DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INTERPOLATION AND APPROXIMATION BY *q*-B-SPLINE FUNCTIONS

by

Gülter BUDAKÇI

November, 2015

İZMİR

INTERPOLATION AND APPROXIMATION BY
q-B-SPLINE FUNCTIONS

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**INTERPOLATION AND APPROXIMATION BY q-B-SPLINE FUNCTIONS** " completed by **GÜLTER BUDAKÇI** under supervision of **PROF. DR. HALİL ORUÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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INTERPOLATION AND APPROXIMATION BY q-B-SPLINE FUNCTIONS

ABSTRACT

In this thesis, from viewpoint of interpolation and approximation, we investigate fundamental properties of q-B-spline functions which are piecewise polynomials whose q-derivatives are continuous. We give a representation of q-B-splines via divided differences of truncated power functions which may be considered as a generalization of classical truncated functions. We derive q-analogue of the de Boor-Fix formula for dual functions and $1/q$-integral of q-B-splines over their support by using recurrence formula for q-B-splines and q-analogue of Marsden identity. A $1/q$-convolution formula for uniform q-B-splines is also obtained. We introduce a q-analogue of Peano kernel theorem. We give applications to polynomial interpolation and construct examples in which classical remainder theory fails whereas q-Peano kernel works. Furthermore, we establish a relation between q-B-splines and divided differences via q-Peano kernel. We show that q-B-splines form a basis for quantum spline spaces. Recurrence relations for $1/q$-integration and $1/q$-differentiation formulas of q-spline functions are derived. Moreover, we demonstrate a way to find the polynomials on each interval of a q-spline function. After all, we give basic theory of interpolation by a q-B-splines and approximation by q-analogue of the Bernstein-Schoenberg operator.

Keywords: q-B-splines, q-Peano kernel, q-integral, q-derivative, q-Taylor formula, q-Bernstein-Schoenberg operator, de Boor-Fix formula

q-B-SPLINE FONKSİYONLARI İLE İNTERPOLASYON VE YAKLAŞIM

ÖZ

Bu tezde, q-türevleri sürekli parçalı polinom olan q-B-spline fonksiyonlarının interpolasyon ve yaklaşım açısından temel özellikleri incelenmiştir. q-B-spline fonksiyonlarının bölünmüş farklarla gösterimi verildi. q-B-spline fonksiyonlarının bilinen yineleme formülü ve Marsden özdeşliğinin q-analoğu kullanılarak dual fonksiyonlar için de Boor-Fix formülünün q-analoğu ve q-B-spline fonksiyonlarının bağ aralığındaki $1/q$-integrali bulundu. Ayrıca tekdüze q-B-spline fonksiyonlarının $1/q$-konvolüsyon formülü elde edildi. Peano çekirdeği teoreminin q-analoğunu tanımlandıktan sonra, polinom interpolasyonuna uygulamalar verildi ve klasik teorinin çalışmadığı ama q-Peano çekirdeğinin çalıştığı örnekler bulundu. q-B-spline fonksiyonları ve bölünmüş farklar arasındaki ilişki q-Peano çekirdeği aracılığıyla kuruldu. q-B-spline fonksiyonlarının kuantum spline uzayının bazları olduğu gösterildi. q-spline fonksiyonlarının $1/q$-integral ve $1/q$-türev formülleri bulundu. Bunlara ek olarak bir q-spline fonksiyonunun her bir aralıktaki polinomları nasıl bulabileceğimiz gösterildi. Tüm bunlardan sonra, q-B-spline fonksiyonlarıyla interpolasyon ve Bernstein-Schoenberg operatörünün q-analoğu ile yaklaşımın temel teorisi verildi.

Anahtar kelimeler: q-B-spline, q-Peano çekirdeği, q-integral, q-türev, q-Taylor formülü, q-Bernstein-Schoenberg operatörü, de Boor-Fix formülü

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CHAPTER ONE
INTRODUCTION

In this chapter, following the books (Phillips, 2003; Powell, 1981) we give definitions and some basic properties of classical splines and Bernstein-Schoenberg operator. The Bernstein-Schoenberg operator may be viewed as a generalization of the Bernstein operator since it replaces polynomials by piecewise polynomials. Quite unusually in an introduction chapter, we provide a new result which generalizes one of the main results of Budakçı (2010) and published in Budakçı & Oruç (2012). Moreover Section 1.3 outlines basic facts about quantum calculus, q-shifted polynomials, q-Taylor Theorem and q-integration.

1.1 Splines

A spline function consists of polynomial pieces on subintervals joined together with certain continuity conditions. Formally, suppose that $k+1$ points $t_0, \ldots, t_k$ are specified satisfying $t_0 < \cdots < t_k$. These points are called knots. Suppose also that an integer $n \geqslant 0$ is prescribed. A spline function of degree n having knots $t_0, \ldots, t_k$ is a function S such that

i. On each interval $[t_{j-1}, t_j)$ is a polynomial of degree $\leqslant n$, $j = 1, \ldots, k$.

ii. S has continuous $(n-1)$st derivative on $[t_0, t_k]$.

The term *spline* comes from the flexible strip of metal used by shipbuilders and drafters to draw smooth shapes. The theory of splines is a good example of an area in mathematics which was developed in response to practical needs. Spline curves were first used as a drafting tool for aircraft and ship building industries. A loft man's spline is a flexible strip of material, which can be clamped or weighted so it will pass through any number of points with smooth deformation (Farin, 2002; Hammerlin & Hoffmann, 1991).

Lobachevsky constructed B-splines as convolutions of certain probability distributions as early as nineteenth century. Spline functions are currently used in diverse domains of numerical analysis: interpolation, data smoothing, computer aided geometric design, numerical solution of differential and integral equations, wavelets, etc.. In 1946, Schoenberg used B-splines for statistical data smoothing, and his paper started the modern theory of spline approximation. Gordon and Reisenfield formally introduced B-splines into computer aided design (Farin, 2002).

Only three kinds of bases for spline spaces have actually been given serious attention; those consisting of truncated power functions, of cardinal splines, and of B-splines. B-splines which are splines with smallest possible support, in other words they are zero on a large set, is the most practical and widely used basis.

1.1.1 B-Splines

Let $\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$ be the knot sequence where $t_{-i} \to -\infty$ as $i \to \infty$ and $t_i \to \infty$ as $i \to \infty$ with $i > 0$.

Definition 1.1.1. The B-splines of degree $zero$ are piecewise constants defined by

$$N_{i,0}(t) = \begin{cases} 1, & t_i \leqslant t < t_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

and linear B-splines are

$$N_{i,1}(t) = \begin{cases} \dfrac{t - t_i}{t_{i+1} - t_i}, & t_i \leqslant t < t_{i+1}, \\ \\ \dfrac{t_{i+2} - t}{t_{i+2} - t_{i+1}}, & t_{i+1} \leqslant t < t_{i+2} \\ \\ 0, & \text{otherwise.} \end{cases}$$

The graphs Figure 1.1 and Figure 1.2 of $N_{i,0}(t)$ and $N_{i,1}(t)$ are shown below.

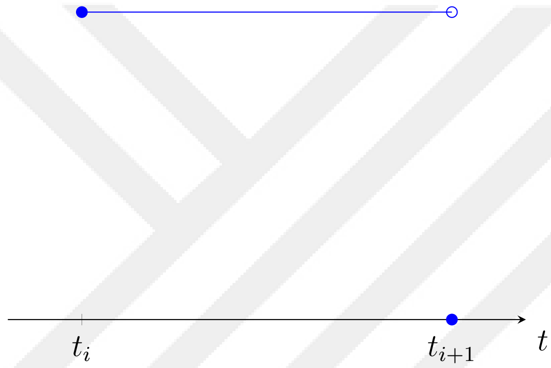

Figure 1.1 Graph of $N_{i,0}(t)$.

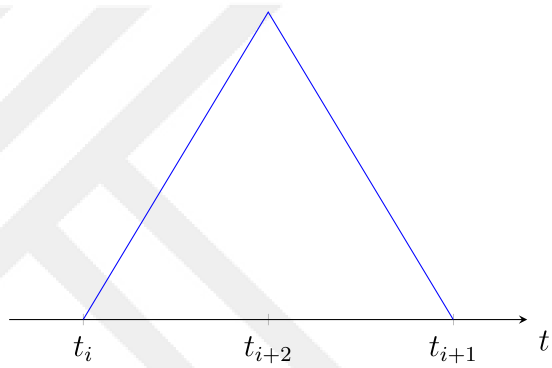

Figure 1.2 Graph of $N_{i,1}(t)$.

Since the original definition of the B-spline basis functions uses the idea of divided differences and is mathematically involved, equivalently we can define B-splines as a multiple of a divided difference of truncated power function which is defined as

$$(x - t)_+^n = \begin{cases} (x - t)^n, & t \leqslant x, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Theorem 1.1.1. *For any $n \geqslant 0$ and all i,*

$$N_{i,n}(t) = (t_{i+n+1} - t_i).[t_i, \ldots, t_{i+n+1}](x - t)_+^n, \tag{1.2}$$

where $[t_i, \ldots, t_{i+n+1}]$ denotes a divided difference operator of order $n + 1$ that is applied to the truncated power $(x - t)_+^n$, regarded as a function of the variable x.

By applying Leibniz's rule for the product of divided differences, Carl de Boor established a recursive relationship for the B-spline basis in de Boor (1972) as follows:

$$N_{i,n}(t) = \left(\frac{t - t_i}{t_{i+n} - t_i}\right) N_{i,n-1}(t) + \left(\frac{t_{i+n+1} - t}{t_{i+n+1} - t_{i+1}}\right) N_{i+1,n-1}(t),$$

starting with $N_{i,0}(t)$.

1.1.2 Properties of B-Splines

Definition 1.1.2. Let S denote a spline defined on the whole real line. The $interval$ $of\ support$ of the spline S is the smallest closed interval outside which S is zero.

Theorem 1.1.2. *The interval of support of the B-spline $N_{i,n}$ is $[t_i, t_{i+n+1}]$, and $N_{i,n}$ is positive in the interior of this interval.*

The next result shows that the derivative of a B-spline function is also a B-spline function.

Theorem 1.1.3. *For $n \geqslant 1$, we have*

$$\frac{d}{dt} N_{i,n}(t) = \left(\frac{n}{t_{i+n} - t_i}\right) N_{i,n-1}(t) - \left(\frac{n}{t_{i+n+1} - t_{i+1}}\right) N_{i+1,n-1}(t) \tag{1.3}$$

for all real t. For $n = 1$, equation (1.3) *holds for all t except at the three knots t_i,t_{i+1}, and t_{i+2}, where the derivative of $N_{i,1}$ is not defined.*

In the remainder of this chapter B-splines are computed with a knot sequence $t_0, \ldots, t_k$ and defined over all $\mathbb{R}$, and the algorithms described are independent of the chosen interval $[a, b]$, with the condition that $t_n \leqslant a$ and $t_{k-n} \geqslant b$. In the algorithms described below, we set $l = k - n - 1$. Notice that the dimension of the spline space is $l + 1$.

We will see that a spline approximation can be represented by

$$S(t) = \sum_{i=0}^{l} a_i N_{i,n}(t), \qquad a_i \in \mathbb{R}, \tag{1.4}$$

a sum of multiples of B-splines of degree n whose interval of support contains one of the subintervals $[t_j, t_{j+1}]$ where $j = n, \ldots, k-n-1$.

1.1.3 Marsden Identity and Its Consequences

The following identity which relates polynomials with B-splines due to Marsden (1970) is frequently used:

$$(z-t)^n = \sum_{j=0}^{l} (z-t_{j+1})(z-t_{j+2})\ldots(z-t_{j+n})N_{j,n}(t) \tag{1.5}$$

for all real or complex z and all real t restricted to the interval

$$I = \{t : \quad t_n \leqslant t \leqslant t_{k-n}\}. \tag{1.6}$$

Note that the summation above may run for all $j \in \mathbb{Z}$ since the B-splines $N_{j,n}(t)$ have compact support.

It is useful to give the definition of elementary symmetric functions since it appears in blossoming and B-splines of the following theorem. One can follow the book Phillips (2003).

Definition 1.1.3. The elementary symmetric function $\sigma_r(x_0, x_1, \ldots, x_m)$, for $r \geqslant 1$, is the sum of all products of r distinct variables chosen from the set

$\{x_0, x_1, \ldots, x_m\}$, and we define $\sigma_r(x_0, x_1, \ldots, x_m) = 1$ for $r = 0$. Namely we have

$$\sigma_r(x_0, \ldots, x_m) = \sum_{0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant m} x_{i_1} \cdots x_{i_r}. \tag{1.7}$$

As a consequence of Definition 1.1.3 we have

$$\sigma_r(x_0, x_1, \ldots, x_m) = 0 \quad \text{if} \quad r > m + 1. \tag{1.8}$$

Since

$$(1 + x_0 x)(1 + x_1 x) \cdots (1 + x_m x) = \sum_{r=0}^{m+1} \sigma_r(x_0, x_1, \ldots, x_m) x^r \tag{1.9}$$

the polynomial $(1 + x_0 x)(1 + x_1 x) \cdots (1 + x_m x)$ is the generating function for elementary symmetric functions.

The following theorem illustrates the relationship between monomials and B-splines. It can be proved easily by using Marsden's Identity, see (Marsden, 1970).

Theorem 1.1.4. *For any given integer $r \geqslant 0$ we can express any monomial t^r as a linear combination of B-splines $N_{i,n}(t)$, for any fixed $n \geqslant r$, in the form*

$$\binom{n}{r} t^r = \sum_{i=0}^{l} \sigma_r(t_{i+1}, \ldots, t_{i+n}) N_{i,n}(t) \tag{1.10}$$

where $\sigma_r(t_{i+1}, \ldots, t_{i+n})$ is the elementary symmetric function of order r in the variables $t_{i+1}, \ldots, t_{i+n}$. Furthermore, if $r = 0$ in (1.10) we have

$$\sum_{i=0}^{l} N_{i,n}(t) = 1 \tag{1.11}$$

and thus the B-splines of degree n form a partition of unity.

Comparing the coefficient of z^n in (1.5) yields

$$\sum_{j=0}^{l} N_{j,n}(t) = 1 \quad \text{if} \quad t \in I, \tag{1.12}$$

and that of z^{n-1} gives

$$\sum_{j=0}^{l} \xi_j N_{j,n}(t) = t \quad \text{if} \quad t \in I, \tag{1.13}$$

where

$$\xi_j = \frac{1}{n}(t_{j+1} + t_{j+2} + \cdots + t_{j+n}) \quad j = 0, 1, \ldots, l, \tag{1.14}$$

known as Greville Abscissae. From (1.14) and $t_0 < \cdots < t_k$, Greville Abscissaes are ordered

$$t_0 < \xi_0 < \xi_1 < \cdots < \xi_l < t_k. \tag{1.15}$$

1.2 Bernstein-Schoenberg Operator

A constructive proof of the Weierstrass' approximation was given in 1912 via Bernstein polynomials. These polynomials are particularly useful for both approximation and curves and surfaces design. Bernstein-Bézier and B-spline techniques are fundamental in computer aided geometric design (CAGD). A striking generalization of the Bernstein polynomials is the Bernstein-Schoenberg operator based on B-splines.

Many of the early results on spline functions are due to Schoenberg. In Schoenberg (1967) B-splines are defined as a basis for spline spaces having smallest possible support. Then he introduced a spline approximation operator which generalized the Bernstein polynomial. The discovery of the recurrence relation of B-splines in de Boor (1972) escalated research on splines. We note that in approximation theory it is often useful to have an approximation operator based on a continuous function f, not only close to f but also having a graph whose shape is similar to that of the graph of f, namely shape preserving operator. Like the Bernstein polynomials, the Bernstein-Schoenberg operator is variation diminishing and therefore has certain shape preserving properties. That is, if the underlying function is monotone so is its Bernstein-Schoenberg spline function. Furthermore it yields a convex function whenever f is convex. Consequently the Bernstein-Schoenberg operator mimics the shape of the function, see (Goodman, 1996; Goodman & Sharma, 1985).

In this section we summarize the properties of Bernstein-Schoenberg operator for general knot sequences and then give the main results briefly on Bernstein Schoenberg operator on q-integers that we derived in Budakçı (2010).

1.2.1 Shape Preserving Properties of The Operator

Goodman discussed in (Goodman, 1996) the advantages of variation diminishing property when designing curves or constructing approximation operators. Like the Bernstein polynomials Bernstein-Schoenberg operator has variation diminishing and therefore has certain shape preserving properties. Goodman and Sharma discussed in (Goodman & Sharma, 1985) the convexity properties for of Bernstein-Schoenberg operator for a special knot sequence.

We call $S_{n,m}$ the *Bernstein-Schoenberg Operator* which maps a continuous function f on $[a, b]$ to a continuous function $S_{n,m}f$.

In the remainder of this section we investigate properties of the operator $S_{n,m}f$ for the functions f defined on the interval $[0, 1]$.

For a function $f(t)$ and $t \in [0,1]$, the operator is given by

$$S_{n,m}(f;t) = \sum_{j=0}^{l} f(\xi_j) N_{j,n}(t), \tag{1.16}$$

where n is the degree of B-splines, m is the number of intervals in $[0,1]$ with $l = n + m - 1$ and

$$\xi_i = \frac{1}{n}(t_{i+1} + \cdots + t_{i+n}),$$

are the Greville abscissae, and we have a nondecreasing (extended) knot sequence

$$t_0 = \cdots = t_n = 0 < t_{n+1} < \cdots < t_{n+m-1} < t_{n+m} = \cdots = t_{l+n+1} = 1.$$

The purpose of repeating end knots $n+1$ times is to make the operator satisfy endpoint interpolation property. It follows from Marsden's identity (1.5) that $S_{n,m}(f;t) = f(t)$ for any linear function $f(t) = at + b$. In Marsden & Schoenberg (1966) it is shown for the special knot sequence

$$t_0 = \cdots = t_n = 0,\ t_{n+1} = \frac{1}{m}, \ldots, t_l = \frac{m-1}{m},\ t_{l+1} = \cdots = t_{l+n+1} = 1$$

that the function $S_{n,m}(t^2;t)$ converges to t^2 uniformly as $n \to \infty$. Thus from the Bohman-Korovkin Theorem, see Phillips (2003), $S_{n,m}f$ converges uniformly to the function f for any $f \in C[0,1]$. The Bohman-Korovkin Theorem is stated as follows:

Theorem 1.2.1. *A sequence of positive linear operators $L_n f$ converges uniformly for every continuous function on a closed interval provided that it converges for the functions $e_0(t) = 1$, $e_1(t) = t$ and $e_2(t) = t^2$.*

The operator $S_{n,m}f$ on $C[0,1]$ is a shape preserving operator. Namely, it mimics the shape of f. We state the underlying facts about shape preservation, (see Goodman (1996)) *i.* $S_{n,m}f$ is variation diminishing: Any line does not cross $S_{n,m}f$ more than f does. In particular, the number of sign changes in $S_{n,m}f$ bounded by the number of sign changes in f.

ii. If a function $f \in C[0,1]$ is increasing (respectively decreasing), then $S_{n,m}f$ is increasing (respectively decreasing).

iii. If f is convex on $[0,1]$, then $S_{n,m}f$ is also convex.

iv. If $f(t)$ is convex on $[0,1]$ then $S_{n,m}(f;t) \geqslant f(t), \quad 0 \leqslant t \leqslant 1.$

1.2.2 Bernstein-Schoenberg Operators on q-Integers

Now we will discuss the properties of Bernstein-Schoenberg operators defined on q-integers, a geometrically spaced knot sequence. We denote this operator on q-integers by $S_{n,m}(f;t,q)$ to emphasize its dependence on an initially given parameter value q. Henceforth we use the knot sequence

$$\begin{gathered} t_0 = \cdots = t_n = 0,\ t_{n+1} = \frac{1}{[m]_q}, \ldots, t_l = \frac{[m-1]_q}{[m]_q}, \\ t_{l+1} = \cdots = t_{l+n+1} = 1. \end{gathered} \tag{1.17}$$

Here $[i]_q$ denotes a q-integer, defined by

$$[i]_q = \begin{cases} (1-q^i)/(1-q), & q \neq 1, \\ i, & q = 1. \end{cases}$$

Koçak & Phillips (1994) studied B-splines based on q-integers, which generalize some particularly simple properties of the uniform B-splines. Since we choose a special knot sequence, the geometric properties of $S_{n,m}(f;t,q)$ will be satisfied for any $q > 0$. Namely, this operator is variation diminishing for any $q > 0$.

Remark. *A very important generalization of Bernstein polynomials has been the q-Bernstein polynomials $B_n(f;t,q)$, see (Phillips, 1997, 2003, 2010; Oruç & Phillips, 2003). These polynomials behave in a very nice way when we vary the parameter q while keeping the degree fixed. Namely, if f is convex on $[0,1]$ and $0 < q \leqslant r \leqslant 1$ then $B_n(f;t,r) \leqslant B_n(f;t,q)$ for all $t \in [0,1]$. On the other hand for fixed parameters $0 < q < r \leqslant 1$ there is no ordering relation between $S_{n,m}(f;t,q)$ and $S_{n,m}(f;t,r)$ for fixed integers n, m. That is none of following inequalities*

$$S_{n,m}(f;t,q) \leqslant S_{n,m}(f;t,r) \text{ or } S_{n,m}(f;t,r) \leqslant S_{n,m}(f;t,q)$$

holds for all $t \in [0,1]$.

