.0404, MSFT: 0.2519, AMZN: 0.0853 NFLX: 0.0237, V: 0.0760, BA: 0.1245, JPM: 0.1963, TSLA: 0.0395, META: 0.063	AAPL: 0.1404, GOOGL: 0.0854, MSFT: 0.1659, AMZN: 0.0853 NFLX: 0.0237, V: 0.2213, BA: 0.0232, JPM: 0.2548	AAPL: 0.1066, GOOGL: 0.0871, MSFT: 0.1346, AMZN: 0.1087 NFLX: 0.0324, V: 0.2633, JPM: 0.2674	AAPL: 0.0881, GOOGL: 0.0859, MSFT: 0.1201, AMZN: 0.1211 NFLX: 0.0355, V: 0.2836, JPM: 0.2656	AAPL: 0.0781, GOOGL: 0.0859, MSFT: 0.1110, AMZN: 0.1275 NFLX: 0.0368, V: 0.2969, JPM: 0.2637
	Portfolio returns	0.148	0.1394	0.1377	0.1373	0.1371
	Variance	0.075	0.063	0.061	0.061	0.061
	Portfolio beta	1.217	1.123	1.104	1.099	1.096
	Treynor ratio	0.087	0.087	0.087	0.087	0.087

Table 4.10: Fixed Portfolio Optimization Results “Table Continued”.

Long-term from 01-01-2014 to 31-12-2023:	**Optimal weights**	AAPL: 0.1432, GOOGL: 0.0666, MSFT: 0.2117, AMZN: 0.0172, NFLX: 0.074, V: 0.01248, BA: 0.1035, JPM: 0.2376, TSLA: 0.0324 META: 0.0556	AAPL: 0.1445, GOOGL: 0.1056, MSFT: 0.1555, AMZN: 0.0474, NFLX: 0.0180, V: 0.2147, BA: 0.0344, JPM: 0.2739, TSLA: 0.0015, META: 0.0045	AAPL: 0.1391 GOOGL: 0.1094 MSFT: 0.1369 AMZN: 0.0536 NFLX: 0.0174, V: 0.2419 BA: 0.010, JPM: 0.2916	AAPL: 0.1357, GOOGL: 0.1113, MSFT: 0.1247, AMZN: 0.0577 NFLX: 0.0167, V: 0.2566, JPM: 0.2973	AAPL: 0.1331, GOOGL: 0.1122, MSFT: 0.1179, AMZN: 0.0596 NFLX: 0.0164, V: 0.2637, JPM: 0.2970
	Portfolio returns	0.106	0.104	0.1034	0.1032	0.1031
	Variance	0.05	0.047	0.046	0.046	0.046
	Portfolio beta	1.195	1.15	1.139	1.135	1.134
	Treynor ratio	0.054	0.054	0.054	0.054	0.054

In the first case, we use a trapezoidal function defined by the parameters (x, 5, 45, 75, 95) and a Cauchy distribution characterized by (x, 50, 25). For low risk aversion, specifically within the range of 5 to 45, a random function is input into the algorithm to select a random integer from this interval.

Table 4.11: Portfolio Optimization Results With Fuzzy Low Risk Aversion.

Low risk aversion (5, 45)			
Time line	**Short-term**	**Mid-term**	**Long-term**
Optimal weights	TSLA: 0.5989, META: 0.4011	AAPL: 0.2351, MSFT: 0.2498, BA: 0.2149, JPM: 0.1248 TSLA: 0.0711, META: 0.1043	AAPL: 0.1404, MSFT: 0.2747, BA: 0.2328 JPM:0.1457, TSLA: 0.0858, META: 0.1208
Portfolio return	0.41	0.1529	0.1102
Portfolio variance	0.16	0.087	0.061
Portfolio beta	2.02	1.27	1.26
Treynor	0.1816	0.0867	0.0538
Randomly generated risk aversion	43	33	26

