

A DESIGN STUDY FOR INVESTIGATING AND IMPROVING
COMPETENCIES IN TEACHING PROOF



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DECLARATION OF ORIGINALITY

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- I am the sole author of this thesis and that I have fully acknowledged and documented in my thesis all sources of ideas and words, including digital resources, which have been produced or published by another person or institution;
- this thesis contains no material that has been submitted or accepted for a degree or diploma in any other educational institution;
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ABSTRACT

A Design Study for Investigating and Improving Competencies in Teaching Proof

This design study aims to develop and implement a program to equip preservice teachers with the knowledge and skills necessary to prepare them to teach proof. The educational policies that guide mathematics education, teacher preparation, and professional development opportunities were examined to shape the designed program. Eight experts in mathematics and mathematics education were interviewed. Preservice teachers' proof knowledge and beliefs on proof were obtained through developed data collection tools. The program, consisting of two modules on content and pedagogical content knowledge, was implemented for eight weeks for two groups of preservice teachers in two cycles. The implementations in the second cycle followed the first cycle one week later. The learning processes of the participants in the program were evaluated by observations of their performances and through quizzes and exams. At the end of the program, the participants designed proof-based activities for geometry objectives and taught geometry proofs to a student. Semi-structured interviews with ten participants provided feedback on the effectiveness of the program and the learning processes. At the end of the program, it was observed that preservice teachers' knowledge about proof and teaching proof improved. Preservice teachers' beliefs about constructing proof and students' proof skills developed positively.

ÖZET

İspat Öğretim Yeterliliklerinin Araştırılması ve Geliştirilmesine Yönelik

Bir Tasarım Çalışması

Bu çalışmanın amacı öğretmen adaylarını ispat öğretmeye hazırlamak için gerekli bilgi ve becerileri sunmak amacıyla bir program hazırlamak ve bu programı uygulamaktır. Tasarlanan programı şekillendirmek amacıyla matematik eğitime, öğretmen yetiştirmeye ve öğretmenlerin profesyonel gelişim fırsatlarına yön veren eğitim politikaları incelenmiştir. Matematik ve matematik eğitimi alanındaki sekiz uzmanla görüşmeler yapılmıştır. Öğretmen adaylarının ispat bilgileri ve ispata yönelik inançları geliştirilen veri toplama araçları aracılığıyla edinilmiştir. Alan bilgisi ve pedagojik alan bilgisi odaklı iki modülden oluşan program sekiz hafta boyunca iki farklı öğretmen adayı grubuna uygulanmıştır. İkinci döngüdeki uygulamalar ilk döngüyü bir hafta sonra takip etmiştir. Programdaki katılımcıların öğrenme süreçleri uygulama sırasındaki performansların gözlemlenmesiyle ve yapılan sınavlarla değerlendirilmiştir. Programın sonunda katılımcılar geometri kazanımlarına yönelik ispat temelli etkinlikler tasarlamış ve bir öğrenciye geometri ispatlarını öğretmişlerdir. On katılımcı ile yapılan yarı yapılandırılmış görüşmeler, programın etkililiği ve öğrenme süreçleri hakkında geri bildirim sağlamıştır. Program sonunda öğretmen adaylarının ispat ve ispat öğretimi konusundaki bilgilerinin geliştiği gözlemlenmiştir. Öğretmen adaylarının ispat yapmaya ve öğrencilerin ispat becerilerine yönelik inançları olumlu yönde gelişim göstermiştir.

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Dedicated to my family

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CHAPTER 1

INTRODUCTION

1.1 Purposes and research questions of the study

Proof is a fundamental skill in mathematics and is sometimes considered synonymous with the act of doing mathematics (Stylianides & Stylianides, 2008). While the standards for learning mathematics prioritize reasoning and proof as crucial skills in all K-12 grade level curriculum objectives, the development of proof skills is primarily associated with the high school level or beyond (Cyr, 2011). However, high school students have challenges in proving since their understanding of proof-related concepts and processes appears more immediate during their high school or college years. It has been argued that introducing proof-related concepts at an earlier stage is crucial for developing a solid basis in proof skills (Stylianides, 2007a). In order to properly prepare students for proof, teachers and preservice teachers need to possess the requisite knowledge, abilities, beliefs, and attitudes on proof. However, not only students but also teachers and preservice teachers lack the required knowledge to prove (Lesseig, 2016). Hence, it is essential to develop preservice teachers' knowledge of proof and help them develop positive beliefs and attitudes toward proving and teaching proof for their future mathematics lessons.

The purpose of this study is to design and implement a program that enhances the development of proof knowledge and skills of preservice mathematics teachers so that they can enable their students to progress toward formal proof in the future. Initially, it was aimed to understand how reasoning and proof skills are reflected in educational policy decisions and practices (such as mathematics curricula, teacher

preparation programs, student examinations, teacher evaluations, professional development opportunities for teachers, and discourses about mathematics education). Secondly, obtaining the opinions and suggestions of people working in the field and specialized in different fields related to mathematics education about the problems of learning and teaching proof was planned. Finally, it was aimed to determine the preservice teachers' knowledge and beliefs about proof and teaching proof. The overview of the study is depicted in Figure 1.

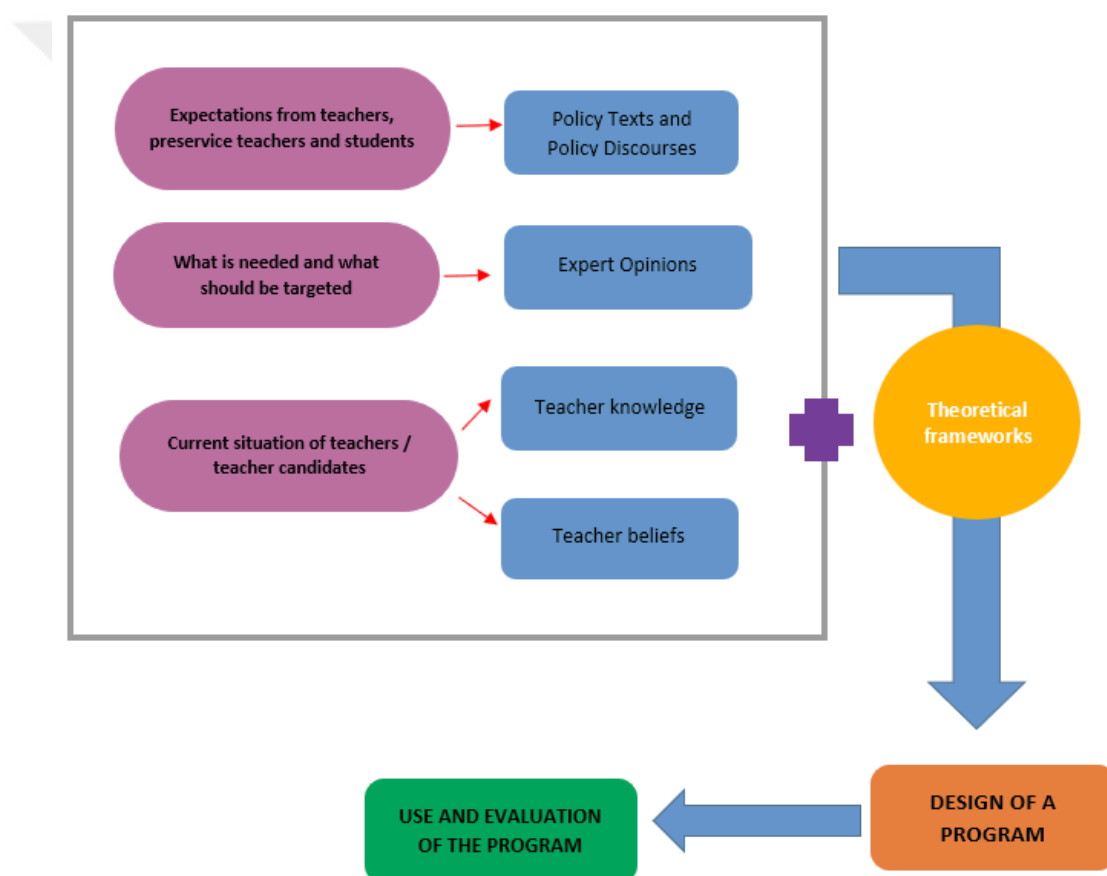


Figure 1. The overview of the study

The research questions of this study, which were handled in two stages in line with the objectives of the study, were formulated for the first part and the design part of the study separately.

The research questions addressed in the preliminary studies conducted for the development of the design study were structured as follows:

1. What is the place of mathematical proof in educational policies that shape curricula and teacher training?
2. What are the opinions and suggestions of experts in mathematics and mathematics education on teaching proof in middle school?
3. What are preservice teachers' levels of knowledge about proof and teaching proof?
4. Which characteristics can explain preservice teachers' beliefs about proof?
5. What are preservice teachers' beliefs about proof and teaching proof?
6. Is there any relationship between preservice teachers' knowledge and beliefs on proof?

The following research questions were addressed for the design study:

7. What are the characteristics of a design program to develop preservice teachers' teaching proof practices (belief/knowledge/attitude) to promote students' reasoning and proof skills?
8. In what ways does the designed program contribute to preservice teachers' teaching-proof practices (belief/knowledge/attitude)?

1.2 The significance of the study

The purpose of this study had several foci; hence, different aspects of learning and teaching proof and reasoning were taken into consideration within the theoretical background. The frameworks on teacher knowledge, teacher beliefs, proof levels-categorization of proofs, ways of reasoning in tasks, activities, and practices were used, where each provided complementary and supportive perspectives for the study.

Theoretical frameworks for teacher knowledge, teacher beliefs, proof and reasoning were studied to understand better and interpret preservice teachers' knowledge, beliefs, and practices on proof in this study. One of the main theoretical frameworks was for understanding teacher knowledge for proving: "Mathematical knowledge for teaching proof" by Ball and her colleagues (2005). This framework consists of four domains: Common content knowledge (CCK), specialized content knowledge (SCK), knowledge of students and content (KSC), and knowledge of teaching and content (KTC). In this framework, CCK as content knowledge for proving was separated from the others, which were merged into pedagogical content knowledge for proving.

To study beliefs on proofs and teaching proof, "Belief theory" by Thompson (1992) was considered, and a blend of dimensions of belief on proof suggested by researchers (Cabassut et al., 2012; Frunghetti & Morselli, 2009; Mingus & Grassl, 1999) was taken into account while studying preservice teachers' beliefs on proving and teaching proof. In order to differentiate proof levels, frameworks by Harel and Sowder (1998) and Balacheff (1988) were considered. Harel and Sowder's framework consists of three broad categories: "external conviction," "empirical proof," and "analytical proof." Although these also involve subcategories, they were not used in this study.

Balacheff's framework consists of four levels of proof, which are "naïve empiricism," "crucial experiment," "generic example," and "thought experiment" levels. Finally, the "Integrated framework for analyzing ways of reasoning" by Sevinc et al. (2022) was used to differentiate ways of reasoning in tasks and activities. In this framework, reasoning forms were categorized as "appealing to authority," "simple (1-step) deductive reasoning," "mathematizing," "reasoning by

analogy,” “reasoning with empirical arguments/ Specific cases,” “developing conclusions/justifying/refuting through deductive reasoning,” and “other.”

This study aims to contribute to the field through the devised tests and scales, the Proof, Reasoning, and Instructional Skills in Mathematics program (PRISM), and their outcomes. Although the knowledge and skills that teachers and preservice teachers should have in teaching proof are stated in the literature under various categories and a general picture is presented, when we set out to measure this knowledge and these skills, the need for a holistic measurement tool and the dimensions that should be focused on still remained. For this reason, a tool to assess teachers’ knowledge of proof was developed. The test, which focused on content knowledge of proof and pedagogical content knowledge of proof, was developed to be able to make inferences about teacher’s knowledge of proof. While the content knowledge of proof focused on principles like whether a counterexample is sufficient to disprove an argument, knowledge of proof methods, and rules of logic (Ball et al., 2002; Sears & Chavez, 2014; Senk, 1989), the pedagogical content knowledge of proof focused on being able to ask questions related to proof skills, evaluating student arguments in terms of whether they are proofs or not, and knowledge of the mathematical proofs that can be presented at the middle school level (Flegas & Charalampos, 2013; Stylianides, 2007).

Although teachers’ beliefs about proof have been studied frequently in the literature, there was a need for a measurement tool that could fully reveal a teacher’s beliefs about proof and its teaching. The statements on beliefs and attitudes towards proving and teaching proof by the participants of various studies and the developed scales in the literature (e.g., Cabassut et al., 2012; Doruk & Güler, 2014; Frunghetti & Morselli, 2009; İmamoğlu, 2010; Keçeli et al., 2015; Kotelawala, 2009; Pekşen-

Sağır, 2013; Ünveren, 2010) were compiled and the findings about teachers' beliefs in the literature were turned into scale items.

Teachers' deficient subject knowledge of proof and knowledge of teaching proof necessitates an intervention for improving teachers' competencies for learning and teaching proof. Likewise, their immature and negative beliefs on proof and teaching proof in middle school require taking action to change these mindsets and to prevent them from being reflected in teaching practices. It was decided that the designed intervention or program needed to follow a comprehensive assessment of the current situation of teachers and preservice teachers, specifically their knowledge and beliefs. Content knowledge, pedagogical content knowledge, and beliefs of teachers about proof should be identified. An analysis of the expectations placed upon teachers, preservice teachers, and students, as well as an evaluation of the available opportunities for them should also be involved. It is crucial to prioritize identifying the specific areas that require improvement to enhance the preparedness of teachers and preservice teachers in teaching proof and developing proof skills.

Another expected contribution of this study was to present an educational policy viewpoint that addresses the expectations placed on teachers, preservice teachers, and students regarding their proof skills and the resources and support provided to enhance these skills. To understand the current situation regarding teaching and learning proof comprehensively, it is essential to thoroughly analyze various aspects such as curriculum analyses, teacher education programs, international and national student and teacher assessment exams, and political discourses on mathematics education, particularly reasoning and proof skills. These factors can provide valuable insights and should be considered when designing a

program to develop the knowledge and beliefs of preservice teachers in proving and teaching proof.

It is crucial to consider both the requirements and the specific objectives that should be focused on within the local setting. If the proposals on the literature are the sole source of information for preparing teachers, preservice teachers, and students for proof, there is a risk of overlooking local problems and conditions. Consulting experts from different but related disciplines can align with existing literature and illuminate challenges and solutions in local educational settings. Solutions developed with an approach that considers these specializations will, therefore, be more pragmatic and practical. This study aimed to bring together the opinions and suggestions of experts working in the field of mathematics and mathematics education on proof practices and teaching of proof.

One of the primary outcomes of this study was a program (Proof, Reasoning, and Instructional Skills in Mathematics- PRISM) designed and implemented for preservice teachers. For a new program to equip teachers or preservice teachers better and bring them closer to achieving demonstrable teaching outcomes, a design-based research perspective offers a comprehensive and evidence based perspective. In design-based research:

- The design emerges from a problem encountered in an authentic learning environment.
- The design takes into account both local practices and theoretical justifications.
- The collective expertise of multiple stakeholders is consolidated.
- Various interdependent variables are considered.

- The study is conducted in several iterations (Anderson & Shattuck, 2012; Barab & Squire, 2004; Wang & Hannafin, 2005).

The design in this study involves collecting data that guide preservice teachers in practicing skills related to proof. The primary objective of the intervention was to design and execute a program that would assist preservice teachers in acquiring knowledge and skills regarding teaching proof. This could be achieved by fostering the development and guidance of preservice teachers' practices and beliefs around the teaching of proof, as well as addressing any negative beliefs they may hold towards this topic in middle school. The design and learning outcomes of the program are significant to the field in that they provide a working example for a solution to overcome preservice teachers' weakness in content and pedagogical content knowledge of proof and to develop positive dispositions towards learning and teaching proof. The scarcity of empirical research on the competencies of preservice teachers in proof underscores the significance of PRISM.

This dissertation is organized into four chapters. In the following Chapter 2, a review of the literature and the theoretical framework that guided the design and analysis of the study are presented. In Chapter 3, the initial studies leading to the design study are explained. The methodology (data collection methods, participants for each data collection tool, data collection and data analysis processes, and procedure), findings, and discussion of findings in light of the literature review and theoretical framework are detailed. In Chapter 4, the design study for developing preservice teachers' proof skills is introduced. The designed program, PRISM, is detailed with the methodology employed, findings of the implementation. Finally the impact of PRISM, and its implications are discussed.

1.3 Definition of relevant terms

In this part, key terms are operationally defined. These definitions and detailed explanations are also given in the relevant part of the Literature Review.

- Proof: Stylianides (2007a) posits that “Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: (1) It uses statements accepted by the classroom community that are true and available without further justification; (2) It employs forms of reasoning that are valid and known to, or within the conceptual reach of the classroom community; and (3) it is communicated with forms of expression that are appropriate and known to, or within conceptual reach of, the classroom community” (p. 291).
- Proof scheme: Proof scheme is defined as a framework used to categorize proof skills. There are several proof schemes in the literature. Among them, the most famous are by Balacheff (1988) and Harel and Sowder (1998).
- Empirical argument: “An argument that aims to establish the truth of a statement by validating it only in a proper subset of all possible cases in its domain” (Stylianides, 2011, p.1)
- Generic example/ Generic argument: Generic arguments are defined as arguments that validate the truth of a statement regarding a set of cases by demonstrating that the statement applies to a specific case within the domain that lacks unique properties, thereby allowing the reasoning employed for that case to be extended to any other case (Stylianides, 2016). The concept was introduced by Balacheff (1998).

- Argumentation: Hanna and de Villiers define argumentation as “a reasoned discourse that is not necessarily deductive but uses arguments of plausibility” (Reiss et al., 2008, p. 455).
- Teacher knowledge of proof: Teacher knowledge of proof is conceptualized as the knowledge required to teach proof. It encompasses different dimensions of knowledge, such as content knowledge, pedagogical content knowledge, and pedagogical knowledge. Among the available frameworks to assess teacher knowledge of proof, the most frequently used one is “Mathematical Knowledge for Teaching (MKT) for proof, developed by Ball and her colleagues (2005).
- Teacher beliefs on proof: It represents dispositions toward proof and teaching proof. Beliefs on proof and teaching proof comprise a variety of dimensions: Beliefs on pedagogical practices on proving, students’ difficulties in proving, the impact of contextual factors, role and place of proofs in mathematics, characteristics of proof in school mathematics, when and how proof should be taught, beliefs about nature of mathematics, teaching and learning mathematics, and teachers’ own abilities in proving (Cabassut et al., 2012; Frunghetti & Morselli, 2009; Mingus & Grassl, 1999).

CHAPTER 2

LITERATURE REVIEW

The literature review of this dissertation consists of eight subheadings. First, the definitions, functions, and importance of proof are detailed. The components and requirements of proving with mathematicians' proof practices are elaborated under the next sub-heading. Proof practices of mathematicians are summarized in the following part. Next, the place of proof in geometry is discussed. The conceptualization of proof by students and their proof practices are specified. In the following part, proof levels of students and the development processes are detailed. The difficulties experienced by students and the reasons behind those difficulties are discussed. The focus is shifted toward teachers and teacher candidates in the next section: Their knowledge, beliefs, and practices regarding proving and teaching proof are elaborated. Last but not least, the suggestions for developing proof skills of students, teachers, and teacher candidates will be discussed.

2.1 Definition, function, and importance of proof

Definitions of proof and proving have been formulated in the literature with distinct concerns and foci. It has been described as a specific form of justification, namely a deductive argument (Conner, 2008; Maher, 2010; Staples & Bartlo, 2010). The proof is shown as a method of explaining and convincing the mathematical community. It involves negotiation and agreement of ideas (Almeida, 2010; Heinze et al., 2008). Heinze et al. (2008) suggested that "Acceptable proof is a bridging process with given and wanted conclusions through inferring rules controlled by coordination

process.” Stylianides (2007a) posits that “ Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: (1) It uses statements accepted by the classroom community that are true and available without further justification; (2) It employs forms of reasoning that are valid and known to, or within the conceptual reach of the classroom community; and (3) it is communicated with forms of expression that are appropriate and known to, or within conceptual reach of, the classroom community (p. 291)”.

Proof serves several functions as “validation, verification, explanation, organization, systematization, discovery, communication, construction of empirical theory, exploration of the meaning of a definition or the consequences of an assumption, and incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective” (Bieda & Lepak, 2014; Maher, 2010; Staples & Bartlo, 2010). Gaining critical thinking skills, the ability to make informed decisions, and detecting flaws not only in academic contexts but also in daily life settings are also shown as other important functions of proving (Almeida, 2010; Ball et al., 2002; Staples & Bartlo, 2010). Mathematics teachers specified that proofs could also be used for the promotion of conceptual understanding, assessing students’ progress and learning in mathematics, and fostering social interactions (Flegas & Charalampos, 2013; Staples & Bartlo, 2010).

The functions of proofs demonstrate the central role of proving in mathematics and mathematics education. Schoenfeld (1994) claimed that “Proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels” (p. 76). The importance of

proving is highlighted both by international education organizations and national mathematics education curricula, either explicitly or implicitly (Knuth et al., 2019). Middle school students are expected to produce deductive proof in some countries (Balacheff, 1991; Heinze et al., 2008). On the other hand, in countries like Türkiye, proving is not explicitly taught at the middle school level, but proof-related abilities are considered in the national mathematics education curricula (Flegas & Charalampous, 2013; MEB [MoNE], 2013). The common argument shared by these curricular emphases is that reasoning and proof should not be viewed as separate activities in mathematics classes. Instead, they must be integral to every mathematics classroom activity (Bieda, 2010).

2.2 Components of and requirements for proving

Construction and evaluation of proofs require several competencies and skills. Some of them are (1) recognizing that proof serves the truth of the statements for all elements in the covered domain, (2) recognizing the need for proof, (3) understanding the role and significance of underlying assumptions, (4) use of different forms of reasoning and implication rules when organizing the arguments, (5) building foundation on already established and accepted body of public knowledge, (6) using mathematical language, (7) looking for structures and invariants, and (8) using algebraic formalism (Ball et al., 2002; Knuth et al., 2002; Stylianides & Stylianides, 2008). Proving is sometimes defined as the act of mathematization or doing mathematics. It involves describing and recognizing patterns, generalizing, making predictions, developing a degree of abstraction, revising conjectures, and final justification as formal proof (Almeida, 2010; Cyr, 2011; Maher, 2010; Stylianides & Stylianides, 2008).

Of the critical elements of proving, assumptions have an important place. They are the set of statements that people agree on or accept. Making sense of proofs and mathematical arguments is impossible if underlying assumptions are not understood or overlooked (Stylianides, 2007a). Students in early grades should understand that the truth of the conclusions depends on the assumptions that support these conclusions. (Stylianides, 2007a). Freudenthal (1973) states that a student should repeatedly ask himself: “What did I actually presuppose?” to understand the meaning of axiomatics and to form sound mathematical arguments (van Dormolen, 1977, p. 31). An example suggested by Stylianides (2007a, p. 380) was: “Consider the truth or the falsity of the statement: ‘Every rectangle is a trapezoid.’” Depending on the definition of a trapezoid (at least one pair of parallel sides vs. exactly one pair of parallel sides), the statement can be verified or refuted. So, an intricate relationship exists between definitions, assumptions, and proof.