Next we will investigate the error function $E_{n,m}$ of the operator defined by

$$E_{n,m}(f;t,q) = S_{n,m}(f;t,q) - f(t).$$

Theorem 1.2.2. *$S_{n,m}(f;t,q)$ converges to f for any $f \in C[0,1]$ uniformly as $n \to \infty$ with the knot sequence in (1.17).*

Proof. From the Bohman-Korovkin Theorem it is sufficient to prove that $S_{n,m}(z^2;t,q)$ converges to t^2 uniformly as $n \to \infty$ since $S_{n,m}(f;t,q)$ is a monotone linear operator reproducing linear functions. The approximating spline function for z^2 is

$$S_{n,m}(z^2;t,q) = \sum_{j=0}^{l} (\xi_j)^2 N_{j,n}(t;q).$$

From Marsden's identity (1.5) by comparing the coefficients of z^{n-2} we have

$$t^2 = \sum_{j=0}^{l} \xi_j^{(2)} N_{j,n}(t;q), \quad \text{where} \quad \xi_j^{(2)} = \frac{1}{\binom{n}{2}} \sum_{j+1 \leqslant r < s \leqslant j+n} t_r t_s.$$

Thus we can write

$$E_{n,m}(z^2;t,q) = \sum_{j=0}^{l} \lambda_j N_{j,n}(t;q),$$

where $\lambda_j = (\xi_j)^2 - \xi_j^{(2)}$. After some computations, we find that

$$\lambda_j = \frac{1}{n^2(n-1)} \sum_{j+1 \leqslant r < s \leqslant j+n} (t_s - t_r)^2. \tag{1.18}$$

Therefore $0 \leqslant \lambda_j \leqslant \frac{1}{2n}$ since $t_{j+n} - t_{j+1} \leqslant 1$ for each j. This inequality shows that $\lambda_j \to 0$ as $n \to \infty$ which implies $S_{n,m}(f;t,q)$ converges to t^2 uniformly. $\square$

The following result exhibits how the error functions behave in approximating f. One may see the proof of the next theorem in Budakçı (2010), Budakçı & Oruç (2012).

Theorem 1.2.3. *If $n \geqslant 2$, we have*

i.) $E_{n,m}(z^2;0,q) = E_{n,m}(z^2;1,q) = 0$, $E_{n,m}(z^2;t,q) > 0$ *for* $0 < t < 1$.

ii.) $E_{n,m}(z^2;t,q) = E_{n,m}(z^2;1-t,1/q)$ *for* $0 \leqslant t \leqslant 1$.

After finishing M.Sc. thesis Budakçı (2010), we discovered a more general theorem than Theorem 1.2.3. This new theorem can be seen in Budakçı & Oruç (2012).

Theorem 1.2.4. *If $f \in C[0,1]$ is symmetric, that is $f(t) = f(1-t)$ for all $t \in [0,1]$, then $E_{n,m}(f;t,q) = E_{n,m}(f;1-t,1/q)$ for $0 \leqslant t \leqslant 1$.*

Proof. Let

$$S_{n,m}(f;1-t,1/q) = \sum_{j=0}^{l} f(\eta_j) N_{j,n}(1-t;1/q) \tag{1.19}$$

and let

$$\eta_j = \frac{1}{n}(t_{j+1} + \cdots + t_{j+n}). \tag{1.20}$$

We will show that $E_{n,m}(f;t,q) - E_{n,m}(f;1-t,1/q) = 0$. It can be seen from (1.19) that

$$S_{n,m}(f;1-t,1/q) = \sum_{j=0}^{l} f(\eta_{l-j}) N_{l-j,n}(1-t;1/q).$$

So, we have

$$E_{n,m}(f;t,q) - E_{n,m}(f;1-t,1/q) = \sum_{j=0}^{l} f(\xi_j) N_{j,n}(t;q) - \sum_{j=0}^{l} f(\eta_{l-j}) N_{l-j,n}(1-t;1/q). \tag{1.21}$$

Since

$$\begin{aligned} N_{j,n}(t;q) &= (u_{j+n+1} - u_j)[u_j, \ldots, u_{j+n+1}](u-t)_+^n \\ &= ((1-t_j) - (1-t_{l+m-j+1}))[1-t_{l+n-j+1}, \ldots, 1-t_{l-j}](u-t)_+^n \\ &= (t_{l+n-j+1} - t_{l-j})[t_{l-j}, \ldots, t_{l+n-j+1}](u-(1-t))_+^n \\ &= N_{l-j,n}(1-t;1/q), \end{aligned}$$

the equation (1.21) becomes

$$E_{n,m}(f;t,q) - E_{n,m}(f;1-t,1/q) = \sum_{j=0}^{l} (f(\xi_j) - f(\eta_{l-j})) N_{j,n}(t;q). \tag{1.22}$$

In (1.20), using $t_{l+n-i+1} = 1 - u_i$ gives $\eta_{l-j} = 1 - \xi_j$ and hence (1.22) becomes

$$E_{n,m}(f;t,q) - E_{n,m}(f;1-t,1/q) = \sum_{j=0}^{l} (f(\xi_j) - f(1-\xi_j)) N_{j,n}(t;q).$$

By the assumption that f is symmetric, $f(\xi_j) = f(1-\xi_j)$ for all j, we obtain $E_{n,m}(f;t,q) = E_{n,m}(f;1-t,1/q)$. □

With such an unusual result above, we deduce the following consequence by taking special knot sequences.

Corollary 1.2.1. *If $f \in C[0,1]$ is symmetric, that is $f(t) = f(1-t)$ for all $t \in [0,1]$,* *then the error function of both Bernstein-Schoenberg operator and Bernstein operator are symmetric.*

Proof. It is enough to show that for a symmetric function the error function of the Bernstein-Schoenberg operator is symmetric since Bernstein operator is just a special case of Bernstein-Schoenberg operator.

Suppose that $f \in C[0,1]$ is symmetric, i.e $f(t) = f(1-t)$ for all $t \in [0,1]$. Now consider the error function $E_{n,m}$ of Bernstein-Schoenberg operator on q-integers for the function f. That is,

$$E_{n,m}(f;t,q) = S_{n,m}(f;t,q) - f(t).$$

From the previous Theorem 1.2.4, we know that for a symmetric function f,

$$E_{n,m}(f;t,q) = E_{n,m}(f;1-t,1/q)$$

holds.

Since we recover the classical Bersntein-Shoenberg operator when $q = 1$, letting $q = 1$ gives

$$E_{n,m}(f;t) = E_{n,m}(f;1-t)$$

and thus the error function of Bernstein-Schoenberg operator for the function f is symmetric. $\square$

1.3 Quantum Calculus

In this section we give essential definitions and properties in q-calculus that we will use in the next chapters. For more details one may see (Ernst, 2012; Kac & Cheung, 2001).

1.3.1 Divided Differences

We begin with divided differences, since both classical and quantum derivatives can be defined in terms of divided differences. Further properties and proofs of divided differences, see (Goldman, 2003; Phillips, 2003; Powell, 1981).

Definition 1.3.1. Let F be an arbitrary function. The divided difference of F at the nodes $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ is defined recursively by

$$
\begin{aligned}
F[t_0] &= F(t_0) && \\
F[t_0, t_1] &= \frac{F(t_1) - F(t_0)}{t_1 - t_0} && t_1 \neq t_0 \\
&= F'(t_0) && t_1 = t_0 \qquad (1.23) \\
&\vdots && \vdots \\
F[t_0, \ldots, t_n] &= \frac{F[t_1, \ldots, t_n] - F[t_0, \ldots, t_{n-1}]}{t_n - t_0} && t_n \neq t_0 \\
&= \frac{F^{(n)}(t_0)}{n!} && t_n = t_0.
\end{aligned}
$$

Notice that the divided difference is a linear operator since the operator satisfies the following properties;

$$(F+G)[t_0,\ldots,t_n] = F[t_0,\ldots,t_n] + G[t_0,\ldots,t_n]$$
$$(cF)[t_0,\ldots,t_n] = c(F[t_0,\ldots,t_n]).$$

There are simple closed formulas for the divided differences of low degree polynomials. We summarize these results in the following theorem.

Theorem 1.3.1. *Let $P(t)$ be a polynomial.*

i) If $degP(t) \leqslant n-1$, then $P[t_0,\ldots,t_n] = 0$.

ii) If $degP(t) = n$, then $P[t_0,\ldots,t_n]$ is the coefficient of t^n in the monomial representation for $P(t)$. Thus in this case, $P[t_0,\ldots,t_n]$ is a constant independent of $t_0,\ldots,t_n$.

We shall use the next theorem for the proof of the representation of the quantum B-splines in terms of divided differences of truncated power functions.

Theorem 1.3.2. *Let $F(t)$ be an arbitrary function, and let $t_0 < t_1 < \cdots < t_n$ be a set of $n+1$ nodes. If*

$$F(t_j) = G(t_j), \quad j = 0,\ldots,n,$$

then

$$F[t_0,\ldots,t_n] = G[t_0,\ldots,t_n].$$

In this section we have used the notation $F[t_0,\ldots,t_n]$ for the divided difference to emphasize the relationship between evaluation and the divided difference. In the remainder of this thesis, however, we shall employ the notation $[t_0,\ldots,t_n]F$ for the divided difference to emphasize that the divided difference is a linear operator.

1.3.2 Quantum Derivatives

Quantum derivatives are classical derivatives without limits. There are many ways to define the classical derivative $D(f)$ of a function f using limits. For example

$$Df(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h},$$
$$Df(t) = \lim_{q\to 1} \frac{f(qt)-f(t)}{(q-1)t}.$$

The h-derivatives are defined for $h \neq 0$ by

$$D_h f(t) = \frac{f(t+h)-f(t)}{h} = [t, t+h]f \tag{1.24}$$
$$D_h^k f(t) = D_h(D_h^{k-1} f(t)).$$

Similarly, for a fixed parameter $q \neq 1$, the q-derivatives are defined by, see (Kac & Cheung (2001)),

$$D_q f(t) = \frac{f(qt)-f(t)}{(q-1)t} = [t, qt]f \tag{1.25}$$
$$D_q^k f(t) = D_q(D_q^{k-1} f(t)).$$

It is easy to verify that

$$D_h^k f(t) = k![t, t+h, \dots, t+kh]f$$

and

$$D_q^k f(t) = [k]_q![t, qt, \dots, q^k t]f,$$

where

$$[k]_q! = [1]_q \cdots [k]_q.$$

Note that if f is a differentiable function, then

$$\lim_{h\to 0} D_h f(x) = Df(x),$$
$$\lim_{q\to 1} D_q f(x) = Df(x).$$

We use the notation q-$C^k[a,b]$ to denote the space of functions whose q-derivatives of order up to k are continuous on a closed bounded interval $[a,b]$.
Quantum derivatives are linear operators

$$D_h(af(x)+bg(x)) = aD_h f(x) + bD_h g(x),$$

$$D_q(af(x)+bg(x)) = aD_q f(x) + bD_q g(x), \tag{1.26}$$

and satisfies the following two product rules

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x), \tag{1.27}$$

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x). \tag{1.28}$$

Next we shall derive some simple properties of the quantum calculus. In each case we will provide proofs only for the q-calculus; the proofs for the h-calculus are much the same.

For any constant c, the simple quantum chain rules are as follows:

<u>*h-calculus:*</u>

$$D_h\left(f(x+c)\right) = D_h\left(f(y)\right)_{y=x+c}$$

<u>*q-calculus:*</u>

$$D_q\left(f(cx)\right) = cD_q\left(f(y)\right)_{y=cx}. \tag{1.29}$$

Lemma 1.3.1. *Let $P(t)$ be a polynomial. Then*

classical calculus:

$$DP(t) \equiv 0 \textit{ if and only if } P(t) \textit{ is a constant}$$

h-calculus:

$$D_h P(t) \equiv 0 \textit{ if and only if } P(t) \textit{ is a constant}$$

q-calculus:

$$D_q P(t) \equiv 0 \textit{ if and only if } P(t) \textit{ is a constant.} \tag{1.30}$$

Proof. Certainly if $P(t)$ is a constant, then $D_q P(t) \equiv 0$. Conversely, suppose that $D_q P(t) \equiv 0$. If $q = 0$, then $P(t) = P(0)$, so the result is obvious. Therefore we assume that $q \neq 0$. Then for all t, we have

$$P(t) = P(qt) = \cdots = P(q^n t) \text{ for all integers } n \geqslant 0.$$

Fix a constant c and define

$$Q(t) = P(t) - P(c).$$

Then $Q(t)$ is a polynomial with infinitely many roots: $c, qc, \ldots, q^n c, \ldots$. Therefore

$$Q(t) \equiv 0,$$

so for all t, $P(t) = P(0)$. Thus $P(t)$ is a constant. □

Observe that Lemma 1.3.1 is valid for q-derivative provided only that the function $P(t)$ is continuous at zero. Thus for the q-derivative (and for the classical derivative) we need not assume that $P(t)$ is a polynomial. However this result does not extend to non-polynomial functions for the h-derivative. Since in this thesis we are interested primarily in polynomials and piecewise polynomials, we content ourselves here with Lemma 1.3.1 for polynomials.

The q-calculus also has an analogue of the classical exponential function, see (Ernst, 2012; Goldman & et al., 2014). The q-exponential function is defined by,

$$e_q^t = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \tag{1.31}$$

is well defined for all $|t| < 1/|1-q|$ if $|q| < 1$ and for all $t \in \mathbb{R}$ if $|q| > 1$ or $q = 1$. Therefore, for any fixed value of $q > 0$, there is an interval in which the series for e_q^t converges. Using the definition of the q-exponential function (1.31), the $q-$derivative of a monomial, and the linearity (1.26) of the operator D_q gives

$$D_q\left(e_q^{at}\right) = ae_q^{at}, \tag{1.32}$$

for any constant a.
There is another form of q-exponential function

$$E_q^t = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{t^j}{[j]_q!}$$

which converges for all t if $0 < q < 1$, and for $|t| < \frac{q}{q-1}$ if $q > 1$. The derivative of E_q^t is not itself but

$$D_q E_q^t = E_q^{qt}.$$

Theorem 1.3.3. *(q-Binomial Theorem)*

$$(x-t)(x-qt)\cdots(x-q^{n-1}t) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} t^j x^{n-j}$$

Proof. See (Andrews, 1998; Kac & Cheung, 2001; Phillips, 2003). □

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are the q-binomial coefficients defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The $q-$binomial coefficients can be computed recursively by the $q-$Pascal identities

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \tag{1.33}$$

or

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q of degree $k(n-k)$, whose coefficients may be regarded as the generating functions for restricted partitions of integers, see Andrews (1998).

We also require shifted products:

$$(x-t)^{n,h} = \begin{cases} 1, & \text{if } n = 0, \\ \\ (x-t)(x-t+h)\cdots(x-t+(n-1)h), & \text{if } n \geqslant 1. \end{cases}$$

$$(x-t)^{n,q} = \begin{cases} 1, & \text{if } n = 0, \\ \\ (x-t)(x-qt)\cdots(x-q^{n-1}t), & \text{if } n \geqslant 1. \end{cases}$$

In the next sections we shall need both the t and x quantum derivatives of these polynomials.

Proposition 1.3.1. *(t-derivative)*

classical calculus:

$$D(x-t)^n = -n(x-t)^{n-1}$$
$$D^k(x-t)^n = (-1)^k \frac{n!}{(n-k)!}(x-t)^{n-k}$$

h-calculus:

$$D_h(x-t)^{n,h} = -n(x-t)^{n-1,h}$$
$$D_h^k(x-t)^{n,h} = (-1)^k \frac{n!}{(n-k)!}(x-t)^{n-k,h}$$

q-calculus:

$$D_{1/q}(x-t)^{n,q} = -[n]_q(x-t)^{n-1,q} \tag{1.34}$$
$$D_{1/q}^k(x-t)^{n,q} = (-1)^k \frac{[n]_q!}{[n-k]_q!}(x-t)^{n-k,q}. \tag{1.35}$$

Proof. By the definition of the $1/q$-derivative we have

$$\begin{aligned} D_{1/q}(x-t)^{n,q} &= \frac{(x-t/q)\cdots(x-q^{n-2}t)-(x-t)\cdots(x-q^{n-1}t)}{t/q-t} \\ &= \frac{(x-t/q)-(x-q^{n-1}t)}{t/q-t}(x-t)\cdots(x-q^{n-2}t) \\ &= -[n]_q(x-t)^{n-1,q}. \end{aligned}$$

Now it follows easily by induction on k that

$$D_{1/q}^k(x-t)^{n,q} = (-1)^k \frac{[n]_q!}{[n-k]_q!}(x-t)^{n-k,q}.$$

□

Proposition 1.3.2. *(x-derivative)*

<u>*classical calculus:*</u>

$$D(x-t)^n = n(x-t)^{n-1}$$
$$D^k(x-t)^n = \frac{n!}{(n-k)!}(x-t)^{n-k}$$

<u>*h-calculus:*</u>

$$D_h(x-t)^{n,h} = n(x-t+h)\cdots(x-t+(n-1)h)$$
$$D_h^k(x-t)^{n,h} = \frac{n!}{(n-k)!}(x-t+kh)\cdots(x-t+(n-1)h)$$

<u>*q-calculus:*</u>

$$D_{1/q}(x-t)^{n,q} = [n]_{1/q}(x-qt)\cdots(x-q^{n-1}t) \tag{1.36}$$
$$D_{1/q}^k(x-t)^{n,q} = \frac{[n]_{1/q}!}{[n-k]_{1/q}!}(x-q^kt)\cdots(x-q^{n-1}t). \tag{1.37}$$

Proof. Applying the definition of $1/q$-derivative yields

$$\begin{aligned}
D_{1/q}(x-t)^{n,q} &= \frac{(x/q-t)\cdots(x/q-q^{n-1}t)-(x-t)\cdots(x-q^{n-1}t)}{x/q-x} \\
&= \frac{\frac{1}{q^n}(x-qt)\cdots(x-q^nt)-(x-t)\cdots(x-q^{n-1}t)}{x/q-x} \\
&= \frac{\frac{1}{q^n}(x-q^nt)-(x-t)}{x/q-x}(x-qt)\cdots(x-q^{n-1}t) \\
&= [n]_{1/q}(x-qt)\cdots(x-q^{n-1}t).
\end{aligned}$$

Now it follows easily by induction on k that

$$D_{1/q}^k(x-t)^{n,q} = \frac{[n]_{1/q}!}{[n-k]_{1/q}!}(x-q^kt)\cdots(x-q^{n-1}t).$$

□

Finally we discuss mixed partial derivatives. In the classical calculus, mixed partials do not always commute, since we need to worry about the order of the limits. But in quantum calculus there are no limits, so the mixed partials always commute.

Proposition 1.3.3. *Let $f(x,t)$ be a function of two variables. Then*

classical calculus:

$$\frac{\partial^2}{\partial x \partial t} f(x,t) = \frac{\partial^2}{\partial t \partial x} f(x,t)$$

h-calculus:

$$\frac{\partial_h^2}{\partial_h x \partial_h t} f(x,t) = \frac{\partial_h^2}{\partial_h t \partial_h x} f(x,t)$$

q-calculus:

$$\frac{\partial_q^2}{\partial_q x \partial_q t} f(x,t) = \frac{\partial_q^2}{\partial_q t \partial_q x} f(x,t)$$

Proof. Applying the definiton of q-derivative gives

$$\begin{aligned}
\frac{\partial_q^2}{\partial_q x \partial_q t} f(x,t) &= \frac{\partial_q}{\partial_q x}\left(\frac{f(x,qt)-f(x,t)}{qt-t}\right) \\
&= \frac{\frac{f(qx,qt)-f(x,qt)}{qx-x} - \frac{f(qx,t)-f(x,t)}{qx-x}}{qt-t} \\
&= \frac{\partial_q}{\partial_q t}\left(\frac{f(qx,t)-f(x,t)}{qx-x}\right) \\
&= \frac{\partial_q^2}{\partial_q t \partial_q x} f(x,t).
\end{aligned}$$

□

1.3.3 Quantum Integrals

Quantum integrals are the analogues of classical integrals for the quantum calculus. Quantum integrals satisfy a quantum version of the fundamental theorem of calculus and there is also a quantum version of integration by parts. The definite h-integral of a function f is defined as a Riemann sum of $f(x)$ on the interval $[a, b]$.

Definition 1.3.2. Let $b - a \in h\mathbb{Z}$. Then the h-integral is defined by

$$\int_a^b f(x)d_hx = \begin{cases} h\left(f(a) + f(a+h) + \cdots + f(b-h)\right) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -h\left(f(a) + f(a+h) + \cdots + f(b-h)\right) & \text{if } a > b. \end{cases}$$

The q-integral is a bit more complicated. Whereas the h-integral is just a finite sum, the q-integral is defined by an infinite sum.