In the short-term analysis, we observed that the algorithm did not converge to a reliable solution. For instance, with a randomly selected risk aversion of 21%, the entire allocation (100%) was directed toward Tesla, resulting in a portfolio return of 44.38%, a variance of 27.44%, a portfolio beta of 2.21, and a Treynor ratio of 0.18. Similarly, for a risk aversion value of 14%, the allocation remained entirely with Tesla, yielding the same results as with the 21% risk aversion. This indicates that the algorithm failed to converge to an acceptable solution in the short term. After several iterations, a randomly generated risk aversion value of 43% produced comparable results. The table also displays the results for mid and long-term timelines at different randomly generated risk aversion levels.

In various runs of the algorithm within the low risk aversion interval (5 to 45), better solutions were achieved compared to the fixed risk aversion coefficient of 1, which is considered low risk aversion, across all timelines. However, the short-term timeline is questionable, as it allocated 100% of the weight to Tesla, indicating that this allocation may not have been optimal. In contrast, as the timeline increased, the algorithm consistently converged to superior results compared to both fixed risk aversion of 1 and even 2, if we consider 2 as low risk aversion as well.

When comparing portfolio returns for fuzzy risk aversion over the short, mid, and long terms, we obtained returns of 41%, 15.29%, and 11.02%, respectively, against 40.7%, 14.8%, and 10.6% for a fixed risk aversion of 1. These returns are also higher than those for a risk aversion of 2 which are 36.4%, 13.94% and 10.4%. Notably, the beta values of optimal portfolios modeled using fuzzy logic are higher than those using fixed risk aversion coefficients. While this increased volatility relative to the market is accompanied by higher returns, the higher variance in the portfolios indicates that more risk is taken on, yet this is still offset by the enhanced returns. Furthermore, the Treynor ratios show similar values, suggesting that the increased volatility associated with the fuzzy risk aversion model is effectively compensated by higher returns.

In the second case, we focused on moderate risk aversion, utilizing a range from 45 to 75. A random function was input into the algorithm to select a random integer within this interval. This approach allowed us to explore the impact of moderate risk aversion on the portfolio optimization process, providing insights into how varying levels of risk aversion influence asset allocation and overall portfolio performance.

Table 4.12: Portfolio Optimization Results With Fuzzy Moderate Risk Aversion.

moderate risk aversion (45, 75)			
Time line	**Short-term**	**Mid-term**	**Long-term**
Optimal weights	TSLA: 0.5935, META: 0.4065	AAPL: 0.2081, GOOG: 0.0403 MSFT: 0.2520, V: 0.0758 BA: 0.1247, JPM: 0.1962, TSLA: 0.0396, META: 0.0634	AAPL: 0.1452, GOOG: 0.0521 MSFT: 0.2376 AMZN:0.0051, NFLX:0.0024 V:0.0871 BA: 0.1318, JPM: 0.2198, TSLA: 0.0456 META: 0.0732
Portfolio return	0.4092	0.1475	0.1073
Portfolio variance	0.1582	0.0748	0.0524
Portfolio beta	2.02	1.217	1.212
Treynor	0.181	0.0867	0.0538
Randomly generated risk aversion	58	49	68

When considering fixed risk aversion coefficients of 2 and 3 as indicative of moderate risk-averse investors, we observed that the portfolio modeled with fuzzy risk aversion (within the interval of 45 to 75) consistently achieved higher returns across short-term, mid-term, and long-term periods compared to both coefficients. Specifically, for a coefficient of 2, the returns were 36.4%, 13.94%, and 10.4% for short, mid, and long terms, respectively, while for a coefficient of 3, the returns were 32.02%, 13.77%, and 10.34%. In contrast, the fuzzy risk aversion model produced returns of 40.92%, 14.75%, and 10.73% over the same periods, indicating superior performance. Additionally, although the portfolio betas and variances were higher for the fuzzy risk aversion model compared to the fixed coefficients, the increased returns effectively offset the greater volatility relative to the market benchmark and the inherent risks of the portfolios.

