T.C.
FIRAT UNIVERSITY
THE INSTITUTE OF NATURAL AND APPLIED
SCIENCES

INVESTIGATIONS OF THE EXACT AND NUMERICAL SOLUTIONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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DEDICATION

This thesis is dedicated to my parents and Engr. Dr. Rabiu Musa Kwankwaso.

Tukur Abdulkadir Sulaiman

Elazığ-2019

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ABSTRACT

Investigations of the Exact and Numerical Solutions of Some Nonlinear Partial Differential Equations

This study investigates a search for analytical solutions, exact and numerical approximations to distinct nonlinear models in distinct nonlinear science areas. The symbolic methods used to explore this search are: the sine-Gordon expansion technique, the extended sinh-Gordon expansion technique, and the finite forward difference process for the exact and numerical approximations. In addition, the extended technique of sinh-Gordon expansion is simplified in order to obtain multiple solitary wave solutions. The nonlinear models we address in this work are; the Korteweg-de Vries (KdV) equation with dual-power law nonlinearity, the coupled nonlinear Maccari's system, the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation, the resonant nonlinear Schrödinger equation with both spatio-temporal and inter-modal dispersions, the conformable space-time fractional second order nonlinear Schrödinger equation, the decoupled nonlinear Schrödinger equation arising in dual-core optical fibers, the long-short-wave interaction system, the (2+1)-dimensional nonlinear Chiral Schrödinger equation, Benjamin-Bona-Mahony equation, and the coupled Boussinesq equation. Various kind of solitary wave solutions are presented. The reported results are compared with the previous results. On the other hand, the numerical and exact approximations to some of the studied models are presented.

Keywords: Analytical Methods; Solitons Solutions; Stability Analysis; Exact and Numerical Approximations.

ÖZET

Bazı Lineer Olmayan Kısmi Diferansiyel Denklemlerin Tam ve Sayısal Çözümlerinin İncelenmesi

Bu çalışma, farklı lineer olmayan bilim alanlarında farklı lineer olmayan modellere analitik çözümler, tam ve sayısal yaklaşımlar için bir araştırmayı incelemektedir. Bu araştırmayı incelemek için kullanılan sembolik yöntemler şunlardır: sine-Gordon genleşme tekniği, genişletilmiş sinh-Gordon genleşme tekniği ve tam ve sayısal yaklaşımlar için sonlu ileri fark işlemi. Ek olarak, katlı solitary dalga çözümleri elde etmek için genişletilmiş sinh-Gordon genleşme tekniği sadeleştirilmiştir. Bu çalışmada ele aldığımız lineer olmayan modeller; dual-kuvvet yasası nonlineerite ile Korteweg-de Vries (KdV) denklemi, bağlı lineer olmayan Maccari sistemi, (2 + 1)- boyutlu Zakharov-Kuznetsov modifiye eş genişlik denklemi, hem uzay-zamanlı hem de inter-modal dağılımlı rezonant lineer olmayan Schrödinger denklemi, conformable uzay-zaman kesirli ikinci mertebeden lineer olmayan Schrödinger denklemi, çift çekirdekli optik fiberlerde ortaya çıkan ayrıştırılmış lineer olmayan Schrödinger denklemi, uzun-kısa-dalga etkileşim sistemi, (2 + 1)-boyutlu-lineer olmayan Chiral Schrödinger denklemi, Benjamin-Bona-Mahony denklemi ve birleşik Boussinesq denklemi. Solitary dalga çözümlerinin çeşitli tipleri sunulmaktadır. Rapor edilen sonuçlar önceki sonuçlarla karşılaştırılmıştır. Öte yandan, çalışılan modellerin bazılarına sayısal ve tam yaklaşımlar sunulmuştur.

Anahtar Kelimeler: Analitik Metodlar; Solitons Çözümleri; Kararlılık Analizi; Tam ve Sayısal Yaklaşımlar.

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LIST OF SYMBOLS

β	: beta
α	: alpha
Γ	: Gamma
μ	: mu
ζ	: zeta
η	: eta
$\boldsymbol{\xi}$	: xi
Φ	: Phi
θ	: theta
λ	: lambda
ψ	: psi
i	: imaginary
ϕ	: phi
ω	: omega
γ	: gamma
ϑ	: vartheta
υ	: nu
δ	: delta
Ψ	: Psi
σ	: sigma
Ω	: Omega
∞	: infinity
ϖ	: varphi

LIST OF ABBREVIATIONS

BBM : Benjamin-Bona-Mahony
Eqs. : Equations
FDM : Finite Difference Method
Figs. : Figures
GVD : Group Velocity Dispersion
IMD : Inter-Modal Dispersion
KdV : Korteweg-de Vries
MEW : Modified Equal Width
NLEE : Nonlinear Evolution Equation
NLSE : Nonlinear Schrödinger Equation
NODE : Nonlinear Ordinary Differential Equation
NPDE : Nonlinear Partial Differential Equation
SGEM : sine-Gordon Expansion Method
ShGEEM : sinh-Gordon Equation Expansion Method
STD : Spatio-Temporal Dispersion
ZK : Zakharov-Kuznetsov

1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) are often utilized to define several complicated nonlinear phenomena that arise in different areas of nonlinear sciences like plasma physics, optical fibers, chemical processes, biological sciences, mathematical physics, etc. Nonlinear partial differential equations have a broad range of roles in applied mathematics, that make it of paramount important to secure their analytical solutions and approximate numerical solutions different kinds of alternatives to this type of equations.

The general form of NPDE may be given as

$$P(u, u_x, u_x u^2, u_{xxy}, u_{xxt}, \ldots) = 0, \tag{1.1}$$

where $u = u(x, y, t)$ is unknown function, P is a polynomial in $u(x, y, t)$ and it is derivatives and the subscripts stand for the partial derivatives of $u(x, y, t)$ with respect to x, y and t.

Nonlinear Schrödinger type equations (NLSEs) are unique types of nonlinear evolution equations which may be utilized to define distinct complicated nonlinear aspects in distinct areas of nonlinear science such as nonlinear optics, water waves, photonics, quantum electronics, electromagnetism, and so on [1-6].

The NLSE is given by [7]

$$iu_t + u_{xx} \pm \beta |u|^2 u = 0. \tag{1.2}$$

The parameter β in Eq. (1.2) is a nonzero constant. The $" + "$ and $" - "$ that are attached to the nonlinear term $|u|^2 u$ are the focusing and de-focusing nonlinearities, respectively. The focusing NLSE provides a pulse-like soliton while a kink-shaped soliton is admitted by the de-focusing one. In nonlinear optics, bright and dark solitons, respectively, are called pulse-like and kink-like solitons[8].

Fractional calculus is a vital and useful mathematics branch with an extensive range of applications in different nonlinear science fields [9]. Fractional calculus can be elucidated as an distension of the approach of a derivative operator from integer order n to arbitrary order α, where α is a real, complex value or a complex-valued function $\alpha = \alpha(x, t)$ [10]. The literature contains different definitions of fractional differential equations, including the definitions Riemann-Liouville, Caputo, Grunwald-Letnikov, Atangana-Baleanu derivative in the Caputo sense, Atangana-Baleanu fractional derivative in the Riemann-Liouville sense, modified Riemann-Liouville fractional derivative [11, 12] and the conformable fractional derivative [13-15]. Recently, the fresh truncated M-fractional derivative was created by Sousa and Oliveira[16]. This fresh fractional derivative generalizes Khalil *et al.* [15] suggested conformable derivative.

Different effective techniques have been developed by multiple academics such as the simple equation method [17], the simplified Hirota's method [18], the Cole-Hopf transformation method [19], the sine-cosine method [20], the tanh method [21], the improved generalized tanh function method [22], the improved F-expansion method with Riccati equation [23, 24], the modified exp $(-\Omega(\xi))$-expansion function method [25, 26], the homogeneous balance method [27], the Homotopy analysis method [28], the simple equation method [29], the extended mapping approach [30], the differential transform method [31], the Ricatti-Bernoulli sub-ODE method [32], the biHamiltonian property [33], the Backlund Transformation [34, 35], the tanh-coth method [36, 37], the solitary wave ansatz [38], the tanh function method [39], the improved F-expansion method [40-42], the modified homogeneous balance method [43], the Homotopy Pade method [44-46], the differential transform method [47], the Ricatti-Bernoulli (RB) sub-ODE method [48], the reductive perturbation technique [49], the Ansatz method [50], the extended generalized Riccati equation mapping method [52], the exp-function method [53], the functional variable method [54], and many other integral techniques [55- 60].

1.1. Aim of the Thesis

The aim of this thesis is to investigate the analytical and numerical solutions to several nonlinear partial differential equations. Various kind of travelling wave solutions will be constructed to these nonlinear partial differential equations by using three integral schemes. On the orther hand, the simplified version and application of one of these integral schemes will be presented. Moreover, the finite different method will also be applied to generate numerical and exact approximations to some of these nonlinear partial differential equations.

1.2. Outline of Thesis

The organization of this thesis is outlined as follows: Part 1 entails the introduction about classical and nonclassical nonlinear partial differential equations. Part 2 of this thesis gives the definetions of some important terms partaining classical/nonclassical differential equations and solitary wave solutions. In part 3 we give the descriptions of the used analytic and numerical techniques in this thesis. Part 4 presents the applications of the decribed analytical and numerical methods to some nonlinear physical models. In part 5 we presents the results and discussion of the reported results in part 4. We present the conclusion of this study in part 6.

2. FUNDAMENTAL DEFINITIONS

Definition 2.1 A differential equation with partial derivatives of one or more dependent variable is called a partial differential equation with regard to more than one independent variable.

Example 2.1 The equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+ku, \quad k \ is \ \text{a constant} \tag{2.1}$$

which governs the nuclear fission, the equation

$$\frac{\partial u}{\partial t}=\beta\frac{\partial^2 u}{\partial x^2}, \quad \beta \ is \ a \ \text{constant} \tag{2.2}$$

which governs the one dimensional heat flow and the equation which governs the motion of vibrating string problems

$$\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2} \tag{2.3}$$

are examples of partial differential equations.

Definition 2.2 The **order** of a differential equation is the order that appears in the equation as the largest derivative.

Example 2.2 The (1+1)-dimensional nonlinear dispersive modified BBM equation

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}}+\frac{\partial \mathrm{u}}{\partial \mathrm{x}}-\alpha \mathrm{u}^2\frac{\partial \mathrm{u}}{\partial \mathrm{x}}+\frac{\partial^3 \mathrm{u}}{\partial \mathrm{t}^3}=0 \tag{2.4}$$

is a third order equation

Definition 2.3 A partial differential equation is said to be **linear** if:

1. the power of the dependent variable and each partial derivative enclosed in the equation is one.
2. the power of the dependent variable and each partial derivative contained in the equation is one. And, its said to be **nonlinear** if any of these conditions failed to satisfy.

Some examples of linear partial differential equation:

1. The heat equation in one dimensional space which is provided by

$$u_t = ku_{xx}, \tag{2.5}$$

where k is nonzero constant.

2. The Laplace equation which is provided by

$$u_{xx} + u_{yy} = 0. \tag{2.6}$$

3. The Telegraph equation which is provided by

$$u_{xx} = au_{tt} + bu_{tt} + cu = 0, \text{where } a \text{ and } b \text{ are constants.} \tag{2.7}$$

Some examples of nonlinear partial differential equation:

1. The Advection equation which is provided by

$$u_t + uu_x = f(x,t). \tag{2.8}$$

2. The modified KdV equation (mKdV) is given by

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{2.9}$$

3. The Boussinesq equation which is given by

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxx} = 0. \tag{2.10}$$

Definition 2.4 In addition, partial differential equations are regarded as homogenous or inhomogeneous. A partial differential equation of any order is referred as homogeneous if each term of the PDE includes the dependent variable u or one of its derivatives.

Examples of homogeneous PDE:

1. $u_t = 4u_{xx}$
2. $u_{xx} + u_{yy} = 0.$

Examples of inhomogeneous PDE:

1. $u_t = u_{xx} + x,$
2. $u_x + u_y = u + 4.$

Definition 2.5 Soliton is a localized nonlinear wave that upholds its shape when moving at steady speed and can communicate heavily with other solitons and preserve unchanged identity (except with exception of phase shift) [61, 62].

Definition 2.6 The dark soliton is a wave solution describing the waves with reduced intensity than the background, the bright soliton is a wave solution describing the waves whose maximum intensity is greater than the background [63].

Definition 2.7 The singular soliton solutions is a solitary wave with discontinuous derivatives; instances of such solitary waves are compactions, which have finite (compact) assistance, and peakons, whose peaks have a discontinuous first derivative [64, 65].

Definition 2.8 A PDE's solution is a function u that meets the equation being discussed and also meets the conditions. In other words, the left side of the PDE and the right side shall be the same when replacing the derived result for u into the equation.

Definition 2.9 Let $g\colon (0, \infty) \to \mathbb{R}$, then the conformable derivative of g of order α is provided as

$$T_\alpha(g)(t) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon t^{1-\alpha}) - g(t)}{\epsilon}, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{2.11}$$

The conformable derivative [15] bears the following properties:

1. $T_\alpha(pg + qh) = pT_\alpha(g) + qT_\alpha(h),\ p, q \in \mathbb{R}$,

2. $T_\alpha(t^\lambda) = \lambda t^{\lambda-\alpha},\ \lambda \in \mathbb{R}$,

3. $T_\alpha(gh) = gT_\alpha(h) + hT_\alpha(g)$,

4. $T_\alpha(\frac{g}{h}) = \frac{hT_\alpha(g) - gT_\alpha(h)}{h^2}$,

5. if g is differentiable, then $T_\alpha(g)(t) = t^{1-\alpha}\frac{dg}{dt}$.

Theorem 2.10 Let $g, h: (0, \infty) \to \mathbb{R}$ be differentiable and also α differentiable functions, then the following guide happens:

$$T_\alpha(g \circ h)(t) = t^{1-\alpha} h'(t) g'(h(t)). \tag{2.12}$$

Definition 2.11 Let $h: [0, \infty) \to \mathbb{R}$, then the new truncated M-derivative of h of order α is provided as

$$\mathcal{D}_M^{\alpha,\beta}\{(h)(t)\} = \lim_{\epsilon \to 0} \frac{h(t\mathbb{E}_\beta(\epsilon t^{1-\alpha})) - h(t)}{\epsilon}, \quad \forall t > 0, \quad 0 < \alpha < 1,\ \beta > 0, \tag{2.13}$$

where $\mathbb{E}_\beta(.)$ is a truncated Mittag-Leffler function of one parameter [16].

Theorem 2.12 Let $0 < \alpha \le 1,\ \beta > 0,\ q,\ r \in \mathbb{R}$, and $g,\ h\ \alpha$-differentiable at a point $t > 0$. Then:

1. $\mathcal{D}_M^{\alpha,\beta}\{(qg + rh)(t)\} = q\mathcal{D}_M^{\alpha,\beta}\{g(t)\} + r\mathcal{D}_M^{\alpha,\beta}\{h(t)\}$.

2. $\mathcal{D}_M^{\alpha,\beta}\{(g\ .\ h)(t)\} = g(t)\mathcal{D}_M^{\alpha,\beta}\{h(t)\} + h(t)\mathcal{D}_M^{\alpha,\beta}\{g(t)\}$.

3. $\mathcal{D}_M^{\alpha,\beta}\{\frac{g}{h}(t)\} = \frac{h(t)\mathcal{D}_M^{\alpha,\beta}\{g(t)\} - g(t)\mathcal{D}_M^{\alpha,\beta}\{h(t)\}}{[h(t)]^2}$.

4. $\mathcal{D}_M^{\alpha,\beta}\{c\} = 0$, where $g(t) = c$ is a constant.

5. If g is differentiable, then $\mathcal{D}_M^{\alpha,\beta}\{g(t)\} = \frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{dg(t)}{dt}$.

3. FUNDAMENTAL CONCEPTS AND METHODOLOGY

3.1. The sine-Gordon Expansion Method

In this chapter, we describe the sine-Gordon expansion technique [67, 68] by illustrating how the technique can be implemented in the search for some fresh results to different nonlinear equations.

Considering the given sine-Gordon equation [69]:

$$u_{xx} - u_{tt} = V^2 sin(u), \tag{3.1}$$

where $u = u(x,t)$ and $V \in \mathbb{R}\backslash\{0\}$.

Utilizing the wave transformation $u = u(x,\ t) = U(\zeta),\ \zeta = \mu(x - kt)$ on Eq. (3.1), provides the following nonlinear ordinary differential equation (NODE):

$$U'' = \frac{V^2}{\mu^2(1-k^2)} sin(U), \tag{3.2}$$

where $U = U(\zeta)$ and ζ stands for the amplitude of the travelling wave and k is the velocity of the travelling wave.

Multiplying both sides of Eq. (3.2) by U', provides

$$U''U' = \frac{V^2}{\mu^2(1-k^2)} U' sin(U). \tag{3.3}$$

Integrating Eq. (3.3), provides

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{V^2}{\mu^2(1-c^2)} sin^2\left(\frac{U}{2}\right) + q, \tag{3.4}$$

where q is the constant of integration.

Placing $q = 0,\ w(\zeta) = \frac{U}{2}$ and $a^2 = \frac{V^2}{\mu^2(1-k^2)}$ into Eq. (3.4), provides

$$w' = asin(w), \tag{3.5}$$

inserting $a = 1$ into Eq. (3.5), provides

$$w' = sin(w). \tag{3.6}$$

One may write Eq. (3.5) in the form

$$\frac{dw}{d\zeta} = sin(w). \tag{3.7}$$

Eq. (3.7) is a variable separable equation.

simplifying Eq. (3.7), provides the following major equations:

$$sin(w) = sin(w(\zeta)) = \frac{2ze^{\zeta}}{z^2e^{2\zeta}+1}, \tag{3.8}$$

$$cos(w) = cos(w(\zeta)) = \frac{z^2e^{2\zeta}-1}{z^2e^{2\zeta}+1}, \tag{3.9}$$

where z is the integration constant resulting from Eq. (3.7) simplification.

At $z = 1$, $sin(w) = sech(\zeta)$ and $cos(w) = tanh(\zeta)$.

To look for the solution of the following nonlinear partial differential equation:

$$P\big(u,\ u_x, u_x u^2,\ u_{xxy},\ u_{xxt},\ \dots\big) = 0, \tag{3.10}$$

we consider,

$$U(\zeta) = \sum_{i=1}^{n} tanh^{i-1}(\zeta)[B_i sech(\zeta) + A_i tanh(\zeta)] + A_0, \tag{3.11}$$

Eq. (3.11) can be re-written by using Eq. (3.8) and (3.9) accordingly as

$$U(w) = \sum_{i=1}^{n} cos^{i-1}(w)[B_i sin(w) + A_i cos(w)] + A_0. \tag{3.12}$$

By using the balancing technique on the extracted NODE, the value n can be derived by taking into account the highest order derivative and nonlinear term. We then assemble a class of algebraic equations by summing the coefficients of the trigonometric functions " $sin^i(w)cos^j(w)$, $(0 \le i \le n, 0 \le j \le n)$ " of the same power and equating each summation to zero. Explicating this class of algebraic equations, provides the values of coefficients; A_i, B_i, μ and k. Finally, when we place the values of these coefficients into Eq. (3.11), we generate up with the new results to Eq. (3.10).

3.2. The Extended ShGEEM

This section discusses the assessment of the extended sinh-Gordon equation expansion method [70-74].

We follow the following steps to apply the ShGEEM:

Step-1: Consider the following NPDE:

$$P(u, u_x, u_x u^2, u_{xxy}, u_{xxt}, \dots) = 0, \tag{3.13}$$

where F is a polynomial in q, the subscripts show that the partial derivative of q with respect to x or t.

Placing the travelling wave transformation

$$q = \Phi(\eta), \quad \eta = x - ct \tag{3.14}$$

into Eq. (3.13), the following NODE is produced:

$$Q(\Phi,\ \Phi',\ \Phi'',\ \Phi^2\Phi',\ \dots) = 0, \tag{3.15}$$

where Q is a polynomial in Φ and the superscripts show that the ordinary derivative of Φ with respect to η.

Step-2: The test result to Eq. (3.15) shall be in the form [75]

$$\Phi(\theta) = \sum_{j=1}^{k} [b_j sinh(\theta) + a_j cosh(\theta)]^j + a_0, \tag{3.16}$$

where $a_0,\ a_j,\ b_j\ (j = 1,\ 2,\ \dots,\ k)$ are constants to be secured later and θ is a function of η which meets the following ordinary differential equation:

$$\theta' = sinh(\theta). \tag{3.17}$$

To get the value of k, the homogeneous balance principle is implemented on the largest derivatives and biggest power nonlinear term in Eq. (3.15).

Eq. (3.17) is derived from the sinh-Gordon equation provided by [76]

$$q_{xt} = \lambda sinh(q). \tag{3.18}$$

Eq. (3.18) earns the following solutions [76]:

$$sinh(\theta) = \pm csch(\eta) \quad or \quad sinh(\theta) = \pm\ i\ sech(\eta) \tag{3.19}$$

and

$$cosh(\theta) = \pm coth(\eta) \quad or \quad cosh(\theta) = \pm\ \ tanh(\eta), \tag{3.20}$$

where $i = \sqrt{-1}$.

Step-3: Placing Eq. (3.16), its derivatives at a set value of k along with Eq. (3.17) into Eq. (3.15), provides a polynomial equation in $\theta' sinh^i(\theta) cosh^j(\theta)$ $(l = 0{,}1\ \ and\ \ i,$ $j = 0,\ 1,\ 2,\ \ldots)$. We generate a class of over-determined nonlinear algebraic equations in $a_0,\ a_j,\ a_j,\ c$ by setting the coefficients of $\theta'^l sinh^i(\theta) cosh^j(\theta)$ to zero.

Step-4: The guaranteed set of over-determined nonlinear algebraic equations is then simplified with the help of symbolic software to secure the values of the parameters $a_0,\ a_j,$ $b_j,\ c$.