Definition 1.3.3. Let $0 < a < b$ and $0 < q < 1$. Then the definite q-integral of a function $f(x)$ is defined by setting

$$\int_0^b f(x)d_qx = (1-q)b\sum_{i=0}^{\infty} q^i f(q^i b)$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

In the limit $q \to 1$, we replace q-integral by ordinary integral.

Definition 1.3.4. The improper q-integral of a function $f(x)$ on $[0, \infty)$ is defined by

$$\int_0^\infty f(x)d_qx = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}}^{q^j} f(x)d_qx \quad \text{for} \quad 0 < q < 1$$

and

$$\int_0^\infty f(x)d_qx = \sum_{j=-\infty}^{\infty} \int_{q^j}^{q^{j+1}} f(x)d_qx \quad \text{for} \quad q > 1.$$

On the whole real line the q-integral is defined by

$$\int_{-\infty}^{\infty} f(x)d_qx = \int_{0}^{\infty} [f(x) + f(-x)]\, d_qx,$$

see Ernst (2012).

Theorem 1.3.4. *The improper q-integral converges if $x^\alpha f(x)$ is bounded in a neighborhood of $x = 0$ with some $\alpha < 1$ and for sufficiently large x with some $\alpha > 1$.*

Proposition 1.3.4. *(Integration by parts)*

classical integral:

$$\int_a^b f(x)Dg(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)Df(x)dx$$

h-integral:

$$\int_a^b f(x)D_hg(x)d_hx = f(b)g(b) - f(a)g(a) - \int_a^b g(x+h)D_hf(x)d_hx$$

q-integral:

$$\int_a^b f(x)D_qg(x)d_qx = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)D_qf(x)d_qx$$

Theorem 1.3.5. *(Fundamental Theorem of Calculus)*

classical calculus:

$$\int_a^b DF(x)dx = F(b) - F(a)$$

h-calculus:

If $b-a \in h\mathbb{Z}$, then

$$\int_a^b D_h F(x) d_h x = F(b) - F(a)$$

q-calculus:

If $F(x)$ is continuous at $x=0$, then

$$\int_a^b D_q F(x) d_q x = F(b) - F(a),$$

where $0 \leqslant a < b \leqslant \infty$.

Next we introduce the quantum analogues of the polynomials $(x-t)^n$. These polynomials replace the polynomials $(x-t)^n$ in the quantum versions of Taylor's Theorem 1.3.6 and in the quantum versions of Marsden's identity.

Theorem 1.3.6. *Let f be $n+1$ times $1/q$-differentiable in the closed interval $[a,b]$. Then*

$$f(x) = \sum_{k=0}^{n} q^{k(k-1)/2} \frac{(D_{1/q}^k f)(q^k a)}{[k]_q!}(x-a)^{k,q} + R_n(f) \tag{1.38}$$

where

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^x (D_{1/q}^{n+1} f)(q^n t)(x-t)^{n,q} d_{1/q} t, \tag{1.39}$$

in particular

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q} t$$

and

$$(x-t)_+^{n,q} = (x-q^{n-1}t)\cdots(x-qt)(x-t)_+.$$

Proof. If we apply integration by parts in equation (1.39), we obtain

$$
\begin{aligned}
R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q! q^n} \Big[&- \left(D^n_{1/q} f\right)(q^n a)(x-a)^{n,q} \\
&+ [n]_q \int_a^x \left(D^n_{1/q} f\right)(q^{n-1}t)(x-t)^{n-1,q} d_{1/q}t \Big] \\
= &- \frac{q^{n(n-1)/2}}{[n]_q!} \left(D^n_{1/q} f\right)(q^n a)(x-a)^{n,q} \\
&+ \frac{q^{n(n-1)/2}}{[n-1]_q!} \int_a^x \left(D^n_{1/q} f\right)(q^{n-1}t)(x-t)^{n-1,q} d_{1/q}t.
\end{aligned}
$$

Therefore

$$
R_n(f) = -\frac{q^{n(n-1)/2}}{[n]_q!} \left(D^n_{1/q} f\right)(q^n a)(x-a)^{n,q} + R_{n-1}(f).
$$

Applying integration by parts to $R_{n-1}(f)$ gives

$$
\begin{aligned}
R_n(f) = -\frac{q^{n(n-1)/2}}{[n]_q!} &\left(D^n_{1/q} f\right)(q^n a)(x-a)^{n,q} \\
&- \frac{q^{(n-1)(n-2)/2}}{[n]_q!} \left(D^{n-1}_{1/q} f\right)(q^{n-1} a)(x-a)^{n-1,q} + R_{n-2}(f).
\end{aligned}
$$

Repeating this argument n times and using

$$
R_0(f) = \int_a^x D_{1/q} f(t) d_{1/q}t = f(x) - f(a),
$$

we get the required result. $\square$

The work Pashaev & Nalci (2014) introduces q-analytic and q-harmonic functions and then solves quantum q-oscillator problem.

The work Rajković & et al. (2002) gives the mean value theorem in q-calculus which will be needed in one of our results.

Theorem 1.3.7. *If F is contiuous and G is $1/q$-integrable and is nonnegative(or nonpositive) on $[a,b]$, then there exists $\tilde{q} \in (1,\infty)$ such that*

$$(\forall q \in (\tilde{q},\infty))\,(\exists \xi \in (a,b)) \quad \int_a^b F(x)G(x)d_{1/q}x = F(\xi)\int_a^b G(x)d_{1/q}x.$$

We also require the following result and q-Hölder inequality, and appropriate notions of distance in q-integrals, see (Anastassiou, 2010; Gauchman, 2004; Tariboon & Ntouyas, 2014) for details.

Definition 1.3.5. For a fixed real q, we will denote by $L_{p,q}([0,b])$, with $1 \leqslant p < \infty$, the set of all functions f on $[0,b]$ such that

$$||f||_{p,q} := \left(\int_0^b |f|^p d_{1/q}t\right)^{\frac{1}{p}} < \infty.$$

Furthermore let $L_{\infty,q}([0,b])$ denote the set of all functions f on $[0,b]$ such that

$$||f||_{\infty,q} := \sup_{x\in[0,b]} |f(x)| < \infty.$$

Theorem 1.3.8. *Let $x \in [0,b]$, $q \in [1,\infty)$ and $p_1, p_2 > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then we have*

$$\int_0^x |f(t)||g(t)|d_{1/q}t \leqslant \left(\int_0^x |f(t)|^{p_1} d_{1/q}t\right)^{\frac{1}{p_1}} \left(\int_0^x |g(t)|^{p_2} d_{1/q}t\right)^{\frac{1}{p_2}}. \qquad (1.40)$$

CHAPTER TWO

QUANTUM B-SPLINE REPRESENTATIONS

As we show in Chapter 1 there are two common types of quantum derivatives h-derivatives and q-derivatives corresponding to h-divided differences and q-divided differences. Quantum splines are piecewise polynomials whose quantum derivatives up to some order agree at the joins. Thus quantum splines allow us to model tolerances, jumps, and even quantum leaps in the derivatives at the joins. In this chapter we give representations of quantum B-splines which forms a basis for quantum spline spaces.

Quantum splines defined using the h-derivative were first introduced in (Mangasarian & Schumaker, 1971, 1973) as solutions to certain minimization problems involving differences. Many properties of these h-splines are derived in Lyche (1975).

Tom Lyche defined h-B-splines (discrete B-splines) in (Lyche, 1975; Schumaker, 2007) in two ways:

Recursively:

$$N_{k,n}(t;h) = \frac{t-(n-1)h-t_k}{t_{k+n}-t_k} N_{k,n-1}(t;h) + \frac{t_{k+n+1}-(t-(n-1)h)}{t_{k+n+1}-t_{k+1}} N_{k+1,n-1}(t;h)$$

Divided differences:

$$N_{k,n}(t;h) = (t_{k+n+1}-t_k)[t_k,\ldots,t_{k+n+1}](x-t)_+^{n,h}$$

Our main focus will be q-B-splines (quantum B-splines) and their fundamental properties from point of view of interpolation and approximation. The work Simeonov & Goldman (2013) mark the start of q-B-splines in the literature.

2.1 Recurrence Relation of q-B-splines

One way of defining q-B-splines of degree n is to consider a basis $\{\Phi_j : j = 1, 2, \ldots, n+m\}$ in the expression

$$S(t;q) = \sum_{j=1}^{n+m} \lambda_j \Phi_j(t;q) \qquad a \leqslant t \leqslant b$$

such that each function $\{\Phi_j(t;q) : j = 1, 2, \ldots, n+m\}$ is identically zero over a large part of the range $a \leqslant t \leqslant b$. Hence we consider how we can find an element of the space $\mathcal{S}(n, t_0, \ldots, t_m)$ that is zero on the intervals $[t_0, t_k]$ and $[t_r, t_m]$, but that is nonzero on (t_k, t_r), where $0 < k < r < m$. If S is such a function it can be expressed in the form

$$\sum_{j=k}^{r} e_j (q^{n-1}t - t_j) \cdots (qt - t_j)(t - t_j)_+, \quad a \leqslant t \leqslant b$$

where the parameters e_j have to satisfy the condition

$$\sum_{j=k}^{r} e_j (q^{n-1}t - t_j) \cdots (qt - t_j)(t - t_j) = 0, \quad t_r \leqslant t \leqslant b.$$

It follows that

$$\sum_{j=k}^{r} e_j t_j^i = 0, \quad i = 0, 1, \ldots, n$$

must hold. These equations have a nonzero solution if $r \geqslant k+n+1$, because then the number of coefficients $\{e_j\}$ is greater than the number of equations. If $r = k+n+1$ then the coefficients

$$e_j = \prod_{\substack{i=k \\ i \neq j}}^{k+n+1} \frac{1}{(t_i - t_j)}, \quad j = k, k+1, \ldots, k+n+1$$

are suitable. Therefore the function

$$S(t;q) = \sum_{j=k}^{k+n+1} \left[\prod_{\substack{i=k \\ i\neq j}}^{k+n+1} \frac{1}{(t_i - t_j)} \right] (q^{n-1}t - t_j) \cdots (qt - t_j)(t - t_j)_+$$

is a q-spline function and

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k) \sum_{j=k}^{k+n+1} \left[\prod_{\substack{i=k \\ i\neq j}}^{k+n+1} \frac{1}{(t_i - t_j)} \right] (t - t_j;q)_+^n, \quad a \leqslant t \leqslant b \quad (2.1)$$

where

$$(t - t_j;q)_+^n = (q^{n-1}t - t_j) \cdots (qt - t_j)(t - t_j)_+$$

is called normalized q-B-spline.

Now, we give q-B-splines recursively. The work by Simeonov & Goldman (2013) shows the recurrence relation via de Boor type algorithm. Here we prove the relation in another way, by induction on n, the degree of polynomial pieces.

Theorem 2.1.1. *Let $n \geqslant 1$ be an integer and $\{t_i; i = k, k+1, \ldots, k+n+1\}$ be a set of distinct real numbers. Then* (2.1) *satisfies the recurrence relation*

$$N_{k,n}(t;q) = \frac{q^{n-1}t - t_k}{t_{k+n+1} - t_k} N_{k,n-1}(t;q) + \frac{t_{k+n+1} - q^{n-1}t}{t_{k+n+1} - t_{k+1}} N_{k+1,n-1}(t;q) \qquad (2.2)$$

Proof. The right hand-side of the equation (2.2) is made up of piecewise polynomials that are joined at the knots $\{t_i; i = k, k+1, \ldots, k+n+1\}$. Notice that q-B-spline $N_{k,n}(t;q)$ is identically zero for $t \leqslant t_k$ and $t \geqslant t_{k+n+1}$. When $t \in [t_k, t_{k+1}]$, the equation (2.1) implies

$$N_{k,n}(t;q) = \frac{q^{n-1}t - t_k}{t_{k+n} - t_k} N_{k,n-1}(t;q)$$

and $N_{k+1,n-1}(t;q)$ is zero. Hence the equation (2.2) is satisfied. Similarly, for

$t \in [t_{k+n}, t_{k+n+1}]$, the equation (2.1) implies

$$N_{k,n}(t;q) = \frac{t_{k+n+1} - q^{n-1}t}{t_{k+n+1} - t_{k+1}} N_{k+1,n-1}(t;q).$$

Now we will show that the equation (2.2) holds for $\{t_j \leqslant t \leqslant t_{j+1}; j = k+1, \ldots, k+n\}$. It is enough to show that for $j \in \{k+1, \ldots, k+n\}$, the change in the right-hand side of the equation (2.2) at t_j is the same as the change in $N_{k,n}$ at t_j. We find the change in right-hand side at t_j as the polynomial $(t - t_j; q)^{n-1}$ multiplied by the factor

$$\begin{aligned}
&(q^{n-1}t - t_k) \prod_{\substack{i=k \\ i \neq j}}^{k+n} \frac{1}{(t_i - t_j)} + (t_{k+n+1} - q^{n-1}t) \prod_{\substack{i=k+1 \\ i \neq j}}^{k+n+1} \frac{1}{(t_i - t_j)} \\
&\qquad = [(q^{n-1}t - t_k)(t_{k+n+1} - t_j) + (t_{k+n+1} - q^{n-1}t)(t_k - t_j)] \prod_{\substack{i=k \\ i \neq j}}^{k+n+1} \frac{1}{(t_i - t_j)} \\
&\qquad = (q^{n-1}t - t_j)(t_{k+n+1} - t_j) \prod_{\substack{i=k \\ i \neq j}}^{k+n+1} \frac{1}{(t_i - t_j)}
\end{aligned}$$

which completes the proof. □

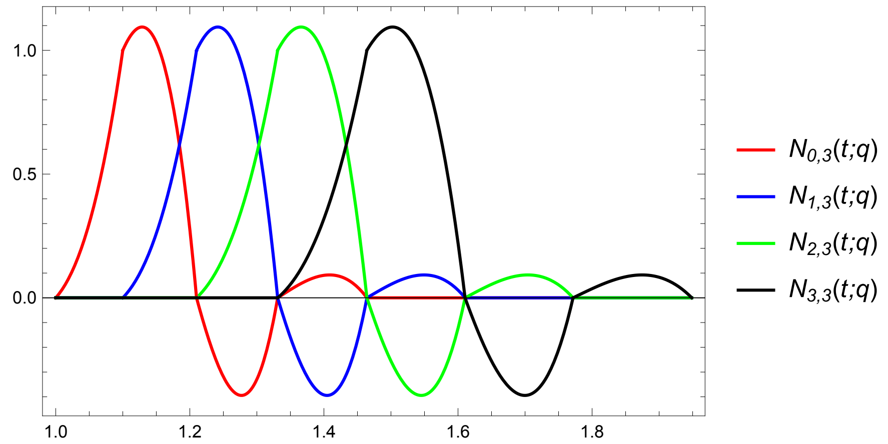

Figure 2.1 Cubic q-B-spline basis with the knot sequence $(1, q, q^2, q^3, q^4, q^5, q^6, q^7)$ with $q = 1.1$.

Throughout this thesis, all figures and examples are generated by Wolfram Mathematica 10 software and codes are based on the recurrence formula for q-B-splines.

2.2 Marsden Identity and Its Consequences

Marsden identity often lie at the core of other fundamental formulas and identities for classical and quantum B-splines. We shall employ the Marsden Identities for the quantum B-splines to derive both the de Boor-Fix form of the dual functionals for the quantum B-splines and the explicit formula for the quantum B-splines in terms of divided differences of certain truncated power functions.

Simeonov & Goldman (2013) states the following theorem whose proof is based on blossoms. Here, we prove it in the classical way.

Theorem 2.2.1.

$$(x-t)^{n,q} = \sum_k (x-t_{k+1})\cdots(x-t_{k+n})N_{k,n}(t;q), \tag{2.3}$$

where the summation extends over all $k \in \mathbb{Z}$.

Proof. Since for $n = 0$ the empty product $(x-t_{k+1})\cdots(x-t_{k+n})$ is equal to 1, we have the partition of unity property of q-B-splines, that is

$$1 = \sum_k N_{k,n}(t;q). \tag{2.4}$$

Now suppose that the equation (2.3) holds for some $n \geqslant 1$. For the next step we multiply both side of (2.3) by the term $(x-q^n t)$. For the right-hand side we write the linear interpolating polynomial at the knots t_k and t_{k+n+1}. It is readily verified that

$$\frac{x-t_{k+n+1}}{t_k-t_{k+n+1}}(t_k-q^n t) + \frac{x-t_k}{t_{k+n+1}-t_k}(t_{k+n+1}-q^n t) = (x-q^n t).$$

Now we have

$$(x-t)^{n+1,q} = \sum_k (x-t_{k+1})\cdots(x-t_{k+n+1})\frac{t_k - q^n t}{t_k - t_{k+n+1}} N_{k,n}(t;q)$$
$$+ \sum_k (x-t_k)\cdots(x-t_{k+n+1})\frac{t_{k+n+1} - q^n t}{t_{k+n+1} - t_k} N_{k,n}(t;q).$$

In the second summation changing k by $k+1$ gives

$$(x-t)^{n+1,q} = \sum_k (x-t_{k+1})\cdots(x-t_{k+n+1})\frac{t_k - q^n t}{t_k - t_{k+n+1}} N_{k,n}(t;q)$$
$$+ \sum_k (x-t_{k+1})\cdots(x-t_{k+n+2})\frac{t_{k+n+2} - q^n t}{t_{k+n+2} - t_{k+1}} N_{k+1,n}(t;q).$$

Hence, by using (2.2) we obtain the required formula

$$(x-t)^{n+1,q} = \sum_k (x-t_{k+1})\cdots(x-t_{k+n+2}) N_{k,n+1}(t;q).$$

This completes the induction. $\square$

We will derive two important new consequences of the Marsden Identity for quantum B-splines: a quantum variant of the de Boor-Fix formula for the dual functionals of the quantum B-splines, and a representation of the quantum B-splines in terms of divided differences of truncated powers. These two formulas are well known for classical B-splines. Here we show how to extend these formulas to quantum B-splines.

Using Marsden identity we can obtain the following representation of monomials, see Simeonov & Goldman (2013) for the proof by using blossoming technique. We prefer classical approach.

Proposition 2.2.1. *Given any integer $j \geqslant 0$, we can express the monomial t^j as a linear combination of q-B-splines for any fixed integer $n \geqslant j$, in the form*

$$t^j = \sum_k \frac{\sigma_j(t_{k+1},\dots,t_{k+n})}{q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q} N_{k,n}(t;q) \tag{2.5}$$

where $\sigma_j(t_{k+1},\dots,t_{k+n})$ *denotes the elementary symmetric function of order* j *in the variables* $t_{k+1},\dots,t_{k+n}$.

Proof. It follows from (1.9) that

$$(1+t_{k+1}t)\dots(1+t_{k+n}t) = \sum_{r=0}^{n} \sigma_r(t_{k+1},\dots,t_{k+n})t^r. \tag{2.6}$$

By replacing t by $-1/x$ and multiplying through x^n, we find that

$$(x-t_{k+1})\dots(x-t_{k+n}) = x^n \sum_{r=0}^{n} \sigma_r(t_{k+1},\dots,t_{k+n}) \left(-\frac{1}{x}\right)^r. \tag{2.7}$$

Combining (2.3) and (2.7) gives

$$(x-t)^{n,q} = \sum_k \left(\sum_{r=0}^{n} (-1)^r \sigma_r(t_{k+1},\dots,t_{k+n}) x^{n-r} \right) N_{k,n}(t;q) \tag{2.8}$$

Using the q-binomial Theorem 1.3.3 and equating the coefficients of x^{n-r} on both sides of the equation (2.8) gives

$$q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q t^r = \sum_k \sigma_r(t_{k+1},\dots,t_{k+n}) N_{k,n}(t;q).$$

Hence

$$t^j = \sum_k \frac{\sigma_j(t_{k+1},\dots,t_{k+n})}{q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q} N_{k,n}(t;q).$$

□

Notice that for $j=0$, we have $\sum_k N_{k,n}(t;q) = 1$, the partition of unity property of q-B-splines.

2.3 The q-Truncated Power Basis

We observe in Section 2.2 that both classical B-splines and quantum B-splines can be constructed from a recurrence. Now we show that both classical B-splines and quantum B-splines can also be represented explicitly in terms of divided differences of appropriate truncated power functions. The first proof of the following theorem appeared in Budakçı & et al. (2015).

Theorem 2.3.1.

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k)[t_k, \ldots, t_{k+n+1}](x-t)^{n,q}(x-t)_+^0.$$

Proof. There are several ways to prove this theorem. For example, we could use Liebniz's rule for divided differences and induction on the degree to show that the divided differences of the truncated powers satisfy the q-B-spline recurrence (2.2).