It is crucial to comprehend that “proof treats the general case” while constructing and evaluating formal proofs (Bieda et al., 2006, p. 395). According to Ellis (2011, p. 314), the activity of generalization consists of at least one of the three actions: “(a) identifying commonality across cases, (b) extending one’s reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases.” Students can have substantial growth in learning to generalize and represent these generalizations when they are allowed the time and cognitive space to make sense of the content they are studying, and this would better prepare them for algebra and proof in middle grades (Blanton et al., 2019). However, it is important to keep in mind that exposure to multiple and similar cases does not suffice for generalizing patterns. Students should be directed to investigate the underlying structure and what is shared and relevant across different cases (Ellis, 2011).

Deductive reasoning is shown to be another critical dimension of formal proofs. It has been defined as logical inferences drawn from a general set of givens or premises (Stylianides & Stylianides, 2008). The definition of proof has been formed by considering deductive reasoning, and the development of proving ability was shown to be aligned with the development of deductive reasoning (Stylianides & Stylianides, 2008). Two forms of deductive reasoning have been specified: Modus Ponens and Modus Tollens. Modus Ponens is explained as: For conditional statements $p \rightarrow q$; if p is true, then q is also true, which is the foundation of direct proof. Modus Tollens is explained as: For $p \rightarrow q$; if q is false, then p is also false (the foundation of indirect proof) (Stylianides & Stylianides, 2008). To connect informal and empirical arguments to formal proofs, students must master deductive reasoning. Hence, deductive reasoning forms the bridge from informal/ empirical mathematical arguments to proof (Cervantes-Barazza et al., 2020).

Argumentation is also shown to be another construct related to proving. Hanna and de Villiers define it as “a reasoned discourse that is not necessarily deductive but uses arguments of plausibility” (Reiss et al., 2008, p. 455). Proof and argumentation are not exact. With argumentation, an agreement is the aim of an interaction. On the other hand, proof deals with the truth or falsity of mathematical statements. Argumentation is an open process in which various means and thinking practices are accepted (Carrascal, 2015; Staples & Newton, 2016). Mathematical proof should provide the knowledge base by the common body of knowledge (e.g., theorems, rules, facts) on which mathematical authorities agree (Balacheff, 1991). Cervantes-Barazza and colleagues (2020, p. 5) stated, “Argumentation is a means by which the proof process begins.” They posited that the emphasis on argumentation in primary years can be attributed to its connection to proof. If argumentation is

fostered in elementary years, students can be in a better position to construct proofs because argumentation activities, especially collaborative ones, support students to justify and think more deeply about mathematical concepts and statements (Cervantes-Barazza et al., 2020; Hoffman et al., 2009). Students may have a variety of warrants in their arguments, such as mathematical definitions, theorems, procedures, and observations, or they may appeal to authority. What is critical is orienting students to use appropriate warrants in constructing proofs (Cervantes-Barazza et al., 2020; Singletary & Conner, 2015). Collaborative argumentation may strengthen proof skills as a social dimension in generating mathematical arguments. Comparison of ideas can strengthen, modify, and consolidate one's arguments. Discussion leads to further thinking about an issue (Maher & Martino, 1996). These social interactions during collaborative argumentation can foster the development of proof practices because they need to reach an agreement on justifications and explanations of their mathematical claims (Balacheff, 1991). Even though acceptance by the classroom community is a requirement for an argument to be counted as proof (Flegas & Charalampos, 2013), being convincing is not enough criterium for an argument to be accepted as proof; there is the danger of regarding empirical arguments as proofs by the classroom community (Stylianides, 2007b, p. 13).

Other prerequisites for constructing proofs are knowledge of mathematical concepts and strategies (declarative knowledge), their application in a problem context (methodological knowledge), the use of metacognitive control strategies, and the proper understanding of the nature of proof (Heinze & Reiss, 2003; Schoenfeld, 1983). The ability to construct intermediary conclusions is also found to be significant for multi-step proof constructions (Heinze et al., 2008). Selecting among

crucial intermediary steps and arguments is a meta-level activity called coordination. It involves monitoring the proof process (Heinze et al., 2008).

Linguistic competence for understanding logical terms (e.g., and, or, not, if, if and only if, some, all) and (2) working memory capacity (being able to consider all possible models) (Stylianides & Stylianides, 2008) have also been shown as essential factors contributing to the development of proof skills.

2.3 Proof practices of mathematicians and students

Proof is a tool for mathematical validation and communication with other mathematicians (Balacheff, 1991). In mathematicians' proving-related activities, examples lead to a deeper understanding of conjectures and insight into developing a proof (Knuth et al., 2019). Epstein and Levy (1995) defend this by stating that "... most significant advances in mathematics have arisen from experimentation with examples . . ." (p. 670). Mathematicians' example use is characterized as spontaneous, productive, and purposeful (Knuth et al., 2019).

The forms of justification practices employed in schools are presented as empirical justifications, visual justifications, narrative justifications/proofs, and formal justification/proofs (Healy & Hoyles, 2000). Some researchers denoted that proving may be taught in a structured manner. For example, two-column proofs have been known to be helpful in the school context and are preferred by some teachers and students (Martin & McCrone, 2003). On the other hand, they have been criticized for "reducing the excitement of proving and negatively influencing students' views of deductive reasoning" (Schoenfeld, 1986, 1991; Sears & Chavez, 2014) and for leading students to focus on structure rather than its function (Ball et al., 2002; Cirillo, 2009). Schoenfeld (1991) states that the two-column format of

proving (“the statements of the proof in steps in a column occupying the left half of the page and . . . the reasons of the statements in steps at the right side of the page, with each reason directly opposite its statement”) becomes a ritual procedure that needs to be practiced and memorized. So, the essence of proof as a logical chain of reasoning becomes withheld/hidden (Stylianides & Stylianides, 2008).

Understanding what mathematicians do in terms of proving is important in mathematics education. Bruner (1960) stated that there needs to be “a continuity between what a scholar does at the forefront of his discipline and what a child does in approaching for the first time” (pp. 27-28). So, mathematics instruction should help students learn how to do authentic mathematics; school mathematics should be a representation of the mathematics discipline (Stylianides, 2007b). His conceptualization of proof establishes a balance between two considerations:

1. Mathematics as a discipline (Proof uses true statements, accepted modes of argumentation, and suitable modes of representations)
2. Students as mathematics learners (Proof depends on what is known, accepted, and accessible to the classroom community) (Stylianides, 2007b).

Viewing proof as an authentic mathematics classroom activity highlights that both academic mathematics and school mathematics should share some activities (Almeida, 2010). These common activities are depicted in the following diagram (see Figure 2).



Figure 2. Activities for authentic mathematics

2.4 Geometry and proof

Euclidean geometry has been the focus of proving contexts as students are first required to write proofs in geometry, and it is built on axiom systems, definitions, and geometry theorems (Martin et al., 2005). Geometric thinking levels proposed by van Hiele (1959) reflect skills required for proving, and they are helpful in understanding students' difficulties and their readiness for proving (Senk, 1985). Van Hiele's (1959) five geometric thinking levels are level 0 (Recognition or Visualization), level 1 (Analysis), level 2 (Ordering or Informal Deductive), level 3 (Deduction or Formal Deductive), and Level 4 (Rigor).

Students at the visualization level can sort shapes by looking at similar physical appearances. At the analysis level, children can list the properties of shapes. However, they cannot comprehend the relationship between these properties, and they cannot notice that some properties imply others. At the informal deductive level, students can formulate meaningful definitions and produce informal deductive arguments. Students at the formal deductive level can understand relationships between properties of shapes and also comprehend relationships between definitions, theorems, axioms, and postulates. They can learn how to construct a formal proof and understand why proof is needed. At the rigor level, children can think within an abstract mathematical system (van Hiele, 1959).

From level 0 to level 4, students show more sophisticated skills in proving. Researchers indicated that level 2 and level 3 were found to be critical for the transition from informal to formal proofs in geometry (Senk, 1989). One of the most important aspects of van Hiele's levels is that the levels are sequential, and to operate at higher levels, students must pass from lower levels. If students are forced to have a higher level of thinking when they are not ready, they will act as if they have a

higher level of thinking by memorizing the required steps for problem-solving or proof-related activities (Flegas & Charalampos, 2013).

2.5 Proof levels of students and development process of proof skills

Researchers have developed several frameworks to investigate students' proof levels for diagnostic purposes. Among them, Balacheff's four levels of proof (Cyr, 2011) and Harel and Sowder's (1998) three levels of proof schemes are the most cited and used frameworks. Balacheff's four levels of proof consist of "naïve empiricism," "crucial experiment," "generic example," and "thought experiment" levels. At the "naïve empiricism level," students state that a statement is proven by the use of a small number of cases. At the "crucial experiment" level, students consider the issue of generalization by working with no particular cases (such as extreme ones). At the "generic example" level, students try to present arguments based on a generic example (a representative example of a set of objects). A generic example is the highest level a student can be without formal teaching and learning of deductive proof. Generic proofs are sometimes expressed as transparent (pseudo) proofs. They are proof attempts involving a particular case that may not be considered a concrete example but can be regarded as a non-specific representative case. They are suggested to be used as a pedagogical tool since they help the development of justification skills. However, the teacher must take extreme care as transparent proofs may direct attention to particular cases rather than generalizations (Ball et al., 2002). At the "thought experiment" level, students formulate their arguments by detaching them from experimentations and example-based statements. They move into more formal and deductive proofs (Knuth & Elliott, 1998).

Harel and Sowder's framework consists of three broad categories: "external conviction," "empirical proof," and "analytical proof." Each of these also involves subcategories. It is noted that these schemes are not mutually exclusive; people can have more than one kind of scheme (Harel & Sowder, 1998). In "external conviction," students only memorize prescriptions; no discovery or creativity is involved, and authority is usually present as the only source of knowledge. "Empirical proofs" are constructed by students' intuitions. Students form their arguments by appealing to some physical facts or sensory experiences. "Analytical proofs" are conducted using logical deductions (Harel & Sowder, 1998).

Waring's (2000) six levels of proof concept development and four levels of proof concept development framework by Bieda and her colleagues (2006) represent more recent attempts at framework development. Waring (2000)'s six levels of proof concept development are:

- Level 0: Students are ignorant of the need for, or existence of, proof.
- Level 1: Students are aware of the notion of proof but consider checking a few cases as sufficient (i.e., Balacheff's naive empiricism).
- Level 2: Students are aware that checking a few cases is not sufficient but are satisfied that either i) checking extreme cases (i.e., Balacheff's crucial experiment) or random cases is proof, or ii) use of a generic example forms a proof for a class of objects (i.e., Balacheff's generic example).
- Level 3: Students are aware of the need for a general argument but cannot produce such arguments by themselves; however, they are likely to understand the generation of such an argument (for example, by a classmate). This also includes the ability to follow a short chain of deductive reasoning.

- Level 4: Students are aware of the need for a general argument, can understand the generation of such an argument, and can produce such arguments by themselves in a limited number of (familiar) contexts.
- Level 5: Students are aware of the need for a general argument, can understand the generation of such an argument (including more formal arguments), and can produce such arguments themselves in a variety of contexts (both familiar and unfamiliar) (Knuth et al., 2002).

Bieda and her colleagues (2006) used four levels of proof concept development in their study: Level 0 – students respond either “I do not know” or give information already presented in the problem. Level 1 – students consider checking a few cases as sufficient. Level 2 – students are aware that checking a few cases is not sufficient and may attempt a general argument, but the argument is on the “wrong track” (i.e., it would not lead to a proof) or is incomplete (it would lead to a proof if completed). Level 3 – students generate an argument that treats the general case of the statement (i.e., a proof).

In all of these frameworks, the levels are developmental, from ignorance of proofs to empirical arguments and then to formal and deductive proofs (Knuth & Elliott, 1998). Students proceed from inductive to deductive and toward greater generality as they move to higher levels (Knuth et al., 2002; Simon & Blume, 1996). Understanding students’ proof levels can foster their transition to more advanced proof skills.

Many researchers indicated that the development of proof skills relies on students’ experiences with proving rather than their age or grade level, so they concluded that “Deductive reasoning can be within elementary students’ grasp” (Cyr, 2011; English, 1991). Lester (1975) stated that students of different ages can employ

several tasks when their complexity is adjusted accordingly, as opposed to what Piaget (1958) suggested. Lester (1975) also indicated that students in the upper elementary grades can be as successful as older students but only need more time to solve proof-related problems.

2.6 Students' conceptions and practices of proof

How students perceive proofs is important for deciding how to approach them and planning for design interventions. Literature shows that most students do not see any reason to prove; hence, they are unwilling to engage in proof-related practices (Balacheff, 1991). It is difficult for students to accept the need for a deductive argument when the truth of a statement is obvious (van Dormolen, 1977). Students' appreciation of proof shapes their attempts to prove it (Nardi & Knuth, 2017). For many, proof is perceived as a ritual rather than its function or meaning (Ball et al., 2002).

Students mostly rely on empirical arguments, trials of many cases, and inductive reasoning when asked to construct proofs (Balacheff, 1991; Bieda & Lepak, 2014; Heinze & Reiss, 2003). They tend to utilize circular arguments when they write proofs (Heinze & Reiss, 2003; Knuth et al., 2002). Most students have a misconception that counterexamples are only particular cases and that one counterexample is not enough to disprove a claim (Stylianides, 2008). Students' approaches while constructing and evaluating proofs show differences. Most of them prefer narrative styles of arguments when constructing proofs, but they prefer formal arguments when evaluating proofs. Formal presentation of arguments is tempting in students' proof evaluations (Heinze & Reiss, 2003). Furthermore, geometric shapes are sometimes considered constraints to developing proofs since students rely on

drawings rather than geometric concepts and facts (Cyr, 2011). Students' proof comprehension skills seem better than their proof-writing skills (Lester, 1975).

2.7 Students' difficulties in proving and their reasons

Students experience difficulties in proving in lots of aspects. It is seen as a challenging endeavor not only by students but also by teachers (Cirillo, 2009). Proof-writing is regarded as one of the most challenging abilities for students to learn (Senk, 1985). Students have problems with creating general arguments and have difficulty in shifting from patterns to their generalizations (Ellis, 2011).

Some cognitive, social, and environmental factors can be attributed to students' problems in proving. Students' difficulties may be due to a lack of awareness of the necessity of proving and a lack of logical maturity (Balacheff, 1991). Limited conceptual understanding, restraints in the scientific thinking processes of students, and several arguments to be combined are important impediments to proof constructing (Heinze et al., 2008). Students' problems in logic and methods of proof, problem-solving skills, use of mathematical language, and conceptual understanding can potentially lead to problems in proving (Moore, 1994). Also, a lack of knowledge of definitions and how to begin proofs and inadequate concept images can lead to difficulties in producing proofs (Moore, 1994). As students learn geometry contents and proof practices together, they may experience cognitive load, which is associated with difficulties in proving as well (Martin et al., 2005; Moore, 1994).

Only conceptual and cognitive limitations cannot explain students' challenges in proving. It is necessary to "move beyond purely cognitive" and look for social factors (Balacheff, 1991, p. 3). Social norms in the classroom can affect students'

approaches to proving. Relying on teacher authority, rather than their mathematical knowledge, on the decision of the validity of mathematical statements would lead students to not make sense of proving activities (Balacheff, 1991; Sears & Chavez, 2014). Hence, Balacheff (1991) suggested the devolution of the responsibility for the validity of mathematical statements from the teacher to the students.

Limited opportunities in students' learning environments are emphasized as one of the primary reasons for students' difficulties in proving. These opportunities are related to teacher practices and how teachers' knowledge and beliefs are reflected in those practices. Lack of understanding of proof and being less prepared to teach proof hinder its promotion in the lessons (Bieda, 2010; Stylianides & Ball, 2008).

2.8 Teacher knowledge, beliefs and practices on proving and teaching proof

Teachers' professional learning about any topic was identified in three fundamental components: Teachers' knowledge, teachers' beliefs, and teacher practices (Lin et al., 2012). The first component, teacher knowledge, is defined and conceptualized in many ways in the literature. Diverse forms of knowledge have been identified and defined so that teachers can successfully transfer the ideas, concepts, and skills to be learned. Professional teacher knowledge was identified in three categories: (i) demonstrating competencies (mathematical content knowledge), (ii) reflecting on proof by didactics (pedagogical content knowledge), and (iii) preference for argumentation and proof in teaching (beliefs). A categorization for teacher knowledge by Shulman (1986) consists of three dimensions of knowledge: Content knowledge, pedagogical content knowledge, and curricular knowledge.

Content knowledge, sometimes referred to as subject matter knowledge, comprises knowledge of the content of a specific discipline (Buchbinder & McCrone, 2020). This knowledge is categorized into two types: (a) substantive

knowledge and (b) syntactical knowledge. In elementary mathematics, substantive knowledge encompasses the comprehension and articulation of fundamental facts, concepts, and principles. Syntactical knowledge refers to the capacity to articulate the structures that underpin mathematical notions; it necessitates comprehension of the syntax, rules, and proofs pertinent to the specific area of focus (Holmes, 2012).

Pedagogical content knowledge is defined as the knowledge of what to teach and how to teach, considering the diverse groups of learners, students' abilities, and instructional materials (Shulman, 1986). It is the amalgamation of what were once considered two separate forms of knowledge: Content knowledge and pedagogical knowledge: "It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1986, p. 8). PCK for mathematics education concentrates on elucidating terms and concepts for students, interpreting students' statements and solutions, evaluating students' mathematical learning and determining subsequent actions, making mathematical and pedagogical assessments regarding students' inquiries, solutions, problems, and insights (both anticipated and unexpected), aiding students in constructing mathematical frameworks, and facilitating students in abstracting and generalizing mathematical concepts (schemas) (Ball et al., 2002; Hill & Ball, 2006). Curricular knowledge was defined as the ability to comprehend and use the instructional materials and the curriculum in order to teach a concept or skill. Elementary teachers must possess skills such as evaluating the mathematical quality of instructional materials and making necessary modifications, assessing and rectifying textbook presentations of specific topics, and integrating mathematical concepts within and across various mathematical domains,

as outlined in Bush's (2005) "Assessing the Mathematics Knowledge of Teachers," a comprehensive checklist (as cited in Holmes, 2012).

Another important categorization of teacher knowledge was done by Ball and her colleagues (2005). In their categorization of mathematical knowledge for teaching, they used four domains: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of students and content (KSC), and knowledge of teaching and content (KTC) (Holmes, 2012). CCK is a mathematical knowledge that any professional can know if taught mathematics. It encompasses the mathematical knowledge that any educated professional should possess, incorporating both procedural and conceptual understanding as outlined by Hiebert and Carpenter (1992). Ball et al. (2005) characterized mathematical knowledge for teaching, or SCK as "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill et al., 2005, p. 3). This specialized teacher knowledge extends beyond mere familiarity with a common body of mathematical content; it encompasses the ability to successfully impart that subject, assess and rectify student comprehension, and conceptually connect the content to practical applications. Mathematics teachers constitute an exclusive group of mathematics users; their requisite knowledge for teaching mathematics surpasses that of other educated people, even mathematicians (Steele & Rogers, 2012). Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK) were identified as a subject matter knowledge domain, whereas, Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) were specified as pedagogical domains (Lesseig, 2011, 2016).

The frameworks for teacher knowledge emphasize the mathematical and pedagogical skills necessary for teachers to foster learning in the classroom, hence

assisting facilitators in designing learning experiences that align with specific learning objectives. They also serve as research instruments, offering terminology to classify teachers' conversations and facilitating comparisons across professional development environments. The frameworks serve as a practitioner tool to establish learning objectives for teachers, plan professional development initiatives to achieve those objectives, and evaluate the outcomes (Lesseig, 2016)

Multiple perspectives can be adopted to measure teachers' mathematical knowledge. One method could be developing conceptual frameworks for mathematical knowledge for teaching (MKT) based on mathematical topics, utilizing research on the pedagogy and comprehension of those topics (Stylianides & Ball, 2004). Assessments, interviews, and interventions can subsequently be formulated to evaluate teachers' topic-specific MKT. As of now, merely two instruments quantitatively assess mathematical knowledge for instructing as delineated by these teacher knowledge frameworks: The Learning for Mathematics Teaching (LMT) program from the University of Michigan, grounded in the frameworks of Ball et al. (2005) and the Diagnostic Teachers' Assessment for Mathematics and Science (DTAMS) from the University of Louisville, which is based on Webb's Depth of Knowledge frameworks.

The increasing recognition of the necessity of integrating proof into all students' mathematics experiences, starting from elementary schools, imposes substantial demands on teachers' understanding of proof. Knowledge of proof for teaching comprises logical and linguistic structures in proof and knowledge of situations for proving (Buchbinder & McCrone, 2020; Stylianides & Ball, 2008). In order to study teacher knowledge in mathematical proof, specific categorizations were formed, and several researchers detailed each category. It is anticipated that

teachers will need to utilize their mathematical expertise in constructing proofs and their understanding of different types of proofs such as special cases or experimental “proofs”, pre-formal proofs, and formal proofs. They will also rely on their pedagogical content knowledge to plan teaching experiences and assess the sufficiency or accuracy of their own and their students’ proofs in different mathematical subject areas (Corleis et al., 2008). Content knowledge for proving has been defined as including the following criteria: Comprehension of theorems, knowledge of theorems, formulating a formal equation, evaluating a preliminary pre-formal proof, assessing the subject-specific sufficiency of a formal proof, carrying out a verified proof (Corleis et al., 2008). Teachers’ proficiency in teaching mathematics is contingent upon the quality of their content knowledge (e.g., Hill et al., 2005); thus, a prerequisite for achieving the stated goal is that teachers at all levels possess a robust comprehension of proof (Stylianides & Ball, 2008; Stylianides & Stylianides, 2009a). Without a comprehensive understanding of proof, teachers cannot be expected to effectively encourage their students to engage in proving (Stylianides & Stylianides, 2009a). Teachers must demonstrate advanced mathematical topic knowledge at the school level and a deep understanding of mathematical proof at the university level. This encompasses the capacity to recognize various proof structures (pre-formal–formal), the capacity to perform proofs at different levels, and the knowledge of alternative specialized mathematical proof processes (Corleis et al., 2008). Teachers should possess the ability to formulate a sound mathematical proof and a comprehensive understanding of the fundamental elements of such proofs. These include understanding that proof is founded on accepted statements, adheres to a logical sequence, and establishes the truth for all objects within the subject. Steele and Rogers (2012), in their study,

provide four elements of subject matter knowledge for proof: Defining, identifying, and constructing proofs, as well as comprehending the functions of proof. However, they do not differentiate between common and specialized knowledge (Lesseig, 2016). Knowledge of logical and linguistic structures in proof was also considered as a part of content knowledge for proof. This knowledge represents the capacity to utilize and comprehend the function of mathematical language in proofs (Ball & Bass, 2000, 2003; Dreyfus, 1999; Zaslavsky & Shir, 2005). In their research of primary mathematics teaching via a disciplinary perspective, Ball and Bass (2000) recognized mathematical language as “a crucial cornerstone of moving toward the construction of publicly shared and validated mathematical knowledge” (p. 221). Focusing on mathematical language necessitates consideration of both the representation of mathematical arguments and the definition of mathematical concepts. When teachers and teacher candidates’ performances in content knowledge of proof were considered, it was observed that most teachers have a limited grasp of proof and its function in mathematics, which has been identified as a significant cause of several shortcomings in students’ proof skills (Dreyfus, 1999; Harel & Sowder, 2007; Jahnke, 2007; Lesseig, 2016; Stylianides & Stylianides, 2009a). In high school geometry, teachers typically restrict the use of proof to the verification of formulas, therefore overlooking the crucial explanatory function that proof can serve in the acquisition of mathematical knowledge at all educational levels (Hanna, 2000; Knuth, 2002; Lesseig, 2016). Teachers often treat empirical justification as proof of a general statement (Harel & Sowder, 2007; Simon & Blume, 1996), have trouble analyzing non-symbolic proofs (Barkai et al., 2009), and view proof only for verification (Knuth, 2002). Teachers at all levels generally possess an insufficient understanding of the logico-linguistic structure of proof (Stylianides & Ball, 2008).