Step-5: Using Eqs. (3.19) and (3.20), respectively, Eq. (3.15) has the following results:

$$\Phi(\eta) = \sum_{j=1}^{k} [\pm i b_j \quad sech(\eta) \pm a_j tanh(\eta)]^j + a_0, \tag{3.21}$$

$$\Phi(\eta) = \sum_{j=1}^{k} [\pm b_j\ \ csch(\eta) \pm a_j coth(\eta)]^j + a_0. \tag{3.22}$$

3.3. The Simplified Extended ShGEEM

We are presenting the overall facts of the suggested technique in this chapter.

The following assessment is performed to reach the fresh traveling wave solutions for a specified NEE using the suggested technique:

Consider this equation

$$P(u,\ u_x, u_x u^2,\ u_{xxy},\ u_{xxt},\ \ldots) = 0, \tag{3.23}$$

where u is a function of the autonomous variable x, y, t and the subscripts show the derivative of u with respect to the independent variables and P is the polynomial of the function u.

Consider the sinh-Gordon equation [76]

$$u_{xt} = \lambda sinh(u), \tag{3.24}$$

where q is the unknown function of x, t and $\lambda \in \mathbb{R}\backslash\{0\}$.

Utilizing the wave transformation below

$$u = \Psi(\xi), \quad \xi = x + \vartheta t \tag{3.25}$$

on Eq. (3.24), the following NODE is revealed:

$$\Psi'' = \frac{\lambda}{\vartheta} sinh(\Psi), \tag{3.26}$$

where $\Psi = \Psi(\xi)$, ξ is the amplitude and ϑ the velocity of the travelling wave respectively.

Multiplying both sides of Eq. (3.26), provides

$$\Psi''\Psi' = \frac{\lambda}{\vartheta}\Psi' sinh(\Psi) \tag{3.27}$$

Integrating Eq. (3.27), provides

$$\left[\left(\frac{\Psi}{2}\right)'\right]^2 = \frac{2\lambda}{\vartheta} sinh^2\left(\frac{\Psi}{2}\right) + \frac{\lambda\delta}{\vartheta}, \tag{3.28}$$

where δ is the integration constant.

Setting $\theta = \frac{\Psi}{2}$ and $\sigma = \frac{2\lambda}{\vartheta} = \frac{\lambda\delta}{\vartheta}$, provides

$$\theta' = \sqrt{\sigma}\ cosh(\theta). \tag{3.29}$$

One may write Eq. (3.29) in the form

$$\frac{d\theta}{d\xi} = \sqrt{\sigma}\ cosh(\theta). \tag{3.30}$$

Eq. (3.30) is a variables separable equation, simplifying it, provides the following important results:

$$cosh(\theta) = tan(\sqrt{\sigma}(\xi + d)) \quad or \quad cosh(\theta) = cot(\sqrt{\sigma}(\xi + d)), \tag{3.31}$$

$$sinh(\theta) = sec(\sqrt{\sigma}(\xi + d)) \quad or \quad sinh(\theta) = csc(\sqrt{\sigma}(\xi + d)), \tag{3.32}$$

where d is the integration constant.

If $\sigma = -1$ and $d = 0$, Eq. (3.3.9) and (3.3.10) change to

$$cosh(\theta) = -i\ tanh(\xi) \quad or \quad cosh(\theta) = i\ coth(\xi), \tag{3.33}$$

and

$$sinh(\theta) = sech(\xi)) \quad or \quad sinh(\theta) = i\ csch(\xi), \tag{3.34}$$

where $i = \sqrt{-1}$.

To find the some new results to Eq. (3.23), the following equation is considered:

$$\Psi(\theta) = \sum_{k=1}^{m} cosh^{k-1}(\theta)[B_k sinh(\theta) + A_k cosh(\theta)] + A_0, \tag{3.35}$$

and the group of series solutions

$$\Psi(\xi) = \sum_{k=1}^{m} tan^{k-1}(\sqrt{\sigma}(\xi + d))[B_k sec(\sqrt{\sigma}(\xi + d)) \pm A_k tan(\sqrt{\sigma}(\xi + d))] + A_0, \tag{3.36}$$

$$\Psi(\xi) = \sum_{k=1}^{m} cot^{k-1}(\sqrt{\sigma}(\xi + d))[B_k csc(\sqrt{\sigma}(\xi + d)) \pm A_k cot(\sqrt{\sigma}(\xi + d))] + A_0, \tag{3.37}$$

$$\Psi(\xi) = \sum_{k=1}^{m} tanh^{k-1}(\xi)[B_k sech(\xi) \pm i\ A_k tanh(\xi)] + A_0, \tag{3.38}$$

$$\Psi(\xi) = \sum_{k=1}^{m} coth^{k-1}(\xi)[i\ B_k csch(\xi) \pm i\ A_k coth(\xi)] + A_0. \tag{3.39}$$

The value of m is secured by balancing the maximum nonlinear power term and the maximum derivative in the transformed NODE. A group of algebraic equations is produced by setting each summation of the coefficients of $\theta' sinh^i(\theta) cosh^j(\theta)$, $(s = 0,\ 1,\ 0 \leq k \leq m,\ 0 \leq j \leq m)$ with the same power to zero. With the help of the Wolfram Mathematical Package, solving this set of algebraic equations yields the values of the coefficients. Inserting the coefficients values acquired into one Eq. (3.36), (3.37), (3.38) or (3.39) together with m, provide some new results to Eq (3.23).

3.4. The Finite Difference Method

The fundamental concept of nearly any numerical technique for approximating differential equations is to approximate the equation through an algebraic system of equations. The algebraic system of equations is cleverly set up so that the appropriate solution gives a good approximation of the solutions to the differential equation. The easiest way to generate such a system is to substitute finite differences for the derivatives in the equation.

Indeed, the fundamental concept of any finite difference scheme arises from a well-known definition ; the definition of a smooth function derivative [77,78]

$$\frac{du(x)}{dx} = \lim_{h\to 0} \frac{u(x+h)-u(x)}{h}. \tag{3.40}$$

This means that h must be small enough to get excellent approximations. The number of unknowns in the algebraic system is typically of order $O(1/h)$. We therefore need to solve very big algebraic system of equations in order to calculate excellent approximations. The differential equation can be considered from this point of view as a linear system of infinitely many unknowns. At the endpoints, the solution is known and the differential equation in the domain of the internal solution is determined.

For this study, we offer the following notes to clarify the finite forward difference technique; Δx is the spatial step, Δt is the time step, $x_i = a + i\Delta x,\ i = 0,1,2,\ldots,N$ points are the coordinates of mesh and $N = \frac{b-a}{\Delta x},\ t_j = j\Delta x,\ j = 0,1,2,\ldots,M$ and $M = \frac{T}{\Delta t}$. The $u(x,t)$ refers to the values of solution at these grid points which are $u(x_i,t_j) \approx u_{i,j}$, where $u_{i,j}$ represents the numerical solutions of the exact value of $u(x,t)$ at the point (x_i,t_j). The difference operators can be provided as

$$H_t u_{i,j} = u_{i,j+1} - u_{i,j}, \tag{3.41}$$

$$H_x u_{i,j} = u_{i+1,j} - u_{i,j}, \tag{3.42}$$

$$H_{xxx} u_{i,j} = u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}. \tag{3.43}$$

Thus, the partial derivatives can be approximated through the finite difference operators as

$$\left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{H_t u_{i,j}}{\Delta t} + O(t), \tag{3.44}$$

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{H_x u_{i,j}}{\Delta x} + O(t), \tag{3.45}$$

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{i,j} = \frac{H_{xxx} u_{i,j}}{2(\Delta x)^3} + O(t), \tag{3.46}$$

4. APPLICATIONS

4.1. The KdV Equation

In this chapter, the sine-Gordon expansion method is implemented in getting the solitary wave solutions to the KdV equation with dual-power law nonlinearity given by [79]

$$u_t + pu_x + quu_x + \alpha u^2 u_x + \beta u_{xxx} = 0. \tag{4.1}$$

Replacing the equation $u = u(x,\ t) = U(\xi),\ \xi = \mu(x - ct)$ into Eq. (4.1), we have

$$6(p - c)U + 3qU^2 + 2\alpha U^3 + 6\beta\mu^2 U'' = 0. \tag{4.2}$$

Balancing the terms U^3 and U'' in (1.1), we get $n = 1$.
With $n = 1$ Eq. (3.12) becomes

$$U(w) = B_1 sin(w) + A_1 cos(w) + A_0. \tag{4.3}$$

Differentiating Eq. (4.3) twice, we get

$$U''(w) = B_1 cos^2(w) sin(w) - B_1 sin^3(w) - 2A_1 sin^2(w) cos(w). \tag{4.4}$$

Placing Eqs. (4.3) and (4.4) into Eq. (4.2), provides

$$\begin{aligned}
-6cA_0 + 6pA_0 + 3qA_0^2 + 2\alpha A_0^3 - 6cA_1 cos(w) + 6pA_1 cos(w)& \\
- 12\beta\mu^2 A_1 sin^2(w) cos(w) + 6qA_0 A_1 cos(w) + 6\alpha A_0^2 A_1 cos(w)& \\
+ 3qA_1^2 cos^2(w) + 6\alpha A_0 A_1^2 cos^2(w) + 2\alpha A_1^3 cos^3(w) - 6cB_1 sin(w)& \\
+ 6pB_1 sin(w) + 6\beta\mu^2 B_1 cos^2(w) sin(w) - 6\beta\mu^2 B_1 sin^3(w)& \\
+ 6qA_0 B_1 sin(w) + 6\alpha A_0^2 B_1 sin(w) + 6qA_1 B_1 sin(w) cos(w)& \\
+ 12\alpha A_0 A_1 B_1 sin(w) cos(w) + 6\alpha A_1^2 B_1 cos^2(w) sin(w) + 3qB_1^2 sin^2(w)& \\
+ 6\alpha A_0 B_1^2 sin^2(w) + 6\alpha A_1 B_1^2 sin^2(w) cos(w) + 2\alpha B_1^3 sin^3(w) = 0.&
\end{aligned}$$

The following group of algebraic equations is obtained by gathering the coefficients of all trigonometric functions of the same power, and by equating the summation to zero:

$$Constant: -6cA_0 + 6pA_0 + 3qA_0^2 + 2\alpha A_0^3 + 3qA_1^2 + 6\alpha A_0 A_1^2 = 0,$$
$$sin(w): -6cB_1 + 6pB_1 - 6\beta\mu^2 B_1 + 6qA_0 B_1 + 6\alpha A_0^2 B_1 + 2\alpha B_1^3 = 0,$$
$$cos(w): -6cA_1 + 6pA_1 + 6qA_0 A_1 + 6\alpha A_0^2 A_1 + 2\alpha A_1^3 = 0,$$
$$sin(w)cos(w): 6qA_1 B_1 + 12\alpha A_0 A_1 B_1 = 0,$$
$$sin^2(w): -3qA_1^2 - 6\alpha A_0 A_1^2 + 3qB_1^2 + 6\alpha A_0 B_1^2 = 0,$$
$$sin^2(w)cos(w): -12\beta\mu^2 A_1 - 2\alpha A_1^3 + 6\alpha A_1 B_1^2 = 0,$$
$$cos^2(w)sin(w): 12\beta\mu^2 B_1 + 6\alpha A_1^2 B_1 - 2\alpha B_1^3 = 0.$$

To get the fresh solitary solution(s), $u(x,\ t)$ to Eq. (4.1), we simplified the above class of equations and replace in each case the secured values of the coefficients into Eq. (3.11).

Figs. 4.1-4.3 present the physical characteristics of the solutions.

Class-1: If

$$A_0 = -\frac{q}{2\alpha}, A_1 = 0, B_1 = -\frac{q}{\sqrt{2}\ \alpha}, \beta = \frac{q^2}{12\alpha\mu^2}, c = p - \frac{q^2}{6\alpha},$$

we get

$$u_1(x,\ t) = -\frac{1}{2\alpha}(q + \sqrt{2}\ q\ \ sech[\mu(x - t(p - \frac{q^2}{6\alpha}))]). \tag{4.5}$$

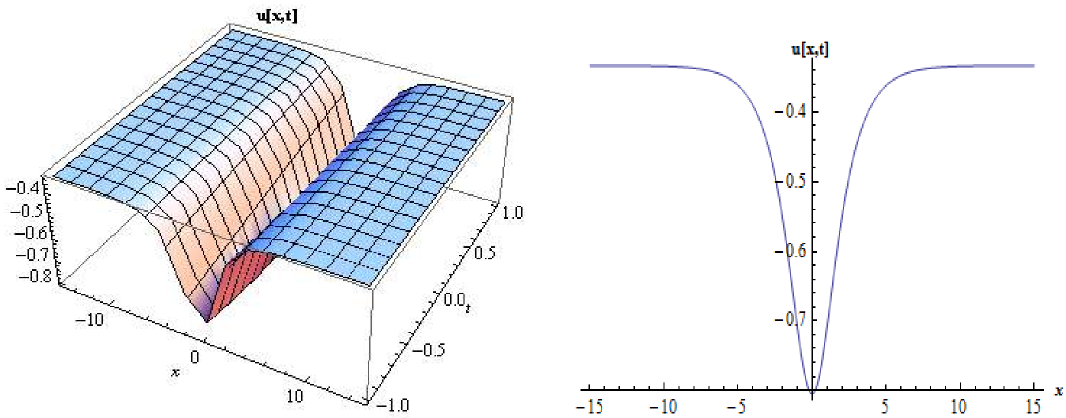


Figure 4.1. The 3D and 2D figures of Eq. (4.5).

Class-2: If

$$A_0 = -\frac{q}{2\alpha}, A_1 = -\frac{q}{2\alpha}, B_1 = -\frac{iq}{2\alpha}, \beta = -\frac{q^2}{6\alpha\mu^2}, c = p - \frac{q^2}{6\alpha},$$

we get

$$u_2(x,\ t) = -\frac{q}{2\alpha}\left(1 + i\ sech\left[\mu\left(x - t\left(p - \frac{q^2}{6\alpha}\right)\right)\right] + tanh\left[\mu\left(x - t\left(p - \frac{q^2}{6\alpha}\right)\right)\right]\right). \quad (4.6)$$

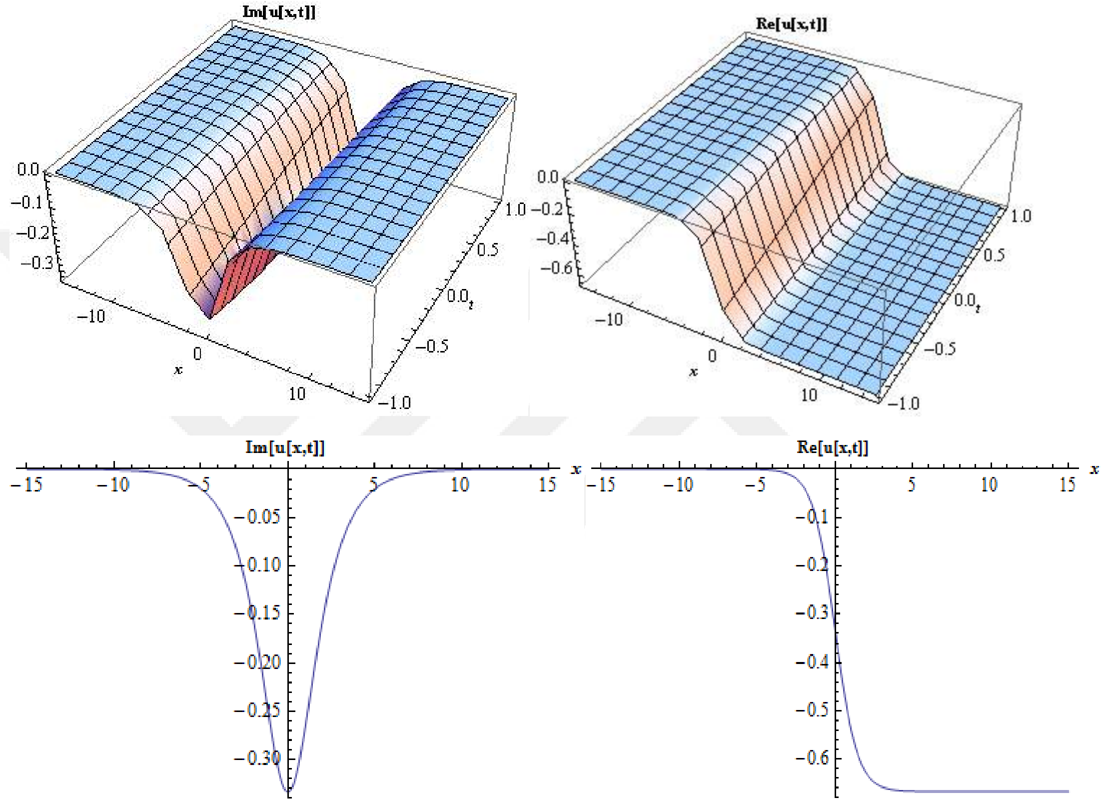


Figure 4.2. The 3D and 2D figures of Eq. (4.6).

Class-3: If

$$A_0 = -\frac{q}{2\alpha}, A_1 = \frac{q}{2\alpha}, B_1 = 0, \beta = -\frac{q^2}{24\alpha\mu^2}, c = p - \frac{q^2}{6\alpha},$$

we get

$$u_3(x,\ t) = \frac{q}{2\alpha}\left(-1 + tanh\left[\mu\left(x - t\left(p - \frac{q^2}{6\alpha}\right)\right)\right]\right). \quad (4.7)$$

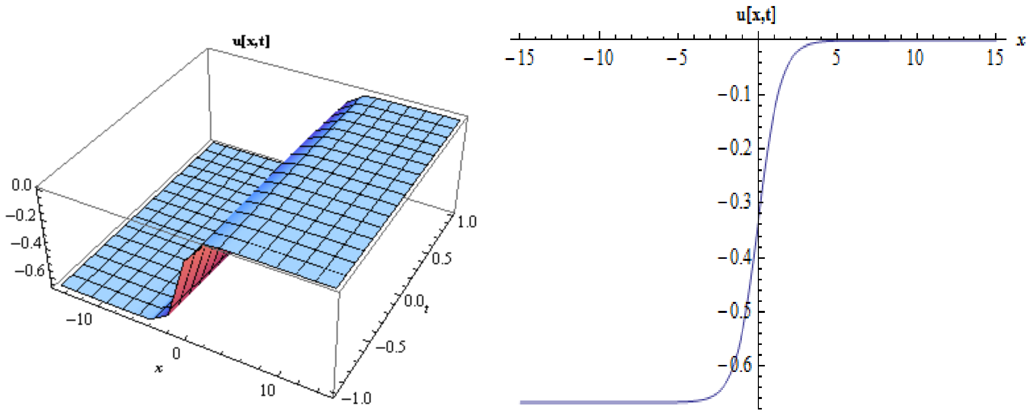


Figure 4.3. The 3D and 2D figures of Eq. (4.7).

4.2. The Coupled Nonlinear Maccari's System

The coupled nonlinear Maccari system provided by [80] in this section

$$\begin{cases} iQ_t + Q_{xx} + RQ = 0, \\ iS_t + S_{xx} + RS = 0, \\ iN_t + N_{xx} + RN = 0, \\ R_t + R_y + (|Q + S + N|^2)_x = 0, \end{cases} \tag{4.8}$$

is investigated by using the SGEM.

Assume that

$$\begin{aligned} &Q(x,\ y,\ t) = u(x,\ y,\ t)e^{\Omega}, \\ &S(x,\ y,\ t) = v(x,\ y,\ t)e^{\Omega}, \\ &N(x,\ y,\ t) = w(x,\ y,\ t)e^{\Omega}, \end{aligned} \tag{4.9}$$

where $\Omega = i(kx + \alpha y + \lambda t + l)$ in which $k,\ \alpha$, and λ are constant to be secured, l is an arbitrary constant.

Placing Eqs. (4.9) into Eq. (4.8), provides

$$\begin{cases} i(u_t + 2ku_x) + u_{xx} - (\lambda + k^2)u + uR = 0, \\ i(v_t + 2kv_x) + v_{xx} - (\lambda + k^2)v + vR = 0, \\ i(w_t + 2kw_x) + w_{xx} - (\lambda + k^2)w + wR = 0, \\ R_t + R_y + (u^2)_x = 0. \end{cases} \tag{4.10}$$

Making the conversion $u = U(\xi)$, $v = V(\xi)$, $w = W(\xi)$, $R = R(\xi)$, $\xi = \mu(x + y - 2kt)$, gives

$$\begin{cases} \mu^2 U'' - (\lambda + k^2)U + UR = 0, \\ \mu^2 V'' - (\lambda + k^2)V + VR = 0, \\ \mu^2 W'' - (\lambda + k^2)W + WR = 0, \\ \mu(1 - 2k)\frac{\partial R}{\partial \xi} + \frac{\partial (U+V+W)^2}{\partial \xi} = 0. \end{cases} \tag{4.11}$$

Integrating the 4 th part of Eq. (4.11), we get

$$R = -\frac{1}{\mu(1-2k)}(U + V + W)^2. \tag{4.12}$$

Placing Eq. (4.12) into the first three parts of Eq. (4.11), provides

$$\begin{cases} \mu^2 U'' - (\lambda + k^2)U - \frac{1}{\mu(1-2k)}(U + V + W)^2 U = 0, \\ \mu^2 V'' - (\lambda + k^2)V - \frac{1}{\mu(1-2k)}(U + V + W)^2 V = 0, \\ \mu^2 W'' - (\lambda + k^2)W - \frac{1}{\mu(1-2k)}(U + V + W)^2 W = 0. \end{cases} \tag{4.13}$$

We finally change Eq. (4.13) to NODE in U by letting

$$V = r_1 U,\ W = r_2 U \tag{4.14}$$

where r_1 and r_2 are arbitrary constants.