<u>First way:</u>

In order to provide a proof that most clearly works in all three cases (classical and two quantum B-splines) with only minor modifications, we employ a technique based on the Marsden Identity similar to the proof given by de Boor for the classical B-splines in de Boor (1972). We begin by recalling the Marsden Identity, equation (2.3), for q-B-splines:

$$(x - q^{n-1}t)\cdots(x-t) = \sum_i (x-t_{i+1})\cdots(x-t_{i+n})N_{i,n}(t;q). \tag{2.9}$$

Multiplying both side of the Marsden Identity by $(x-t)_+^0$ yields

$$(x - q^{n-1}t)\cdots(x-t)(x-t)_+^0 = \sum_i (x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0 N_{i,n}(t;q).$$

Applying the divided difference operator with respect to x to both sides with the nodes

$t_j, \ldots, t_{j+n+1}$, we find that

$$[t_j, \ldots, t_{j+n+1}](x - q^{n-1}t) \cdots (x - t)(x - t)_+^0 =$$

$$\sum_i [t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n})(x - t)_+^0 N_{i,n}(t; q). \tag{2.10}$$

Our aim is to show that all the terms in the summation on the right hand side of equation (2.10) are zero except for $i = j$. Thus we investigate the behavior of the divided differences for the terms

$$(x - t_{i+1}) \cdots (x - t_{i+n})(x - t)_+^0 N_{i,n}(t; q).$$

We consider five cases.

Case 1. $\underline{i < j - n}$:

For $t \in (t_i, t_{i+n+1}) = supp(N_{i,n}(t; q))$, the support of $N_{i,n}(t; q)$, and $x \in \{t_j, \ldots, t_{j+n+1}\}$, we have $(x - t)_+^0 = 1$. Therefore

$$[t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n})(x - t)_+^0$$

$$= [t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n}).$$

Since $(x - t_{i+1}) \cdots (x - t_{i+n})$ is a polynomial of degree n, it follows from Theorem 1.3.1 that

$$[t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n}) = 0.$$

So

$$[t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n})(x - t)_+^0 N_{i,n}(t; q) = 0 \quad \text{for } i < j - n.$$

Case 2. $\underline{j-n \leqslant i < j}$:

Let $i = j-k$ for $1 \leqslant k \leqslant n$. Consider the functions $(x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0$ and $(x-t_{i+1})\cdots(x-t_{i+n})$. Both are zero for $x \in \{t_j, \ldots, t_{j+n-k}\}$. Moreover these two functions have the same value for $x \in \{t_{j+n-k+1}, \ldots, t_{j+n+1}\}$ because $t \in (t_i, t_{i+n+1}) = supp(N_{i,n}(t;q))$. So for these values of x and t, we have $(x-t)_+^0 = 1$. Hence, by Theorem 1.3.2

$$[t_j, \ldots, t_{j+n+1}](x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0$$

$$= [t_j, \ldots, t_{j+n+1}](x-t_{i+1})\cdots(x-t_{i+n}).$$

Therefore by Theorem 1.3.1

$$[t_j, \ldots, t_{j+n+1}](x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0 N_{i,n}(t;q) = 0 \quad \text{for } j-n \leqslant i < j.$$

Case 3. $\underline{i > j+n}$:

For $t \in (t_i, t_{i+n+1}) = supp(N_{i,n}(t;q))$ and $x \in \{t_j, \ldots, t_{j+n+1}\}$, we have $(x-t)_+^0 = 0$. Therefore

$$[t_j, \ldots, t_{j+n+1}](x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0 = 0$$

and

$$[t_j, \ldots, t_{j+n+1}](x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0 N_{i,n}(t;q) = 0 \quad \text{for } i > j+n.$$

Case 4. $\underline{j < i \leqslant j+n}$:

Let $i = j+k$ for $1 \leqslant k \leqslant n$. Then the product $(x-t_{i+1})\cdots(x-t_{i+n})$ is zero for $x \in \{t_{j+k+1}, \ldots, t_{j+n+1}\}$. Moreover for $x \in \{t_j, \ldots, t_{j+k}\}$ we have $(x-t)_+^0 = 0$ because $t \in (t_i, t_{i+n+1}) = supp(N_{i,n}(t;q))$ and $i = j+k$. Therefore $(x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0$ agrees with the zero function at the nodes $\{t_j, \ldots, t_{j+n+1}\}$. Therefore by Theorem 1.3.2

$$[t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n})(x - t)_+^0 = 0,$$

hence

$$[t_j, \ldots, t_{j+n+1}](x - t_{i+1}) \cdots (x - t_{i+n})(x - t)_+^0 N_{i,n}(t; q) = 0 \text{ for } j < i \leqslant j + n.$$

Case 5. $\underline{i = j}$:

For $i = j$ and $t \in (t_j, t_{j+n+1}) = (t_i, t_{i+n+1}) = supp(N_{i,n}(t; q))$ consider

$$[t_j, \ldots, t_{j+n+1}](x - t_{j+1}) \cdots (x - t_{j+n})(x - t)_+^0 N_{i,n}(t; q).$$

Notice that the functions $(x - t_{j+1}) \cdots (x - t_{j+n})(x - t)_+^0$ and

$$\frac{x - t_j}{t_{j+n+1} - t_j}(x - t_{j+1}) \cdots (x - t_{j+n})$$

coincide at the nodes $t_j, \ldots, t_{j+n+1}$. Indeed for $x = t_{j+1}, \ldots, t_{j+n}$ both functions have the value zero. Moreover for $x = t_j$, $(x - t)_+^0 = 0$ and $\frac{x - t_j}{t_{j+n+1} - t_j} = 0$ and for $x = t_{j+n+1}$, $(x - t)_+^0 = 1$ and $\frac{x - t_j}{t_{j+n+1} - t_j} = 1$. Therefore the divided differences of these two functions are same at the nodes $t_j, \ldots, t_{j+n+1}$. Hence by Theorem 1.3.2

$$[t_j, \ldots, t_{j+n+1}](x - t_{j+1}) \cdots (x - t_{j+n})(x - t)_+^0 N_{j,n}(t; q) =$$

$$[t_j, \ldots, t_{j+n+1}]\frac{x - t_j}{t_{j+n+1} - t_j}(x - t_{j+1}) \cdots (x - t_{j+n}) N_{i,n}(t; q).$$

On the other hand, by Theorem 1.3.1

$$[t_j,\dots,t_{j+n+1}]\frac{x-t_j}{t_{j+n+1}-t_j}(x-t_{j+1})\cdots(x-t_{j+n})=\frac{1}{t_{j+n+1}-t_j}.$$

Thus from equation (2.10) we conclude that

$$[t_j,\dots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)(x-t)_+^0$$

$$\begin{aligned}
&=\sum_i [t_j,\dots,t_{j+n+1}](x-t_{i+1})\cdots(x-t_{i+n})(x-t)_+^0 N_{i,n}(t;q)\\
&=[t_j,\dots,t_{j+n+1}](x-t_{j+1})\cdots(x-t_{j+n})(x-t)_+^0 N_{j,n}(t;q)\\
&=[t_j,\dots,t_{j+n+1}]\frac{x-t_j}{t_{j+n+1}-t_j}(x-t_{j+1})\cdots(x-t_{j+n})N_{i,n}(t;q)\\
&=\frac{N_{j,n}(t;q)}{t_{j+n+1}-t_j}
\end{aligned}$$

which is equivalent to

$$N_{j,n}(t;q)=(t_{j+n+1}-t_j)[t_j,\dots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)_+.$$

<u>Second way:</u>

The proof is based on induction on n. Firstly let us show that it is true for $n=1$.

$$\begin{aligned}
N_{j,1}(t;q)&=(t_{j+2}-t_j)[t_j,t_{j+1},t_{j+2}](x-t)_+\\
&=[t_{j+1},t_{j+2}](x-t)_+-[t_j,t_{j+1}](x-t)_+
\end{aligned}$$

$$
= \begin{cases} \dfrac{t-t_j}{t_{j+1}-t_j}, & t_j \leqslant t < t_{j+1} \\ \\ \dfrac{t_{j+2}-t}{t_{j+2}-t_{j+1}}, & t_{j+1} \leqslant t < t_{j+2} \\ \\ 0, & \text{otherwise.} \end{cases}
$$

So it is satisfied for $n = 1$.

Then we assume that it holds for $n \geqslant 1$, that is

$$
N_{j,n}(t;q) = (t_{j+n+1}-t_j)[t_j,\ldots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)_+
$$

Let us use the recurrence relation (2.2) to write

$$
N_{j,n+1}(t;q) = \frac{tq^n - t_j}{t_{j+n+1}-t_j} N_{j,n}(t;q) + \frac{t_{j+n+2}-tq^n}{t_{j+n+2}-t_{j+1}} N_{j+1,n}(t;q). \tag{2.11}
$$

Now apply Liebniz Rule to give

$$
\begin{aligned}
[t_j,\ldots,t_{j+n+1}](x-q^n t)\cdots(x-t)_+ = (t_j - q^n t)[t_j,\ldots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)_+ \\
+[t_j,\ldots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)_+. \quad (2.12)
\end{aligned}
$$

The first term on the right of (2.11), say α_j, is equivalent to

$$
\alpha_j = (q^n t - t_j)[t_j,\ldots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)_+.
$$

Then by (2.12) we have

$$
\alpha_j = [t_j,\ldots,t_{j+n+1}](x-q^{n-1}t)\cdots(x-t)_+ - [t_j,\ldots,t_{j+n+1}](x-q^n t)\cdots(x-t)_+.
$$

Similarly, the second term on the right of (2.11), say β_j, is equivalent to

$$\beta_j = (t_{j+n+2} - tq^n)[t_{j+1}, \ldots, t_{j+n+2}](x - q^{n-1}t) \cdots (x - t)_+.$$

Now we can write

$$\beta_j = \{(t_{j+1} - tq^n) + (t_{j+n+2} - t_{j+1})\}[t_{j+1}, \ldots, t_{j+n+2}](x - q^{n-1}t) \cdots (x - t)_+$$

Applying (2.12) to β_j gives

$$\begin{aligned}\beta_j &= [t_{j+1}, \ldots, t_{j+n+2}](x - q^n t) \cdots (x - t)_+ - [t_{j+2}, \ldots, t_{j+n+2}](x - q^{n-1}t) \cdots (x - t)_+ \\ &\quad + (t_{j+n+2} - t_{j+1})[t_{j+1}, \ldots, t_{j+n+2}](x - q^n t) \cdots (x - t)_+\end{aligned}$$

Now adding α_j to β_j and then simplifying gives

$$\begin{aligned}N_{j,n+1}(t;q) &= [t_{j+1}, \ldots, t_{j+n+2}](x - q^n t) \cdots (x - t)_+ - [t_j, \ldots, t_{j+n+1}](x - q^n t) \cdots (x - t)_+ \\ &= (t_{j+n+2} - t_j)[t_j, \ldots, t_{j+n+2}](x - q^n t) \cdots (x - t)_+\end{aligned}$$

which completes the proof. $\square$

CHAPTER THREE

q-B-SPLINES AND DIVIDED DIFFERENCES

Blossoming is a powerful technique in design of curves and surfaces. It is first introduced in Ramshaw (1989) and has been a unifying structure for Bézier, B-spline curves and surfaces. It employs symmetric polynomials. Only one type of blossom existed until the work Simeonov & et al. (2011) which introduced h-blossoming. But now we have a third kind of blossoming introduced in Simeonov & et al. (2012); the q-blossoming.

In this chapter, "(quantum) spline" refers to all three types of splines; q-spline, h-spline and classical spline. We provide an explicit formula for the blossoms in terms of the bracket operator for the de Boor-Fix form of the dual functionals. The results of this chapter appeared in Budakçı & et al. (2015). We begin by recalling the definition of the q-blossom.

3.1 q-Blossom

Definition 3.1.1. Let $P(t)$ be a polynomial of degree n. The *blossom* of $P(t)$ is the unique symmetric, multiaffine function $p(u_1, \ldots, u_k, \ldots, u_n)$ that reduces to $P(t)$ along a diagonal.

Thus the blossom satisfies the following axioms.

- *Symmetry*

$$p(u_1, \ldots, u_n) = p(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$$

for every permutation σ of $\{1, \ldots, n\}$.

- *Multiaffine*

$$p(u_1,\dots,(1-\lambda)v_k+\lambda u_k,\dots,u_n)=(1-\lambda)p(u_1,\dots,v_k,\dots,u_n)$$
$$+\lambda p(u_1,\dots,u_k,\dots,u_n) \quad (3.1)$$

for each variable.

These symmetry and multiaffine axioms are the same for all forms of the blossom, but the diagonal axiom depends on the type of blossoms.

- *Diagonal Property*

$$p(t,qt,\dots,q^{n-1}t;q)=P(t).$$

For $q=1$, the above expression reduces to the classical blossom.

Example 3.1.1. Consider the q-blossoms of the monomials 1, t, t^2 and t^3 as cubic polynomials:

$$P(t)=1 \ \rightarrow p(t_1,t_2,t_3;q)=1$$

$$P(t)=t \ \rightarrow p(t_1,t_2,t_3;q)=\frac{t_1+t_2+t_3}{1+q+q^2}$$

$$P(t)=t^2 \rightarrow p(t_1,t_2,t_3;q)=\frac{t_1t_2+t_1t_3+t_2t_3}{q(1+q+q^2)}$$

$$P(t)=t^3 \rightarrow p(t_1,t_2,t_3;q)=\frac{t_1t_2t_3}{q^3}.$$

Then any cubic polynomial $P(t)=c_0+c_1t+c_2t^2+c_3t^3$ can be written by q-blossoms as

$$p(t_1,t_2,t_3;q)=c_0+c_1\frac{t_1+t_2+t_3}{1+q+q^2}+c_2\frac{t_1t_2+t_1t_3+t_2t_3}{q(1+q+q^2)}+c_3\frac{t_1t_2t_3}{q^3}.$$

3.2 de Boor-Fix Formula for The Dual Functionals

One of the most important consequences of the Marsden Identity is the de Boor-Fix formula. Here we state the de Boor-Fix Formula for quantum B-splines.

We begin with the definition of the bracket operator.

Definition 3.2.1. Let $P(t)$ and $Q(t)$ be n times quantum differentiable functions.

$$[Q,P]_n^q(\tau) = \sum_{k=0}^{n} \frac{(-1)^k}{[n]_q!} q^{(n-k)(n-k-1)/2} D_{1/q}^k Q(\tau)(D_{1/q}^{n-k}P)(q^{n-k}\tau).$$

Lemma 3.2.1. *Let $P(t)$ and $Q(t)$ be polynomials of degree n. Then $[Q,P]_n^q(\tau)$ is constant independent of the choice of τ.*

Proof. By Lemma 1.3.1, to show that $[Q,P]_n^q(\tau)$ is constant independent of the choice τ, it is enough to verify that the $\frac{1}{q}$-derivative of $[Q,P]_n^q(\tau)$ with respect to τ vanishes. Differentiating the right hand side of $[Q,P]_n^q(\tau)$ and applying both the product rule (1.27) and the chain rule (1.29) yields

$$\sum_{k=0}^{n} \frac{(-1)^k}{[n]_q!} q^{(n-k)(n-k-1)/2} \Big\{ D_{1/q}^k Q(\tau)(D_{1/q}^{n-k+1}P)(q^{n-k}\tau)q^{n-k} \\ + (D_{1/q}^{n-k}P)(q^{n-k-1}\tau)D_{1/q}^{k+1}Q(\tau) \Big\}.$$

Rearranging the terms gives

$$\sum_{k=1}^{n} \frac{(-1)^k}{[n]_q!} q^{(n-k)(n-k-1)/2} D_{1/q}^k Q(\tau)(D_{1/q}^{n-k+1}P)(q^{n-k}\tau)q^{n-k} \\ - \sum_{k=1}^{n} \frac{(-1)^k}{[n]_q!} q^{(n-k)(n-k+1)/2} D_{1/q}^k Q(\tau)(D_{1/q}^{n-k+1}P)(q^{n-k}\tau) = 0$$

□

Theorem 3.2.1.

$$P(x) = \left[(x-t)^{n,q}, P(t)\right]_n^q. \tag{3.2}$$

$$D_{1/q}^r P(x) = \frac{[n]_{1/q}!}{[n-r]_{1/q}!}\left[(x-q^r t)\cdots(x-q^{n-1}t), P(t)\right]_n^q, \tag{3.3}$$

for $r = 0, 1, \ldots, n-1$.

Proof. Recall that by equation (1.35) we have

$$\begin{aligned}\left.\frac{\partial_{1/q}^k}{\partial_{1/q}t^k}(x-t)^{n,q}\right|_{t=\tau} &= (-1)^k \frac{[n]_q!}{[n-k]_q!}(x-\tau)\cdots(x-q^{n-k-1}\tau) \\ &= (-1)^k \frac{[n]_q!}{[n-k]_q!}(x-\tau)^{n-k,q}. \end{aligned} \tag{3.4}$$

Since $[(x-t)^{n,q}, P(t)]_n^q$ is a constant independent choice of τ, we can choose $\tau = x$. Then in equation (3.4) each derivative of $(x-t)^{n,q}|_{t=\tau}$ is zero at $\tau = x$ except for $k = n$. When $k = n$ we have $\frac{\partial_{1/q}^n}{\partial_{1/q}t^n}(x-t)^{n,q}\Big|_{t=x} = (-1)^n[n]_q!$. Thus from the summation in Definition 3.2.1 we have

$$\sum_{k=0}^{n} \frac{(-1)^k}{[n]_q!} q^{(n-k)(n-k-1)/2} \left.\frac{\partial_{1/q}^k (x-t)^{n,q}}{\partial_{1/q}t^k}\right|_{t=\tau} (D_{1/q}^{n-k}P)(q^{n-k}\tau) = \frac{(-1)^n}{[n]_q!}(-1)^n[n]_q!P(x)$$

which is equal to $P(x)$.

Now we continue with the proof of equation (3.3). From equation (3.2) we have

$$P(x) = \left[(x-t)^{n,q}, P(t)\right]_n^q.$$

Since in the bracket only the function $(x-t)^{n,q}$ depends on the variable x, and since by Proposition 1.3.3 mixed partials of polynomials commute

$$D_{1/q}^r P(x) = \left[\frac{\partial_{1/q}^r (x-t)^{n,q}}{\partial_{1/q}x^r}, P(t)\right]_n^q.$$

Therefore by Proposition 1.3.2 we have

$$D_{1/q}^{r}P(x)=\left[\frac{[n]_{1/q}!}{[n-r]_{1/q}!}(x-q^{r}t)\cdots(x-q^{n-1}t),P(t)\right]_{n}^{q}.$$

Recall that the bracket operation is bilinear. Hence

$$D_{1/q}^{r}P(x)=\frac{[n]_{1/q}!}{[n-r]_{1/q}!}\left[(x-q^{r}t)\cdots(x-q^{n-1}t),P(t)\right]_{n}^{q}.$$

□

We are now ready to invoke the Marsden Identity to derive the de Boor-Fix formula for the dual functionals.

Corollary 3.2.1. *(de Boor Fix Formula for the Dual Functionals)*
Let $\Psi_{i,n}(t)=(t-t_{i+1})\cdots(t-t_{i+n})$. *Then*

$$[N_{i,n}(t;q),\Psi_{j,n}(t)]_{n}^{q}=\delta_{i,j}.$$

Proof. We apply Theorem 3.2.1 together with the Marsden Identity. By Theorem 3.2.1 we have

$$\Psi_{j,n}(x)=[(x-t)^{n,q},\Psi_{j,n}(t)]_{n}^{q}. \tag{3.5}$$

But by the equation (2.3)

$$(x-t)^{n,q}=\sum_{i}\Psi_{i,n}(x)N_{i,n}(t;q).$$

Now expanding equation (3.5) by the bilinear property of the bracket operator yields

$$\Psi_{j,n}(x)=\sum_{i}[N_{i,n}(t;q),\Psi_{j,n}(t)]_{n}^{q}\Psi_{i,n}(x).$$

Since the polynomials $\{\Psi_{k,n}(x)\}$ are linearly independent, therefore

$$[N_{i,n}(t;q), \Psi_{j,n}(t)]_n^q = \delta_{i,j}.$$

$\square$

Corollary 3.2.2. *Let $S(t)$ be a (quantum) spline with knots $\{t_i\}$, and let*

$$\Psi_{i,n}(t) = (t - t_{i+1}) \cdots (t - t_{i+n}).$$

Then the control points P_j relative to the (quantum) B-spline bases are given by the following bracket operators:

$$P_j = [S(t), \Psi_{j,n}(t)]_n^q.$$

Proof. Consider $S(t) = \sum_k P_k N_{k,n}(t;q)$. Then the result follows from Corollary 3.2.1 and the bilinearity of the bracket operator:

$$\begin{aligned}
[S(t), \Psi_{j,n}(t)]_n^q &= \left[\sum_k P_k N_{k,n}(t;q), \Psi_{j,n}(t)\right]_n^q \\
&= \sum_k P_k \left[N_{k,n}(t;q), \Psi_{j,n}(t)\right]_n^q \\
&= \sum_k P_k \delta_{k,j} \\
&= P_j.
\end{aligned}$$

$\square$

Corollary 3.2.3.

$$p(u_1, \ldots, u_n; q) = [q^{-n(n-1)/2}(u_1 - q^{n-1}t) \cdots (u_n - q^{n-1}t), P(t)]_n^q. \tag{3.6}$$

Proof. The right hand side of equation (3.6) is clearly symmetric and multiaffine.