Pedagogical content knowledge for proving is defined by the following abilities: Engaging in systematic analysis of proving in mathematics teaching and being well acquainted with various types of proofs (Corleis et al., 2008). Teachers need to master the concept of various definitions and understand how an argument can be reorganized based on acceptable definitions. Enhancing students' capacity to demonstrate proof also requires teachers to possess proficiency in a range of visual, symbolic, or verbal techniques that can be employed to articulate a general argument. Teachers should possess a comprehensive understanding of several argument types, such as validating a claim about a limited number of cases using a systematic list or effectively supporting a general claim with a generic example (Buchbinder & McCrone, 2020; Lesseig, 2016).

Schwarz et al. (2008) described three characteristics to possess for teaching proof as “(1) proving competencies (mathematical content knowledge), (2) didactical reflections about proof (pedagogical content knowledge), and (3) affinity to argumentation and proof in teaching (beliefs)”. Lesseig et al. (2011) studied Mathematical Knowledge for Teaching (MKT) for proof based on the categorization of Ball and her colleagues (2005). CCK embodies the fundamental knowledge and abilities expected of students (e.g., Buchbinder & McCrone, 2020; McCrone & Martin, 2009; National Council of Teachers of Mathematics, 2000), which include the understanding that a proof relies on accepted claims, adheres to a logical sequence, and establishes truth for all elements within the domain. To facilitate teaching, essential concepts of proof must be cultivated and articulated clearly. This expanded knowledge, or Specialized Content knowledge (SCK), constitutes the mathematical information required explicitly for teaching proof. This knowledge type comprises understanding several definitions and how an argument's structure

may vary based on the accepted definitions. Facilitating students' proof skills requires teachers to be adept in various visual, symbolic, or verbal techniques for articulating a general argument. Teachers must possess an understanding of many argument types, including the ability to verify a claim regarding a finite number of cases through a systematic enumeration, as well as the use of a generic example to prove a statement (Lesseig, 2016).

Knowledge of Content and Students (KCS) pertains to the comprehension of students' common beliefs or misconceptions regarding proof, and an awareness of developmental trajectories. It also encompasses the knowledge of how students conceptualize proof and how different categories exist to categorize these conceptualizations (e.g., Balacheff, 1988; Sowder & Harel, 1998; Stylianides, 2011). Teachers should recognize that although neither external conviction nor empirical proof schemes align with mathematical standards of proof, both are frequently seen in student proof attempts (Harel & Sowder, 2007; Kuchemann & Hoyles, 2009; Martin & Harel, 1989). A comprehensive understanding of this taxonomy, or a comparable classification based on research into student cognition, would enhance teachers' capacity to analyze student thought processes and determine subsequent teaching strategies (Lesseig, 2016).

KCT for proof encompasses techniques for representing, elucidating, or linking proof concepts, in addition to strategies for addressing student input. To facilitate students' progression from authoritarian or empirical justification to more advanced concepts of proof, teachers must first acknowledge the connection between their questioning techniques and the proof schemes that students formulate (Harel & Rabin, 2010; Lesseig, 2016). To promote proof in their classrooms, teachers must identify circumstances that necessitate proof, discern significant mathematical

distinctions among these situations, and create suitable opportunities for students to engage in proving. If teachers possess a restricted understanding of a proving task, they would probably focus their instruction solely on the kind of tasks with which they are acquainted. Consequently, students would likely be deprived of exposure to many essential proving challenges (Stylianides & Ball, 2008). Employing diverse proving tasks is essential in the sense that they provide students with opportunities to cultivate a grasp of various proving procedures and reasoning skills (Stylianides, 2011). To explore the mathematical knowledge pertinent to teachers' decisions on transforming a standard problem into a proving task, one must examine the many types of proving tasks and the correlation between these tasks and proving activities (Stylianides & Ball, 2008). Teachers' decisions on sequencing students' work on related proving problems is also crucial. The matter is determining a suitable sequence for the class to accomplish these tasks (Stylianides & Ball, 2008).

Teachers' performances on PCK for proof revealed that primary teachers typically have deficiencies in the pedagogy of proving. Even secondary teachers possessing robust subject matter knowledge, when assigned suitable tasks, failed to capitalize on such possibilities. This condition indicates a deficiency, possibly in understanding how to execute tasks or in a belief in the significance of proofs in mathematical education (Lin et al., 2012). In addition to a restricted comprehension of visual representations, teachers also lack proficiency with generic example proofs (Karunakaran et al., 2014). Classroom research demonstrates that generic examples facilitate students' transition from inductive to deductive reasoning (Lannin et al., 2006; Pedemonte & Buchbinder, 2011), highlighting the need to provide teachers with the opportunity to enhance this knowledge (Lesseig, 2016).

Knowledge types for teaching proof can sometimes be intertwined. For instance, interjections of the representations available to students (KCS for proof) were combined with evaluations of whether an argument represented a valid proof of the general claim (CCK). Comprehending and differentiating between empirical and deductive types of argumentation can be considered in both content knowledge and pedagogical content knowledge in the same manner (Stylianides, 2011; Stylianides & Ball, 2008). Similarly, the ability to assess the validity of proof constitutes a component of both CCK and PCK (Lin et al., 2012).

Effective teaching in proof necessitates interconnecting knowledge types. A comprehensive comprehension of students' concepts of proof requires a solid grasp of the mathematical principles that underlying these beliefs. A teacher's capacity to apply pedagogical methods in the classroom to aid students in developing their notions of proof necessitates a comprehensive understanding of the students' conceptions and the more sophisticated forms these may take knowledge (Stylianides, 2011).

Belief theory has been widely used in mathematics education and teacher professional development research (Leder et al., 2002; Thompson, 1992). Philipp (2007) characterizes beliefs as "the lenses through which one interprets the world" (p. 258). Richardson (1996) presents a domain-neutral definition of beliefs grounded in a comprehensive understanding. She defines beliefs as "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103). Beliefs comprise cognitive and affective characteristics, shaping individuals' perceptions and interactions with their environment (Furinghetti & Pehkonen, 2002). In contrast to professional knowledge, which is universally

accepted, beliefs can differ markedly among individuals (Buchholtz & Vollstedt, 2024; Philipp, 2007).

Beliefs may be peculiar to particular domains (Törner, 2002; Eichler & Erens, 2015) or even specific situations (Schoenfeld, 2010). Concerning the long-term development of beliefs, it is posited that they are reasonably stable in terms of restructuring and can function as psychological filters and/or barriers to some degree (Fives & Buehl, 2012). Conversely, more peripheral beliefs are more readily adaptable to new information and are closely associated with the cultivating expertise (De Vries et al., 2014). Beliefs may evolve during teachers' professional development (Eichler & Erens, 2015) or be affected by the competencies of the students being instructed (Beswick, 2018; Safrudiannur, 2020). Beliefs have been shown to be indistinguishable from knowledge (Thompson, 1992). In some studies, beliefs are considered as part of knowledge (e.g., Frunghetti & Morselli, 2009). Furinghetti and Pehkonen (2002) stressed that beliefs, conceptions, and knowledge are mutually related terms and may be used interchangeably in relevant research studies. This might imply that investigating beliefs might also inform researchers about teachers' knowledge and conceptions of proof by implication.

Three primary components of teachers' beliefs in mathematics were stated as: The essence of mathematics, the nature of teaching mathematics, and the process of learning mathematics (Furinghetti & Morselli, 2009). Teacher beliefs constitute a fundamental affective element of teachers' professional competence (Shulman, 1986). The incorporation of proof in mathematics lessons and its utilization to enhance students' mathematical reasoning is significantly influenced by teachers' beliefs regarding the significance and place of proof within mathematics and mathematics education, among other variables (Frunghetti & Morselli, 2009;

Kotelawala, 2009; Pehkonen, 1994; Thompson, 1984). Beliefs underpin the rationale for teachers' decisions and are connected to the practices (Frunghetti & Morselli, 2009). Hence, their effect on student learning is non-negligible (Pehkonen, 1994). Leatham (2006) stated that when a teacher behaves in a manner that appears incongruent with inferred beliefs, either the implications of the belief were misinterpreted or another belief was dominant in that specific context (p. 95). Teaching proof is influenced not only by particular beliefs regarding proof but by a broad spectrum of beliefs. Some teachers are profoundly affected by their ideas regarding the essence of mathematics, while others are more impacted by their beliefs concerning mathematics teaching and learning (Furinghetti & Morselli, 2009).

Beliefs can be contextual and can be affected by external factors (Buchholtz & Vollstedt, 2024). Current research regarding the correlation between teachers' beliefs and their instructional practices remains ambiguous: Some researchers have demonstrated consistent beliefs and practices (e.g., Staub & Stern, 2002; Safrudiannur & Rott, 2020), whereas others have exhibited inconsistencies (e.g., Cross Francis, 2015; Furinghetti & Pehkonen, 2002; Li & Yu, 2010). One potential interpretation of these discrepancies is that teachers may possess differing attitudes based on the specific situation or context (Safrudiannur & Rott, 2020, 2021). External factors, such as support from school administration or educational policy regulations, may result in discrepancies between teachers' beliefs and behaviors (Buehl & Beck, 2015). Buchholtz and Vollstedt (2024) specified that teacher beliefs are typically influenced by individual experiences in mathematics education and the university admission process. While the impact of external factors on beliefs has been shown, the effect of teaching experience on teacher beliefs is debatable

(Buchholtz & Vollstedt, 2024). Page and Clark (2010) demonstrate that a teacher's experiences, both in teaching mathematics and as a mathematics learner, profoundly impact their attitudes and instructional methodologies. In contrast, extensive investigations yield contradictory findings. Safrudiannur and Rott (2020) assert that teaching experience, often measured in years, may not influence teachers' attitudes and behaviors as previously assumed. Nisbet and Warren (2000) discovered no substantial correlation between the length of teaching experience and educators' opinions toward mathematics instruction. Drageset (2010) noted that the duration of teaching experience does not correspond with opinions about the significance of rules, accurate answers, logic, or justification in mathematics lessons.

Beliefs on proof and teaching proof comprise a variety of dimensions to be considered. To be able to examine teacher beliefs on proving, one might look at their beliefs on pedagogical practices on proving, on students' difficulties in proving, on the impact of contextual factors, on role and place of proofs in mathematics, on characteristics of proof in school mathematics, on when and how proof should be taught, on beliefs about nature of mathematics, about teaching and learning mathematics, and on their own abilities in proving (Cabassut et al., 2012; Frunghetti & Morselli, 2009; Mingus & Grassl, 1999).

While literature acknowledges particular positive and constructive views on teaching proof, the negative ones garner greater attention due to their higher prevalence among teachers. Teachers clearly appreciate the importance of students engaging in exploration, investigation, and reasoning; nevertheless, they are either uncertain about middle school students' capacity to construct proofs or consider that only more capable students could achieve this (Furinghetti & Morselli, 2009; Iscimen, 2011; Knuth, 2002). Most of the time, they indicate uncertainty whether

middle school children could independently formulate proofs. A significant number of secondary teachers disregarded the importance of proving and expressed doubts regarding their students' preparedness to engage with the concept of proving (Furinghetti & Morselli, 2009; Iscimen, 2011; Kotelawala, 2009). A significant number of teachers expressed disagreement regarding the expectation that students should be the ones to provide proof or justification (Kotelawala, 2009). Most participants concurred on the significance of fostering logical thinking skills. Nevertheless, a significant portion did not believe that the theorems taught in high school necessitated formal proof (Cabassut et al., 2010; Kotelawala, 2009). Many secondary school teachers reported that they would accept empirical arguments in their mathematics classes. Some teachers argue that an argument that is not acceptable as a mathematical proof would nonetheless suffice for middle school students (Furinghetti & Morselli, 2009; Knuth, 2002). It was also reported that some teachers view proof as a subject to be taught separately rather than treating it as a process skill (Furinghetti & Morselli, 2009; Knuth, 2002).

To address concerns regarding the pedagogy of mathematics, it is imperative to gain insight into the perspectives and methodologies that teachers employ in their approach to proofs (Kotelawala, 2009), and revealing teachers' prior implicit beliefs regarding the objectives of proof carries significant implications for teaching and knowledge construction (Chua et al., 2010). Teacher beliefs regarding teaching proof have been investigated through different data collection methods. Some studies used qualitative methods (e.g., case studies, a variety of interviews) (Frunghetti & Morselli, 2009; Iscimen, 2011) by suggesting the reconstruction of beliefs based on their expressions and behaviors (Furinghetti & Morselli, 2009); in some others, questionnaires and scales have been developed to examine teacher beliefs on

teaching proofs and related concepts (e.g., Chua et al., 2010; Kotelawala, 2009; Mingus & Grassl, 1999). It is also possible and meaningful to combine these diverse methods to better comprehend teachers' beliefs on teaching proof.

Examining teacher beliefs and comprehending their roots is essential to discern their impact on teaching proof. The exploration of teacher beliefs and the potential of changing negative beliefs have been widely studied in mathematics education research (Thomson, 1992). Especially during the time of curriculum innovations, teachers have been at the center of the attempts to change (Hoyles, 1992, as cited in Pehkonen, 1994). This change can only be realized through altering beliefs profoundly, not superficially. Changing those beliefs immediately does not end up with successful results. In order to achieve this change, teachers should understand and internalize the need for change; they should be provided with the means to realize the change, and the contextual factors should not become obstacles. Teachers' negative and unproductive beliefs on teaching proof, their thinking, and actions on it should be challenged by providing support for them and presenting situations of conflict or perturbations for the desired change.

Teacher beliefs of students' abilities also impact the effectiveness of proof teaching as their expectancies shape their actions in mathematics lessons (Bieda, 2010). When teachers consider proof as an isolated topic, view students as passive subjects of specific proof formats, and perceive themselves as the only authority to judge the correctness of proof attempts, this turns into a debilitating effect on students' proof competencies (Boyle et al., 2015). When teachers promote the belief that a couple of examples or a graphical representation is enough to be counted as justification, students' proof skills are hindered (Stylianides, 2008). Sometimes, teachers may intentionally prefer less cognitively demanding tasks because they

think they are more achievable by students (Bieda, 2010; Sears & Chavez, 2014; Stylianides, 2008).

Knowledge and beliefs influence not only practice but also one another (Lin et al., 2012). Knowledge and beliefs can be altered with new experiences as they frequently derive from their prior experiences for both teachers and students. Therefore, it is essential to provide teachers with novel experiences to improve their proof skills. Studies indicate that collaboration between researchers and educators can positively influence teachers' knowledge, beliefs, and practices. In a particular study, teachers assessing students' arguments, who initially disregarded generic examples as merely empirical, ultimately acknowledged this form of argument and recognized the underlying general logic (Healy et al., 2009).

Teachers' plans and implementation of mathematical activities tremendously shape students' proof-related skills (Bieda, 2010). Setting up proof-related tasks is insufficient to establish standards for mathematical proof. Some teachers do not provide required feedback, do not promote enough discussion about the validity of statements, do not give importance to building a critical mathematical community, and give importance to the clarity and completeness of mathematical justifications (Bieda, 2010). Also, teachers' excessive guidance or promotion of limited opportunities contribute to students' difficulties in proving. Few and low-quality proof tasks implemented in the class also have a debilitating effect on students' proving skills (Sears & Chavez, 2014). Integrating proofs and proving in classroom practice requires a significant transformation in the pre-existing mathematical habits of mind. Cirillo (2009) contended that the teacher's prior familiarity with demanding proofs in collegiate courses and the existing curriculum materials impeded the implementation of a proving practice in his classroom. Also, the objectives specified

in written curricular materials may not align with the teacher's agenda for proofs and proving.

The literature on teachers' knowledge, beliefs, and practices have many implications for both the training of teachers and the mathematical courses prospective teachers undertake at university (Schwarz et al., 2008). A design including instructional sequences for prospective teachers' knowledge and beliefs for proof can be developed, implemented, and analyzed as done by Stylianides and Stylianides (2009). The question posed by Hanna et al. (2008) "How can we design opportunities for student teachers to acquire the knowledge (skills, understandings, and dispositions) necessary to provide effective instruction about proof and proving?", should be considered for such designs by combining all the relevant and implementable suggestions in the literature (Schwarz et al., 2008).

2.9 Suggested interventions for overcoming ongoing problems in proving

The problems with proving have been continuously shown in relevant research studies. It was stated that they continue to exist at an international level even after explicit instructions on proving. Some issues related to learning to prove remain unsolved and pose problems to further learning the area of proof in school mathematics (Stylianides & Stylianides, 2008, 2018). Stylianides et al. (2017, p. 41) also note that "Research thus far has offered inadequate support to teachers, teacher educators, and curriculum developers about how they might address problems of practice in the area of proof."

Several studies point out the necessity of early intervention on proving as a gradual approach or a transition phase to promote a continuum between elementary and secondary school curricula and to reduce cognitive load for upper-level courses

(Cyr, 2011; Moore, 1994). Sowder and Harel (1998) stated that “It seems clear that to delay exposure to reason . . . and to expect at that point an instant appreciation for the more sophisticated mathematical justifications is an unreasonable expectation” (p. 674) (as cited in Knuth et al., 2002). According to Perrin-Glorian (2003), it is necessary to take action before teaching proof-writing in secondary school. The transition from practical to theoretical geometry can be best achieved before secondary school (Cyr, 2011). Lester (1975) indicated that whenever students’ intellectual development suffices, proof should be introduced into the agenda of teaching mathematics as early as possible.

Multiple ideas were suggested in the literature for a systematic change regarding teaching and learning proof. These suggestions are mainly about teacher practices and their roles in the classroom because of teachers’ influence on students’ conceptualization of justification (Staples & Bartlo, 2010). Nardi and Knuth (2017) stated that three factors should be considered for effective interventions on proof: (1) teacher preparation and professional development, (2) intentionally designed curricular materials for proving, (3) instructional practices that engage students in proof, and proof-related activities such as conjecturing, exploring and justifying. Teachers play three critical roles in the development of students’ mathematical reasoning: Firstly, the selection of tasks to raise the need for mathematical reasoning and proof and then scaffolding of the mathematical language and knowledge and making it public, and finally, the establishment of a classroom culture that brings about respecting, being interested in, and attending to others’ ideas (Ball et al., 2002). Institutionalization of proofs means that the constructed proofs are used for future work in proof attempts. This can be initiated by the mathematics teacher (Stylianides, 2007b). Teacher decisions such as choice of tasks, maintaining the

demand of tasks, modeling methods, instructional strategies, and expectations from students shape their practices in the classroom; therefore, these should be advanced according to suggestions given in the literature (Ball & Bass, 2003; Martin et al., 2005; Meta-Pereira & da Ponte, 2017; Stein & Lane, 1996; Wilhelm, 2014).

Teacher questioning is shown to be important for the contribution of reasoning and proof skills. Teachers should use effective questioning techniques and suitable classroom discussion approaches to challenge students critically and better prepare them for proving practices. Students should be expected to question the reasoning behind the arguments, explanations, and justifications and then ask for clarifications (Almeida, 2010). In his study, Maher (2010) found that questioning resulted in more elaborate explanations and arguments among students and led students to develop more effective organizational schemes.

Teachers should provide students with proof instructions based on cognitive conflict approaches. The main foci of the cognitive conflict approach are on three issues: (1) students need to reflect on their current understandings, (2) they should confront some contradictions, and (3) they need to recognize that to be able to resolve this contradiction they should modify their understandings (Stylianides & Stylianides, 2009b).

Students need to be provided opportunities to mix deduction and experimentation in proving processes (Ball et al., 2002). Proofs depend on formal definitions, and formal definitions depend on informal concept images. Before understanding the formal definitions, these images improve with diagrams, graphs, and examples. As students practice more, their reliance on concept images diminishes, and they become more comfortable with mathematical notation, grammar, and logical structure of proofs (Moore, 1994). Formal proof is not the only

means to form the foundation of justification and verification of mathematical statements (Ellis, 2011). Encouraging students to justify and clarify their ideas, asking them to share their arguments publicly, directing students' attention to mathematical relationships, asking for building on an idea, and predicting activities are found to be supportive in the generalizing process (Ellis, 2011).

Asking students to generate their own examples about concepts may have some pedagogical benefits even though empirical arguments are stated to be problematic in proving activities. Example-based learning is highly beneficial for beginning a skill acquisition process (Renkl & Atkinson, 2003) because it provides a model on how to solve proof problems (Reiss et al., 2008) when the examples lead to noticing the structure and pattern behind the problem. Ineffective learning may be due to students' passive and shallow use of examples (Renkl, 1997). To increase effectiveness, students should be given guidance to study productive examples (Reiss & Renkl, 2002). Researchers who suggest examples as pedagogical tools assert that examples can increase engagement with formal theory, can lead to a better understanding of concepts and ideas, and can base the foundation of expressing generality (Iaonne et al., 2011). Heuristic worked-out examples have been shown as an auxiliary tool for teaching proof. They encompass all solution steps and heuristic strategies that guide the process of problem-solving. Heuristic worked-out examples help learners learn independently as they scaffold students through the proving process and foster their own activity. They involve exploration of the problem at hand, building conjectures, monitoring available information about the problem situation, developing the idea of proof, constructing the proof for the conjecture, and finally looking back to the process. They also involve "explicit explanations from a meta-perspective." A mixture of scaffolding and individual learning opportunities

make heuristic worked-out examples valuable means to initiate robust learning processes (Reiss & Renkl, 2002; Reiss et al., 2008, p. 460).

Proof skills are highly related to limited working memory capacity that does not increase with age. However, practice results in efficient use of operating space and leaving more capacity for storage space for proving. To improve students' performance in proof-related tasks, teachers can (1) prevent unnecessary usage of students' working memory with the help of scaffolded proving tasks by excluding irrelevant premises and focusing on the logical structure of the proof. After some time, the ability to identify the relevant premises should be desired, but in the earlier steps of proving this scaffolding would be beneficial. Teachers can also (2) help students develop strategies for effectively managing working memory capacity by integrating efficient use of visual diagrams and sets of givens (Stylianides & Stylianides, 2008).

There exists a dilemma regarding teaching and learning proof: Since the shared knowledge basis of students is different from that of mathematicians, a variety of proofs should be accepted in the classroom. On the other hand, students should practice authentic mathematics. This can be solved through the authentic but less rigorous nature of proof activities (Heinze & Reiss, 2003). A balance between mathematics as a discipline and students as mathematical learners is also emphasized by the notion of "intellectual honesty" by Bruner (1960). According to the intellectual honesty principle, proof in school mathematics should be honest to mathematics as a discipline and honor students as learners. So, for proof instructions, the normative aspect of proof and learner characteristics should be considered (proof should be conceptually accessible to students) (Stylianides, 2007b). The aspect of mathematics as a discipline encompasses using true statements, valid modes of

argumentation, and appropriate modes of representation, as in the field of mathematics. Consideration of students as learners is about paying attention to what is known, accepted, and conceptually accessible to the classroom community and also their existing levels of competencies (Stylianides & Ball, 2008).