Placing Eq. (4.14) into Eq. (4.13), provides

$$\mu^2 U'' - (\lambda + k^2)U - \frac{1}{\mu(1-2k)}(1 + r_1 + r_2)^2 U^3 = 0. \tag{4.15}$$

Balancing the terms U'' and U^3, we get $n = 1$.
with $n = 1$ and Eq. (3.12), we reveal

$$U(w) = B_1 sin(w) + A_1 cos(w) + A_0, \tag{4.16}$$

and

$$U''(w) = B_1 cos^2(w) sin(w) - B_1 sin^3(w) - 2A_1 sin^2(w) cos(w). \tag{4.17}$$

Placing Eqs. (4.16) as well as (4.17) into Eq. (4.15), provides a polynomial of trigonometric functional powers. We secure a group of nonlinear algebraic equations by collecting the coefficients of trigonometric functions based on their distinct strengths. We acquire the values of the parameters engaged in simplifying the group of algebraic equations. So we get the following results instances:

Class-1: If

$$A_0 = 0, A_1 = 0, B_1 = -\frac{\sqrt{2}\sqrt{(2k-1)\mu^3}}{\sqrt{(1+r_1+r_2)^2}}, \lambda = \mu^2 - k^2,$$

we have

$$u_1(x,\ y,\ t) = -\frac{\sqrt{2}\sqrt{(2k-1)\mu^3}\ sech[\mu(x+y-2kt)]}{\sqrt{(1+r_1+r_2)^2}}, \tag{4.18}$$

$$v_1(x,\ y,\ t) = -\frac{\sqrt{2}\ r_1\sqrt{(2k-1)\mu^3}\ sech[\mu(x+y-2kt)]}{\sqrt{(1+r_1+r_2)^2}}, \tag{4.19}$$

$$w_1(x,\ y,\ t) = -\frac{\sqrt{2}\ r_2\sqrt{(2k-1)\mu^3}\ sech[\mu(x+y-2kt)]}{\sqrt{(1+r_1+r_2)^2}}. \tag{4.20}$$

Thus,

$$Q_1(x,\ y,\ t) = -\frac{\sqrt{2}\ e^{i(kx+\alpha y+(\mu^2-k^2)t+l)}\sqrt{(2k-1)\mu^3}\ sech[\mu(x+y-2kt)]}{\sqrt{(1+r_1+r_2)^2}}, \tag{4.21}$$

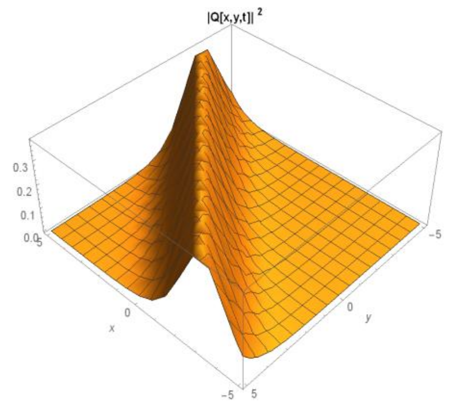
$$S_1(x,\ y,\ t) = -\frac{\sqrt{2}\ r_1\ e^{i(kx+\alpha y+(\mu^2-k^2)t+l)}\sqrt{(2k-1)\mu^3}\ sech[\mu(x+y-2kt)]}{\sqrt{(1+r_1+r_2)^2}}, \tag{4.22}$$

$$N_1(x,\ y,\ t) = -\frac{\sqrt{2}\ r_2\ e^{i(kx+\alpha y+(\mu^2-k^2)t+l)}\sqrt{(2k-1)\mu^3}\ sech[\mu(x+y-2kt)]}{\sqrt{(1+r_1+r_2)^2}}, \tag{4.23}$$

$$R_1(x,\ y,\ t)) = \frac{(1+r_1+r_2)\varpi_1}{\mu(1-2k)}\ [sech[f_1(x,\ y,\ t)]], \tag{4.24}$$

where $\varpi_1 = \frac{\sqrt{2}\sqrt{(2k-1)\mu^3}}{\sqrt{(1+r_1+r_2)^2}}$, $f_1(x,\ y,\ t) = \mu(x+y.$

Fig. 4.4 gives the physical features of Eq. (4.21).



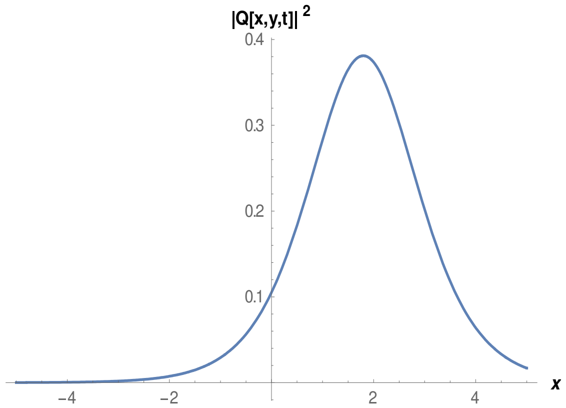


Figure 4.4. The 3D and 2D figures of Eq. (4.21).

4.3. The (2+1)-Dimensional ZK-MEW Equation

Consider the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation given by [81]

$$u_t + 3au^2u_x + b(u_{xxt} + u_{xyy}) = 0. \tag{4.24}$$

Substituting the wave transformation $u = U(\zeta)$, $\zeta = \mu(x + y - 2kt)$ into Eq. (4.24), gives

$$b\mu^2(1 - 2k)U'' + aU^3 - 2kU = 0. \tag{4.25}$$

Balancing the terms U'' and U^3, gives $n = 1$.

with $n = 1$ and Eq. (3.12), we have

$$U(w) = B_1 sin(w) + A_1 cos(w) + A_0, \tag{4.26}$$

differentiating Eq. (4.26) twice, yields

$$U'' = B_1 cos^2(w)sin(w) - B_1 sin^3(w) - 2A_1 sin^2(w)cos(w). \tag{4.27}$$

Placing Eq. (4.26) as well as (4.27) into Eq. (4.25), provides a polynomial of trigonometric functional powers. By summing the coefficients of the trigonometric functions of the same power, we obtain a class of algebraic equations and equate each summation to zero. To achieve the values of the parameters concerned, we solve the class of algebraic equations. We therefore have the following results:

Class-1: If

$$A_0 = 0, A_1 = -\frac{\sqrt{b}\ \mu}{\sqrt{a(b\mu^2 - 2)}}, B_1 = -\frac{\sqrt{b}\ \mu}{\sqrt{a(2 - b\mu^2)}}, k = \frac{1}{2} + \frac{1}{(b\mu^2 - 2)},$$

we have

$$u_1(x,\ y,\ t) = \sqrt{b}\ \mu\Big(-\frac{sech[\mu(x+y-2t(\frac{1}{2}+\frac{1}{(b\mu^2-2)}))]}{\sqrt{a(2-b\mu^2)}} - \frac{tanh[\mu(x+y-2t(\frac{1}{2}+\frac{1}{(b\mu^2-2)}))]}{\sqrt{a(b\mu^2-2)}}\Big). \tag{4.28}$$

Figs. 4.5-4.6 provide the physical features of Eqs. (4.28) and (4.29).

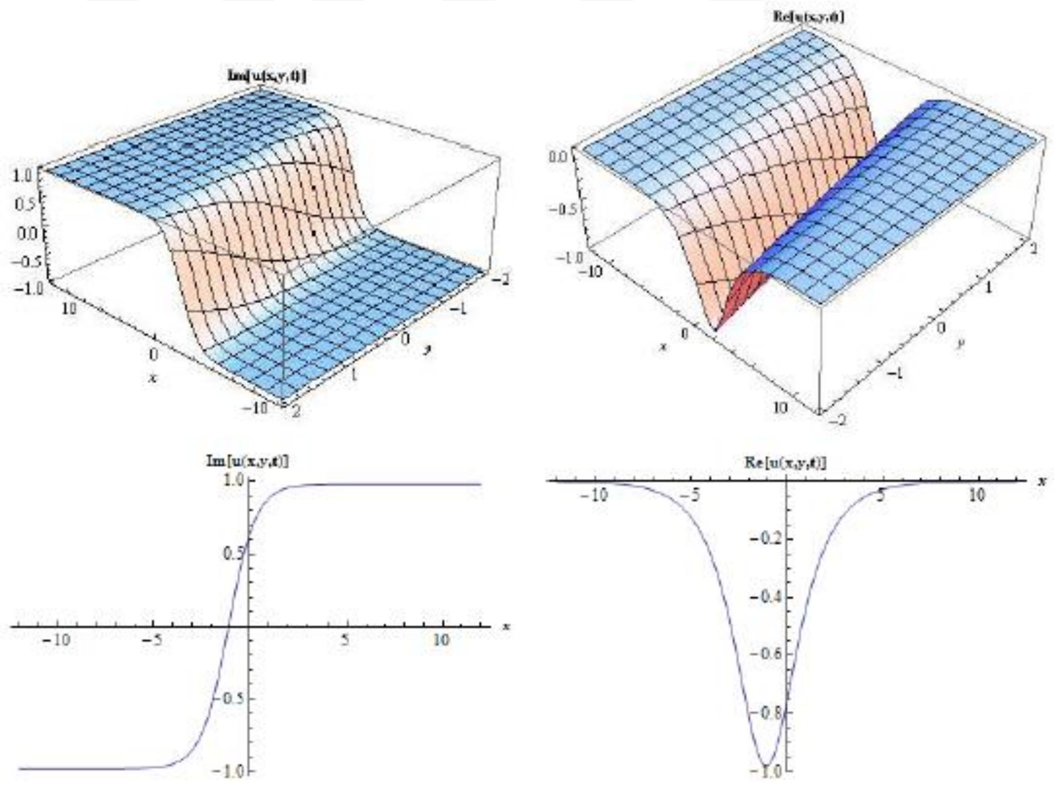


Figure 4.5. The kink-type and soliton figures of Eq. (4.28).

Class-2: If

$$A_0 = 0, A_1 = 0, B_1 = -\frac{\sqrt{2}\sqrt{b}\ \mu}{\sqrt{a(1 + b\mu^2)}}, k = \frac{b\mu^2}{2(1 + b\mu^2)},$$

we have

$$u_2(x,\ y,\ t) = -\frac{\sqrt{2}\sqrt{b}\ \mu\ sech[\mu(x+y-\frac{2bt\mu^2}{2(1+b\mu^2)})]}{\sqrt{a(1+b\mu^2)}}. \tag{4.29}$$

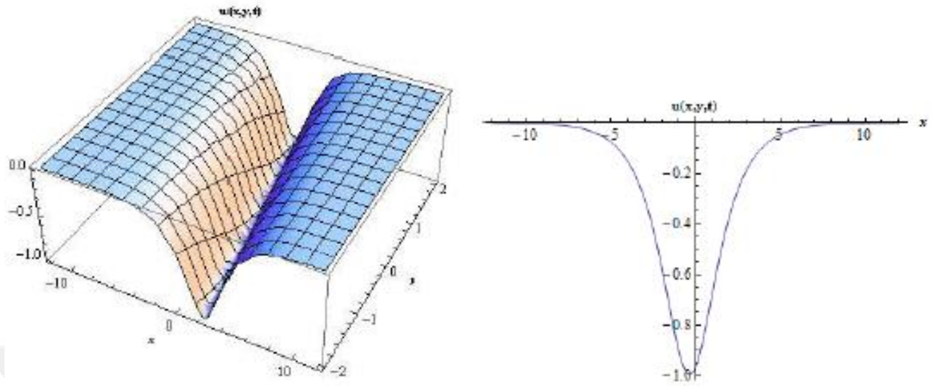


Figure 4.6. The non-topological soliton surface of Eq. (4.29).

4.4. The Resonant Nonlinear Schrödinger

In this section, the optical soliton solutions to the resonant nonlinear Schrödinger are built using the extended ShGEEM with both spatio-temporal and inter-modal dispersions under Kerr law nonlinearity.

The the resonant nonlinear Schrödinger is given by [82]

$$i(\psi_t - \delta\psi_x) + \alpha\psi_{xx} + \beta\psi_{xt} + \lambda F(|\psi|^2)\psi + \gamma(\frac{|\psi|_{xx}}{|\psi|})\psi = 0, \tag{4.30}$$

where $\psi(x,t)$ is the complex wave profile, x and t are the spatial and temporal variables, respectively. In Eq. (4.30), α, β and δ represent for the coefficients of group-velocity dispersion, STD and IMD, respectively, whereas the coefficients λ and γ represent non-Kerr law nonlinearity and resonant nonlinearity correspond to the coefficients of non-Kerr law [82]

Inserting the wave transformation

$$\psi(x,t) = \Phi(\eta)e^{i\varphi}, \quad \eta = x - ct, \quad \varphi = -\vartheta x + \omega t + \phi, \tag{4.31}$$

into Eq. (4.30), we get

$$(\alpha - \beta c + \gamma)\Phi'' + (\omega(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2)\Phi + \lambda F(\Phi^2)\Phi = 0 \tag{4.32}$$

from the real part and the relation below:

$$c = \frac{\delta + 2\alpha\vartheta - \beta\omega}{\beta\vartheta - 1} \tag{4.33}$$

from the imaginary part.

For Kerr law nonlinearity, $F(\Phi^2) = \Phi^2$, this turns Eq. (4.32) to

$$(\alpha - \beta c + \gamma)\Phi'' + (\omega(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2)\Phi + \lambda\Phi^3 = 0. \tag{4.34}$$

Balancing Φ'' and Φ^3 in Eq. (4.34), yields $k = 1$.

For $k = 1$, Eqs. (3.16), (3.21) and (3.22) take the shapes

$$\Phi(\theta) = b_1 sinh(\theta) + a_1 cosh(\theta) + a_0, \tag{4.35}$$

$$\Phi(\eta) = \pm i b_1 \; sech(\eta) \pm a_1 tanh(\eta) + a_0 \tag{4.36}$$

and

$$\Phi(\eta) = \pm b_1 \; csch(\eta) \pm a_1 coth(\eta) + a_0. \tag{4.37}$$

Placing Eq. (4.35) and Eq's (4.35) second derivative along with (3.17) into Eq. (4.34), provides a polynomial of hyperbolic functional powers. The summation of the coefficients of the same power of hyperbolic functions and the equation of each summation to zero results in a set of algebraic equations. To get the values of the parameters, the set of algebraic equations is simplified. For each case, replacing the secured values of the parameters into Eqs. (4.36) and (4.37), provides the new results to Eq. (4.30).

Class-1: If

$$a_0 = 0, \qquad a_1 = -\sqrt{\frac{\omega + \vartheta(\alpha\vartheta + \delta - \beta\omega)}{\lambda}}, \quad b_1 = 0,$$

$\gamma = (\alpha(2 + \vartheta^2 + 2\vartheta\beta - \vartheta^3\beta) + \delta(\vartheta + 2\beta - \vartheta^2\beta) + \omega + \beta\omega(-2\vartheta + \beta(\vartheta^2 - 2)))(2(\vartheta\beta - 1))^{-1}$, we have

$$\psi_{1.1}(x,t) = \pm\sqrt{\frac{\omega+\vartheta(\alpha\vartheta+\delta-\beta\omega)}{\lambda}}\,tanh[\eta], \tag{4.38}$$

$$\psi_{1.2}(x,t) = \pm\sqrt{\frac{\omega+\vartheta(\alpha\vartheta+\delta-\beta\omega)}{\lambda}}\,coth[\eta], \tag{4.39}$$

where $\lambda(\omega + \vartheta(\alpha\vartheta + \delta - \beta\omega)) > 0$ for legitimate solitons to exist.

Class-2: If

$$a_0 = 0, \qquad a_1 = 0, \quad b_1 = -\sqrt{\frac{2(-\vartheta(\alpha\vartheta + \delta) + \omega(\vartheta\beta - 1))}{\lambda}},$$

$\gamma = (\alpha(1 + \vartheta(\beta + \vartheta(\vartheta\beta - 1))) + \delta(\beta + \vartheta(\vartheta\beta - 1)) - \omega(1 + \beta(\beta + \vartheta(\vartheta\beta - 2))))(\vartheta\beta - 1)^{-1}$, we have

$$\psi_{2.1}(x,t) = \pm\sqrt{\frac{2(\vartheta(\alpha\vartheta+\delta)-\omega(\vartheta\beta-1))}{\lambda}}\,sech[\eta], \tag{4.40}$$

where $2\lambda(\vartheta(\alpha\vartheta + \delta) - \omega(\vartheta\beta - 1)) > 0$ or $2\lambda(\vartheta(\alpha\vartheta + \delta) - \omega(\vartheta\beta - 1)) < 0$ for legitimate solitons to exist.

$$\psi_{2.2}(x,t) = \pm\sqrt{\frac{2(-\vartheta(\alpha\vartheta+\delta)+\omega(\vartheta\beta-1))}{\lambda}}\,csch[\eta], \tag{4.41}$$

where $\lambda(\omega(\vartheta\beta - 1) - \vartheta(\alpha\vartheta + \delta)) > 0$ or $\lambda(\omega(\vartheta\beta - 1) - \vartheta(\alpha\vartheta + \delta)) > 0$ for legitimate solitons to exist.

Class-3: When

$$a_0 = 0, \qquad a_1 = -\sqrt{\frac{\omega + \vartheta(\alpha\vartheta + \delta - \beta\omega)}{\lambda}}, \quad b_1 = a_1,$$

$\gamma = (\alpha(1 + \vartheta(2\vartheta + \beta - 2\vartheta^2\vartheta)) + 2\omega + 2\vartheta(\delta - 2\beta\omega) + \beta(\delta - \beta\omega) + 2\vartheta^2\beta(\beta\omega - \delta))(\vartheta\beta - 1)^{-1}$, we have

$$\psi_{3.1}(x,t) = \sqrt{\frac{\omega+\vartheta(\alpha\vartheta+\delta-\beta\omega)}{\lambda}}\,(\pm i\ \ sech[\eta] \pm tanh[\eta]), \tag{4.42}$$

where $\lambda(\omega + \vartheta(\alpha\vartheta + \delta - \beta\omega)) > 0$ for legitimate solitons to exist.

Figs. 4.7-4.10 provide the physical features of Eqs. (4.38), (4.40), (4.41) and (4.42).

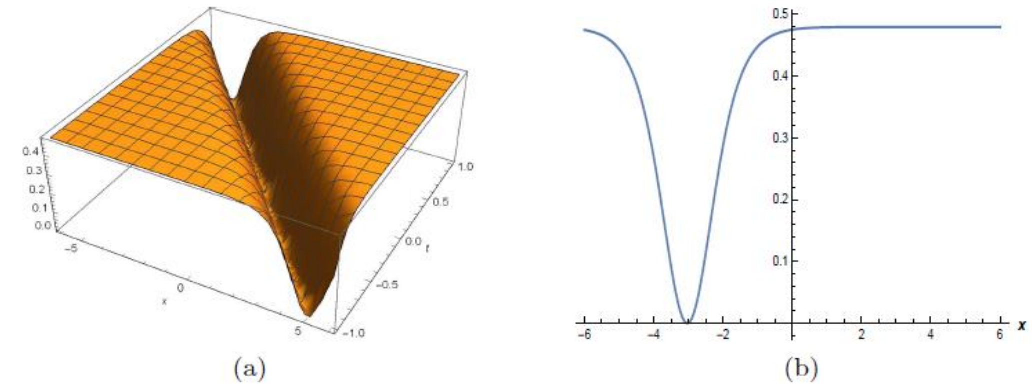


Figure 4.7. The 3D and 2D figures of Eq. (4.38).

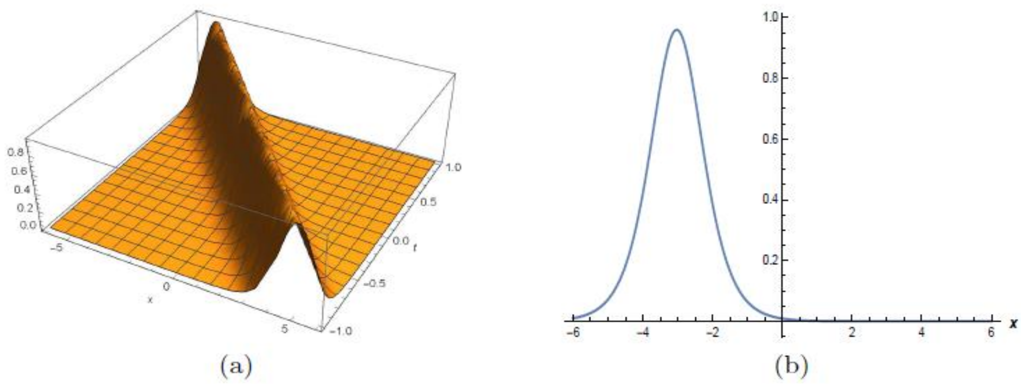


Figure 4.8. The 3D and 2D figures of Eq. (4.40).

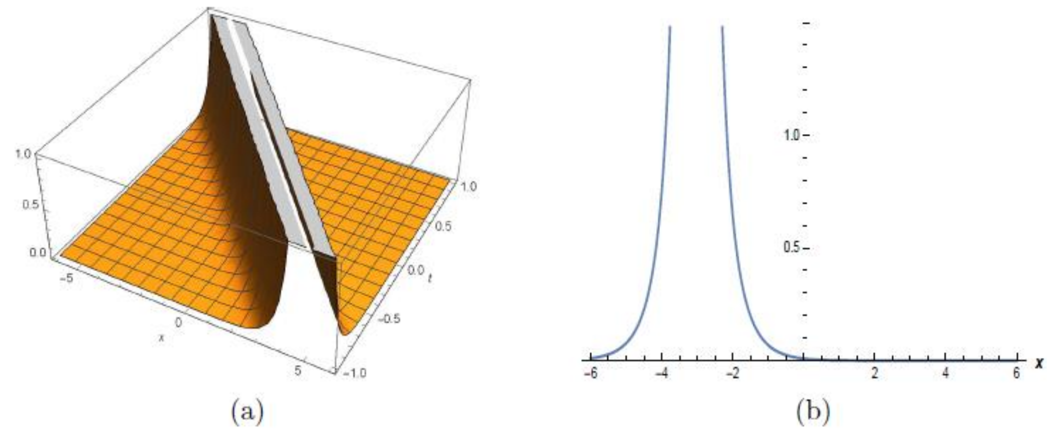


Figure 4.9. The 3D and 2D figures of Eq. (4.41).