Moreover by Theorem 3.2.1 the right hand side reduces to $P(x)$ along the q-diagonal. Namely, $p(x, qx, \ldots, q^{n-1}x; q) = [q^{-n(n-1)/2}(x - q^{n-1}t)\cdots(q^{n-1}x - q^{n-1}t), P(t)]_n^q$. Therefore $p(u_1, \ldots, u_n; q) = [q^{-n(n-1)/2}(u_1 - q^{n-1}t)\cdots(u_n - q^{n-1}t), P(t)]_n^q$. □

Corollary 3.2.4. *The control points P_j of a polynomial $P(t)$ relative to the quantum B-spline bases with knots $\{t_i\}$ are given by the quantum blossom of $P(t)$ evaluated at consecutive knots:*

$$P_j = p(t_{j+1}, \ldots, t_{j+n}; q)$$

Proof. This result follows immediately from Corollaries 3.2.2 and 3.2.3. □

Corollary 3.2.5. *Let $S(t)$ be a quantum spline of degree n with knots $\{t_i\}$, and let $S_k(t)$ be the polynomial of degree n that represents the quantum spline $S(t)$ over the interval $[t_k, t_{k+1}]$. Then the control points P_j of the quantum spline $S(t)$ relative to the quantum B-spline bases with knots $\{t_i\}$ are given by the blossom of $S_k(t)$ for $j \leqslant k \leqslant j+n$ evaluated at consecutive knots:*

$$P_j = s_k(t_{j+1}, \ldots, t_{j+n}; q) \qquad j \leqslant k \leqslant j+n$$

Proof. This is a consequence of Corollary 3.2.4. □

Now we can extend blossoming to splines in the following fashion. Let $S(t)$ be a quantum spline of degree n with knots $\{t_i\}$, and let $S_k(t)$ be the polynomial that represents the quantum spline $S(t)$ over the interval $[t_k, t_{k+1}]$. Define the blossom of $S(t)$ evaluated at the knots $t_{j+1}, \ldots, t_{j+n}$ by setting

$$s(t_{j+1}, \ldots, t_{j+n}; q) = s_k(t_{j+1}, \ldots, t_{j+n}; q) \quad \text{for} \quad j \leqslant k \leqslant j+n$$

Then by Corollary 3.2.5 this definition is well-defined, and we have the following consequence.

Corollary 3.2.6. *Let $S(t)$ be a quantum spline of degree n with knots $\{t_i\}$. Then the control points of the quantum spline $S(t)$ relative to the quantum B-spline bases with knots $\{t_i\}$ are given by the blossom of $S(t)$ evaluated at consecutive knots:*

$$P_j = s(t_{j+1}, \ldots, t_{j+n}; q).$$

The following three figures depict cubic q-B-spline curves with seven control points and different q values.

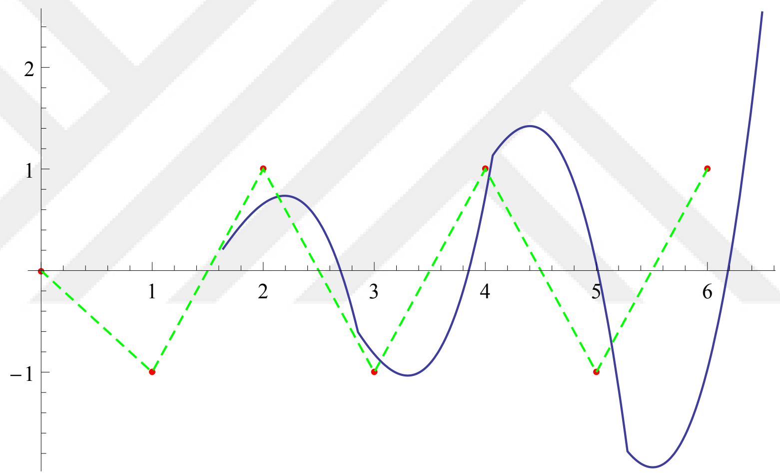

Figure 3.1 Knot sequence $(0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1)$, $t \in [0.3, 0.7]$ and $q = 1.2$.

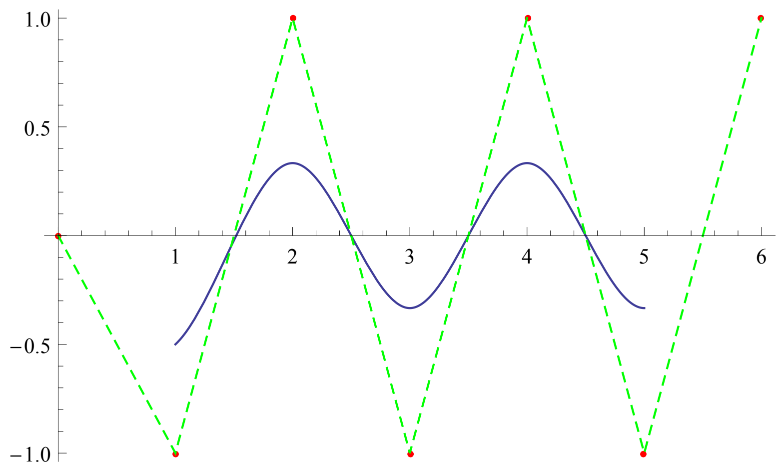

Figure 3.2 Knot sequence $(0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1)$, $t \in [0.3, 0.7]$ and $q = 1$.

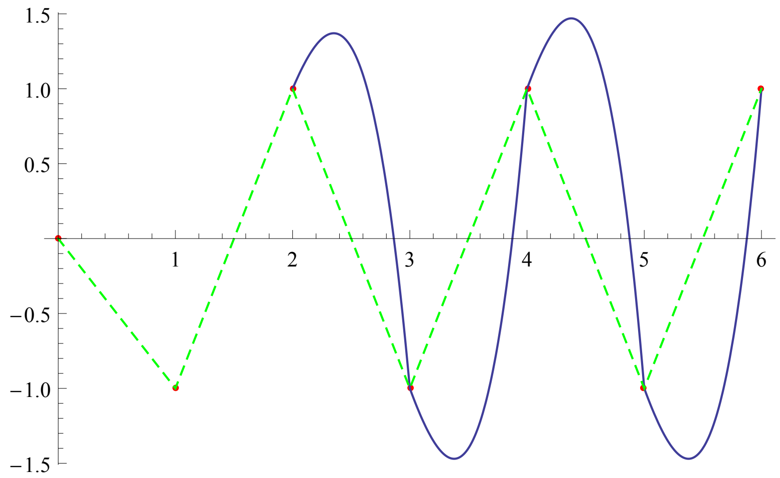

Figure 3.3 Knot sequence $(1, q, q^2, q^3, q^4, q^5, q^6, q^7, q^8, q^9, q^{10})$, $t \in [q^3, q^7]$ and $q = 1.2$.

3.3 Integration Formulas for Quantum B-Splines

In this section we derive three important quantum integration formulas for quantum B-splines: a formula for arbitrary divided differences as integrals of products of B-splines with high order quantum derivatives, a simple closed formula for the integrals of the B-splines over their support, and a convolution formula for the uniform B-splines. Again these three formulas are well known for classical B-splines. Here we show how to extend these formulas to quantum B-splines.

3.3.1 Integral Representations of Divided Differences

The next theorem plays a central role in deriving properties of the divided difference operator by using properties of the B-splines and vice versa.

Theorem 3.3.1. *(B-splines and Divided Differences) For $F \in q\text{-}C^{n+1}[t_k, t_{k+n+1}]$,*

$$\frac{1}{q^{n(n+1)/2}}[t_k, \ldots, t_{k+n+1}]F = \int_{t_k}^{t_{k+n+1}} \left\{ \frac{N_{k,n}(t;q)}{t_{k+n+1} - t_k} \right\} \left\{ \frac{\left(D_{1/q}^{n+1}F\right)(q^n t)}{[n]_q!} \right\} d_{1/q}t. \tag{3.7}$$

Proof. We proceed by induction on n. Our main tools are integration by parts and two-term recurrence formulas for the derivatives of the B-splines in terms of lower order B-splines. We can easily show that for $n = 0$, this equality is satisfied, since by the Fundamental Theorem of q-Calculus (Theorem 1.3.5)

$$[t_k, t_{k+1}]F = \frac{F(t_{k+1}) - F(t_k)}{t_{k+1} - t_k} = \int_{t_k}^{t_{k+1}} \frac{\left(D_{1/q}F\right)(t)}{t_{k+1} - t_k} d_{1/q}t.$$

Now assume that the result is true for $n - 1$ and apply integration by parts for q-integrals. By the Chain Rule (equation 1.29)

$$D_{1/q}\left(\left(D_{1/q}^{n}F\right)(q^n t)\frac{1}{q^n}\right) = \left(D_{1/q}^{n+1}F\right)(q^n t).$$

Therefore integrating by parts the right-hand side of (3.7), we have

$$\int_{t_k}^{t_{k+n+1}} \frac{N_{k,n}(t;q)}{t_{k+n+1} - t_k} \left\{ \frac{\left(D_{1/q}^{n+1}F\right)(q^n t)}{[n]_q!} \right\} d_{1/q}t$$

$$= \int_{t_k}^{t_{k+n+1}} \left\{ \frac{N_{k,n}(t;q)}{t_{k+n+1} - t_k} \right\} \frac{1}{q^n [n]_q!} D_{1/q}\left(D_{1/q}^{n}F\right)(q^n t) d_{1/q}t$$

$$= \left\{ \frac{N_{k,n}(t;q)\left(D_{1/q}^{n}F\right)(q^n t)}{q^n [n]_q!(t_{k+n+1} - t_k)} \right\}_{t_k}^{t_{k+n+1}} - \frac{1}{t_{k+n+1} - t_k} \int_{t_k}^{t_{k+n+1}} \frac{\left(D_{1/q}^{n}F\right)(q^{n-1} t)}{q^n [n]_q!} D_{1/q}N_{k,n}(t;q) d_{1/q}t.$$

But $N_{k,n}(t;q)$ vanishes at the end points of its support, so

$$\left\{ \frac{N_{k,n}(t;q)\left(D_{1/q}^{n}F\right)(q^n t)}{q^n [n]_q!(t_{k+n+1} - t_k)} \right\}_{t_k}^{t_{k+n+1}} = 0.$$

Furthermore, by $1/q$-derivative of q-B-splines

$$D_{1/q}N_{k,n}(t;q) = [n]_q \left(\frac{N_{k,n-1}(t,q)}{t_{k+n}-t_k} - \frac{N_{k+1,n-1}(t;q)}{t_{k+n+1}-t_{k+1}} \right).$$

Therefore, using the inductive hypothesis we deduce that

$$\int_{t_k}^{t_{k+n+1}} \frac{N_{k,n}(t;q)}{t_{k+n+1}-t_k} \left\{ \frac{\left(D_{1/q}^{n+1}F\right)(q^n t)}{[n]_q!} \right\} d_{1/q}t$$

$$= \frac{-1}{t_{k+n+1}-t_k} \int_{t_k}^{t_{k+n+1}} \frac{\left(D_{1/q}^{n}F\right)(q^{n-1} t)}{q^n [n]_q!} D_{1/q}N_{k,n}(t;q) d_{1/q}t$$

$$= \frac{-1}{t_{k+n+1}-t_k} \int_{t_k}^{t_{k+n+1}} [n]_q \left(\frac{N_{k,n-1}(t;q)}{t_{k+n}-t_k} - \frac{N_{k+1,n-1}(t;q)}{t_{k+n+1}-t_{k+1}} \right) \left\{ \frac{\left(D_{1/q}^{n}F\right)(q^{n-1} t)}{q^n [n]_q!} \right\} d_{1/q}t$$

$$= \frac{1}{q^n}\frac{1}{t_{k+n+1}-t_k} \left[\int_{t_{k+1}}^{t_{k+n+1}} \frac{N_{k+1,n-1}(t;q)}{t_{k+n+1}-t_{k+1}} \left\{ \frac{\left(D_{1/q}^{n}F\right)(q^{n-1} t)}{[n-1]_q!} \right\} d_{1/q}t \right.$$
$$\left. - \int_{t_k}^{t_{k+n}} \frac{N_{k,n-1}(t;q)}{t_{k+n}-t_k} \left\{ \frac{\left(D_{1/q}^{n}F\right)(q^{n-1} t)}{[n-1]_q!} \right\} d_{1/q}t \right]$$

$$= \frac{1}{q^{n(n+1)/2}} \frac{[t_{k+1},\dots,t_{k+n+1}]F - [t_k,\dots,t_{k+n}]F}{t_{k+n+1}-t_k}$$

$$= \frac{1}{q^{n(n+1)/2}} [t_k,\dots,t_{k+n+1}]F.$$

The last step of the proof follows from the recurrence relation of the divided differences (1.23). □

The next result shows that the average value of a q-B-spline $N_{k,n}(t;q)$ over its support with respect to $\frac{1}{q}$-integration is independent both of the index k and the choice of the knots $\{t_j\}$.

Corollary 3.3.1.

$$\int_{t_k}^{t_{k+n+1}} \frac{N_{k,n}(t;q)}{t_{k+n+1}-t_k} d_{1/q}t = \frac{1}{[n+1]_q}.$$

Proof. Let $F(t) = (t-a)^{n+1,q}$ in equation (3.7). Then by Theorem 1.3.1 on the left-hand side of equation (3.7) we have $[t_k, \ldots, t_{k+n+1}]F = 1$, since F is a polynomial of degree $n+1$ and the coefficient of t^{n+1} is 1. On the right-hand side of equation (3.7), Proposition 1.3.2 gives

$$D_{1/q}^{n+1} F = [n+1]_{1/q}! = \frac{[n+1]_q!}{q^{n(n+1)/2}}.$$

Substituting these results into equation (3.7) and rearranging the terms, we conclude that

$$\int_{support} \frac{N_{k,n}(t;q)}{support} d_{1/q}t = \frac{1}{[n+1]_q}.$$

□

The above result Corollary 3.3.1 resembles q-integration of q-Bernstein polynomials, see (Oruç, 1998; Goldman & et al., 2014). In particular if $|q| < 1$, then

$$\int_0^1 B_k^n(t;q) d_q t = \begin{cases} \dfrac{q^{k+1}}{[n+1]_q} & k = 0, \ldots, n-1, \\ \dfrac{1}{[n+1]_q} & k = n. \end{cases}$$

If $|q| > 1$, then

$$\int_0^1 B_k^n(t;q) d_{1/q}t = \frac{q^k}{[n+1]_q}, \quad k = 0, \ldots, n$$

where $B_k^n(t;q) = t^k \prod_{s=0}^{n-k-1} (1 - q^s t)$, for $k = 0, 1, \ldots, n$ and $t \in [0,1]$, and an empty product denotes 1.

3.3.2 Convolution for Uniform q-B-Splines

We can generate B-splines with arbitrary knots either from a two term recurrence or explicitly using divided differences of truncated powers. Here we explore yet another way to generate B-splines with uniform knots by repeated convolution with the characteristic function on the unit interval.

Let $f(t)$ and $g(t)$ be integrable functions defined for all values of t. The classical convolution of f and g denoted by $(f * g)(t)$ is defined by setting:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx.$$

Similarly we define the q-convolution $(f *_q g)(t)$ by setting

$$(f *_q g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)d_qx.$$

The following theorem and its consequences show that q-B-splines with uniform knots can be generated by using $1/q$-convolutions.

Theorem 3.3.2. *(Uniform Knots)*
Let $t_k = k$, $k \in \mathbb{Z}$ be the knot sequence. Then

$$N_{k,n}(t;q) = \frac{[n]_q}{n} \int_0^1 N_{k,n-1}(t-x;q) d_{1/q}x, \quad n \geqslant 1.$$

Proof. Let

$$L_{k,n}(t;q) = \frac{[n]_q}{n} \int\limits_0^1 N_{k,n-1}(t-x;q) d_{1/q}x.$$

We use induction on n to show that $L_{k,n}(t;q) = N_{k,n}(t;q)$. For $n = 1$ we have

$$L_{k,1}(t;q) = \int_0^1 N_{k,0}(t-x;q) d_{1/q}x.$$

Since

$$N_{k,0}(t-x;q) = \begin{cases} 1, & k \leqslant t-x < k+1 \\ 0, & \text{otherwise,} \end{cases}$$

it follows that

$$\int_0^1 N_{k,0}(t-x;q) d_{1/q}x = \begin{cases} \int_k^t d_{1/q}x, & k \leqslant t < k+1 \\ \int_{t-1}^{k+1} d_{1/q}x, & k+1 \leqslant t < k+2 \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, by the recurrence relation (2.2)

$$\begin{aligned} N_{k,1}(t;q) &= (t-k)N_{k,0}(t;q) + (k+2-t)N_{k+1,0}(t;q) \\ &= \begin{cases} t-k, & k \leqslant t < k+1 \\ k+2-t, & k+1 \leqslant t < k+2 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $L_{k,1}(t;q) = N_{k,1}(t;q)$. Now suppose $L_{k,n-1}(t;q) = N_{k,n-1}(t;q)$. We shall show that $L_{k,n}(t;q) = N_{k,n}(t;q)$. Consider the $1/q$-derivative of $L_{k,n}(t;q)$. Applying the inductive hypothesis along with $1/q$-derivative of the q-B-splines in terms of lower order q-B-splines, we find that

$$\begin{aligned}
D_{1/q}L_{k,n}(t;q) &= \frac{[n]_q}{n}\int_0^1 D_{1/q}N_{k,n-1}(t-x;q)d_{1/q}x \\
&= \frac{[n]_q}{n}\frac{[n-1]_q}{n-1}\left(\int_0^1 N_{k,n-2}(t-x;q)d_{1/q}x - \int_0^1 N_{k+1,n-2}(t-x;q)d_{1/q}x\right) \\
&= \frac{[n]_q}{n}\left(N_{k,n-1}(t;q) - N_{k+1,n-1}(t;q)\right) \\
&= D_{1/q}N_{k,n}(t;q).
\end{aligned}$$

Therefore by Lemma 1.3.1, $L_{k,n}(t;q)$ and $N_{k,n}(t;q)$ differ by at most a constant. Substituting $t = k$ into the definition of $L_{k,n}(t)$, we see that

$$L_{k,n}(k;q) = \frac{[n]_q}{n}\int_0^1 N_{k,n-1}(k-x;q)d_{1/q}x = 0 = N_{k,n}(k;q)$$

since $N_{k,n-1}(k-x;q) = 0$ for $0 \leqslant x \leqslant 1$. Hence $L_{k,n}(t;q) = N_{k,n}(t;q)$. $\square$

Corollary 3.3.2. *Let* $t_k = k \in \mathbb{Z}$ *be the knot sequence and let* $\chi_{[0,1)}$ *denote the characteristic function on* $[0,1)$. *Then*

$$N_{k,n}(t;q) = \left(N_{k,n-1}(;q) *_{1/q} \chi_{[0,1)}\right)(t)$$

Proof. From Theorem 3.3.2,

$$\begin{aligned}
N_{k,n}(t;q) &= \int_0^1 N_{k,n-1}(t-x;q)d_{1/q}x \\
&= \int_{-\infty}^{\infty} N_{k,n-1}(t-x;q)\chi_{[0,1)}(x)d_{1/q}x \\
&= \left(N_{k,n-1}(;q) *_{1/q} \chi_{[0,1)}\right)(t).
\end{aligned}$$

$\square$

Corollary 3.3.3.

$$N_{0,n}(t;q) = (\underbrace{\chi_{[0,1)} *_{1/q} \cdots *_{1/q} \chi_{[0,1)}}_{n+1\, \textit{factors}})(t)$$

Proof. If we set $k = 0$ in Corollary 3.3.2, then we have

$$N_{0,n}(t;q) = \left(N_{0,n-1}(;q) *_{1/q} \chi_{[0,1)}\right)(t).$$

Repeating last equation recursively, we get

$$N_{0,n}(t;q) = \left(N_{0,0}(;q) *_{1/q} \underbrace{\chi_{[0,1)} *_{1/q} \cdots *_{1/q} \chi_{[0,1)}}_{n \text{ factors}}\right)(t).$$

Observe that

$$N_{0,0}(;q) = \chi_{[0,1)}.$$

Hence

$$N_{0,n}(t;q) = (\underbrace{\chi_{[0,1)} *_{1/q} \cdots *_{1/q} \chi_{[0,1)}}_{n+1 \text{ factors}})(t).$$

□

So far Corollaries 3.3.2 and 3.3.3 had no analogues for h-B-splines due to the lack of convolution for h-integration.

CHAPTER FOUR

PEANO KERNEL AND ITS q-ANALOGUE

This chapter deals with classical Peano kernel theorem and its q-analogue. We derive a q-analogue of the Peano kernel (q-Peano kernel). Furthermore, we present a simple way to find the kernel under some conditions. Then we demonstrate how the q-Peano kernel is used to find the error of Lagrange interpolation problem. Moreover, we discuss the error bounds of quadrature formula on the remainder. Finally, we establish a relation between q-B-splines and q-Peano kernel. The results in this chapter establish the core of the paper Budakçı & Oruç (2015).

4.1 The Peano Kernel

The Peano kernel theorem provides a powerful technique for computing the errors of approximations such as interpolation, quadrature rules and B-splines. Therefore it is useful for numerical solutions of differential equations. The errors are represented by a linear functional which operates on functions $f \in C^{n+1}[a,b]$ and annihilates all polynomials of degree at most n.