The successful intervention should be accessible and usable by teachers; hence, it is essential to support teacher learning alongside student learning of proof (Stylianides & Stylianides, 2008). Implementation of complex topics could result better if teachers are provided with guidance that goes beyond the solution of proof tasks. Teachers should be guided with teachers' edition of the textbooks and other curriculum materials for several reasons: (1) teachers have a limited understanding of proof, (2) teachers have limited understanding of what it means to promote proof in the classroom, (3) proof tasks are challenging to implement because of their high cognitive demand and students' difficulties with them. In the teachers' edition, solutions for the proof tasks are given with three additional guidance: (1) explanations on the importance of engagement in proof tasks, (2) important points about what to expect as students approach and how to deal with them. This guidance helps strengthen pedagogical content knowledge in the proof domain and (3) ideas to promote content knowledge on proof (Stylianides, 2008).

Some learning environments are better in promoting opportunities with conjecturing and justifications: Supportive and safe collaborative learning environment, open-ended tasks to be investigated in sufficient time, directing students to build representations, providing tools to build models, expecting students to explain and justify their mathematical claims, stressing sense-making, carefully designed teacher interventions, and fostering mathematical discourse in the classroom constitute some of the conditions for supportive learning environments for

proving and reasoning. Safe and supportive environments improve students' motivation to work on tasks, voice their ideas, and build solutions as a community, hence developing multiple forms of reasoning for students (Mueller, 2009).

Some studies have been conducted to prepare teachers for teaching proof. For example, in her study, Cirillo (2014) worked with middle school mathematics teachers to help them teach their students how to construct geometry proofs and scaffold the introduction to proof. She developed PD sessions that involved discussion of articles on geometry and proof, reflection on videos on teaching practices, discussions on classroom discourse, and thinking about alternative teaching methods. While the PD did not help alone, she also developed a tool, "MPT (Mathematical Proof Tool)," that divides the proof practices into more manageable proving activities. By breaking down the proof practices, sub-goals to achieve were built, and it was found that the tool helped teachers improve their learning and teaching of proof. In their study, Buchbinder and McCrone (2020) developed four instructional modules in a college course to teach teacher candidates logical aspects of proof and ask them to plan and implement lessons while considering those aspects to develop preservice teachers' mathematical knowledge for teaching proof. The modules focused on quantified statements, conditional statements, direct proof and indirect reasoning. The modules helped preservice teachers to develop lesson plans that integrate what they learned in four modules. They could also incorporate them into their teaching practices, but had issues in maintaining the cognitive demand of proof activities. In another study, Lin et al. (2012) stated that teachers' professional learning on teaching proof comprises three elements: knowledge of proof, practice of proof, and beliefs about proof. In their theoretical work, they suggested three mechanisms to be involved in helping teachers develop knowledge on proof:

“establishing conviction,” “the role of the teacher educator,” and “the notion of cognitive conflict” (p. 13)



CHAPTER 3

PART 1: PRELIMINARY PARTS LEADING TO THE DESIGN STUDY

In this chapter, preliminary studies that shape the design study are elaborated. In order to develop preservice teachers' knowledge and skills in learning and teaching proof, it is important to understand the current situation of preservice teachers and what they need to be able to teach proof. It is important to examine the experiences of preservice teachers that shape their knowledge and beliefs, the opportunities available to them, and the extent to which educational policies equip students and teachers to learn proof. In order to better understand the situations and problems encountered with proof skills and to compile solutions, the opinions of experts in the field are of great importance.

The studies that guided the design, which aimed to improve preservice teachers' knowledge and beliefs about proof and its teaching, consisted of (i) educational policy analysis, (ii) compilation of expert opinions, and (iii) examination of preservice teachers' knowledge and beliefs about proof and its teaching.

This chapter first describes the methodology of these studies. Then, the findings from these studies are presented. Finally, the findings are discussed in the light of the relevant literature.

3.1 Methodology

3.1.1 Research design

This dissertation study aims to investigate how to support preservice mathematics teachers in acquiring the necessary knowledge and skills for teaching proof and

related skills to middle-grade students. For this purpose, the study began by exploring the expectations from teachers, preservice teachers, and students through various policy documents and discourses. Stakeholders and experts were consulted to find solutions to common problems and improve existing conditions on proving and teaching proof. Then, the current situation of preservice teachers in terms of knowledge and beliefs regarding proof and teaching proof were investigated. The design of the preliminary studies is summarized in Table 1.

Table 1. The Summary of Preliminary Studies

Research Focus	Data Sources	Data Collection Tools	Participants
Expectations from teachers, preservice teachers, and students	Policy Analysis	Curricula, Undergraduate programs, Student and Teacher Assessments, and their Reports	-
Needs and aims for teaching proof	Expert Opinions	Interviews with Experts (Semi-structured interviews)	2 Mathematicians, 2 Mathematics Educators, 2 Middle School Mathematics Teachers, 2 High School Mathematics Teachers
Current situation of teacher candidates	Preservice teacher beliefs, preservice teacher knowledge	Teacher beliefs scale, teacher knowledge test	57 teachers and preservice teachers in the pilot study, 347 preservice teachers in the main study Preservice teachers in the main study 20 preservice teachers in the pilot study, 305 preservice teachers in the main study

3.1.2 Instruments

Data collection tools for examining how policy decisions are shaped around reasoning and proof, experts' opinions on learning and teaching proof, preservice

teachers' knowledge for proof, and preservice teachers' beliefs on proving were used as instruments throughout the study. The details for each data collection tool are presented below separately.

3.1.2.1 Policy documents and discourse

The examination of policy statements was achieved through the analysis of policy documents and policy discourses related to reasoning and proof in specific and mathematics education in general. In this study, first, middle school (MEB [MoNE], 2018a) and high school mathematics education curricula (MEB [MoNE], 2018b) were investigated in terms of the promotion of reasoning and proof. To investigate mathematics teacher education programs and courses offered to middle school and high school mathematics teacher candidates, programs developed by Yükseköğretim Kurumu (YÖK) [Higher Education Council] were analyzed (YÖK, 2018a, 2018b). Also, national and international tests (TIMSS, PISA, ABİDE) were analyzed according to the extent to which they include proof and reasoning (Kasap, 2019; MEB, 2017; Suna et al., 2019, 2020). Assessments and professional development programs for mathematics teachers were investigated through documents and announcements made by the MEB. The projects and discourses of the minister of national education on mathematics education were analyzed through interviews and news reports in the press (Kasap, 2022; “Matematik seferberliği başladı”, 2022; “Matematik Dijital Eğitim platformu açıldı”, 2022).

3.1.2.2 Interview protocols with experts

Semi-structured interviews were conducted with two mathematicians, two teacher educators, and four mathematics teachers (two middle school teachers and two high

school teachers). Interview questions were formulated based on the relevant literature (e.g., Isler, 2015) and the participants' expertise. The questions can be seen in Appendix A.

3.1.2.3 Teacher knowledge test on proof

A 6-item proof test was developed to measure the knowledge of teachers and preservice teachers about proof and teaching proof. The first three items in this test are related to content knowledge of proof, and the other three are related to pedagogical content knowledge of proof. While some of the items in the test were taken from the literature, some of them were developed and organized by the researcher in line with the purpose of the study. The first item is about counterexamples (Ball et al., 2002). The second item concerns direct proof (Sears & Chavez, 2014). The third item is about logic rules in mathematical proof (Senk, 1989). The second part of the survey aims to explore teachers' pedagogical content knowledge in proof. The fourth item is about promoting questions to reveal students' understanding of geometry proofs (Flegas & Charalampos, 2013). The fifth item is about evaluating students' arguments on proof terms (Flegas & Charalampos, 2013; Stylianides, 2007). The sixth and final item of the survey is about judging the suitability of two proofs on the same mathematical argument for the middle school level (Reiss & Renkl, 2002).

Evidence for content validity was obtained by asking two researchers working in the field of mathematics education to evaluate the content and format of the devised test. They were asked to analyze the test to determine whether it covers adequate dimensions of the construct that the test seeks to measure and knowledge of proof. In addition, they were asked about the appropriateness of the language,

directions, and space allocated for the responses, details related to the test format (Fraenkel et al., 2011).

The scale was prepared in both paper and online format, preserving the same content and structure. The teacher knowledge test in English and Turkish can be found in Appendix B and Appendix C, respectively.

3.1.2.4 Teacher beliefs scale on proof

A Likert-type 40-item beliefs scale was developed and administered in pilot and main studies. From the current literature and the existing scales (Cabassut et al., 2012; Doruk & Güler, 2014; Frunghetti & Morselli, 2009; İmamoğlu, 2010; Keçeli et al., 2015; Kotelawala, 2009; Pekşen-Sağır, 2013; Ünveren, 2010), items were selected, adapted, translated, and constructed by considering different aspects of teacher beliefs on proving. In the studies cited above, beliefs and attitudes towards proof and its teaching were generally addressed on a single or few dimensions. In order to examine preservice teachers' and teachers' beliefs about proof and teaching of proof more comprehensively, a new scale was developed by considering all possible dimensions.

Initially, 91 items were developed for the scale. After a thorough examination and consideration of the characteristics of scales (e.g., number of items in each dimension and in total, relevance of items, clarity of items, avoiding repetitions), several items were discarded, and 70 items remained. These items were sorted into different dimensions, and experts were asked to rate their appropriateness as (i) quite appropriate, (ii) appropriate, (iii) acceptable, and (iv) not appropriate in the Expert Opinion Form (see Appendix D). The responses for each item by each expert were quantified as 4, 3, 2, and 1, respectively, and the average score was obtained for

every item. Experts were given space to record their notes and recommendations in the form. Based on the evaluations and suggestions by 10 experts in mathematics education, several items were revised, and some were discarded. Expressions were clarified linguistically. Items with a low average score and items that could not be corrected based on the suggestions were discarded. The latest version of the scale consists of 40 items. The scale is a 5-point Likert-type scale, which includes responses from strongly agree to strongly disagree.

After the evaluations of experts, a finalized “teacher beliefs scale on proof” was conducted in a pilot study. The scale was prepared in an online survey platform and shared with the graduates of middle and high school mathematics teaching programs of a public university and preservice teachers from another public university. In the pilot study, 22 teachers and 35 preservice teachers responded to the survey. In the main study, the scale was administered to 347 preservice teachers from 10 universities in Türkiye to respond to the research questions of “Which characteristics can explain preservice teachers’ beliefs about proof?” and “What are preservice teachers’ beliefs about proof and teaching proof?”

To identify the dimensions of teacher beliefs on proving and teaching proof, Exploratory Factor Analysis (EFA) and Confirmatory Factor Analysis (CFA) were conducted. EFA analysis was realized after the reliability analysis of the scale, where deletion of seven items (Items 6, 11, 13, 14, 16, 28, 32) was suggested. The Kaiser-Meyer-Olkin test was done to determine if the data was suitable for factor analysis. The overall KMO measures of sampling adequacy were calculated as 0.87, where scores from 0.80 to 0.90 are interpreted as meritorious (Kaiser, 1974). Bartlett’s Test of Sphericity was conducted ($\chi^2=37$, $p < .001$) to test the hypothesis that the correlation matrix is an identity matrix. A p-value less than 0.05 shows that the

correlation matrix is not an identity matrix and can proceed with factor analysis. The principal axis factoring extraction method was used in combination with a varimax rotation. A 5-factor structure was obtained, which explains 40.2 % of the variance.

Factors and the number of items for each factor could be specified as follows:

- Beliefs on the role and importance of proof (11 items)
- Beliefs on the need for proof (3 items)
- Beliefs on the mastery in proof related skills (3 items)
- Beliefs on the place of proof in middle school (11 items)
- Beliefs on the ability to do proof (5 items)

Factor loadings are given in Appendix E, and factor statistics are detailed in Table 2.

Table 2. Factor Statistics of the Beliefs Scale on Proof

Factor	SS Loadings	% of Variance	Cumulative %
1	4.51	13.66	13.7
2	3.24	9.83	23.5
3	2.59	7.84	31.3
4	1.55	4.70	36.0
5	1.36	4.13	40.2

The model test shows that a significant and small chi-square indicates a good fit ($\chi^2 = 591, p < .001$). Also, RMSEA needs to be less than 0.06 for a good fit (0.0421). TLI needs to be higher than 0.9 for a good fit (0.903). Model fit measures can be seen in Table 3.

Table 3. Model Fit Measures

RMSEA 90% CI					Model Test		
RMSEA	Lower	Upper	TLI	BIC	χ^2	df	p
0.0421	0.0358	0.0486	0.903	-1569	591	373	< .001

CFA was also conducted to confirm the suggested factor structure. While the chi-square test for exact fit points out model fit ($\chi^2 = 984$, $df = 485$, $p < .001$), RMSEA (0.0544) indicates a moderate fit, CFI (.0860) and TLI (.847) show no fit, both need to be greater than 0.9 (see the model fit measures in Table 4).

Table 4. Model Fit Indices for CFA

CFI	TLI	RMSEA	RMSEA 90% CI	
			Lower	Upper
0.860	0.847	0.0544	0.0495	0.0593

After the reliability and factor analyses, some items were excluded from the scale, and the analyses were conducted with the remaining 33 items. With the suggested deletion of items, the reliability of the scale (Cronbach's alpha) was calculated to be 0.87.

The scale was prepared in both paper and pencil and online format, preserving the same content and structure. The teacher beliefs scale on proof in English and Turkish can be found in Appendix F and Appendix G, respectively.

3.1.3 Participants

Different groups of participants contributed to different parts of this study. Each group is detailed in the following subheadings.

3.1.3.1 Participants of the interview with experts

Four mathematics teachers (two were middle school mathematics teachers, the other two were high school mathematics teachers), two teacher educators, and two mathematicians were interviewed. Experts in the field of mathematics and mathematics education were chosen purposefully (Fraenkel et al., 2011). Their demographic information can be seen in Table 5.

Table 5. Demographic Information of the Interviewed Experts

Expert Code	Field	Gender	Experience
M-1	Mathematician (professor)	Female	> 20 years
M-2	Mathematician (Professor)	Female	> 20 years
TE-1	Teacher educator (Assist. Prof.)	Female	> 10 years
TE-2	Teacher educator (Assist. Prof.)	Female	> 10 years
MT-1	Mathematics teacher (Middle school)	Male	> 10 years
MT-2	Mathematics teacher (High school)	Female	> 40 years
MT-3	Mathematics teacher (High school)	Female	> 20 years
MT-4	Mathematics teacher (Middle school)	Female	> 20 years

3.1.3.2 Participants of the teacher knowledge test on proof

A pilot study and then the main study were conducted for the teacher knowledge test. 20 (15 female and five male) preservice teachers completed the survey for the pilot study. Participants were senior students from two public universities. Participants completed the survey in one of the online survey platforms.

In the main study, the survey was administered to 305 preservice teachers from 9 universities in Türkiye to respond to the research question, “What are preservice teachers’ knowledge about proof and teaching proof?”. Most participants who completed the teacher beliefs scale on proof also completed the teacher knowledge test on proof. The two data collection tools were presented to the same participants. However, since some of the participants did not complete the knowledge test, they were not included in the data analysis. The demographic information of the participants can be seen in Table 6. Both in the pilot study and the main study, the convenience sampling method was used (Fraenkel et al., 2011).

3.1.3.3 Participants of the teacher beliefs survey on proof

A pilot study and then a main study were conducted for the teacher beliefs scale on proof. The participants of the pilot study were the graduates of middle and high school mathematics teaching programs of a public university and preservice teachers from another public university. Twenty-two teachers and 35 preservice teachers responded to the survey. Preservice teachers who participated in this study had a minimum GPA of 1.75 out of 4.00 and a maximum of 3.67 ($M = 3.04$, $SD = 0.41$). All except one indicated that they took courses that included and required proof skills. Participants’ demographic information is given in Table 7.

Table 6. Demographic Information of the Participants for the Knowledge Test

University	f	%
A	56	18.18
B	82	26.62
C	1	0.32
D	31	10.07
E	1	0.32
F	101	32.8
G	32	10.39
I	3	0.97
J	1	0.33
Department	f	%
Teaching Elementary School	272	88.31
Teaching High School	36	11.69
Grade	f	%
Junior	166	53.9
Senior	142	46.1

In the main study, the scale was administered to 347 preservice teachers from 10 universities in Türkiye to respond to the research question of “What are preservice teachers’ beliefs about proof and teaching proof?”. All participants who completed the teacher knowledge test on proof also completed the teacher beliefs scale on proof. The two data collection tools were presented to the participants at the same time, and while some of the participants did not choose to complete the knowledge test, all participants completed the belief scale on proof. The demographic information of the participants can be seen in Table 8.

Table 7. Demographic Information for the Beliefs Scale- Pilot Study

	Frequency (n)	Percentage (%)
Gender		
Female	43	75.4
Male	14	24.6
School type		
Public school	12	21.05
Private school	11	19.3
Does not work/ University student	34	59.65
School level		
Middle school	15	26.3
High school	8	14.0
Does not work/ University student	34	59.7
Years of experience at work		
0 (college student)	34	59.6
1-5 years	14	24.6
6-10 years	5	8.8
11-15 years	2	3.5
16 years or more	2	3.5

Table 8. Demographic Information of the Participants for the Beliefs Scale

University	f	%
A	68	19.6
B	82	23.6
C	3	0.9
D	31	8.9
E	4	1.2
F	101	29.1
G	34	9.8
H	13	3.7
I	7	2
J	4	0.3
Department	f	%
Teaching Elementary school	311	89.6
Teaching High school	36	10.4
Grade	f	%
Junior	183	52.7
Senior	164	47.3

Both in the pilot study and in the main study, the convenience sampling method was used. Sampling was convenient because participants who were available at the time were expected to participate in the study on a voluntary basis (Fraenkel et al., 2011).

3.1.4 Data collection procedure

Data collection was realized in different timelines for each data collection tool.

Initially, the teacher beliefs scale on proof and teacher knowledge test on proof were constructed. Before the pilot studies, the items were analyzed by several experts.

After getting the approval of the ethics board of the graduate institute and the Ministry of National Education (see Appendix H), pilot studies for both tools were conducted through an online data collection platform. The initial analyses led to the final forms of these two data collection tools. Finalized tools were applied to preservice teachers through visits to colleges or online forms (if face-to-face interaction was not possible due to physical distances) after they consented to participate in the study.

Experts were asked to participate in a 30-40-minute interview meeting. These meetings were scheduled based on the experts' availability, and the platform for the meeting (face-to-face or online) was selected according to the participants' choices.

The timeline of the preliminary studies is shown in Table 9.

Table 9. The Timeline of the Preliminary Studies

Timeline	Activity
June 2021 - Jan 2022	Extensive Literature Review and Specifying the Problem Situation
Jan 2022 - Jan 2023	Preparation of the Knowledge Test (Expert opinions and pilot study)
Jan 2022 - Jan 2023	Preparation of the Beliefs Scale (Expert opinions and pilot study)
Jan 2023 - Oct 2023	Implementation of the Knowledge Test and Beliefs Scale
June 2023 - Sept 2023	Analysis of Policy Documents
Jan 2023 - Oct 2023	Interview with Experts
Oct 2023 - Dec 2023	Compiling the findings from the knowledge test, beliefs scale, policy analyses, and interviews with experts

3.1.5 Data analysis

Research questions in the study and the multiple data sources necessitated different data analysis methods. The study consists of both quantitative and qualitative data analysis approaches.

The teacher knowledge test on proof was analyzed through developed rubrics (see Appendix I). The open-ended items in the test were also analyzed qualitatively through emerging codes and categories (Yıldırım & Şimşek, 2018). The dimensions obtained from the factor analyses for the teacher beliefs scale on proof were used to calculate the scores of participants for each dimension. All the reverse items in the scale were re-coded. The items were scored by considering the ranges of the Likert scale and the reverse items. Accordingly, one could get a minimum of 40 and a maximum of 200 points from the scale. The items for each dimension can be found in Appendix J.

Interviews with the experts were audio-recorded or video-recorded, depending on the medium of the meeting. Records were transcribed, and categories were formed based on the codes from participants' responses.

For the analysis of policy texts and discourse, middle school and high school mathematics curricula were investigated. For both curriculum documents, expressions that include the following keywords were searched: proof, inference, judgment, reasoning, generalization, assumption, justification, verification, proposition, evidence, cause-effect. Those expressions are then analyzed in terms of the promotion of reasoning and proof. Similarly, national and international student exams (e.g., TIMSS, PISA, ABIDE) were analyzed considering the proof related-skills and specific outcomes of those exams including reasoning and proof. When examining teacher professional development opportunities, the list of courses was

checked for mathematical reasoning, critical thinking, reasoning and proof skills, and scientific thinking. Assessments for teacher appointment or promotion were investigated by examining the contents of the exams and items of the published exams. Middle school and high school mathematics teacher preparation programs were developed by the Higher Education Council and used in almost all faculty of education programs in Türkiye. The content of those programs was analyzed through the lens of the promotion of reasoning and proof skills. Each course offered in the two programs was reviewed by looking at keywords related to proof.

3.1.6 Validity and reliability

The validity and reliability of this study were addressed separately through the quantitative and qualitative parts of the study. For the quantitative parts of the study, content validity, external validity, and reliability were examined. For the qualitative part of the study, the trustworthiness of data was reported.

3.1.6.1 Content and construct validity

The validity of the study was ensured as all tools were devised and constructed through the ideas presented in the relevant literature. To establish content validity in this study, the instruments must adequately and thoroughly include the domain or elements they claim to represent. It concerns the coverage, relevance, and representativeness of the items in the tools (Cohen et al., 2018). The data collection tool needs to cover the intended content with appropriate depth and breadth to ensure fairness and adequacy without exceeding the boundaries of the content (Cohen et al., 2018).

To provide content validity, experts were consulted regarding the different tools used in the study. The knowledge test was evaluated by two researchers in the field of mathematics education before the test was administered. Based on expert opinions, items were changed in structure and language and expressions were revised. The appropriateness of the items for the construct being investigated was discussed, and the test was finalized.

After developing the beliefs scale, expert opinion was obtained for the appropriateness of the items and the match between items and dimensions in the scale for construct validity. Ten experts, mathematics education researchers, compared the dimensions and the items assigned to those dimensions and provided feedback on these matches, linguistics of the statements, and clarity of the expressions. After analyzing the responses, some items were discarded, and some were revised based on the suggestions, and the scale was finalized.

3.1.6.2 External validity

External validity is explained as the degree to which attributes of a situation seen in one context may be applied or generalized to another one. In other words, it refers to research findings applicable to different contexts (Cohen et al., 2018). In this study, generalizations from the sample were ensured through explicitly described independent variables, detailed demographic information of participants, adequately operationalized dependent variables, and consideration of extraneous factors while collecting and analyzing the data (Cohen et al., 2018).

3.1.6.3 Reliability

Reliability encompasses dependability, consistency, and replicability across time, instruments, and response groups (Cohen et al., 2018). Three primary aspects of reliability are stability, equivalence, and internal consistency (Carmines & Zeller, 1979). Ary et al. (2002) propose that the reliability of a data-collection tool depends the instrument's length (e.g., a test), the heterogeneity of the investigated group (greater heterogeneity correlates with increased reliability), the participants' abilities, the reliability testing methods employed; and the characteristics of the variable being measured or examined.

Cooper and Schindler (2001) propose that reliability can be enhanced by reducing external sources of variation through the standardization and control of data collection and measurement conditions, training researchers to ensure consistency (inter-rater reliability), expanding the number of items on a specific topic, and omitting extreme responses from data analysis (e.g., outliers).

