Skeleton Decomposition, Conditional Speed of Branching Brownian Motion, and Application to Random Obstacles

by

Mehmet Öz

Thesis submitted to the

Graduate School of Sciences and Engineering

in Partial Fulfillment of the Requirements for

the Degree of

Doctor of Philosophy

in

Mathematics

Koç University

July 30, 2015

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a Ph.D. thesis by

Mehmet Öz

and found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee members:

Mine Çağlar, Ph.D. (advisor)

Ali Ülger, Ph.D.

Süleyman Özekici, Ph.D.

Atilla Yılmaz, Ph.D.

Serdar Altok, Ph.D.

Date:

Abstract

We study a branching Brownian motion Z in $\mathbb{R}^d$, among obstacles scattered according to a Poisson random measure. Obstacles are balls with constant radius and each one works as a trap for the whole process when hit by a particle. Considering a general offspring distribution for branching, we find the decay rate of the annealed probability that none of the particles of Z hits a trap, asymptotically in time t. This problem motivates the proof of a more general result about the speed of branching Brownian motion conditioned on non-extinction. On the way, we make use of the skeleton decomposition of the Galton-Watson process that underlies Z, and show that the skeleton alone is responsible for both the conditional speed of branching Brownian motion and the asymptotic decay rate of the annealed trap-avoiding probability of the system, that is, the non-skeleton particles play no role.

Özet

Bu tezde, Poisson rassal ölçüme göre dağılmış engeller arasında $\mathbb{R}^d$ üzerinde seyreden dallanan Brown süreci Z çalışılmıştır. Engeller, sabit yarıçapa sahip toplardır ve her biri bir parçacık tarafından değildiği zaman tüm sistem için bir tuzak görevi görür. Dallanma için genel bir dağılım ele alınarak, Z'nin hiçbir parçacığının bu tuzaklara düşmeme olasılığının zaman sonsuza giderken asimptotik azalma hızı hesaplanmıştır. Bu problemin çözümüne yönelik olarak ayrıca, dallanan Brown hareketinin soyu tükenmemesi şartı altında hızı bulunmuş ve ispatlanmıştır. Yapılan ispatta, dallanan Brown sürecinin altında yatan Galton-Watson sürecinin iskelet ayrışmasından faydalanılmış ve sonunda hem soyu tükenmeme koşuluyla dallanan Brown sürecinin hızına, hem de sistemin asimptotik olarak tuzaklara düşmeme olasılığına sadece iskeletin katkı yaptığı, bir başka deyişle iskelete mensup olmayan parçacıkların bir katkısı olmadığı gösterilmiştir.

Acknowledgements

It has been a very long Ph.D. journey for me. Part 1 in Illinois, an intermission in the Turkish Armed Forces, and then Part 2 in İstanbul at Koç University. So naturally, there is a lot of people to acknowledge. Also, naturally, parts of these acknowledgements will be in Turkish. What I write below concerns Part 2 of my Ph.D. journey unless otherwise stated.

Firstly, I would like to express my special thanks and gratitude to my advisor Dr. Mine Çağlar for her support, guidance, sincerity and above all her encouragement throughout my Ph.D. Her encouragement and optimism has always helped me when I felt disheartened upon getting stuck in research. She has always been constructive and willing to listen and help. I remember many occasions where I was stuck with something for a long while, and somehow solved the puzzle on the spot while discussing the matter with her. I must say that I will miss our weekly meetings, where we not only discussed mathematics, but also talked about the issues of everyday life. Hocam, herşey için çok teşekkür ederim, haftalık görüşmelerimizi gerçekten özleyeceğim.

I am grateful to Prof. Ali Ülger for being on my Ph.D. committee in the past three and a half years, and for teaching me analysis in the three courses that I took from him. It was a real pleasure to listen to and learn from his lectures, I wish I had the chance of taking more courses from him. Hocam bölümdeki arkadaşlara hep söylerim, Ali Hoca ders açsa da gidip dinlesek, matematik dersi olmasına bile gerek yok, coğrafya dersi açsa da gidip dinlerim. Size de herşey için teşekkür ederim.

I would like to thank Prof. Süleyman Özekici for being on my Ph.D. committee in the past three and a half years, and for his constructive remarks during my thesis progress meetings. I would also like to thank Dr. Atilla Yılmaz for valuable discussions concerning my thesis topic, and for accepting to be on my Ph.D. committee, and to Dr. Serdar Altok for accepting to be on my Ph.D. committee.

I owe special thanks to Dr. János Engländer for writing the papers from which my thesis originated, for his very friendly attitude and valuable discussions during his visit to İstanbul in 2013, for the joint work that we did along with my advisor, and for sharing with me his deep expertise in our research field.

I also owe special thanks to Prof. Yoshitsugu Oono, who was my advisor at Illinois in the first part of my Ph.D. journey, and who has always been very helpful and friendly with me, and willing to share all his deep expertise. Also, special thanks to Prof. Calvin Stubbins, who was my advisor at my undergraduate years at Franklin and Marshall College, to Prof. Alan Levine, who introduced me to the field of stochastic processes, to Prof. Frank den Hollander for his hospitality and valuable discussions during my final year of Ph.D. when I was a guest at the Mathematical Institute in Leiden, and to Independent University of Moscow, where back in the spring of 2004, I first came to like the realm of mathematics.

I would like to acknowledge the support of TÜBİTAK (Türkiye Bilimsel ve Teknik Araştırma Kurumu) by TÜBİTAK Fellowship 2211 (Yurt İçi Lisansüstü Burs Programı) and TÜBİTAK Fellowship 2214-A (Yurt Dışı Doktora Sırası Araştırma Burs Programı) during my Ph.D.

In all those happy and not so happy times during my Ph.D. journey, several friends played a central role in my life and helped me survive through the journey.

From the Illinois times, three names will never be forgotten, Cesar Chialvo, Stanimir Kondov, and Matias Negrete Pincetic. Guys, those times in Loomis and Castle, all those funny and obscure moments, and above all your camaraderie will never be forgotten. By the way, Cesar, I put your name in the acknowledgements, what more do you want me to do? At this point the reader suspects that the author is so close with these guys that he will probably not even thank them.

And from the İstanbul years, Pınar Kılıçer, Neslihan Güğümcü and Nicat Aliyev receive special recognition here.

Koç Üniversitesi'ndeki ilk yılımda üç kere konuşmuşumdur kendisiyle, sonra bir anda bir iki muhabbet oldu, bir de baktık epey bir arkadaş olmuşuz bu Pınar'la, Sinan Ağabey'e gidip maç seyretmişiz, bizi Hayri'ye götürmüş aralarda derelerde. Uzaklara gitti sonra doktora yapmaya ama kısmet işte, Leiden'e düştü yolum doktoramın son senesinde, bu son on ayda iyice bir sıkı fıkı olduk, iyice bir laubalileştik.

Neslihan ofise havuz gözlüğüyle geldiği gün ofisteki otoritesini kaybetti ama sonra çok iyi arkadaş olduk. Sonra Neslihan ofise pencere çizdi ve esrarengiz olaylar birbiri ardına gelişti. Ofiste Starless dinledik, Space akımı getirdik ve Gezi'ye gittik. Bir gece 2'de dürüm söylemeye çalıştık. Buralardan gidip Atina'ya yelken açtı sonra ama biz yine de Filotea'yla Xalanthri'yi karıştırdığımız yetmezmiş gibi üstüne bir de Cambaz'da dilektaşı heyecanı yaşadık.

Nicat'a Sabri diyorum ben, o da bana Sabri diyor, böyle de tuhaf adamlarız, belki de bu yüzden çok iyi anlaşmışızdır. Şunu yaptık bunu yaptık demem manasız olacak zira yediğimiz içtiğimiz ayrı gitmez lafının tam tecelli etmiş haliydik doktora boyunca. Kongrede president olmalar, simplexler, soğukta dışarda manasızca beklemeler, kutular mutular, çok şey var yazacak ama burası yeri değil tabii.

Doktora serüvenim boyunca, doktoradan önce olduğu ve bundan sonra da hep olacağı gibi ailem, yani annem, kardeşim, anneannem, dayım ve doktora serüvenimin başlarında aramızdan ayrılan rahmetli dedem, ne olursa olsun hep yanımda oldular. Bunun farkındayım.

Contents

Nomenclature

d	spatial dimension
$\mathbb{N}$	set of natural numbers including zero
$\mathbb{R}^d$	d-dimensional Euclidean space
$\vec{0}$	origin in $\mathbb{R}^d$
$\mathbb{B}(\mathbb{R}^d)$	Borel σ-algebra on $\mathbb{R}^d$
t	time
supp	support of a function
$B(x, r)$	open ball with radius r centered at x
$\bar{B}(x, r)$	closed ball with radius r centered at x
$Z = (Z(t))_{t \geq 0}$	branching Brownian motion (BBM)
$X = (X(t))_{t \geq 0}$	continuous-time Markov branching process
$N = (N(n))_{n \in \mathbb{N}}$	Galton-Watson process (GWP)
β	branching rate
$(p_k)_{k \in \mathbb{N}}$	offspring distribution
f	probability generating function for the offspring distribution
μ	mean number of offspring
m	mean number of offspring $- 1$
P	probability law for branching process
E	expectation corresponding to branching process

τ	extinction time
$\mathcal{E}$	event of extinction
q	probability of extinction
$R(t)$	range of BBM at time t
Π	Poisson random measure (PRM)
ν	mean measure for Poisson random measure
K	random trap configuration
$\mathbb{P}$	probability measure for PRM
$\mathbb{E}$	expectation corresponding to PRM
T	first trapping time
$M(t)$	radius of the minimal ball containing the range of BBM

Chapter 1

Introduction

1.1 Motivation

A typical class of trapping problems involves a randomly moving system of particles in space, which contains a collection of randomly located traps. The main object of interest in such problems is the probability that the system of particles avoids the traps up to time t averaged over all trap configurations. The particles and traps are usually spheres or points. Trapping problems of this sort find various applications in chemistry and physics (see [21]), and in some models the particles branch or reproduce randomly as they move in space.

Branching Brownian motion (BBM) among random obstacles functioning as traps has been studied recently in [14, 15, 17, 28]. This thesis concerns a BBM that evolves in $\mathbb{R}^d$, where a random field of Poissonian traps are present. We are mainly interested in the trap-avoiding probability of the system up to time t asymptotically as t tends to infinity, where the Galton-Watson process (GWP) underlying the BBM has a general offspring distribution. The problem in this thesis originates from two articles, Engländër [14], and Engländër and den Hollander [15], where the same problem was solved for the case of strictly dyadic branching for the underlying GWP. We aim at extending their results to BBMs with general offspring distribution. Investigation of this problem leads us to finding the speed of BBM conditioned on non-extinction, which is of independent interest.

1.2 Problem description and notation

Our model involves two major random components: a branching Brownian motion, which is an extensively studied example of a spatial branching process, and a Poissonian field of random traps in $\mathbb{R}^d$, which makes a popular setting for contemporary random media problems [41].

Let $Z = (Z(t))_{t\geq 0}$ denote a d-dimensional BBM with branching rate $\beta > 0$ and offspring distribution λ, where λ is a probability measure on $\mathbb{N} = \{0, 1, 2, \ldots\}$. The process starts with a single particle at the origin, which performs a Brownian motion in $\mathbb{R}^d$ for a random time which is distributed exponentially with constant parameter β. Then, the particle dies and simultaneously gives birth to a random number of particles distributed according to λ. Similarly, each offspring particle repeats the same procedure independently of all others and of the history of the process up to her birth, starting from the position of her parent. In this way, one obtains a measure-valued Markov process $Z = (Z(t))_{t\geq 0}$, where $Z(t)$ can be identified as a particle configuration for given $t \geq 0$. By assumption, $Z(0) = \delta_{\vec{0}}$, where $\vec{0}$ denotes the origin. The total mass process $|Z| = (|Z(t)|)_{t\geq 0}$ is a continuous-time Markov branching process with rate β and offspring distribution λ, and the number of particles in generation n of Z is a (discrete-time) Galton-Watson process $N = (N(n))_{n\in\mathbb{N}}$ with offspring distribution λ. The particle present at $t = 0$ makes up the 0^{th} generation, the offspring of the initial particle make up the 1^{st} generation, and so forth. We denote the extinction time of the process $|Z|$ by τ, formally written as $\tau = \inf\{t \geq 0 : |Z(t)| = 0\}$, where we use the convention that $\inf\emptyset = \infty$. We then denote the event of extinction of the process $|Z|$ by $\mathcal{E}$, and formally write $\mathcal{E} = \{\tau < \infty\}$. We use the term *non-extinction* for the event $\mathcal{E}^c$. Let P be the probability law for the process Z, and E the corresponding expectation.

We now create a random environment in $\mathbb{R}^d$ via a Poisson random measure (PRM), which will be the second major random component of our model. Let $\mathbb{B}(\mathbb{R}^d)$ be the collection of d-dimensional Borel sets, and let Π denote a PRM on $\mathbb{B}(\mathbb{R}^d)$, with a boundedly finite mean measure ν such that $\mathrm{d}\nu/\mathrm{d}x$ exists, where $\mathrm{d}x$ is for the Lebesgue measure. In this work, two instances of ν will be considered. Firstly, as standard, we will work with a uniform field of traps, which corresponds to

$$\mathrm{d}\nu/\mathrm{d}x = v \tag{1.1}$$

for some constant $v > 0$. Then, we will consider a radially decaying field of traps, where $\mathrm{d}\nu/\mathrm{d}x$ is continuous on $\mathbb{R}^d$ and

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} \sim \frac{l}{|x|^{d-1}}, \quad |x| \to \infty, \quad l > 0. \tag{1.2}$$

It is the PRM with mean measure as in (1.2) that makes our problem much richer, and leads us to finding the speed of BBM conditioned on non-extinction. The random trap configuration with radius a attached to Π on $\mathbb{R}^d$ is defined as

$$K := \bigcup_{x_i \in \mathrm{supp}(\Pi)} \bar{B}(x_i, a), \tag{1.3}$$

where $\bar{B}(x,a)$ is the closed ball of radius $a > 0$ centered at $x \in \mathbb{R}^d$. Let $\mathbb{P}$ be the probability measure for the PRM, and $\mathbb{E}$ the corresponding expectation.

The branching Brownian motion Z is assumed to live in the random environment consisting of Poissonian traps. For $t \geq 0$, let

$$R(t) = \bigcup_{s \in [0,\mathrm{t}]} \mathrm{supp}(Z(s)) \tag{1.4}$$

be the range of Z up to time t. Note that $R(t)$ is the totality of all points in $\mathbb{R}^d$ that are hit by Z up to time t. Now let T be the first time that Z hits a trap,

$$T = \inf\left\{t \geq 0 : R(t) \cap K \neq \emptyset\right\}. \tag{1.5}$$

Then, the event of trap-avoiding up to time t of Z among the Poissonian field of traps is given by $\{T > t\}$. Our main quantity of interest in this work is

$$(\mathbb{E} \times P)(T > t),$$

the annealed (averaged) trap-avoiding probability of the system up to time t, for a BBM with a general offspring distribution λ.

1.3 History of the problem

The history of the present problem dates back to 1975, when Donsker and Varadhan considered a single particle performing Brownian motion in $\mathbb{R}^d$, and studied the asymptotic behavior of the volume of its associated Wiener sausage [11]. The Wiener sausage with radius a is defined as

$$S_t^a := \bigcup_{0 \leq s \leq t} B(W(s), a),$$

where $W = (W(s))_{s \geq 0}$ stands for the Wiener process (Brownian motion) in $\mathbb{R}^d$, and $B(x,a)$ is the open ball with radius a centered at x. The asymptotic distribution of the volume of the Wiener sausage is characterized by the result

$$\lim_{t \to \infty} t^{-\frac{d}{d+2}} \log E \exp(-v|S_t^a|) = -c(d,v), \tag{1.6}$$

where E is the expectation corresponding to the Wiener process, $c(d,v)$ is a constant depending on $v>0$ and the dimension d, and $|S_t^a|$ is the volume of the Wiener sausage accumulated by time t.

By Tonelli's theorem, it is easy to see that

$$E\exp(-v|S_t^a|) = E\left[e^{-v|S_t^a|}\frac{(v|S_t^a|)^0}{0!}\right] = (E\times\mathbb{P})(\bar{T}>t) = (\mathbb{E}\times P)(\bar{T}>t), \quad (1.7)$$

where $\bar{T}$ is the first time the Brownian particle hits a uniform field of Poissonian traps with intensity $v>0$. In (1.7), $\mathbb{P}$ and $\mathbb{E}$ stand respectively for the probability and expectation corresponding to the PRM as before. In 1998, in view of the equivalence in (1.7), the work of Sznitman [41] provided an alternative proof of (1.6) in the context of Brownian survival among Poissonian traps. Note that (1.7) casts the original problem of Donsker and Varadhan as a trapping problem similar to our problem, but without branching, so we may regard (1.6) as a solution to our problem when $p_1=1$, and see that the tail of the annealed probability of trap-avoiding decays subexponentially (as reflected by the factor $t^{-d/(d+2)}$ in (1.6)) when there is no branching.

In 2000, Engländer added branching to the problem of Brownian survival in a Poissonian field of traps [14], where he considered a uniform PRM with mean measure as in (1.1). He studied the case of strictly dyadic branching, where each particle produces precisely two offspring upon branching, which corresponds to $p_2=1$ according to our notation. We emphasize that henceforth, the notation that we use will be consistent with the notation described in the previous section. Engländer's result from [14] is

$$\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\times P)(T>t) = -\beta \quad \text{for} \quad d\geq 2. \quad (1.8)$$

The key feature of the result above is that the asymptotic decay of the annealed trap-avoiding probability has become exponential as a result of branching, and moreover the dominating effect of branching over spatial motion is clear, since the result depends only on a branching parameter.

In 2003, in search for extending the result (1.8) to $d=1$, Engländer and den Hollander [15] considered a radially decaying field of traps, where the trap points were given as the atoms of a PRM with mean measure as in (1.2). Their result is as follows:

$$\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\times P)(T>t) = -I(l,\beta,d), \quad (1.9)$$

where I is given as a variational formula, and depends on the trap intensity $l>0$, branching rate $\beta > 0$, and dimension $d \geq 1$. Note that a uniform mean measure is a special case of (1.2) when $d = 1$, hence the result in (1.9) is an extension of (1.8) when $d = 1$. Similar to the case of a uniform field, in this case of radially decaying traps, we see that the tail of the annealed trap-avoiding probability decays exponentially. As opposed to the case of a uniform field, now the motion component has also become important in that the rate of decay, namely I, depends on the spatial dimension and the trap intensity, as well as on the rate of branching.

The starting point of the present work is the observation that in both works described above, namely in [14, 15], the branching was assumed to be strictly dyadic, so there was no randomness coming from branching. We wish to allow a general branching mechanism for particles, thereby to introduce a new level of randomness into the problem.

The reader is referred to [16] for an interesting survey article on the topic of BBM among Poissonian traps, and to [17, 28] for various related problems.

1.4 Outline

The thesis is organized as follows. In Chapter 2, we focus on the Galton-Watson process and the continuous-time Markov branching process, underlying the branching Brownian motion. We first provide some general theory on Galton-Watson processes and continuous-time Markov branching processes, and then building on this general theory, prove two results (Proposition 2.9 and Proposition 2.14) that are central in the proof of Theorem 3.9, Theorem 4.1 and Theorem 4.4. Proposition 2.9 provides a decomposition of supercritical GWPs into two sets of particles, one of which is composed of 'skeleton' particles, which have infinite lines of descent and constitute the 'skeleton' of the process, and the other set is composed of particles with finite line of descent, the so-called 'doomed' particles. Proposition 2.14 shows that the decomposed process can be regarded as a two-type GWP. This chapter is written to be mostly self-contained with the exception of a few technical results, which are stated without proof.

Chapter 3 is devoted to finding the speed of a BBM, conditioned on non-extinction, where the underlying GWP has a general offspring distribution. First, we introduce the notion of 'speed' for a BBM, followed by some historical development of the subject. Then, we give the famous result of McKean [30, 31], which holds when $d = 1$ and the underlying GWP has $p_0 = 0$, as a proposition, followed by its proof for the sake of completeness. We generalize McKean's result to all dimensions in Proposition 3.8, and finally to the case where $p_0 > 0$ in Theorem 3.9, where we condition the BBM on non-extinction. Theorem 3.9 is the main result of Chapter 3, whose proof is based heavily on the skeleton decomposition of supercritical GWPs, and on the interpretation of this decomposition as a two-type GWP, both of which were studied in Chapter 2. In the proof, it is shown by a comparison argument that the doomed particles do not contribute to the conditional speed. We emphasize that the problem of the conditional speed of BBM emerges naturally on the way to proving Theorem 4.4.

Chapter 4 brings the BBM in contact with the random environment composed of a Poissonian field of traps in $\mathbb{R}^d$. We always work with a BBM with a general offspring distribution. We first consider the simple case of a random environment with a uniform field of Poissonian traps, and prove a result similar to (1.8) in Theorem 4.1. Next, we consider the more complicated case of a radially decaying trap field, with corresponding mean measure given by (1.2), and prove a result similar to (1.9) in Theorem 4.4, which arises as a non-trivial application of Theorem 3.9. In both instances of the Poisson random measure, we show that, conditioned on non-extinction, the doomed particles of the decomposition of the supercritical GWP do not contribute on an exponential scale to the asymptotic decay rate of the trap-avoiding probability. Finally, we prove Theorem 4.5, which is a technical result identifying the rate function of Theorem 4.4 through analysis.

Chapter 2

Skeleton Decomposition of a Supercritical Galton-Watson Process

In this chapter, we first define a Galton-Watson process, which is a discrete-time Markov chain on the nonnegative integers. We state and prove the extinction criterion for GWPs. Then, we define a continuous-time Markov branching process, which can be thought of as the continuous-time analogue of a GWP. After proving two results regarding continuous-time Markov branching processes, we give the central result of this chapter, the skeleton decomposition of a supercritical GWP. Finally, we introduce the notion of a multitype GWP, and translate the aforementioned decomposition into the language of multitype GWPs. This chapter is intended to present the relevant background on branching processes that underlie a BBM. The material in this chapter is standard except the computation of the mean matrices (2.38) and (2.39), and in general could be found in standard monographs on branching processes such as [20] and [4]. The main results of this chapter are central in proving Theorem 3.9, Theorem 4.1, and Theorem 4.4.

2.1 Galton-Watson process

Galton-Watson processes are the simplest example of branching processes. They were introduced by Francis Galton in 1873 as a model for the evolution of last names, after which he published [19] with Henry William Watson. We refer the reader to [20] and [4] for a detailed treatment on GWPs.

Let λ be a probability measure on $\mathbb{N} = \{0, 1, 2, \ldots\}$, and set $p_k = \lambda(k)$ for $k \in \mathbb{N}$. Throughout this thesis, $(p_k)_{k\in\mathbb{N}}$ will be used to denote the offspring distribution, and

P and E will denote the probability law and corresponding expectation, respectively, for branching processes unless otherwise stated.

Definition 2.1. Let ξ be a random variable having the probability distribution $P(\xi{=}k){=}p_k$, $k \in \mathbb{N}$. A Galton-Watson process with offspring distribution $(p_k)_{k\in\mathbb{N}}$ is a Markov chain $N{=}(N(n))_{n\in\mathbb{N}}$ on the nonnegative integers, which evolves according to the following recursion:

$$N(0) = 1, \quad N(n+1) = \sum_{k=1}^{N(n)} \xi_k^n, \tag{2.1}$$

where ξ_k^n, $k \geq 1$, $n \geq 0$, are independent random variables that are identically distributed copies of ξ. If $N(n) = 0$, then $N(n+1) = 0$ with probability 1.