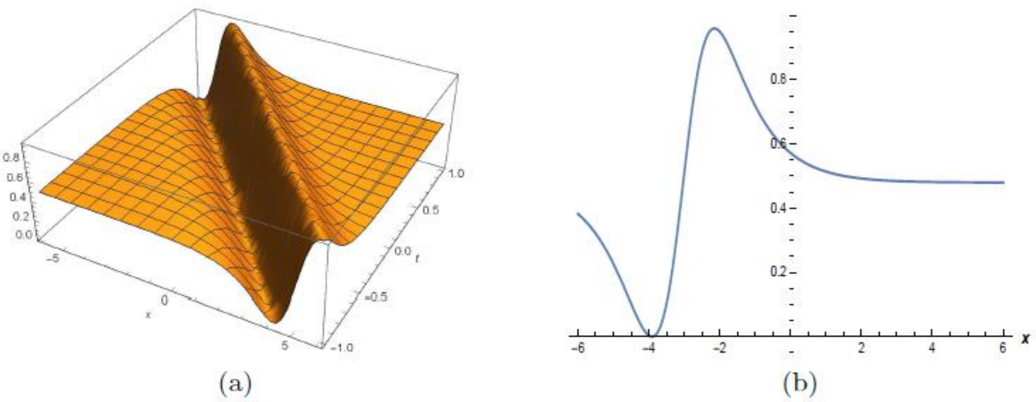


Figure 4.10. The 3D and 2D figures of Eq. (4.42).

4.5. The Second-Order Spatiotemporal Dispersion

Consider the space-time fractional second order NLSE with coefficient of group speed dispersion (GSD) and spatiotemporal dispersion (STD) provided by [83, 84].

$$i(\delta D_t^{\alpha}\psi + D_x^{\beta}\psi) + \nu D_t^{2\alpha}\psi + \gamma D_x^{2\beta}\psi + |\psi|^2\psi, \quad \alpha,\beta \in (0,1). \tag{4.43}$$

The extended ShGEEM is used in this section to build different optical solitons to Eq. (4.43).

Inserting the wave transformation

$$\psi = \Psi(\xi)e^{i\phi}, \quad \phi = p\frac{x^{\beta}}{\beta} + r\frac{t^{\alpha}}{\alpha}, \quad \xi = \frac{x^{\beta}}{\beta} + \upsilon\frac{t^{\alpha}}{\alpha} \tag{4.44}$$

into Eq. (4.43), provides the following NODE:

$$(\gamma\Delta + \nu(2p\gamma + 1)^2)\Psi'' + \Delta\Psi^3 - \Delta(r(\delta + r\nu) + p(p\gamma + 1))\Psi \tag{4.45}$$

from the real part, where $\Delta = (\delta + 2r\nu)^2$, and the relationship below:

$$\upsilon = -\frac{(2p\gamma+1)}{\delta+2r\nu} \tag{4.46}$$

from the imaginary part.

Balancing the terms Ψ'' and Ψ^3 in Eq. (4.45), provides $m = 1$.

with $m = 1$, Eqs. (3.16), (3.21) and (3.22) have the shapes

$$\Psi(\theta) = b_1 sinh(\theta) + a_1 cosh(\theta) + a_0, \tag{4.47}$$

$$\Psi(\xi) = \pm ib_1 \quad sech(\xi) \pm a_1 tanh(\xi) + a_0, \tag{4.48}$$

$$\Psi(\xi) = \pm b_1 \; csch(\xi) \pm a_1 coth(\xi) + a_0. \tag{4.49}$$

Placing Eq. (4.47) and its second derivative into Eq. (4.45), generates a polynomial in powers of hyperbolic functions. The summation of each coefficient of the same power of hyperbolic functions and the equation of each summation to zero results in an algebraic class of equations. The class of algebraic equations is solved to secure the values of the parameters. Insert the guaranteed parameters' values into Eq. (4.48) and/or Eq. (4.49) , the solutions to Eq. (4.43) are produced.

Class-1: If

$$a_0 = 0, \qquad a_1 = \sqrt{-\frac{(\nu + \gamma\delta^2)}{(4\nu(2r^2\nu + \gamma) + 8r\nu\delta + 2\delta^2)}}, \qquad b_1 = a_1,$$

$$p = -((2v(2r^2v+\gamma)+4rv\delta+\delta^2 - \sqrt{-(2rv+\delta)^2(2v(2r^2v+\gamma)+4rv\delta+\delta^2)(-1+2\gamma(\gamma+2r(rv+\delta)))}) / (2\gamma(2v(2r^2v+\gamma)+4rv\delta+\delta^2))),$$

Eq. (4.43) admits a combination of dark and bright optical soliton

$$\psi_1(x,t) = \sqrt{-\frac{(v+\gamma\delta^2)}{(4v(2r^2v+\gamma)+8rv\delta+2\delta^2)}}(\pm i\ sech[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}] \pm tanh[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}])e^{i(p\frac{x^\beta}{\beta}+r\frac{t^\alpha}{\alpha})} \tag{4.50}$$

and the paired singular soliton

$$\psi_2(x,t) = \sqrt{-\frac{(v+\gamma\delta^2)}{(4v(2r^2v+\gamma)+8rv\delta+2\delta^2)}}(\pm coth[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}] \pm csch[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}])e^{i(p\frac{x^\beta}{\beta}+r\frac{t^\alpha}{\alpha})}, \tag{4.51}$$

where $(v+\gamma\delta^2)(4v(2r^2v+\gamma)+8rv\delta+2\delta^2) < 0$ and $((2v(2r^2v+\gamma)+4rv\delta+\delta^2)(-1+2\gamma(\gamma+2r(rv+\delta)))) < 0$ for legitimate solitons to exist.

Class-2: If

$$a_0 = 0, \qquad a_1 = -\sqrt{-\frac{2(v+\gamma\delta^2)}{4r^2v^2+8v\gamma+4rv\delta+\delta^2}}, \qquad b_1 = 0,$$

$$p = -((4r^2v^2+8v\gamma+4rv\delta+\delta^2 - \sqrt{-(2rv+\delta)^2(4r^2v^2+8v\gamma+4rv\delta+\delta^2)(-1+4\gamma(2\gamma+r(rv+\delta)))})/(2\gamma(4r^2v^2+8v\gamma+4rv\delta+\delta^2))),$$

Eq. (4.43) admits the dark optical soliton

$$\psi_3(x,t) = \pm\sqrt{-\frac{2(v+\gamma\delta^2)}{4r^2v^2+8v\gamma+4rv\delta+\delta^2}}\, tanh[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}]e^{i(p\frac{x^\beta}{\beta}+r\frac{t^\alpha}{\alpha})} \tag{4.52}$$

and the singular soliton

$$\psi_4(x,t)=\pm\sqrt{-\frac{2(\nu+\gamma\delta^2)}{4r^2\nu^2+8\nu\gamma+4r\nu\delta+\delta^2}}coth[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}]e^{i(p\frac{x^\beta}{\beta}+r\frac{t^\alpha}{\alpha})}, \tag{4.53}$$

where $(\nu+\gamma\delta^2)(4r^2\nu^2+8\nu\gamma+4r\nu\delta+\delta^2)<0$ and $-(2r\nu+\delta)^2(4r^2\nu^2+8\nu\gamma+4r\nu\delta+\delta^2)(-1+4\gamma(2\gamma+r(r\nu+\delta)))<0$ for legitimate solitons to exist.

Class-3: If

$$a_0=0,\qquad a_1=0,\qquad b_1=\sqrt{\frac{2(\nu+\gamma\delta^2)}{4r^2\nu^2-4\nu\gamma+4r\nu\delta+\delta^2}},$$

$$\begin{aligned}p\\ &=-((4r^2\nu^2-4\nu\gamma+4r\nu\delta+\delta^2\\ &-\sqrt{-(2r\nu+\delta)^2(4r^2\nu^2-4\nu\gamma+4r\nu\delta+\delta^2)(-1+4\gamma(-\gamma+r(r\nu+\delta)))})/(2\gamma(4r^2\nu^2\\ &-4\nu\gamma+4r\nu\delta+\delta^2))),\end{aligned}$$

Eq. (4.43) admits the bright optical soliton

$$\psi_5(x,t)=\pm\sqrt{\frac{2(\nu+\gamma\delta^2)}{4r^2\nu^2-4\nu\gamma+4r\nu\delta+\delta^2}}sech[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}]e^{i(p\frac{x^\beta}{\beta}+r\frac{t^\alpha}{\alpha})} \tag{4.54}$$

and the singular soliton

$$\psi_6(x,t)=\pm\sqrt{\frac{2(\nu+\gamma\delta^2)}{4r^2\nu^2-4\nu\gamma+4r\nu\delta+\delta^2}}csch[\frac{x^\beta}{\beta}+\upsilon\frac{t^\alpha}{\alpha}]e^{i(p\frac{x^\beta}{\beta}+r\frac{t^\alpha}{\alpha})}, \tag{4.55}$$

where $(2r\nu+\delta)^2(4r^2\nu^2-4\nu\gamma+4r\nu\delta+\delta^2)(-1+4\gamma(-\gamma+r(r\nu+\delta)))<0$ for legitimate solitons to exist.

4.6. The Decoupled NSE in Dual-Core Optical Fibers

This section uses the extended ShGEEM to address the decoupled NSE.

The decoupled NSE is given by [85]

$$\begin{aligned} i(u_x+\lambda_1 q_t)+\lambda_2 u_{tt}+\lambda_3|u|^2u+\lambda_4 q &= 0,\\ i(q_x+\lambda_1 u_t)+\lambda_2 q_{tt}+\lambda_3|q|^2q+\lambda_4 u &= 0,\end{aligned} \tag{4.56}$$

where u and q are field envelopes, x is the proliferation co-ordinate, $\frac{1}{\lambda_1}$ is the group speed mismatch, λ_2 is the group speed dispersion, λ_4 is the linear coupling coefficient and λ_3 is provided as $\lambda_3 = \frac{2\pi m_2}{\kappa B_{eff}}$, where m_2 is the nonlinear refractive index, κ is the wavelength and B_{eff} is effective mode area of each wave length [85].

Consider the wave transformation

$$u = \Phi(\eta)e^{i\Psi},\ q = \psi(\eta)e^{i\Psi},\ \eta = x - ct,\ \Psi = -\mu x + \omega t + p, \tag{4.57}$$

where Ψ is the soliton phase component, μ is the frequency of the soliton, ω is the soliton wave number, p is the phase constant and c is the soliton speed.

Placing Eq. (4.57) into Eq. (4.56), provides

$$\begin{aligned} (\mu - \omega^2\lambda_2)\Phi + \lambda_3\Phi^3 + \lambda_4\psi - \lambda_1\omega\psi + c^2\lambda_2\Phi'' &= 0, \\ (\mu - \omega^2\lambda_2)\psi + \lambda_3\psi^3 + \lambda_4\Phi - \lambda_1\omega\Phi + c^2\lambda_2\psi'' &= 0 \end{aligned} \tag{4.58}$$

from the real part, and

$$\begin{aligned} (2c\omega\lambda_2 - 1)\Phi' + \lambda_1 c\psi' &= 0, \\ (2c\omega\lambda_2 - 1)\psi' + \lambda_1 c\Phi' &= 0 \end{aligned} \tag{4.59}$$

from the imaginary part.

Integrating Eq. (4.59) once, provides

$$\begin{aligned} (2c\omega\lambda_2 - 1)\Phi + \lambda_1 c\psi &= 0, \\ (2c\omega\lambda_2 - 1)\psi + \lambda_1 c\Phi &= 0 \end{aligned} \tag{4.60}$$

As Φ and ψ functions of η satisfy both Eqs. (4.58), (4.59) and (4.60), we provide the following relation from Eq. (4.60):

$$\frac{2c\omega\lambda_2 - 1}{\lambda_1 c} = \frac{\lambda_1 c}{2c\omega\lambda_2 - 1}. \tag{4.61}$$

Solving Eq. (4.62) for c, provides

$$c = \frac{1}{2\omega\lambda_2 - \lambda_1}. \tag{4.62}$$

Balancing the terms Φ^3, Φ'', ψ^3 and ψ'' in Eq. (4.58), provides $k = 1$.

For the value $k = 1$, Eqs. (3.16), (3.21) and (3.22) with respect to Φ and ψ have the shapes

$$\Phi(\theta) = b_1 sinh(\theta) + a_1 cosh(\theta) + a_0, \tag{4.63}$$

$$\psi(\theta) = d_1 sinh(\theta) + c_1 cosh(\theta) + c_0, \tag{4.64}$$

$$\Phi(\eta) = \pm i b_1\ sech(\eta) \pm a_1 tanh(\eta) + a_0, \tag{4.65}$$

$$\psi(\eta) = \pm i d_1\ sech(\eta) \pm c_1 tanh(\eta) + c_0, \tag{4.66}$$

$$\Phi(\eta) = \pm b_1\ csch(\eta) \pm a_1 coth(\eta) + a_0 \tag{4.67}$$

and

$$\psi(\eta) = \pm d_1\ csch(\eta) \pm c_1 coth(\eta) + c_0, \tag{4.68}$$

respectively.

Placing Eqs. (4.63), (4.64) and their second derivatives into Eq. (4.58), provides a polynomial in powers of hyperbolic functions. The summation of each hyperbolic function coefficient of the same power and the equating each summation to zero provide a class of algebraic equations. The class of algebraic equations is solved to reveal the values of the parameters. For each case, putting the revealed values of the parameters into Eqs. (4.65)-(4.68), provides the results to Eq. (4.58).

Class-1: If

$$a_0 = 0, \quad a_1 = -\frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}, \quad c_1 = -a_1, \quad c_0 = 0, \quad b_1 = 0,$$

$$d_1 = 0,$$

$$\mu = \frac{2\lambda_2 + (\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1)}{(\lambda_1 - 2\omega\lambda_2)^2},$$

we get the dark optical solitons

$$u_1(x,t) = -\frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}tanh[x - ct]e^{i(-\frac{(2\lambda_2 + (\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1))}{(\lambda_1 - 2\omega\lambda_2)^2}x + \omega t + p)}, \quad (4.69)$$

$$q_1(x,t) = \frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}tanh[x - ct]e^{i(-\frac{(2\lambda_2 + (\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1))}{(\lambda_1 - 2\omega\lambda_2)^2}x + \omega t + p)} \quad (4.70)$$

where $2\lambda_2\lambda_3 < 0$ for legitimate solitons to exist, and the singular solitons

$$u_2(x,t) = -\frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}coth[x - ct]e^{i(-\frac{(2\lambda_2 + (\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1))}{(\lambda_1 - 2\omega\lambda_2)^2}x + \omega t + p)}, \quad (4.71)$$

$$q_2(x,t) = \frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}coth[x - ct]e^{i(-\frac{(2\lambda_2 + (\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1))}{(\lambda_1 - 2\omega\lambda_2)^2}x + \omega t + p)}, \quad (4.72)$$

where $2\lambda_2\lambda_3 < 0$ for legitimate solitons to exist.

Class-2: If

$$a_0 = 0, \quad a_1 = 0, \quad c_1 = 0, \quad c_0 = 0, \quad b_1 = \frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}, \quad d_1 = -b_1,$$

$$\mu = \frac{(\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1) - \lambda_2}{(\lambda_1 - 2\omega\lambda_2)^2},$$

we get the bright optical solitons

$$u_3(x,t) = \frac{1}{\lambda_1 - 2\omega\lambda_2}\sqrt{\frac{2\lambda_2}{\lambda_3}}sech[x - ct]e^{i(-\frac{((\lambda_1 - 2\omega\lambda_2)^2(\omega^2\lambda_2 + \lambda_4 - \omega\lambda_1) - \lambda_2)}{(\lambda_1 - 2\omega\lambda_2)^2}x + \omega t + p)}, \quad (4.73)$$

$$q_3(x,t) = -\frac{1}{\lambda_1-2\omega\lambda_2}\sqrt{\frac{2\lambda_2}{\lambda_3}}sech[x-ct]e^{i(-\frac{((\lambda_1-2\omega\lambda_2)^2(\omega^2\lambda_2+\lambda_4-\omega\lambda_1)-\lambda_2)}{(\lambda_1-2\omega\lambda_2)^2}x+\omega t+p)} \quad (4.74)$$

where $2\lambda_2\lambda_3 > 0$ for legitimate solitons to exist, and the singular solitons

$$u_4(x,t) = \frac{1}{\lambda_1-2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}csch[x-ct]e^{i(-\frac{((\lambda_1-2\omega\lambda_2)^2(\omega^2\lambda_2+\lambda_4-\omega\lambda_1)-\lambda_2)}{(\lambda_1-2\omega\lambda_2)^2}x+\omega t+p)}, \quad (4.75)$$

$$q_4(x,t) = -\frac{1}{\lambda_1-2\omega\lambda_2}\sqrt{-\frac{2\lambda_2}{\lambda_3}}csch[x-ct]e^{i(-\frac{((\lambda_1-2\omega\lambda_2)^2(\omega^2\lambda_2+\lambda_4-\omega\lambda_1)-\lambda_2)}{(\lambda_1-2\omega\lambda_2)^2}x+\omega t+p)}, \quad (4.76)$$

where $2\lambda_2\lambda_3 < 0$ for legitimate solitons to exist.

Class-3: If

$$a_0 = 0, \quad a_1 = \frac{\sqrt{\lambda_2}}{\sqrt{-2\lambda_3(\lambda_1-2\omega\lambda_2)^2}}, \quad c_1 = -a_1, \quad c_0 = 0, \quad b_1 = a_1,$$

$$d_1 = -a_1,$$

$$\lambda_4 = -\frac{(\lambda_2 - 2(\lambda_1-2\omega\lambda_2)^2(\mu+\omega(\lambda_1-\omega\lambda_2)))}{2(\lambda_1-2\omega\lambda_2)^2},$$

we get the combination of the dark and bright optical solitons

$$u_5(x,t) = \frac{\sqrt{\lambda_2}}{\sqrt{-2\lambda_3(\lambda_1-2\omega\lambda_2)^2}}(i\ sech[x-ct] + tanh[x-ct])e^{i(-\mu x+\omega t+p)}, \quad (4.77)$$

$$q_5(x,t) = -\frac{\sqrt{\lambda_2}}{\sqrt{-2\lambda_3(\lambda_1-2\omega\lambda_2)^2}}(i\ sech[x-ct] + tanh[x-ct])e^{i(-\mu x+\omega t+p)}, \quad (4.78)$$

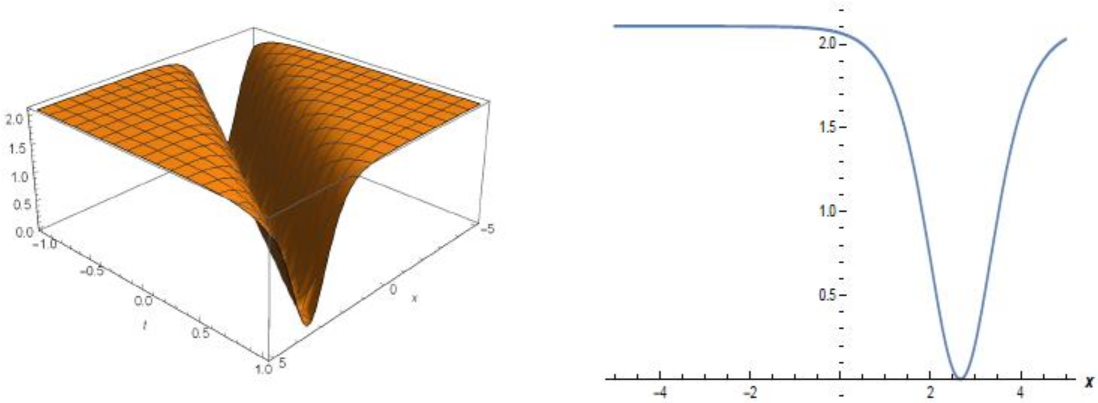
where $2\lambda_2\lambda_3(\lambda_1-2\omega\lambda_2)^2 < 0$ for legitimate solitons to exist, and the mixed singular solitons

$$u_6(x,t) = -\frac{\sqrt{\lambda_2}}{\sqrt{-2\lambda_3(\lambda_1-2\omega\lambda_2)^2}}(coth[x-ct] + csch[x-ct])e^{i(-\mu x+\omega t+p)}, \quad (4.79)$$

$$q_6(x,t) = \frac{\sqrt{\lambda_2}}{\sqrt{-2\lambda_3(\lambda_1-2\omega\lambda_2)^2}}(coth[x-ct] + csch[x-ct])e^{i(-\mu x+\omega t+p)}, \quad (4.80)$$

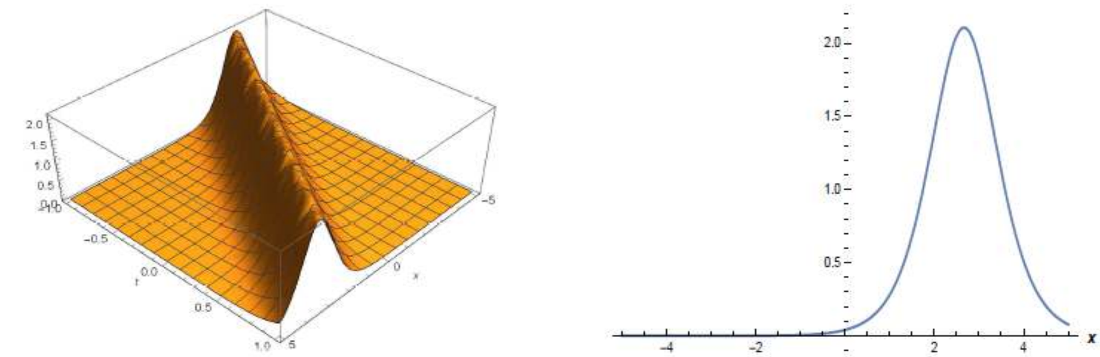
where $2\lambda_2\lambda_3(\lambda_1-2\omega\lambda_2)^2 < 0$ for legitimate solitons to exist.

Figs. 4.11-4.13 provide the physical features of Eqs. (4.70), (4.73) and (4.75).