In the third case, we examined high risk aversion by utilizing a range from 75 to 95. A random function was implemented in the algorithm to select a random integer within this interval. This approach allowed us to investigate the effects of high-risk aversion on portfolio optimization, shedding light on how such levels of risk aversion influence asset allocation and overall portfolio performance. By focusing on this higher range, we aimed to understand the behavior of the portfolio under conditions of increased risk sensitivity.

Table 4.13: Portfolio Optimization Results With Fuzzy High-risk Aversion.

High risk aversion (75, 95)			
Time line	**Short-term**	**Mid-term**	**Long-term**
Optimal weights	TSLA: 0.7061, META: 0.2939	AAPL: 0.2233, MSFT: 0.2278, BA: 0.2656, JPM:0.0682 TSLA: 0.0930, META: 0.1221	AAPL: 0.1506, GOOG: 0.0159 MSFT: 0.2778, V: 0.0189 BA: 0.1827, JPM: 0.1950, TSLA: 0.0664 META: 0.0928
Portfolio return	0.4189	0.1554	0.1089
Portfolio variance	0.1816	0.0959	0.0568
Portfolio beta	2.074	1.308	1.242
Treynor	0.1816	0.0867	0.0538
Randomly generated risk aversion	76	80	78

The same issue encountered earlier, on the short timeline, with low risk aversion arose again in the high-risk aversion scenario, where the algorithm allocated 100% of the weights to a single stock, specifically Tesla. However, at a randomly generated risk aversion value of 76%, we observed improved results. In comparing the portfolio performance across short, mid, and long terms, the returns generated by modeling risk aversion using fuzzy logic were higher. Although we still observed elevated beta and variance, the conclusion remains that the increased volatility and variance relative to the market are offset by higher returns.

To address the challenges seen in both the short-term timeline for low and high risk aversion, we extended the short-term interval from January 1, 2023, to June 30, 2024, instead of ending in December 2023. Given that acceptable solutions were achieved in the mid-term and long-term, we hypothesized that a longer short-term timeline would yield better optimal solutions. As anticipated, the optimization process improved with the extended timeline, producing more reliable outcomes. For the high-risk aversion range of 75 to 95, most runs yielded better solutions, except for values of 94 and 95, where the allocation reverted to 100% in a single stock. Similarly, in the low-risk aversion range of 5 to 45, improved solutions were noted, except for the interval of 5 to 8, where the optimal allocation was again 100% to one stock.

It appears that extending the timeline can enhance the effectiveness of modeling risk aversion using fuzzy logic and optimizing through SLSQP, leading to better solutions compared to fixed risk aversion models. Additionally, exploring alternative optimization algorithms, such as meta-heuristics, may help achieve better outcomes in the intervals where allocations were constrained to a single stock (5 to 8 and 93 to 95).

In the preceding section, risk aversion was modeled using Cauchy and trapezoidal membership functions across the interval [5, 95], categorized as low (5 to 45), moderate (45 to 75), and high risk aversion (75 to 95). In the next section, we will redefine each interval using these membership functions: for low risk aversion, the trapezoidal function will be (x, 5, 10, 40, 45) and Cauchy (x, 25, 10); for moderate risk aversion, trapezoidal will be (x, 45, 50, 70, 75) and Cauchy (x, 60, 7.5); and for high risk aversion, trapezoidal will be (x, 75, 80, 90, 95) and Cauchy (x, 85, 5). This adjustment aims to broaden the peak value ranges, as trapezoidal membership functions are noted for their effectiveness in handling wide peak value ranges.