Equation (2.1) is called the branching process equation, and gives the transition probabilities

$$P_{ij} := P(N(n+1) = j | N(n) = i), \quad i, j, n = 0, 1, 2, \ldots$$

for the Markov chain N, provided that the conditional probability above is well defined, i.e., $P(N(n) = i) > 0$. If $P(N(n) = i) = 0$, then $P_{ij} = \delta_{0j}$.

We interpret the process N as a population evolution model as follows. Consider a population of particles, where $N(n)$ represents the population size in the nth generation. The population starts at time 0 with 1 particle, and each particle in the nth generation gives birth to a random number of offspring, distributed according to $(p_k)_{k\in\mathbb{N}}$, independent of the history of the process and all other particles existing at the present. The recursion (2.1) states that the population size at the $(n+1)$st generation is the sum of the number of offspring of the particles in the nth generation, where each particle in the nth generation produces offspring according to the distribution $(p_k)_{k\in\mathbb{N}}$, independently of all others.

Let $f : [0, 1] \to [0, 1]$ be the probability generating function (p.g.f.) for ξ,

$$f(s) = \sum_{k=0}^{\infty} p_k s^k,$$

and let μ be the mean number of offspring for a particle,

$$\mu := \sum_{k=0}^{\infty} k p_k = f'(1-).$$

We suppose that $|s| \leq 1$ for all probability generating functions considered henceforth, and interpret $0^0 = 1$ so that $f(0) = p_0$.

Note that if $N(n) = 0$ for some $n > 0$, then $N(m) = 0$ for all $m \geq n$, that is, if the population has died out by generation n, then it will not reappear after generation n. Define the extinction time for N as

$$\tau := \inf \{n > 0 : N(n) = 0\},$$

and let $\mathcal{E}$ be the event of extinction, formally defined as $\mathcal{E} := \{\tau < \infty\}$.

If $\mathcal{E}$ occurs, we say that the process has become extinct. Let $q := P(\mathcal{E})$. Then,

$$\begin{aligned} q &= P(N(n) = 0 \text{ for some } n) \\ &= P\left(\cup_{n=1}^{\infty} \{N(n) = 0\}\right) \\ &= \lim_{n\to\infty} P(N(n) = 0), \end{aligned} \tag{2.2}$$

where we have used the continuity from below of measures in passing to the second equality (note that $\{N(n) = 0\} \nearrow \mathcal{E}$ as events). We now give a preparatory lemma, followed by the extinction criterion for GWPs. Generating function methods will be used in the proof. Let $f_n : [0, 1] \to [0, 1]$, given by

$$f_n(s) = \sum_{k=0}^{\infty} P(N(n) = k)s^k, \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

be the p.g.f. for the number of particles in the nth generation.

Lemma 2.2.

$$f_{n+1}(s) = f_n(f(s)) = f(f_n(s)) \quad \forall\ n \in \mathbb{N}. \tag{2.4}$$

Proof. Using the definition of f_{n+1} and conditioning on the number of particles in the nth generation,

$$\begin{aligned} f_{n+1}(s) &= \sum_{k=0}^{\infty} P(N(n+1) = k)s^k \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P(N(n+1) = k|N(n) = j)P(N(n) = j)s^k \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P(N(n) = j)P(\xi_1^n + \ldots + \xi_j^n = k)s^k \\ &= \sum_{j=0}^{\infty} P(N(n) = j) \sum_{k=0}^{\infty} P(\xi_1^n + \ldots + \xi_j^n = k)s^k, \end{aligned} \tag{2.5}$$

where we have used (2.1) in passing to the third equality, and changed the order of summations (since all terms are nonnegative) in passing to the fourth equality. Since ξ_r^n, $r = 1, \ldots, j$, are independent identically distributed random variables with common probability generating function f, the sum $\xi_1^n + \ldots + \xi_j^n$ has the probability generating function f^j, so that (2.5) becomes

$$\begin{aligned} f_{n+1}(s) =& \sum_{j=0}^{\infty} P(N(n) = j)[f(s)]^j \\ =& f_n(f(s)), \end{aligned}$$

where we have used the definition of f_n in passing to the last equality. Then, by induction, it follows that

$$f_{n+1}(s) = f(f_n(s)).$$

□

Proposition 2.3 (Extinction criterion)**.** *Assume that $p_1 \neq 1$. Then, the probability of extinction for a GWP with mean μ is 1 if $\mu \leq 1$ and less than 1 if $\mu > 1$.*

Proof. Suppose that $p_0 + p_1 = 1$. Then, there is at most 1 particle in any generation. Therefore, $\lim_{n\to\infty} P(N(n) = 0) = \lim_{n\to\infty}(1 - P(N(n) = 1)) = 1 - \lim_{n\to\infty}(p_1)^n = 0$, which implies that extinction is certain, i.e., $q = 1$. Now assume that $p_0 + p_1 < 1$. Then, f is strictly convex in $(0, 1)$ since $f''(s) = \sum_{k=2}^{\infty} k(k-1)p_k s^{k-2} > 0$.

First, we show that q of (2.2) solves the equation

$$f(s) = s. \tag{2.6}$$

This follows since, in order for the process to become extinct, either the initial particle should have 0 offspring, or all of the subprocesses initiated by the offspring of the initial particle should become extinct. Conditioning the process on $N(1)$, by independence of particles, this translates to

$$q = p_0 + p_1 q + p_2 q^2 + \ldots = f(q).$$

Next, we show that q is the smallest nonnegative root of (2.6). Let x be a nonnegative root of (2.6). Then, as each f_n is nondecreasing,

$$f_n(0) \leq f_n(x) = x, \tag{2.7}$$

where Lemma 2.2 is used $n-1$ times to conclude that $f_n(x) = x$. Taking the limit as $n \to \infty$ on both sides of (2.7), and using (2.2), we conclude that $q \leq x$. So far, we have proved that q is the smallest nonnegative root of (2.6).

Recall that as $p_0 + p_1 < 1$, f is strictly convex, hence has strictly increasing first derivative on $(0, 1)$. Also, note that 1 is always a solution to (2.6), i.e., $f(1) = 1$. If

$\mu = f'(1-) \leq 1$, then $f(x) = x$ for some $x \in [0, 1)$ together with $f(1) = 1$ would imply by the mean value theorem that there exists a point $y \in (x, 1)$ such that $f'(y) = 1$, which contradicts f' being strictly increasing on $(0, 1)$ (since $\lim_{x \to 1^-} f'(x) = f'(1-)$ by Abel's theorem). Hence, if $\mu = f'(1-) \leq 1$, then $q = 1$.

Suppose that $\mu > 1$. If $p_0 = 0$, then clearly 0 is the smallest nonnegative root of (2.6), and $q = 0$. Now assume that $p_0 > 0$. As $\mu = f'(1-) > 1$, there exists a point x_0 in the left neighborhood of 1 such that $f(x_0) < x_0$. Together with $f(0) = p_0 > 0$, this implies, by the intermediate value theorem, that there exists a point $x_1 \in (0, x_0)$ such that $f(x_1) = x_1$. This shows that $q < 1$ when $\mu > 1$. $\square$

We refer the reader to the books of Harris [20], and Athreya and Ney [4] for more results and historical notes on GWPs.

2.2 Continuous-time Markov branching process

We would like to define a time-homogeneous Markov process $X = (X(t))_{t \geq 0}$ that is a natural extension to continuous-time of a GWP described in the previous section. The description below follows [20, Chapter 5] closely. Let X be a random function on $\mathbb{N}$, representing the size of a population, with the properties that for any t_0, if $X(t_0) = i$, then

- the subsequent history of the population depends only on i, and not on the history of the population before t_0, and not on the absolute time t_0,
- the families initiated by the particles at time t_0 are independent of one another.

The first property makes X a time-homogeneous Markov process, and implies that all particles are of the same type (i.e., they behave in the same way probabilistically). We must impose that the lifetimes of particles are independent and identically distributed exponential random variables in order to satisfy the first property. The second property is the branching property that singles out X from other Markov processes. As the process is time-homogeneous, it is enough to consider the transition probabilities $P_{ij}(t) := P(X(t) = j | X(0) = i)$ for $i, j \in \mathbb{N}$. The branching property is equivalent to the condition that

$$P_{ij}(t) = \sum_{r_1 + r_2 + \ldots + r_i = j} P_{1r_1}(t) P_{1r_2}(t) \cdots P_{1r_i}(t), \tag{2.8}$$

which, along with time-homogeneity, says that for any $t_0, t \geq 0$, the population size at time $t_0 + t$, starting with i particles at time t_0, is distributed as the sum of i independent populations, each starting with one particle.

As before, let λ be a probability measure on $\mathbb{N}$, $p_k = \lambda(k)$ for $k \in \mathbb{N}$, and let $0 < \beta < \infty$ be a real number. A continuous-time Markov branching process is constructed as follows. It starts with a single particle. If a particle is alive at a certain time, then its additional lifetime is an exponential random variable with parameter β. Upon death, it gives birth to k offspring with probability p_k, $k \in \mathbb{N}$, independent of other particles that are alive and of the history of the process. Hence, if i particles are present at a given time, the rate at which a death occurs is βi, and the new population size is $j \geq i-1$ with probability p_{j-i+1}. The process thus obtained is called the minimal process. Now we are ready to formally define a continuous-time Markov branching process.

Definition 2.4. A continuous-time Markov branching process with rate $0 < \beta < \infty$ and offspring distribution $(p_k)_{k\in\mathbb{N}}$ is a time-homogeneous Markov chain on $\mathbb{N}$ whose transition probabilities $(P_{ij}(t) : i, j \in \mathbb{N})$ satisfy the Kolmogorov forward equations:

$$\frac{d}{dt}P_{ij}(t) = -j\beta P_{ij}(t) + \beta \sum_{k=1}^{j+1} kP_{ik}(t)p_{j-k+1}, \quad P_{ij}(0+) = \delta_{ij}, \tag{2.9}$$

where δ_{ij} is the Kronecker delta symbol.

It is proved in [20] that the forward equations (2.9) are solved by a unique set of transition probabilities, and that these transition probabilities indeed satisfy the branching property (2.8). However, without any restriction on the offspring distribution, it is possible that the unique set of solutions to (2.9) satisfy $\sum_j P_{ij}(t) < 1$ for some $t > 0$. In this case, with positive probability, the process produces infinitely many particles in finite time, i.e., explodes. In order to avoid an explosion, we must have $\sum_j P_{ij}(t) = 1$ for all t. The following proposition provides a sufficient condition for $\sum_j P_{ij}(t) = 1$. The proof may be found in [20, Section 5.9].

Proposition 2.5. *A continuous-time Markov branching process is non-explosive if the mean number of offspring for a particle is finite, that is, if* $\mu := \sum_{k=0}^{\infty} kp_k < \infty$. *In this case,* $P(X(t) < \infty) = 1$, *where* X *is the minimal process.*

In the rest of this manuscript, to avoid explosions, we assume that $\mu < \infty$ for all branching processes considered. Also, note that so far we have defined a continuous-time Markov branching process only in the sense that we have given its transition probabilities. In order to complete the definition, we note that we may and always do assume that the path of the process $X = (X(t))_{t\geq 0}$ is a right-continuous step function.

Next, we give the extinction criterion for continuous-time Markov branching processes. As is probabilistically obvious by considering the number of individuals in generation n, the result is identical to that of GWPs. A rigorous proof may be found in [20, Section 6.5].

Proposition 2.6. *Assume that $p_1 \neq 1$. Then, the probability of extinction for a continuous-time Markov branching process with rate $0 < \beta < \infty$ and offspring mean μ is 1 if $\mu \leq 1$ and less than 1 if $\mu > 1$.*

When a particle in the population gives birth to exactly 1 offspring, the state of the process does not change, hence this scenario cannot be included in the underlying jump chain. It is probabilistically clear that a nonzero p_1 can be absorbed into the branching rate β. We see how to do this in the following lemma. Throughout this thesis, we assume that $0 < \beta < \infty$.

Lemma 2.7. ***(Reduction to** $p_1 = 0$**)*** *A continuous-time Markov branching process with rate β and offspring distribution $(p_k)_{k\in\mathbb{N}}$ is equal in law to a continuous-time Markov branching process with rate $\beta(1-p_1)$ and offspring distribution $(\bar{p}_k)_{k\in\mathbb{N}}$, where $\bar{p}_1 = 0$ and $\bar{p}_j = p_j/(1-p_1)$ for $j \neq 1$.*

Proof. Using standard theory of Markov chains, the q-matrix, where $q_{ij} = P'_{ij}(0+)$, for the process is obtained from the forward equation (2.9), and is given as follows:

$$q_{ij} = \begin{cases} 0 & \text{if } j < i-1, \\ -i\beta(1-p_1) & \text{if } j = i, \\ i\beta p_{j-i+1} & \text{if } j \geq i-1, j \neq i. \end{cases} \tag{2.10}$$

Let T_i be the sojourn time in state $i \in \mathbb{N}$ and let $P_{ij} := P(X(T_i) = j | X(0) = i)$, $i, j \in \mathbb{N}$ be the transition probabilities for the underlying jump chain. Using $P_{ij} = \frac{q_{ij}}{-q_{ii}}$ and (2.10), we have

$$P_{ij} = \begin{cases} \frac{p_{j-i+1}}{1-p_1} & \text{if } j \geq i-1, j \neq i, i \geq 1 \\ 0 & \text{otherwise }. \end{cases} \tag{2.11}$$

Since all particles in the population are of the same type and independent of each other, it suffices to consider the particle present at time $t = 0$. Setting $i = 1$ yields

$$\begin{aligned} -q_{11} =& \beta(1-p_1), \\ P_{1j} =& \begin{cases} \frac{p_j}{1-p_1} & \text{if } j \geq 0, j \neq 1 \\ 0 & \text{if } j = 1. \end{cases} \end{aligned}$$

The claim follows. $\square$

Above, we have given an analytic proof of Lemma 2.7. Probabilistically, the process has the following structure. The probability P_{ij} that the process makes a transition from state i to state j, $i \neq j$, is the same as the probability that a particle gives birth to $j-i+1$ new particles, conditioned on not giving birth to exactly 1 particle. Hence, we obtain the jump probabilities in (2.11). As for the sojourn time in state 1, it is not the same as a particle lifetime since a particle may have exactly 1 offspring. The sojourn time in state 1 is a geometric sum with parameter $(1-p_1)$ of independent exponential random variables, all with the same parameter β. In other words, if we let T to be the sojourn time in state 1, then

$$T = T_1 + \ldots + T_M, \text{ where } M \sim Geo(1-p_1),$$

where each T_j is an exponential random variable with parameter β, and $Geo(p)$ is written for the geometric distribution with parameter p. It is well known that a geometric sum with parameter p of exponential random variables, all with same parameter β is also exponential, with parameter βp. Hence, the conclusion follows. We may and do assume henceforth that $p_1 = 0$ for all continuous-time Markov branching processes considered.

The next result gives the expected population size for a continuous-time Markov branching process at any given time.

Lemma 2.8. ***(Expected population size)*** *Let $X = (X(t))_{t\geq 0}$ be a continuous-time Markov branching process with rate β and offspring distribution $(p_k)_{k\in\mathbb{N}}$, with corresponding probability generating function f. Let $\mu = \sum_{k=0}^{\infty} kp_k$ be the mean number of offspring. Then,*

$$E[X(t)] = e^{\beta m t}, \tag{2.12}$$

where $m = \mu - 1$.

Proof. Let $p_k(t) = P(X(t)=k)$, and let $\phi(s,t) = \sum_{k=0}^{\infty} p_k(t)s^k$ be the p.g.f. for $X(t)$. Also, let $\sigma(t) = E[X(t)]$. Conditioning on the first branching time, we have

$$p_k(t) = \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \left[p_0\delta_{0k} + \sum_{j=1}^{\infty} p_j \sum_{k_1\geq 0,\ldots,k_j\geq 0}^{k_1+\ldots+k_j=k} p_{k_1}(t-\tau)\cdots p_{k_j}(t-\tau) \right].$$

By definition of ϕ, we have

$$\begin{aligned}
\phi(s,t) &= \sum_{k=0}^{\infty} s^k \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \left[p_0 \delta_{0k} + \sum_{j=1}^{\infty} p_j \sum_{k_1\geq 0,\ldots,k_j\geq 0}^{k_1+\ldots+k_j=k} p_{k_1}(t-\tau)\cdots p_{k_j}(t-\tau) \right] \\
&= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \sum_{k=0}^{\infty} s^k \left[p_0 \delta_{0k} + \sum_{j=1}^{\infty} p_j \sum_{k_1\geq 0,\ldots,k_j\geq 0}^{k_1+\ldots+k_j=k} p_{k_1}(t-\tau)\cdots p_{k_j}(t-\tau) \right] \\
&= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \left[p_0 + \sum_{j=1}^{\infty} p_j \sum_{k=0}^{\infty} s^k \sum_{k_1\geq 0,\ldots,k_j\geq 0}^{k_1+\ldots+k_j=k} p_{k_1}(t-\tau)\cdots p_{k_j}(t-\tau) \right],
\end{aligned}$$

where, in passing to the second equality we have changed the order of summation and integration, and in passing to the third equality we have changed the order of two summations, since all quantities involved are positive. Observe that the coefficients of $\sum_{k=0}^{\infty} s^k \sum_{k_1\geq 0,\ldots,k_j\geq 0}^{k_1+\ldots+k_j=k} p_{k_1}(t-\tau)\cdots p_{k_j}(t-\tau)$ are the j-fold convolution of the power series $\phi(s,t-\tau) = \sum_{k=0}^{\infty} p_k(t-\tau)s^k$ with itself. Hence, we have

$$\begin{aligned}
\phi(s,t) &= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \left(p_0 + \sum_{j=1}^{\infty} p_j [\phi(s,t-\tau)]^j \right) \\
&= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} f\left(\phi(s,t-\tau)\right), \qquad (2.13)
\end{aligned}$$

where we have used the definition of f in passing to the second equality. We have thus obtained an integral equation (2.13) for $\phi(s,t)$.

Now observe that

$$\left.\frac{\partial\phi(s,t)}{\partial s}\right|_{s=1} = \sum_{k=1}^{\infty} k p_k(t) = \sigma(t). \qquad (2.14)$$

By (2.13),

$$\begin{aligned}
\left.\frac{\partial\phi(s,t)}{\partial s}\right|_{s=1} &= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} f'(\phi(1,t-\tau)) \left.\frac{\partial\phi(s,t-\tau)}{\partial s}\right|_{s=1} \\
&= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} f'(1) \left.\frac{\partial\phi(s,t-\tau)}{\partial s}\right|_{s=1} \\
&= \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \mu \left.\frac{\partial\phi(s,t-\tau)}{\partial s}\right|_{s=1}, \qquad (2.15)
\end{aligned}$$

where f' is the derivative of f, and we have used that $\phi(1,t-\tau) = 1$ (the condition $\sum_j P_{ij}(t) = 1$ is used here) and $f'(1) = \mu$ in passing from the first to second, and second to third equality, respectively. From (2.14) and (2.15), it follows that

$$\sigma(t) = \mu \int_0^t \mathrm{d}\tau \beta e^{-\beta\tau} \sigma(t-\tau). \qquad (2.16)$$

Multiplying both sides of (2.16) by $e^{\beta t}$ and making the change of variables $u = t - \tau$ yield

$$\sigma(t)e^{\beta t} = \mu\beta \int_0^t \mathrm{d}u e^{-\beta u}\sigma(u).$$

Differentiating both sides and simplifying, we arrive at

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = (\mu - 1)\beta\sigma(t). \tag{2.17}$$

Recall that $\mu - 1 = m$. Solving the first order differential equation (2.17) with the initial condition $\sigma(0) = 1$ gives

$$\sigma(t) = e^{\beta m t}.$$

□

We note that an alternative shorter proof of Lemma 2.8 may be given using the Kolmogorov forward equations (2.9), however we prefer to give the proof above as it is more direct.

2.3 Galton-Watson process conditioned on non-extinction

In this section, we describe a GWP conditioned on non-extinction. Recall that $N = (N(n))_{n\in\mathbb{N}}$ denotes a GWP with offspring distribution $(p_k)_{k\in\mathbb{N}}$, and extinction is the event $\{N(n) \to 0\}$. We say that a particle is of infinite line of descent if the family initiated by that particle does not go extinct, and call such particles *skeleton* particles. Following the terminology in [40], particles which are not of infinite line of descent will be called *doomed*. It is clear that the event of non-extinction occurs if and only if the initial particle is a skeleton particle. In other words, if we let $N^*(n)$ be the number of skeleton particles in generation n, then $N^*(0) = \mathbb{1}_{\{N(n)\not\to 0\}}$, where $\mathbb{1}_A$ is the indicator function for event A.

Note that if $\mu \leq 1$, then extinction is certain, and all particles are doomed. On the other hand, if $p_0 = 0$, then each particle is certain to give at least 1 offspring, so all particles are of infinite line of descent. The following lemma provides a decomposition for GWPs conditioned on non-extinction in the non-trivial case, where the process is supercritical (i.e., $\mu > 1$) and $p_0 > 0$. Note that $\mu > 1$ and $p_0 > 0$ together are equivalent to $0 < q < 1$, where q is the probability of extinction as before. The proposition below and its proof are mostly taken from [29].

Proposition 2.9 (Decomposition of the supercritical GWP)**.** *Let $N = (N(n))_{n\in\mathbb{N}}$ be a supercritical GWP with offspring distribution $(p_k)_{k\in\mathbb{N}}$, where $p_0 > 0$, and f be the corresponding p.g.f. Let q be the probability of extinction for N, and define $\bar{q} := 1 - q$. Let $N^* = (N^*(n))_{n\in\mathbb{N}}$ be the process, where $N^*(n)$ is the number of particles in generation n that have an infinite line of descent. Let ξ and ξ^* be random variables that are identical copies of $N(1)$ and $N^*(1)$, respectively.*

(a) The joint p.g.f. of $\xi - \xi^*$ and ξ^* is

$$E\left[s^{\xi-\xi^*}t^{\xi^*}\right] = f(qs + \bar{q}t). \tag{2.18}$$

(b) The law of N given extinction is the same as that of a GWP with p.g.f.

$$\tilde{f}(s) := f(qs)/q. \tag{2.19}$$

(c) The law of N^* given non-extinction is the same as that of a GWP with p.g.f.

$$f^*(s) := [f(q + \bar{q}s) - q]/\bar{q}. \tag{2.20}$$

(d) The law of N given non-extinction is the same as that of a process $\overline{N}$ generated as follows: To each particle of N^* having n_0 children, add n_1 more children, that each start an independent GWP with p.g.f. $\tilde{f}$, where n_1 has the p.g.f.

$$f_{n_1}(s) := \frac{(D^{n_0}f)(qs)}{(D^{n_0}f)(q)},$$

where D is the differentiation operator, and all n_1 and all GWPs added are mutually independent given N^*. The resultant process is $\overline{N}$.

(e) Let $X = (X(t))_{t\geq 0}$ be a continuous-time Markov branching process with rate β and offspring distribution $(p_k)_{k\in\mathbb{N}}$, and let $X^*(t)$ be the number of skeleton particles of X present at time t. Then, the process $X^* = (X^*(t))_{t\geq 0}$ given non-extinction is equal in law to a continuous-time Markov branching process with rate $\beta(1 - f'(q))$ and offspring distribution, whose p.g.f. is

$$\bar{f} := [f^*(s) - f'(q)s]/(1 - f'(q)). \tag{2.21}$$

Remark. The reader should note that in (2.20), positive mass at 1 is allowed.