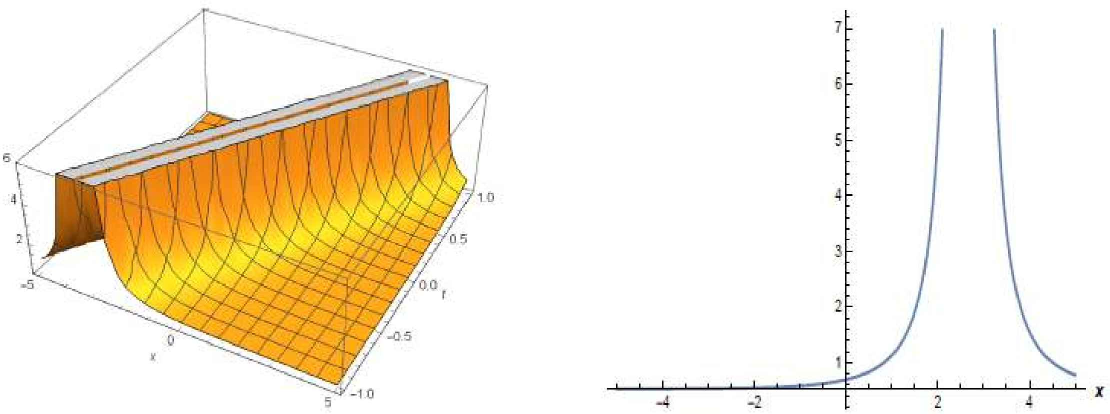
(a) (b)

Figure 4.11. The 3D and 2D figures of Eq. (4.70).



(a) (b)

Figure 4.12. The 3D and 2D figures of Eq. (4.73).



(a) (b)

Figure 4.13. The 3D and 2D figures of Eq. (4.75).

4.7. The Long-Short-Wave Interaction System

Consider the long-short-wave interaction system is given by [86]

$$\begin{aligned} iq_t + q_{xx} - qv &= 0, \\ v_t + v_x + (|q|^2)_x &= 0, \end{aligned} \tag{4.81}$$

where q is a complex function, v is a real function, x is the spatial coordinate, and t is the time respectively.

Eq. (4.71) defines the interaction in a generalized elastic medium between one longitudinal wave and one brief transverse wave. The v feature depicts the longitudinal wave and q is the slowly variable shorter transverse wave envelope [84].

Using the wave transformation:

$$q = \Psi(\xi)e^{i\Omega},\ v = V(\xi),\ \xi = x - \vartheta t,\ \Omega = px + rt. \tag{4.82}$$

Placing Eq. (4.82) into Eq. (4.81), the relation $\vartheta = 2p$ is secured from the imaginary part and the following single NODE is obtained from the real part:

$$(1 - 2p)\Psi'' + \Psi^3 - (1 - 2p)(p^2 + r)\Psi = 0. \tag{4.83}$$

Balancing the terms Ψ'' and Ψ^3 in Eq. (4.83), we get $m = 1$.

Thus, Eq. (3.35), (3.36), (3.37), (3.38) and (3.39) have the following shapes:

$$\Psi(\theta) = B_1 sinh(\theta) + A_1 cosh(\theta) + A_0, \tag{4.84}$$

$$\Psi(\xi) = B_1 sec(\sqrt{\sigma}(\xi + d)) \pm A_1 tan(\sqrt{\sigma}(\xi + d)) + A_0, \tag{4.85}$$

$$\Psi(\xi) = B_1 csc(\sqrt{\sigma}(\xi + d)) \pm A_1 cot(\sqrt{\sigma}(\xi + d)) + A_0, \tag{4.86}$$

$$\Psi(\xi) = B_1 sech(\xi) \pm i\ A_1 tanh(\xi) + A_0, \tag{4.87}$$

$$\Psi(\xi) = i\ B_1 csch(\xi) \pm i\ A_1 coth(\xi) + A_0. \tag{4.88}$$

Placing Eq. (4.84) and its second derivative into Eq. (4.83), we reach a hyperbolic function equation. We secure a class of algebraic equations after creating some hyperbolic function identity substitutions by equating each summation of the coefficients of hyperbolic functions with the same power to zero. To get the results to Eq. (4.81), we put the secured values of the coefficients into one of Eq. (4.85), (4.86), (4.87) and (4.88).

Class-I: If

$$A_0 = 0, \qquad A_1 = -\sqrt{\sigma(p - \frac{1}{2})}, \qquad B_1 = \sqrt{\sigma(p - \frac{1}{2})}, \qquad r = \frac{1}{2}(\sigma - 2p^2).$$

Class-II: If

$$A_0 = 0, \qquad A_1 = -\sqrt{(p^2 + r)(2p - 1)}, \qquad B_1 = \sqrt{(p^2 + r)(2p - 1)},$$
$$\sigma = 2(p^2 + r).$$

For each situation, the following set of results are acquired:

Set-1: Singular periodic waves solutions.

Placing the coefficients in case-I into Eq. (4.85) and (4.86), we reach

$$q_1(x,t) = \sqrt{\sigma(p - \frac{1}{2})}(sec[\sqrt{\sigma}(x - 2pt + d)] \pm tan[\sqrt{\sigma}(x - 2pt + d)])e^{i(px + \frac{1}{2}(\sigma - 2p^2)t)}, \tag{4.89}$$

$$v_1(x,t) = -\frac{\sigma(p - \frac{1}{2})}{1 - 2p}(sec[\sqrt{\sigma}(x - 2pt + d)] \pm tan[\sqrt{\sigma}(x - 2pt + d)])^2. \tag{4.90}$$

$$q_2(x,t) = \sqrt{\sigma(p - \frac{1}{2})}tan[\frac{1}{2}\sqrt{\sigma}(x - 2pt + d)]e^{i(px + \frac{1}{2}(\sigma - 2p^2)t)}, \tag{4.91}$$

$$v_2(x,t) = -\frac{\sigma(p - \frac{1}{2})}{1 - 2p}tan^2[\frac{1}{2}\sqrt{\sigma}(x - 2pt + d)]. \tag{4.92}$$

$$q_3(x,t) = \sqrt{\sigma(p - \frac{1}{2})}cot[\frac{1}{2}\sqrt{\sigma}(x - 2pt + d)]e^{i(px + \frac{1}{2}(\sigma - 2p^2)t)}, \tag{4.93}$$

$$v_3(x,t) = -\frac{\sigma(p - \frac{1}{2})}{1 - 2p}cot^2[\frac{1}{2}\sqrt{\sigma}(x - 2pt + d)], \tag{4.94}$$

where $\sigma > 0$ for legitimate solitons to exist.

Placing the coefficients in case-II into Eq. (4.85) and (4.86), we reach

$$\begin{aligned} q_4(x,t) = {} & \sqrt{(p^2 + r)(2p - 1)}(sec[\sqrt{2(p^2 + r)}(x - 2pt + d)] \\ & \pm tan[\sqrt{2(p^2 + r)}(x - 2pt + d)])e^{i(px + rt)}, \end{aligned} \tag{4.95}$$

$$v_4(x,t) = -\frac{(p^2+r)(2p-1)}{1-2p}(sec[\sqrt{2(p^2+r)}(x-2pt+d)] \pm tan[\sqrt{2(p^2+r)}(x-2pt+d)])^2, \quad (4.96)$$

$$q_5(x,t) = \sqrt{(p^2+r)(2p-1)}tan[\sqrt{\frac{p^2+r}{2}}(x-2pt+d)]e^{i(px+rt)}, \quad (4.97)$$

$$v_5(x,t) = -\frac{(p^2+r)(2p-1)}{1-2p}tan^2[\sqrt{\frac{p^2+r}{2}}(x-2pt+d)], \quad (4.98)$$

$$q_6(x,t) = \sqrt{(p^2+r)(2p-1)}cot[\sqrt{\frac{p^2+r}{2}}(x-2pt+d)]e^{i(px+rt)}, \quad (4.99)$$

$$v_6(x,t) = -\frac{(p^2+r)(2p-1)}{1-2p}cot^2[\sqrt{\frac{p^2+r}{2}}(x-2pt+d)], \quad (4.100)$$

where $2(p^2+r) > 0$ for legitimate solitons to exist.

Set-2: The soliton solutions

Putting the coefficients in case-I into Eq. (4.87) and (4.88), we reach

$$q_7(x,t) = \sqrt{p-\frac{1}{2}}(i\ sech[2pt-x] \pm tanh[2pt-x])e^{i(px-\frac{1}{2}(1+2p^2)t)}, \quad (4.101)$$

$$v_7(x,t) = -\frac{(p-\frac{1}{2})}{1-2p}(i\ sech[2pt-x] \pm tanh[2pt-x])^2. \quad (4.102)$$

$$q_8(x,t) = \sqrt{p-\frac{1}{2}}tanh[\frac{1}{2}(x-2pt)]e^{i(px-\frac{1}{2}(1+2p^2)t)}, \quad (4.103)$$

$$v_8(x,t) = -\frac{(p-\frac{1}{2})}{1-2p}tanh^2[\frac{1}{2}(x-2pt)]. \quad (4.104)$$

Placing the coefficients in case-II into Eq. (4.87) and (4.88), we reach

$$q_9(x,t) = \frac{1}{\sqrt{2}}\sqrt{-1+\sqrt{-2-4r}}(i\ sech[\sqrt{2(-1-2r)}\ t-x]$$
$$\pm tanh[\sqrt{2(-1-2r)}\ t-x])e^{i(\frac{\sqrt{-1-2r}}{\sqrt{2}}x+rt)}, \tag{4.105}$$

$$v_9(x,t) = -\frac{(-1+\sqrt{-2-4r})}{2(1-\sqrt{2(-1-2r)})}(i\ sech[\sqrt{2(-1-2r)}\ t-x]$$
$$\pm tanh[\sqrt{2(-1-2r)}\ t-x])^2, \tag{4.106}$$

$$q_{10}(x,t) = \frac{1}{\sqrt{2}}\sqrt{-1+\sqrt{-2-4r}}\ tanh[\tfrac{1}{2}(x-\sqrt{2(-1-2r)}\ t)]e^{i(\frac{\sqrt{-1-2r}}{\sqrt{2}}x+rt)}, \tag{4.107}$$

$$v_{10}(x,t) = -\frac{(-1+\sqrt{-2-4r})}{2(1-\sqrt{2(-1-2r)})}tanh^2[\tfrac{1}{2}(x-\sqrt{2(-1-2r)}\ t)], \tag{4.108}$$

where $r \leq -1$ for legitimate solitons to exist.

Set-3: The combination of singular soliton solutions

Placing the coefficients in case-I into Eq. (4.88), we reach

$$q_{11}(x,t) = \sqrt{p-\frac{1}{2}}(coth[2pt-x]+csch[2pt-x])e^{i(px-\frac{1}{2}(1+2p^2)t)}, \tag{4.109}$$

$$v_{11}(x,t) = -\frac{(p-\frac{1}{2})}{1-2p}(coth[2pt-x]+csch[2pt-x])^2. \tag{4.110}$$

Putting the coefficients in case-II into Eq. (4.7.8), we reach

$$q_{12}(x,t) = \frac{1}{\sqrt{2}}\sqrt{-1+\sqrt{-2-4r}}(coth[\sqrt{2(-1-2r)}\ t-x]$$
$$+csch[\sqrt{2(-1-2r)}\ t-x])e^{i(\frac{\sqrt{-1-2r}}{\sqrt{2}}x+rt)}, \tag{4.111}$$

$$v_{12}(x,t) = -\frac{(-1+\sqrt{-2-4r})}{2(1-\sqrt{2(-1-2r)})}(coth[\sqrt{2(-1-2r)}\ t-x]$$
$$+csch[\sqrt{2(-1-2r)}\ t-x])^2, \tag{4.112}$$

where $r \leq -1$ for legitimate solitons to exist.

Figs. 4.14-4.17 provide the physical features of Eqs. (4.89), (4.101), (4.103) and (4.109).

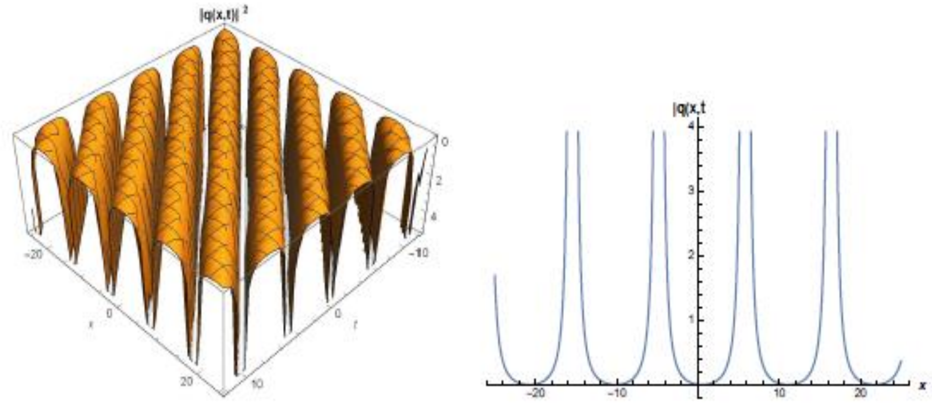


Figure 4.14. The 3D and 2D figures of Eq. (4.89).

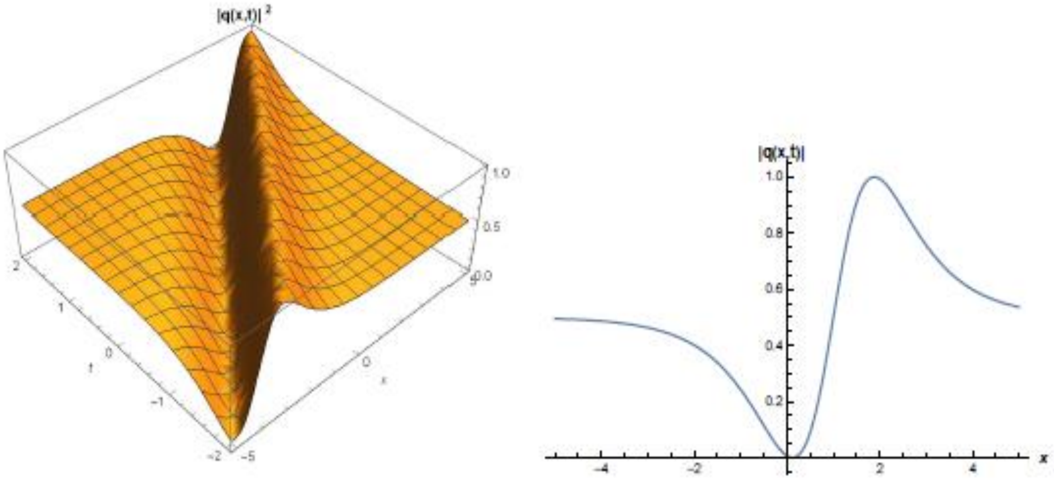


Figure 4.15. The 3D and 2D figures of Eq. (4.101).

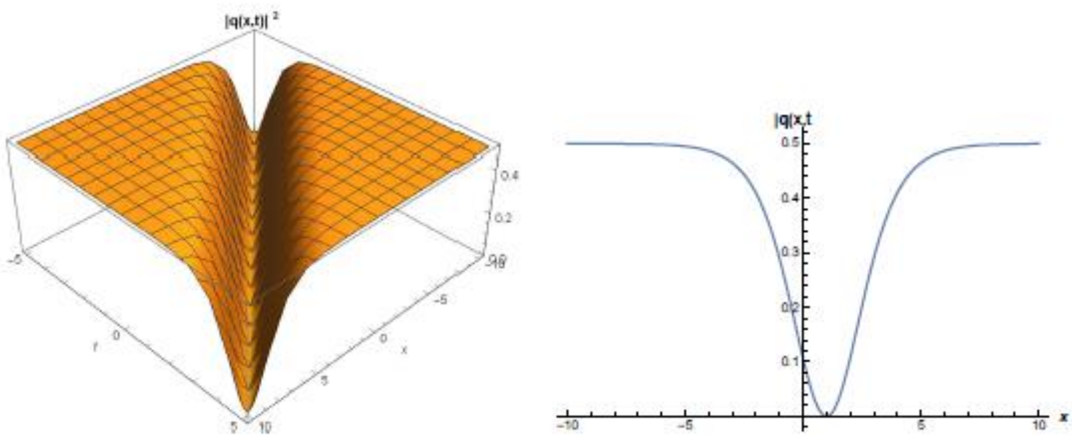


Figure 4.16. The 3D and 2D figures of Eq. (4.103).

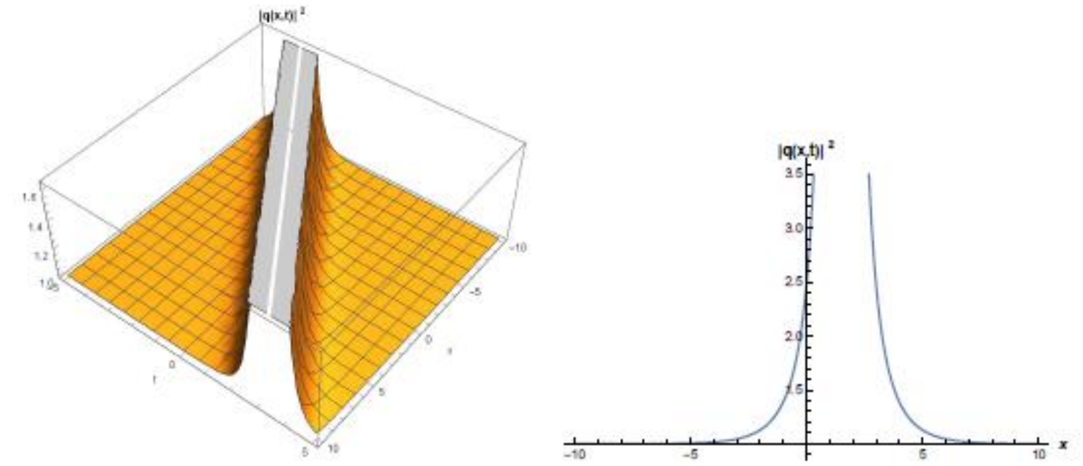


Figure 4.17. The 3D and 2D figures of Eq. (4.109).

4.8. The (2+1)-dimensional Chiral NSE

Consider the (2+1)-dimensional Chiral nonlinear Schrödinger equation is provided by [87]

$$iq_t + a(q_{xx} + q_{yy}) + i(b_1(qq_x^* - q^*q_x) + b_2(qq_y^* - q^*q_y))q = 0, \tag{4.113}$$

where q is the complex function of x and t, a is the coefficient of the dispersion terms and $b_1,\ b_2$ are nonlinear coupling constants.

The (2 + 1)-dimensional Chiral nonlinear Schrödinger equation defines the edge countries of the fractional impact of the quantum hall [87] Several studies on these models have been carried out.

Consider the following complex wave transformation:

$$q = \Psi(\xi)e^{i\Omega},\ \xi = \alpha x + \beta y - \vartheta t,\ \Omega = rx + sy + \omega t + \varphi \tag{4.114}$$

into Eq. (4.113), we get the relation

$$\vartheta = 2a(\alpha r + \beta s) \tag{4.115}$$

from the imaginary part and

$$a(\alpha^2 + \beta^2)\Psi'' + 2(rb_1 + sb_2)\Psi^3 - (a(r^2 + s^2) + \omega)\Psi = 0 \tag{4.116}$$

from the real part.

Balancing the terms Ψ'' and Ψ^3 in Eq. (4.116), provides $m = 1$.

Thus, the solutions of Eq. (4.113) take the same shapes with the solutions of Eq. (4.81) submitted in subsection **4.7** (that is Eq. (4.85), (4.86), (4.87) and (4.88)).

Continuing as before, we have the following cases:

Case-I:

$$A_0 = 0, \qquad A_1 = B_1 = -\frac{\sqrt{-\sigma\omega(\alpha^2+\beta^2)}}{\sqrt{2(-2s^2-2r^2+\sigma(\alpha^2+\beta^2))(rb_1+sb_2)}},$$

$$a = \frac{2\omega}{\sigma(\alpha^2+\beta^2)-2s^2-2r^2}.$$

Case-II: $A_0 = 0,\ A_1 = B_1 = -\frac{\sqrt{a^{\frac{3}{2}}\sigma(\alpha^2+\beta^2)\left(\sqrt{2\left(a\left(-2s^2+\sigma(\alpha^2+\beta^2)\right)-2\omega\right)}\ b_1+2sb_2\sqrt{a}\right)}}{2\sqrt{\left(a\left(-2s^2+\sigma(\alpha^2+\beta^2)\right)-2\omega\right)b_1^2-2as^2b_2^2}},$

$$r = -\frac{\sqrt{a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega}}{\sqrt{2a}}.$$

When these coefficients with Eq. (4.85), (4.86), (4.87) and (4.88)) are utilized, we pursue the following set of solutions:

Set-1: The singular periodic wave solutions

Placing the coefficients in case-I into Eq. (4.85) and (4.86), provide

$$\begin{aligned} q_1(x,y,t) &= \frac{\sqrt{-\sigma\omega(\alpha^2+\beta^2)}}{\sqrt{2(-2s^2-2r^2+\sigma(\alpha^2+\beta^2))(rb_1+sb_2)}} \\ &\times (\pm tan[\sqrt{\sigma}(d+\alpha x+\beta y-\frac{4\omega(\alpha r+\beta s)}{\sigma(\alpha^2+\beta^2)-2s^2-2r^2}t)] \\ &-sec[\sqrt{\sigma}(d+\alpha x+\beta y-\frac{4\omega(\alpha r+\beta s)}{\sigma(\alpha^2+\beta^2)-2s^2-2r^2}t)])e^{i(rx+sy+\omega t+\varphi)}, \end{aligned} \tag{4.117}$$

$$\begin{aligned} q_2(x,y,t) &= -\frac{\sqrt{-\sigma\omega(\alpha^2+\beta^2)}}{\sqrt{2(-2s^2-2r^2+\sigma(\alpha^2+\beta^2))(rb_1+sb_2)}} \\ &\times cot[\tfrac{1}{2}\sqrt{\sigma}(d+\alpha x+\beta y-\frac{4\omega(\alpha r+\beta s)}{\sigma(\alpha^2+\beta^2)-2s^2-2r^2}t)]e^{i(rx+sy+\omega t+\varphi)}, \end{aligned} \tag{4.118}$$

$$\begin{aligned} q_3(x,y,t) &= -\frac{\sqrt{-\sigma\omega(\alpha^2+\beta^2)}}{\sqrt{2(-2s^2-2r^2+\sigma(\alpha^2+\beta^2))(rb_1+sb_2)}} \\ &\times tan[\tfrac{1}{2}\sqrt{\sigma}(d+\alpha x+\beta y-\frac{4\omega(\alpha r+\beta s)}{\sigma(\alpha^2+\beta^2)-2s^2-2r^2}t)]e^{i(rx+sy+\omega t+\varphi)}, \end{aligned} \tag{4.119}$$

where $\sigma > 0$ for legitimate solitons to exist.