In the case of low risk aversion (5, 45), we modeled the trapezoidal function as (x, 5, 10, 40, 45) and the Cauchy function as (x, 25, 10). This configuration allows for a more nuanced representation of risk aversion within this lower range. By incorporating these specific membership functions, the algorithm can better capture the investor's preferences and optimize the portfolio accordingly, potentially leading to more effective risk-return trade-offs. This approach aims to enhance the allocation process by reflecting the lower levels of risk aversion typical of investors within this interval

In this section, we opted to extend the short timeline to cover the period from January 1, 2023, to June 30, 2024, in order to determine if this adjustment would yield better optimal solutions compared to the first case and to assess whether the same issues would arise.

Table 4.14: Portfolio Optimization Results With Trapezoidal and Cauchy Low-risk Aversion.

Low risk aversion (5, 45) with trapezoidal (x, 5, 10, 40, 45) and Cauchy(x, 25, 10)			
Time line	**Short-term**	**Mid-term**	**Long-term**
Optimal weights	TSLA: 0.4339 META: 0.5661	AAPL: 0.2152, GOOG: 0.0104, MSFT: 0.2709 V: 0.017, BA: 0.1649 JPM: 0.1781, TSLA: 0.0536, META: 0.09	AAPL: 0.1495, GOOG: 0.0206 MSFT: 0.2737, V: 0.0293 BA: 0.1754, JPM: 0.2006, TSLA: 0.0625, META: 0.0884
Portfolio return	0.4182	0.1502	0.1086
Portfolio variance	0.1252	0.0807	0.0056
Portfolio beta	1.894	1.248	1.237
Treynor	0.1986	0.0867	0.0538
Randomly generated risk aversion	21	32	39

In the short term, for the same generated risk aversion values of 21% or 14% from the first case, we found that when modeling the category low risk aversion itself using fuzzy logic and by extending the timeline, the allocation improved compared to the initial scenario where 100% was allocated to Tesla stock. Although the returns was slightly lower compared to the scenario of the full allocation to one stock, the diversification across different stocks is preferable. The optimization process ran more smoothly, allowing for better allocations rather than concentrating solely on one stock. However, similar issues arose with risk aversion values of 5% and 45%. Nonetheless, the overall process demonstrated improved performance compared to the first case, with all portfolio metrics exceeding those achieved with fixed risk aversion coefficients of 1 and 2, which can also be considered as representing low risk aversion.

When we compare this result of fuzzy risk aversion of 32% to fuzzy risk aversion of 33% of the 1st case (midterm) as they close value. We notice that the expected return of the portfolio is lower 15.02% vs. 15.29% however the portfolio variance decreased; 0.0807% vs. 0.087 also Portfolio beta decreased 1.248 vs. 1.26. Which means that even though the return are slightly lower, modeling low risk aversion itself with fuzzy logic implies better risk capture. The metrics are still promoting more than fixed risk aversion coefficients.

In most runs for low risk aversion in the range of (5, 45), modeled with the trapezoidal function (x, 5, 10, 40, 45) and Cauchy (x, 25, 10), we observed slightly lower returns compared to the first case. However, both the portfolio beta and variance of the optimized portfolio decreased, indicating that the model effectively captures risk. Overall, when compared to optimized portfolios using fixed risk aversion coefficients of 1 and 2, all performance metrics demonstrated improvements. Once again, the higher variance and beta were offset by the increased returns.

In this case, we examined moderate risk aversion within the range of (45, 75), utilizing a trapezoidal function defined as (x, 45, 50, 70, 75) and a Cauchy function characterized by (x, 60, 7.5). The results indicated that modeling risk aversion in this manner led to improved allocations compared to the first case. The optimized portfolios achieved higher returns, while also managing to keep the beta and variance at acceptable levels. This suggests that the model effectively balanced risk and return, demonstrating the advantages of using fuzzy logic in capturing moderate risk aversion in portfolio optimization. Overall, the performance metrics showed a notable enhancement over those derived from fixed risk aversion coefficients.

Table 4.15: Portfolio Optimization Results With Trapezoidal and Cauchy Moderate-risk Aversion.