Proof. Let $Y_{i,j}^n$ be the indicator that the jth child of the ith particle in generation n is of infinite line of descent. Observe that $(Y_{i,j}^n)_{i,j\geq 1}$ are independent copies of a generic random variable Y for each $n \in \mathbb{N}$, where $P(Y=1)=\bar{q}$ and $P(Y=0)=q$. Let ξ_i^n denote the number of offspring of the ith particle at generation n, where ξ_i^{*n} of them are skeleton particles, so that

$$\xi_i^{*n} = \sum_{j=1}^{\xi_i^n} Y_{i,j}^n, \quad n = 0,1,2,\dots, \quad i = 1,2,\dots.$$

(a) Let Y_j, $j = 1,2,\dots$ be independent copies of Y. Then,

$$\begin{aligned} E\left[s^{\xi-\xi^*}t^{\xi^*}\right] =& E\left[E\left[s^{\xi-\xi^*}t^{\xi^*} \mid \xi\right]\right] \\ =& E\left[E\left[s^{\sum_{j=1}^{\xi}(1-Y_j)}t^{\sum_{j=1}^{\xi}Y_j} \mid \xi\right]\right] \\ =& E\left[E\left[s^{1-Y}t^{Y}\right]^{\xi}\right] = E\left[(qs+\bar{q}t)^{\xi}\right] \\ =& f(qs+\bar{q}t). \end{aligned}$$

(b) We show that, conditional on the event $\{N(k) \to 0\}$, given the σ-algebra $\mathscr{F}_{n,i}$ generated by ξ_k^n, $k \neq i$ and ξ_k^m, $m < n$, $k \geq 1$, the p.g.f. of ξ_i^n is $\tilde{f}$. Indeed,

$$\begin{aligned} E\left[s^{\xi_i^n} \mid \mathscr{F}_{n,i}, N(k) \to 0\right] =& E\left[s^{\xi_i^n} \mid \xi_i^{*n} = 0\right] \\ =& E\left[s^{\xi}\mathbb{1}_{\{\xi^*=0\}}\right]/q = E\left[s^{\xi-\xi^*}0^{\xi^*}\right]/q \\ =& f(qs+\bar{q}0)/q, \end{aligned}$$

where we have used that $0^x = \mathbb{1}_{\{0\}}(x)$ in the third equality, and part (a) in the last equality. Note that conditioning on $\mathscr{F}_{n,i}$ is necessary to show that particles behave independently of each other.

(c) The proof is similar to that of part (b). We show that conditional on the event $\{\xi_i^{*n} \neq 0\}$, given the σ-algebra $\mathscr{F}_{n,i}^*$ generated by ξ_k^{*n}, $k \neq i$ and ξ_k^{*m}, $m < n$, $k \geq 1$, the p.g.f. of ξ_i^{*n} is f^*. Indeed,

$$\begin{aligned} E\left[s^{\xi_i^{*n}} \mid \mathscr{F}_{n,i}^*, \xi_i^{*n} \neq 0\right] =& E\left[s^{\xi_i^{*n}} \mid \xi_i^{*n} \neq 0\right] \\ =& E\left[s^{\xi^*}(1-\mathbb{1}_{\{\xi^*=0\}})\right]/\bar{q} = \left[E\left[1^{\xi-\xi^*}s^{\xi^*}\right] - P(\xi^*=0)\right]/\bar{q} \\ =& (f(q1+\bar{q}s)-q)/\bar{q}, \end{aligned}$$

where we have again used part (a) in the last equality.

(d) Since N^* contains all the skeleton particles, any particle that is added to N^* must be of finite line of descent, and hence starts an independent GWP with p.g.f. $\tilde{f}$, as shown in part (b). Therefore, it suffices to prove that

$$E\left[s^{\xi-\xi^*}|\xi^*=n_0\right]=\frac{(D^{n_0}f)(qs)}{(D^{n_0}f)(q)}.$$

By conditioning on ξ^*, we have

$$E\left[s^{\xi-\xi^*}t^{\xi^*}\right]=\sum_{k=0}^{\infty}t^kE\left[s^{\xi-k}\mid\xi^*=k\right]P(\xi^*=k). \tag{2.22}$$

Differentiating both sides of (2.18) k times with respect to t, setting $t=0$ and using (2.22) yield

$$E\left[s^{\xi-\xi^*}\mid\xi^*=k\right]=c(D^kf)(qs), \tag{2.23}$$

where $c>0$ is a constant. Substituting $s=1$ in (2.23) gives $c=1/(D^kf)(q)$. Note that in (2.22), we have implicitly assumed that $P(\xi^*=k)>0$ for each $k\in\mathbb{N}$. Otherwise they can be excluded from the sum, and this does not affect the argument.

(e) We apply Lemma 2.7 to f^*. Recall that $p_1=0$ and $p_0>0$ by assumption.

$$\begin{aligned}f^*(s)&=\frac{1}{\bar{q}}\left[(p_0+p_2(q+\bar{q}s)^2+p_3(q+\bar{q}s)^3+\ldots)-q\right]\\&=\frac{1}{\bar{q}}\left[p_0+p_2q^2+p_3q^3+\ldots-q+(2p_2q\bar{q}+3p_3q^2\bar{q}+4p_4q^3\bar{q}+\ldots)s+\ldots\right].\end{aligned}$$

Note that the constant term vanishes as q solves the equation $x=\sum_{j=0}^{\infty}p_jx^j$. The coefficient of the linear term is the probability of giving 1 offspring for the process X^* and equals $f'(q)$. Hence the result follows by Lemma 2.7. We check that $f'(q)<1$. To see this, note that by assumption, $0<q<1$. Since f' is strictly increasing on $(0,1)$ and $\lim_{x\to1}f'(x)=f'(1)>1$, we conclude that $f'(q)<1$.

□

2.4 A multitype Galton-Watson process

So far we have considered branching processes of a single type, that is, each particle in the process had the same probabilistic behavior. A natural extension is to consider a process with a finite number of types, say k, where a particle of type i, $1\leq i\leq k$, behaves probabilistically in the same way as other particles of type i, but differently than particles of type $1 \leq j \leq k$, $j \neq i$. As we did previously, we first define the discrete-time version of such branching processes, and then proceed with the continuous-time Markov version. Our main interest lies in translating the decomposition of the supercritical branching process of the previous section into a two-type continuous-time Markov branching process, and finding its mean matrix (defined below). Hence, beyond the definitions of multitype branching processes, we will only consider the mean matrix, and give a recipe as to how to find it, rather than proving results. The interested reader is referred to the books of Harris [20], and Athreya and Ney [4] for a detailed treatment on multitype branching processes.

Let T_k be the set of all k-dimensional vectors whose components are nonnegative integers. Let $\mathbf{e}_i$, $1 \leq i \leq k$, be the vector whose ith component is 1 and other components are 0.

Definition 2.10. A multitype (k-type) Galton-Watson process with offspring probabilities $p^i(r_1, \ldots, r_k)$, $1 \leq i \leq k$, $r_j \in \mathbb{N}$ for $1 \leq j \leq k$ is a time-homogeneous Markov chain $\mathbf{N} = (\mathbf{N}(n))_{n \in \mathbb{N}}$ on T_k, which evolves according to the following recursion. If $\mathbf{N}(0) = \mathbf{e}_i$, then $\mathbf{N}(1)$ has the p.g.f.

$$f^i(s_1, \ldots, s_k) = \sum_{r_1, \ldots, r_k = 0}^{\infty} p^i(r_1, \ldots, r_k) s_1^{r_1} \cdots s_k^{r_k}, \quad 0 \leq s_j \leq 1, \quad j = 1, \ldots, k$$

where $p^i(r_1, \ldots, r_k)$ is the probability that a particle of type j has r_1 children of type 1, r_2 children of type 2, ..., r_k children of type k. If $\mathbf{N}(n) = (m_1, \ldots, m_k) \in T$, then similar to the recursion (2.1) for a single type GWP, $\mathbf{N}(n+1)$ is the sum of $m_1 + \ldots + m_k$ independent random vectors, m_j of which are distributed according to $(p^j(r_1, \ldots, r_k) : r_1, \ldots, r_k \in \mathbb{N})$ for $1 \leq j \leq k$. If $\mathbf{N}(n) = 0$, then $\mathbf{N}(n+1) = 0$, where 0 denotes here the zero vector of size k.

Note that the transition probabilities $P_{\mathbf{rm}} := P(\mathbf{N}(n+1) = \mathbf{m} \mid \mathbf{N}(n) = \mathbf{r})$, $\mathbf{r}, \mathbf{m} \in T_k$, for the process is given by the recursion described above in the same way as in the single type case. Now let $\mathbf{r} = (r_1, \ldots, r_k) \in T_k$. In terms of coordinates, we write $\mathbf{N}(n) = (N_1(n), \ldots, N_k(n))$. We write $\mathbf{N}^{\mathbf{r}}(n) = (N_1^{\mathbf{r}}(n), \ldots, N_k^{\mathbf{r}}(n))$ if the process starts in state $\mathbf{r}$, where $N_j^{\mathbf{r}}(n)$ is the number of particles of type j at generation n if the process starts in state $\mathbf{r}$. Furthermore, let $\mathbf{N}^i(n) = \mathbf{N}^{\mathbf{e}_i}(n)$.

Definition 2.11. Let $\mathcal{M}_{ij} = E\mathbf{N}_j^i(1)$ be the expected number of particles of type j that a single particle of type i produces in one generation. The mean matrix $\mathcal{M}$ for the multitype GWP is defined to be the $k \times k$ matrix whose (i,j)th entry is $\mathcal{M}_{ij}$, that is, $\mathcal{M} = (\mathcal{M}_{ij})_{i,j=1,\ldots,k}$.

Let $\mathbf{s} = (s_1, \ldots, s_k)$. It is easy to see that $\mathcal{M}_{ij} = \left.\frac{\partial f^i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=1}$.

Now we define the continuous-time multitype Markov branching process. The passage from discrete-time to continuous-time is performed in the same way as before, where each particle of type j lives an exponential lifetime with rate β_j.

Definition 2.12. A continuous-time multitype (k-type) Markov branching process with rate vector $\beta = (\beta_1, \ldots, \beta_k)$ and offspring probabilities $p^i(r_1, \ldots, r_k)$, $1 \le i \le k$, $r_j = 0, 1, 2, \ldots$ for $1 \le j \le k$ is a time-homogeneous Markov chain $\mathbf{X} = (\mathbf{X}(t))_{t \ge 0}$ on T, whose transition probabilities $(P_{\mathbf{rm}}(t) : \mathbf{r}, \mathbf{m} \in T_k)$ satisfy the branching property

$$P_{\mathbf{rm}}(t) = \sum_{\mathbf{s}_1 + \cdots + \mathbf{s}_{r_1 + \cdots + r_k} = \mathbf{m}} \left(\prod_{i=1}^{r_1} P_{1\mathbf{s}_i}(t) \prod_{i=r_1+1}^{r_1+r_2} P_{2\mathbf{s}_i}(t) \cdots \prod_{i=r_1+\cdots+r_{k-1}+1}^{r_1+\cdots+r_k} P_{k\mathbf{s}_i}(t) \right),$$

where $\mathbf{r} = (r_1, \ldots, r_k)$, $\mathbf{m} = (m_1, \ldots, m_k)$, $P_{j\mathbf{s}_i}(t) := P_{e_j\mathbf{s}_i}(t)$ along with the Kolmogorov forward equations

$$\begin{aligned}
\frac{d}{dt} P_{i\mathbf{m}}(t) = &- \sum_{j=1}^{k} \beta_j m_j P_{i\mathbf{m}}(t) \\
&+ \sum_{j=1}^{k} \beta_j \sum_{\substack{1 \le \rho_j \le m_j+1 \\ 0 \le \rho_l \le m_l,\ l \ne j}} \rho_j P_{i\rho}(t) p^i(m_1 - \rho_1, \ldots, m_j - \rho_j + 1, \ldots, m_k - \rho_k), \\
P_{i\mathbf{m}}(0+) = \delta_{\mathbf{e}_i\mathbf{m}}, &\quad 1 \le i \le k.
\end{aligned}$$

The corresponding minimal process is constructed as follows. We assume that it starts with a single particle of a given type. If a particle of type i is alive at a certain time, then its additional lifetime is an exponential random variable with parameter β_i, at the end of which, it gives birth to offpring of types $1 \le \ldots \le k$ according to the distribution $p^i(\mathbf{r})$, $\mathbf{r} = (r_1, \ldots, r_k) \in T_k$, independent of other particles and the history of the process.

In terms of coordinates, similarly as before, we write $\mathbf{X}(t) = (X_1(t), \ldots, X_k(t))$, and if the process starts in state $\mathbf{r}$, we write $\mathbf{X}^{\mathbf{r}}(t) = (X_1^{\mathbf{r}}(t), \ldots, X_k^{\mathbf{r}}(t))$, where $X_j^{\mathbf{r}}(t)$ is the number of particles of type j at time t if the process starts in state $\mathbf{r}$. Furthermore, let $\mathbf{X}^i(t) = \mathbf{X}^{\mathbf{e}_i}(t)$.

In order to avoid the process from producing infinitely many particles in finite time, we assume that

$$\left.\frac{\partial f^i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=1} < \infty \quad \text{for all} \quad i, j. \tag{2.24}$$

Note that this is a sufficient condition, which also ensures that $\sum_{\mathbf{m} \in T_k} P_{\mathbf{rm}}(t) = 1$ for all t. The mean matrix for the continuous-time process is defined as follows.

Definition 2.13. Let $M_{ij}(t) = E\mathbf{X}_j^i(t)$ be the expected number of particles of type j at time t given that the process starts with a single particle of type i at $t=0$. The mean matrix $M(t)$ for the continuous-time multitype Markov branching process is defined to be the $k \times k$ matrix whose (i,j)th entry is $M_{ij}(t)$, that is, $M(t) = (M_{ij})_{i,j=1,\ldots,k}$.

One can show that (2.24) implies that $M_{ij}(t) < \infty$ for any $i, j, t > 0$. Furthermore, it can be shown from the Kolmogorov equations that $M = (M(t))_{t\geq 0}$ forms a strongly continuous semigroup when regarded as acting on the Banach space $\mathbb{R}^k$, that is,

$$\begin{aligned} M &: \mathbb{R}_+ \to L(\mathbb{R}^k) \\ M(t) &: \mathbb{R}^k \to \mathbb{R}^k \quad \text{for every} \quad t \geq 0, \end{aligned}$$

where $L(\mathbb{R}^k)$ is the Banach space of all bounded linear operators on $\mathbb{R}^k$, and M satisfies

$$\begin{aligned} M(0) &= I, \\ M(s+t) &= M(s)M(t), \quad s,t \geq 0 \\ \lim_{t\to 0^+} M(t)v &= v \quad \text{for each} \quad v \in \mathbb{R}^k, \end{aligned}$$

where I is the identity operator ($k \times k$ identity matrix in this case). Since $\mathbb{R}^k$ is a finite dimensional Banach space, any strongly continuous semigroup is necessarily uniformly continuous, and hence there exists a bounded linear operator A on $\mathbb{R}^k$ such that $M(t)$ can be represented as

$$M(t) = \exp(At) = \sum_{r=0}^{\infty} \frac{A^r t^r}{r!}, \tag{2.25}$$

where the series converges in the norm. The linear operator A can be identified as the matrix $A = (a_{ij})$, where

$$a_{ij} = \beta_i b_{ij}, \quad b_{ij} = \left.\frac{\partial f^i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=1} - \delta_{ij}. \tag{2.26}$$

Note that (2.25) and (2.26) give a recipe as to how to compute the mean matrix.

Now we translate the decomposition of supercritical GWPs in Section 2.3 into the language of multitype GWPs, and show that the decomposed process can be interpreted as a two-type GWP, where the types are skeleton and doomed particles.

Proposition 2.14. *Let N be a supercritical Galton-Watson process with offspring p.g.f. f and extinction probability $0 < q < 1$, and let N^* be the process generated by the skeleton particles. Let $\bar{q} = 1 - q$. Then, the process N conditioned on non-extinction is a two-type Galton-Watson process $\mathbf{N} = (N^*, N - N^*)$, with types composed of skeleton and doomed particles, and with offspring p.g.f.s*

$$f^1(s,t) = [f(\bar{q}s + qt) - f(qt)]/\bar{q}, \tag{2.27}$$

$$f^2(s,t) = f(qt)/q, \tag{2.28}$$

respectively for skeleton and doomed particles, and where $\mathbf{N}(0) = (1, 0)$.

Proof. Since $N^*(0) = \mathbb{1}_{\{N(n)\not\to 0\}}$, it follows that conditioning N on non-extinction is equivalent to imposing that $\mathbf{N}(0) = (1, 0)$. Parts (c) and (d) of Proposition 2.9 imply that, conditioned on non-extinction, for any $n \in \mathbb{N}$, $\mathbf{N}(n+1)$ depends only on $\mathbf{N}(n)$, and is independent of $\mathbf{N}(m)$ for $m < n$, and of n. Hence, $\mathbf{N}$ is a time-homogeneous Markov chain on T_2.

Let f^1, f^2 be the offspring p.g.f.s for skeleton and doomed particles, respectively. The p.g.f. for doomed particles has been previously computed in part (b) of Proposition 2.9. Following the notation in the proof of Proposition 2.9, the p.g.f. for skeleton particles is the same as the joint p.g.f. of $\xi - \xi^*$ and ξ^* conditioned on the event $\{\xi^* \geq 1\}$. We have

$$\begin{aligned} E[s^{\xi^*} t^{\xi-\xi^*} | \xi^* \geq 1] &= \frac{E[s^{\xi^*} t^{\xi-\xi^*} \mathbb{1}_{\{\xi^* \geq 1\}}]}{P(\xi^* \geq 1)} \\ &= \frac{E[s^{\xi^*} t^{\xi-\xi^*}] - E[s^{\xi^*} t^{\xi-\xi^*} \mathbb{1}_{\{\xi^*=0\}}]}{\bar{q}} \\ &= \frac{f(\bar{q}s + qt) - f(qt)}{\bar{q}}, \end{aligned} \tag{2.29}$$

where we have used that $E[s^{\xi^*} t^{\xi-\xi^*}] = f(\bar{q}s + qt)$ in the second equality, and part (b) of Proposition 2.9 in the last equality.

In order to show that $\mathbf{N} = (N^*, N - N^*)$ is a two-type GWP, we show that $f^1_{n+1}(s,t) = f^1(f^1_n(s,t), f^2_n(s,t))$ for $n \in \mathbb{N}$, where f^i_n is the p.g.f. of $\mathbf{N}(n)$ when $\mathbf{N}(0) = (\delta_{1i}, \delta_{2i})$ (see [20, Theorem 3.1]). One can show using part (a) of Proposition 2.9 that

$$E[s^{N^*(n)} t^{(N-N^*)(n)}] = f_n(\bar{q}s + qt), \tag{2.30}$$

where we use the notation introduced before Lemma 2.2. Then,

$$
\begin{aligned}
f^1_{n+1}(s,t) &:= E[s^{N^*(n+1)}t^{(N-N^*)(n+1)} \mid \mathbf{N}(0)=(1,0)] \\
&= \frac{E[s^{N^*(n+1)}t^{(N-N^*)(n+1)}\mathbb{1}_{\{\mathbf{N}(0)=(1,0)\}}]}{P(\mathbf{N}(0)=(1,0))} \\
&= \frac{1}{\bar{q}}\left[E[s^{N^*(n+1)}t^{(N-N^*)(n+1)}] - E[t^{N(n+1)}\mathbb{1}_{\{\mathbf{N}(0)=(0,1)\}}]\right] \\
&= \frac{1}{\bar{q}}\left[f_{n+1}(\bar{q}s+qt) - f_{n+1}(qt)\right] \qquad (2.31) \\
&= \frac{1}{\bar{q}}\left[f(f_n(\bar{q}s+qt)) - f(f_n(qt))\right] \\
&= \frac{1}{\bar{q}}\left[f(\bar{q}(f_n(\bar{q}s+qt)-f_n(qt))/\bar{q} + qf_n(qt)/q) - f(qf_n(qt)/q)\right] \\
&= \frac{1}{\bar{q}}\left[f(\bar{q}f^1_n(s,t)+qf^2_n(s,t)) - f(qf^2_n(s,t))\right] \qquad (2.32) \\
&= f^1(f^1_n(s,t), f^2_n(s,t)),
\end{aligned}
$$

where we have used Lemma 2.2 throughout, (2.30) and part (b) of Proposition 2.9 in passing to (2.31), (2.28) and (2.31) in passing to (2.32), and (2.27) in the last equality. □

We now give the mean matrix $\mathcal{M}$ for the discrete-time two-type process $\mathbf{N}$ using the formula $\mathcal{M}_{ij} = \frac{\partial f^i(\mathbf{s})}{\partial s_j}|_{\mathbf{s}=1}$. As we have a two-type process, let us set $\mathbf{s}=(s,t)$. Then, setting $s=1$ in (2.29) yields the p.g.f. for the number of doomed offspring for a skeleton particle:

$$E[t^{\xi-\xi^*}|\xi^* \geq 1] = \frac{f(\bar{q}+qt)-f(qt)}{\bar{q}}. \qquad (2.33)$$

Differentiating (2.33) with respect to t and setting $t=1$ gives the desired expectation

$$\left.\frac{\partial f^1(\mathbf{s})}{\partial t}\right|_{\mathbf{s}=1} = (f'(1)-f'(q))\frac{q}{\bar{q}}. \qquad (2.34)$$

It is clear from parts (b) and (c) of Proposition 2.9 that

$$\left.\frac{\partial f^1(\mathbf{s})}{\partial s}\right|_{\mathbf{s}=1} = f'(1), \qquad (2.35)$$

$$\left.\frac{\partial f^2(\mathbf{s})}{\partial s}\right|_{\mathbf{s}=1} = 0, \qquad (2.36)$$

$$\left.\frac{\partial f^2(\mathbf{s})}{\partial t}\right|_{\mathbf{s}=1} = f'(q). \qquad (2.37)$$

Setting $\mu = f'(1)$, $\kappa = f'(q)$, it follows that

$$\mathcal{M} = \begin{pmatrix} \mu & (\mu-\kappa)\frac{q}{\bar{q}} \\ 0 & \kappa \end{pmatrix}. \tag{2.38}$$

It is clear that if $X = (X(t))_{t\geq 0}$ is a continuous-time Markov branching process with rate β such that the number of particles in generation n of X is $N(n)$ for $n \in \mathbb{N}$, and if $X^*(t)$ is the number of skeleton particles of X present at time t, then the process $X = (X(t))_{t\geq 0}$ given non-extinction is a continuous-time two-type Markov branching process $\mathbf{X} = (X^*, X - X^*)$ with rate vector $\beta = (\beta, \beta)$ and offspring p.g.f.s (2.27) and (2.28) for skeleton and doomed particles, respectively.

We conclude this section by computing the mean matrix $M(t)$ for process $\mathbf{X}$. In accordance with the recipe given in (2.25) and (2.26), and using (2.38), we obtain

$$M(t) = \begin{pmatrix} e^{\beta t(\mu-1)} & \frac{q}{\bar{q}}\left[e^{\beta t(\mu-1)} - e^{\beta t(\kappa-1)}\right] \\ 0 & e^{\beta t(\kappa-1)} \end{pmatrix}, \tag{2.39}$$

where $M_{ij}(t)$ above is the expected size of the population of type j at time t given that the process starts with a single particle of type i.

Upon concluding this chapter, we note that the decomposition of supercritical GWPs, the mean matrix $\mathcal{M}$ computed above, and the mean matrix $M(t)$ for two-type continuous-time Markov branching processes will be central tools in the next chapters in the proof of our main theorems.

Chapter 3

Conditional Speed of Branching Brownian Motion

A branching Brownian motion is a spatial branching process, which is a continuous-time Markov branching process in space where the initial particle starts her motion from the origin and each particle that is alive performs a Brownian motion independently of others during her lifetime. We define Brownian motion as follows.