Placing the coefficients in case-II into Eq. (4.85) and (4.86), provide

$$\begin{aligned}
q_4(x,y,t) = {} & \frac{\sqrt{a^{3/2}\sigma(\alpha^2+\beta^2)(\sqrt{2(a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega)}\; b_1+2sb_2\sqrt{a})}}{2\sqrt{(a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega)b_1^2-2as^2b_2^2}} \\
& \times (\pm tan[\sqrt{\sigma}(d+\alpha x+\beta y-2a(\beta s-\frac{\alpha\sqrt{a(\sigma(\alpha^2+\beta^2)-2s^2)-2\omega}}{\sqrt{2a}})t)] \\
& -sec[\sqrt{\sigma}(d+\alpha x+\beta y-2a(\beta s-\frac{\alpha\sqrt{a(\sigma(\alpha^2+\beta^2)-2s^2)-2\omega}}{\sqrt{2a}})t)]) \\
& \times e^{i(sy-\frac{\sqrt{a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega}}{\sqrt{2a}}x+\omega t+\varphi)},
\end{aligned} \tag{4.120}$$

$$\begin{aligned}
q_5(x,y,t) = {} & -\frac{\sqrt{a^{3/2}\sigma(\alpha^2+\beta^2)(\sqrt{2(a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega)}\; b_1+2sb_2\sqrt{a})}}{2\sqrt{(a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega)b_1^2-2as^2b_2^2}} \\
& \times cot[\frac{1}{2}\sqrt{\sigma}(d+\alpha x+\beta y-2a(\beta s-\frac{\alpha\sqrt{a(\sigma(\alpha^2+\beta^2)-2s^2)-2\omega}}{\sqrt{2a}})t)] \\
& \times e^{i(sy-\frac{\sqrt{a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega}}{\sqrt{2a}}x+\omega t+\varphi)},
\end{aligned} \tag{4.121}$$

$$\begin{aligned}
q_6(x,y,t) = {} & -\frac{\sqrt{a^{3/2}\sigma(\alpha^2+\beta^2)(\sqrt{2(a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega)}\; b_1+2sb_2\sqrt{a})}}{2\sqrt{(a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega)b_1^2-2as^2b_2^2}} \\
& \times tan[\frac{1}{2}\sqrt{\sigma}(d+\alpha x+\beta y-2a(\beta s-\frac{\alpha\sqrt{a(\sigma(\alpha^2+\beta^2)-2s^2)-2\omega}}{\sqrt{2a}})t)] \\
& \times e^{i(sy-\frac{\sqrt{a(-2s^2+\sigma(\alpha^2+\beta^2))-2\omega}}{\sqrt{2a}}x+\omega t+\varphi)},
\end{aligned} \tag{4.122}$$

where $\sigma,\ 2a(a(\sigma(\alpha^2+\beta^2)-2s^2)-2\omega) > 0$ for legitimate solitons to exist.

Set-2: Combination of dark and bright soliton solutions

Placing the coefficients in case-I into Eq. (4.87), provides

$$q_7(x,y,t) = \frac{\sqrt{\omega(\alpha^2+\beta^2)}}{\sqrt{2(2s^2+2r^2+\alpha^2+\beta^2)(rb_1+sb_2)}}$$
$$\times (i\ sech[\alpha x + \beta y + \frac{4(\alpha r+\beta s)}{2s^2+2r^2+\alpha^2+\beta^2} t] \tag{4.123}$$
$$\pm tanh[\alpha x + \beta y + \frac{4(\alpha r+\beta s)}{2s^2+2r^2+\alpha^2+\beta^2} t])e^{i(rx+sy+\omega t+\varphi)},$$

where $2\omega(\alpha^2+\beta^2)(2s^2+2r^2+\alpha^2+\beta^2)(rb_1+sb_2) > 0$ for legitimate solitons to exist.

Placing the coefficients in case-II into Eq. (4.87), provides

$$q_8(x,y,t) = \frac{\sqrt{-a^{3/2}(\alpha^2+\beta^2)(\sqrt{-2a(2s^2+\alpha^2+\beta^2)-4\omega}\ b_1+2sb_2\sqrt{a})}}{2\sqrt{(a(2s^2+\alpha^2+\beta^2)+2\omega)b_1^2+2as^2b_2^2}}$$
$$\times (i\ sech[\alpha x + \beta y - 2a(\beta s - \frac{\alpha\sqrt{-a(2s^2+\alpha^2+\beta^2)-2\omega}}{\sqrt{2a}})t] \tag{4.124}$$
$$\pm tanh[\alpha x + \beta y - 2a(\beta s - \frac{\alpha\sqrt{-a(2s^2+\alpha^2+\beta^2)-2\omega}}{\sqrt{2a}})t])$$
$$\times e^{i(sy-\frac{\sqrt{-a(2s^2\alpha^2+\beta^2)}}{\sqrt{2a}}x+\omega t+\varphi)},$$

where $a^{3/2}(\alpha^2+\beta^2) < 0,\ (2a(2s^2+\alpha^2+\beta^2)+4\omega) < 0,\ 2a(a(2s^2+\alpha^2+\beta^2)+2\omega) < 0, (a(2s^2+\alpha^2+\beta^2)+2\omega)b_1^2+2as^2b_2^2 > 0\ \ and\ \ a > 0$ for legitimate solitons to exist.

Set-3: The dark soliton solutions

Placing the coefficients in case-I into Eq. (4.88), provides

$$q_9(x,y,t) = \frac{\sqrt{\omega(\alpha^2+\beta^2)}}{\sqrt{2(2s^2+2r^2+\alpha^2+\beta^2)(rb_1+sb_2)}}$$
$$\times tanh[\frac{1}{2}(\alpha x + \beta y + \frac{4\omega(\alpha r+\beta s)}{2s^2+2r^2+\alpha^2+\beta^2} t)]e^{i(rx+sy+\omega t+\varphi)}, \tag{4.125}$$

where $2\omega(\alpha^2+\beta^2)(2s^2+2r^2+\alpha^2+\beta^2)(rb_1+sb_2) > 0$ for legitimate solitons to exist.

Placing the coefficients in case-II into Eq. (4.88), provides

$$q_{10}(x,y,t) = \frac{\sqrt{-a^{3/2}(\alpha^2+\beta^2)(\sqrt{-2a(2s^2+\alpha^2+\beta^2)-4\omega}\ b_1+2sb_2\sqrt{a})}}{2\sqrt{(a(2s^2+\alpha^2+\beta^2)+2\omega)b_1^2+2as^2b_2^2}}$$
$$\times\, tanh[\tfrac{1}{2}(\alpha x+\beta y-2a(\beta s-\tfrac{\alpha\sqrt{-a(2s^2+\alpha^2+\beta^2)-2\omega}}{\sqrt{2a}})t)] \tag{4.126}$$
$$\times\, e^{i(sy-\frac{\sqrt{-a(2s^2\alpha^2+\beta^2)}}{\sqrt{2a}}x+\omega t+\varphi)},$$

where $a^{3/2}(\alpha^2+\beta^2)<0$, $(2a(2s^2+\alpha^2+\beta^2)+4\omega)<0$, $2a(a(2s^2+\alpha^2+\beta^2)+2\omega)<0$, $(a(2s^2+\alpha^2+\beta^2)+2\omega)b_1^2+2as^2b_2^2>0$ and $a>0$ for legitimate solitons to exist.

Set-4: Singular soliton solutions

Placing the coefficients in case-I into Eq. (4.88), provides

$$q_{11}(x,y,t) = -\frac{\sqrt{\omega(\alpha^2+\beta^2)}}{\sqrt{2(2s^2+2r^2+\alpha^2+\beta^2)(rb_1+sb_2)}}$$
$$\times\, coth[\tfrac{1}{2}(\alpha x+\beta y+\tfrac{4\omega(\alpha r+\beta s)}{2s^2+2r^2+\alpha^2+\beta^2}t)]e^{i(rx+sy+\omega t+\varphi)}. \tag{4.127}$$

Placing the coefficients in case-I into Eq. (4.7.8), provides

$$q_{12}(x,y,t) = -\frac{\sqrt{-a^{3/2}(\alpha^2+\beta^2)(\sqrt{-2a(2s^2+\alpha^2+\beta^2)-4\omega}\ b_1+2sb_2\sqrt{a})}}{2\sqrt{(a(2s^2+\alpha^2+\beta^2)+2\omega)b_1^2+2as^2b_2^2}}$$
$$\times\, coth[\tfrac{1}{2}(\alpha x+\beta y-2a(\beta s-\tfrac{\alpha\sqrt{-a(2s^2+\alpha^2+\beta^2)-2\omega}}{\sqrt{2a}})t)] \tag{4.128}$$
$$\times\, e^{i(sy-\frac{\sqrt{-a(2s^2\alpha^2+\beta^2)}}{\sqrt{2a}}x+\omega t+\varphi)},$$

where $2a(a(2s^2+\alpha^2+\beta^2)+2\omega)<0$ for legitimate solitons to exist.

4.9. The Benjamin-Bona-Mahony Equation

The approximate numerical and exact solutions to the equation Benjamin-Bona-Mahony are provided in this section. The Benjamin-Bona-Mahony equation is provided by [88]

$$u_t+pu_x+quu_x+ru^2u_x+su_{xxx}=0, \tag{4.129}$$

where p,q,r and s are arbitrary nonzero constants.

Eq. (9.9.1) can be provided in the form of the finite difference method's operator as

$$\frac{H_t u_{i,j}}{\Delta t} + p\frac{H_x u_{i,j}}{\Delta x} + q u_{i,j}\frac{H_x u_{i,j}}{\Delta x} + r(u_{i,j})^2\frac{H_x u_{i,j}}{\Delta x} + s\frac{H_{xxx} u_{i,j}}{2(\Delta x)^3} = 0. \tag{4.130}$$

Placing Eq. (3.41), (3.42) and (3.43) into Eq. (4.129), provides the following indexed form.

$$u_{i+1,j} = \frac{1}{2(\Delta t)(q^2 - r(\Delta x)^2(p + q u_{i,j} + r u_{i,j}^2))}$$
$$(2r(\Delta x)^3(u_{i,j+1} - u_{i,j}) -$$
$$(\Delta t)(q^2 u_{i-2,j} - 2q^2 u_{i-1,j} + 2pr(\Delta x)^2 u_{i,j} + 2qr(\Delta x)^2 u_{i,j}^2 + 2r^2(\Delta x)^2 u_{i,j}^3 - q^2 u_{i+2,j}))$$
$$(4.131)$$

where the initial values $u_{i,0} = u_0(x_i)$.

4.9.1. Von-Neumann Stability Analysis

Here, we study the stability of the Benjami-Bona-Mahony equation by utilizing the Von-Neumann stability analysis. When the Fourier method of analyzing stability is utilized, and ξ^n is taken as the amplification factor, the growth factor of a typical Fourier mode is provided as:

$$u_m^n = \xi^n e^{im\varphi}, \tag{4.132}$$

where $i = \sqrt{-1}$.

To check the stability of the numerical technique, the nonlinear term $(u + u^2)u_x$ in the Benjamin-Bona-Mahony equation has been linearized by taking the quantity $u + u^2$ a local constant. Thus the nonlinear term in the equation changes into $(\hat{u}^2 + \hat{u})u_x$ in this case Eq. (4.129) changes to

$$u_t + p u_x + q\hat{u} u_x + r\hat{u}^2 u_x + s u_{xxx} = 0. \tag{4.133}$$

Placing the Fourier mode (4.132) into Eq. (4.133), provides

$$\frac{\xi^n e^{im\varphi} - \xi^n e^{im\varphi}}{(\Delta t)} + p\left(\frac{\xi^n e^{i(m+1)\varphi} - \xi^n e^{im\varphi}}{(\Delta x)}\right) + q\hat{u}\left(\frac{\xi^n e^{i(m+1)\varphi} - \xi^n e^{im\varphi}}{(\Delta x)}\right)$$
$$r\left(\frac{\xi^n e^{i\varphi} - 2\xi^n + \xi^n e^{-i\varphi}}{(\Delta x)^2}\right) + s\left(\frac{\xi^n e^{i2\varphi} - 2\xi^n e^{i\varphi} + 2\xi^n e^{-i\varphi} - \xi^n e^{-2i\varphi}}{2(\Delta x)^2}\right) = 0. \tag{4.134}$$

We therefore reach the following expression by letting $\xi^{n+1} = \xi\xi^{n}$ and assuming that $\xi = \xi(\varphi)$ does not depend on the time "t"

$$\xi = 1 + \frac{2(\Delta t)}{n^2}(p(\Delta x) + q\hat{u}(\Delta x) + r\hat{u}^2(\Delta x))sin^2\frac{\varphi}{2}$$
$$+i\frac{(\Delta t)}{n^2}sin(\varphi)(8s\ \ sin^2(\varphi) - (p(\Delta x) + q\hat{u}(\Delta x) + r\hat{u}^2(\Delta x))) \tag{4.135}$$

Hence

$$\xi = X_1 + iX_2 \tag{4.136}$$

According to the Fourier stability, for the provided scheme to be stable, the condition $|\xi| \le 1$ must be satisfied so if the following inequality holds, the scheme is unconditionally stable.

$$|\xi|^2 = X_1^2 + X_2^2. \tag{4.137}$$

We can investigate other schemes following a similar way.

4.9.2. L $_2$ and L $_\infty$ Error Norms

The numerical approximations of the equation were achieved for the sample issue used in the current research and all calculations in this research were achieved using the mathematica program. To demonstrate how close to each order we use the analytical approximations and numerical approximations L_2 and L_∞ error norms. The error L_2 norms provided by [89]

$$L_2 = \left\|u^{exact} - u^{numeric}\right\|_2 = \sqrt{h\sum_{j=0}^{N}\left|u_j^{exact} - u_j^{numeric}\right|^2},$$

and L_∞ error norm defined as

$$L_\infty = ||u^{exact} - u^{numeric}||_\infty = \ {}^{Max}_{\ j}|u_j^{exact} - u_j^{numeric}|.$$

4.9.3. Exact and Numerical Approximations

Consider the exact solutions of Eq. (4.129) presented in Yokus *et al.* [90]

$$u(x,\ t) = -\frac{q}{2r}(1 + \sqrt{2}\ sech[(p - \frac{q^2}{6r})t - x]). \tag{4.138}$$

Placing the datum $r = 1, p = 1, q = 1,\ 0 < x < 1$ and $0 < t < 1$ into Eq. (4.138), provides

$$u_0(x) = -\frac{1}{2}(1 + \sqrt{2}sech[x]) \tag{4.139}$$

Utilizing the supposition above, the exact solution of Eq. (4.138) turns to

$$u(x,\ t) = -\frac{1}{2}(1 + \sqrt{2}sech[\frac{5t}{6} + x]) \tag{4.140}$$

For the finite difference method, we provide the following indexed form by placing the values $r = 1, p = 1, q = 1,$ and $(\Delta x) = (\Delta t) = 0.02$ into Eq. (4.131):

$$\begin{aligned} u_{i+1,j} = -50(&1250u_{i-2,j} - 2500u_{i-1,j} + 2u_{i,j} + u_{i,j}^2 + u_{i,j}^3 \\ &-u_{i,j+1} + 2499u_{i+1,j} - u_{i,j}u_{i+1,j} - u_{i,j}^2u_{i+1,j}^2 - 1250u_{i+2,j}). \end{aligned} \tag{4.141}$$

Fig. 4.18 provides the physical feature of Eq. (4.138), and Fig. 4.19 presents the comparision between exact and numerical approximations. Tables 4.1 and 4.2 provide the approximate numerical/exact values of Eq. (4.138) and its numerical errors, respectively.

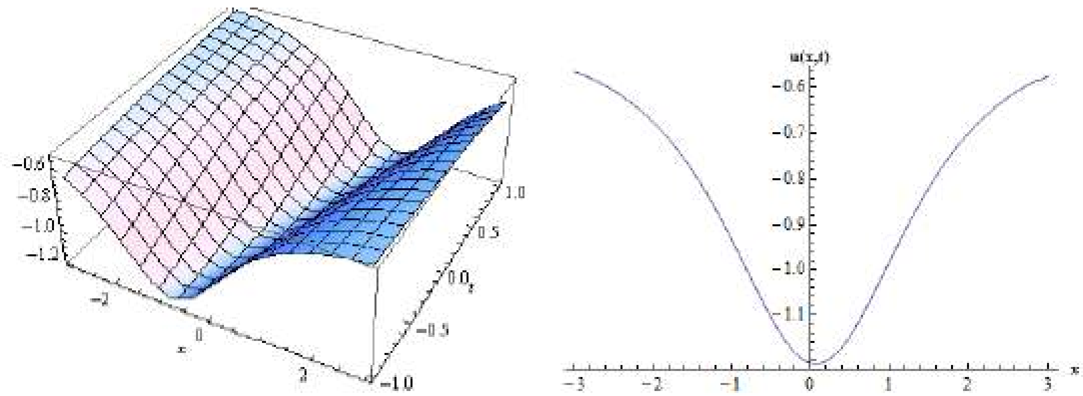


Figure 4.18. The 3D and 2D figures of Eq. (4.138).

We provide in the Table 1 and 2 below, the comparison between the exact and numerical solutions and the L_2 and L_∞ error norms tables reached by ustiling the finite difference method, respectively

Table 4.1. Numerical and exact solutions of equation (1) and absolute errors $\Delta(x) = 0.02$ and $0 \le x \le 1$.

x_i	t_j	Numerical Solution	Exact Solution	Error
0	0.01	-1.20728	-1.20701	2.74945×10^{-4}
0.01	0.01	-1.20608	-1.20710	1.01901×10^{-3}
0.02	0.01	-1.20461	-1.20691	2.30727×10^{-3}
0.03	0.01	-1.20286	-1.20644	3.58359×10^{-3}
0.04	0.01	-1.20085	-1.20569	4.84184×10^{-3}
0.05	0.01	-1.19858	-1.20466	6.07605×10^{-3}
0.06	0.01	-1.19607	-1.20335	7.28046×10^{-3}

Table 4.2. L_2 and L_∞ error norm when $0 \le h \le 1$ and $0 \le x \le 1$ From Table 1 and 2 above, we may simply observe that the numerical results are in closed agreement with the reached exact solution

$x_i = t_j$	L_2	L_∞
0.2	0.128064	0.171159
0.1	0.071052	0.093571
0.05	0.036889	0.048321
0.01	0.007552	0.009848
0.001	0.000759	0.000988

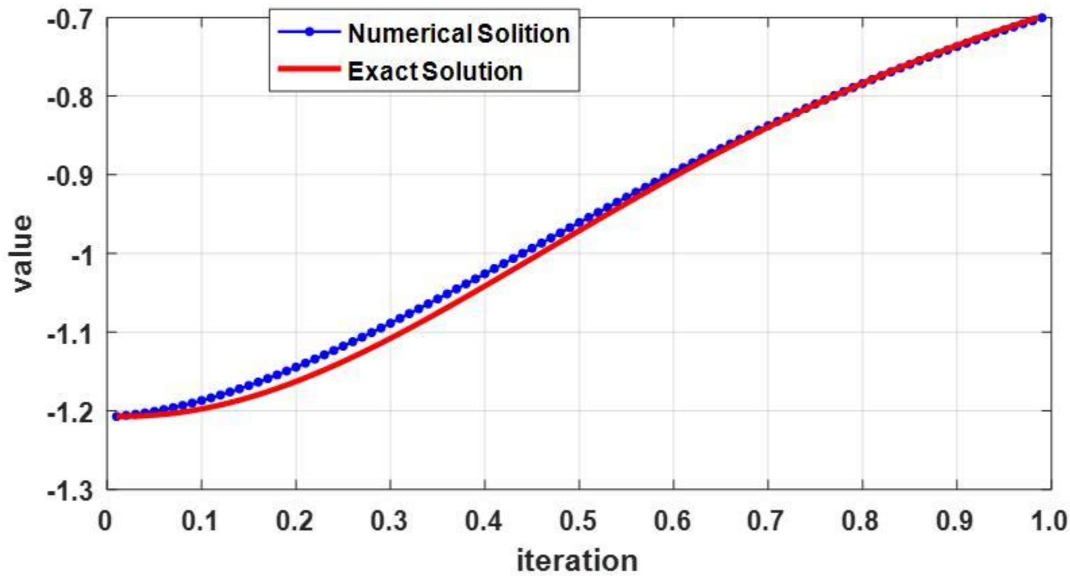


Figure 4.19. Illustrates the solution's physical conduct and shows that the precise value of the solution is near the numerically calculated values. We acknowledge that the decision is based on the truncation error of the Δx and Δt. There are appointed close to zero indicates that the truncation error is very small.

4.10. The Coupled Boussinesq Equation

The exact and numerical approximations of the Boussinesq equation combined are provided in this section. The equation of Boussinesq is provided by [91]

$$\begin{aligned} v_t + w_x + v v_x &= 0, \\ w_t + (wv)_x + v_{xxx} &= 0. \end{aligned} \tag{4.142}$$

Boussinesq equations are utilized to model the dynamics of shallow waves of water that occur in various locations such as rivers, lakes and beaches of the sea [91, 92]. The combined Boussinesq equation occurs for two layered fluid flow in the shallow water waves. This happens whenever a ship's accidental oil spill causes a layer of oil to float above the water layer [92].