To obtain more reliable results, increasing the number of items and the length of the measurement tool were suggested (Ary et al., 2002; Cooper & Schindler, 2001). In this study, the number of items of the data collection tools was tried to be kept manageable in a way to cover the content of the construct to be measured and to allow the participants to respond to the entire scale within the specified time. When deciding on the number of items, attention was paid to the tools being inclusive as well as being not too long enough to be answered by the participants. In this way, it was aimed to prevent participants from leaving some or all parts of the tools blank or withdrawing from the study (attrition) (Cohen et al., 2018). Items that were too easy and too difficult were avoided in the teacher knowledge test on proof. Scoring was done in an objective manner. Evaluations were carried out using the developed

rubrics or available theoretical frameworks so that consistent analyses could be achieved. For the teacher beliefs scale on proof, Croanbach's alpha coefficient was calculated as 0.87, as evidence for internal consistency. For the knowledge test, a researcher with a PhD degree in mathematics education examined %10 percent of the test along with the researcher. Cohen's Kappa was calculated between the two scores as 0.89; the inconsistencies were discussed, and conflicts were resolved afterward. The researcher and the co-coder reached an agreement at the end; interrater reliability was attained.

3.1.6.4 Trustworthiness

Trustworthiness was ensured for both the validity and reliability of the qualitative measures of the study. The trustworthiness of a research study is contingent upon the methodologies employed to ensure validity and reliability (Patton, 2002). Trustworthiness is studied with the four terms: Credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985).

3.1.6.4.1 Credibility

Credibility refers to the authenticity and truth value of findings in qualitative research, and it is related to internal validity (Lincoln & Guba, 1985). To ensure the study's credibility, the researcher had prolonged engagement in the study, triangulation of data sources and methods, persistent observation, and detailed descriptions of the study (Lincoln & Guba, 1985; McKenney & Reeves, 2018).

This research study utilized prolonged engagement as a method to establish credibility (Lincoln & Guba, 1985). This technique entailed fostering sustained engagement with the participants, facilitating a more profound comprehension of

their behaviors and dialogues (Cresswell & Miller, 2000). Data triangulation was also utilized to ensure credibility. The data triangulation procedure entailed cross-checking the findings using many data collection tools, such as observations, field notes, written documents, and recordings. This facilitated a thorough comprehension of the participants' reasoning and proving competencies. Rich descriptions entail giving thorough and contextualized explanations of the participants, procedures, and study background (Shenton, 2004). The methods section offers a thorough and exhaustive account of the planning, implementation, and analysis processes. The incorporation of detailed explanations enables readers to comprehend the study's methodologies.

3.1.6.4.2 Transferability

Transferability pertains to external validity (Lincoln & Guba, 1985) and generalizability (Bakker & van Eerde, 2015), which refers to the applicability of the study findings to other situations. To address transferability, researchers need to provide a “thick description” and detailed research context (Bakker & van Eerde, 2015; Gravemeijer & Cobb, 2006; Lincoln & Guba, 1985; McKenney & Reeves, 2018). These descriptions should entail participants’ reactions, design processes and “failures and successes, procedures followed, the conceptual framework used, and the reasons for certain choices” (Bakker & van Eerde, 2015, p. 445). In this study, frameworks (teacher knowledge and beliefs, proof skills, etc.), procedures, and the progress of the study were detailed to provide transferability.

3.1.6.4.3 Dependability

The dependability is related to the consistency, replicability, and reliability of the findings in qualitative studies (Lincoln & Guba, 1985). It is provided through triangulation of data, prolonged engagement in the field, peer review, and audit trials.

To address dependability and to examine the concepts and phenomena in the study in depth, different and diverse data sources (observations, written materials, interviews, etc.) were used. In this way, triangulation was achieved. Peer review was also utilized to increase the dependability of the study. The researcher consulted academics specialized in mathematics education and the dissertation advisor throughout the study. An audit trail was utilized where the researcher documented memos throughout the data collection phase to capture the coding process, emphasizing the merged codes and their corresponding themes.

3.1.6.4.4 Confirmability

Confirmability is related to objectivity and neutrality. It addresses the question, "To what extent are the findings influenced by the participants and conditions rather than researcher bias?" (Lincoln & Guba, 1985). It can be provided through audio or video recording during data collection. The triangulation of data collection methods can enhance confirmability. Audit trails can address the confirmability of results as well (Lincoln & Guba, 1985; McKenney & Reeves, 2018). In this study, a variety of data collection tools was used. Observations were recorded and interviews with experts were either audio or video recorded. Transcripts of the interactions or expressions of participants were presented when necessary.

3.2 Findings

In this part, findings related to the analysis of policy documents and discourses on reasoning and proof were detailed under the heading “Expectations from teachers, preservice teachers, and students.” Curricula, undergraduate programs, assessments for students, teacher assessments and PD for mathematics teachers, discourses by important political figures were analyzed in turn.

Later, expert opinions were discussed under the heading “Understanding the needs and formulating the targets.” There, experts’ reflections on the current situation of learning and teaching proof, views on the challenges to teaching and learning proof, ideas on teaching proof at the middle school level, and suggestions to make proof a part of the curriculum were elaborated.

Next, the findings of the knowledge test on proof were detailed first quantitatively, then qualitatively for each item. The performances of preservice teachers on the beliefs scale on proof were elaborated and the relationship between participants’ scores on the knowledge test and beliefs scale was presented. These findings obtained from the preliminary studies informed the design of the study and they were discussed in detail.

3.2.1 Expectations from teachers, preservice teachers and students

To respond to the research question: “What is the place of mathematical proof in educational policies that shape curricula and teacher training?” several policy documents were investigated. In order to understand the reasoning and proof skills of students and teachers (also teacher candidates), it is important to reveal and examine what is offered to and expected from them in the current policy actions. Education policies are goal-directed and reflect the viewpoint/approach of the government as it

is the main decision-making actor (although sometimes some private and voluntary sectors are involved in the decision and the implementation processes) (Adams, 2014). For almost all education policy acts, the agenda is decided by one authority, and ground-level actors such as teachers are expected to put it into the ground (Trowler, 1999), which is sometimes referred to as a “top-down policy approach” (Adams, 2014). The multiple actors and complexity of education policies result in diverse interpretations and impact the implementation as it requires sense-making, interpretation, negotiation, and change (Trowler, 1999). Hence, most of the time, conflict is observed between who makes the policy and who puts it into practice.

Policies can be analyzed through discourses and texts (Adams, 2014; Trowler, 1999). Policy as discourse reflects forms and expressions of communicating policies. In comparison, the policy as text indicates the documentation of policy decisions in formal reports or other written materials (Trowler, 1999). In this study, several policy documents and discourses were reviewed to comprehensively understand the mathematics education policy in terms of promoting reasoning and proof-related skills.

3.2.1.1 Curriculum analysis

First, the mathematics curricula for middle school (MEB [MoNE], 2018a) and high school (MEB [MoNE], 2018b) were examined. In the middle school mathematics curriculum, the aim was to raise individuals who are critical thinkers, problem solvers, and who can use mathematics functionally. For primary school students, developing a basic level of verbal, numerical, and scientific reasoning was emphasized. For middle school students and high school students, competencies expressed in the "Turkish Qualifications Framework" were highlighted. In that

framework, one of the qualifications is classified and defined as: “Mathematical competence and core competencies in science/technology: Mathematical competence is the development and application of mathematical thinking to solve a range of problems encountered in everyday life. Emphasis is placed on processes, activities, and knowledge that build on a sound arithmetic skill. Mathematical competence includes the ability and willingness to use mathematical modes of thinking (logical and spatial thinking) and representation (formulas, models, constructions, graphs and tables) to varying degrees. Competence in science refers to the availability of knowledge and the ability and desire to utilize methodology to explain the natural world in order to define questions and produce evidence-based conclusions” (MEB [MoNE], 2018a).

It is stated in ministry documents that with the General Objectives and Basic Principles set out in the Basic Law of National Education, the general objectives that the Mathematics Curriculum seeks to achieve can be listed as follows: “. . . (students) will be able to express their own thoughts and reasoning easily in the problem-solving process and will be able to see the deficiencies or gaps in the mathematical reasoning of others.”, “. . . (students) will be able to use mathematical terminology and language correctly to explain and share their mathematical ideas in a logical way.” (MEB [MoNE], 2018a).

In all domains of mathematics, expressions such as “. . . to understand the basic properties”, “. . . to make sense of the concepts”, and “. . . to calculate” were used. Furthermore, in the directions for the implementation of the program, algebra was given a special place: Teachers were directed to implement the objectives developed for the algebra topic as planned in the program. There is no such

suggestion for other domains or process skills in the curriculum as for algebra (MEB [MoNE], 2018a).

In the high school curriculum, with the General Objectives of Turkish National Education and the Basic Principles of Turkish National Education as stated in the Basic Law of National Education, students were aimed to “Develop problem-solving skills by looking at problems from different perspectives,” “To gain mathematical thinking and application skills,” and “to use mathematics in a correct, effective and useful way” (MEB [MoNE], 2018b).

In the ninth grade, logic appeared as a topic and the main title was given as "Propositions and Compound Propositions". The objectives were elaborated as: “Explains the proposition, the truth value of the proposition, the equivalence of two propositions and the not of the proposition”, “Explains the compound proposition with examples, shows the properties of compound propositions formed with the conjunctions “and”, or, “or” and De Morgan rules using a truth table.”, “Explains conditional proposition and two-way conditional proposition.”, “Explain the quantifiers every ($\forall$) and some ($\exists$) with examples.”, and “Explains the concepts of definition, axiom, theorem and proof.”. There is no further topic related to reasoning and proof in the high school curriculum. Also, logic and proof do not take any place in the following high school objectives (MEB [MoNE], 2018b).

It can be seen that middle school students are expected to reason mathematically, but the emphasis is much more on problem-solving rather than proof-related skills. Even though high school students are exposed to the rules of logic and mathematical reasoning in more detail, the concepts taught in the ninth grade are not maintained throughout high school.

3.2.1.2 Undergraduate program analyses

In the next step, suggested undergraduate programs by Yükseköğretim Kurumu (YÖK) [Higher Education Council] for Middle school and High school mathematics teacher education were investigated (YÖK, 2018a, 2018b).

For the Primary Mathematics Teaching program, three proof and reasoning-related courses are offered: Abstract Mathematics, Logical Reasoning, and Critical and Analytical thinking. In Abstract Mathematics, the topics are listed as: Symbolic logic and proof techniques, sets, algebra of sets, partitions of sets, product sets, relations, the inverse of a relation, conjunction of relations, equivalence relations, and equivalence classes, order relations, partially ordered sets, fully ordered sets, functions, one-to-one and overlapping functions, conjunction of functions, inverse of functions, permutations, operations.

In the Logical Reasoning course, the topics are identified as: Defending the accuracy and validity of inferences, making logical generalizations and inferences, explaining and using mathematical patterns and relationships when analyzing a mathematical situation, making predictions about the outcome of operations and measurements using strategies such as rounding, grouping appropriate numbers, using first and last digits, or self-developed strategies; making predictions about measurement by considering a specific reference point.

In Critical and Analytical Thinking course, the topics are specified as basic concepts and definitions; the brain as an organ of thinking, ways of thinking and grouping of thinking; involuntary thinking and its characteristics; voluntary thinking and its characteristics; methods of voluntary thinking; critical and analytical thinking; basic characteristics and criteria of critical and analytical thinking, stages of critical and analytical thinking; factors affecting critical and analytical thinking;

scope of critical and analytical thinking; critical and analytical reading; critical and analytical listening; critical and analytical writing.

For the High School Mathematics Teaching program, more courses related to proof and reasoning were offered: Abstract Mathematics 1-2, Euclidean Geometry, Teaching Geometry, and Philosophy of Mathematics.

While abstract mathematics courses were similar to the ones offered in Middle school programs, in the Euclidean Geometry course, the topics were stated as: Axiomatic structure of geometry; the concept of congruence and congruence of triangles, the fundamental proportion theorem and applications, the concept of similarity and similarity of triangles, Seva and Menelaus theorems and applications, right triangle and applications, geometric drawings, the concept of polygon and special quadrilaterals, the properties of angle, beam, tangent, and intercept in a circle, the concept of force in a circle, obtaining the circumference of a circle and area of a circle and applications, space geometry and basic axioms of space geometry, properties of objects in space, applications related to area and volume of solids.

In the Geometry Teaching course, Piaget, Van Hiele, Fischbein, and Duval's models explain geometric thinking and their reflections on learning environments, geometric thinking habits, spatial visualization and spatial orientation, geometric drawings, geometric space, proofs of geometry theorems in different ways, using dynamic geometry software (DGS) in geometry teaching were listed as topics.

In Philosophy of Mathematics courses, ontology and epistemology of mathematics, mathematical concepts such as numbers, sets, functions, etc., and the meaning of propositions and mathematical expressions; foundations of mathematics, methods, and philosophical problems concerning the nature of mathematics, objectivity in mathematics and applicability to the real world; works of pioneers in

the philosophy of mathematics such as Frege, Russel, Hilbert, Brouwer, and Gödel; the concept of plane and dimension, fundamental theories in the philosophy of mathematics: Logicism, Formalism and Intuitionism, quasi-experimentalists and Lakatos, the relation of philosophy of mathematics to mathematics education, social groups in the philosophy of mathematics education were given as suggested topics.

It is observed that primary school mathematics teachers were suggested to have fewer opportunities for learning proof and how to teach proof in the offered courses.

3.2.1.3 Student assessments

After these curricula and program analyses, the assessment documents for students and teachers were examined. Türkiye participates in Trends in International Mathematics and Science Study (TIMSS) with fourth and eighth grade students every four years. Three cognitive domains are considered in this examination: Knowing, Applying, and Reasoning (with different numbers of items and weights). The Reasoning dimension consists of 6 subdimensions as analysis, synthesis, evaluation, deduction (deducing ideas based on the information and evidence), generalization (forming expressions on relations to be more general and widely applicable), and verification (presenting mathematical arguments for supporting a strategy or a solution). The TIMSS 2019 results on the reasoning domain show that Turkish fourth graders perform poorer in Reasoning items than in other domains. Whereas eighth graders performed better in Reasoning when compared to Applying. When comparisons with other countries are made, the Turkish fourth graders are in 35. order out of 56 participant countries/regions, and eighth graders are in 24. order out of 44 participant countries/regions in average Reasoning scores. For both cases,

approximately half of the participant countries performed better than Türkiye (Suna et al., 2019, 2020).

The Programme for International Student Assessment (PISA) is another international exam that 15-year-old Turkish students enter every three years. Mathematical literacy, flexible use of mathematics in daily life, mathematical reasoning, and foundational mathematical skills take place in the PISA examination. One such skill in mathematical processes was defined as “Reasoning and Proof,” which was conceptualized as: “Reasoning and showing evidence include the processes of identifying the elements that constitute problems, relating these elements, making inferences, verifying what is given, proposing and ensuring the correctness of solutions.” Furthermore, the competence levels for mathematical literacy were distinguished by some factors, including mathematical thinking and reasoning capacity. The last report published in 2018 showed that Türkiye ranks forty-second in mathematics among 79 countries participating in PISA 2018 and thirty-third among 37 OECD countries (Suna et al., 2019).

Other than international exams, the Ministry of National Education (MoNE) sets out to examine fourth and eighth-grade students’ academic skills in school subjects, including mathematics in Akademik Becerilerin İzlenmesi ve Değerlendirilmesi (ABİDE [Monitoring and Assessment of Academic Skills]) examination. In this exam, five competence levels (sub-basic, basic, medium, above-medium, high) were defined. Reasoning skills and generalization were mentioned in the above-medium level as “Can perform the necessary operations in geometry or probability problems that involve more than one situation and whose solution requires reasoning and strategy.”, “Reflect on their own actions (solution path) and formulations, find the general term of patterns.”, and “Can establish a relationship

between inferences based on their own interpretation and reasoning.”. At the high level, showing evidence and verification is given as decisive skills and defined as “Can produce mathematical proofs/arguments to defend the correctness and validity of a solution, inference or strategy.” A 2016 report for eighth graders shows that 8.2% of the students are in the above-medium level, whereas only 3.1% of the students are in the high level. In the 2019 report, 11.3% of eighth graders were in the above-medium level, and 3% of them were in the high level. In the same year, 24.6% of fourth graders were in the above-medium level, and 4.7% of them were in the high level. In the other national placement exams, there is no emphasis on reasoning and proof. The focus is on knowing and applying the mathematics rules and formulas in general (Kasap, 2019; MEB, 2017).

Although all the abovementioned exams emphasize mathematical reasoning and proof, their results show that students’ performance in related tasks is underdeveloped.

3.2.1.4 Teacher assessments and in-service teacher training opportunities

For assessing teachers, some currently used and critical exams were detailed. In 1999, to select the civil officers, an exam called DMS (Devlet Memuru Sınavı [Civil Officer Examination] was held. In 2002, the title of the exam changed to KPSS (Kamu Personeli Seçme Sınavı) [Public Personnel Selection Examination]. Teachers who wish to work in public schools have to take this exam and have high scores to be appointed. The exam covers items on mathematics content knowledge and pedagogical content knowledge on mathematics. The items rely on knowledge of analysis, algebra, geometry, and applied mathematics. Items do not target reasoning and proof skills for mathematics teacher candidates.

In-service teachers are not evaluated throughout their careers, but they are subject to several in-service trainings, which are given as professional development activities by the Ministry of National Education. In the 2023 plan, courses like “Dynamic Mathematics Software (GeoGebra),” “Inquiry-Based Mathematics Education Seminar,” “Training Course on the Use of Mathematics Course Materials,” and “Critical Thinking Seminar” were offered to mathematics teachers. The scope and details of the courses are not shared with the public.

With a recent regulation, teachers can become specialized teachers or head teachers with the program and examination developed by the MoNE. These exams are the only evaluation opportunities for in-service teachers, but the content of the exam does not cover any specialized subject; rather, it encompasses general education topics like “Learning and Teaching Processes,” “Inclusion in Education,” “Digital Competence,” etc. For the selection of head teachers, “Cognitive Thinking Skills” is also mentioned in the objectives, but no item in the published exam booklet matched this objective (MEB[MoNE], 2018c).

These processes point out that we do not have any clue and data-based evidence on how teachers perform and how they are given professional development opportunities in (teaching) mathematical reasoning and proof.

3.2.1.5 Policy makers’ discourse

The Ministry of National Education announced the Initiative of Mathematics Mobilization (Matematik Seferberliği) with the aim of both facilitating the learning of mathematics by adapting it to daily life skills and ensuring that students love it from an early age. The Ministry of National Education, at the time, indicated that the initiative comprised mathematics workshops, material development, teacher training,

and a digital platform for teacher training. The minister stated that “. . . the keystone of equal opportunity in education is the teaching of mathematics . . . because we all need mathematics.” and “Mathematics provides an instrument that is necessary for every human being; for the individual to understand, interpret and navigate life in a rational way.” (“Matematik seferberliği başladı”, 2022).

The Mathematics Digital Education Platform, which has been prepared within the scope of Initiative of Mathematics Mobilization, aims to present the place of mathematics in daily life for our students with animations, videos, and interactive content and to support their mathematics discovery processes. It also supports in-class learning processes with summaries of subjects and lecture videos (“Matematik Dijital Eğitim platformu açıldı”, 2022).

Minister Özer who was in office (06.08.2021-04.06.2023) shared details about the work carried out by the Ministry within the scope of Initiative of Mathematics Mobilization and pointed out that although improvement efforts have been carried out for years, the results have not been at the desired level (Kasap, 2022):

When you look at the latest Trends in International Mathematics and Science Study (TIMSS) and Program for International Student Assessment (PISA) surveys, we can see that there has been a very significant improvement in science literacy, but we have seen that the improvement in mathematics is not at the level we want . . . For this purpose, our Ministry, together with all higher education institutions, universities, TUBITAK, academics, and scientists, has been working intensively for about 3 months to make it easier for our children to learn and to associate mathematics with daily life skills. Different applications and math workshops have started to be established in our provinces (Kasap, 2022).

Minister Özer said that the current goal is to make children and young people love mathematics and make it the most important instrument in their hands to help them

in their daily lives, to encourage them to think analytically, to read data, to make projections:

Our aim is to show students, especially in the first stages of education starting from pre-school, how what they encounter in their daily lives, which they see as not related to mathematics, is actually related to mathematics, and to strengthen the connection of this knowledge with life, rather than abstract. (Kasap, 2022).

Minister Özer underlined that they want to show students at all levels of education that mathematics is indispensable for life, that it is not something to be afraid of, and that in fact the codes of life can be learned much more easily when mathematical literacy is increased (Kasap, 2022).

This initiative focused on the development of mathematical literacy and analytical thinking to make citizens cope with daily life skills that require mathematics. Although the increased emphasis on mathematics education in the discourses and in the activities planned to be put into action was pleasing, the lack of emphasis on reasoning skills in those discourses and activities was disappointing.

3.2.2 Understanding the needs and formulating the targets

In order to conduct design-based research on proof-based instructions, not only expressions in the literature but also the current situation in real learning should be investigated through different measures. One such measure is interviewing professionals who are supposed to use mathematical proofs in their work. For this reason, two mathematicians (M1 and M2), two teacher educators (TE1 and TE2), and four teachers (MT1, MT2, MT3, and MT4) were directed to some questions to understand the phenomenon of proof-based instructions at different learning levels. Their responses revealed several themes about proving and teaching/learning proof: (i) reflections on the current situation, (ii) proof-related skills to be learned, (iii)

teaching proof at the middle school, and (iv) suggestions to make proof a meaningful part of the school curriculum. The participants shared their experiences, observations, and their knowledge about the literature on teaching and learning proof.

Interviewees had diverse proof practices in their professional lives. M1 was a professor of mathematics, and she stated that the proof dominated her work:

“Nothing I can’t prove is true for me. My area of expertise is abstract algebra.

Therefore, I cannot continue without proving everything I have stated in its general form and writing the correctness of the mathematical explanation of the statement I wrote. Of course, I think through examples to understand the feeling; it already guides us. To give that intuition is to understand the feeling, you know, where can I go from there, we do it on natural examples. However, I wonder if this is a generalization, then try it again, I have to write the proof for it.” TE1 was a mathematics teacher educator who said that she would cover logic, proof methods, and relevant proofs in different compulsory and elective courses: “[Other than teaching geometry course,] I use proofs in the Teaching algebra course,” “. . . for instance, I have an elective course called Mathematical thinking. There, we emphasize not only proof methods but also problem-solving.” MT1 was a middle school mathematics teacher who did not have much experience in teaching proof in his classroom. When asked how he would feel if proof was included in the curriculum, he stated “I would feel worried and terrified for myself . . . I would investigate what and how to teach to be ready . . . Proving is difficult for me, as I encountered it in high school.” MT2 was an experienced high school math teacher. She had also taught at the middle school level in the past. She found proving very pleasing, like working on a puzzle. Her experiences in proving began in middle

school in geometry classes. She stated that the inclusion of proof-related skills into the curriculum and its proper application would make her very happy.