Definition 3.1 (one-dimensional Brownian motion)**.** A real-valued stochastic process $B = (B(t))_{t\geq 0}$ is called a Brownian motion started at $x \in \mathbb{R}$ if:

1. $B(0) = x$,

2. B has independent increments, i.e., for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the increments $B(t_n) - B(t_{n-1})$, $B(t_{n-1}) - B(t_{n-2})$, ..., $B(t_2) - B(t_1)$ are independent random variables,

3. B has stationary increments that are normally distributed with mean 0 and variance that equals the length of the increment, i.e., for all $t > 0$ and $s \geq 0$, $B(s+t) - B(s)$ has probability density function given by

$$p_t : x \longmapsto \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},$$

4. the function $t \longmapsto B(t)$ is almost surely continuous.

We refer the reader to the monograph by Mörters and Peres [32] for the proof of existence of Brownian motion, its sample path properties, and much more on the subject. By a d-dimensional Brownian motion, we mean the following.

Definition 3.2 (d-dimensional Brownian motion)**.** A vector-valued stochastic process $B = (B(t))_{t \geq 0}$ taking values in $\mathbb{R}^d$ is called a d-dimensional Brownian motion started at $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ if

$$B(t) = (B_1(t), \ldots, B_d(t)),$$

where $B_1, \ldots, B_d$ are independent Brownian motions started at $x_1, \ldots, x_d$, respectively.

Henceforth, when we say Brownian motion, we mean a d-dimensional Brownian motion started at the origin unless otherwise stated.

A detailed informal description of BBM was given in Section 1.2. For a formal treatment, we refer the reader to the series of three papers by Ikeda, Nagasawa and Watanabe [22], where the process of BBM was formally introduced, and for a simpler treatment to [33], where the concept of a 'marked tree' was introduced.

A natural question to ask is how much a BBM spreads out in space as time grows, which leads to the concept of speed of the process. The first result on the speed of BBM was given by McKean in [30, 31], where the problem was linked to a well-known partial differential equation, called the F-KPP equation, which was introduced in [18] by Fisher, and in [25] by Kolmogorov, Petrovskii, and Piskunov. Later, the speed of BBM proved to be a rich topic in that many refinements and extensions of McKean's result were obtained [7, 8, 10, 27, 26, 1]. However, the case where the particles could give zero offspring was left uncovered.

In Section 3.1, we define the concept of *speed* of a BBM, and give a brief survey of previous work. In Section 3.2, we state and prove the well-known convergence in probability result due to McKean [30, 31], which holds when $d = 1$ and when each particle is certain to give at least one offspring, i.e., when $p_0 = 0$. We then extend this result to dimension $d \geq 2$ in Section 3.3, and finally to the case where $p_0 > 0$ in Section 3.4. The results of this section are central in proving Theorem 4.1.

3.1 Definition and history

Consider a one-dimensional branching Brownian motion $Z = (Z(t))_{t \geq 0}$ in $\mathbb{R}$ with constant branching rate $\beta > 0$, offspring distribution $(p_k)_{k \in \mathbb{N}}$, and initial measure $Z(0) = \delta_0$. Let $X^+(t)$ be the position of its rightmost particle at time t. In accordance with the classical formula '*distance* $=$ *speed* $\times$ *time*' from physics, the *speed* at time t of BBM is defined as

$$c^+(t) := \frac{X^+(t)}{t}, \tag{3.1}$$

where c is used to denote speed. This definition can be modified to cover all $d \geq 1$ by writing

$$c(t) := \frac{X(t)}{t}, \tag{3.2}$$

where $X(t) := \inf\left\{r \geq 0 : \mathrm{supp}(Z(t)) \subseteq B(\vec{0}, r)\right\}$ is the radius of the minimal ball centered at $\vec{0}$ containing the support of the BBM at time t. Studies concerning the quantity in (3.1) date back to 1975, where McKean [30, 31] showed that $P(X^+(t) \leq x)$ satisfies the partial differential equation of Fisher and Kolmogorov-Petrovskii-Piskunov (F-KPP) [18, 25], and showed that the speed c^+ converges to $\sqrt{2\beta m}$ in probability as $t \to \infty$, under the assumption that $p_0 = 0$ and $p_1 < 1$. Recall that $m = \mu - 1$, where μ is the expected number of offspring for a particle. After McKean, Bramson [7, 8] proved a stronger result, showing that c^+ converges to $\sqrt{2\beta m}$ almost surely as $t \to \infty$. Later, Chauvin and Rouault [10] studied the probability of existence in the sub-critical area, and found asymptotic results for $P(Z(t)(ct, \infty) > 0)$ for $c > c^+$. Kyprianou's work [26] extended the existing results to the multidimensional case, where he showed that c, defined in (3.2), converges to $\sqrt{2\beta m}$ almost surely in any dimension. We emphasize that in all the works mentioned above, it was assumed that $p_0 = 0$.

The definition of speed that we use in this work is slightly different than the classical definition given in (3.2). As the distance by which a BBM spreads out in space, instead of considering the support of the process at a given time, we consider the range (see (1.4)) of the process starting from $t = 0$ up to the given time. In other words, we define the speed at time t of a BBM as

$$v(t) := \frac{M(t)}{t}, \tag{3.3}$$

where $M(t) = \inf\left\{r \geq 0 : R(t) \subseteq B(\vec{0}, r)\right\}$ is the radius of the minimal ball centered at $\vec{0}$ that contains the range of the BBM up to time t. Note that the definition in (3.3) makes sense for all spatial dimensions. Also, by comparing the two definitions, (3.2) and (3.3), one immediately sees that $c(t) \leq v(t)$ for all $t \geq 0$.

In the next section, for the sake of completeness, we provide a proof of McKean's result on the convergence in probability of the speed of BBM with $p_0 = 0$ and $d = 1$, as we use this result in the proof of Theorem 3.9.

3.2 McKean's result

Consider the partial differential equation on $\mathbb{R} \times [0, \infty)$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + g(u), \tag{3.4}$$

where

$$g(u) = \beta \left(\sum_{j=1}^{\infty} p_j u^j - u \right), \quad u \in [0, 1], \tag{3.5}$$

with

$$\beta > 0, \quad \sum_{j=1}^{\infty} p_j = 1, \quad p_j \geq 0, \quad p_1 < 1$$

subject to Heaviside initial condition

$$u(x, 0) = \mathbb{1}_{[0,\infty)}(x) \tag{3.6}$$

Equation (3.4) has been first studied by Kolmogorov, Petrovskii and Piskunov [25], and Fisher [18] in the more general case, where $g \in C^1[0, 1]$ is such that

$$g(0) = g(1) = 0, \quad g(u) < 0 \quad \text{for} \quad u \in (0, 1)$$

and

$$g'(1) > 0, \quad g'(u) \leq g'(1) \quad \text{for} \quad u \in [0, 1].$$

Observe that the function in (3.5) satisfies these conditions since $\sum_{j=1}^{\infty} jp_j > 1$.

Lemma 3.3. *Let u be the distribution function of the position of the rightmost particle of a one-dimensional BBM with rate β and offspring distribution $(p_k)_{k\in\mathbb{N}}$, i.e., $u(x, t) = P(X^+(t) \leq x)$. Suppose that $p_0 = 0$ and $p_1 < 1$. Then u solves the initial value problem (3.4) with g in (3.5) and initial condition (3.6).*

Proof. By assumption, the BBM starts with a single particle at the origin. Hence, the initial condition (3.6) is trivially satisfied. Let $W(t)$ denote the displacement of the initial particle at time t, and T be the time that it branches. Conditioning on T, we have for $t > 0$

$$\begin{aligned} u(x,t) =& P(T > t)P(W(t) \leq x) \\ &+ \int_0^t P(T \in \mathrm{d}s) \int_{-\infty}^{\infty} P(W(s) \in x - \mathrm{d}y) \sum_{j=1}^{\infty} p_j u^j(y, t-s) \\ =& e^{-\beta t} P(W(t) \leq x) + \int_0^t \beta e^{-\beta s} \int_{-\infty}^{\infty} p_s(x, y) \sum_{j=1}^{\infty} p_j u^j(y, t-s) \mathrm{d}y \mathrm{d}s, \end{aligned}$$

where $p_s(x,y) = 1/(\sqrt{2\pi s})\exp(-\frac{(y-x)^2}{2s})$ is the Brownian transition density. Substituting s for $t-s$ yields

$$u(x,t) = e^{-\beta t}P(W(t) \leq x) + \sum_{j=1}^{\infty} p_j \beta e^{-\beta t} \int_0^t e^{\beta s} \int_{-\infty}^{\infty} p_{t-s}(x,y) u^j(y,s) \mathrm{d}y \mathrm{d}s. \quad (3.7)$$

Note that p_t is differentiable for every $t > 0$. Differentiating (3.7) with respect to t and using the Leibniz rule, we obtain

$$\begin{aligned} \dot{u}(x,t) =& -\beta e^{-\beta t}P(W(t) \leq x) + e^{-\beta t}\frac{\partial}{\partial t}P(W(t) \leq x) \\ &- \sum_{j=1}^{\infty} p_j \beta^2 e^{-\beta t} \int_0^t e^{\beta s} \int_{-\infty}^{\infty} p_{t-s}(x,y) u^j(y,s) \mathrm{d}y \mathrm{d}s \\ &+ \sum_{j=1}^{\infty} p_j \beta e^{-\beta t} \int_0^t e^{\beta s} \frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} p_{t-s}(x,y) u^j(y,s) \mathrm{d}y\right] \mathrm{d}s + \beta \sum_{j=1}^{\infty} p_j u^j(x,t). \quad (3.8) \end{aligned}$$

Note that $P(W(t) \leq x)$ satisfies

$$\frac{\partial}{\partial t}P(W(t) \leq x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}P(W(t) \leq x) \quad \text{for} \quad x \in \mathbb{R},\ t > 0, \quad (3.9)$$

which is a special case of a more general result, which goes as follows. Let $B = (B(t))_{t\geq 0}$ be a Brownian motion, and E be the corresponding expectation. Then, for any bounded Borel function $f : \mathbb{R} \mapsto \mathbb{R}$,

$$\frac{\partial}{\partial t}E[f(x + W(t))] = \frac{1}{2}\frac{\partial^2}{\partial x^2}E[f(x + W(t))] \quad \text{for} \quad x \in \mathbb{R},\ t > 0, \quad (3.10)$$

Equation (3.9) follows from (3.10) since for $x \in \mathbb{R}$ and $t > 0$,

$$P(W(t) \leq x) = \int_{-\infty}^{\infty} g(y) p_t(x,y) \mathrm{d}y = E[g(x + W(t))],$$

with

$$g(y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0. \end{cases}$$

Note that $g : \mathbb{R} \mapsto \mathbb{R}$ is bounded and Borel. For a full proof of (3.10), refer to [23, p.209-210]. The crucial ingredient in the proof is that

$$\frac{\partial}{\partial t}p_t(x,y) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p_t(x,y) \quad \text{for} \quad x \in \mathbb{R},\ t > 0. \quad (3.11)$$

As a historical note, the realization that the Brownian transition density solves the heat equation (3.11) dates back to 1905, where Einstein [13] derived this result from physical principles coming from the molecular-kinetic theory of heat. Continuing the proof, (3.10) implies that

$$\begin{aligned}\dot{u}(x,t) =& -\beta e^{-\beta t} P(W(t) \leq x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(e^{-\beta t} P(W(t) \leq x)\right) \\ & - \sum_{j=1}^{\infty} p_j \beta^2 e^{-\beta t} \int_0^t e^{\beta s} \mathrm{d}s \int_{-\infty}^{\infty} p_{t-s}(x,y) u^j(y,s) \mathrm{d}y \\ & + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\sum_{j=1}^{\infty} p_j \beta e^{-\beta t} \int_0^t e^{\beta s} \mathrm{d}s \int_{-\infty}^{\infty} p_{t-s}(x,y) u^j(y,s) \mathrm{d}y\right\} + \beta \sum_{j=1}^{\infty} p_j u^j(x,t).\end{aligned} \tag{3.12}$$

Grouping the first and third, and second and fourth terms of (3.12), and comparing with (3.7), we arrive at

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + \beta\left(\sum_{j=1}^{\infty} p_j u^j(x,t) - u(x,t)\right).$$

□

Now let $\bar{u} := 1 - u$ and $f(u) = \sum_{j=1}^{\infty} p_j u^j$. From (3.4), (3.5) and (3.6), it follows that $\bar{u}(x,t)$ satisfies the partial differential equation

$$\frac{\partial \bar{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial x^2} + h(\bar{u}), \tag{3.13}$$

where

$$h(\bar{u}) = \beta(1 - \bar{u} - f(1-\bar{u})), \tag{3.14}$$

subject to the initial condition

$$\bar{u}(x,0) = \mathbb{1}_{(-\infty,0)}(x). \tag{3.15}$$

Since $f(0) = 0$ and $f'(1) > 1$, the function h satisfies the following properties:

$$\begin{aligned} h(0) &= 1 = h(1), \\ h &\in C^1[0,1], \\ h(u) &> 0 \quad \forall\, u \in (0,1), \\ h'(0) &> 0, \\ h'(u) &< h'(0) \quad \forall\, u \in (0,1]. \end{aligned} \tag{3.16}$$

It follows from the results in [2, 3] that

$$c > \sqrt{2h'(0)} \quad \Rightarrow \quad \lim_{t\to\infty} \bar{u}(ct,t) = 0 \tag{3.17}$$

$$|c| < \sqrt{2h'(0)} \quad \Rightarrow \quad \lim_{t\to\infty} \bar{u}(ct,t) = 1. \tag{3.18}$$

Recall that $\bar{u} = 1 - u$ so that by Lemma 3.3, $\bar{u}(x,t) = P(X^+(t) > x)$. Define $m := f'(1) - 1 = h'(0)/\beta$. Then, equation (3.17) implies that

$$\forall\, c > \sqrt{2\beta m}, \quad P\left(\frac{X^+(t)}{t} > c\right) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.19}$$

On the other hand, let $-\sqrt{2\beta m} < c < \sqrt{2\beta m}$. Then, Lemma 3.3 and equation (3.18) imply that $P\left(\frac{X^+(t)}{t} < c\right) \to 0$ as $t \to \infty$. By monotonicity of the event $\left\{\frac{X^+(t)}{t} < c\right\}$ in $c \in \mathbb{R}$, this implies that

$$\forall\, c < \sqrt{2\beta m}, \quad P\left(\frac{X^+(t)}{t} < c\right) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.20}$$

Equations (3.19) and (3.20) imply the following convergence in probability for the speed of BBM, which we state as a proposition.

Proposition 3.4 (McKean)**.** *Let $c^+(t)$ be the speed (as defined in (3.1)) at time t of a BBM evolving in $\mathbb{R}$ with rate $\beta > 0$, and offspring law $(p_k)_{k\in\mathbb{N}}$ having $p_0 = 0$ and $p_1 < 1$, and let f be the corresponding p.g.f. Let $m := f'(1) - 1$. Then*

$$c^+(t) \xrightarrow{P} \sqrt{2\beta m} \quad \textit{as} \quad t \to \infty, \tag{3.21}$$

where $\xrightarrow{P}$ denotes convergence in probability.

Remark. Note that m is the expected net growth per particle since a particle dies upon branching. Also note that if $p_1 = 1$, then the system has a single Brownian particle. In this case, as Brownian motion is diffusive, one can easily show that for any $\varepsilon > 0$, $P(X(t) > \varepsilon t) \to 0$ as $t \to \infty$. This implies that $c^+ = 0$ when $p_1 = 1$, consistent with the proposition above, since in this case $m = \mu - 1 = 0$.

Section 3.3 and Section 3.4 below are based on [35].

3.3 Extension to higher dimension

In this section, we extend McKean's result to higher dimension, i.e., $d \geq 2$. First, a preparatory lemma is given on Brownian hitting times in $\mathbb{R}^d$, extending the well-known one-dimensional result (see for example [24, Eqn.7.3.3]) which holds when

$d = 1$. Then, with the help of this lemma, McKean's result is extended to $d \geq 2$ in Proposition 3.8.

Lemma 3.5 (Brownian hitting times in $\mathbb{R}^d$). *Let $B = (B(t))_{t \geq 0}$ be a Brownian motion in $\mathbb{R}^d$ starting at the origin, with probability law P_0, and let $k > 0$. Then, as $t \to \infty$,*

$$P_0\left(\sup_{0 \leq s \leq t} |B(s)| \geq kt\right) = \exp\left\{-\frac{k^2 t}{2}(1 + o(1))\right\}. \tag{3.22}$$

Furthermore, for $d = 1$, even the following stronger statement is true. Let $m(t) = \inf_{0 \leq s \leq t} B(s)$ and $M(t) = \sup_{0 \leq s \leq t} B(s)$. (The process $M(t) - m(t)$ is called the range process of B.) Then, as $t \to \infty$,

$$P_0(M(t) - m(t) \geq kt) = \exp\left\{-\frac{k^2 t}{2}(1 + o(1))\right\}. \tag{3.23}$$

Proof. For $d = 1$, recall that [24, Eqn.7.3.3] for any $a > 0$,

$$P_0\left(\sup_{0 \leq s \leq t} |B(s)| \geq a\right) = \sqrt{\frac{2}{\pi}} \int_{a/\sqrt{t}}^{\infty} \exp(-y^2/2) \mathrm{d}y. \tag{3.24}$$

Setting $a = kt$ above, we see via l'Hôpital's rule that

$$P_0\left(\sup_{0 \leq s \leq t} |B(s)| \geq kt\right) = \exp\left\{-\frac{k^2 t}{2}(1 + o(1))\right\}. \tag{3.25}$$

We now extend this result to $d \geq 2$. Note that the lower bound holds trivially since the projection of a d-dimensional Brownian motion onto the first coordinate axis is a one-dimensional Brownian motion. In order to prove the upper bound, we first prove (3.23).

Let $d = 1$. Define

$$\theta_c = \inf\left\{s \geq 0 : M(s) - m(s) = c\right\}.$$

According to [9, p.199] and references therein, the Laplace transform of θ_c satisfies

$$E \exp\left(-\frac{\lambda^2}{2}\theta_c\right) = \frac{2}{1 + \cosh(\lambda c)}, \quad \lambda > 0.$$

Hence, by the exponential Markov inequality,

$$P_0(\theta_c \leq t) \leq \exp\left(\frac{\lambda^2}{2}t\right) E \exp\left(-\frac{\lambda^2}{2}\theta_c\right) = \exp\left(\frac{\lambda^2}{2}t\right) \frac{2}{1 + \cosh(\lambda c)}.$$

Taking $c = kt$, one obtains

$$P_0(M_t - m_t \geq kt) = P_0(\theta_{kt} \leq t) \leq \exp\left(\frac{\lambda^2}{2}t\right)\frac{2}{1+\cosh(\lambda kt)}.$$

Optimizing over λ, the estimate is the sharpest when $\lambda = k$, in which case, we obtain the desired upper bound. Since $P_0\left(\sup_{0\leq s\leq t}|B_s| \geq kt\right) \leq P_0(M_t - m_t \geq kt)$, the lower bound coincides with the upper bound, which implies (3.23).

For $d \geq 2$, let us consider more generally the Brownian motion started at x, denoted by B^x, $x \in \mathbb{R}^d$, and let $r = |x|$. We will use the well known fact [37, Example 8.4.1] that if $R = |B^x|$ and $r > 0$, then the process R, called the *d-dimensional Bessel process* satisfies the integral equation

$$R(t) = r + \int_0^t \frac{d-1}{2R(s)}\mathrm{d}s + W(t) \tag{3.26}$$

on $[0,\infty)$ almost surely, where W is the one-dimensional Brownian motion started at 0. Using the strong Markov property of Brownian motion, applied at the first hitting time of the ρ-sphere (we call $B(0,\rho)$ the ρ-sphere), it is clear that in order to verify the lemma, it suffices to prove it when B is replaced by B^x, with $|x| = \rho > 0$. This follows, since for every fixed ρ such that $0 < \rho < kt$, we have

$$P_0\left(\sup_{0\leq s\leq t}|B(s)| \geq kt\right) \leq P_0\left(\sup_{0\leq s\leq t+\tau}|B(s)| \geq kt\right) = P_\rho\left(\sup_{0\leq s\leq t}R(s) \geq kt\right), \tag{3.27}$$

where τ is the first hitting time of B on the ρ-sphere, and P_ρ is the probability law for $R = |B^x|$.

Now let P be the probability law for W, consider $R = |B^x|$ (where $|x| = \rho > 0$) and define the sequence of stopping times $0 =: \tau_0 < \sigma_0 < \tau_1 < \sigma_1 < \cdots$ with respect to the filtration generated by R (hence, also with respect to the one generated by W) as follows:

$$\sigma_0 := \inf\{s > 0 : R(s) = \rho/2\},$$

and for $i \geq 1$,

$$\tau_{i+1} := \inf\{s > \sigma_i : R(s) = \rho\}, \quad \sigma_{i+1} := \inf\{s > \tau_{i+1} : R(s) = \rho/2\}.$$

Note that by (3.26), for $i \geq 0$ and $s \in [\tau_i, \sigma_i]$,

$$R(s) = \rho + \int_{\tau_i}^s \frac{d-1}{2R(z)}\mathrm{d}z + W(s) - W(\tau_i) \leq \rho + \rho^{-1}(d-1)\Delta_i^s + W(s) - W(\tau_i), \tag{3.28}$$

where $\Delta_i^s := s - \tau_i$.

Since $R(s) \le \rho$ for $\sigma_i \le s \le \tau_{i+1}$ and $i \ge 0$, it is also clear that for $t > \rho/k$, the relation $\sup_{0\le s\le t} R(s) \ge kt$ is tantamount to

$$\sup_{i\ge 0} \sup_{\tau_i\wedge t<s\le\sigma_i\wedge t} R(s) \ge kt.$$

Putting this together with (3.28), it follows that $P_\rho(\sup_{0\le s\le t} R(s) \ge kt)$ can be estimated from above by

$$P\left(\exists\, i \ge 0, \exists\, \tau_i \wedge t < s \le \sigma_i \wedge t : W(s) - W(\tau_i) \ge kt - \rho^{-1}(d-1)\Delta_i^s - \rho\right).$$

This can be further estimated from above by

$$P\left(M(t) - m(t) \ge [k - \frac{d-1}{\rho} - \delta]t\right),$$

for any $\delta > 0$, as long as $t \ge \rho/\delta$. To complete the proof, fix ρ, $\delta > 0$, let $t \to \infty$, and use the already proved relation (3.23), along with (3.27). Finally let $\delta \to 0$ and $\rho \to \infty$. $\square$

Lemma 3.6 (Overproduction)**.** *Let $\delta > 0$. Then for any $t \ge 0$*

$$P(|Z(t)| > e^{\beta mt+\delta}) \le e^{-\delta t}.$$

Proof. By Lemma 2.8, we have $E|Z(t)| = e^{\beta mt}$. The result follows by applying the Markov inequality to $P(|Z(t)| > e^{\beta mt+\delta})$. $\square$

Lemma 3.7 (Comparison)**.** *Suppose that $p_0 = 0$. Let $B \subset \mathbb{R}^d$ be open or closed. Let $x \in B$. Let P_x be the law of Brownian motion, denoted by W, starting from x, and let P_{δ_x} be the law of BBM starting from δ_x. Define the first exit times from B as*

$$\begin{aligned} \psi_B &= \inf\{t \ge 0 : W(t) \in B^c\} \\ \hat{\psi}_B &= \inf\{t \ge 0 : R(t) \cap B^c \ne \emptyset\}. \end{aligned} \tag{3.29}$$

Then for any $k \in \mathbb{N}$ and $t \ge 0$

$$P_{\delta_x}(\hat{\psi}_B > t \,|\, |Z(t)| \le k) \ge [P_x(\psi_B > t)]^k.$$

Proof. See the corresponding proof in [15], which is written for strictly dyadic branching, and note that the proof can easily be extended to cover for the case of BBM whose offspring distribution $(p_k)_{k\in\mathbb{N}}$ has $p_0 = 0$, since the population does not decrease for such processes, that is, for any $t \geq 0$, $|Z(s)| \leq |Z(t)|$ for all $s \in [0,t]$. The idea behind the proof is that, comparing a system with k independent Brownian particles to a BBM conditioned on the BBM having at most k particles throughout $[0,t]$, the probability of escape in $[0,t]$ is smaller for the BBM due to correlations between paths of descent of different particles. Each particle of BBM has a Brownian path of descent, but the paths of descent of different particles are not independent. $\square$

Recall that $M(t) = \inf\left\{r > 0 : R(t) \subseteq B(\vec{0}, r)\right\}$ is the radius of the minimal ball centered at the origin containing $R(t)$.