Consider the following difference operators: For $v(x,t)$:

$$H_t v_{i,j} = v_{i,j+1} - v_{i,j}, \tag{4.143}$$

$$H_x v_{i,j} = v_{i+1,j} - v_{i,j}, \tag{4.144}$$

$$H_{xxx}v_{i,j} = v_{i+2,j} - 2v_{i+1,j} + 2v_{i-1,j} - v_{i-2,j}. \tag{4.145}$$

For $w(x,t)$:

$$H_t w_{i,j} = w_{i,j+1} - w_{i,j}, \tag{4.146}$$

$$H_x w_{i,j} = w_{i+1,j} - w_{i,j}, \tag{4.147}$$

Thus, one may approximate the partial derivatives into the finite difference operators as

For $v(x,t)$:

$$\left.\frac{\partial v}{\partial t}\right|_{i,j} = \frac{H_t v_{i,j}}{\Delta t} + O((\Delta t)), \tag{4.148}$$

$$\left.\frac{\partial v}{\partial x}\right|_{i,j} = \frac{H_x v_{i,j}}{\Delta x} + O((\Delta x)), \tag{4.149}$$

$$\left.\frac{\partial^3 v}{\partial x^3}\right|_{i,j} = \frac{H_{xxx} v_{i,j}}{(\Delta x)^3} + O((\Delta x)^2), \tag{4.150}$$

For $w(x,t)$:

$$\left.\frac{\partial w}{\partial t}\right|_{i,j} = \frac{H_t w_{i,j}}{\Delta t} + O((\Delta t)), \tag{4.151}$$

$$\left.\frac{\partial w}{\partial x}\right|_{i,j} = \frac{H_x w_{i,j}}{\Delta x} + O((\Delta x)). \tag{4.152}$$

One may rewrite Eq. (4.142) in the finite forward difference operator's form as

$$\begin{aligned} \frac{H_t v_{i,j}}{\Delta t} + \frac{H_x w_{i,j}}{\Delta x} + v_{i,j}\frac{H_x v_{i,j}}{\Delta x} &= 0, \\ \frac{H_t w_{i,j}}{\Delta t} + w_{i,j}\frac{H_x v_{i,j}}{\Delta x} + v_{i,j}\frac{H_x w_{i,j}}{\Delta x} + \frac{H_{xxx} v_{i,j}}{(\Delta x)^3} &= 0. \end{aligned} \tag{4.153}$$

We provide the following indexed shapes by placing Eq. (4.148-4.151) into Eq. (4.142):

$$\begin{aligned} v_{i+1,j} = &-\frac{1}{2(\Delta t)(1+(\Delta x)^2 v_{i,j}^2 - (\Delta x)^2 w_{i,j})}((\Delta t)v_{i-2,j} - 2(\Delta t)v_{i-1,j} - 2(\Delta x)^3 v_{i,j}^2 \\ &-2(\Delta t)(\Delta x)^2 v_{i,j}^3 + 2(\Delta x)^3 v_{i,j} v_{i,j+1} - (\Delta t)v_{i+2,j} + 2(\Delta x)^3 w_{i,j} \\ &+2(\Delta t)(\Delta x)^2 v_{i,j} w_{i,j} - 2(\Delta x)^3 w_{i,j+1}), \end{aligned} \tag{4.154}$$

$$w_{i+1,j} = -\frac{1}{2(\Delta t)(1+(\Delta x)^2 v_{i,j}^2 - (\Delta x)^2 w_{i,j})}(-2(\Delta x)v_{i,j} - (\Delta t)v_{i-2,j}v_{i,j}$$
$$+2(\Delta t)v_{i-1,j}v_{i,j} - 2(\Delta t)v_{i,j}^2 + 2(\Delta x)v_{i,j+1} + (\Delta t)v_{i,j}v_{i+2,j} - 2(\Delta t)w_{i,j}$$
$$-2(\Delta t)(\Delta x)^2 v_{i,j}^2 w_{i,j} - 2(\Delta x)^3 v_{i,j+1}w_{i,j} + 2(\Delta t)(\Delta x)^2 w_{i,j}^2 + 2(\Delta x)^3 v_{i+j}w_{i,j+1}), \tag{4.155}$$

where the initial values $v_{i,0} = v_0(x_i)$ and $w_{i,0} = w_0(x_i)$.

4.10.1. Von-Neumann Stability Analysis

Here, the stability of the numerical scheme is evaluated using the Fourier Von-Neumann stability analysis with the combined Boussinesq equation. We consider ζ^n as the amplification factor. The growth factor of a typical Fourier mode may be provided as follows:

$$v_m^n = P\zeta^n e^{i\beta}, w_m^n = W\zeta^n e^{i\beta}, \tag{4.156}$$

where $i = \sqrt{-1}$.

To examine the stability of the numerical scheme, the nonlinear terms in the coupled Boussinesq equation vv_x, wv_x and vw_x must be linearized by taking v and w local constants. Thus, the nonlinear terms vv_x, wv_x and vw_x changes Av_x, Bv_x and Aw_x respectively.

The finite difference operator form of these linearized terms is provided as

$$Av_x = A\frac{H_x v_{i,j}}{\Delta x}, \quad Bv_x = B\frac{H_x v_{i,j}}{\Delta x}, \quad Aw_x = A\frac{H_x w_{i,j}}{\Delta x}, \tag{4.157}$$

where $A = v_m^n$ and $B = w_m^n$

Employing these changes on Eq. (4.155), provides

$$\begin{aligned} \frac{H_t v_{i,j}}{\Delta t} + \frac{H_x w_{i,j}}{\Delta x} + A\frac{H_x v_{i,j}}{\Delta x} &= 0, \\ \frac{H_t w_{i,j}}{\Delta t} + B\frac{H_x v_{i,j}}{\Delta x} + A\frac{H_x w_{i,j}}{\Delta x} + \frac{H_{xxx} v_{i,j}}{(\Delta x)^3} &= 0. \end{aligned} \tag{4.158}$$

Now, placing Eq. (4.157) into Eq. (4.158), provides

$$P(\frac{1}{(\Delta t)+(A^2-B)(\Delta t)(\Delta x)^2}((-A^2+B)(\Delta t)(\Delta x)^2+A(\Delta x)^3(\zeta-1)$$
$$+(A^2-B)(\Delta t)(\Delta x)^2 cos[\beta]+i\Delta t(2+(A^2-B)\Delta x^2-2cos[\beta]sin[\beta])) \quad (4.159)$$
$$-W(\frac{\Delta x^3(\zeta-1)}{\Delta t+(A^2-B)\Delta t\Delta x^2}))=0$$

$$P((-A\Delta t-\Delta x(-1+B\Delta x^2)(\zeta-1)+A\Delta t cos[\beta]-iA\Delta t sin[\beta]+2iA\Delta t cos[\beta]sin[\beta])$$
$$\times(\Delta t+(A^2-B)\Delta t\Delta x^2)^{-1})+W((-A^2\Delta t\Delta x^2+\Delta t(1-B\Delta x^2)+A\Delta x^3(\zeta-1)$$
$$+\Delta t cos[\beta]+(A^2-B)\Delta t\Delta x^2 cos[\beta]$$
$$+i\Delta t sin[\beta]+i(A^2-B)\Delta t\Delta x^2 sin[\beta])(\Delta t+(A^2-B)\Delta t\Delta x^2)^{-1})=0$$
(4.160)

where $A=v_m^n, B=w_m^n$ Next, Let $\zeta^{n+1}=\zeta\zeta^n$ and supposing that ζ is independent of time.

Then, we may reach the following class of algebraic equations.

Eq. (4.159) and (4.160) are not obvious to P and W for at least one solution necessary and sufficient condition determinant of coefficients matrix of system take the value zero. Therefore

$$\zeta_1=\frac{1}{\Delta x^4\cos[\frac{\beta}{2}]-i\Delta x^4 sin[\frac{\beta}{2}]}(\Delta x^4 cos[\frac{\beta}{2}]-2iA\Delta t\Delta x^3 sin[\frac{\beta}{2}]-i\Delta x^4 sin[\frac{\beta}{2}]-$$
$$2(((\Delta t^2\Delta x^4-B\Delta t^2\Delta x^6)cos[\frac{\beta}{2}]^2 sin[\frac{\beta}{2}]^2-\Delta t^2\Delta x^4 cos[\frac{\beta}{2}]cos[\frac{3\beta}{2}]sin[\frac{\beta}{2}]^2-$$
$$i\Delta t^2\Delta x^4 cos[\frac{\beta}{2}]sin[\frac{\beta}{2}]^3+i\Delta t^2\Delta x^4 cos[\frac{3\beta}{2}]sin[\frac{\beta}{2}]^3-B\Delta t^2\Delta x^6 sin[\frac{\beta}{2}]^4))^{\frac{1}{2}})$$
(4.161)

$$\zeta_2=\frac{1}{\Delta x^4\cos[\frac{\beta}{2}]-i\Delta x^4 sin[\frac{\beta}{2}]}((\Delta x^4 cos[\frac{\beta}{2}]-2iA\Delta t\Delta x^3 sin[\frac{\beta}{2}]-i\Delta x^4 sin[\frac{\beta}{2}]+$$
$$2(((\Delta t^2\Delta x^4-B\Delta t^2\Delta x^6)cos[\frac{\beta}{2}]^2 sin[\frac{\beta}{2}]^2-\Delta t^2\Delta x^4 cos[\frac{\beta}{2}]cos[\frac{3\beta}{2}]sin[\frac{\beta}{2}]^2-$$
$$i\Delta t^2\Delta x^4 cos[\frac{\beta}{2}]sin[\frac{\beta}{2}]^3+i\Delta t^2\Delta x^4 cos[\frac{3\beta}{2}]sin[\frac{\beta}{2}]^3-B\Delta t^2\Delta x^6 sin[\frac{\beta}{2}]^4)^{\frac{1}{2}})$$
(4.162)

According to the Fourier stability, numerical scheme is unconditionally stable as $|\zeta_1|\le 1$, $|\zeta_1|\le 1$.

4.10.2. L $_2$ and L $_\infty$ Error Norms

To demonstrate how near the exact and numerical solutions are, we use L_2 and L_∞ error norms.

The L_2 and L_∞ error norms are defined as [89]

$$L_2 = \left\|u^{exact} - u^{numeric}\right\|_2 = \sqrt{h \sum_{j=0}^{N} \left|u_j^{exact} - u_j^{numeric}\right|^2}, \tag{4.163}$$

$$L_\infty = ||u^{exact} - u^{numeric}||_\infty = \max_j |u_j^{exact} - u_j^{numeric}|. \tag{4.164}$$

4.10.3. Exact and Numerical Approximations of Coupled Boussinesq Equation

Consider the exact solutions of Eq. (4.142) presented in Sulaiman *et al.* [93]

$$v_{1.1}(x,t) = \sqrt{\lambda^2 - 4\mu} - \lambda + \frac{4\mu}{\lambda + \sqrt{\lambda^2 - 4\mu}\ \ tanh[\psi_{1.1}(x,t)]}, \tag{4.165}$$

$$w_{1.1}(x,t) = 2\mu(\lambda^2 - 4\mu)/(\lambda cosh[\psi_{1.1}(x,t)] + \sqrt{\lambda^2 - 4\mu} sinh[\psi_{1.1}(x,t)])^2, \tag{4.166}$$

where

$$\psi_{1.1}(x,t) = \frac{1}{2}(E + x - \sqrt{\lambda^2 - 4\mu}\ \ t) \times \sqrt{\lambda^2 - 4\mu}.$$

Placing the datum $k = 1, \lambda = 3, \mu = 2, E = 2.5$ into Eq. (4.165) and (4.166), provides the following special exact solutions for the approximations:

$$v(x,t) = -2 + \frac{8}{3 + tanh[\frac{1}{2}(2.5 - t + x)]}, \tag{4.167}$$

$$w(x,t) = 4\left(3\cosh\left[\frac{1}{2}(2.5 - t + x)\right] + \sinh\left[\frac{1}{2}(2.5 - t + x)\right]\right)^{-2}. \tag{4.168}$$

At $t = 0$, Eq. (4.167) and (4.167) turns to

$$v_0(x) = -2 + \frac{8}{3 + tanh[\frac{1}{2}(2.5 + x)]}, \tag{4.169}$$

$$w_0(x) = 4(3cosh[\frac{1}{2}(2.5+x)] + sinh[\frac{1}{2}(2.5+x)])^{-2}. \quad (4.170)$$

Placing $(\Delta x) = (\Delta t) = 0.01$ into Eq. (4.154) and (4.155), provides

$$v_{i+1,j} = (9999.10 + v_{i,j}^2 - w_{i,j})^{-1}(-4999.10v_{i-2,j} + 9999.10v_{i-1,j} \\ +v_{i,j}^2 + v_{i,j}^3 - v_{i,j}v_{i,j+1} + 4999.10v_{i+2,j} - w_{i,j} - v_{i,j}w_{i,j} + w_{i,j+1}), \quad (4.171)$$

$$w_{i+1,j} = (9999.10 + v_{i,j}^2 - w_{i,j})^{-1}(9999.10v_{i,j} + 4999.10v_{i-2,j}v_{i,j} \\ -9999.10v_{i-1,j}v_{i,j} + 9999.10v_{i,j}^2 - 9999.10v_{i,j+1} - 4999.10v_{i,j}v_{i+2,j} \\ +9999.10w_{i,j} + v_{i,j}^2w_{i,j} + v_{i,j+1}w_{i,j} - w_{i,j}^2 - v_{i,j}w_{i,j+1}), \quad (4.172)$$

respectively.

Fig. 4.20 provide the physical feature of Eqs. (4.165) and (4.166), Fig. 4.21 gives the exact and numerical comparison of Eq. (4.165), Fig. 4.22 gives the exact and numerical comparison of Eq. (4.166), Fig. 4.23 presents the absolute error graph of Eq. (4.165) and Fig. 4.24 absolute error graph of Eq. (4.166). Tables 4.3 and 4.4 provide the approximate numerical/exact values of Eq. (4.142) via Eq. (4.165) and (4.166), respectively. Table 4.5 give the L_2 and L_∞ error norms of from Eqs. (4.165) and (4.166).

The exact and numerical approximations of Eq. (4.142) are therefore presented in table 1 and 2 below and L_2 and L_∞ error norms in table 3.

Table 4.3. Numerical and exact approximations of Eq. (4.142) and absolute errors under Eq. (4.165).

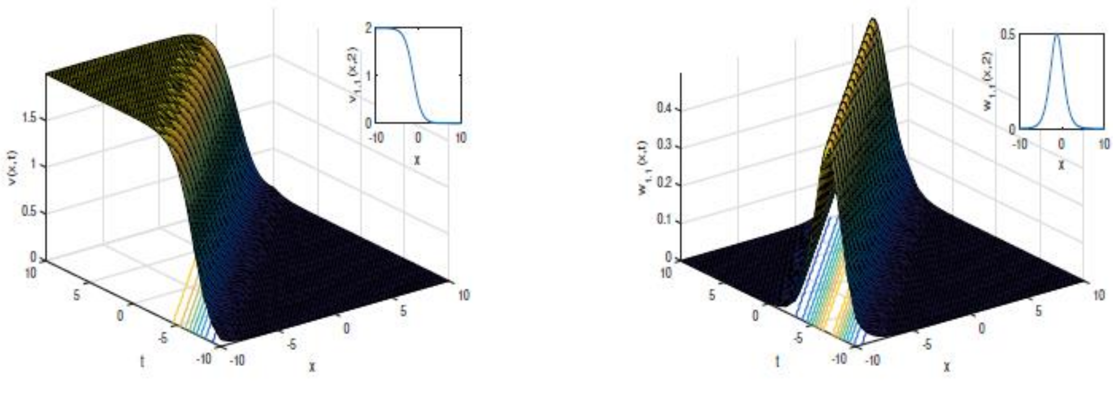
x_i	t_j	Numerical	Exact	Error
0	0.01	0.07960	0.07961	6.69266×10^{-6}
0.01	0.01	0.07884	0.07885	6.63951×10^{-6}
0.02	0.01	0.07809	0.07809	6.58665×10^{-6}
0.03	0.01	0.07734	0.07735	6.53408×10^{-6}
0.04	0.01	0.07660	0.07661	6.48180×10^{-6}
0.05	0.01	0.07587	0.07587	6.42982×10^{-6}
0.06	0.01	0.07514	0.07515	6.37813×10^{-6}
0.07	0.01	0.07442	0.07443	6.32674×10^{-6}
0.08	0.01	0.07371	0.07371	6.27564×10^{-6}
0.09	0.01	0.07300	0.07301	6.22483×10^{-6}

Table 4.4. Numerical and exact approximations of Eq. (4.142) and absolute errors under Eq. (4.166)

x_i	t_j	Numerical	Exact	Error
0	0.01	0.07644	0.07644	3.42592×10^{-6}
0.01	0.01	0.07574	0.07574	3.39847×10^{-6}
0.02	0.01	0.07504	0.07505	3.37117×10^{-6}
0.03	0.01	0.07435	0.07436	3.34400×10^{-6}
0.04	0.01	0.07367	0.07367	3.31699×10^{-6}
0.05	0.01	0.07210	0.07210	3.29012×10^{-6}
0.06	0.01	0.07232	0.07232	3.26340×10^{-6}
0.07	0.01	0.07165	0.07166	3.23682×10^{-6}
0.08	0.01	0.07099	0.07010	3.21039×10^{-6}
0.09	0.01	0.07034	0.07034	3.18412×10^{-6}

Table 4.5. L_2 and L_∞ error norm under $0 \leq h \leq 1$ and $0 \leq x \leq 1$

$x_i = t_j$	$L_2(v(x,t))$	$L_\infty(v(x,t))$	$L_2(w(x,t))$	$L_\infty(w(x,t))$
0.1	0.0004379358	0.0006196246	0.0002236631	0.0003188607
0.01	0.0000000321	0.0000001112	0.0000000090	0.0000000302
0.001	0.0000000465	0.0000000668	0.0000000237	0.0000000346
0.0001	0.0000000005	0.0000000007	0.0000000235	0.0000000832



(a) (b)

Figure 4.20. The (a) kink-type and (b) soliton figures of Eq. (4.165) and (4.166).

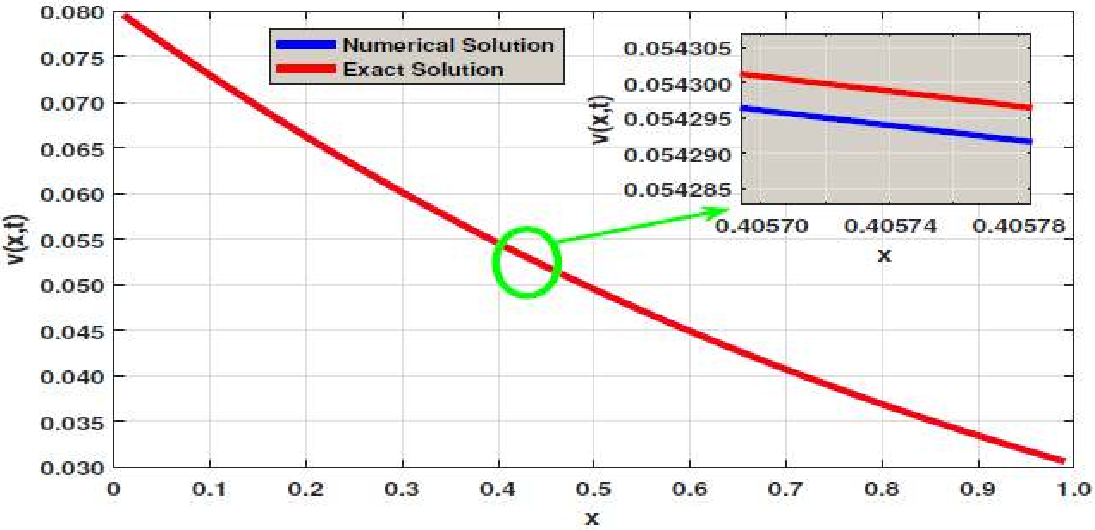


Figure 4.21. Numerical and exact approximations graph of Eq. (4.142) via Eq. (4.165).

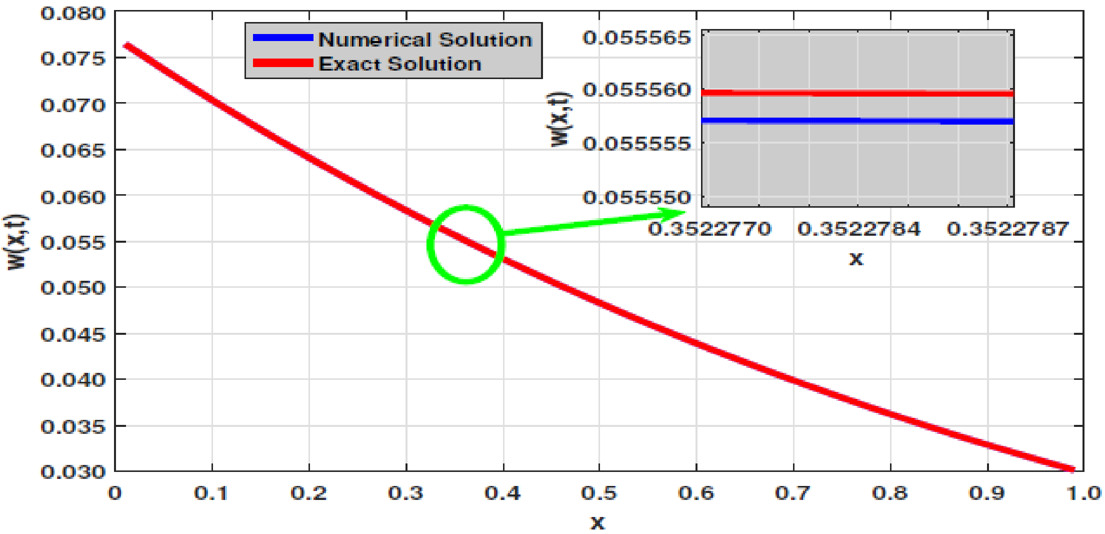


Figure 4.22. Numerical and exact approximations graph of Eq. (4.142) via Eq. (4.166).