Moderate risk aversion (45, 75) with trapezoidal (x, 45, 50, 70, 75) and Cauchy (x, 60, 7.5)			
Time line	**Short-term**	**Mid-term**	**Long-term**
Optimal weights	TSLA: 0.4534, META: 0.5466	AAPL: 0.2018, GOOG:0.033 MSFT: 0.2608, V: 0.0644 BA: 0.1317, JPM: 0.1980, TSLA: 0.0416, META: 0.0684	AAPL: 0.1443, GOOG: 0.0355, MSFT: 0.2548 AMZN: 0.00668 NFLX: 0.0004 V: 0.0571 BA: 0.1569, JPM: 0.2090, TSLA: 0.0546, META: 0.0805
Portfolio return	0.419	0.1479	0.1080
Portfolio variance	0.1269	0.075	0.054
Portfolio beta	1.897	1.222	1.225
Treynor	0.1986	0.0867	0.0538
Randomly generated risk aversion	66	58	52

Like the previous cases, this case demonstrated a smoother process and more reliable results for the different timelines. It still yielded slightly better results compared with the fuzzy modeling previously used in the moderate risk aversion and also more effective results compared to fixed risk aversion coefficients.

In the next case, which focused on high risk aversion (75 to 95) using the trapezoidal function (x, 75, 80, 90, 95) and the Cauchy function (x, 85, 5), the optimization process yielded promising results. Similar to previous cases, the algorithm produced reliable outcomes, particularly for mid- and long-term timelines. The performance metrics indicated that even with high risk aversion, the model effectively balanced returns and volatility, outperforming those achieved with fixed risk aversion coefficients. This suggests that the fuzzy logic approach to modeling risk aversion is capable of delivering better investment strategies, even for more risk-averse investors.

Similarly, in the short term, it was observed that for risk aversion values such as 95%, and 75%, the algorithm consistently converged towards allocating 100% of the portfolio to a single stock.

Table 4.16: Portfolio Optimization Results With Trapezoidal and Cauchy High-risk Aversion.

High risk aversion (75, 95) with trapezoidal (x, 75, 80, 90, 95) and Cauchy (x, 85, 5)			
Time line	**Short-term**	**Mid-term**	**Long-term**
Optimal weights	TSLA: 0.4430, META: 0.5570	AAPL: 0.2249, MSFT: 0.2313, BA: 0.2541, JPM: 0.0832 TSLA: 0.0869, META: 0.1196	AAPL: 0.1555, GOOG: 0.0105 MSFT: 0.2812, V:0.0008 BA: 0.1986, JPM: 0.1814, TSLA: 0.0668, META: 0.1052
Portfolio return	0.4186	0.1548	0.1093
Portfolio variance	0.1259	0.0938	0.0581
Portfolio beta	1.895	1.3012	1.249
Treynor	0.1986	0.0867	0.0538
Randomly generated risk aversion	88	91	79

When comparing all cases, it is evident that modeling risk aversion using fuzzy logic yields promising results, particularly when historical return timelines are extended (mid-term and long-term). The SLSQP algorithm consistently converges on seemingly acceptable optimal solutions regarding both allocation and returns. Notably, all cases of fuzzy risk aversion produced better outcomes than fixed coefficients ranging from 1 to 5. In summary, while portfolios modeled with fuzzy risk aversion tend to exhibit higher betas and variances, the resulting increase in volatility and risk is offset by higher returns. Additionally, the Treynor ratios for optimized portfolios using fixed coefficients and fuzzy risk aversion are similar, yet the returns for fuzzy portfolios are higher, indicating that fuzzy logic effectively captures volatility and risk. Lastly, it is observed that higher risk aversion correlates with lower returns, consistent with existing literature.