Proposition 3.8 (Speed of BBM). *Suppose that $p_0 = 0$. Then, for $d \geq 1$, $\frac{M(t)}{t}$ converges to $\sqrt{2\beta m}$ in probability as $t \to \infty$.*

Remark. It is easy to see via Lemma 2.7 that the quantity βm is invariant under changing the setting from the 'canonical one' (that is, putting zero mass at 1 ($p_1 = 0$)) to a non-canonical one (that is, assigning positive mass to 1).

Proof. Let $d \geq 1$. Since the projection of Z onto the first coordinate axis is a one-dimensional BBM with the same parameters as Z, the lower estimate for convergence in probability follows from Proposition 3.4 and the inequalities

$$P(M(t)/t > \sqrt{2\beta m} - \varepsilon) \geq P^*(M(t)/t > \sqrt{2\beta m} - \varepsilon) \geq P^*(X^+(t)/t > \sqrt{2\beta m} - \varepsilon)$$

for every $\varepsilon > 0$ and for every $t \geq 0$, where P^* denotes the law of the one-dimensional projection of Z. To prove the upper estimate for convergence in probability, let $\varepsilon > 0$ and let $B = B(0, (\sqrt{2\beta m} + \varepsilon)t)$. Pick $\delta > 0$ such that

$$\frac{1}{2}(\sqrt{2\beta m} + \varepsilon)^2 > \beta m + \delta. \tag{3.30}$$

Recall the definition of exit times from (3.29), and see that the following events are identical: $\left\{M(t)/t > \sqrt{2\beta m} + \varepsilon\right\} = \left\{\hat{\psi}_B \leq t\right\}$. Estimate

$$\begin{aligned} P\left(M(t)/t > \sqrt{2\beta m} + \varepsilon\right) \leq P\left(|Z(t)| > \lfloor e^{(\beta m+\delta)t}\rfloor\right) \; + \\ P\left(\hat{\psi}_B \leq t \mid |Z(t)| \leq \lfloor e^{(\beta m+\delta)t}\rfloor\right). \end{aligned} \tag{3.31}$$

By Lemma 3.6, the first term on the right-hand side tends to zero exponentially fast. Now consider the second term. By Lemma 3.7, we have the estimate

$$P_{\delta_0}\left(\hat{\psi}_B > t \big| |Z(t)| \leq \lfloor e^{(\beta m+\delta)t}\rfloor\right) \geq [P_0(\psi_B > t)]^{\lfloor e^{(\beta m+\delta)t}\rfloor}.$$

By Lemma 3.5,

$$[P_0(\psi_B > t)]^{\lfloor e^{(\beta m+\delta)t} \rfloor} = \left[1 - \exp\left(-\frac{[(\sqrt{2\beta m}+\varepsilon)t]^2}{2t}\right)\exp(o(t))\right]^{\lfloor e^{(\beta m+\delta)t} \rfloor}. \quad (3.32)$$

By (3.30), using the binomial expansion, the right-hand side of (3.32) tends to 1 exponentially fast as $t \to \infty$, so that (3.31) yields the desired upper estimate

$$P\left(M(t)/t > \sqrt{2\beta m} + \varepsilon\right) \to 0 \quad \text{as} \quad t \to \infty,$$

which completes the proof. □

3.4 Extension to possibility of zero offspring

In this section, we allow the particles of the BBM to give zero offspring with positive probability, namely, we put $p_0 > 0$. In doing so, the probability of extinction for the GWP that underlies the BBM becomes positive, hence we must condition on non-extinction before we prove results concerning the speed of BBM. We now state the first major result of this thesis.

Theorem 3.9 (Conditional speed of BBM)**.** *Suppose that the GWP underlying the BBM is supercritical, i.e., $m > 0$. Then, for $d \geq 1$, conditioned on non-extinction, $\frac{M(t)}{t}$ converges to $\sqrt{2\beta m}$ in probability as $t \to \infty$.*

For $d \geq 1$ and $p_0 = 0$, we have already found the speed in Proposition 3.8. Theorem 3.9 covers the general case where p_0 can be nonzero. In this result, we see that conditioned on non-extinction, the speed remains $\sqrt{2\beta m}$, which can be explained as follows. When the supercritical GWP that underlies the BBM is decomposed into skeleton and doomed particles (see Proposition 2.9), the doomed particles do not contribute to the speed, and the skeleton particles alone form another GWP with the same offspring mean as the original (undecomposed) process before conditioning on non-extinction.

We first give two lemmas that are needed in order to prove Theorem 3.9. Recall the definitions of $\tilde{f}$ and f^* from (2.19) and (2.20), respectively, and recall that $\bar{q} = 1 - q$, where q is the probability of extinction.

Lemma 3.10. *If $s \in [0, 1]$, then $f^*(s) \leq \tilde{f}(s)$.*

Proof. By (2.19) and (2.20), we need to show that if $s \in [0, 1]$, then

$$[f(q + \bar{q}s) - q]/\bar{q} \leq f(qs)/q. \quad (3.33)$$

The left-hand side of (3.33) is equal to $[f(s+q(1-s))-q]/(1-q)$. Since f is convex, by Jensen's inequality, it is enough to show that

$$[s+(1-s)f(q)-q]/(1-q) \leq f(qs)/q.$$

Since $f(x) \geq x$ when $x \in [0,q]$, $s \in [0,1]$ implies that $f(qs) \geq qs$. We now have, using $f(q) = q$, that

$$\frac{s+(1-s)f(q)-q}{1-q} = \frac{s+(1-s)q-q}{1-q} = s \leq \frac{f(qs)}{q}.$$

□

Lemma 3.11. *Let $Z_1 = (Z_1(t))_{t\geq 0}$ and $Z_2 = (Z_2(t))_{t\geq 0}$ be two BBMs with the same branching rate $\beta > 0$, and offspring distributions with probability generating function ϕ_1 and ϕ_2, respectively. Let $P^1_{s,x}$ and $P^2_{s,x}$ be the corresponding probabilities for processes that start at time s with a single particle at position $x \in \mathbb{R}^d$. Assume that $\phi_1 \leq \phi_2$ on $[0,1]$. Then, for any Borel set B, and any fixed time $t \geq s$,*

$$P^1_{s,x}(R^1(t) \subset B) \leq P^2_{s,x}(R^2(t) \subset B), \tag{3.34}$$

where $R^j(t)$ is the range of process Z_j up to time t for $j = 1,2$.

Proof. Clearly, if $x \notin B$, then $P^1_{s,x}(R^1(t) \subset B) = 0 = P^2_{s,x}(R^2(t) \subset B)$, so the inequality in (3.34) is trivially satisfied. Now, assume that $x \in B$. Define

$$u^j(s,x,t) := P^j_{s,x}(R^j(t) \subset B), \quad s \leq t, \quad \text{for} \quad j = 1,2. \tag{3.35}$$

Let τ^j, $j = 1,2$, be the first branching time after s, for the initial particles of Z_1 and Z_2, respectively. Then, τ^j is exponentially distributed with parameter β for $j = 1,2$. Conditioning on the event $\tau^j \in [s,t]$ and using the strong Markov property along with the branching property, we obtain for $j = 1,2$, that

$$\begin{aligned} u^j(s,x,t) = \int_s^t & g(r,s)\pi(r-s,x,B)\, \mathbb{E}^{(r)}_{s,x}\phi_j\left(u^j(r,W_r,t)\right) \mathrm{d}r \\ & + e^{-\beta(t-s)}\pi(t-s,x,B), \end{aligned} \tag{3.36}$$

where $g(r,s) = (\beta e^{-\beta(r-s)})/(1-e^{-\beta(t-s)})$, and $\pi(t-s,x,B)$ is the probability that a single Brownian particle, with position given by $W = (W_r)_{r\geq s}$ that starts at time s and position $x \in \mathbb{R}^d$ remains in B until time t, while $\mathbb{E}^{(r)}_{s,x}$ denotes the expectation corresponding to W conditioned on the event that W remains in B within the time interval $[s,r]$. Thus, $\phi_1 \le \phi_2$ implies that

$$(u^1 - u^2)(s,x,t) \le \int_s^t g(r,s)\pi(r-s,x,B)\times \\ \mathbb{E}^{(r)}_{s,x}\left[\phi_1\left(u^1(r,W_r,t)\right) - \phi_1\left(u^2(r,W_r,t)\right)\right] \mathrm{d}r. \tag{3.37}$$

Set $s=0$, abbreviate $\pi(r) := \pi(r,x,B)$ and $g(r) := g(r,0)$ to arrive at

$$(u^1 - u^2)(0,x,t) \le \int_0^t g(r)\pi(r)\mathbb{E}^{(r)}_{0,x}\left[\phi_1\left(u^1(r,W_r,t)\right) - \phi_1\left(u^2(r,W_r,t)\right)\right] \mathrm{d}r. \tag{3.38}$$

Letting $v^j(x,t) := u^j(0,x,t),\ j=1,2$, and making the substitution $r \mapsto t-r$,

$$(v^1 - v^2)(x,t) \le \int_0^t g(t-r)\pi(t-r)\mathbb{E}^{(t-r)}_{0,x}\left[\phi_1\left(v^1(W_{t-r},r)\right) - \phi_1\left(v^2(W_{t-r},r)\right)\right] \mathrm{d}r. \tag{3.39}$$

Now let $w := (v^1 - v^2) \vee 0$, where $\vee$ denotes maximum. Recall that ϕ_1 is monotone nondecreasing and convex on $[0,1]$, having a bounded derivative. Hence $\phi_1(v^1) - \phi_1(v^2) \ge 0$ holds if and only if $v^1 - v^2 \ge 0$, in which case $\phi_1(v^1) - \phi_1(v^2) \le C(v^1 - v^2)$ for some $C > 0$. Therefore,

$$w(x,t) \le C\int_0^t g(t-r)\pi(t-r)\mathbb{E}^{(t-r)}_{0,x} w(W_{t-r},r)\,\mathrm{d}r. \tag{3.40}$$

Note that if $w(x,t) = 0$, then (3.40) holds since the right-hand side is nonnegative, and if $w(x,t) \ge 0$, then (3.40) holds by definition of w and by (3.39). Define $F(r) := \mathbb{E}^{(t-r)}_{0,x} w(Y_{t-r},r)$, and note that $F(t) = w(x,t)$. We have

$$F(t) \le C\int_0^t g(t-r)\pi(t-r)F(r)\,\mathrm{d}r,$$

and, by Gronwall's inequality, we conclude that $F(t) \le 0$. Hence, $w(x,t) \le 0$. But $w(x,t) \ge 0$ by definition, therefore $w(x,t) = 0$, that is $v^1 - v^2 \le 0$, which means that $u^1 \le u^2$, as claimed. □

3.4.1 Proof of Theorem 3.9

We emphasize that in this proof, the branching rate for all types of particles for all processes considered is constant and equal to β, and all probabilities written should be understood as conditioned on the non-extinction of the GWP that underlies the BBM.

First, use the decomposition of supercritical GWPs into two sets of particles as described in Proposition 2.9 (particles with infinite line of descent, which are called skeleton particles; and particles with finite line of descent, which are called doomed particles) to decompose the BBM conditioned on non-extinction into skeleton and doomed particles. Then, interpret the decomposed BBM conditioned on non-extinction as a two-type process as described in Proposition 2.14, where we let the skeleton particles to be of type 1, and the doomed particles to be of type 2. We have the following decomposition for the BBM:

$$(Z(t))_{t\geq 0} = (Z^1(t), Z^2(t))_{t\geq 0},$$

where Z^1 is the process consisting of the skeleton particles, and Z^2 is the one consisting of the doomed particles. Note that Z^1 by itself is a BBM, however, Z^2 is not. Let $|Z^j(t)|$, $j=1,2$ be the number of particles of type j existing at time t. Recall that by Proposition 2.14, the offspring p.g.f.s for skeleton and doomed particles are

$$\begin{aligned} f^1(s,t) =& [f(\bar{q}s+qt) - f(qt)]/\bar{q}, \quad \text{and} \\ f^2(s,t) =& f(qt)/q, \end{aligned}$$

respectively. It is clear that skeleton particles can produce both skeleton and doomed particles, whereas a doomed particle can only produce particles of its own kind. Recall that conditioning Z on non-extinction is equivalent to the initial condition $(|Z^1(0)|, |Z^2(0)|) = (1,0)$, which says that the process starts with one skeleton particle, hence never dies out. For $d \geq 1$, let us define the range of Z^1 as R^1, and the range of Z^2 as R^2. Similarly, define

$$\begin{aligned} M^1(t) :=& \inf\left\{r>0 : R^1(t) \subseteq B(\vec{0}, r)\right\}, \\ M^2(t) :=& \inf\left\{r>0 : R^2(t) \subseteq B(\vec{0}, r)\right\}. \end{aligned}$$

Recall the definitions of f^* and $\tilde{f}$ from (2.20) and (2.19), and observe that $f^1(\cdot,1) = f^*(\cdot)$ and $f^2(1,\cdot) = \tilde{f}(\cdot)$. It is easy to see that $(f^*)'(1) = f'(1) = \mu$ and $\tilde{f}'(1) = f'(q)$. Define $\kappa := f'(q)$ and $k =: \kappa - 1$. Now since $M(t) = \max\{M^1(t), M^2(t)\}$, $f^*(0) = 0$, $(f^*)'(1) - 1 = m$, and βm is invariant under how one describes the branching (see the remark after Proposition 3.8), Proposition 3.8 immediately gives the desired lower bound for the speed: $\forall \varepsilon > 0$,

$$P\left(\frac{M(t)}{t} > \sqrt{2\beta m} - \varepsilon\right) \geq P\left(\frac{M^1(t)}{t} > \sqrt{2\beta m} - \varepsilon\right) \to 1 \quad \text{as} \quad t \to \infty. \tag{3.41}$$

By the same reasoning as above, the process Z^1 satisfies the desired upper bound, so in order to obtain the desired upper bound for Z, we need to control the spread of doomed particles in the system, and to show that $\forall \varepsilon > 0$,

$$P\left(\frac{M^2(t)}{t} > \sqrt{2\beta m + \varepsilon}\right) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.42}$$

The idea is to control the number of doomed particles at not just time t, but in the entire interval $[0, t]$. The reason is as follows. We wish to compare the spatial spread of doomed particles in the time interval $[0, t]$ to the spread of as many independent Brownian particles as there are doomed particles present in the system at time t, hence we do not want to have an instant in $[0, t]$, where the number of doomed particles are more than that number at time t. To be precise, we wish the following condition to hold:

$$|Z^2(s)| \le |Z^2(t)| \quad \text{for all} \quad s \in [0, t]. \tag{3.43}$$

Since the doomed particles of Z give zero offspring with positive probability, the condition (3.43) does not hold. The strategy now is to introduce a new two-type BBM, which is comparable to the original two-type process Z in a sense that will be clear shortly, prove the equivalent of (3.42) for the new process, and finally conclude that (3.42) holds for the original process by comparison. We introduce a new BBM as follows. Let $\bar{Z} = (\bar{Z}^1, \bar{Z}^2)$ be a two-type BBM, $\bar{P}$ be the corresponding probability, and $\bar{f}^1$, $\bar{f}^2$ be the offspring p.g.f.s for particles of type 1 and type 2 respectively, such that for every $0 \le s \le 1$ and every $0 \le t \le 1$,

$$\bar{f}^1(s, t) = f^1(s, t), \tag{3.44}$$

$$\bar{f}^2(s, t) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} p^1(j, i)\right) t^j, \tag{3.45}$$

where $p^1(j, i)$ is the probability that a particle of type 1 of the process Z gives j offspring of its own type and i offspring of type 2 upon branching. Equations (3.44) and (3.45) say that particles of type 1 of $\bar{Z}$ and that of Z behave exactly alike, whereas a particle of type 2 of $\bar{Z}$ produces no offspring of type 1 similar to Z, but produces offspring of her own type just as a particle of type 1 of Z produces offspring of type 1, with mean $\mu > 1$. Hence, the condition (3.43) is satisfied for particles of type 2 of the new process. We proceed by finding the mean matrix at time t, denoted by $\bar{A}(t)$ (see Section 2.4) for $\bar{Z}$. Following the recipe in (2.25) and (2.26), we obtain

$$\bar{A}(t) := \begin{pmatrix} e^{\beta t m} & \frac{q}{\bar{q}}(m-k)\beta t e^{\beta t m} \\ 0 & e^{\beta t m} \end{pmatrix}. \tag{3.46}$$

Define similarly as before the range of $\bar{Z}^1$ as $\bar{R}^1$, and the range of $\bar{Z}^2$ as $\bar{R}^2$. Further define

$$\overline{M}^1(t) := \inf\left\{r > 0 : \overline{R}^1(t) \subseteq B(\vec{0}, r)\right\},$$
$$\overline{M}^2(t) := \inf\left\{r > 0 : \overline{R}^2(t) \subseteq B(\vec{0}, r)\right\}.$$

Since particles of type 1 of Z and $\bar{Z}$ behave probabilistically alike, and particles of type 2 of $\bar{Z}$ produce offspring according to f^* while particles of type 2 of Z produce offspring according to $\tilde{f}$, Lemma 3.10 and Lemma 3.11 imply that for any $\varepsilon > 0$ and any $t \geq 0$,

$$\bar{P}\left(\frac{\overline{M}^2(t)}{t} > \sqrt{2\beta m} + \varepsilon\right) \geq P\left(\frac{M^2(t)}{t} > \sqrt{2\beta m} + \varepsilon\right). \tag{3.47}$$

Then, to complete the proof it suffices to show that

$$\bar{P}\left(\frac{\overline{M}^2(t)}{t} > \sqrt{2\beta m} + \varepsilon\right) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for any} \quad \varepsilon > 0. \tag{3.48}$$

We proceed as in the proof of Proposition 3.8. Let $B = B(0, (\sqrt{2\beta m + \varepsilon})t)$, and let $\delta > 0$ satisfy (3.30) as before. Define $\bar{\psi}_B = \inf\left\{t \geq 0 : \overline{R}^2(t) \cap B^c \neq \emptyset\right\}$, and estimate

$$\begin{aligned}\bar{P}\left(\frac{\overline{M}^2(t)}{t} > \sqrt{2\beta m} + \varepsilon\right) \leq\ & \bar{P}\left(|\overline{Z}^2(t)| > \lfloor e^{(\beta m+\delta)t}\rfloor\right) \\ & + \bar{P}\left(\bar{\psi}_B \leq t \mid |\overline{Z}^2(t)| \leq \lfloor e^{(\beta m+\delta)t}\rfloor\right).\end{aligned} \tag{3.49}$$

By the fact that $\bar{A}_{12}(t) = \frac{q}{\bar{q}}(m-k)\beta t e^{\beta t m}$ and the Markov inequality, the first term on the right-hand side of (3.49) tends to zero. The second term on the right-hand side tends to zero by an extension of Lemma 3.7, via comparison of type 2 particles of $\bar{Z}$ with $\lfloor e^{(\beta m+\delta)t}\rfloor$ many independent Brownian particles that start motion at $t = 0$ from the origin as in the proof of Proposition 3.8. Note that since each particle of type 2 of $\bar{Z}$ produces at least one particle of its own type (i.e., $\bar{f}^2(s, 0) = 0$), we may extend Lemma 3.7 to apply to particles of type 2 of $\bar{Z}$. This completes the proof of Theorem 3.9. $\square$

Chapter 4

Branching Brownian Motion Among Poissonian Traps

In this chapter, we find the trap-avoiding asymptotics for a BBM evolving in a Poissonian field of traps in $\mathbb{R}^d$, for the cases of uniform trap intensity and a radially decaying trap intensity. The result for the case of radially decaying field of traps is a direct application of Theorem 3.9. Section 4.1 is based on [34, 36], and Section 4.2 is based on [35]. In the paragraph below, we briefly describe the model. We refer the reader to Section 1.2 for details regarding the model of BBM in a Poissonian field of traps, and to Section 1.3 for a brief history of the problem.

Throughout this chapter, we consider a branching Brownian motion Z living in a random environment consisting of Poissonian traps, which is constructed as follows. Let $\mathbb{B}(\mathbb{R}^d)$ be the collection of d-dimensional Borel sets, and let Π denote a Poisson random measure on $\mathbb{B}(\mathbb{R}^d)$ with a boundedly finite mean measure ν such that $\mathrm{d}\nu/\mathrm{d}x$ exists, where $\mathrm{d}x$ stands for the Lebesgue measure. A random trap configuration K with radius a on $\mathbb{R}^d$ is defined as

$$K = \bigcup_{x_i \in \mathrm{supp}(\Pi)} \bar{B}(x_i, a),$$

where $\bar{B}(x, a)$ is the closed ball of radius $a > 0$ centered at x. Letting $\mathbb{P}$ be the probability measure for the Poisson random measure, $\mathbb{E}$ the corresponding expectation, and T the first time that Z hits a trap, i.e.,

$$T = \inf \{t \geq 0 : R(t) \cap K \neq \emptyset\},$$

where R stands for the range of the BBM. Then, the event of trap-avoiding up to time t of Z among the trap field is given by $\{T > t\}$. In this chapter, we are after the asymptotic behavior as time tends to infinity of the annealed trap-avoiding probability

$$(\mathbb{E} \times P)(T > t),$$

for a BBM with a general offspring distribution $(p_k)_{k\in\mathbb{N}}$.

4.1 Uniform trap intensity

In this section, we consider a uniform field of traps, distributed in space according to a PRM with uniform mean measure ν, where $\nu(\mathrm{d}x) = v\ \mathrm{d}x$ and $v > 0$. Below, we state the main result for this section, and then give its proof.

4.1.1 Main result

Suppose that $p_1 < 1$ so that branching phenomenon is present in the system. Given this assumption, recall from Lemma 2.7 that we may assume without loss of generality that $p_1 = 0$. Also, recall that f is the p.g.f. for the offspring distribution, μ is the mean number of offspring, $m = \mu - 1$, and assume that $\mu < \infty$. Let $q = P(\mathcal{E})$ be the probability of extinction for the GWP underlying the BBM as before, and let

$$\alpha = 1 - f'(q).$$

Note that $p_0 = 0$ implies $q = 0$ since two or more offspring is produced each time a particle branches. When $p_0 = 0$, it then follows that $\alpha = 1$ as $f'(0) = p_1$, and $p_1 = 0$ by assumption. Also, $q = 1$ when $\mu \leq 1$ as shown in Proposition 2.3, and $q \in (0, 1)$ when $p_0 > 0$ and $\mu > 1$.

Theorem 4.1. *The trap-avoiding asymptotics for the problem of BBM among uniform field of Poissonian traps is as follows.*

1. *If $p_0 = 0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) = -\beta \quad \text{for} \quad d \geq 2 \tag{4.1}$$

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) = \begin{cases} -\beta & \text{for} \quad d = 1,\ v > v_{cr}, \\ -\beta \frac{v}{v_{cr}} & \text{for} \quad d = 1,\ v \leq v_{cr}, \end{cases} \tag{4.2}$$

where $v_{cr} := \frac{\alpha}{2}\sqrt{\frac{\beta}{2m}}$ is the critical trap intensity for $d = 1$.