Namely, if $Lf = 0$ for all $f \in \mathcal{P}_n$, then

$$Lf = \int_a^b f^{(n+1)}(t)K(x,t)dt,$$

where $K(x,t) = \dfrac{1}{n!}L\left((x-t)_+^n\right)$.

An important application of this result is the Kowalewski's interpolating polynomial remainder. Let $t_0, t_1, \ldots, t_n \in [a,b]$ be fixed and distinct, and

$$Lf = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x)$$

where $l_{nk}(x) = \prod_{\substack{v=0 \\ v \neq k}}^{n} \frac{x - t_v}{t_k - t_v}$. If $f \in C^{m+1}[a, b]$, then

$$Lf = \frac{1}{m!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^m f^{(m+1)}(t) dt, \quad \text{for each } \ m = 0, 1, \ldots, n$$

is the error functional, see Hammerlin & Hoffmann (1991).

Our purpose is to extend Peano kernel when classical derivatives are replaced by q-derivatives. It is worth pointing out that there are certain functions whose q-derivatives exist while classical derivatives may fail.

4.2 The q-Analogue of Peano Kernel Theorem

In this section we derive a generalization of the Peano kernel theorem. It is based on the q-Taylor expansion analogous to classical Peano kernel Theorem. A detailed treatment of classical Peano Kernel theorem can be found in (Hammerlin & Hoffmann, 1991; Phillips, 2003; Powell, 1981).

Theorem 4.2.1. *Let $g_t(x) = (x-t)_+^{n,q}$ and let L be a linear functional which commutes with the operation of q-integration and also satisfies the conditions: $L(g_t)$ exists and $L(f) = 0$ for all $f \in \mathcal{P}_n$. Then for all $f \in 1/q - C^{n+1}[a, b]$*

$$L(f) = \int_a^b (D_{1/q}^{n+1} f)(q^n t) K(x, t) d_{1/q} t,$$

where

$$K(x, t) = \frac{q^{n(n+1)/2}}{[n]_q!} L(g_t).$$

Proof. Recall that here the function $(x-t)_+^{n,q}$ is a function of t and x behaves as a parameter. When we say $L(g_t)$ we mean that L is applied to the truncated power function, regarded as a function of x with t as a parameter. Hence we find real number that depends on t. We apply L to the equation (1.38). Since L is linear and annihilates polynomials, we have

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} L\left(\int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q}t\right).$$

Since L commutes with the operation of q-integration,

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D_{1/q}^{n+1} f)(q^n t) L\left((x-t)_+^{n,q}\right) d_{1/q}t.$$

□

Corollary 4.2.1. *If the conditions in Theorem 4.2.1 are satisfied and also the kernel $K(x,t)$ does not change sign on $[a,b]$, then*

$$L(f) = \frac{\left(D_{1/q}^{n+1} f\right)(\xi)}{[n+1]_q!} q^{n(n+1)/2} L(x^{n+1})$$

for some ξ in (a,b).

Proof. Since $D_{1/q}^{n+1} f$ is continuous and $K(x,t)$ does not change sign on $[a,b]$, we can apply the Mean Value Theorem 1.3.7. Thus we have

$$L(f) = \left(D_{1/q}^{n+1} f\right)(\xi) \int_a^b K(x,t) d_{1/q}t, \quad a < \xi < b.$$

Replacing $f(x)$ by x^{n+1} gives

$$L(x^{n+1}) = \frac{[n+1]_q!}{q^{n(n+1)/2}} \int_a^b K(x,t) d_{1/q}t,$$

and

$$\int_a^b K(x,t)d_{1/q}t = \frac{q^{n(n+1)/2}}{[n+1]_q!}L(x^{n+1}),$$

and this completes the proof. □

4.3 Applications to Polynomial Interpolation

Suppose $t_0, t_1, \ldots, t_n \in [a,b]$ are distinct points, and for a fixed $x \in [a,b]$, define the corresponding error functional as

$$Lf = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x).$$

Since $\sum_{k=0}^{n} l_{nk}(x) = 1$, by q-Peano kernel Theorem 4.2.1 we obtain,

$$\begin{aligned}\frac{[m]_q!}{q^{m(m+1)/2}} K(x,t) = L\left((x-t)_+^{m,q}\right) &= (x-t)_+^{m,q} - \sum_{k=0}^{n}(t_k - t)_+^{m,q} l_{nk}(x) \\ &= \sum_{k=0}^{n}\left[(x-t)_+^{m,q} - (t_k-t)_+^{m,q}\right] l_{nk}(x).\end{aligned}$$

From the fact that

$$\begin{aligned}&\int_a^b \left[(x-t)_+^{m,q} - (t_k-t)_+^{m,q}\right]\left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t \\ &= \int_a^x \left[(x-t)^{m,q} - (t_k-t)^{m,q}\right]\left(D_{1/q}^{m+1} f\right)(q^m t) + \int_{t_k}^x (t_k - t)^{m,q}\left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t\end{aligned}$$

we have

$$\frac{[m]_q!}{q^{m(m+1)/2}} \int_a^b K(x,t)\left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t$$

$$= \int_a^x \left(D_{1/q}^{m+1} f\right)(q^m t) \sum_{k=0}^{n} \left[(x-t)^{m,q} - (t_k - t)^{m,q}\right] l_{nk}(x) d_{1/q} t$$

$$+ \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t.$$

For each $m \leqslant n$, since the interpolation operator is a projection, it reproduces polynomials and the term in the square brackets vanishes in the last equation for $f(x) = (x-t)^{m,q}$. Accordingly,

$$Lf = \int_a^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t$$

$$= \frac{q^{m(m+1)/2}}{[m]_q!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t$$

for each $m = 0, 1, \ldots, n$.

Now we give examples which show how we can find the q-Peano kernel. Note that classical Peano kernel fail because of lack of smoothness.

Example 4.3.1. Suppose that we interpolate a function $f \in 1/q - C^3[-1,1]$ by a polynomial $p \in \mathcal{P}_2$. Here $n = 2$ and $m = 2$. Let $t_0 = -1,\ t_1 = 0,\ t_2 = 1$. Then the error function becomes

$$Lf = \frac{q^3}{[2]_q!} \sum_{k=0}^{2} l_{2k}(x) \int_{t_k}^{x} (t_k - t)^{2,q} \left(D_{1/q}^3 f\right)(q^2 t) d_{1/q} t$$

with $l_{20}(x) = \frac{1}{2}x(x-1)$, $l_{21}(x) = (1-x^2)$, $l_{22}(x) = \frac{1}{2}x(x+1)$. Then,

$$\frac{[2]_q!}{q^3} Lf = l_{20}(x) \int_{-1}^{x} (-1-t)^{2,q} \left(D_{1/q}^3 f\right)(q^2 t) d_{1/q} t + l_{21}(x) \int_{0}^{x} (-t)^{2,q} \left(D_{1/q}^3 f\right)(q^2 t) d_{1/q} t$$

$$+ l_{22}(x) \int_{1}^{x} (1-t)^{2,q} \left(D_{1/q}^3 f\right)(q^2 t) d_{1/q} t.$$

Now if $x \leqslant 0$, then

$$\frac{[2]_q!}{q^3}Lf = l_{20}(x)\int_{-1}^{x}(-1-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t - l_{21}(x)\int_{x}^{0}(-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t$$
$$-l_{22}(x)\int_{x}^{0}(1-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t - l_{22}(x)\int_{0}^{1}(1-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t.$$

Hence,

$$Lf = \frac{q^3}{[2]_q!}\int_{-1}^{1}K(x,t)\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t$$

where

$$K(x,t) = \begin{cases} l_{20}(x)(-1-t)^{2,q}, & -1 \leqslant t < x \\ -l_{21}(x)(-t)^{2,q} - l_{22}(x)(1-t)^{2,q}, & x \leqslant t < 0 \\ -l_{22}(x)(1-t)^{2,q}, & 0 \leqslant t \leqslant 1. \end{cases}$$

Similarly for $x \geqslant 0$,

$$\frac{[2]_q!}{q^3}Lf = l_{20}(x)\int_{-1}^{0}(-1-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t + l_{20}(x)\int_{0}^{x}(-1-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t$$
$$+l_{21}(x)\int_{0}^{x}(-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t - l_{22}(x)\int_{x}^{1}(1-t)^{2,q}\left(D_{1/q}^{3}f\right)(q^2t)d_{1/q}t$$

and the q-Peano kernel becomes

$$K(x,t) = \begin{cases} l_{20}(x)(-1-t)^{2,q}, & -1 \leqslant t < 0 \\ l_{20}(x)(-1-t)^{2,q} + l_{21}(x)(-t)^{2,q} - l_{22}(x)(1-t)^{2,q}, & 0 \leqslant t < x \\ -l_{22}(x)(1-t)^{2,q}, & x \leqslant t \leqslant 1. \end{cases}$$

Example 4.3.2. Let

$$f(x) = \begin{cases} \dfrac{q^3x^3}{6}, & 0 \leqslant x < 1 \\[2em] \dfrac{1}{6}\left(4 - 4[3]_q x + 4q[3]_q x^2 - 3q^3x^3\right), & 1 \leqslant x < 2 \\[2em] \dfrac{1}{6}\left(-44 + 20[3]_q x - 8q[3]_q x^2 + 3q^3x^3\right), & 2 \leqslant x < 3 \\[2em] -\dfrac{1}{6}(-4+x)(-4+qx)(-4+q^2x), & 3 \leqslant x < 4 \\[2em] 0, & \text{otherwise.} \end{cases}$$

It is obvious that for $q \neq 1$, $f \in C[0,4]$ but $f \notin C^1[0,4]$. However, one may check that $f \in 1/q - C^2[0,4]$. Because classical error functional cannot work, we may find the error via q-Peano kernel theorem. Let $t_0 = 0$, $t_1 = 2$ and $t_2 = 4$. Then the error functional

$$Lf = q\sum_{k=0}^{2} l_{2k}(x) \int_{t_k}^{x} (t_k - t)\left(D_{1/q}^2 f\right)(qt)d_{1/q}t,$$

where $l_{20}(x) = \frac{1}{8}(x-2)(x-4)$, $l_{21}(x) = -\frac{1}{4}x(x-4)$ and $l_{22}(x) = \frac{1}{8}x(x-2)$. It follows that

$$\begin{aligned} \frac{1}{q}Lf = \; & l_{20}(x)\int_{0}^{x}(-t)\left(D_{1/q}^2 f\right)(qt)d_{1/q}t + l_{21}(x)\int_{2}^{x}(2-t)\left(D_{1/q}^2 f\right)(qt)d_{1/q}t \\ & + l_{22}(x)\int_{4}^{x}(4-t)\left(D_{1/q}^2 f\right)(qt)d_{1/q}t. \end{aligned}$$

Now we will find the kernel. If $0 \leqslant x < 2$, then

$$K(x,t) = \begin{cases} -l_{20}(x)t, & 0 \leqslant t < x \\ \\ l_{21}(x)(2-t) - l_{22}(x)(4-t), & x \leqslant t < 2 \\ \\ -l_{22}(x)(4-t), & 2 \leqslant t < 4. \end{cases}$$

Similarly, for $2 \leqslant x < 4$,

$$K(x,t) = \begin{cases} -l_{20}(x)t, & 0 \leqslant t < 2 \\ \\ -l_{20}(x)t + l_{21}(x)(2-t), & 2 \leqslant t < x \\ \\ l_{21}(x)(2-t) - l_{22}(x)(4-t), & x \leqslant t < 4. \end{cases}$$

The function $f(x)$ given in Example 4.3.2 is indeed a cubic q-B-spline.

Example 4.3.3. Consider the $1/q$-integral of a function f on the interval $[a,b]$. We want to evaluate the integral approximately using linear interpolant formula. That is,

$$\int_a^b f(x) d_{1/q}x \approx \frac{b-aq}{[2]_q} f(a) + \frac{bq-a}{[2]_q} f(b)$$

Let us define the operator L as

$$L(f) = \int_a^b f(x) d_{1/q}x - \frac{b-aq}{[2]_q} f(a) - \frac{bq-a}{[2]_q} f(b).$$

Since $L(f)=0$ for all functions $f\in\mathcal{P}_1$, for all $f\in 1/q-C^2[a,b]$ we have

$$L(f)=\int\limits_a^b \left(D_{1/q}^2 f\right)(qt)K(t)d_{1/q}t$$

and

$$K(x,t)=qL((x-t)_+).$$

What follows we find the kernel $K(x,t)$. First,

$$K(x,t)=q\left\{\int\limits_a^b (x-t)_+ d_{1/q}x-\frac{b-aq}{[2]_q}(a-t)_+-\frac{bq-a}{[2]_q}(b-t)_+\right\}.$$

Then for $t\in[a,b]$,

$$\int\limits_a^b (x-t)_+ d_{1/q}x=\int\limits_t^b (x-t)d_{1/q}x,\quad (a-t)_+=0\quad\text{and}\quad (b-t)_+=(b-t).$$

Thus,

$$\begin{aligned}K(x,t)&=q\left\{\int\limits_t^b (x-t)d_{1/q}x-\frac{bq-a}{[2]_q}(b-t)\right\}\\&=q\left\{\frac{(b-t)(b-\frac{t}{q})}{[2]_{1/q}}-\frac{bq-a}{[2]_q}(b-t)\right\}\\&=q\left\{\frac{q(b-t)(b-\frac{t}{q})}{[2]_q}-\frac{bq-a}{[2]_q}(b-t)\right\}\\&=\frac{q}{[2]_q}(b-t)(a-t)\end{aligned}$$

for $a\leqslant t\leqslant b$.

Since $K(x,t)<0$ on $[a,b]$, we can apply Mean Value Theorem 1.3.7. So, we have

$$L(f)=\frac{D_{1/q}^2 f(\xi)}{[2]_q!}qL(x^2),$$

where

$$
\begin{aligned}
L(x^2) &= \int\limits_a^b x^2 d_{1/q}x - \frac{b-aq}{[2]_q}a^2 - \frac{bq-a}{[2]_q}b^2 \\
&= \frac{b^3}{[3]_q!} - \frac{a^3}{[3]_q!} - \frac{b-aq}{[2]_q}a^2 - \frac{bq-a}{[2]_q}b^2 \\
&= \frac{-(b-a)(bq-a)(b-aq)}{[3]_q!}.
\end{aligned}
$$

Finally, we derive

$$
\begin{aligned}
L(f) &= \int\limits_a^b f(x)d_{1/q}x - \frac{b-aq}{[2]_q}f(a) - \frac{bq-a}{[2]_q}f(b) \\
&= \frac{-q(b-a)(bq-a)(b-aq)}{[3]_q!}D^2_{1/q}f(\xi),
\end{aligned}
$$

where $a < \xi < b$.

When $q = 1$, the last equation reduces to the well-known trapezoidal rule, see Phillips (2003).

We now discuss error bounds of quadrature formula on the remainder given by

$$
R_n(f;q) = \int\limits_0^b f(x)d_{1/q}x - \sum_{k=0}^{n} \gamma_{nk} f(t_{nk})
$$

which appear in numerical integration. Assuming $f \in 1/q - C^{m+1}[0,b]$ and $R_n(f;q) = 0$ for all $f \in \mathcal{P}_m$ with $m \leqslant n$, we can apply the q-Peano kernel theorem and hence

$$
R_n(f;q) = \int\limits_0^b K(x,t)\left(D^{m+1}_{1/q}f\right)(q^m t)d_{1/q}t.
$$

By applying q-Hölder's inequality (1.40), we have

$$|R_n(f;q)| \leqslant \left[\int_0^b \left|\left(D_{1/q}^{m+1} f\right)(q^m t)\right|^{p_1} d_{1/q}t\right]^{\frac{1}{p_1}} \left[\int_0^b |K(x,t)|^{p_2}\, d_{1/q}t\right]^{\frac{1}{p_2}}$$

for all $1 \leqslant p_1, p_2 \leqslant \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Since the second integral in the last equation is independent of f, by choosing coefficients and nodes appropriately we can minimize the remainder.

(i) For $p_1 = \infty$ and $p_2 = 1$,

$$|R_n(f;q)| \leqslant ||D_{1/q}^{m+1} f||_\infty \int_0^b |K(x,t)| d_{1/q}t.$$

(ii) For $p_1 = p_2 = 2$,

$$|R_n(f;q)| \leqslant ||D_{1/q}^{m+1} f||_2 \left[\int_0^b |K(x,t)|^2 d_{1/q}t\right]^{\frac{1}{2}}.$$

The q-Peano kernel $K(x,t)$ can be written as

$$K(x,t) = q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1,q}}{[m+1]_q!} - s(t;q),$$

where $s(t;q) = \dfrac{q^{m(m+1)/2}}{[m]_q!} \displaystyle\sum_{k=0}^{n} \gamma_{nk}(t_{nk} - t)_+^{m,q}$ is a quantum spline with the knot sequence $\{t_{nv}\}_{v=0,\ldots,n}$. Eventually, the problem of minimizing the q-integral

$$\left[\int_0^b |K(x,t)|^{p_2} d_{1/q}t\right]^{\frac{1}{p_2}}$$

is equivalent to finding the best approximation of the polynomial

$q^{m(m+3)/2}\dfrac{(b-\frac{t}{q})^{m+1,q}}{[m+1]_q!}$ in t by a quantum spline with respect to the norm $||.||_{p_2}$.

This argument can be interpreted as a generalization of the result in (Hammerlin & Hoffmann, 1991, chap. 5).

4.4 Connection between q-Peano Kernel and q-B-Splines

In this section we establish certain relations with q-B-splines and q-Peano kernel. When $q = 1$, the following result reduces to its classical counterpart that can be found in (Powell, 1981, Chap. 19).

Let us recall the fact that a divided difference $f[t_0, t_1, \ldots, t_{n+1}]$ can be represented as symmetric sum of $f(t_j)$, see Powell (1981),

$$f[t_0, t_1, \ldots, t_{n+1}] = \sum_{i=0}^{n+1} f(t_i) / \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j). \tag{4.1}$$

Hence we can readily derive from

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k)[t_k, \ldots, t_{k+n+1}](x-t)_+^{n,q}$$

that

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k) \sum_{i=k}^{k+n+1} (t_i - t)_+^{n,q} / \prod_{\substack{j=k \\ j\neq i}}^{k+n+1} \frac{1}{(t_i - t_j)}.$$

Theorem 4.4.1.

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b \frac{N_{0,n}(t;q)}{t_{n+1} - t_0} \left(D_{1/q}^{n+1} f\right)(q^n t) d_{1/q} t.$$

Proof. We first set our linear functional L as

$$\begin{aligned} f[t_0, t_1, \ldots, t_{n+1}] =& \sum_{i=0}^{n+1} f(t_i) / \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j) \\ =& L(f). \end{aligned}$$

We see that, for any fixed and distinct points $\{t_i : i = 0, 1, \ldots, n+1\}$, L is a bounded linear operator. From the q-Peano Kernel Theorem 4.2.1, we have

$$L(f) = \int_a^b K(x,t)(D_{1/q}^{n+1} f)(q^n t) d_{1/q} t,$$

where

$$\begin{aligned} K(x,t) &= \frac{q^{n(n+1)/2}}{[n]_q!} L\left((x-t)_+^{n,q}\right) \\ &= \frac{q^{n(n+1)/2}}{[n]_q!} \sum_{i=0}^{n+1} (t_i - t)_+^{n,q} \,/\, \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j). \end{aligned}$$

Thus

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} \frac{N_{0,n}(t;q)}{t_{n+1} - t_0}.$$

Combining the last equation with equation (4.1) we derive

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b \frac{N_{0,n}(t;q)}{t_{n+1} - t_0} \left(D_{1/q}^{n+1} f\right) (q^n t) d_{1/q} t.$$

□

CHAPTER FIVE

QUANTUM SPLINE SPACES

In this chapter we begin with definition of quantum splines and then give some properties of q-B-splines to show that q-B-splines form a basis for quantum spline spaces. Also we derive algorithmic formula for $1/q$-integration and $1/q$-differentiation for q-spline functions. Moreover, we show a way to find the polynomials on each interval of a q-spline function. After all, we give basic theory of interpolation by q-B-splines and approximation by q-Bernstein-Schoenberg operator.

5.1 Quantum Splines

Let $t_0 < t_1 < \cdots < t_m$ be the knots and let n be a nonnegative integer and $q \neq 0$ be a fixed real. A q-spline function of degree n having knots $t_0 < t_1 < \cdots < t_m$ is a function S such that

i) S is a polynomial of degree up to n on each interval $[t_{i-1}, t_i)$

ii) S is quantum continuous of order $n-1$ at the knots.