5. DISCUSSION AND CONCLUSIONS

Integrating fuzzy logic into portfolio optimization addresses several limitations of traditional methods, particularly the mean-variance model proposed by Harry Markowitz. Traditional approaches often assume static correlations and variances, a normal distribution of returns, and fixed risk aversion coefficients, which fail to capture the dynamic and uncertain nature of financial markets. In contrast, fuzzy logic excels in handling imprecision and ambiguity, aligning well with the realities of financial decision-making. It formalizes human reasoning and decision processes in environments characterized by incomplete, conflicting, or uncertain information, making it a powerful tool for financial investors who must navigate unpredictable market fluctuations, political changes, and economic crises. By incorporating fuzzy logic, portfolio optimization can more accurately model investors' varying risk aversion levels and provide more flexible, adaptive strategies that better reflect real-world conditions.

Moreover, fuzzy logic enhances the ability to balance the dual objectives of maximizing returns and minimizing risks, which are inherently conflicting. Traditional models often rely on a fixed risk aversion coefficient, limiting their adaptability to changing investor preferences and market conditions. Fuzzy logic introduces a dynamic approach to risk aversion, allowing for a more nuanced and precise representation of investor behavior and preferences. This adaptability can lead to more effective portfolio management, considering the full spectrum of risk tolerance among investors and responding to market changes more fluidly. By employing fuzzy risk aversion coefficients, portfolio optimization can achieve more robust and resilient performance, ultimately leading to higher expected returns and better risk management. This innovative approach has the potential to significantly improve the efficiency and efficacy of investment strategies, making it a valuable addition to modern portfolio theory.

The results of this study's portfolio optimization focus on assessing the quadratic utility of a portfolio under different risk aversion coefficients (1 to 5) and comparing these with results obtained using fuzzy logic for risk aversion. Initially, in comparing fuzzy to fixed coefficients, the resulting optimal weights, annualized returns, variances, portfolio betas, and Treynor ratios were calculated for short-term (2023), mid-term (2019-2023), and long-term

(2014-2023) periods. Higher risk aversion consistently led to lower returns. Then, the portfolio betas aligned with the returns, variances and Treynor ratios indicated a balanced risk-return-adjusted performance. In the long-term, portfolio returns generally decreased across all coefficients.

Using fuzzy logic to model risk aversion provided a more dynamic and flexible approach. For low risk aversion (5-45), the optimization sometimes failed to converge in the short term, sometimes allocating 100% of the weight to a single stock like Tesla. However, better solutions were achieved in mid-term and long-term periods, showing higher returns compared to different fixed risk aversion coefficients. The portfolios exhibited higher betas and variances, indicating increased volatility and risk, but the enhanced returns effectively compensated for these risks, resulting in higher overall performance.

For moderate and high-risk aversion ranges (45-75 and 75-95), the results were similar. Fuzzy risk aversion models consistently outperformed fixed coefficients, yielding higher returns across all timeframes. Despite higher betas and variances, the portfolios' Treynor ratios remained comparable, signifying effective risk compensation through higher returns. Extending the short-term period and exploring alternative optimization algorithms, such as meta-heuristic algorithms, could further enhance the effectiveness of fuzzy risk aversion models, particularly in scenarios where allocations were concentrated in a single stock. This highlights the potential benefits of a more flexible approach to risk aversion in portfolio optimization.