2. *If $p_0 > 0$ and $\mu \leq 1$, then*

$$\lim_{t\to\infty}(\mathbb{E}\times P)(T>t) = (\mathbb{E}\times P)(T>\tau) > 0, \tag{4.3}$$

where $\tau := \inf\{t \geq 0 : |Z(t)| = 0\}$ is the extinction time for the BBM.

3. *If $p_0 > 0$ and $\mu > 1$, then*

$$\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\times P)(T>t|\mathcal{E}^c) = -\beta\alpha \quad \textit{for} \quad d\geq 2 \tag{4.4}$$

$$\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\times P)(T>t|\mathcal{E}^c) = \begin{cases} -\beta\alpha & \textit{for} \quad d=1,\ v>v_{cr}, \\ -\beta\alpha\frac{v}{v_{cr}} & \textit{for} \quad d=1,\ v\leq v_{cr}. \end{cases} \tag{4.5}$$

Remark. Note that the most interesting case is when

$$\mu > 1,\ p_0 > 0 \quad \text{and} \quad d \geq 2. \tag{4.6}$$

In this case, from (4.4), we see that allowing the particles to give zero offspring with positive probability contributes an extra factor α to the result, which naturally depends on f. This extra factor appears due to the modified p.g.f. for the offspring distribution of the skeleton, which is a part of the decomposition of the supercritical GWP conditioned on non-extinction (see Proposition 2.9). It is easy to see that $0 < \alpha < 1$ when $\mu > 1$ and $p_0 > 0$.

4.1.2 Preparations

In this section we give preparatory results that will be needed in the proof of Theorem 4.1. Proposition 4.2 is a well-known general result coming from the spectral theory of compact self-adjoint operators, whereas Lemma 4.3 is a special result concerning the confinement of Brownian particles to balls, and is used directly in the proof of Theorem 4.1.

Spectral decomposition of the killed Brownian transition density

For a Borel set $B \subset \mathbb{R}^d$, let $p_B(t,x,y)$ be transition densities for Brownian motion killed upon exiting B, and let P_t^B be linear operators acting on the Hilbert space $L^2(B)$ as follows:

$$P_t^B : L^2(B) \to L^2(B),$$
$$P_t^B f(x) = \int_B p_B(t,x,y)f(y)dy \quad \text{for every} \quad t \geq 0.$$

Suppose that B has finite non-zero Lebesgue measure. Then, it is well known that P_t^B is a compact self-adjoint operator on $L^2(B)$, and by the spectral theorem admits an eigenvalue expansion in terms of the eigenvalues and eigenfunctions of the Dirichlet Laplacian on B. In more detail, consider the eigenvalue problem for the Dirichlet Laplacian,

$$\begin{aligned} -\tfrac{1}{2}\Delta\phi &= \lambda\phi \quad \text{on} \quad B, \\ \phi &= 0 \quad \text{on} \quad \partial B. \end{aligned} \tag{4.7}$$

Proposition 4.2. *If B has finite Lebesgue measure, there exists a discrete set of real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots < \infty$ and a corresponding complete orthonormal set of eigenfunctions ϕ_i in $L^2(B)$ that satisfy (4.7). Furthermore, if we let μ be the restriction of the Lebesgue measure to B, then*

(a) the densities $p_B(t,x,y)$ admit the eigenfunction expansion

$$p_B(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y), \tag{4.8}$$

for μ^2-almost every pair (x,y), where the convergence is absolute, and takes place in $L^\infty(B \times B)$,

(b) if $f \in L^2(B)$, then

$$P_t^B f = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle f, \phi_i \rangle \phi_i, \quad \mu - a.e., \tag{4.9}$$

where the convergence is absolute, and takes place in $L^\infty(B)$, and $\langle f, g\rangle$ is used to denote $\int f(x)g(x)dx$,

(c) if for some $t > 0$ we have $p_B(t,x,y) > 0$ for μ^2-almost every pair (x,y), then $\lambda_1 < \lambda_2$ and $\phi_1 > 0$ μ-almost everywhere.

Proposition 4.2 is taken from [5]. We refer the reader to [5] and references therein for a proof of parts (a), (b) and (c), and more on the theory of compact self-adjoint operators on Hilbert spaces. Note that part (c) is applicable when B is an open connected set.

The lemma below is a standard result on Brownian confinement to balls. The proof presented here is similar to the proof of [11, Theorem 2.1].

Lemma 4.3 (Lower bound on probability of confinement)**.** *Let $B = B(0,r)$ be the open ball in $\mathbb{R}^d$ with radius r, centered at the origin. Let W be the standard Brownian motion, and $(P_x,\ x \in \mathbb{R})$ be the corresponding probabilities for processes that start* *at $x \in \mathbb{R}$. Then,*

$$P_0(W[0,t] \in B) \geq c(d)\ e^{-\frac{\lambda}{r^2}t},$$

where $W[0,t] := \{W(s) : 0 \leq s \leq t\}$, λ is the principal Dirichlet eigenvalue for the open unit ball in $\mathbb{R}^d$, and $c(d)$ is a constant that depends on the dimension.

Proof. Let $p_B(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$ be the eigenfunction expansion for transition densities of Brownian motion killed upon exiting B, and $\mathbb{1}_B$ be the indicator function for the set B. Observe that $(P_t^B \mathbb{1}_B)(x) = \int_B p_B(t,x,y)dy$ is the probability that a Brownian particle starting at $x \in B$ remains in B in the period $[0,t]$, that is, $(P_t^B \mathbb{1}_B)(x) = P_x(W[0,t] \in B)$. By part (a) of Proposition 4.2 and the orthonormality of the eigenfunctions, we have

$$\int_B p_B(t,x,y)\phi_1(y)dy = e^{-\lambda_1 t}\phi_1(x) \quad \text{for a.e.} \quad x \in B. \tag{4.10}$$

By part (c) of Proposition 4.2, which is applicable here since B is open and connected, $\phi_1 > 0$ μ-almost everywhere on B. Along with (4.10), this implies that there exists $x_0 \in B$ such that $|\phi_1(x_0)| =: c_1 \neq 0$ and

$$\int_B p_B(t,x_0,y)\phi_1(y)dy = e^{-\lambda_1 t}\phi_1(x_0). \tag{4.11}$$

Let $q(t,x,y)$ be unkilled Brownian transition densities. Since $p_B(t,x,y) \leq q(t,x,y)$, it follows that $p_B(t,x,y)$ is supported on $y \in B$, where it is bounded for each $t > 0$. Then,

$$\left| \int_B p_B(t,x,y)\phi_1(y)dy \right| \leq c_2(t) \left| \int_B \phi_1(y)dy \right| \leq c_3(t), \tag{4.12}$$

where $c_2(t)$ and $c_3(t)$ are time-dependent constants, and we have used that $\phi_1 \in L^2(B)$ implies that $\phi_1 \in L^1(B)$ since B has finite volume. It follows from (4.10) and (4.12) that $\phi_1 \in L^\infty(B)$. From (4.11), we have

$$c_1 e^{-\lambda_1 t} \leq \operatorname*{ess\,sup}_{x \in B} |\phi_1(x)| \int_B p_B(t,x_0,y)dy.$$

Using $\int_B p_B(t,x,y)dy = P_x(W[0,t] \in B)$, we conclude that

$$P_{x_0}(W[0,t] \in B) \geq \frac{c_1}{\operatorname*{ess\,sup}_{x \in B} |\phi_1(x)|} e^{-\lambda_1 t}.$$

Finally, use that $P_0(W[0,t] \in B) \geq P_0(W[0,t+\tau] \in B) = P_{x_0}(W[0,t] \in B)$, where τ is the first hitting time of the $|x_0|$-sphere by a Brownian particle starting at the origin, along with the fact that the principal Dirichlet eigenvalue $\lambda_1(r)$ for $B(0,r)$

scales as $\lambda_1(r) = \lambda/r^2$. $\square$

4.1.3 Proof of Theorem 4.1

We prove case by case.

1. Let $p_0 = 0$ and $d \geq 2$. Recalling that $p_1 = 0$ by assumption, and comparing Engländer's result (1.8), which holds when $p_2 = 1$, with the general case of $p_0 = p_1 = 0$ yields

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) \leq -\beta \tag{4.13}$$

since introducing extra independent particles to the system decreases the survival probability for every $t > 0$ and for every trap configuration. Note that one way for the system to survive is to suppress branching while the single Brownian particle avoids the traps. As the branching and motion mechanisms are independent of each other, it follows that for every $t > 0$,

$$(\mathbb{E} \times P)(T > t) \geq e^{-\beta t}\ (\mathbb{E} \times P)(\bar{T} > t), \tag{4.14}$$

where $e^{-\beta t}$ is the probability that the initial particle does not branch before time t, and $\bar{T}$ is the first trapping time of a Brownian particle. Since $(\mathbb{E} \times P)(\bar{T} > t)$ decays subexponentially as $t \to \infty$ according to (1.6) and (1.7), we find from (4.14) that

$$\liminf_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) \geq -\beta. \tag{4.15}$$

Now, (4.13) and (4.15) together imply the result for $d \geq 2$. The result for $d = 1$ is a special case of Theorem 4.4, which will be proved later.

2. Firstly, see that $\lim_{t\to\infty} (\mathbb{E} \times P)(T > t)$ exists by the Bolzano-Weierstrass theorem since $(\mathbb{E} \times P)(T > t)$ is non-increasing in t and bounded below by 0. Now, let us bound this probability from below by a positive number. For instance, consider the following event: the initial Brownian particle starting at the origin remains in $B(0, 1)$ until the time of first branching which occurs in the time interval $[0, 1]$ and gives zero offspring, and there are no traps in $B(0, 1 + a)$. Since the mechanisms for branching, motion, and trap configuration are independent of each other, this event has positive probability, and the inequality in (4.3) follows.

We now prove the equality. If $p_0 > 0$ and $\mu \leq 1$, then $P(\mathcal{E}) = 1$, so that $\tau < \infty$ almost surely. Fix a trap configuration ω and let P^ω denote the probability law for Z, given this trap configuration in $\mathbb{R}^d$. First we find $\lim_{t\to\infty} P^\omega(T > t)$, and to that end write for every $t > 0$,

$$P^\omega(T > t) = P^\omega(T > t, \tau > t) + P^\omega(T > t \geq \tau). \tag{4.16}$$

Note that the first term on the right-hand side in (4.16) is bounded by $P^\omega(\tau > t)$, and that $\lim_{t\to\infty} P^\omega(\tau > t) = 0$ by monotone convergence of measures since $\tau < \infty$ almost surely. We now concentrate on the second term. The following holds for every $t > 0$:

$$P^\omega(T > \tau) = P^\omega(t \geq T > \tau) + P^\omega(T > t \geq \tau) + P^\omega(T > \tau > t). \tag{4.17}$$

We see that the left-hand side of (4.17) is a constant, free of t, so remains the same when we take the limit as $t \to \infty$. The third term on the right-hand side is bounded above by $P^\omega(\tau > t)$, hence tends to zero as $t \to \infty$. The first term on the right-hand side is $P^\omega(t \geq T, T > \tau) = P^\omega(t \geq T | T > \tau) P^\omega(T > \tau)$, and we see that $P^\omega(t \geq T | T > \tau) = 0$ since $T > \tau$ implies that $T = \infty$ almost surely. Since we have shown that the first and third terms on the right-hand side of (4.17) tend to zero as $t \to \infty$, (4.16) yields

$$\lim_{t\to\infty} P^\omega(T > t) = P^\omega(T > \tau). \tag{4.18}$$

Now, observing that $P^\omega(T > t)$ for each t is a random variable on the probability space for traps, and applying bounded convergence theorem, we arrive at

$$\lim_{t\to\infty} (\mathbb{E} \times P)(T > t) = (\mathbb{E} \times P)(T > \tau).$$

3. The result for $d = 1$ is a special case of Theorem 4.4, and will be proved later. Now let $d \geq 2$. We obtain the desired upper bound on the annealed survival probability conditioned on non-extinction by considering the skeleton of Z alone, and applying part (e) of Proposition 2.9. Next, consider the trap-avoiding scenario below, which yields the desired lower bound on $(\mathbb{E} \times P)(T > t | \mathcal{E}^c)$.

(a) Recall that the first particle must be of infinite line of descent since the process is conditioned on non-extinction. Suppress the branching so that there is precisely one skeleton particle in the system up to time t. This is equivalent to requiring that whenever a skeleton particle branches within the period $[0, t]$, it gives precisely one offspring of its own kind along with perhaps some doomed offspring. Therefore, by part (e) of Proposition 2.9, this event has probability

$$\exp[-\beta\alpha t].$$

(b) Empty the ball $B(0, r(t))$, where $t \mapsto r(t)$ is a nonnegative mapping such that $r(t) = o(t^{1/d})$, and $\lim_{t\to\infty} r(t) = \infty$. By emptying a region, we mean clearing a region from trap points. By the upper bound above on $r(t)$, the definition of a Poisson random measure gives that the probability to empty $B(0, r(t))$ is $\exp[o(t)]$.

(c) Confine the single skeleton particle to the smaller ball $B(0, r(t)/2)$ up to time t. Since $r(t) \to \infty$ as $t \to \infty$, by Lemma 4.3, this event has probability $\exp[o(t)]$ as well.

(d) Confine all the doomed particles that are created within the period $[0, t]$ to the larger ball $B(0, r(t))$ up to time t. Since the number of occurrences of branching up to time t along a single ancestral line is a Poisson process with mean βt, and by (2.38) a skeleton particle on average produces $(\mu - \kappa)q/(1 - q)$ doomed particles every time it branches, it follows that at most $\lfloor \beta t(\mu-\kappa)q/(1-q) \rfloor$ doomed particles are produced along the single skeletal line up to time t with probability $\exp[o(t)]$. Let the radii[1] of the subtrees initiated by these $\lfloor \beta t(\mu - \kappa)q/(1 - q) \rfloor =: n(t)$ doomed particles be $\rho_1, \rho_2, \ldots, \rho_{n(t)}$. If $K(t) := \max\left\{\rho_1, \rho_2, \ldots, \rho_{n(t)}\right\}$, then by independence of the subtrees,

$$P(K(t) < r(t)/2) = P(\rho_1 < r(t)/2)^{n(t)}. \tag{4.19}$$

Now, since a doomed subtree is almost surely finite, $\lim_{t\to\infty} r(t)/2 = \infty$ implies that $P(\rho_1 < r(t)/2) \to 1$, which in turn implies that the probability in (4.19) is $\exp[o(t)]$. We have already confined the single skeleton particle to the smaller ball $B(0, r(t)/2)$ up to time t. Since each doomed particle that is produced along the single skeletal line in the period $[0, t]$ must be produced within the smaller ball $B(0, r(t)/2)$, and the dimensions of the larger ball $B(0, r(t))$ all exceed the corresponding dimensions of the smaller ball by $r(t)/2$, we conclude that all of the doomed particles that are created within the period $[0, t]$ stay inside the larger ball with probability $\exp[o(t)]$.

Since the motion, branching and trap formation mechanisms are independent of each other, the desired lower estimate for $(\mathbb{E} \times P)(T > t|\mathcal{E}^c)$ follows from combining the steps listed above. Note that the survival scenario described above is composed of only one large deviation event, that is, suppressing the skeletal branching for a linear time. The remaining components are not large deviation events and do not have exponential probability costs.

[1]By radius we mean the radius of the smallest ball centered at the root of the subtree, containing the (finite) range of the subtree.

4.2 Radially decaying trap intensity

In this section, we consider a radially decaying field of traps, distributed in space according to a PRM with mean measure ν such that $\mathrm{d}\nu)/\mathrm{d}x$ exists, is continuous on $\mathbb{R}^d$, and satisfies the following asymptotic equivalence,

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} \sim \frac{l}{|x|^{d-1}}, \quad |x| \to \infty, \quad l > 0. \tag{4.20}$$

For $r, b \geq 0$, define

$$g_d(r,b) = \int_{B(\vec{0},r)} \frac{\mathrm{d}x}{|x + be_1|^{d-1}}, \tag{4.21}$$

where $e_1 = (1, 0, \ldots, 0)$ is the unit vector in the direction of the first coordinate. Finally, let

$$l^*_{cr} = l^*_{cr}(m, \beta, d) = \frac{1}{s_d}\sqrt{\frac{\beta}{2m}}, \tag{4.22}$$

where s_d is the surface area of the d-dimensional unit ball ($s_1 = 2$, $s_2 = 2\pi$, $s_3 = 4\pi$, etc.). As before, let f denote the p.g.f. of the offspring distribution of the underlying GWP. Below, we state the main results for this section, and then give their proof.

4.2.1 Main results

Theorem 4.4 (Variational formula). *The trap-avoiding asymptotics for the problem of BBM in a radially decaying field of Poissonian traps, where the corresponding PRM has density as in (4.20) is as follows. Define*

$$I(l, f, \beta, d) = \min_{\eta \in [0,1],\ c \in [0, \sqrt{2\beta}]} \left\{ \beta\alpha\eta + \frac{c^2}{2\eta} + l g_d(\sqrt{2\beta m}(1-\eta), c) \right\}. \tag{4.23}$$

(For $\eta = 0, c = 0$, set $c^2/2\eta = 0$, and for $\eta = 0, c > 0$, set $c^2/2\eta = \infty$.)

1. *If $p_0 = 0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P)\,(T > t) = -I(l, f, \beta, d).$$

2. *If $p_0 > 0$ and $m > 0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P)\,(T > t | \mathcal{E}^c) = -I(l, f, \beta, d).$$

3. If $p_0 > 0$ and $m \leq 0$, then

$$\lim_{t\to\infty} (\mathbb{E} \times P)\,(T > t) = (\mathbb{E} \times P)\,(T > \tau) > 0,$$

where $\tau = \inf\{t \geq 0 : |Z(t)| = 0\}$ is the extinction time and $\mathcal{E}$ is the event of extinction for the BBM.

Remark. In comparison with the variational result given in [15, Theorem 1.1] for dyadic branching, Theorem 4.4 contributes two extra factors in the expression for the rate function (4.23), where both factors naturally depend only on f. The extra factor α in the first term appears when $p_0 > 0$ (if $p(0) = 0$, then $\alpha = 1$). It is a consequence of the modified p.g.f. for the offspring distribution of the skeleton. The extra factor m inside g_d appears, because the speed of BBM is now no longer $\sqrt{2\beta}$ but $\sqrt{2\beta m}$. Since $\alpha = 1 = m$ for strictly dyadic branching, the result in [15] is recovered by (4.23) as a special case.

The next result is purely analytic, and solves the variational problem for the rate function $I(l, f, \beta, d)$. We generalize the corresponding result in [15]. Recall that l is the constant in the definition of ν in (4.20), and a is the constant trap radius in (1.3).

Theorem 4.5 (Crossover). *Fix f,β,a.*

1. For $d \geq 1$ and all $l \neq l_{cr}$, the variational problem in (4.23) has a unique minimizing pair, denoted by $\eta^* = \eta^*(l, f, \beta, d)$ and $c^* = c^*(l, f, \beta, d)$.

2. For $d = 1$,

$$\begin{aligned} l \leq l_{cr} : &\quad I(l, f, \beta, d) = \beta \frac{l}{l^*_{cr}}, \\ l > l_{cr} : &\quad I(l, f, \beta, d) = \beta\alpha, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} l < l_{cr} : &\quad \eta^* = 0,\ c^* = 0, \\ l > l_{cr} : &\quad \eta^* = 1,\ c^* = 0, \end{aligned} \tag{4.25}$$

where $l_{cr} = \alpha l^*_{cr} = \frac{\alpha}{2}\sqrt{\frac{\beta}{2m}}$.

3. For $d \geq 2$,

$$\begin{aligned} l \leq l_{cr}: &\quad I(l,f,\beta,d) = \beta \frac{l}{l^*_{cr}}, \\ l > l_{cr}: &\quad I(l,f,\beta,d) < \beta(\alpha \wedge \frac{l}{l^*_{cr}}), \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} l < l_{cr}: &\quad \eta^* = 0,\ c^* = 0, \\ l > l_{cr}: &\quad 0 < \eta^* < 1,\ 0 < c^* < \sqrt{2\beta}, \end{aligned} \tag{4.27}$$

where $l_{cr} = \gamma_d l^*_{cr}$, with

$$\gamma_d = \frac{-1 + \sqrt{1 + 4m\alpha M_d^2}}{2mM_d^2} \in (0,\alpha), \tag{4.28}$$

where

$$M_d = \frac{1}{2s_d m} \max_{R \in (0,\infty)} [g_d(R,0) - g_d(R,1)]. \tag{4.29}$$

Furthermore, $c^* > \sqrt{2\beta m}(1-\eta^*)$ when $l > l_{cr}$.

Remark. Comparing this theorem to the corresponding one in [15], we see that in this case the extra factors α and m appear ubiquitously in the formulas. Again, we note that the formulas above reduce to the corresponding ones in [15] upon setting $\alpha = 1$, $m = 1$.

4.2.2 Preparations

Here, we state and prove two lemmas that will be needed in the proof of Theorem 4.4. Lemma 4.6 is about the expected number of trap points contained in balls according to the trap intensity in (4.20), and Lemma 4.7 is about Brownian displacement in $\mathbb{R}^d$.

Lemma 4.6. *Let $r > 0$ and $b > 0$. Then,*

$$\lim_{t\to\infty} \frac{1}{lt} \nu(B(bte_1, rt)) = g_d(r,b).$$

Proof. By substituting $y = x/t$, we see that

$$g_d(rt,bt) = \int_{B(\vec{0},rt)} \frac{\mathrm{d}x}{|x + bte_1|^{d-1}} = t \int_{B(\vec{0},r)} \frac{\mathrm{d}y}{|y + be_1|^{d-1}} = tg_d(r,b). \tag{4.30}$$

Note that by assumption $\mathrm{d}\nu/\mathrm{d}x$ is locally finite, so it will not blow up near the origin. Recall that integrals of asymptotically equivalent positive continuous functions are asymptotically equivalent if one of the integrals diverges. Apply this result to the radial integral to justify the asymptotic similarity below in (4.31). By assumption, $\mathrm{d}\nu/\mathrm{d}x$ is continuous on $\mathbb{R}^d$. As the d-dimensional volume element in spherical coordinates has radial component $|x|^{d-1}\mathrm{d}|x|$, the integrands of both radial integrals below are continuous on $\mathbb{R}_+$. Hence, we have

$$\nu(B(bte_1, rt)) = \int_{B(bte_1,rt)} \frac{\mathrm{d}\nu}{\mathrm{d}x}\mathrm{d}x \sim \int_{B(bte_1,rt)} \frac{l}{|x|^{d-1}}\mathrm{d}x = lg_d(rt, bt), \tag{4.31}$$

where the first and third equalities hold by definition of ν and g_d, respectively. Combining (4.30) and (4.31) yields $\lim_{t\to\infty} \frac{\nu(B(bte_1,rt))}{ltg_d(r,b)} = 1$, which implies the result. □

Lemma 4.7. ***(Brownian displacement in*** $\mathbb{R}^d$***)*** *The probability that a Brownian particle starting from the origin moves to a distance* $[ct, ct + g(t)]$ *from the origin during time* t *in* $\mathbb{R}^d$ *is*

$$\exp\left[-\frac{c^2}{2}t + o(t)\right],$$

if $g(t) \geq 1$ *for all* t, *and* $g(t) = o(t)$ *as* $t \to \infty$.