Then, S is a continuous piecewise polynomial of degree at most n and quantum continuous of order $n-1$ at the knots. Here "quantum continuous of order $n-1$ at the knots" means the quantum derivatives $D^k_{1/q}$, $k = 0, \ldots, n-1$ of adjacent polynomial segments agree at the knots. Quantum splines of degree *zero* are continuous piecewise constants. One can write a quantum spline of degree *zero* as

$$S(t;q)=\begin{cases} p_0(t)=c_0, & t\in[t_0,t_1) \\ p_1(t)=c_1, & t\in[t_1,t_2) \\ \vdots & \vdots \\ p_{m-1}(t)=c_{m-1}, & t\in[t_{m-1},t_m) \end{cases}$$

and quantum spline of degree 1 as

$$S(t;q)=\begin{cases} p_0(t)=a_0t+b_0, & t\in[t_0,t_1) \\ p_1(t)=a_1t+b_1, & t\in[t_1,t_2) \\ \vdots & \vdots \\ p_{m-1}(t)=a_{m-1}t+b_{m-1}, & t\in[t_{m-1},t_m). \end{cases}$$

Since on each subinterval S is a polynomial of degree 1, we must have $S(t_i;q) = S(t_{i+1};q)$ for each interior knot.

Example 5.1.1. The figure below shows two quadratic quantum spline functions $S(t;q)$ with $q=2$ and $q=1/2$.

$$S(t;q)=\begin{cases} \frac{1}{2}(-1+t)(-1+qt), & 1\leqslant t<2 \\ \frac{1}{2}\left(-11+5(1+q)t-2qt^2\right), & 2\leqslant t<3 \\ \frac{1}{2}(-4+t)(-4+qt), & 3\leqslant t<4 \\ 0, & \text{otherwise.} \end{cases}$$

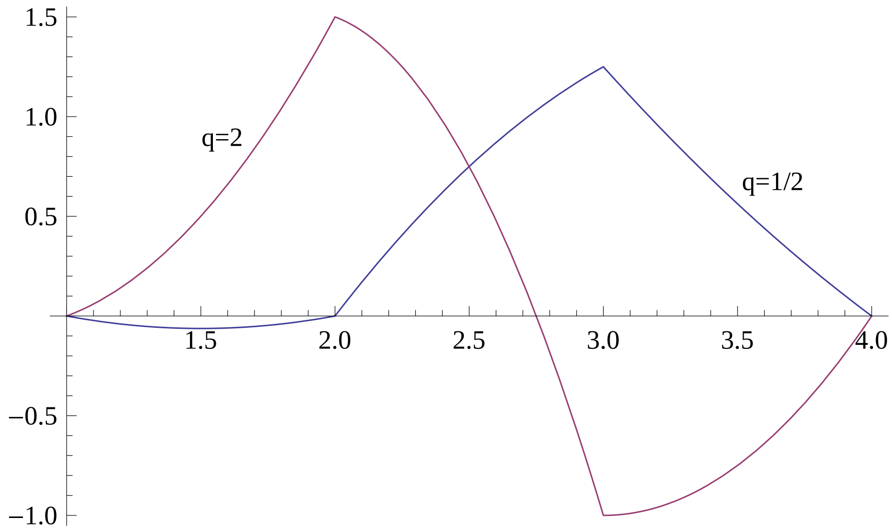

Figure 5.1 Spline functions for $q = 1/2$ and $q = 2$.

5.1.1 Properties of q-B-Splines

In the classical theory, basis splines, B-splines came out after two decades of introduction of spline functions by I.J. Schoenberg. Nowadays, B-splines, $N_{k,n}$ as in Chapter 1 is the most widely used basis for spline spaces. Let us recall q-B-splines given in Chapter 2.

The following result is given in the work Simeonov & Goldman (2013).

Proposition 5.1.1. *The interval of support of the q-B-spline* $N_{k,n}(t;q) \subseteq [t_k, t_{k+n+1}]$.

The classical B-spline $N_{k,n}(t)$ is nonnegative for each k, n and all $t \in [t_k, t_{k+n+1}]$. However, the following new result shows that q-B-splines $N_{k,n}(t;q)$ may take negative values.

Theorem 5.1.1. $N_{k,n}(t;q) \geqslant 0$ *for all* k, n *and* $t \in [t_k, t_{k+n+1}]$ *if and only if* $q = 1$.

Proof. If $q = 1$, $N_{k,n}(t;q)$ becomes classical B-spline. Hence it is obvious that for all k, n and t q-B-spline $N_{k,n}(t;q)$ is non-negative for $q = 1$.

For the converse we prove its contraposition: If $q \neq 1$, then there exist k, n and t such that $N_{k,n}(t;q) < 0$. It is easy to prove this statement. For example take $q = 2$ and the knot sequence as $(1, 2, 3, 4)$. Then

$$N_{0,2}(t;q) = \begin{cases} \frac{1}{2}(t-1)(2t-1), & 1 \leqslant t < 2 \\ \frac{1}{2}\left(-4t^2 + 15t - 11\right), & 2 \leqslant t < 3 \\ (t-4)(t-2), & 3 \leqslant t < 4. \end{cases}$$

But $N_{0,2}(t;q) < 0$ for $t \in (\frac{11}{4}, 4)$. This completes the proof.

The figure below depicts $N_{0,2}(t;q)$ with the knot sequence $(1, 2, 3, 4)$.

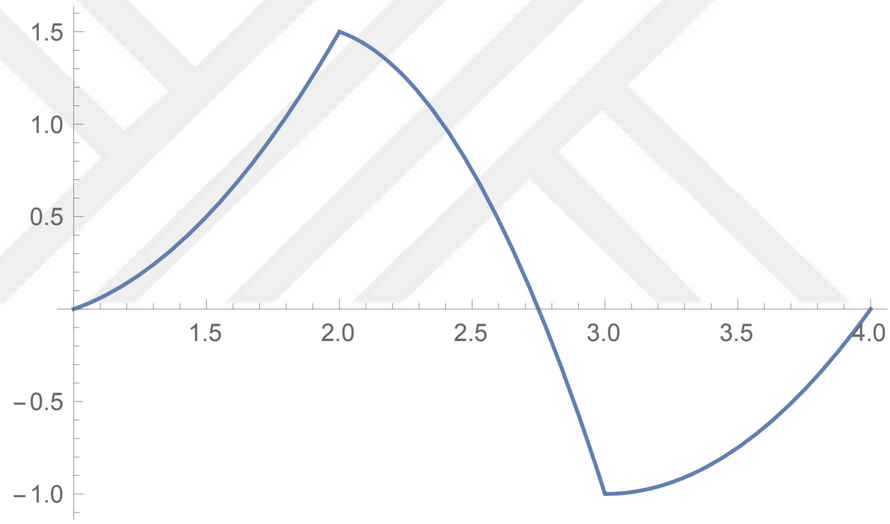

Figure 5.2 The graph of $N_{0,2}(t;q)$ with the knot sequence $(1, 2, 3, 4)$ and $q = 2$.

□

The next Proposition 5.1.2, proved in Simeonov & Goldman (2013) via blossoms gives a formula of the $1/q$-derivative of q-B-spline of degree n. Although Lemma 5.1.1 is also proved in Simeonov & Goldman (2013), here we give an alternative proof.

Proposition 5.1.2. *For any integer $n \geqslant 2$, we have*

$$D_{1/q}N_{k,n}(t;q) = [n]_q \left(\frac{N_{k,n-1}(t;q)}{t_{k+n} - t_k} - \frac{N_{k+1,n-1}(t;q)}{t_{k+n+1} - t_{k+1}} \right).$$

Lemma 5.1.1. *For $n \geqslant 1$, the q-B-splines $N^q_{k,n}$ belong to $q - C^{n-1}(\mathbb{R})$.*

Proof. It is clear that $N^q_{k,1}$ is continuous i.e. $N^q_{k,1} \in q - C^0(\mathbb{R})$. Now, suppose that $N^q_{k,n} \in q - C^{n-1}(\mathbb{R})$. By Proposition 5.1.2 it follows that $D_{1/q} N^q_{k,n+1} \in q - C^{n-1}(\mathbb{R})$ because it is written as a linear combination of $N^q_{k,n}$ and $N^q_{k+1,n}$. Thus, $N^q_{k,n+1} \in q - C^n(\mathbb{R})$.

□

This result provide us $1/q$-derivative of q-B-splines are also q-B-splines.

From the Lemma 5.1.1 and the Proposition 5.1.2 we can write the coefficients of $1/q$-derivative of a q-B-spline function from its initial coefficients:

$$D_{1/q}\left(\sum_k c_k N_{k,n}(t;q)\right) = [n]_q \sum_k \left(\frac{c_k - c_{k-1}}{t_{k+n} - t_k}\right) N_{k,n-1}(t;q) \quad n \geqslant 2. \tag{5.1}$$

Given a function f and a subset K of its domain let $f|K$ denote the restriction of f to K. That is

$$(f|K)(t) = f(t) \quad (t \in K).$$

This notation will be useful for the next lemmas. We use the restriction of $N^q_{k,n}$ to any single interval between knots.

Although the linear independence of q-B-splines is used in Simeonov & Goldman (2013), its proof was missing. So it is better to prove it.

Lemma 5.1.2. *The set of q-B-splines $\{N^q_{i,n}, N^q_{i+1,n}, \ldots, N^q_{i+n,n}\}$ is linearly independent on (t_{i+n}, t_{i+n+1}).*

Proof. For $n = 0$, it is obvious that the set is linearly independent. Let $n \geqslant 1$ and assume that the Lemma holds for $n - 1$. Let $S = \sum\limits_{k=0}^{n} c_{i+k} N^q_{i+k,n}$ and suppose that

$S|(t_{i+n}, t_{i+n+1}) = 0$. By equation (5.1) we have

$$0 = D_{1/q}S|(t_{i+n}, t_{i+n+1}) = [n]_q \sum_{k=1}^{n} \left(\frac{c_{i+k} - c_{i+k-1}}{t_{i+k+n} - t_{i+k}} \right) N^q_{k,n-1}|(t_{i+n}, t_{i+n+1}). \quad (5.2)$$

Since $N_{i+n+1,n-1}(t;q) = 0$ and $N_{i,n-1}(t;q) = 0$ on (t_{i+n}, t_{i+n+1}), by induction hypothesis $\{N^q_{i+1,n-1}, \dots, N^q_{i+k,n-1}\}$ is linearly independent on the interval (t_{i+n}, t_{i+n+1}). Therefore, we must have all the coefficients in equation (5.2) zero, that is, $c_i = c_{i+1} = \cdots = c_{i+n}$, say all of them equal to value λ. So by partition unity property of q-B-splines we have $S(t;q) = \lambda$ on (t_{i+n}, t_{i+n+1}). Hence by the assumption we have $\lambda = 0$ which shows The set of q-B-splines $\{N^q_{i,n}, N^q_{i+1,n}, \dots, N^q_{i+n,n}\}$ is linearly independent on (t_{i+n}, t_{i+n+1}). ☐

Lemma 5.1.3. *The set of q-B-splines* $\{N^q_{-n,n}, N^q_{-n+1,n}, \dots, N^q_{m-1,n}\}$ *is linearly independent on* (t_0, t_m).

Proof. Let $S = \sum_{k=-n}^{m-1} c_k N^q_{k,n}$ and suppose $S|(t_0, t_1) = 0$. Since only $\{N^q_{-n,n}, N^q_{-n+1,n}, \dots, N^q_{0,n}\}$ are non-zero on the interval (t_0, t_1) we have

$$0 = S|(t_0, t_1) = \sum_{k=-n}^{0} c_k N^q_{k,n}|(t_0, t_1) \quad (5.3)$$

From the previous lemma, the set $\{N^q_{-n,n}, N^q_{-n+1,n}, \dots, N^q_{0,n}\}$ is linearly independent on (t_0, t_1). Hence the coefficients $c_k = 0$ for $-n \leqslant k \leqslant 0$ in equation (5.3). If all the c_k's are zero then there is nothing to prove. Suppose to the contrary that not all the c_k's are zero, let i be the index that $c_i \neq 0$. Assume that $i \geqslant 1$, and $(t_j, t_{j+1}) \subset (t_0, t_1)$. For any $t \in (t_j, t_{j+1})$, we obtain a contradiction

$$0 = S(t;q) = \sum_{k=-n}^{0} c_k N_{k,n}(t;q) = c_i N_{i,n}(t;q) \neq 0.$$

Therefore, all the c_k's are zero and this implies that the set $\{N^q_{-n,n}, N^q_{-n+1,n}, \dots, N^q_{0,n}\}$ is linearly independent. ☐

5.2 A Basis for The Space $\mathscr{S}_m^{n,q}$

We first emphasize that q-spline space denoted by $\mathscr{S}_m^{n,q}$ depends on the degree of polynomial pieces, knot sequence and an initial parameter q.
Next theorem shows that q-B-splines form a basis for the quantum spline space of degree n with the knot sequence $t_0 < t_1 < \cdots < t_m$.
We note that although the work Simeonov & Goldman (2013) investigates broadly q-B-splines via blossoming, the following theorem and its proof were not mentioned.

Theorem 5.2.1. *A basis for the space $\mathscr{S}_m^{n,q}$ is*

$$1, t, t^2, \ldots, t^n, (t-t_1;q)_+^n, (t-t_2;q)_+^n, \ldots, (t-t_{m-1};q)_+^n$$

Consequently, the dimension of $\mathscr{S}_m^{n,q}$ is $n+m$.

Proof. It is obvious that each $N_{i,n}^q|[t_0, t_m]$ for $-n \leqslant i \leqslant m-1$ is in the space $\mathscr{S}_m^{n,q}$. Now we prove that the dimension of $\mathscr{S}_m^{n,q}$ is at most $n+m$. Firstly, we show that each element in $\mathscr{S}_m^{n,q}$ can be written in the form

$$S_m^n(t;q) = \sum_{i=0}^{n} a_i t^i + \sum_{i=1}^{m-1} b_i (t-t_i;q)_+^n, \tag{5.4}$$

where the q-truncated powers are defined as

$$(t-t_i;q)_+^n = (q^{n-1}t - t_i)\cdots(qt-t_i)(t-t_i)_+$$

and

$$(t-t_i)_+ = \begin{cases} t-t_i, & \text{if } t \geqslant t_i \\ 0, & \text{if } t < t_i. \end{cases}$$

Since in the interval $[t_0, t_1]$ all the truncated powers vanish, $S_m^n(t;q)$ is a polynomial of degree n, say p_0. Thus we have $p_0(t;q) = \sum\limits_{i=0}^{n} a_i t^i$ which determines all the coefficients a_i. In the interval $[t_1, t_2]$, $S_m^n(t;q)$ is another polynomial, say p_1. According to the quantum continuity at the knots, we have at t_1,

$$D_{1/q}^r (p_1 - p_0)(t_1) = 0, \quad 0 \leqslant r \leqslant n-1.$$

Since $p_1 - p_0$ is a polynomial of degree at most n, we have $(p_1 - p_0)(t;q) = b_1(t - t_1;q)^n$ for some b_1. Hence, we can write

$$S_m^n(t;q) = \sum_{i=0}^{n} a_i t^i + b_1(t-t_1;q)_+^n, \quad t_0 \leqslant t \leqslant t_2$$

When we repeat the same argument at the other internal knots $t_2, t_3, \ldots, t_{m-1}$ we obtain the equation (5.4). Thus any spline of degree n on the interval $[t_0, t_m]$ with intermediate knots $t_1, t_2, \ldots, t_{m-1}$, may be written as a sum of multiples of $n+m$ functions

$$1, t, t^2, \ldots, t^n, (t-t_1;q)_+^n, (t-t_2;q)_+^n, \ldots, (t-t_{m-1};q)_+^n.$$

Since these functions are linearly independent, they form a basis for the this spline space and hence the dimension of the space $S_m^n(t;q)$ is $n+m$. □

We have shown that q-B-splines form a basis for q-splines. Now let us investigate how we can find the quantum derivatives of the spline functions which is proved by de Boor (1972) for classical spline functions.

Theorem 5.2.2. *Let S be a given spline function such that $S(t;q) = \sum_{i=-n}^{k} a_i N_{i,n}(t;q)$. Then for all $j \in \{1, 2, \ldots, n-1\}$ and all $t \in [a,b]$*

$$D_{1/q}^j S(t;q) = \sum_{i=-n+j}^{k} a_i^{(j)} N_{i,n-j}(t;q), \tag{5.5}$$

where

$$a_i^{(j)} = \begin{cases} a_i, & \text{if } j = 0 \\ [n+1-j]_q \dfrac{a_i^{(j-1)} - a_{i-1}^{(j-1)}}{t_{i+n+1-j} - t_i}, & \text{if } j > 0. \end{cases} \tag{5.6}$$

Proof. We begin with $j = 1$.

$$\begin{aligned} D_{1/q}S(t;q) &= \sum_{i=-n}^{k} a_i D_{1/q} N_{i,n}(t;q) \\ &= \sum_{i=-n}^{k} a_i [n]_q \left(\frac{N_{i,n-1}(t;q)}{t_{i+n} - t_i} - \frac{N_{i+1,n-1}(t;q)}{t_{i+n+1} - t_{i+1}} \right) \\ &= \sum_{i=-n}^{k} [n]_q \left(\frac{a_i}{t_{i+n} - t_i} N_{i,n-1}(t;q) - \frac{a_i}{t_{i+n+1} - t_{i+1}} N_{i+1,n-1}(t;q) \right) \\ &= \sum_{i=-n+1}^{k} [n]_q \left(\frac{a_i - a_{i-1}}{t_{i+n} - t_i} \right) N_{i,n-1}(t;q). \end{aligned}$$

This shows that equation (5.6) is true for $j = 1$. Repeating the same argument shows that the formula (5.6) is true for each $j = 1, 2, \ldots, n-1$. □

Here we derive a formula for computing the $1/q$-integral of a given q-spline.

Theorem 5.2.3. *Let S be a given spline function such that $S(t;q) = \sum_{i=-n}^{k} a_i N_{i,n}(t;q)$. Then for all $t \in [a,b]$,*

$$\int_{t_{-n}}^{t} S(x;q) d_{1/q}x = \sum_{i=-n}^{k} \left(\sum_{j=-n}^{i} a_j \frac{t_{j+m+1} - t_j}{[n+1]_q} \right) N_{i,n+1}(t;q). \tag{5.7}$$

Proof. Define a function $\tilde{S}$ by

$$\tilde{S}(t;q) = \int_{t_{-n}}^{t} S(x;q) d_{1/q}x.$$

This is a quantum spline of degree $n+1$ and so $\tilde{S}$ can be written as

$$\tilde{S}(t;q) = \sum_{i=-n}^{k} \tilde{a}_i N_{i,n+1}(t;q).$$

Using (5.5) and (5.6) for all $t \in [a,b]$ gives

$$S(t;q) = D_{1/q}\tilde{S}(t;q) = \sum_{i=-n}^{k} [n+1]_q \frac{\tilde{a}_i - \tilde{a}_{i-1}}{t_{i+n+1} - t_i} N_{i,n}(t;q)$$

where $\tilde{a}_{-m-1} = 0$. It follows by comparing the coefficient of the basis that

$$a_i = \frac{[n+1]_q}{t_{i+n+1} - t_i}\tilde{a}_i \quad \text{for } i = -n$$

and

$$a_i = \frac{[n+1]_q}{t_{i+n+1} - t_i}\left(\tilde{a}_i - \tilde{a}_{i-1}\right) \quad \text{for } i = -n+1, \ldots, k.$$

Therefore

$$\tilde{a}_i = \sum_{j=-n}^{i} a_j \frac{t_{j+m+1} - t_j}{[n+1]_q} \quad \text{for } i = -n, -n+1, \ldots, k$$

which completes the proof. □

Proposition 5.2.1. *Let $S(t;q)$ be a spline function such that*

$$S(t;q) = \sum_{i=-n}^{k} a_i N_{i,n}(t;q), \quad t \in [a,b]$$

and P_j be the polynomial restricted to $S|_{[t_j,t_{j+1})}$ for $j = 0, \ldots, k$. Then

$$P_j(t;q) = \sum_{r=0}^{n} \frac{1}{[r]_q!}\left(D^r_{1/q}S(t_j;q)\right)(t-t_j)(qt-t_j)\cdots(q^{r-1}t-t_j), \quad t \in [t_j, t_{j+1})$$

where we can compute the values $D^r_{1/q}S(t_j;q)$, $r = 0, \ldots, n$ by using formula (5.2).

Proof. Since $S(t;q)$ is a polynomial of degree n on the interval $[t_j, t_{j+1})$ for each $j = 0, 1, \ldots, k$, we can write

$$P_j(t;q) = \sum_{r=0}^{n} w_{rj}(t - t_j; q)^r,$$

where w_{rj} for $r = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, k$ are the unknowns. Let us recall the formula (5.5). Then it is easily seen that by taking $1/q$-derivative of $S(t;q)$ r times and substituting $t = t_j$ gives

$$w_{rj} = \frac{1}{[r]_q!} D^r_{1/q} S(t_j; q).$$

Hence we derive

$$P_j(t;q) = \sum_{r=0}^{n} \frac{1}{[r]_q!} D^r_{1/q} S(t_j; q)(t - t_j; q)^r, \quad t \in [t_j, t_{j+1}).$$

$\square$

5.3 Interpolation

Cubic Quantum Spline

This section follows an analogous technique Kincaid & Cheney (1996) to construct cubic quantum spline function which interpolates a given table of values. Namely knots are the abscissas of interpolation points in $\mathbb{R}^2$.

t	t_0	t_1	$\cdots$	t_m
y	y_0	y_1	$\cdots$	y_m

Let $S_i(t;q)$ be the cubic polynomial which is a part of $S(t;q)$ on the interval $[t_i, t_{i+1}]$.