3.2.2.1 Reflections on the current situation

All participants reflected on their students' proof skills through their experiences in the field. All pointed out their students' lack of proof skills. M1 stated, "When they first arrive at university, students don't have proof skills at all. They are not taught proof in high school. That is why, proof becomes something new for them.", "When they are asked to prove, they begin with examples.", "If we give the proof directly, they try to imitate and memorize it." TE1 also shared the same opinions. She also added that teacher candidates have difficulties teaching proof to younger students, even when they are competent as learners. MT2 explained her experiences with high school students who had never encountered proof instructions previously. In their proof attempts, memorization prevails. They tend to construct example-based justifications. Students lack understanding and internalizing logic rules when asked to transfer them to written/verbal expressions. A mathematics professor, M2, summarized this situation in her interview: "They did it in high school as 1-0 [for truth tables in logic; 1 refers to true statements and 0 refers to false statements. Students tend to decide on the truthness of the given statements (conjunctions, disjunctions, conditional statements, biconditional statements) by applying the rules memorized in high school using ones and zeros]. Okay, I have no problem with that. But when I ask them to prove it, he writes one here and zero there, he writes something, he proves it. However, I don't understand anything he writes. Therefore, [I expect them to provide the answer] not as a truth table, but knowing the logic

behind.” and she continued as, “In high school, they don’t teach the mathematics we want, or at least the child doesn’t come with that mindset”.

What is stated as goals in the curriculum is not sufficient to develop proof skills. Although proof has a place in International Baccalaureate (IB) programs in some schools, most of them follow the national curriculum, which lacks emphasis on proof. This deficiency stems from high cognitive demand and too abstract ideas in proof tasks. In the first year of the high school curriculum, logic rules are given in an isolated manner; the ideas are not used in any mathematics topic until the end of high school. The middle school teacher MT4 shares the limitations of the curriculum for promoting reasoning: “When the eighth grade comes, when we are going to teach negative powers, we show the power in the form of a pattern, but since we don’t show it in the sixth grade, I tell them like this: There is an explanation for why the zeroth power of every number except zero is one, there is a proof of this, but it is too early for me to show you now. Just trust me, know that this is the case because I feel guilty when I teach it like that because the child will memorize it and the child will not get up and say why, but I say wait for the eighth grade so that I can explain to you that when the power goes less and less . . . ”

The “instruction” aspect of the problems is uttered frequently. It is stated that they are not ready to teach proof in middle schools, as mentioned by TE2, a teacher educator at a public university: “Most teachers are aware that the proof is actually not mentioned at all in middle school. Or maybe they don’t know how to handle it, even though it is relatively late in the programs like reasoning skills . . . ” Neither in-service nor preservice teachers have enough knowledge of proof and teaching proof. Teacher candidates do not consider using proof in their future practices. The scarcity of sources and guides provided to teachers was also indicated by TE2: “Usually there

used to be manuals, resources used to be good. They [teachers] indicate their needs. But in terms of teacher training, we are probably weak; we need to improve in that regard.” Proof is considered an isolated topic rather than a process that is applicable to any mathematics subject. This view is reflected in teachers’ complaints about the dense curriculum. Teacher authority and autonomy were also mentioned in relation to what curricula offer to teachers. MT4 reflected that she does not see herself in a position to “change” what the curriculum suggests or requires. She briefly stated that, “But I don’t think I have the authority to give formulas that are beyond his level or that he shouldn’t know to prove it. [In that case] Then things get more complicated. I mean, the curriculum itself determines our framework. I don’t think I have the authority to go beyond that.” and she continued, “I definitely follow the textbook . . . I mean it is the limit. Because I think I don’t have the authority to change anything, to be honest. Are there things we criticize? Yes. But if I don’t have the authority to change what I criticize, I don’t want to teach differently and confuse the children.”

Other than curriculum and resources, the entrance exams and parental expectations regarding preparation for these set a limitation on proof-oriented lessons. Students’ success in high-stakes exams is a primary target for parents and their children. Students have instrumental goals rather than mastery goals for learning mathematics. It is rather challenging to convince parents and students of the necessity of proof. Hence, students cannot be given the opportunity to explore. Rather, they are asked to prove already proven statements when they are presented with proof questions.

University students and teacher candidates are not provided courses emphasizing proof in their mathematics courses. One such criticism was that geometry proofs were not covered in college courses. Also, TE-2 stated that teacher

candidates were offered courses that focused on the authority in the learning environment, criteria for the truthfulness of ideas, justification, socio-mathematical norms, and Common Core standards, but not proofs specifically. These observations also reflected the interviewee's conceptualization of proof-related activities as they resemble constructivist mathematics education: Proving can be considered a constructivist act because the individual builds her knowledge on her existing knowledge repertoire (Driscoll, 2014; Schunk, 2012).

3.2.2.2 Proof-related skills to be learned

When participants were asked to identify skills related to proof that can be taught to learners to prepare them to prove, they listed many ideas that could be fruitful while planning instructional designs for improving proof-related skills. Convincing others and mathematical language were given as important skills by M1: “. . . you are describing something; you have to tell the other party what you are describing.

Explaining that proof is, above all, a spelling/writing beyond mathematics, I think, is very important because it is a language in which you can express your own ideas to your reader mathematically. In fact, it is the whole of mathematics, but in the proof, when you argue that, when you have a debate, your text to defend an idea is that proof. The better you write that text in a way that can be understood by everyone, the more convincing it will be. This is to give this to the other side. Actually, I think, to explain that mathematics and literature are so intertwined. Because mathematics is a language, and if we do not express a language with the right words, what we want to say will not be understood correctly, and what we do will not be correct.”, and by M2 “Mathematics, in particular, is really a language and we are trying to teach that

language. Language also has rules; they need to learn the rules so that they can use it correctly.”, “. . . he has to be able to convey his thoughts to the reader.”

Not only knowing but also internalizing the logic rules also suggested by M1 and TE1: “I mean, I can’t say that she can’t answer the questions about logic, answering the questions is not the same as understanding the subject” and “Knowing the rules of logic, while explaining logic, they make truth tables: p implies q , q implies p , etc. It seems like a simple matter, but they don’t think much about how they use them while proving later on. Logic and proof methods should be associated more”.

Thinking about rationales between steps and argumentation skills were suggested by M1: “Some of them just order the steps. It is very difficult to read it. Especially when you have divided the proof into parts; if you do not explain to the other party, your reader, why you are doing these steps, and if you do not give a preliminary preparation, then the proof is incomplete. All of a sudden, you are describing the characters; you are not describing how the characters relate to each other. So it should be set up like a novel. If it is constructed correctly, the proof is done correctly. Therefore, they should never forget the other side; they write it themselves, but they have to write it by thinking about what someone else will understand when he reads it. The person who made it does a perfect proof.”

Problem-solving and developing and following strategies for the solution were suggested by TE1 as an important skill related to proof: “They both encompass following a logical path. Student needs to look back to the solution after identifying a strategy. Proof steps are also similar to that. I see proving like problem-solving.”.

Establishing cause-effect relationships, understanding the starting point of the presented ideas, deducing things from given information, and building connections

between pieces of information were stated by MT1. He and MT2 also pointed out the importance of analysis and synthesis skills for proving.

Other than some teachable and learnable skills some characteristics were also stated as important for learning proof. These were about talent, predisposition, and curiosity. MT4 shared her thoughts as: “But I expect everyone to know, for example, multiplication, addition, subtraction, and division, but can we expect everyone to be able to do everything in their proofs, professionally speaking, it may be a bit utopian at this grade [middle school] level.”

3.2.2.3 Challenges in teaching and learning mathematical proof

TE1 stated several teaching problems experienced by teachers and teacher candidates. First, she stated that neither teachers nor candidates consider using proof in their teaching practices. She expressed: “Especially in middle school, they (primary school mathematics teacher candidates) do not have such a thing [teaching with proof]. Not so many in high school, too. It may appear only in the form of deducing some simple rules.” They treat proof as a subject, not as a process that needs to be prevalent in all contents. TE1 explained this as: “As if it were something additional. When you include proof-oriented skills, your teaching will be easier and more permanent.” Teachers complain about the intensity of the curriculum (as an excuse not to include proof in lectures), but students learn a few at the end of the term. They do not see the proof necessary and doable in the class. They judge their proof skills as insufficient.

The curriculum does not give place to proof either in middle or in high school. MT2 expressed that only International Baccalaureate (IB) programs in some private schools provide opportunities for proving. TE1 touched on this subject as: “In

one of the previous curricula, proof concepts were given, but I believe they were excluded due to its abstract nature, which was thought to be difficult for students.”. These indicate the lack of proof content provided in the prevalent curriculum.

MT1 specified that in order to implement decisions to include proof practices in the classroom, teachers should be trained first because they may not be ready to teach proof when we talk about developing proof skills. They also find proving difficult for themselves. Another factor was shown as the high school entrance exam. MT1 indicates this as: “. . . to do this, there should not be the exam that we call LGS.” TE1 also supported that: “Unfortunately, even in the best schools, it is a bit exam-oriented, you know, test-oriented . . .” Their problems, such as preparing the students for the exam, raising the subject due to the expectations of the parents of the students, and preparation for the exam, come to the fore. Don’t they ever do it [proof]? There are lessons made, there are teachers who do. On the other hand, students are always waiting to see different types of questions. No matter how high the student capacity is and no matter how curious he is.” The exam puts lots of anxiety on schools, teachers, and parents. In private schools, teachers hesitate to take the initiative to teach concepts with proof because parents’ expectancies are towards the entrance exam preparation and testing skills, as suggested by MT2: “Middle school teachers might have a hard time because it’s the parent’s expectation. However, it is not so easy for the Ministry of National Education to assemble it very well, otherwise it is not so easy for them to try to do it on their own because they think about the money they gave to the school and expect things in return.”

Presented proofs are not given with all the required explanations in between proof steps, as clarified by M1: “. . . we do not write the words that the reader can understand. I think they are the explanations. Those explanations are necessary for

you to understand it and internalize that proof. That's the persuasive part. So I think we don't just write the proof for experts." Proofs that explain is not prevalent in the sources they have.

Students are not equipped with proof-related skills in their previous academic years. In other words, they are not used to them. Almost all of the activities were on computations and following procedures in the available classroom practices. Lack of previous exposure was explained in the following excerpt by MT2: "... we used to teach with proofs and the concepts and proofs were very well formed when they come to high school. After that, the system changed. We continue that way now. So, of course, these students came from very different locations; their background are not in the form of proof, unfortunately. This impacts the students' proof capabilities and our instructional fluency very badly."

TE1 pointed out the missing skills of exploration in the proving process of students/ teacher candidates: They are suddenly expected to prove already proven statements. The obvious formulas or rules interfere with the necessity of proving: "Why do we want him to learn proof? Yes, this is how mathematicians do mathematics, but theorems never come in front of mathematicians to be proven. Mathematicians come up with those theorems themselves. I mean, we kind of seem to have taken away the beautiful, fun parts. When we say, "Let's prove it now", it is clear where he will end up. This doesn't seem interesting. It has already been proven; it has already been determined to be true. So what does he do there most? He examines the existing evidence and understands his steps. Then, when asked for a similar proof, he follows the same steps and proves it. This is also a skill that needs to be acquired, but it's like we're missing something. You know, in teaching methods courses, we say that students should "do mathematics" just like

mathematicians; doing mathematics is important. In actually doing mathematics, there is a discovery phase that we skip in math lessons, so it doesn't start with what I'm going to prove . . . It starts with what's going on here."

Lastly, it is stated that proofs are complicated by their nature. They are both new to learners and necessitate the monitoring and organizing of thinking. Proofs require a lot of cognitive demand (Stein & Lane, 1996; Wilhelm, 2014). Also, they are formed around the synthesis of pieces of relevant information, hence inadequate knowledge of the subject prevents successful proof constructions.

3.2.2.4 Teaching proof at middle school

All participants agreed on the teachability of proof at the middle school level, but they also indicated some conditions for successfully achieving this. M1 suggested: "I think it can be given [in middle school]; you can tell anything at any age, but it's important to figure out how to explain it. Don't explain logic as you did in university without boring him, but you can explain logic in the same way you covered it in ninth grade . . . It is up to us, educators, to find the method that can be used. I guess it is up to all of us." M1 continued to defend the teaching of proof at early ages: "It is about raising awareness, right? Even if the student doesn't understand, he will at least hear about it. I think we should not be afraid, let's not be afraid of getting it wrong. You can fix it. It's not such a terrible thing to correct a misunderstanding. We all go through a lot of misunderstandings in life. They should also be exposed to proofs from a young age. On accepting proofs in verbal expressions without symbolic/ algebraic elements." M1 clarified her thoughts as: ". . . But now there is this. You wouldn't write in a very famous article that odd plus odd is even. The cognitive level must be considered. But you do this in elementary school. Therefore,

it seems to be a matter of dressing for any occasion.” TE1 and MT3 also supported this argument as: “Proof in primary school might be something different than proof in secondary school. I think there must be other steps that get some students there before formal proof.” “There certainly is, but now the proof we teach at the university level and the proof we teach at the middle school level may not actually have the same format.”

In his explanations, MT1 approached the teaching of proof in middle school critically. He shared his hesitations about teaching proof to low achievers:

“Spreading it to the whole school like this, applying proof-based lectures in all school, we can further alienate students who have poor mathematical and thinking skills.” But he continued his argument with the possibility of successful integration of proof in middle school: “But sometimes it can be something like that, that the person who teaches proof can actually go down to that level [of students], and sometimes it can cause mathematics to be liked more. Therefore, we need to be very careful if we are going to apply the proof-based lessons; we need to pay attention to the developmental level of the child and the class.”

On the use of symbolic language and algebraic expressions, TE1 shared her views as they are not necessary at the middle school level, and students can be allowed to use verbal expressions and use daily life instances: “. . . without using the symbolic language too much in middle school, for example, in geometry, every square is a rectangle, but not every rectangle is a square and so on.” And “It’s too abstract, let’s throw it out, they can’t do it [on the existence of proof in the curriculum]. It sounds too abstract to students. This means that if we give these ideas before they actually start abstract mathematics, you don’t have to say that p implies q . The student can also construct sentences from daily life.”

Both mathematics teachers in this study mentioned that the grade level to start teaching proof ideas could be seventh or eighth grade. Their explanations relied on their own experiences in school, as MT2 stated that in her long teaching experiences, she had introduced proof in seventh-grade geometry classes and observed successful results when students began high school. Also, these teachers considered the developmental characteristics of students at those ages and the readiness of students in terms of conceptual knowledge they have. M1 commented: “If we think middle school, seventh grade and beyond, maybe seventh grade because after you get basic math skills, you can switch to proof ideas. When I say basic, what I mean is this: A student who can easily solve an equation can think a little more abstractly.”

The two-column proof was suggested to be suitable for initiating proof skills. MT2 stated that students get used to the structure of proof with the help of this form. Contrary to criticisms shared in the literature, she continued: “Yes, the more mechanical it is [on two-column proof structure], the better for that age. The child gets used to and starts to like proof with this. You can start there”. M1 shared an episode from her middle school years as her teacher introduced a proof-based geometry lesson in the form of two-column proof: “Our teacher made up something called “xyz” geometry. And he said the axioms of this geometry are these . . . There were statements written there that you can prove with these axioms. Proofs were constructed step-by-step, such as " definition, first axiom, second axiom, the statement I will deduce, finished" It's like we are starting to accept the hypothesis. Second, how I went from step one to step two because I have axiom 1. Here is how I got there: definition. How I got from there to there: axiom 3. Finally, he finishes the proof. So, you have proof with two columns, what you wrote in one, and the reasons

in the other.” She found presenting the idea of proof with this form very influential and long-lasting.

3.2.2.6 Suggestions to make proof a meaningful part of school curriculum

The participants were asked about how to teach proofs in schools better. They reflected on different ideas that they could incorporate into the classes. One such response was about practicing proof skills. Here, not only exposure to but also continuous practicing to proving was mentioned by M1 as: “I think giving a lot of homework is important at this point. I also have other teacher friends who think like this in my department. Therefore, the more we get them to practice in this lesson, the more we will improve their proof skills.” Similarly, M2 pointed out the importance of intense practice to better learn proof skills: “In order to get a good grasp of proof in mathematics, you actually need to solve a lot of questions, practice, and study regularly.”

All respondents highlighted the importance of sustained questioning, discussions, and asking “why” and “how” questions. M1 added that focusing on comprehending proof ideas and ownership of the proving process by students. M1 and MT2 suggested prioritizing geometry proofs because they demand simpler algebra knowledge and contain more visual elements. Then, students can have more opportunities to ask questions about a familiar topic.

In terms of exposure to proof skills, all interviewees agreed on the early introduction to proof ideas. One such suggestion by MT4 was: “Why are we waiting until nine [ninth grade] or why do we start with p and q in nine and scare people because first of all, there would be such p and q ’s, you know, those propositions: if, and, or . . . they were always taught it by memorizing. For example, we can use

weather forecasts from daily life: “If the air pressure is low, it will rain.” Students can follow these examples. That way, maybe we can prepare the ground for ninth grade logic . . . with a small introduction from middle school, from eighth grade, before making it formal and scaring the child. I would be very pleased on my own behalf.”

It was suggested that teachers need training (a PD, in-service training etc.) on proving and on teaching proof. MT1 specifically mentioned: “In my opinion, the teacher should be specially trained in this regard. When we say let’s apply the proof as it is, maybe they need to be trained, I don’t know.” TE2 also stated that, “They need guidance . . . I feel like we need to make them feel the need for it in in-service training . . .”

Participants suggested that in order to create change, there needs to be a long-term teaching plan starting from very early ages with systematic and holistic work. MT1 claimed: “This is not a skill that can be given once in a month. This is actually a combination of all these, starting from birth, what he ate and drank, what kind of games he played as a child, who he made friends with, and his social environment. All these trigger each other. Therefore, it is not something like: “Let me give you the following foundations so that you can learn how to prove”. I think the proof requires deep thinking, so we should start from primary school at a very early age. If you do not begin like this, and change your focus at the middle school, it becomes difficult when you say let’s start proving. We need to think as a whole.”

MT-1 stated that changes in the curriculum and the books are required. TE-1 also suggested that the number of subjects should be reduced, but the concepts should be taught more deeply in order to practice proof-related skills. This may allow the integration of proof skills into the lecture more effectively. Books are thought to

be the sources from which both teachers and students benefit. They might be aligned with the intention of including proof skills so that they can guide both teachers and students.

Before beginning their teaching practices, teacher candidates need to be equipped well in terms of teaching proof. Some respondents suggested providing more opportunities for studying proof. For instance, M1 suggested that geometry proofs and related exercises are incorporated into the Introduction to Mathematical Structures course, which is offered to freshmen at the university. TE1 also proposed to increase the number of pedagogy courses that focus on discussing math concepts with proofs and problem-solving skills.

MT2 articulated that teachers need to be convinced on the necessity and doability of proof-based instructions and be allowed some time to digest new concepts and regulations, if any. She stated that: “It is not pleasant to force teachers to take action.” and “. . . because they [MoNE] sometimes invited us to national education meetings. “You’ll do it, you’ll do that, we’ll come check it out” and stuff like that, when it starts like that, no one wants to do them anyway, and eventually all the efforts become useless.”

Overall, the findings regarding what preservice teachers know and believe in terms of proving and teaching proof, what current curricula, policy documents, discourses, and other relevant educational policy actions bring about, and what experts and practitioners in the mathematics and mathematics education field reflected on and suggested were examined in detail. The essential and manipulatable aspects were considered in the design ideas. In Chapter 4, findings regarding the design are detailed.

3.2.3 Findings from the knowledge test on proof

The test was administered to 305 preservice teachers from nine universities in Türkiye to respond to the research question “What are preservice teachers’ knowledge about proof and teaching proof?”

The first part of the test was on content knowledge of proof. The average score of participants in this part was 5.57 ($SD = 2.93$), where one could get a maximum of 10 points. In the first item (see Figure 3), participants were asked to construct a mathematical argument for a problem.

1. Item

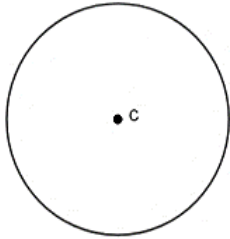
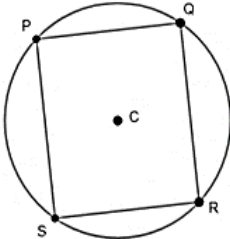
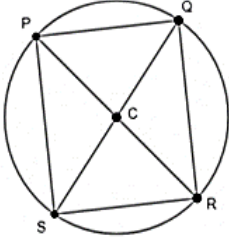
Deniz sketches a circle, and he calls the center C.	He then draws a quadrilateral PQRS, whose corners lie on the circle.	He then draws the diagonals of the quadrilateral.
		
Deniz says: “Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross at the center of the circle”.		
Is Deniz right?	YES / NO	
Explain your answer:		

Figure 3. The first item in the knowledge test

The majority of the participants’ explanations were correct but relied on empirical justifications. Approximately one-fourth of the explanations were false, and a small percentage of participants could only provide correct and formal explanations. Table 10 shows the scoring of the first item and the detailed classification of responses.

Figure 4 shows sample responses for the first item.

Table 10. Frequency of Scores for the First Item of Knowledge Test

Score	<i>f</i>	%
0	77	25
1	39	12.66
2	12	3.9
3	148	48.05
4	32	10.39

Note. i) 0 pt: The response is false, 1 pt: The response is correct.

ii) If the response is correct (in addition to 1 pt); 0 pt: No explanation/explanation is false, 1pt: The explanation is correct but relies only on repetition of authority, 2 pt: The explanation is correct and based on empirical justifications, 3 pt: The explanation is correct and based on formal justifications.

(1)	Deniz haklı çünkü köşegenler her türlü çemberin çapı oluyor ve çemberde ağırlıklı olarak sonsuz sayıda çapı merkezden geçmek zorundadır.
(2)	Bu durum dörtgenden dörtgene değişiklik gösterebilir. Bu nedenle daha fazla bilgiye ihtiyac duyuyoruz.
(3)	 Böyle bir dörtgen çizilirse Benzin fikrinin her zaman doğru olmadığı anlaşılır.
(4)	PQRS dörtgeninde karşılıklı açıların toplamı 180° olmalıdır. Çapı gören çevre açısı 90° dir. Bu nedenle eğer bu dörtgende tüm açılar 90° olursa köşegenler merkezde kesişir ve köşegenler çapa eşit olur. Eğer bir çevre açısı 90° olmazsa çizilen köşegen merkezden geçmek zorunda değildir.

Figure 4. Sample responses for the first item in the knowledge test

Response (1); “Deniz is right because the diagonals are the diameter of any circle, and the infinite number of diameters we draw in a circle must pass through the center,” shows that the statement was misunderstood: The property of diameters, of

passing through the center, was considered only. In response (2), “This may vary from quadrilateral to quadrilateral. This is why we need more information.”, it is seen that the respondent did not provide any explanation for the statement to be true or false. In response (3), “Drawing such a quadrilateral shows that Deniz’s idea is not always correct.”, an example was provided with a drawing, which can be considered as an empirical argument. In response (4), “In the PQRS quadrilateral, the sum of the opposite angles must be 180 degrees. The central angle facing the diameter is 90 degrees. Therefore, if all angles in this quadrilateral are 90 degrees, the diagonals will intersect at the center, and the diagonals will be equal to the diameter. If a central angle is not 90 degrees, the diagonal drawn does not have to pass through the center.” The conditions of the statement to be true and cases when the statement does not hold are explained with rules, theorems, and facts. Hence, it can be regarded as a formal justification.

One of the most frequent responses for the first item was, “If the polygon is not regular, then diagonals will not cross at the center.” The explanation was false; the rectangle can be such a quadrilateral. This response was observed frequently, and it reflects the misconceptions or misuse of the term “regular.”