Proof. Recall that the transition density for d-dimensional Brownian motion is

$$p(t, x, y) := (2\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

It follows that the desired probability is

$$k(d)\int_{ct}^{ct+g(t)} (2\pi t)^{-d/2} \exp\left(-\frac{r^2}{2t}\right) r^{d-1}\mathrm{d}r,$$

where $k(d) = \exp[o(t)]$ is a constant that depends on the dimension, coming from the angular integral. The radial integral is bounded as follows:

$$\begin{aligned} g(t)e^{\frac{-(ct+g(t))^2}{2t}}(ct)^{d-1}(2\pi t)^{-d/2} &\leq \int_{ct}^{ct+g(t)} (2\pi t)^{-d/2} \exp\left(-\frac{r^2}{2t}\right) r^{d-1}\mathrm{d}r \\ &\leq g(t)e^{\frac{-c^2 t}{2}}(ct + g(t))^{d-1}(2\pi t)^{-d/2}. \end{aligned}$$

Now since $1 \leq g(t) = o(t)$, it follows that $g(t) \subseteq \exp[o(t)]$. The desired result then follows since $t^\alpha \subseteq \exp[o(t)]$ for any $\alpha \in \mathbb{R}$. □

4.2.3 Proof of Theorem 4.4 and Theorem 4.5

Proof of Theorem 4.4

Part 3 of the theorem follows since $\tau < \infty$ almost surely for a non-supercritical process, by an argument identical to the one given in the proof of the corresponding case of Theorem 4.1. Since $p_0 = 0$ implies that $\alpha = 1$ and $P(\mathcal{E}^c) = 1$, conditioning the process on $\mathcal{E}^c$ is the same as not conditioning it when $p_0 = 0$. Hence, we may extend part 2 to cover for the case $p_0 = 0$, and prove the first two parts of the theorem together. Note that the key ingredients in the proof are results concerning the decomposition of supercritical GWPs and the conditional speed of BBM, namely Proposition 2.9 and Theorem 3.9.

Proof of the lower bound

Let $m > 0$. Fix β, d, f, η and c. The trap-avoiding scenario below yields the desired lower bound for the annealed survival probability conditioned on non-extinction, i.e., $(\mathbb{E} \times P)\,(T > t|\mathcal{E}^c)$.

1. Similar to the corresponding part of the proof of Theorem 4.1, the initial particle is of infinite line of descent since the process is conditioned on $\mathcal{E}^c$. Suppress the branching so that there is precisely one skeleton particle in the system up to time ηt. This event has probability

$$\exp[-\beta\alpha\eta t].$$

2. For $d \geq 2$, empty the two-sided cylinder ("tube") T_t, defined by

$$T_t = \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : |x_1| \leq kt + h(t),\ \sqrt{x_2^2 + \ldots + x_d^2} \leq r(t) + h(t)\right\},$$

where $k > c$, and $t \mapsto r(t)$ and $t \mapsto h(t)$ are nonnegative mappings that are picked such that $\lim_{t\to\infty} r(t)/t = 0$, $\lim_{t\to\infty} r(t) = \infty$ and $\lim_{t\to\infty} h(t)/t = 0$, $\lim_{t\to\infty} h(t) = \infty$. By the upper bounds above on $r(t)$ and $h(t)$, Lemma 4.6 gives that the probability to empty T_t is $\exp[o(t)]$ for $d \geq 2$. For $d = 1$, empty the line $T_t^1 := \{x_1 \in \mathbb{R} : |x_1| \leq 2h(t)\}$, which again costs a probability of $\exp[o(t)]$.

3. For $d \geq 2$, move the single skeleton particle (see the remark after Proposition 2.9) during the period $[0, \eta t]$ so that it is at a distance $ct + o(t)$ from the origin at time ηt (assume without loss of generality that the specific site has

first coordinate $ct + o(t)$ and all other coordinates zero), and confine it to the smaller tube $\hat{T}_t$, defined by

$$\hat{T}_t = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d : |x_1| \leq kt, \ \sqrt{x_2^2 + \ldots + x_d^2} \leq r(t) \right\},$$

up to time ηt.

Let P_0 be the probability law for d-dimensional Brownian motion, W^1 be the projection of the d-dimensional Brownian motion onto the first coordinate axis (W^1 lives in $\mathbb{R}^d$), and decompose the Brownian motion into an independent sum $W = W^1 + W^{d-1}$, where W^{d-1} is hence implicitly defined. Define the events

$$\begin{aligned} A_t &= \left\{ ct \leq |W^1(\eta t)| \leq ct + o(t) \right\}, \\ B_t &= \left\{ |W^1(s)| \leq kt \ \ \forall\, 0 \leq s \leq \eta t \right\}, \\ C_t &= \left\{ |W^{d-1}(s)| \leq r(t) \ \ \forall\, 0 \leq s \leq \eta t \right\}. \end{aligned}$$

Note that the event $A_t \cap B_t \cap C_t$ is exactly the desired scenario for this part. Lemma 4.7 gives $P_0(A_t) = \exp[-\frac{c^2}{2\eta}t + o(t)]$, and Lemma 3.5 gives $P_0(B_t^c) = \exp[-\frac{k^2}{2\eta}t + o(t)]$, so it follows that

$$\begin{aligned} \exp\left[-\frac{c^2}{2\eta}t + o(t)\right] &= P_0(A_t) \geq P_0(A_t \cap B_t) \geq P_0(A_t) - P_0(B_t^c) \\ &= \exp\left[-\frac{c^2}{2\eta}t + o(t)\right] - \exp\left[-\frac{k^2}{2\eta}t + o(t)\right] \\ &= \exp\left[-\frac{c^2}{2\eta}t + o(t)\right], \end{aligned}$$

which yields

$$P_0(A_t \cap B_t) = \exp\left[-\frac{c^2}{2\eta}t + o(t)\right]. \tag{4.32}$$

Since $r(t) \to \infty$, Lemma 4.3 implies that $P_0(C_t) = \exp[o(t)]$. Combining this with (4.32) and using the independence of W^1 and W^{d-1}, we arrive at

$$P_0(A_t \cap B_t \cap C_t) = \exp\left[-\frac{c^2}{2\eta}t + o(t)\right].$$

For $d = 1$, it turns out that the function in (4.23) is minimized when $c = 0$ (see the proof of Theorem 4.5). Confine the single skeleton particle to the one-dimensional tube $\hat{T}_t^1 := \{x_1 \in \mathbb{R} : |x_1| \leq h(t)\}$ up to time ηt. Since the length of the tube tends to infinity as t tends to infinity, Lemma 4.3 implies that the probability of this event is $\exp[o(t)]$.

4. Confine all the doomed particles that are created within the period $[0, \eta t]$ to the larger tubes T_t and T_t^1 for $d \geq 2$ and $d = 1$, respectively, up to time t. Since the number of occurrences of branching up to time t along a single ancestral line is a Poisson process with mean βt, and by (2.38) a skeleton particle on average produces $(\mu - \kappa)q/(1-q)$ doomed particles every time it branches, it follows that at most $\lfloor \beta\eta t(\mu - \kappa)q/(1-q) \rfloor$ doomed particles are produced along the single skeletal line up to time ηt with probability $\exp[o(t)]$. Let the radii of the subtrees initiated by these $\lfloor \beta\eta t(\mu - \kappa)q/(1-q) \rfloor =: n(t)$ doomed particles be $\rho_1, \rho_2, \ldots, \rho_{n(t)}$. If $K(t) := \max\left\{\rho_1, \rho_2, \ldots, \rho_{n(t)}\right\}$, then
$$P(K(t) < h(t)) = P(\rho_1 < h(t))^{n(t)}. \tag{4.33}$$
Now, since a doomed subtree is almost surely finite, $\lim_{t\to\infty} h(t) = \infty$ implies that $P(\rho_1 < h(t)) \to 1$, which implies that the probability in (4.33) is $\exp[o(t)]$. We have already confined the single skeleton particle to the smaller tube $\hat{T}_t$ for $d \geq 2$ ($\hat{T}_t^1$ for $d = 1$). By an argument identical to the corresponding one in the previous section, we conclude that all of the doomed particles that are created within the period $[0, \eta t]$ stay inside the larger tube with probability $\exp[o(t)]$.

5. Empty a $\sqrt{2\beta m}(1-\eta)t$-ball around the position of the single skeleton particle at time ηt. By Lemma 4.6, this event has probability
$$\exp\left[-lg_d(\sqrt{2\beta m}(1-\eta), c)t + o(t)\right].$$

6. Require the branching system initiated by the single skeleton particle present at time ηt to stay inside the $\sqrt{2\beta m}(1-\eta)t$-ball during the remaining time $(1-\eta)t$. When $m > 0$, by Theorem 3.9, the probability of this event conditioned on non-extinction is $\exp[o(t)]$.

Since the motion, branching and trap formation mechanisms are independent of each other, minimizing the cost of all these events over the parameters η and c provides us with the desired lower estimate for $(\mathbb{E} \times P)(T > t | \mathcal{E}^c)$. Note that the survival scenario described above is composed of three large deviation events: suppressing the skeletal branching for a linear time, moving the Brownian particle to a linear distance, and emptying a ball with linear radius. The remaining three components, namely parts two, four and six are not large deviation events and do not have exponential costs.

Remark. The Poisson intensity in (4.20), namely $\frac{\mathrm{d}\nu}{\mathrm{d}x} \sim \frac{l}{|x|^{d-1}}$ is interesting, because it creates a competition between the costs of suppressing the branching for a linear time, and emptying a ball with radius linearly growing with time. A ball with radius linearly growing with time is relevant, because a BBM travels a linear displacement with time. In other words, if we use any other exponent than $d-1$ in (4.20), the competition mentioned above will be lost, and the optimal survival scenario will involve exactly one of suppressing the branching in the entire period $[0,t]$, or emptying a ball with radius equal to the displacement of the "free" BBM in $[0,t]$.

Proof of the upper bound

Fix β, d, f. Let $0<\varepsilon<1$. Recall that $|Z(t)|$ is the number of particles in the system at time t, $|Z^*(t)|$ of which are skeleton particles. For $t>1$, define

$$\eta_t = \sup\left\{\eta \in [0,1] : |Z^*(\eta t)| \le \lfloor t^{d+\varepsilon}\rfloor\right\}.$$

Before we proceed, we prove the following lemma:

Lemma 4.8.

$$P\left(\frac{i}{n} \le \eta_t \mid \mathcal{E}^c\right) = \exp[-\beta\alpha\frac{i}{n}t + o(t)].$$

Proof. Observe that $\left\{\frac{i}{n} \le \eta_t\right\} = \left\{|Z^*(\frac{i}{n}t)| \le \lfloor t^{d+\varepsilon}\rfloor\right\}$. By part (e) of Proposition 2.9, we know that the process $|Z^*|$ conditioned on non-extinction of $|Z|$ is equal in law to a continuous-time branching process with rate $\beta(1-f'(q)) = \beta\alpha$ and offspring distribution $(\bar{p}_k)_{k\in\mathbb{N}}$, where $\bar{p}_0 = \bar{p}_1 = 0$. On the event $\mathcal{E}^c$, since the first particle is of infinite line of descent and $t>1$, the inequality $P(\frac{i}{n} \le \eta_t \mid \mathcal{E}^c) \ge \exp(-\beta\alpha\frac{i}{n}t)$ for every $t>1$ follows trivially.

Now consider a strictly dyadic branching Brownian motion $\widetilde{Z}$. We have $P(|\widetilde{Z}(t)| = k) = e^{-\beta t}(1-e^{-\beta t})^{k-1}$ for every $k\in\mathbb{N}$ and for every $t\ge 0$, see [24]. It follows that $P(|\widetilde{Z}(t)| \ge k) = (1-e^{-\beta t})^k$. Using the binomial expansion $(1-x)^n = 1 - nx + \binom{n}{2}x^2 - \ldots + (-1)^n x^n$, we obtain

$$\begin{aligned}
&P(|\widetilde{Z}(t)| \le \lfloor t^{d+\varepsilon}\rfloor)\\
&=1-(1-e^{-\beta t})^{\lfloor t^{d+\varepsilon}\rfloor}\\
&=e^{-\beta t}\left(\lfloor t^{d+\varepsilon}\rfloor - \binom{\lfloor t^{d+\varepsilon}\rfloor}{2}e^{-\beta t} + \ldots + (-1)^{\lfloor t^{d+\varepsilon}\rfloor - 1}e^{-\beta t\lfloor t^{d+\varepsilon}\rfloor - 1}\right)\\
&\le e^{-\beta t}\lfloor t^{d+\varepsilon}\rfloor = \exp[-\beta t + o(t)].
\end{aligned}$$

The result follows by comparison with the strictly dyadic case, since $\bar{p}_0 = \bar{p}_1 = 0$. $\square$

Henceforth, all probabilities written should be understood as conditioned on $\mathcal{E}^c$. Now, for every $n \in \mathbb{N}$, conditioning on η_t, we have

$$\begin{aligned}
&(\mathbb{E} \times P)(T > t) \\
&= \sum_{i=0}^{n-1} (\mathbb{E} \times P) \left(\{T > t\} \cap \left\{ \frac{i}{n} \leq \eta_t < \frac{i+1}{n} \right\} \right) + (\mathbb{E} \times P) \left(\{T > t\} \cap \{\eta_t = 1\} \right) \\
&\leq \sum_{i=0}^{n-1} \exp\left[-\beta\alpha \frac{i}{n} t + o(t) \right] (\mathbb{E} \times P_t^{(i,n)}) \{T > t\} + \exp[-\beta t + o(t)], \qquad (4.34)
\end{aligned}$$

where we use Lemma 4.8 in passing to the inequality, and introduce the conditional probabilities

$$P_t^{(i,n)}(\cdot) = P\left(\cdot \;\middle|\; \frac{i}{n} \leq \eta_t < \frac{i+1}{n} \right), \quad i = 0, 1, \ldots, n-1.$$

See that $\{\eta_t < (i+1)/n\} = \left\{ |Z^*(t(i+1)/n)| > \lfloor t^{d+\varepsilon} \rfloor \right\}$. Let $A_t^{(i,n)}$, $i = 0, 1, \ldots, n-1$ denote the event that at most $\lfloor t^{d+\varepsilon} \rfloor$ out of $|Z^*(t(i+1)/n)|$ skeleton particles alive at time $t(i+1)/n$ are such that the ball with radius

$$\rho_t^{(i,n)} := (1 - \varepsilon)\sqrt{2\beta m}\left(1 - \frac{i+1}{n}\right)t \qquad (4.35)$$

around the particle is non-empty, i.e., contains a trap point. Estimate

$$(\mathbb{E} \times P_t^{(i,n)})(T > t) \leq (\mathbb{E} \times P_t^{(i,n)})(A_t^{(i,n)}) + (\mathbb{E} \times P_t^{(i,n)})(T > t \mid [A_t^{(i,n)}]^c). \qquad (4.36)$$

On the event $[A_t^{(i,n)}]^c$ there are more than $\lfloor t^{d+\varepsilon} \rfloor$ balls containing a trap point, and by Theorem 3.8 and (4.35), the sub-BBM emanating from the center of each ball exists this ball in the remaining time $(1 - (i+1)/n)t$ with a probability tending to 1 as $t \to \infty$. Hence, with negligible error, we may assume that each of $\lfloor t^{d+\varepsilon} \rfloor$ sub-BBMs exit their respective balls that contain a trap point, in the remaining time. By geometric considerations, upon exiting this ball, each sub-BBM hits a trap inside this ball with a probability at least $C_1/[\rho_t^{(i,n)}]^{d-1}$, where $C_1 > 0$ is a constant depending on dimension and on a, which is the fixed radius of each trapping ball. Hence, the second term in the right-hand side of (4.36) is bounded above by

$$\left[1 - \frac{C_1}{[\rho_t^{(i,n)}]^{d-1}} \right]^{\lfloor t^{d+\varepsilon} \rfloor} \leq \exp[-C_2 t^{1+\varepsilon}] \qquad (4.37)$$

uniformly in all parameters, where $C_2 > 0$ is another constant. Note that we use the elementary estimate $(1+\frac{x}{n})^n \leq e^x$ in passing to the inequality (4.37), which implies that the second term on the right-hand side of (4.36) is superexponentially small (SES) in t.

Now consider the first term on the right-hand side of (4.36). Since $g_d(rt, bt) = tg_d(r, b)$ for $r, b > 0$, the cost of clearing a ball with radius linear in t is exponentially small in t for large t. Then, for large t, we have

$$(\mathbb{E}\times P_t^{(i,n)})({A_t}^{(i,n)} \mid |Z^*(t(i+1)/n)| = \lfloor t^{d+\varepsilon}\rfloor + j)$$
$$\geq (\mathbb{E}\times P_t^{(i,n)})(A_t^{(i,n)} \mid |Z^*(t(i+1)/n)| = \lfloor t^{d+\varepsilon}\rfloor + (j+1))$$

for every $j \in \{1, 2, \ldots\}$. It follows that

$$(\mathbb{E}\times P_t^{(i,n)})({A_t}^{(i,n)}) \leq (\mathbb{E}\times P_t^{(i,n)})(A_t^{(i,n)} \mid |Z^*(t(i+1)/n)| = \lfloor t^{d+\varepsilon}\rfloor + 1). \quad (4.38)$$

Having further conditioned on $\left\{|Z^*(t(i+1)/n)| = \lfloor t^{d+\varepsilon}\rfloor + 1\right\}$, we continue to estimate from above as follows. Each skeleton particle alive at time $t(i+1)/n$ is at a random point, whose spatial distribution is identical to that of $W(t(i+1)/n)$, where W denotes the standard Brownian motion. Let x_0 be the center of the empty ball at time $t(i+1)/n$ that is closest to the origin. Let $0 \leq c < \infty$, $\delta > 0$, and define B_j, $j = 1, \ldots, \lfloor t^{d+\varepsilon}\rfloor + 1$ to be the event that jth skeleton particle at time $t(i+1)/n$ is at a distance $ct \leq r < (c+\delta)t$ from the origin and that the $\rho_t^{(i,n)}$-ball around it is empty. Then,

$$\begin{aligned}
(\mathbb{E}\times &P_t^{(i,n)})(A_t^{(i,n)} \cap \{ct \leq |x_0| < (c+\delta)t\} \;\mid |Z^*(t(i+1)/n)| = \lfloor t^{d+\varepsilon}\rfloor + 1)\\
&\leq (\mathbb{E}\times P_t^{(i,n)})(B_1 \cup \ldots \cup B_{\lfloor t^{d+\varepsilon}\rfloor + 1})\\
&\leq (\lfloor t^{d+\varepsilon}\rfloor + 1)(\mathbb{E}\times P_t^{(i,n)})(B_1)\\
&= (\lfloor t^{d+\varepsilon}\rfloor + 1)\exp\left[-\frac{c^2}{2(i+1)/n}t + o(t)\right]\\
&\qquad \times \exp\left[-lg_d\left((1-\varepsilon)\sqrt{2\beta(m-1)}(1-\frac{i+1}{n}), c\right)t + o(t)\right], \qquad (4.39)
\end{aligned}$$

where we have used the union bound in passing to the second inequality, and Lemma 3.5, Lemma 4.6, along with the independence of BBM and trap mechanisms in passing to the equality. Indeed, on the event $A_t^{(i,n)}$ conditioned on $|Z^*(t(i+1)/n)| = \lfloor t^{d+\epsilon}\rfloor + 1$, there is at least 1 skeleton particle with an empty ball around it. From (4.36),(4.38) and (4.39), conditioning on $|x_0|$, we obtain

$$
\begin{aligned}
&(\mathbb{E}\times P_t^{(i,n)})(T>t)\\
&\leq \sum_{j=0}^{n-1}(\mathbb{E}\times P_t^{(i,n)})\left(\left\{\frac{j}{n}\sqrt{2\beta}t \leq |x_0| < \frac{j+1}{n}\sqrt{2\beta}t\right\}\cap A_t^{(i,n)}\right)\\
&\qquad + (\mathbb{E}\times P_t^{(i,n)})\left(\left\{|x_0|\geq \sqrt{2\beta}t\right\}\cap A_t^{(i,n)}\right) + \text{SES}\\
&\leq \sum_{j=0}^{n-1}(\lfloor t^{d+\varepsilon}\rfloor + 1)\exp\left[-\frac{\beta j^2/n^2}{(i+1)/n}t + o(t)\right]\\
&\qquad \times \exp\left[-lg_d\left((1-\varepsilon)\sqrt{2\beta m}(1-\frac{i+1}{n}), \frac{j}{n}\sqrt{2\beta}\right)t + o(t)\right]\\
&\qquad + \exp[-\beta t + o(t)] + \text{SES}. \qquad (4.40)
\end{aligned}
$$

Here, the SES comes from the second term in the right-hand side of (4.36). Also, in passing to the last inequality, we have used that the probability of the event $\left\{|x_0|\geq\sqrt{2\beta}t\right\}$ is bounded above by the probability that a single Brownian particle is at a distance more than $\sqrt{2\beta}t$ at time t, which is $\exp[-\beta t + o(t)]$. Substituting (4.40) into (4.34), and optimizing over $i, j \in \{0, 1, \ldots, n-1\}$ gives

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\times P)(T>t)\\
&\leq -\min_{i,j\in\{0,1,\ldots,n-1\}}\left\{\frac{\beta\alpha i}{n} + \frac{\beta j^2/n^2}{(i+1)/n} + lg_d\left((1-\varepsilon)\sqrt{2\beta m}(1-\frac{i+1}{n}), \frac{j}{n}\sqrt{2\beta}\right)\right\}.
\end{aligned}
$$

Now let $\eta = i/n$, $c = j/n\sqrt{2\beta}$, let $n\to\infty$, use the continuity of the functional form from which the minimum is taken (in order to extend the domain under which the minimization is performed from points with rational coordinates of $[0,1]\times[0,\sqrt{2\beta}]$ to the entire $[0,1]\times[0,\sqrt{2\beta}]$), and finally let $\varepsilon\to 0$ to obtain the desired upper bound,

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\times P)(T>t) \leq -I(l,f,\beta,d).$$

Note that $c > \sqrt{2\beta}$ cannot minimize

$$\left\{\beta\alpha\eta + \frac{c^2}{2\eta} + lg_d(\sqrt{2\beta m}(1-\eta), c)\right\}$$

since this makes it greater than β, which can be improved by setting $c = 0$ and $\eta = 1$. This completes the proof of Theorem 4.4.

Proof of Theorem 4.5

Let

$$G_d(\eta, c) = \begin{cases} \beta\alpha\eta + \frac{c^2}{2\eta} + lg_d(\sqrt{2\beta m}(1-\eta), c) & \text{for} \quad \eta \in (0,1],\ c \in [0, \sqrt{2\beta}], \\ ls_d\sqrt{2\beta m} & \text{for} \quad \eta = 0,\ c = 0, \\ \infty & \text{for} \quad \eta = 0,\ c \in (0, \sqrt{2\beta}] \end{cases} \tag{4.41}$$

so that

$$I(l, f, \beta, d) = \min_{\eta\in[0,1],\ c\in[0,\sqrt{2\beta}]} G_d(\eta, c). \tag{4.42}$$

To see the existence of the minimizers η^*, c^* of (4.41), note that G_d is lower semi-continuous on the compact subset $[0,1] \times [0, \sqrt{2\beta}]$ of $\mathbb{R}^2$ since

$$\lim_{(\eta,c)\to(0,0)} \Big(\beta\alpha\eta + lg_d(\sqrt{2\beta m}(1-\eta), c)\Big) = ls_d\sqrt{2\beta m}$$

and $\frac{c^2}{2\eta} \geq 0$. We refer the reader to [15] for the proof of the uniqueness of the minimizers when $l \neq l_{cr}$.