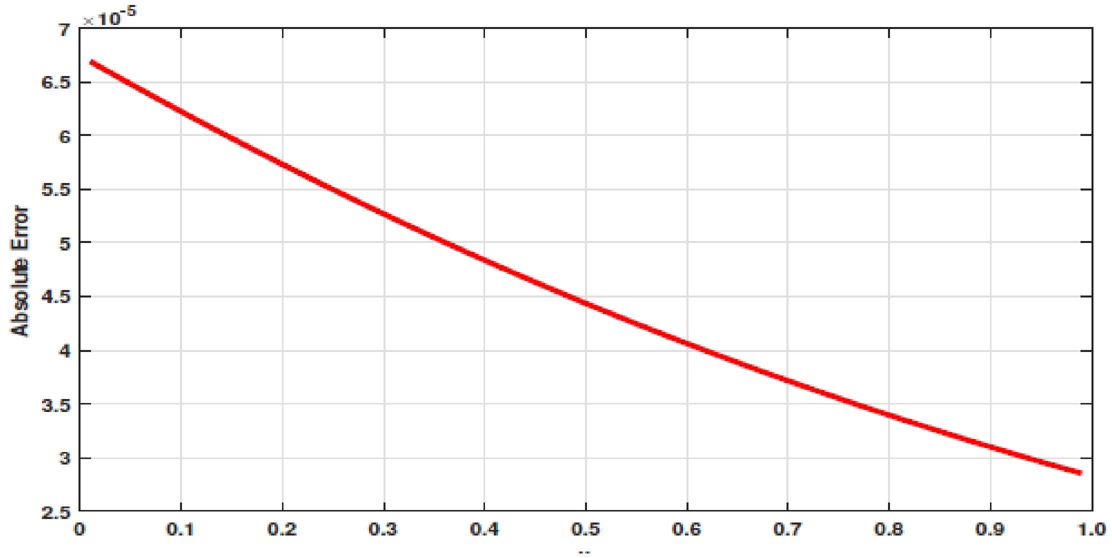


Figure 4.23. Absolute error graph under Eq. (4.165).

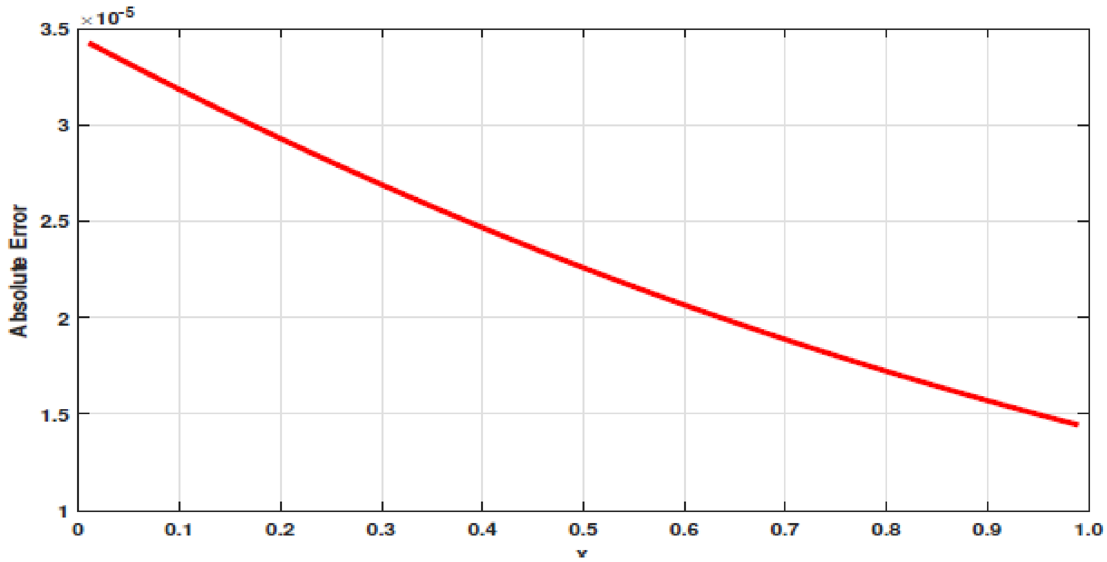


Figure 4.24. Absolute error graph under Eq. (4.166).

5. RESULTS AND DISCUSSION

This study focuses on building distinct solitary wave solutions and computing the approximate accurate and numerical solutions for distinct types of nonlinear mathematical models. On the other side, to achieve multiple types of solitary wave solutions, one of the analytical methods used in this research has been simplified.

The reported solutions for KdV equation with dual-power law nonlinearity by secured by using the SGEM are in the form of hyperbolic, trigonometric, complex and rational function. Hyperbolic functions are of different physical meanings. For instance, the hyperbolic sine comes from the gravitational potential of a cylinder. The hyperbolic cosine function is the design of a hanging cable. The hyperbolic tangent comes from the calculation and rapidity of special relativity. The hyperbolic secant comes from the profile of a laminar jet. The hyperbolic cotangent arises in the Langevin function for magnetic polarization [94]. It is estimated that these solutions are related to physical features of hyperbolic functions when we consider the solutions reported in this study.

Lie group analysis as one of the well-known approach for reaching exact solutions to different kind of NLEEs has been employed by Khalique *et al.* [81] in finding the solutions of the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation and some soliton, singular soliton and singular periodic waves solutions with the trigonometric and hyperbolic function structures were reported. With the sine-Gordon expansion method, we revealed some new soliton and kink-type solutions with various hyperbolic functions structure.

Several analytical techniques which have been developed by using the concept of the sinh-Gordon equation and tell the differences with our newly presented simplified approach. Under the choice of suitable values of the parameters, we also present the two- and three-dimensional plots to some of the obtained solutions reported in this study.

Yan [95] proposed a sinh-Gordon equation expansion method which obtains the Jacobi elliptic function solutions to some nonlinear wave equations. In Yan [95], Eq. (3.24) is transformed into nonlinear ordinary differential equation (Eq. (3.26)) by using Eq. (2.25). When simplifying Eq. (3.26) by Yan [95], the constant of integration is considered to be nonzero and the following Jacobi elliptic functions are obtained:

$$sinh(\theta) = cs(\xi; m) \quad and \quad cosh(\theta) = ns(\xi; m). \tag{5.1}$$

It is known that when $m \to 1,\ cs(\xi; m) \to csch(\xi)$ and $ns(\xi; m) \to coth(\xi)$ and when $m \to 0,\ cs(\xi; m) \to cot(\xi)$ and $ns(\xi; m) \to csc(\xi)$. For details, see Yan [93].

Xian-Lin and Jia-Shi [96] extended sinh-Gordon equation expansion method developed by Yan [93]. In the extended sinh-Gordon equation expansion method trigonometric and hyperbolic functions solutions can directly be secured. When simplifying Eq. (3.26), Xian-Lin and Jia-Shi [96] also considered the constant of integration to be zero in the following way:

In Eq. (3.28), Xian-Lin and Jia-Shi [96] considered $\frac{2\lambda}{\vartheta} = a$ and $\frac{\lambda\delta}{\vartheta} = b$ so that the following equation is secured:

$$\theta' = \sqrt{a\ sinh^2(\theta) + b}. \tag{5.2}$$

Thus, Xian-Lin and Jia-Shi [94] solved Eq. (5.2) considering the following two cases:

Case-I: When $a = 1,\ b = 0$, Eq. (5.2) becomes

$$\theta' = sinh(\theta). \tag{5.3}$$

Case-II: When $b = 1,\ a = 1$, Eq. (5.2) becomes

$$\theta' = cosh(\theta). \tag{5.4}$$

Case-I is solved to give hyperbolic functions solutions and Case-II is solved to give trigonometric functions solutions. This leads to having two individual polynomial equations in the power of hyperbolic functions $(\theta'^s sinh^k(\theta) cosh^j(\theta))$, $(s = 0,1 \ \ and \ \ k,\ j = 0,\ 1,\ 2,\ \ldots)$ as θ' varies for each case. For details, see Xian-Lin and Jia-Shi [94].

In this study, we transform the same sinh-Gordon equation Eq. (3.24) as in Yan [93], Xian-Lin and Jia-Shi [94] by using Eq. (3.25). We also consider a nonzero constant of integration when simplifying Eq. (3.26), but in different way with Yan [95], Xian-Lin and

Jia-Shi [96], as it can be seen from Eq. (3.29). Doing this, we provide Eq. (3.30). Eq. (3.30) is a variable separable equation, we simplify it and obtain the following trigonometric function solutions:

$$cosh(\theta) = tan(\sqrt{\sigma}(\xi + d)) \quad or \quad cosh(\theta) = cot(\sqrt{\sigma}(\xi + d)), \tag{5.5}$$

$$sinh(\theta) = sec(\sqrt{\sigma}(\xi + d)) \quad or \quad sinh(\theta) = csc(\sqrt{\sigma}(\xi + d)). \tag{5.6}$$

When we consider $\sigma = -1$ and $d = 0$ in Eq. (5.5) and (5.6), we directly obtain the following hyperbolic function solutions:

$$cosh(\theta) = -i \ tanh(\xi) \quad or \quad cosh(\theta) = i \ coth(\xi), \tag{5.7}$$

$$sinh(\theta) = sech(\xi)) \quad or \quad sinh(\theta) = i \ csch(\xi). \tag{5.8}$$

Placing these solutions together, we have the forms of solutions to Eq. (3.1) provided in Eq. (3.36), (3.37), (3.38) and (3.39) which can be used to obtain different travelling wave solutions to various nonlinear evolution equations.

It can be seen that in this study, all the solutions can be generated from the equation ($\sqrt{\sigma} \ cosh(\theta)$) which overcome the use of two seprate polynomial equations in the power of hyperbolic functions ($\theta'^{s} sinh^{k}(\theta) cosh^{j}(\theta)$), ($s = 0{,}1 \ \ and \ \ k, \ j = 0, \ 1, \ 2, \ \ldots$) as we have only one θ'. Thus, one single polynomial equation can be used in collecting the system of algebraic equations and this is enough to obtained the solutions of the considered nonlinear model after obtaining the values of the parameters involved by solving the system of algebraic equations.

The well-known numerical scheme, namely; the finite forward difference method is used in obtaining the approximate exact and numerical solutions to the coupled Boussinesq equation. We observed that as $\Delta x = \Delta t$ are getting smaller the approximations are approaching zero (see Fig. 4.21 and 4.22).

6. CONCLUSION

This study developed several solitary wave solutions for different types of nonlinear mathematical models using different computational methods. The technique of finite forward difference is also used to discover approximate solutions for some nonlinear. Topological, non-topological, combined topological-non-topological and singular soliton solutions are reported for the Korteweg-de Vries equation with dual-power law nonlinearity, the coupled nonlinear Maccari's system, the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation and the long-short-wave interaction system. Dark, bright, combined dark-bright, singular, combined singular solitons and singular periodic wave solutions are constructed for the resonant nonlinear Schrödinger equation with both spatio-temporal and inter-modal dispersions, the conformable space-time fractional second order nonlinear Schrödinger equation, the decoupled nonlinear Schrödinger equation arising in dual-core optical fibers and the (2+1)-dimensional nonlinear Chiral Schrödinger equation. Secondly, the extended sinh-Gordon equation expansion method is simplified. The forward finite difference scheme is used to report the exact and numerical approximations to the Benjamin-Bona-Mahony equation and the coupled Boussinesq equation. The mathematical techniques used in this study are powerful and easy-handling to address various kind of nonlinear models. The simplified extended sinh-Gordon equation expansion method provides a simplicity in the tedious work involves in the extended sinh-Gordon equation expansion method. The finite forward difference method provides a better approximation to various nonlinear models when used.

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CURRICULUM VITAE

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PERSONAL DETAILS

- DATE OF BIRTH: Born on 03/04/1986.
- NATIONALITY: Nigerian
- PLACE OF BIRTH: Kiru, Kano Nigeria.
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RESEARCH INTEREST

Stochastic optimization, analytical and numerical solutions of linear/nonlinear classical and fractional differential equations.

EDUCATION

- Ph.D. candidate in Applied Mathematics, Firat University, Elazig, Turkey. GPA: 94/100.

  PhD Thesis title: Investigations of the Exact and Numerical Solutions of Some Nonlinear Partial Differential Equations (Thesis completed). Anticipating graduation in December 2019.

  PhD Seminar title: On the Comparison Between SGEM and IBSEFM for the Solutions of the (2+1)-dimensional Broer-Kaup-Kupershmidt System

- Master of Science in Mathematics, Jordan University of science and Technology, Irbid, Jordan. Completed in 2014 GPA: 83/100.

  Msc Thesis title: Using Simulated Annealing for Solving Multi-Objective Optimization Problems
- Bachelor of Science in Mathematics, Bayero University Kano, Kano, Nigeria. Completed in 2009 GPA:3.81/5.

Computer Languages

Expert in Matlab, Wolfram Mathematica, Arena simulation software, C++ programming. Sound knowledge in LaTeX and Microsoft word typesetting.

Others

Creative, innovative, excellent teaching skills.

SCIENTIFIC ACTIVITIES

- The member of Local Committee, 2nd International Conference on Computational Mathematics and Engineering Sciences (CMES-2017), 20th-22nd May, 2017, Istanbul/Turkey.
- The member of Local Committee, 3rd International Conference on Computational Mathematics and Engineering Sciences (CMES-2018), 4th-6th May, 2018, Girne/ Turkish Republic of Northern Cyprus (TRNC).
- The member of Local Committee, 4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019), to be held 20th-22nd April, 2019, Antalya/ Turkey.

EMPLOYMENT, FEDERAL UNİVERSİTY DUTSE, NİGERİA

Lecturer II of Mathematics *Spring 2015 to present*

I was employed as a Graduate Assistant in March 2013 with my bachelor degree. I was upgraded to the position of Assistant Lecturer in July 2014 after the completion of my master degree. I was promoted to the position of Lecturer II in October 2015. Since then, my major work has been preparation and presentation of lectures, supervision of group work, writing and grading tests and quizzes, grading projects and papers, preparing and grading the final exam for the courses. I also served as a departmental examination officer, level coordinator, and secretary to the committee on accreditation matters.Below is a list of thought courses for undergraduate:

MTH 103(120 students 2 groups)	Elementary Mathematics II	Summer 2014, Fall 2014
MTH 205 (82 students)	Linear Algebra I	Summer 2014, Fall 2014,

		spring 2015,
MTH 301 (25 students)	Abstract Algebra I	Summer 2014, spring 2015
MTH 102 (115 students)	Pre-calculus I	spring 2018
MTH 204 (57 students)	Linear Algebra II	spring 2018
MTH 204 (65 students)	Linear Algebra I	spring 2018

CONFERENCE TALKS

1. The Second International Conference on Mathematics and Statistics (AUS-ICMS'15), 2-5 April, 2015, Sharjah, UAE *April, 2015*
   Multi-Objective Stochastic Optimization Via Simulated Annealing.
2. Computational Management Science Conference, 31 May-2 June, 2016, Salamanca, Spain *May-June, 2016*
   Designing Health care appointment systems using GA and SA.
3. 6th International Youth Sciences Forum, 24-26 July, 2016 Lviv, Ukraine *July 2016*
   On The Hyperbolic Simulations to the Camassa-Holm Equation.

4. 6th International Youth Sciences Forum, 24-26 July, 2016 Lviv, Ukraine *July 2016*
   On The New Hyperbolic Function Solutions to the (2+1)-dimensional BKK Systems.

5. International Conference on Mathematics and Mathematics Education (ICMME-2017), 11-13 May, 2017, Sanliurfa, Turkey*May, 2017*
   A Note on Solutions of the Benjamin-Bona-Mahony Equation.

6. International Conference on Mathematics and Mathematics Education (ICMME-2017), 11-13 May, 2017, Sanliurfa, Turkey *May, 2017*
   Investigation of Various Travelling Wave Solutions to the Extended (2+1)-dimensional Quantum ZK Equation.
7. 2nd International Conference on Computational Mathematics and Engineering Sciences (CMES-2017), 20-22 May, 2017, Istanbul, Turkey *May, 2017*
   On the New Solutions to the (3+1)-dimensional Modified KdV-Zakharov-Kuznetsov Equation.
8. 2nd International Conference on Computational Mathematics and Engineering Sciences (CMES-2017), 20-22 May 2017, Istanbul, Turkey *May, 2017*
   Novel hyperbolic behaviors to some important models arising in quantum science.

9. International Conference on Current Scenario in Pure and Applied Mathematics (ICCSPAM-2018), 14-16 February, 2018, Tamil Nadu, India *February, 2018*
   Dark, bright optical and other solitons with conformable space-time fractional secondorder spatiotemporal dispersion.

10. International Conference on Current Scenario in Pure and Applied Mathematics (ICCSPAM-2018), 14-16, February 2018, Tamil Nadu, India *February, 2018*

    Optical solitons to the resonant nonlinear Schrödinger equation with both spatiotemporal and inter-modal dispersions under Kerr law nonlinearity.
11. 3rd International Conference on Computational Mathematics and Engineering Sciences (CMES-2018), 4-6 May, 2018, Girne, TRNC *May, 2018* Exact and numerical solutions to the time-fractional combined KdV-mKdV equation.

12. 3rd International Conference on Computational Mathematics and Engineering Sciences (CMES-2018), 4-6 May, 2018, Girne, TRNC *May, 2018* On the wave solutions to the TRLW equation.

13. International Conference on Mathematical and Related Sciences (ICRMS-

2018), April 30-May, 4 2018, Antalya, Turkey *April-May, 2018*

Designing Appointment System by Multi Objective Simulated Annealing.

14. The 6th International Conference on Control and Optimization with Industrial Applications (COIA-2018), 11-13 July, 2018, Baku, Azerbaijan

    July, 2018

    Optical Solitons to the Conformable Time-Fractional Perturbed Radhakrishnan-KunduLakshmanan Equation.
15. 4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019), 20-22 April, 2019, Anlatla, Turkey *April, 2019*

    Solitary Wave Solutions and Convergence Analysis to the Local M-Fractional KdV Equation with Dual Power Law Nonlinearity.
16. 4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019), 20-22 April, 2019, Anlatla, Turkey *April, 2019*

    Solitary Wave Solutions and Convergence Analysis to the Local M-Fractional Simplified MCH Equation.

REVIEW AND EDITORIAL ACTIVITIES

Reviewer/Editorial Board Member for several international journals including the following:

1. Journal of Ocean Engineering and Sciences (Elsevier Journal). **(Emerging Sources Citation Index)**.

2. Journal of King Saud University-Science (Elsevier Journal), **(ISI: Science Citation Index)**.

3. Results in Physics (Elsevier Journal), **(ISI: Science Citation Index)**.

4. Indian Journal of Physics (Springer Journal), **(ISI: Science Citation Index Expanded)**.

5. ITM Web of Conferences (CMES-2018 Conference Proceedings), **(Emerging Sources Citation Index)**.

6. Modern Physics Letters B (World Scientific), **(ISI: Science Citation Index)**.
7. Basic Research, King Fahd University of Petroleum and Minerals, Saudi Arabia.
8. Advances in Mathematical Physics (Hindawi), **(ISI: Science Citation Index Expanded)**.
9. Editorial board member, International Journal of Engineering, Mathematics, and Physics.
10. Journal of Statistics Application and Probability (Natural Sciences Publishing).
11. Applied Mathematics and & Information Sciences (Natural Sciences Publishing).
12. Optik-International Journal for Light and Electron Optics (Elsevier Journal), **(ISI: Science Citation Index)**.
13. Chaos, Solitons and Fractals (Elsevier Journal), **(ISI: Science Citation Index)**.
14. The European Physical Journal Plus (Springer Journal), **(ISI: Science Citation Index)**.
15. Optical and Quantum Electronics (Springer Journal), **(ISI: Science Citation Index)**.
16. Physica A (Elsevier Journal), **(ISI: Science Citation Index)**.
17. Physica Scriptica (IOP Journal), **(ISI: Science Citation Index)**.
18. Pramana (Springer Journal), **(ISI: Science Citation Index)**.
19. Modern Physics Letters A (World Scientific), **(ISI: Science Citation Index)**.
20. Arab Journal of Basic and Applied Sciences (Tailor and Francis), **(ISI: Science Citation Index)**.
21. Physics Letters A (Elsevier Journal), **(ISI: Science Citation Index)**.
22. Applied Mathematics and Nonlinear Sciences.
23. International Journal of Physical Sciences.
24. Communications in Nonlinear Science and Numerical Simulation (Elsevier Journal), **(ISI: Science Citation Index)**.
25. International Journal for Simulation and Multidisciplinary Design Optimization.
26. International Journal of Modern Physics B (World Scientific), **(ISI: Science Citation Index)**.
27. Mathematics (American Institute of Mathematical Sciences), **(ISI: Science Citation Index)**.
28. International Journal of Computer Mathematics (Tailor and Francis), **(ISI: Science Citation Index)**.

CITATION REPORT

My citation report as at today 8/12/2019 via google scholar citation report reads **853** with a **H-index of 19**. Details can be found at "**Tukur Abdulkadir Sulaiman google scholar citations**".

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2. Optik **131** 1036-1043 (2017).
3. Optik **135** 327-336 (2017).
4. Indian Indian J Phys **91(10)** 1237-1243 (2017).
5. Eur. Phys. J. Plus **132** 350 (2017).
6. Numer Methods Partial Differential Eq. **34** 211-227 (2018).
7. Opt Quant Electron **49** 349 (2017).
8. ITM Web of Conferences **13** 01019 (2017).
9. An International Journal of Optimization and Control: Theories & Applications **7(3)**, 240-247 (2017).
10. Eur. Phys. J. Plus **132** 459 (2017).
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12. Nonlinear Dyn **91** 1985-1991 (2018).
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15. Journal of Electromagnetic Waves and Applications **32(9)** 1093-1105 (2017).
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17. Superlattices and Microstructures **115** 19-29 (2018).
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