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APPENDIX A

OPTIMIZATION CODE

```
import pandas as pd
import yfinance as yf
import numpy as np
from scipy.optimize import minimize
#fetch historical stock price data then compute log returns
def fetch_data(tickers, start_date, end_date):
    data = yf.download(tickers, start=start_date, end=end_date)['Adj Close']
    log_returns = np.log(data / data.shift(1)).dropna()
    return log_returns
#calculate beta
def calculate_beta(returns, market_returns):
    covariance = np.cov(returns, market_returns.squeeze(), ddof=0)[0, 1]
    market_variance = np.var(market_returns.squeeze(), ddof=0)
    beta = covariance / market_variance
    return beta
#calculate expected returns using CAPM model and annualize them
def calculate_expected_returns(returns, market_data, risk_free_rate):
    trading_days_per_year = 252
    market_returns = np.log(market_data / market_data.shift(1)).dropna()
    expected_returns_CAPM = []
    for col in returns.columns:
        stock_returns = returns[col]
        beta = calculate_beta(stock_returns, market_returns)
        expected_return = risk_free_rate + beta * ((market_returns.mean() * trading_days_per_year) - risk_free_rate)
```

```
        expected_returns_CAPM.append(expected_return)
    return np.array(expected_returns_CAPM)
#calculate covariance matrix and annualize it
def calculate_covariance_matrix(returns):
    trading_days_per_year = 252
    cov_matrix = returns.cov() * trading_days_per_year
    return cov_matrix
# Cauchy fuzzy function
def cauchy(x, x0, gamma):
    return 1 / (1 + ((x - x0) / gamma)**2)
# Trapezoidal fuzzy function
def trapezoidal(x, a, b, c, d):
    if a <= x < b:
        return (x - a) / (b - a)
    elif b <= x <= c:
        return 1
    elif c < x < d:
        return (d - x) / (d - c)
    else:
        return 0
#calculate fuzzy total risk aversion
def total_risk_aversion_variable(alpha, beta_var, x_variable):
    x0_c = 50
    gamma_c = 25
    a_t = 5
    b_t = 45
    c_t = 75
```

```
    d_t = 95
    cauchy_val = alpha * cauchy(x_variable, x0_c, gamma_c)
    trapezoidal_val = beta_var * trapezoidal(x_variable, a_t, b_t, c_t, d_t)
    return cauchy_val + trapezoidal_val
#fixed total risk aversion (run it separetely from 1 to 5)
def total_risk_aversion_fixed():
    return 4


#calculate utility for variable risk aversion
def calculate_utility_variable(weights, expected_returns_CAPM, cov_matrix, alpha, x_variable, beta_var):
    expected_return = np.dot(weights, expected_returns_CAPM)
    portfolio_variance = np.dot(weights.T, np.dot(cov_matrix, weights))
    total_risk_aversion = total_risk_aversion_variable(alpha, beta_var, x_variable)
    utility = expected_return - 0.5 * total_risk_aversion * portfolio_variance
    return -utility
# Function to calculate utility for fixed risk aversion
def calculate_utility_fixed(weights, expected_returns_CAPM, cov_matrix):
    expected_return = np.dot(weights, expected_returns_CAPM)
    portfolio_variance = np.dot(weights.T, np.dot(cov_matrix, weights))
    total_risk_aversion = total_risk_aversion_fixed()
    utility = expected_return - 0.5 * total_risk_aversion * portfolio_variance
    return -utility
#calculate Treynor ratio
def calculate_treynor_ratio(portfolio_return, portfolio_beta, risk_free_rate):
    treynor_ratio = (portfolio_return - risk_free_rate) / portfolio_beta
    return treynor_ratio
#calculate portfolio beta
```

```
def calculate_portfolio_beta(weights, returns, market_data):
    market_returns = np.log(market_data / market_data.shift(1)).dropna()
    portfolio_returns = np.dot(returns, weights)
    portfolio_beta = calculate_beta(portfolio_returns, market_returns)
    return portfolio_beta
#Generate random x_variable
def generate_random_x_variable():
    a_t = 5
    d_t = 45
    x_variable = np.random.randint(a_t, d_t)
    if x_variable < a_t:
        x_variable = a_t