Consider $d = 1$. Since $g_1(r, b) = 2r$, the minimum over c in (4.41) is taken at $c^* = 0$, so that (4.42) reduces to

$$\begin{aligned} I(l, f, \beta, 1) &= \min_{\eta\in[0,1]} \left\{\beta\alpha\eta + 2l\sqrt{2\beta m}(1-\eta)\right\} \\ &= \min_{\eta\in[0,1]} \left\{\eta(\beta\alpha - 2l\sqrt{2\beta m}) + 2l\sqrt{2\beta m}\right\}. \end{aligned}$$

This proves (4.24) and (4.25), and identifies l_{cr} as the solution to $\beta\alpha = 2l\sqrt{2\beta m}$.

Now consider $d \geq 2$. Set $c^2/2\eta = 0$ when $\eta = 0$. We have

$$G_d(\eta, c) - G_d(0,0) = \beta\alpha\eta + \frac{c^2}{2\eta} + l[g_d(\sqrt{2\beta m}(1-\eta), c) - g_d(\sqrt{2\beta m}, 0)], \tag{4.43}$$

where the last term in the right-hand side is less than or equal to zero, with equality if and only if $(\eta, c) = (0,0)$ (this follows from the definiton of g_d). Suppose that $(\eta, c) = (0,0)$ is a minimizer when $l = l_0$. Then, by uniqueness of minimizers, the right-hand side of (4.43) is strictly positive for all $(\eta, c) \neq (0,0)$ when $l = l_0$. As the last term in the right-hand side of (4.43) is strictly negative for all $(\eta, c) \neq (0,0)$, it follows that for all $l < l_0$, the right-hand side of (4.43) is zero when $(\eta, c) = (0,0)$ and strictly positive otherwise. Therefore, there must exist an $l_{cr} \in (0, \infty)$ such that

1. $(\eta, c) = (0,0)$ is the minimizer when $l < l_{cr}$,

2. $(\eta, c) = (0,0)$ is not the minimizer when $l > l_{cr}$.

Henceforth, we will take this as the definition of l_{cr} for $d \geq 2$. This proves the first parts of (4.26) and (4.27). Note that $l_{cr} \in (0,\infty)$, because the last term in the right-hand side of (4.43) decreases without bound as $l \to \infty$ and tends to zero as $l \to 0$ for fixed $(\eta, c) \neq (0,0)$.

Now let $d \geq 2$ and $l > l_{cr}$. We know that $(\eta, c) = (0,0)$ is not a minimizer. The combination $\eta^* = 0$, $c^* > 0$ is not possible because $G(0,c) = \infty$ for all $c > 0$. The combination $\eta^* > 0$, $c^* = 0$ is ruled out as well because $G(\eta, 0)$ takes its minimum either at $\eta = 0$ or $\eta = 1$, so $\eta^* > 0$ would imply that $\eta^* = 1$. However, $\eta^* = 1$ can be excluded via [15, Lemma 4.1]. Also, $c^* = \sqrt{2\beta}$ is not possible since this yields $G_d > \beta$, whereas $(\eta, c) = (1,0)$ is not a minimizer yet yields $G_d \leq \beta$ since $\alpha \in (0,1]$. Hence, we conclude that the minimizers for $d \geq 2$ and $l > l_{cr}$ appear in the interior of $[0,1] \times [0, \sqrt{2\beta}]$. This proves the second parts of (4.26) and (4.27).

In the rest of the proof, we find l_{cr} for $d \geq 2$. For $R \geq 0$, let

$$g_d(R) = \int_{B(\vec{0},R)} \frac{\mathrm{d}x}{|x + e_1|^{d-1}}. \tag{4.44}$$

Then, we may write (4.41) as

$$G_d(\eta, c) = \beta\alpha\eta + \frac{c^2}{2\eta} + lcg_d\left(\frac{\sqrt{2\beta m}(1-\eta)}{c}\right). \tag{4.45}$$

The stationary points are the solutions of the equations $\frac{\partial G_d}{\partial \eta} = 0$ and $\frac{\partial G_d}{\partial c} = 0$, which yield

$$\begin{aligned} 0 =& \beta\alpha - \beta m v^2 - l\sqrt{2\beta m} g_d'(u) \\ 0 =& \sqrt{2\beta m} v + l[g_d(u) - u g_d'(u)] \end{aligned} \tag{4.46}$$

upon setting

$$u = \frac{\sqrt{2\beta m}(1-\eta)}{c}, \quad v = \frac{c}{\sqrt{2\beta m}\eta}. \tag{4.47}$$

Eliminating g_d' in (4.46) gives

$$\frac{l}{\sqrt{2\beta m}} g_d(u) = -v + \frac{1}{2m} u(\alpha - mv^2). \tag{4.48}$$

It follows by (4.45), (4.47) and (4.48) that for $d \geq 2$ and $l > l_{cr}$,

$$I(l, f, \beta, d) = G_d(u^*, v^*) = \beta(\alpha - mv^{*2}). \tag{4.49}$$

Since the minimizer (u^*, v^*) must satisfy (4.48) (note that there may be more than one pair (u, v) that satisfies (4.48)), we conclude by (4.49) that v^* is the maximal value of $v > 0$ on the curve in the (u, v)-plane given by (4.48).

The following lemma is taken from [15], where a proof is given as well.

Lemma 4.9. *$R \mapsto g_d'(R)$ is strictly increasing on $(0, 1)$, infinite at 1, and strictly decreasing on $(1, \infty)$.*

Using (4.22), we may write (4.48) as

$$\frac{g_d(u)l}{2s_d m l_{cr}^*} = -v + \frac{1}{2m}u(\alpha - mv^2). \tag{4.50}$$

Since $g_d(u) \sim s_d u$ as $u \to \infty$ by (4.44), we obtain from Lemma 4.9 that the left-hand side of (4.50) is strictly convex on $(0, 1)$, has infinite slope at 1, is strictly concave on $(1, \infty)$, and has limiting derivative $l/(2ml_{cr}^*)$ (see [35, Figure 1]).

Firstly, see that $l_{cr} \leq \alpha l_{cr}^*$. Indeed, assume the contrary. Then, there exists $\bar{l} \in (\alpha l_{cr}^*, l_{cr})$, which implies that $(\eta, c) = (0, 0)$ is the unique minimizer for G_d when $l = \bar{l}$. However, $G_d(0, 0) = \beta \bar{l}/l_{cr}^* > \beta\alpha = G_d(1, 0)$, which contradicts $(\eta, c) = (0, 0)$ being the minimizer. This implies that $l_{cr} \leq \alpha l_{cr}^*$. We claim that l_{cr} can be identified by the following formula:

$$l_0 := \sup\left\{ l > 0 : \frac{l}{l_{cr^*}} \frac{1}{2s_d m} g_d(u) > -\sqrt{\frac{1}{m}\left(\alpha - \frac{l}{l_{cr^*}}\right)} + \frac{1}{2m}\frac{l}{l_{cr^*}}u \quad \forall u \in (0, \infty) \right\}. \tag{4.51}$$

In order to prove our claim, i.e., $l_0 = l_{cr}$, we need to show that $(\eta, c) = (0, 0)$ is the minimizer for G_d when $l < l_0$, and $(\eta, c) = (0, 0)$ is not the minimizer when $l > l_0$. Equivalently, as $G_d(0, 0) = \beta l/l_{cr}^*$, we need to show via (4.49) that for $l > l_0$ there exists a $v > \sqrt{1/m\,(\alpha - l/l_{cr^*})}$ such that v satisfies (4.50) for some $u > 0$, and that for $l < l_0$ there is no such v. As we know that $l_{cr} > 0$ and $(\eta, c) = (0, 0)$ is the unique minimizer when $l < l_{cr}$, there must exist an $l_1 > 0$ such that the inequality in (4.51) holds for all $u \in (0, \infty)$ when $l = l_1$, hence $0 < l_1 < l_0$. Now take any $0 < l_2 < l_1$, and see that the inequality in (4.51) holds for all $u \in (0, \infty)$ also for $l = l_2$. This implies that if $l < l_0$, then the curve on the left-hand side of (4.50) and the line on the right-hand side do not intersect when $v = \sqrt{1/m\,(\alpha - l/l_{cr^*})}$. As both terms on the right-hand side of (4.50) decrease as v increases and the left-hand side of (4.50) does not depend on v, the curve on the left-hand side of (4.50) and the line on the right-hand side do not intersect for $v > \sqrt{1/m\,(\alpha - l/l_{cr^*})}$ as well when $l < l_0$.

Now consider $l > l_0$. By definition of l_0, there exists $u \in (0, \infty)$, say $\bar{u}$, such that

$$\frac{l_0}{l_{cr^*}} \frac{1}{2s_d m} g_d(u) = -\sqrt{\frac{1}{m}\left(\alpha - \frac{l_0}{l_{cr^*}}\right)} + \frac{1}{2m}\frac{l_0}{l_{cr^*}} u. \tag{4.52}$$

Note that since $g_d(u) < s_d u$ for all $u \in (0, \infty)$, (4.52) implies that we have the strict inequality $l_0 < \alpha l^*_{cr}$. Now take any $l_1 \in (l_0, \alpha l^*_{cr})$, and observe that at $u = \bar{u}$, the right-hand side of (4.52) becomes greater than the left-hand side if we replace l_0 by l_1. Also, since $g_d(u) \sim s_d u$, the left-hand side of (4.52) must be greater than the right-hand side for large u if we replace l_0 by l_1. As the functions on both sides of (4.52) are continuous with respect to u, it follows by the intermediate value theorem that there exists $u \in (0, \infty)$, say $\tilde{u}$, such that (4.52) holds, with l_0 replaced by l_1. This in turn implies that $G_d(u, v) = G_d(\tilde{u}, \sqrt{1/m(\alpha - l_1/l_{cr^*})}) = \beta l_1/l^*_{cr}$, but since the minimizer is unique, this shows that $(\eta, c) = (0, 0)$ is not the minimizer for $l_0 < l_1 < l^*_{cr}$. As we have shown that $(\eta, c) = (0, 0)$ is not the minimizer when $l \in (l_0, l^*_{cr})$, we conclude by definition of l_{cr} that $(\eta, c) = (0, 0)$ is not the minimizer for G_d for all $l > l_0$. Hence, $l_0 = l_{cr}$.

Now put

$$\hat{g}_d(u) = s_d u - g_d(u), \quad M_d = \frac{1}{2s_d m} \max_{u \in (\infty)} \hat{g}_d.$$

Then, (4.51) reads

$$l_{cr} = \sup\left\{ l > 0 : \sqrt{\frac{1}{m}\left(\alpha - \frac{l}{l^*_{cr}}\right)} > \frac{l}{l^*_{cr}} M_d \right\}.$$

This implies that

$$\sqrt{\frac{1}{m}\left(\alpha - \frac{l_{cr}}{l^*_{cr}}\right)} = \frac{l_{cr}}{l^*_{cr}} M_d,$$

which completes the proof of (4.28) and (4.29).

We finally show that $u^* \in (0, 1)$ when $l > l_{cr}$. Note that for a fixed v, if the line on the right-hand side of (4.50) cuts the curve on the left-hand side at more than one point, then by (4.49), this v cannot be v^* since the minimizer is unique. Hence, we look for a v value such that (4.50) is satisfied for exactly one u value, and this pair is (u^*, v^*). This implies that when $v = v^*$, the line on the right-hand side of (4.50) is tangent to the curve on the left-hand side at $u = u^*$. Since $v^* > \sqrt{1/m(\alpha - l/l_{cr^*})}$, the slope of the line on the right-hand side of (4.50) is less than $1/(2m)(l/l^*_{cr})$. Then, since the left-hand side of (4.50) has infinite slope at $u = 1$, is concave on $(1, \infty)$, and decreases asymptotically to $1/(2m)(l/l^*_{cr})$, it follows that $u^* \in (0, 1)$. By (4.47), we conclude that $c^* > \sqrt{2\beta m}(1 - \eta^*)$. This completes the proof of Theorem 4.5.

Chapter 5

Conclusion and Future Work

This thesis has originated from a trapping problem in which a randomly moving and branching system of particles, namely a branching Brownian motion, is evolving in a field of randomly located traps in $\mathbb{R}^d$. The particles and traps are taken to be pointlike and spherical, respectively. The quantity of interest is the large time trap-avoiding probability of a system with a general branching mechanism. The main mathematical object of interest of this thesis is a branching Brownian motion, which is the most studied example of spatial branching processes, and which is built on two simpler Markov processes, namely, a Galton-Watson process and a continuous-time Markov branching process.

In search for finding the asymptotic rate of decay of the trap-avoiding probability in a trap field that is distributed according to a Poisson random measure, we have been led to the problem of the speed of BBM. We have decomposed the branching processes that are supercritical, and where the particles give zero offspring with positive probability, into skeleton and doomed particles, and then have shown that the process could be interpreted as a two-type branching process.

In particular, we have found the asymptotic speed of BBM with a general offspring distribution, conditioned on non-extinction. The challenging case is when the process is supercritical, and the probability of giving zero offspring for particles is nonzero. In the proof, the skeleton decomposition of the GWP and its interpretation as a two-type GWP have been centrally used, and it has been shown via a comparison argument that the skeleton alone determines the asymptotic speed, whereas the doomed particles play no role.

Coming back to our original trapping problem, we have first found the relevant trap-avoiding asymptotics in the case of a uniformly distributed field of traps in $\mathbb{R}^d$, where the skeleton decomposition of the branching Brownian motion is again used, along with some properties of Brownian motion. Our result extends the previous work, where the branching is assumed to be strictly dyadic. As an application of the conditional speed of branching Brownian motion, we have then studied the case of radially decaying field of traps, and have extended previous results where the branching was again strictly dyadic. The rate function, which is expressed as a variational formula, has two parameters coming from the general branching mechanism: one coming from the conditional speed of the branching Brownian motion, and the other coming from the effective rate of branching for the skeleton. Once more, our results show that the skeleton alone is responsible for the large-time asymptotics of the trap-avoiding probability.

Several variations of the problem can be considered as future work. Some open problems are as follows.

Branching Brownian motion in a moving trap field

A popular class of trapping problems is concerned with mobile traps as opposed to frozen traps. One natural model is a uniform field of Poissonian traps in $\mathbb{R}^d$, where each atom is moving under Brownian motion independently of others. This model is an instance of a time dependent random medium, and is called a *dynamic Boolean model* [6] in the context of stochastic geometry. In this model, a fundamental problem is the time until a moving target point is caught by the traps, which is called *detection*. If the target particle is Brownian, and its motion is independent of the moving trap field, then this can be viewed as an extension of the problem in [11] in the sense that the Brownian particle is trying to survive in a randomly moving field of traps as opposed to frozen traps. Several results about the trap-avoiding asymptotics for this problem have been obtained in [38, 39] for dimensions $d = 1, 2$, and in the discrete setting of random walks in [12]. However, in both discrete and continuous settings, the problem remains open when $d \geq 3$, where only upper bounds on the trap-avoiding probability are obtained as time tends to infinity. Once the case of a single target particle is resolved, one can then allow the target particle to branch, and study the survival of an entire branching Brownian motion among the moving trap field.

Optimal survival strategies

In the context of trapping problems, an *optimal survival strategy* is a set of events indexed by time t, that are bound to happen as $t \to \infty$, given that the system avoids the traps up to time t. In the setting of BBM among Poissonian traps in $\mathbb{R}^d$, it is natural to seek results regarding three types of optimal survival strategies, concerning the population growth of the system, position and radius of trap-free regions, and the range of the BBM. We believe that existing results for the case with Poisson intensity (4.20) on optimal survival strategies concerning the population growth of the system (see ([15])) are not sharp, and more restrictive bounds could be obtained.

Quenched asymptotics

There are two sets of results in random media problems. Results that hold for almost every environment are called *quenched* results, while results that are obtained after averaging over all possible environments are called *annealed* results. It would be interesting to find quenched analogues of the survival asymptotics found in this thesis, where only annealed results are obtained. On the other hand, in the setting of the dynamic Boolean model with a Brownian target particle, [38] and [39] study annealed asymptotics only. One could possibly find quenched results in the same spirit as in [12], where the same problem, appropriately adapted to the discrete setting of random walks, has been considered in both the annealed and quenched settings.

Soft obstacles

Instead of killing the BBM when it first falls into a trap (traps of this sort are called *hard obstacles*), we may think of traps as regions where the branching rate of the BBM is smaller than outside of traps, and investigate the growth of the total population size in this setting. Traps that tend to suppress growth rather than kill the process are called *soft obstacles*. The effect of reduced branching rate in the traps on the total population size is a non-trivial question, as studied in [17] for a strictly dyadic BBM. For this problem, it would be interesting to consider general branching, in particular to allow particles to give zero offspring, since in this case the traps do not necessarily suppress population growth. The techniques employed in this thesis regarding the skeleton decomposition and conditional speed of BBM may be useful in investigating the soft obstacle problem with general branching.

Bibliography

[1] E. Aïdékon, J. Berestycki, É. Brunet and Z. Shi, Branching Brownian motion seen from its tip, *Probability Theory and Related Fields*, 157, 405 – 451, 2013.

[2] D. G. Aronson and H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, *Lect. Notes Math.*, 446, 5 – 49, Springer, Berlin, 1975.

[3] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Advances in Mathematics*, 30, 33 – 76, 1978.

[4] K. Athreya and P. Ney, *Branching Processes*, Springer, Berlin, 1972.

[5] R. F. Bass and K. Burdzy, Eigenvalue expansions for Brownian motion with an application to occupation times, *Electronic Journal of Probability*, 1, 1 – 19, 1996.

[6] J. van den Berg, R. Meester and D. J. White, Dynamic Boolean model *Stochastic Processes and their Applications*, 69, 247 – 257, 1997.

[7] M. Bramson, Maximal displacement of branching Brownian motion, *Communications on Pure and Applied Mathematics*, 31 (5), 531 – 581, 1978.

[8] M. Bramson, Convergence of solutions of the Kolmogorov equation to travelling waves, *Memoirs of the American Mathematical Society*, 44, no. 285, 1983.

[9] L. Chaumont and M. Yor, *Exercises in Probability, A Guided Tour from Measure Theory to Random Processes, via Conditioning*, 2nd edition, Cambridge University Press, New York, 2012.

[10] B. Chauvin and A. Rouault, KPP equation and supercritical branching brownian motion in the subcritical speed area. Application to spatial trees, *Probability Theory and Related Fields*, 80, 299 – 314, 1988.

[11] M. D. Donsker and S. R. S. Varadhan, Asymptotics for the Wiener sausage, *Communications in Pure and Applied Mathematics*, 28 (4), 525 – 565, 1975.

[12] A. Drewitz, J. Gärtner, A. F. Ramirez and R. Sun, Survival probability of a random walk among a Poisson system of moving traps, *Probability in Complex Physical Systems*, Springer Proceedings in Mathematics 11, 119 – 158, 2012.

[13] A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Annalen der Physik*, 322 (8), 549 – 560, 1905.

[14] J. Engländer, On the volume of the supercritical super-Brownian sausage conditioned on survival, *Stochastic Processes and their Applications*, 88, 225 – 243, 2000.

[15] J. Engländer and F. den Hollander, Survival asymptotics for branching Brownian motion in a Poissonian trap field, *Markov Processes and Related Fields*, 9 (3), 363 – 389, 2003.

[16] J. Engländer, Branching diffusions, superdiffusions and random media, *Probability Surveys*, 4, 303 – 364, 2007.

[17] J. Engländer, Quenched law of large numbers for branching Brownian motion in a random medium, *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 44 (3), 490 – 518, 2008.

[18] R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics*, 7, 355 – 369, 1937.

[19] F. Galton and H. W. Watson, On the probability of extinction of families, *Journal of the Anthropological Institute of Great Britain*, 4, 138 – 144, 1874.

[20] T. E. Harris, *The Theory of Branching Processes*, Springer, Berlin, 1963.

[21] F. den Hollander and G. H. Weiss, Aspects of trapping in transport processes, in *Contemporary Problems in Statistical Physics*, G. H. Weiss (ed.), Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 147 – 203, 1994.

[22] N. Ikeda, M. Nagasawa and S. Watanabe, Branching Markov processes I,II,III, *J. Math. Kyoto Univ.*, 8, 233-278, 365-410; 9, 95 – 160, 1968,1969.

[23] F. John, *Partial Differential Equations*, 4th edition, Springer, Berlin, 1991.

[24] S. Karlin and M. Taylor, *A First Course in Stochastic Processes*, Academic Press, New York, 1975.

[25] A. Kolmogorov, I. Petrovskii and N. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Moscow Univ. Bull. Math.*, 1, 1 – 25, 1937.

[26] A. E. Kyprianou, Asymptotic radial speed of the support of supercritical branching Brownian motion and super-Brownian motion in R^d, *Markov Processes and Related Fields*, 11 (1), 145 – 156, 2005.

[27] S. P. Lalley and T. Sellke, A conditional limit theorem for the frontier of a branching Brownian motion, *Annals of Probability*, 15 (3), 1052 – 1061, 1987.

[28] J. F. Le Gall and A. Véber, Escape probabilities for branching Brownian motion among mild obstacles, *Journal of Theoretical Probability*, 25, 505 – 535, 2012.

[29] R. Lyons, Random walks, capacity and percolation on trees, *Annals of Probability*, 20 (4), 2043 – 2088, 1992.

[30] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, *Communications in Pure and Applied Mathematics*, 28, 323 – 331, 1975.

[31] H.P. McKean, A Correction to "Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov", *Communications in Pure and Applied Mathematics*, 29, 553 – 554, 1976.

[32] P. Mörters and Y. Peres, *Brownian Motion*, Cambridge University Press, New York, 2010.

[33] J. Neveu, Arbres et processus de Galton-Watson, *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 22 (2), 199 – 207, 1986.

[34] M. Öz and M. Çağlar, Tail probability of avoiding Poisson traps for branching Brownian motion, *Statistics and Probability Letters*, 83 (9), 2034 – 2038, 2013.

[35] M. Öz, M. Çağlar and J. Engländer, Conditional speed of branching Brownian motion, skeleton decomposition and application to random obstacles, (submitted).

[36] M. Öz, Survival of Branching Brownian Motion in a Uniform Trap Field, (submitted).

[37] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 6th edition, Springer, Berlin, 2003.

[38] Y. Peres, A. Sinclair, P. Sousi and A. Stauffer, Mobile geometric graphs: detection, coverage and percolation, *Probability Theory and Related Fields*, 156 (1-2), 273 – 305, 2012.

[39] Y. Peres and P. Sousi, An isoperimetric inequality for the Wiener sausage, *Geometric and Functional Analysis*, 22 (4), 1000 – 1014, 2012.

[40] S. Sagitov and A. Shaimerdenova, Decomposition of Supercritical Linear-Fractional Branching Processes, *Applied Mathematics*, 4 (2), 352 – 359, 2013. doi: 10.4236/am.2013.42054.

[41] A. Sznitman, *Brownian motion, obstacles and random media*, Springer Monographs in Mathematics, Springer, Berlin, 1998.

Vita

Mehmet Öz was born in İstanbul, Turkey in 1983. After finishing high school at Robert College, İstanbul in 2001, Mehmet entered Franklin and Marshall College in Lancaster, PA. In the spring semester of 2004, he studied at the Independent University of Moscow, Russia through the Math in Moscow program. He received a Bachelor of Arts with majors in mathematics and physics from Franklin and Marshall College in May 2005. He received a Master of Science in physics in October 2007 and a Master of Science in mathematics in December 2008 from the University of Illinois at Urbana-Champaign. He served in the Turkish Armed Forces from August 2009 to January 2010. He entered the Graduate School of Koç University in September 2010 for doctoral studies in mathematics. From September 2014 to July 2015, he continued doctoral studies at the Mathematical Institute of Leiden University through the TUBITAK international doctoral research fellowship. He received a Doctor of Philosophy in mathematics from Koç University in August 2015.