In the second item, participants were required to provide proof for a given statement. Their responses were judged based on the existence of justification, the completeness of the argument/proof, the correctness of the assumptions, and the correctness of terms and concepts. The item can be seen in Figure 5 below.

2. Item

Prove the statement below.

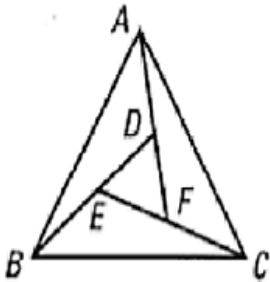
Given: $\triangle ABC$ is equilateral,	
$\angle CAD \cong \angle ABE \cong \angle BCF$	
Prove: $\triangle DEF$ is equilateral.	

Figure 5. The second item in the knowledge test

In this item, the majority of the participants got zero points, followed by those who got four points. Those who got the full score of five points were approximately a quarter of the participants. Those who could not get the full score from this item either could not provide justifications for each step, could not complete the proof, or misused the concepts or rules such as the side-angle-side (SAS) property of congruence where the congruent sides and angle were not specified. There were statements like “The sides opposite the congruent angle must be congruent.” Here, it was not clear which angles or which sides were meant. Some examples of the responses are shown in Figure 6.

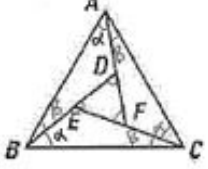
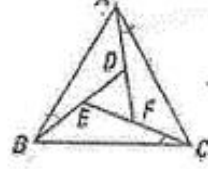
(1)	Verilenler: $\triangle ABC$ eşkenardır ve $\angle CAD = \angle ABE = \angle BCF$ İspat: $\begin{cases} D \text{ açısı} \\ E \text{ açısı} \\ F \text{ açısı} \end{cases} \alpha + \beta = 60^\circ \text{ dir.}$ İspatlanacak: $\triangle DEF$ eşkenardır.  Dolayısıyla $\triangle DEF$ eşkenardır.
(2)	 Bu bu örnekte eşkenar üçgen çizip $\angle ADB$, $\angle AFC$ ve $\angle BEC$ üçgeni keserek üst köşe koydum. ve açılarını eşitliğini gösterirdim.
(3)	Aşağıdaki ifadeyi ispatlayınız. Verilenler: $\triangle ABC$ eşkenardır ve $\angle CAD = \angle ABE = \angle BCF$ İspat: İspatlanacak: $\triangle DEF$ eşkenardır.  Bir kenarları eşit ve bir açısı eşit olan üç üçgenin diğer açıları da birbirine eşit diyebiliriz. Bu nedenden dolayı $\angle ADB = \angle AFC = \angle BEC$ açısı eşit olacaktır. Bu nedenden dolayı üçgenin dış açısı 360° olduğundan dolayı $a + a + a = 360$ $a = 120 \rightarrow$ Üçgenin iç açısı 60° olur. Bu açıya a derssek $\triangle DEF$ eşkenardır.
(4)	Aşağıdaki ifadeyi ispatlayınız. Verilenler: $\triangle ABC$ eşkenardır ve $\angle CAD = \angle ABE = \angle BCF$ İspat: $\triangle ABC$'nin eşkenar olduğunu ve verilen açıların aynı olduğunu biliyoruz. O halde aynı açılara a derssek, komşu açılarda $60 - a$ deriz çünkü eşkenar üçgenin bir açısı 60° dir. İki iç açının ölçüsü bir dış açısı eşit olacağından $\angle EDF = 60 - a + a = 60^\circ$ $m(\angle EDF) = 60 - a + a = 60^\circ$ $m(\angle DFE) = 60 - a + a = 60^\circ$ olur. Bu durumda $\triangle DEF$ üçgeninin tüm açıları 60° geldiğinden $\triangle DEF$ eşkenar olur. 

Figure 6. Sample responses for the second item of the knowledge test

In response (1), it is seen that the participant placed labels for the measures of angles (beta) and deduced other congruent angles (alpha) without stating the rationale.

Another deduction was made for the sum of the interior angles to reach the remote exterior angle without stating the reason. In response (2), the respondent claimed that

she would cut the triangles ADB, AFC, and BEC, put them on top of each other and show the congruency of the angles. In this explanation formal justification is missing, and it could not be considered complete proof. In response (3), the respondent claimed that if a side length and an angle are congruent, three triangles are congruent, which was a wrong mathematical argument. The respondent seemed to overgeneralize/mislearn what she had learned about the congruency rules for triangles. Then, she concludes this argument with the congruence of three exterior angles of triangle DEF. By then, she stated that as they are congruent and add up to 360 degrees, each should be 120, so all interior angles of triangle DEF are 60. Most of the explanations were missing in this response, and most importantly, it was based on wrong assumptions. In response (4), the respondent began with the given information of the item and, step by step, reached the desired solution by using relevant theorems and stating required justifications.

In the third item, the last item for CK, participants were given a multiple-choice question where two statements were given as converses of each other. The item required participants to know that a statement is not equivalent to its converse and that a counterexample is enough to refute a statement. Most of the participants selected the correct option: To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other. The item can be seen in Figure 7.

3. Item

Here are two statements.

- I. If a figure is a rectangle, then its diagonals bisect each other.
- II. If the diagonals of a figure bisect each other, the figure is a rectangle.

Which is correct?

- A) To prove I is true, it is enough to prove that II is true.
- B) To prove II is true, it is enough to prove that I is true.
- C) To prove II is true, it is enough to find one rectangle whose diagonals bisect each other.
- D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.
- E) None of A-D is correct.

Figure 7. The third item in the knowledge test

The second part of the test comprised items on pedagogical content knowledge of proof. The average score of participants in this part was 4.36 ($SD = 1.70$), where one could get a maximum of 10 points. In the first item of the pedagogical content knowledge part, which was also the fourth item of the test, participants were asked to provide students with questions that promote mathematical reasoning and build relationships between the geometric properties of figures mentioned in the question. The item can be seen in Figure 8. Analysis of the item was summarized in Table 11.

4. Item

QUESTION: "Nalan says that if a figure has four sides and the opposite sides are parallel, it must be a square. Do you agree or disagree with Nalan? You can use geometrical shapes in your answer" (adapted from NCTM, 2005, p.78)

The response given by Cem for this question was: "I do not agree with Nalan because a square has 4 equal sides and 2 pairs of opposite sides that are parallel."

What kind of questions should a teacher ask to Cem based on his response?

Figure 8. The fourth item in the knowledge test

Table 11. The Analysis of Item 4 of the Knowledge Test

Response Code	<i>f</i>
The questions are based on remembering rules and facts only (memorization) and do not promote student reasoning and justifications.	62
The questions emphasize the relationship between the properties of polygons but do not promote student reasoning and justifications.	183
The questions are built around the relationship between the properties of polygons, and they promote student reasoning and justifications.	63

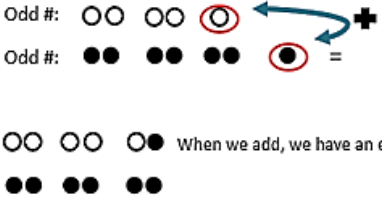
Most of the participants directed their attention toward the properties of quadrilaterals but overlooked mathematical reasoning in their responses. Only a small portion of the participants satisfied two conditions together. A lot of preservice teachers either could not promote reasoning and justification in their questions, could not provide a response, or their questions were just based on recalling some rules or facts. The definition provided by Nalan can be satisfied by a parallelogram, rectangle, square, and rhombus. Cem's opposition was on the selection of one of them based on a new assumption: Equal side lengths. In most of the responses, Cem's response was also found incorrect, but most reactions were towards Nalan. Also, rhombus was forgotten in most of the responses as an option in the given statement. Sample responses can be seen in Figure 9.

of the students regarding the properties of quadrilaterals. At the same time, she could help students to notice the problematic parts through proper questions.

In the fifth item, students were required to compare and evaluate different student arguments on the expression “The sum of two odd numbers is an even number.” They were asked to judge the correctness of each method and score them from highest to lowest (see Figure 10).

5. Item:

Evaluate the student arguments for proving the following expression: “The sum of two odd numbers is an even number”

A. Ayşe x is an odd number. y is an odd number. $2x+1$ ve $2y+1$ are odd numbers. $2x+1+2y+1=2(x+y+1)$ is an even number.	B. Burak When we group odd numbers by 2, we have 1 left. When we add two odd numbers, leftover ones will form a new group. That is why our sum will be an even number.
C. Ceren $3+3=6$ $3+5=8$ $3+7=10$ $5+5=10$	D. Derya All odd numbers can be written as $2n+1$ for every integer n. When we add two odd numbers, we get: $(2k+1)+(2m+1)=(2k+2m)+(1+1)$ $=2k+2m+2=2(k+m+1)$. This is an even number.
E. Emre Odd #: ○○ ○○ ○ Odd #: ●● ●● ● 	F. Fatma $1+3=4$ $19+91=110$ $1011+509=1620$ $1000001+7=1000008$

- Which student(s)' response(s) would you accept as correct?
- If this was an exam question, how would you order students' responses from the highest grade to the lowest grade?

Figure 10. The fifth item in the knowledge test

The majority of the participants accepted only some of the formal options (one or two options from the responses by Burak, Derya, Emre) as such one of them could not be perceived as proof. This was followed by those who accepted all formal options as correct. A considerable number of participants stated they would accept some formal and informal options together when they consider “the middle school level.” When they were asked to order these arguments, most of the responses showed that formal arguments were given higher grades. The formal responses without algebraic expressions (Burak, Emre) were evaluated skeptically. Some participants did not consider them as formal proofs, and some were accepted but put less emphasis on them. The pictorial proof was found to be confusing and incomprehensible by some participants. Also, the argument by Emre was considered as an empirical argument. Hence, it was treated similarly to the ones provided by Ceren and Fatma. An argument provided by Ayşe was formed differently from Derya’s as in the former one; the argument started with an assumption that x and y are odd by themselves. Only a small number of participants could capture this. Most of the participants viewed it as equivalent to another algebraic argument. Some sample responses can be seen in Figure 11.

(1)	a) Hangi öğrenci cevabını/cevaplarını doğru olarak kabul ederdiniz? → Eğin bunu en çok kabul seviyesine sunuyordu hepini kabul ederdim. → Lisan düzeyinde soruydum deriyati kabul ederdim. b) Bu bir sınav sorusu olsaydı en yüksek değerlendirmeden en düşük değerlendirmeye öğrenci cevaplarını nasıl sıralardınız? Derya > Ayşe > Burak > Emre > Fatma > Ceren
(2)	a) Hangi öğrenci cevabını/cevaplarını doğru olarak kabul ederdiniz? A, B, D, b) Bu bir sınav sorusu olsaydı en yüksek değerlendirmeden en düşük değerlendirmeye öğrenci cevaplarını nasıl sıralardınız? D > A > B > E > F > C
(3)	a) Hangi öğrenci cevabını/cevaplarını doğru olarak kabul ederdiniz? Ayşe, Derya, ~~Emre~~, Emre, Burak. b) Bu bir sınav sorusu olsaydı en yüksek değerlendirmeden en düşük değerlendirmeye öğrenci cevaplarını nasıl sıralardınız? Derya, Ayşe, Emre, Burak, Fatma, Ceren

Figure 11. Sample responses for the fifth item of the knowledge test

In response (1), the respondent stated that if the statement is asked to middle school students, then she would accept all of the answers. This means that informal and empirical arguments were accepted as the correct justification for the given statement. When asked to order the student arguments, she listed them to give fewer points to empirical arguments but still gave high scores to the argument that looked like a formal one: Ayşe. In response (2), the argument by Emre was not counted as proof. But at the same time, the respondent gave higher scores to Emre's argument than those by Fatma and Ceren (empirical arguments). This showed that she

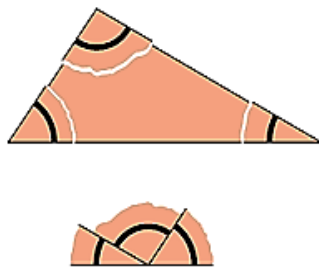
evaluated it as more formal but still relied on examples. In response (3), the most remarkable observation was considering the argument by student A as formal proof and assigning a high score for it.

In the sixth item, the last item for the PCK part, participants were given a formal and an informal argument for the sum of the interior angles of a triangle and were asked to explain the most suitable option for middle school students with rationales. The item can be seen in Figure 12.

6. Item

Identify the suitable method(s) for middle school students from the proofs developed for “The sum of the interior angles of a triangle is 180° ”. Explain the pros and cons of these methods.

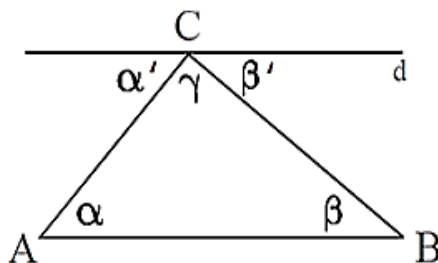
1. Method:



Cutting the angles of a triangle from the paper triangle, assembling the angles to form a straight angle and having students to explore that the sum of the interior angles of a triangle is 180° .

2. Method:

ABC is a triangle and α , β , and γ are the interior angles of this triangle. Hence, we will try to prove that $\alpha + \beta + \gamma = 180^\circ$.



Draw the line d , that is parallel to AB and passes from point C . Name the angles α' and β' as in the drawing. α and α' and also β and β' are congruent since they are alternate interior angles. Hence, α can be substituted for α' and β can be substituted for β' . Since d is a straight line, we get $\alpha + \beta + \gamma = 180^\circ$.

Figure 12. The sixth item in the knowledge test

In the majority of the responses, either the informal argument was favored over the formal one, or both were found to be equally valuable. Only a small percentage of participants favored the formal proof. The responses are summarized in Table 12.

Table 12. Distribution of Responses for the Sixth Item of the Knowledge Test

Response Code	Score	<i>f</i>
No answer/ A method was stated without a rationale/ No option was selected; only pros and cons are listed.	0	70
Informal argument is favored without considering proof skills.	1	169
Formal argument is favored (considering factors other than proof skills)	2	24
Formal argument is favored (considering proof skills)	3	45

Although the first method is not proof, just a demonstration, its simplicity, and the “fun” factor were shown as criteria in most of the responses to choose it as the most suitable method for middle schoolers. Some participants reported that cutting the angles and assembling them may not be perfect and may result in misinterpretations. For the second and formal method, participants stated the algebraic expressions as the obstacle for this argument to be understood by middle school students. Those who favored the second argument indicated that it is the one that could lead to generalizations and improve students’ proof skills in future years. Some also reported that the first method would require a lot of time; the second is advantageous in the sense that it is not that time-consuming. Some sample responses are shown in Figure 13.

(1)	→ 1. yöntemde mantıklı ve öğretici. 1. yöntem için görsel ve dokunmaya dayalı bir örnek olduğu için öğrencilerde kalıcılık artar. Aynı zamanda etkinlik yapmak onları heyecanlandırır. Tek dumsuza noktası ders içinde bunu yaptıkça süre sıkıntı olabilir. → 2. yöntem daha ispata dayalıdır. Yani işlemsel ispat için matematik kısmı için gereklidir. Ancak kâğıt kesme kadar öğrenciler eğlenmez bu da kalıcılığı etkileyebilir. Süre açısından bunu göstermek derste daha uygundur.
(2)	I. yöntem orta okul öğrencisi için uygunken II. yöntemi uygun olarak görmüyorum. Çünkü I. yöntem orta okul öğrencileri için daha somut bir ispat yöntemi sunuyor. I. yöntem ise daha soyut ancak bilinmeyen değişkenler daha en planda ve bu öğrencilerin kafasını karıştırabilir. Bazı öğrenciler bu yöntemi de anlayabilir belki ama sınıfta herkesi hitap ederken I. yöntemi gösterip ardından II. yöntemi de gösterebilir.
(3)	Her yöntemde etkili olabilir fakat matematiksel olarak 2. yöntem daha uygundur. Öğrencilerin abillorunda kalıcılığı da daha fazladır. Orta okul seviyesinde ki bir öğrenci 2. yöntemi verimlilik bir şekilde anlayabileceklerdir (7 ve 8. sınıf). Fakat 5 ve 6. sınıflar için 1. yöntemin daha etkili olabileceğini düşünüyorum. Uyguların görsel yollarından yutarak bahsedim. Faydalı yollar ve 1. yöntemde ki kesme ve birleştirme işlemi öğrenciler tam anlamıyla gerçekleştirilmezse kafa karışıklığı yaratacaktır. Aşının 100 derece olduğu konusunda kararsız kalacaktır.

Figure 13. Sample responses for the sixth item of the knowledge test

The response (1) can be translated as: “Both methods are logical and instructive. For the first method, since it is a visual and tactile example, students’ retention increases. At the same time, doing an activity excites them. The only drawback is that there

may be a shortage of time when doing this in the classroom. Method 2 is more proof-based. So it is good for the math part for procedural proof. However, students do not have as much fun as cutting paper, which may affect retention. In terms of time, it is more appropriate to show this in class.” In this explanation, factors related to fun, retention, and application time were put forward. In response (2), it is stated that: “While method 1 is suitable for middle school students, I do not see method 2 as suitable because method 1 offers a more concrete method of proof for middle school students. Method 2 is more abstract because unknown variables are at the forefront, which can confuse students. Some students may be able to understand this method as well, but while addressing everyone in the class, method 1 can be shown and then method 2 can be shown.” This explanation was based mostly on students’ cognitive development. The respondent believes that the formal proof is beyond most of the students’ cognitive skills, and it is very abstract for them. Algebraic expressions were stated as a reason for student confusion. The response (3) is translated as: “Both methods can be effective, but method 2 is mathematically more appropriate. It is also more retentive in students’ minds. A student at the middle school level will be able to understand method 2 efficiently (seventh and eighth grade). I have mentioned the strengths of the application above. The weakness is that if the student does not fully realize the cutting and combining process in the first method, he/she will experience confusion. He/she will be undecided about how many degrees the angle is.” This response was more proof-oriented and showed that the respondent held positive beliefs on the place of proof in the middle school level. The informal argument was overlooked because of its complex procedure rather than because it was a demonstration, not proof.

As the CK scores were higher than the PCK scores of participants, it was questioned whether this difference was statistically significant. Paired sample t-test was computed, and a statistically significant difference with a small practical effect size was found ($t = 3.511$, $p < .001$, $d = 2.824$).

3.2.4 Findings form the beliefs scale on proof

After obtaining the factor structure and the items for each factor/dimension, scores from the scale were calculated, and the average scores of dimensions were computed to be able to compare them with a diverse number of items. The results showed that the belief score for “the role and importance of proof” was highest among other dimensions ($M = 4.09$, $SD = 0.506$), showing that participants of the scale appreciated proof and its function in mathematics and mathematics education. They also had high scores in “the need for proof” ($M = 3.78$, $SD = 0.769$). The positive views on the necessity of proof and its teaching are important for this study. The opinions about the need for proof may prepare a ground to facilitate the inclusion of proof in school mathematics. Participants’ beliefs about being competent in skills related to proof present a positive picture ($M = 3.68$, $SD = 0.633$). The skills clustered in this dimension consist of generalizing, detecting patterns, knowing theorems, axioms and definitions, which are related to proof, rather than directly proving. On the other hand, participants had the lowest scores at beliefs on the “ability to do proof” ($M = 2.54$, $SD = 0.692$), which indicates that participants believed that they have poor proof construction skills. This was followed by the belief in “the place of proof in middle school” ($M = 3.37$, $SD = 0.567$). This pointed out participants’ hesitations in including proof in secondary school mathematics lessons. These findings suggest that the design should focus specifically on

increasing teachers' beliefs in their ability to do proofs and their ability to present proofs at the middle school level (see Table 13).

Table 13. Descriptives of Beliefs Scale for Each Dimension

	<i>M</i>	Min.	Max.	<i>SD</i>
Role and importance of proof	4.09	1.82	5	0.506
The need for proof	3.78	1	5	0.769
Mastery in proof related skills	3.68	1.33	5	0.633
Place of proof in middle school	3.37	1.36	4.64	0.567
Ability to do proof	2.54	1	4.60	0.692

3.2.5 Relationship between knowledge and beliefs on proof

To be able to respond to the research question: “Is there any relationship between teachers’/ preservice teachers’ knowledge and beliefs on proof?”, scores from the knowledge test and beliefs scale from the participants who filled out both surveys were analyzed together. 305 preservice teachers provided responses for both surveys. The correlation matrix is shown in Appendix K.

The total knowledge score of participants, total beliefs scores, and scores of the dimensions of beliefs scale were used to see whether a correlation exists between any two. The results showed that the total knowledge score was moderately correlated with the total belief score, belief in the role and importance of proof, and belief in the place of proof in school mathematics. A moderate correlation between CK and PCK was also found. Correlation coefficients with CK were found to be

higher than correlation coefficients with PCK. Beliefs on the ability to do proof had weak and negative correlations with two types of knowledge.

3.3 Discussion

3.3.1 Overview of the study and discussion of research findings

The study began with exploring current policy decisions and implementations that shape mathematics education in general and learning and teaching reasoning and proof in particular. To analyze the current situation regarding learning and teaching proof, experts in the field of mathematics and mathematics education were consulted. Later, teachers' knowledge and beliefs on proving and teaching proof were examined. The findings obtained before the implementation of the PRISM both informed the design of this study and shed light on the current situation of students, preservice teachers, and teachers regarding their proof competencies.

One way to understand the current state of proof and the teaching of proof for teachers, teacher candidates, and students at various levels of schooling is to examine the policy documents and policy discourse shaping mathematics education. The reflections of educational policies can be observed in curricula, teacher education programs, the content of exams in which students are assessed, teacher training and evaluations, and official statements to inform the public. In this study, these policy texts and discourses were analyzed.

The analyses revealed that although the middle school mathematics curriculum (MEB [MoNE], 2018a) does not directly address proof skills, related skills (such as logical thinking, drawing conclusions based on data, using mathematical language and terminology correctly, reasoning, expressing one's own reasoning, and evaluating the reasoning of others) are mentioned superficially. Proof

and reasoning related goals took part only in the introduction part of the curriculum text but were not reflected in the course objectives. The skill that was emphasized was problem-solving, but the role of proof and reasoning skills in problem-solving was disregarded even though it was very effective for developing problem-solving skills. In the high school curriculum (MEB [MoNE], 2018b), while logic is included in the ninth-grade program, there are no other objectives related to proof in the high school curriculum. This leads both teachers and students to see proof as an isolated subject rather than a skill that can be included in all subjects (Knuth, 2002). After 2018 curricula, middle school and high school curricula were changed in 2024. In the document introducing the middle school curriculum (MEB [MoNE], 2024a), higher-order thinking skills are included under the heading of conceptual skills. It is stated that the systematic, rational, analytical, coherent, and relational structure of mathematical thinking is taken into consideration for curriculum design: “In the program, which prioritizes students to have the skills that enable them to reach mathematical knowledge instead of acquiring more knowledge, and to be able to construct their new knowledge as a whole by questioning the relationships between the knowledge they have acquired, the content that supports only procedural knowledge and performance has been limited as much as possible. Important components of mathematical thinking such as problem-solving, conjecture, generalization, and verification are included in line with the holistic approach of the program.”