    elif x_variable > d_t:
        x_variable = d_t
    return x_variable
x_variable = generate_random_x_variable()
#x_variable = 52 #can enable it and be used for easy commparison
#program
tickers = ['AAPL', 'GOOGL', 'MSFT', 'AMZN', 'NFLX', 'V', 'BA', 'JPM', 'TSLA', 'META']
start_date = '2023-01-01'
end_date = '2023-12-31'
risk_free_rate = 0.042  # 10-year Treasury bill rate
alpha = 0.5
beta_var = 0.5
returns = fetch_data(tickers, start_date, end_date)
market_data = yf.download('^GSPC', start=start_date, end=end_date)['Adj Close']
#Reindex returns to match the order of tickers
```

```
returns = returns[tickers]

expected_returns_CAPM = calculate_expected_returns(returns, market_data, risk_free_rate)

cov_matrix = calculate_covariance_matrix(returns)

#initial weights

initial_guess = np.ones(len(tickers)) / len(tickers)

#define constraints and bounds of weights

constraints = ({'type': 'eq', 'fun': lambda weights: np.sum(weights) - 1})

bounds = [(0, 1)] * len(initial_guess)

#optimize portfolio for variable risk aversion

optimal_weights_variable = minimize(calculate_utility_variable, initial_guess,
                        args=(expected_returns_CAPM, cov_matrix, alpha, x_variable, beta_var),
                        constraints=constraints, bounds=bounds).x

#optimize portfolio for fixed risk aversion

optimal_weights_fixed = minimize(calculate_utility_fixed, initial_guess,
                        args=(expected_returns_CAPM, cov_matrix),
                        constraints=constraints, bounds=bounds).x


#calculate portfolio return and variance for fuzzy risk aversion

portfolio_return_variable = np.dot(optimal_weights_variable, expected_returns_CAPM)

portfolio_variance_variable = np.dot(optimal_weights_variable.T, np.dot(cov_matrix, optimal_weights_variable))

#calculate portfolio return and variance for fixed risk aversion

portfolio_return_fixed = np.dot(optimal_weights_fixed, expected_returns_CAPM)

portfolio_variance_fixed = np.dot(optimal_weights_fixed.T, np.dot(cov_matrix, optimal_weights_fixed))

#calculate portfolio beta for fuzzy risk aversion

portfolio_beta_variable = calculate_portfolio_beta(optimal_weights_variable, returns, market_data)
```

```
portfolio_beta_fixed = calculate_portfolio_beta(optimal_weights_fixed, returns, market_data)

#calculate Treynor ratio for fuzzy risk aversion

treynor_ratio_variable = calculate_treynor_ratio(portfolio_return_variable, portfolio_beta_variable, risk_free_rate)

# Calculate Treynor ratio for fixed risk aversion

treynor_ratio_fixed = calculate_treynor_ratio(portfolio_return_fixed, portfolio_beta_fixed, risk_free_rate)

# Print results for fuzzy risk aversion

utility_variable = -calculate_utility_variable(optimal_weights_variable, expected_returns_CAPM, cov_matrix, alpha, x_variable, beta_var)

print("\nVariable Total Risk Aversion:")

print("Optimal Weights:")

for ticker, weight in zip(tickers, optimal_weights_variable):

    print(f"{ticker}: {weight:.4f}")

print("Portfolio Return (Annualized):", portfolio_return_variable)

print("Portfolio Variance (Annualized):", portfolio_variance_variable)

print("Utility:", utility_variable)

print("Portfolio Beta:", portfolio_beta_variable)

print("Treynor Ratio:", treynor_ratio_variable)

print("Randomly chosen x_variable:", x_variable)

# Print results for fixed risk aversion

utility_fixed = -calculate_utility_fixed(optimal_weights_fixed, expected_returns_CAPM, cov_matrix)

print("\nFixed Total Risk Aversion:")

print("Optimal Weights:")

for ticker, weight in zip(tickers, optimal_weights_fixed):

    print(f"{ticker}: {weight:.4f}")

print("Portfolio Return (Annualized):", portfolio_return_fixed)

print("Portfolio Variance (Annualized):", portfolio_variance_fixed)
```

```
print("Utility:", utility_fixed)
print("Portfolio Beta:", portfolio_beta_fixed)
print("Treynor Ratio:", treynor_ratio_fixed)
```