Hence we have

$$S(t;q) = \begin{cases} S_0(t;q), & t \in [t_0, t_1] \\ S_1(t;q), & t \in [t_1, t_2] \\ \vdots & \\ S_{m-1}(t;q), & t \in [t_{m-1}, t_m]. \end{cases}$$

Notice that by continuity of $S(t;q)$ we have

$$S_{i-1}(t_i;q) = y_i = S_i(t_i;q),\ 1 \leqslant i \leqslant m-1.$$

Also $D_{1/q}S$ and $D^2_{1/q}S$ are continuous and we use these conditions to derive cubic quantum spline function. As in classical case we need $4m$ conditions but we have $4m-2$ conditions. So we need two more conditions to determine the quantum spline function.

Let us define $z_i = D^2_{1/q}S(t_i;q)$. For $0 \leqslant i \leqslant n$, z_i exists and satisfy

$$\lim_{t \to z_i^-} D^2_{1/q}S(t;q) = z_i = \lim_{t \to z_i^+} D^2_{1/q}S(t;q)$$

since $D^2_{1/q}S(t;q)$ is continuous at each interior knot. We know that $S_i(t;q)$ is a cubic polynomial on $[t_i, t_{i+1}]$, so $D^2_{1/q}S_i(t;q)$ is a linear polynomial satisfying $D^2_{1/q}S_i(t_i;q) = z_i$ and $D^2_{1/q}S_i(t_{i+1};q) = z_{i+1}$ hence $D^2_{1/q}S_i(t;q)$ is a straight line between z_i and z_{i+1}, i.e,

$$D^2_{1/q}S_i(t;q) = \frac{z_i}{h_i}(t_{i+1} - t) + \frac{z_{i+1}}{h_i}(t - t_i), \tag{5.8}$$

where $h_i = t_{i+1} - t_i$. If we q-integrate the equation (5.8) twice, we derive

$$S_i(t;q) = \frac{z_i}{[3]_q!h_i}(t_{i+1} - t)^3_q - \frac{z_{i+1}}{[3]_q!h_i}(t_i - t)^3_q + C_1(t - t_i) + C_2(t_{i+1} - t) \tag{5.9}$$

We determine the constants C_1 and C_2 by using the conditions $S_i(t_i;q) = y_i$ and

$S_i(t_{i+1};q) = y_{i+1}$. From the first condition we find

$$S_i(t_i;q) = \frac{z_i}{[3]_q!h_i} h_i(t_{i+1} - qt_i)(t_{i+1} - q^2t_i) + C_2h_i = y_i$$

which gives

$$C_2 = \frac{y_i}{h_i} - \frac{z_i(t_{i+1} - qt_i)(t_{i+1} - q^2t_i)}{[3]_q!h_i}.$$

From the second condition we find

$$S_i(t_{i+1};q) = \frac{z_{i+1}}{[3]_q!h_i} h_i(t_i - qt_{i+1})(t_i - q^2t_{i+1}) + C_1h_i = y_{i+1}$$

hence

$$C_1 = \frac{y_{i+1}}{h_i} - \frac{z_{i+1}(t_i - qt_{i+1})(t_i - q^2t_{i+1})}{[3]_q!h_i}.$$

Now our aim is to find z_i for $1 \leqslant i \leqslant n-1$. Since $D_{1/q}S$ is continuous, we use the equality $D_{1/q}S_{i-1}(t_i;q) = D_{1/q}S_i(t_i;q)$ at the interior knots. First, let's find $D_{1/q}S_i(t;q)$. We have

$$\begin{aligned} D_{1/q}S_i(t;q) = &-\frac{z_i}{[3]_q!h_i}[3]_q(t_{i+1} - t)(t_{i+1} - qt) \\ &+\frac{z_{i+1}}{[3]_q!h_i}[3]_q(t_i - t)(t_i - qt) \\ &-\frac{y_i}{h_i} + \frac{z_i(t_{i+1} - qt_i)(t_{i+1} - q^2t_i)}{[3]_q!h_i} \\ &+\frac{y_{i+1}}{h_i} - \frac{z_{i+1}(t_i - qt_{i+1})(t_i - q^2t_{i+1})}{[3]_q!h_i}. \end{aligned}$$

Using the equality $D_{1/q}S_{i-1}(t_i;q) = D_{1/q}S_i(t_i;q)$ and rearranging the terms give

$$\begin{aligned} &-(t_{i+1} - t_i)(t_i - qt_{i-1})_q^2 z_{i-1} \\ &+[2]_q \left\{(t_i - qt_{i+1})(t_{i+1} - qt_i)(t_i - t_{i-1}) + (t_i - qt_{i-1})(t_{i-1} - qt_i)(t_{i+1} - t_i)\right\} z_i \\ &+(qt_{i+1} - t_i)(q^2t_{i+1} - t_i)(t_i - t_{i-1})z_{i+1} \\ &= [3]_q! \left\{h_i(y_i - y_{i-1}) - h_{i-1}(y_{i+1} - y_i)\right\}. \end{aligned} \tag{5.10}$$

The linear system of equations (5.10) for $1 \leqslant i \leqslant m-1$ with $z_0 = z_m = 0$ yields the following tridiagonal matrix system.

$$\begin{pmatrix} u_1 & r_1 & & & & \\ a_2 & u_2 & r_2 & & & \\ & a_3 & u_3 & r_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{m-2} & u_{m-2} & r_{m-2} \\ & & & & a_{m-1} & u_{m-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{m-2} \\ z_{m-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-2} \\ v_{m-1} \end{pmatrix}$$

where

$$\begin{aligned} u_i &= [2]_q \left\{ (t_i - qt_{i+1})(t_{i+1} - qt_i)(t_i - t_{i-1}) + (t_i - qt_{i-1})(t_{i-1} - qt_i)(t_{i+1} - t_i) \right\} \\ a_i &= -(t_{i+1} - t_i)(t_i - qt_{i-1})_q^2 \\ r_i &= (qt_{i+1} - t_i)(q^2 t_{i+1} - t_i)(t_i - t_{i-1}) \\ v_i &= [3]_q! \left\{ h_i(y_i - y_{i-1}) - h_{i-1}(y_{i+1} - y_i) \right\}. \end{aligned}$$

After determining $z_0, z_1, \ldots, z_m$, any value of the the cubic quantum spline function can be computed easily from equation (5.9).

The following proposition introduces cardinal q-splines (i.e. q-B-splines which have knots in geometric progression) which is found in Simeonov & Goldman (2013). Its proof is given by using q-blossom technique implicitly for the case $q > 1$. But here we give another proof of the proposition for $0 < q < 1$ using induction.

Proposition 5.3.1.

For $q > 1$;

Suppose that the knots are in geometric progression with ratio q, that is $t_{k+1} = qt_k$ for all k. Then

$$N_{k,n}(t_j;q) = \begin{cases} 0, & j \neq k+1 \\ 1, & j = k+1. \end{cases}$$

For $0 < q < 1$;

Suppose that the knots are in geometric progression with ratio $1/q$, that is $t_{k+1} = \frac{1}{q} t_k$ for all k. Then

$$N_{k,n}(t_j;q) = \begin{cases} 0, & j \neq k+n \\ 1, & j = k+n. \end{cases}$$

Proof. Recall that we have

$$N_{k,0}(t;q) = \begin{cases} 1, & t_k \leqslant t < t_{k+1} \\ 0, & \text{otherwise.} \end{cases}$$

Hence for the knot sequence $t_{k+1} = \frac{1}{q} t_k$, $N_{k,0}$ satisfy

$$N_{k,0}(t_j;q) = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

Now suppose that $N_{k,n}(t_j;q)$ also satisfy the property, that is,

$$N_{k,n}(t_j;q) = \begin{cases} 1, & j = k+n \\ 0, & j \neq k+n. \end{cases}$$

Let us use recurrence relation (2.2) to write $N_{k,n+1}$ in terms of the lower degree q-B-splines:

$$N_{k,n+1}(t;q) = \frac{tq^n - t_k}{t_{k+n+1} - t_k} N_{k,n}(t;q) + \frac{t_{k+n+2} - tq^n}{t_{k+n+2} - t_{k+1}} N_{k+1,n}(t;q)$$

to show $N_{k,n+1}(t_j;q)$ satisfy the property, firstly suppose that $j = k+n+1$. Thus we have

$$N_{k,n+1}(t_{k+n+1};q) = \frac{t_{k+n+1}q^n - t_k}{t_{k+n+1} - t_k} N_{k,n}(t_{k+n+1};q) + \frac{t_{k+n+2} - t_{k+n+1}q^n}{t_{k+n+2} - t_{k+1}} N_{k+1,n}(t_{k+n+1};q).$$

By assumption we know that $N_{k,n}(t_{k+n+1};q) = 0$ and $N_{k+1,n}(t_{k+n+1};q) = 1$. So we must only check if

$$\frac{t_{k+n+2} - t_{k+n+1}q^n}{t_{k+n+2} - t_{k+1}} = 1. \tag{5.11}$$

Because $t_{k+1} = \frac{1}{q}t_k$ for each k, it follows that $t_{k+n+1} = \frac{1}{q^n}t_{k+1}$. Therefore the equation (5.11) becomes

$$\frac{t_{k+n+2} - \frac{1}{q^n}t_{k+1}q^n}{t_{k+n+2} - t_{k+1}} = 1.$$

For $j = k+n+1$, $N_{k,n}(t_j;q) = 1$.

It only remains to show that for $j \neq k+n+1$, $N_{k,n}(t_j;q) = 0$. Since

$$N_{k,n}(t_j;q) = \begin{cases} 1, & j = k+n \\ 0, & j \neq k+n \end{cases} \quad \text{and} \quad N_{k+1,n}(t_j;q) = \begin{cases} 1, & j = k+n+1 \\ 0, & j \neq k+n+1. \end{cases}$$

It is enough to show that for $j = k+n$, $N_{k,n}(t_j;q) = 0$. We have

$$\begin{aligned} N_{k,n+1}(t_{k+n};q) &= \frac{t_{k+n}q^n - t_k}{t_{k+n+1} - t_k} N_{k,n}(t_{k+n};q) + \frac{t_{k+n+2} - t_{k+n}q^n}{t_{k+n+2} - t_{k+1}} N_{k+1,n}(t_{k+n};q) \\ &= \frac{t_k\frac{1}{q^n}q^n - t_k}{t_{k+n+1} - t_k} \\ &= 0 \end{aligned}$$

Thus,

$$N_{k,n+1}(t_j;q) = \begin{cases} 1, & j = k+n+1 \\ 0, & j \neq k+n+1. \end{cases}$$

$\square$

Proposition 5.3.2. *Suppose that the knots are in geometric progression with ratio q, that is $t_{k+1} = qt_k$ for all k and we have a data set $\{(v_i, f_i)\}_{i=0}^m$ where*

$$v_i = \frac{t_{i+1} + t_{i+2} + \cdots + t_{i+n}}{[n]_q} \quad \textit{for} \ \ i = 0, 1, \ldots, m.$$

Let $\sum_{j=0}^m P_j N_{j,n}(t; q)$ be the interpolating polynomial such that

$$f_i = \sum_{j=0}^m P_j N_{j,n}(v_i; q)$$

for $i = 0, 1, \ldots, m$. Then

$$f_i = P_i, \quad \textit{for} \ \ i = 0, 1, \ldots, m.$$

Proof. Let $f_i = \sum_{j=0}^m P_j N_{j,n}(v_i; q)$ for $i = 0, 1, \ldots, m$ then we have the following system of equations.

$$\begin{pmatrix} N_{0,n}(v_0; q) & N_{1,n}(v_0; q) & \cdots & N_{m,n}(v_0; q) \\ N_{0,n}(v_1; q) & N_{1,n}(v_1; q) & & N_{m,n}(v_1; q) \\ & & \vdots & \\ N_{0,n}(v_{n-1}; q) & N_{1,n}(v_{n-1}; q) & \cdots & N_{m,n}(v_{n-1}; q) \\ N_{0,n}(v_m; q) & N_{1,n}(v_m; q) & \cdots & N_{m,n}(v_m; q) \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ p_{m-1} \\ P_m \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{m-1} \\ f_m \end{pmatrix}. \tag{5.12}$$

Since $v_i = t_{i+1}$ for all $i = 0, 1, \ldots, m$, from Proposition 5.3.1 we have $N_{i,n}(v_j) = \delta_{ij}$. Hence the matrix of q-B-splines in equation (5.12) becomes an identity matrix. Thus we find

$$P_i = f_i \quad \text{for} \ \ i = 0, 1, \ldots, m.$$

$\square$

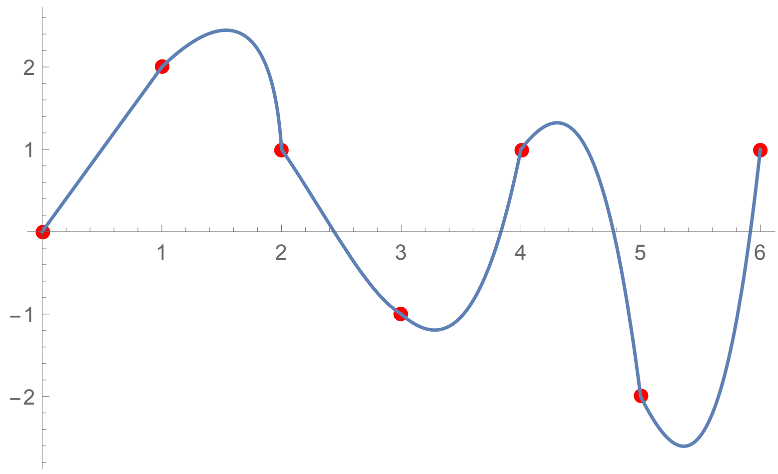

Figure 5.3 q-B-spline interpolation with knot sequence $(1, q, q^2, q^3, q^4, q^5, q^6, q^7, q^8, q^9, q^{10})$ and $q = 1.2$

5.4 q-Bernstein-Schoenberg Operator

In this section we define a generalization of the Bernstein-Schoenberg operator and give basic properties of the operator. We denote it by $\mathcal{S}_{m,n}$. If $f(t)$ is defined on the interval $[a, b]$, the operator is given by

$$\mathcal{S}_{m,n}(f, t; q) = \sum_{j=0}^{l} f(\xi_j) N_{j,n}(t; q)$$

where

- n is the degree of q-B-splines
- m is the number of the intervals on $[a, b]$
- Given a knot sequence $t_0, t_1, \ldots, t_m$ set

$$\xi_j = \frac{t_{j+1} + \cdots + t_{j+n}}{[n]_q}$$

which we may call q-Greville abscissae.

Properties of Generalized Bernstein-Schoenberg Operator

i. $\mathcal{S}_{m,n}$ is a linear operator, i.e, $\mathcal{S}_{m,n}(\lambda f + \beta g) = \lambda \mathcal{S}_{m,n}(f) + \beta \mathcal{S}_{m,n}(g)$

ii. If f is a linear function then $\mathcal{S}_{m,n} f = f$.

iii. If $q = 1$, $\mathcal{S}_{m,n}(f, t; q)$ reduces to classical Bernstein-Schoenberg operator.

Proof.

i. We have

$$\begin{aligned}
\mathcal{S}_{m,n}(\lambda f + \beta g, t; q) &= \sum_{j=0}^{l} (\lambda f + \beta g)(\xi_j) N_{j,n}(t; q) \\
&= \sum_{j=0}^{l} (\lambda f)(\xi_j) N_{j,n}(t; q) + \sum_{j=0}^{l} (\beta g)(\xi_j) N_{j,n}(t; q) \\
&= \lambda \sum_{j=0}^{l} f(\xi_j) N_{j,n}(t; q) + \beta \sum_{j=0}^{l} g(\xi_j) N_{j,n}(t; q) \\
&= \lambda \mathcal{S}_{m,n}(f, t; q) + \beta \mathcal{S}_{m,n}(g, t; q).
\end{aligned}$$

ii. Let $f(t) = at + b$ be a linear function with constant a and b, so

$$\mathcal{S}_{m,n}(f, t; q) = \sum_{j} (a\xi_j + b) N_{j,n}(t; q).$$

Since $\mathcal{S}_{m,n}$ is a linear operator we can write the last equation as

$$\mathcal{S}_{m,n}(f, t; q) = a \sum_{j} \xi_j N_{j,n}(t; q) + b \sum_{j} N_{j,n}(t; q).$$

Now consider the first term of right of the equation. From proposition (2.2.1) we have

$$a \sum_{j} \xi_j N_{j,n}(t; q) = at,$$

and the second term of right of the equation,

$$b \sum_j N_{j,n}(t; q) = b$$

which follows from proposition 2.4. Consequently we obtain

$$\mathcal{S}_{m,n}(f, t; q) = at + b = f(t)$$

and this completes the proof.

iii. For $q = 1$, q-B-splines and q-Greville abscissae reduce to classical B-splines and classical Greville abscissae respectively. Hence if $q = 1$, $\mathcal{S}_{m,n}(f, t; q)$ reduces to classical Bernstein-Schoenberg operator.

Proposition 5.4.1. *Given a convex function f on $[a, b]$. Then $\mathcal{S}_{n,m} f$ is convex for all n, m and knot sequence (t_k) if and only if $q = 1$.*

Proof. We have showed that if $q = 1$, $\mathcal{S}_{m,n}(f, t; q)$ reduces to classical Bernstein-Schoenberg operator. Thus, if f is convex on $[a, b]$, then $\mathcal{S}_{n,m} f$ is also convex for all n, m and knot sequence (t_k). This is proved in Goodman & Sharma (1985).
For the converse, our goal is to show that if $q \neq 1$, then there exist n, m and a knot sequence (t_k) such that when f is convex, $\mathcal{S}_{n,m} f$ is not convex.

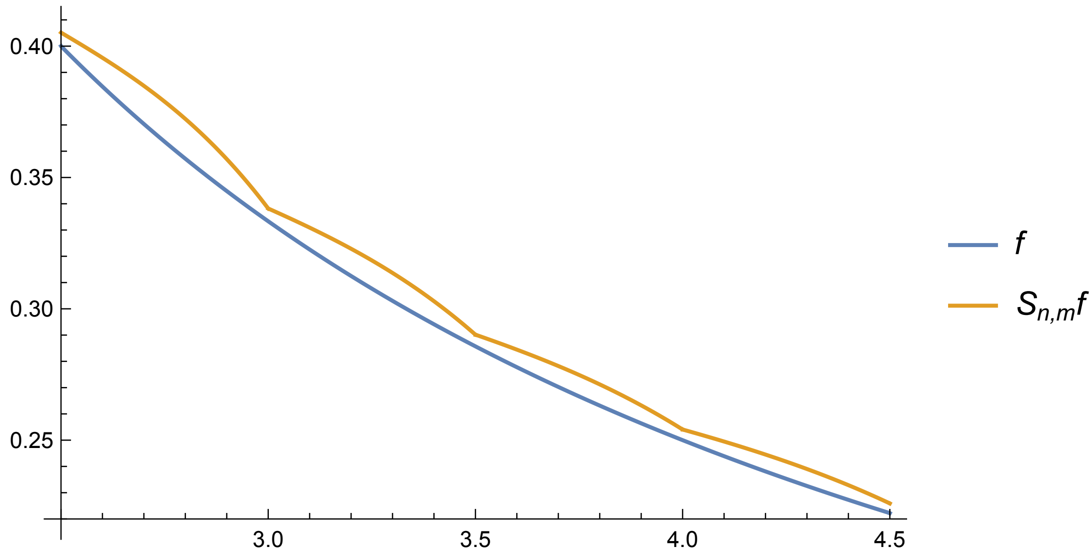

Figure 5.4 $f(t) = \frac{1}{t}$, $q = 1.6$, $n = 3$, knot sequence $(1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6)$ and $t \in [2.5, 4.5]$

The above counter example completes the proof. $\square$

CHAPTER SIX

CONCLUSION

Splines are piecewise polynomials whose derivatives up to some order agree at the joins. Spline curves and surfaces, which are used widely in Computational Science and Engineering such as industrial design and manufacture, car bodies, ship hulls, and even entertainment, computer animated movies and computer games, are useful for both interpolation and approximation. Splines are preferable since they are easy to store and fast to compute. In real life, the boundaries between bones and soft tissues, between atoms in molecules, etc. are not precisely smooth so we need tolerances, jumps, and quantum leaps to model. Thus quantum splines allow us to model tolerances, jumps and quantum leaps in the derivatives at the joins. Hence, in this thesis we aimed to develop the theory of quantum splines. The fundamental properties q-B-splines such as representation of q-B-splines in terms of divided differences, q-analogue of de Boor-Fix formula for dual functions, $1/q$-integral of q-B-splines over their support, $1/q$-convolution formula of uniform q-B-splines were derived. We gave basic theory of interpolation by q-B-splines and approximation by a q-analogue of the Bernstein-Schoenberg operator.

The Peano kernel theorem provides a powerful technique for computing the errors of approximations such as interpolation, quadrature rules and B-splines. Therefore it is useful for numerical solutions of differential equations. Hence, we introduced a q-analogue of Peano kernel theorem and give a relation between q-B-splines and q-Peano kernel.

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