The 2024 middle school mathematics curriculum, unlike previous curricula, emphasizes mathematical reasoning very frequently. In the document of 2024 middle school mathematics curriculum, mathematical reasoning skill is explained as consisting of four main skills and their process components: analyzing, interpreting,

inferring, mathematical verification, and/or proof. It was stated that many mathematical skills such as pattern-seeking, generalization, estimation, making propositions, making use of different representations, and association were included in the context of the process components of reasoning skills. At the seventh grade level, it is seen that proof is directly included in the objectives of the topic of Algebra operations. There are many statements related to proof skills under the titles of enrichment and support for students in learning outcomes, learning-teaching practices: “Provides mathematical justifications for verifying the propositions by using the properties of the circle”, “Euclid’s book ‘Elements’ can be introduced and students can be asked to examine Euclid’s first proposition”, “A worksheet consisting of open-ended questions requiring the evaluation of different proofs of a proposition, the completion of incomplete proofs or the proof of a given proposition can be used”, “Students can be asked to research different proof methods of the Pythagorean relation and present the justifications for their conclusions”.

At the high school level (MEB [MoNE], 2024b), mathematical justification and proof skills are much more intense than in previous curricula. For most of the content, mathematical language and symbolism are planned to support the effective use of logic connectives and quantifiers. Unlike in the previous curriculum, the learning outcomes in the current program/ curriculum related to proof are not limited to ninth grade; proof skills are aimed at all high school grades. Since these positive developments regarding proof and reasoning skills in the current mathematics education curriculum are very recent, it is not yet possible to observe their reflections. These innovations in the curricula related to reasoning and proof should be studied and addressed in the future in terms of readiness, implementation, and the effects of the curriculum on learning and teaching mathematics and proof.

Teacher training programs in Faculties of Education are prepared in line with the policy decisions made by the Higher Education Council and conform to its principles. When the mathematics teaching courses and their aims in these programs were analyzed, it was observed that the courses directly related to proof and reasoning were quite limited for primary school and middle school mathematics teachers preparation programs. This situation is slightly different for students studying at the high school mathematics teaching programs. The plan was to provide more opportunities for proof and reasoning for those who would become high school math teachers. Another striking finding was that courses such as Euclidean geometry and geometry teaching were only given place in high school programs but not in middle school mathematics teacher education programs. However, geometry courses, by their very nature, provide a fertile ground for teaching and practicing proof (Martin et al., 2005). It is probable that not offering these opportunities to prospective middle school mathematics teachers negatively affect both their knowledge of and their approaches to proof and its teaching (Bieda, 2010; Stylianides & Ball, 2008).

Exams such as PISA and TIMMS, which are applied at the international level, address the current situation of students in certain age groups in a country and provide the opportunity to make comparisons with other countries. These international exams play an important role in shaping the educational policies of countries, having an impact on the quality and intensity of the courses that the students of that country will take in the following periods. This impact can often be realized through curriculum changes, teacher training, and teacher professional development. When the performances of students participating in TIMSS in Türkiye were evaluated in terms of reasoning, it was observed that they were below the

average of the participating countries. Similarly, PISA findings with groups of 15-year-old students showed that the students who took the test in Türkiye performed below the average among the participating countries both in general and in the “Reasoning and Proof” dimension. In addition to these international exams, a national test, ABIDE, was prepared and administered to differentiate students’ skill levels. The findings of this test were in line with the findings of TIMSS and PISA. All these findings suggest that action should be taken as soon as possible to improve students’ reasoning and proof skills.

Another area where we could see the impact of education policy on the learning and teaching of proof would be in teacher evaluations and in-service training opportunities offered to teachers. In Türkiye, teacher evaluations are conducted for teacher appointments or promotions rather than monitoring teachers’ performance. Although mastery of the field is at the forefront of these evaluations, knowing or being able to teach reasoning and proof has never been taken into account sufficiently. For this reason, national data to show the proof knowledge and proof teaching knowledge of new or prospective senior mathematics teachers could not be obtained. The only evidence for this comes from small-scale qualitative or quantitative studies by researchers.

The themes that emerged in the professional development programs mainly were on popular topics such as digital mathematics learning tools and STEM education. These topics are very important for teacher education to keep up to date. Reasoning and proof skills are as important as the given subjects. Furthermore, there is no indication that proof and reasoning are included in the topics covered in PD programs. On the other hand, proof and reasoning cannot be considered separately from the given subjects; they should be considered as an integral part of all subjects

(Bieda, 2010; Boyle et al., 2015). When considered and handled separately, proof and reasoning skills are considered more abstract and as an isolated subject rather than a process skill. This is seen as one of the most important obstacles to teaching proof (Knuth, 2002).

The plans and the policy discourse regarding mathematics education by the Minister of National Education and other state officials involved in education policy regarding mathematics education are also important indicators for understanding education policy. In the statements made by the minister of the period on mathematics education, it was stated that the purpose of the Initiative of Mathematics Mobilization (Matematik Seferberliği) was to improve students' ability to use mathematics in daily life. In this study, which emphasizes the place of mathematics education in children's lives and mathematical literacy, it is aimed to overcome students' deficiencies in mathematics subjects with a learning platform (Mathematics Digital Education Platform) and to facilitate students' inquiry into mathematics with interactive content. In fact, proof and reasoning skills could have found a place in line with these goals. Because skills such as critical thinking, looking for a generalization in the light of existing knowledge, identifying the structure and pattern in a problem situation, making a conjecture based on this structure, and testing the conjectures cannot be considered only in the context of reasoning and proof, they should also be considered as skills and actions that need to be intensively involved in mathematics and real life (Almeida, 2010; Ball et al., 2002; Staples & Bartlo, 2010). However, none of the points mentioned in the explanations and the content of the learning platform specifically addressed these skills. Policy decisions and actions are reflected in classroom practices (Trowler, 1999), so when the emphasis on proof and

reasoning is missing in policy texts and decisions, teachers may fail to include them in their instructions naturally.

Experts in the fields of mathematics and mathematics education were consulted in order to compile the experiences, conditions, and problems of students and teachers at different levels of education regarding proof and its teaching. The experiences and evaluations of experts in different fields are important in terms of bringing a holistic, permanent, and meaningful solution to a problem and creating more effective designs by taking different perspectives into account.

In this study, interviews with mathematics educators and mathematicians, as well as mathematics teachers, provided us with information on both students' learning from primary school to university and the instructional experiences of prospective teachers to university professors. The similarities as well as the differences between these groups came to the fore. The common themes observed for different groups are important for recognizing recurring problems and intervening in them at an early stage. On the other hand, issues about learning proof at different levels may show a variety of situations. For example, some issues that change over time, disappear, or come to the agenda for the first time become more meaningful when evaluated in a certain process. It was observed that all experts had positive views about the importance of teaching proof. Their experiences with proof seemed to affect their views and beliefs about proof and its teaching. They generally agreed on the problems stated in the literature and drew on the experience of their own expertise to come up with ideas on how to solve them. When experts were asked how much they included proof in their teaching, their responses were useful in better understanding their experiences and how this framed their responses to other questions in the interview. For mathematicians, proof is an essential element of their

work, whereas, for mathematics educators, proof is considered more in some courses. For mathematics teachers, it was expressed as a skill that was not included in the intended learning outcomes much. The intensity of the curriculum both in higher education and in middle and high schools was cited as a reason. The experts who presented this rationale for themselves sometimes emphatically offered it for teachers and prospective teachers as well. Similarly, when asked how they would feel if proof was a compulsory part of the mathematics curriculum, they reflected their own past experiences in their answers. Therefore, in order for teachers and preservice teachers to be more prepared for teaching proof, it is very important to take into account their past negative experiences, if any, and to provide them with proof learning and teaching experiences that will make them neglect these negative experiences (Lin et al., 2012; Stylianides & Ball, 2008).

The answers given by the experts to the question about students' proof skills are in line with the problems frequently encountered in the literature. When students of all age groups were expected to construct proofs, the result was expressed as that these students did not know how to do proof (Balacheff, 1991). Students are unfamiliar with proof (Furinghetti & Morselli, 2009; Mingus & Grassl, 1999), attempt to prove mostly through examples and trial and error (Balacheff, 1991; Bieda & Lepak, 2014; Heinze & Reiss, 2003), tend to imitate or memorize proofs (Ball et al., 2002), and try to follow the procedure without understanding its logic (Ball et al., 2002; Heinze & Reiss, 2003).

Another dimension that shaped the experts' views on the current situation was curricula. It was argued that the mathematics education curriculum is insufficient in emphasizing proof and reasoning. Even in the high school mathematics curriculum, where proof is included, it is seen as a concept that is taught only at the beginning of

high school, not associated with any subject afterward, and presented not as a skill but as an isolated subject. The fact that proof, which is left to be taught later, has no continuity and is more challenging in terms of cognitive load than other skills, is not properly included in the curriculum negatively affects teacher practices. When the intensity of the curriculum is added to this situation, teachers' not including proof in their teaching has started to be justified on the one hand. However, the proof process involves not only writing proofs as a product but also includes many cognitive skills that will lead to proof (Almeida, 2010; Cyr, 2011; Maher, 2010; Stylianides & Stylianides, 2008). These skills are quite possible during any subject or acquisition (Mingus & Grassl, 1999). All these approaches again stem from the fact that proof is not seen as a process but as a subject (Knuth, 2002).

Another issue that experts agree on is that teachers do not have enough knowledge to teach proof. For this reason, teachers avoid taking the initiative to teach proof. This situation brought up the issue of teacher autonomy in the classroom during the interviews. It is not clear to what extent teachers have authority over whether they can apply the teaching method they wish within the framework of the curriculum during the lesson. Even though it is thought that there will be room for proof practices provided that they do not deviate from the learning outcomes, the fact that the school administration and parents are more involved in the teaching process than usual creates a real obstacle for some teachers. At the same time, for students who are in the preparation period for high school, spending time in mathematics lessons on proof skills that will not be included in the exam is seen as a waste of time. Moreover, even in scenarios where the teacher is the only absolute authority in the classroom, there are teachers who do not feel competent or free to go beyond the curriculum or textbooks. In light of these findings, what needs to be done is to clarify

more clearly what teachers understand from proof practices. For those who conceptualize proof in a more limited framework, it should be stated that the elements that prepare for proof in many dimensions are also important in this process. Then, they should be made to see that they can do more proof-oriented work than they had anticipated within the framework of the curriculum. Another important point is the need to strengthen teachers and preservice teachers in terms of content knowledge and pedagogical content knowledge. In this context, the experts in this study emphasized the lack of resources that teachers may need for teaching proof.

Identifying the skills related to proof is very important for determining the conceptualization of proof. When experts were asked about the skills related to proof, they mentioned using the language of mathematics correctly, convincing others, justifying each step mathematically, following the rules of logic, determining and following a strategy, and establishing cause-and-effect relationships (Almeida, 2010; Ball et al., 2003; Cyr, 2011; Knuth et al., 2002; Maher, 2010; Stylianides & Stylianides, 2008). Moreover, the compilation of these skills is valuable in terms of providing a framework for an intervention or program to be designed.

Another point that all experts agreed on was the idea that middle school students should be at a level where they can learn proof. What needs to be done for this is to introduce students to proof as early as possible, taking into account their knowledge and cognitive levels. What should be noted here is that the practices of proof at the secondary school and university level should differ. This should be done without going beyond the nature of proof and mathematics. For example, for students who do not know algebra, it is possible to follow the other steps of the proof appropriately without resorting to algebraic expressions. Geometry class has been shown to be a fertile ground for introducing proof ideas (Martin et al., 2005).

Moreover, the two-column proof format has also been one of the suggestions by experts to improve students' proof skills. In the literature, there are those who recommend the use of two-column proofs (e.g., Martin & McCrone, 2003) as well as those who claim that their use would hinder the learning of proofs (Ball et al., 2002; Cirillo, 2009; Schoenfeld, 1986; Sears & Chavez, 2014). While the proponents of this structure state that it is an important tool in constructing proof steps, the opponents claim that this structure reduces the proof to a form and distracts from the essence of the proof (Ball et al., 2002; Cirillo, 2009). The experts in this study, like the first group, did not think that it had a negative effect; in fact, they considered it as a scaffold that would facilitate the proof process and would be abandoned over time and replaced by paragraph proofs.

The last dimension compiled from the experts' opinions was the practices suggested to make proof an indispensable part of mathematics education and to teach proof better. The most prominent view was that proof skills should be practiced a lot and that students should be exposed to proof-related applications more frequently. This is a behavioristic approach, based on the principle of frequent repetition to reinforce newly learned skills (Schunk, 2012). In the acquisition of cognitively demanding skills such as proof, repetition can make a significant difference for learners. Another practice was "asking questions". It has been argued that every piece of information learned and the steps of every procedure followed should be accompanied by why and how questions. This practice is very important as it will strengthen the skill of justification, which is an important skill for proof. Moreover, asking questions should not be considered as an action to be performed only by the teacher. Students should also internalize questioning and learn to ask questions to themselves and their friends in the process of learning mathematics and proof. This

can be realized in learning environments where data-based justifications are made by the mathematics teacher, where the authority in the classroom is not in the individual but in mathematics itself (Balacheff, 1991; Sears & Chavez, 2014).

The opinions that comprehensive changes are needed to realize proof education at the secondary school level in a good way were quite prominent. These changes include the scope and content of the curriculum, in-service trainings and professional development opportunities in which teachers should be more involved, the content of preservice teacher trainings, and the need for a long-term action plan. Many of these suggestions can be considered and evaluated in the context of educational policies.

Teaching proof necessitates knowledge of proving and knowledge of ways to implement proof practices in the classroom (Ball et al., 2005). These two broad knowledge types are represented as content knowledge and pedagogical content knowledge in proof (Corleis et al., 2008; Lesseig, 2016; Stylianides & Ball, 2008). Content knowledge in proof encompasses comprehension of theorems, knowledge of theorems, formulating a formal equation, evaluating a preliminary pre-formal proof, assessing the subject-specific sufficiency of a formal proof, and carrying out a verified proof (Corleis et al., 2008). The pedagogical content knowledge of the participants showed that teacher candidates have difficulties in prompting questions that aim for reasoning and proof as such they direct questions that target conceptual or procedural understanding only. This situation was observed in the relevant literature as well (Stylianides & Ball, 2008). Teacher candidates also need assistance to distinguish students' arguments as being proof or non-proof. They treated generic examples as non-proofs (Lesseig, 2016; Lin et al., 2012), some relied on algebraic expressions, and concluded that arguments containing those expressions are formal

proofs (Lesseig, 2016). Almost all of the participants evaluated empirical arguments as non-proofs as suggested by the relevant literature (Harel & Sowder, 2007; Kuchemann & Hoyles, 2009; Martin & Harel, 1989). On the other hand, in one example most of the participants showed a tendency to accept or practice empirical arguments for justification at the middle school level. This finding leads us to two important conclusions: Not only in proof constructions but also in proof evaluations the proof schemes may vary; they are contextual (Harel & Sowder, 1998). Furthermore, factors other than “mathematics” itself, like being simple and fun and requiring less implementation time, can be stated as the reasons for orienting to informal arguments. Participants showed poor performance in differentiating between proof and demonstration (Lesseig, 2016; Stylianides & Ball, 2008). What students can understand and achieve in mathematics (or specifically in geometry) classes was underestimated (Furinghetti & Morselli, 2009).

The overall result of the knowledge test indicated that participants who completed the test had better scores in content knowledge than pedagogical content knowledge. This finding was not attained in previous studies. Based on the experiences of participants on proving, this was an expected outcome. They took some courses that involved proof construction, but they did not have an opportunity to think about how to implement proof and formal reasoning in mathematics classes.

Beliefs have an important impact on teaching practices, hence students’ proof competencies. That is why, exploring the current situation by examining beliefs on proving and teaching proof is essential (Frunghetti & Morselli, 2009; Pehkonen, 1994; Thompson, 1984). Investigating preservice teacher beliefs through the developed scale showed that they held positive beliefs on the role and importance of proof, which was a desired outcome in the literature. If teachers believe that proving

is an unnecessary endeavor, they would reflect those beliefs in their teaching practices (Bieda, 2010). It was pleasing to observe that teacher candidates achieved high scores on the need for proof as the beliefs on the necessity of proof bring about a positive attitude to give place proof in classroom implementations. While they held more positive beliefs on the mastery in proof-related skills (generalizing, seeking patterns, etc.), the lowest score was observed in the beliefs on the ability to do proof. This finding aligns with what the relevant literature suggests: Teachers and teacher candidates believe that their proof construction skills are underdeveloped (e.g., Knuth, 2002).

The correlation between knowledge and beliefs scores of participants showed that knowledge of proof is positively correlated with total beliefs score and dimensions of the beliefs survey except for beliefs on the ability to do proof. This shows an acknowledged relationship between knowledge and beliefs in the proof and teaching proof context (Corleis et al., 2008; Lin et al., 2012). Although the participants have some knowledge about proof, their weak belief in their ability to do proof can be explained by the difficulties of the proof practices they encountered at the university and their prejudices about being able to do proof. This may indicate the impact of prior experiences on the attitude and performance in proving and teaching proof (Buchholtz & Vollstedt, 2024; Lin et al., 2012; Mingus & Grassl, 1999).

3.3.2 Implications for practice

The scales used in this study were initially developed by the researcher because the data collection tools presented in the literature on proof knowledge and beliefs towards the proof scale did not meet the needs of the study. These scales, which were

developed in the light of the findings of research in the field, the ideas put forward by researchers who have worked in the field for many years, the theoretical frameworks they have developed, and the data obtained from expert opinions, can be used by mathematics education researchers. In this way, the data obtained from different participants in different time periods in future studies can lead to more reliable and comprehensive findings about preservice teachers' knowledge on proof and beliefs on proof. It will be valuable to examine whether the findings obtained from different countries will also differ culturally, and if so, to examine the reasons for the differences and to make comparative analyses.

Another finding of the study is that the current curriculum and exams should emphasize more on proof and reasoning skills in order to teach proof and include proof practices in the classroom. Since many teachers do not go beyond the curriculum and think that it is necessary to follow the directives set by the ministry one-to-one, changes towards a proof-oriented curriculum and exam content made by politicians will push teachers to make lesson practices accordingly. The recent changes in the mathematics education curricula will require intensive teacher training, which in turn will require education policymakers to engage in a well-planned and well-organized program change effort over a large period of time. This change will be a very logical move for raising individuals who can reason better, make more logical inferences, analyze current problems well, and develop solutions both in the short term and in the long term.

3.3.3 Limitations of the study

The data groups, data collection tools, and context of this study brought some limitations in the interpretation of the data. Teacher knowledge test on proof and

teacher belief scale on proof were collected from preservice teachers who agreed to participate in the study. Although the validity and reliability of these scales were ensured and reported, the findings obtained were evaluated within the framework of the answers provided by the participants. Even though the number of participants is sufficient for statistical analyses, there may be preservice teachers with different demographic characteristics, which has brought a slight limit to generalization. Similarly, the people contacted for expert opinions were selected according to the principle of convenience sampling. Since it was observed that the obtained data accumulated after a certain stage, interviews were not conducted with more people. Although the different findings reflect the differences in the opinions of the experts, it is also possible that there are experts with different perspectives. Therefore, the information presented is limited to the perspectives of the experts accessed. When developing the knowledge test on proof and belief scale on proof, the constructs were prepared in the broadest sense to ensure content validity, but the number of questions and items were reduced in order to prevent the participants from leaving the data collection tools halfway through, to prevent them from returning the scales without answering, and to collect data from as many people as possible, and to collect data without tiring and boring the participants. The fact that digital versions of the scales (prepared as online forms) were developed and used to collect data from preservice teachers in universities from different cities can be considered as another limitation. One of the limitations observed in the scales on the digital platform, which were prepared in exactly the same way as the written format, was that mathematical representations, which are easy to express when writing by hand but more difficult to express on a computer, tablet or phone, were tried to be expressed in a more verbal

way, taking care to preserve the meaning. Similarly, the participants' ability to use these technological devices and their writing habits may have shaped their answers.

3.3.4 Suggestions for future research

The data collection process carried out to determine preservice teachers' knowledge and beliefs on proof and its teaching can be collected from a wider sample of preservice teachers. Similarly, data can be collected from middle and high school mathematics teachers about their knowledge and beliefs on proof using the same scales. Moreover, teachers' knowledge and beliefs on proof can be evaluated together with their teaching practices. The data to be collected from teachers can be compared in terms of the institution they work in, their teaching experience, and whether they work at the middle school or high school level.

Evaluating the first reflections of the recently changed mathematics education curriculum can make an important contribution to the field. The implications of the new curriculum, which places more emphasis on proof and reasoning skills, on teachers' mathematics teaching practices, on their content knowledge and pedagogical content knowledge about proof, and on their beliefs about proof and its teaching, can be examined. Similarly, mathematicians and mathematics educators could be asked to evaluate the changing curriculum. They can be asked whether this change will have an impact on their own practice.

Quantitative data collection tools can be developed to determine students' knowledge of proof and their beliefs about proof. Except for the limited number of scales for students (Pesen, 2018) and the theoretical approaches expressed in the literature, qualitative studies have generally come to the fore. The findings of the knowledge and belief scales on proof to be developed and applied in parallel with the

scales applied to preservice teachers within the scope of this study will contribute a lot to the literature: It will help to evaluate both students and preservice teachers from the same perspective, a very comprehensive data will be obtained for proof learning and teaching. In this way, a large data set on the same dimensions can be obtained from a wide range of groups, from middle school students to preservice teachers and even teachers.

3.3.5 Use of preliminary findings in shaping the design study, PRISM

In the design, which was prepared for preservice mathematics teachers to learn how to construct proof better and acquire the knowledge to teach proof at the middle school level, preliminary studies that were reported to facilitate the understanding of the current situation were utilized in various ways. Proof skills, which were not included in the curriculum before, have become more visible with the new curriculum. This shows the importance of training teachers who can transfer and teach these skills. In order to teach proof skills to students, it is necessary to develop preservice teachers' content and pedagogical content knowledge about proof. Although the program by YÖK designed to train teachers is not implemented in the same way in every university, it can be considered as a reflection of what preservice teachers learn. Accordingly, preservice teachers should have more experience in learning and teaching proof.

The expert opinions in this study show that teachers and preservice teachers need to undergo comprehensive training in order to teach proof at the middle school level. Mathematics lessons need to be equipped with reasoning and proof skills from planning to implementation and then to evaluation processes. The preliminary findings showed that the preservice teachers needed to be intervened in two modules:

Proof content knowledge (e.g., rules of logic, proof methods) and pedagogical content knowledge (e.g., categorizing student skills, preparing and implementing proof tasks, teaching skills needed in classroom practice).

The findings used in the preparation of the content for developing CK for proof are as follows: The content of the courses offered to preservice teachers in their undergraduate programs is insufficient to develop their proof skills. In the interviews with the experts, it was found that students from all levels had difficulties with the rules of logic, and there were deficiencies in the mathematical arguments they constructed for proof. In the teacher knowledge test on proof, it was observed that the preservice teachers' knowledge of logic should be improved. The missing justifications in some of the participants' proofs are something that needs to be changed. It is necessary to teach valid forms of proof to preservice teachers who present and accept proofs only in narrative form. Since the preservice teachers' beliefs about their own ability to do proof in the teacher belief scale on proof showed a negative picture, it was aimed to change this negative situation by exposing them to diverse proof practices as suggested by the interviewed experts and as stated in the relevant literature.

With the interpretations of the findings of these preliminary studies, the CK of proof in the design study was shaped by the following considerations: Conjunctions, disjunctions, conditional and biconditional statements, and truth tables needed to be revisited. Comprehension of logically equivalent statements is critical, especially for distinguishing between a statement and its converse and using contrapositives of the statements in case it is difficult to proceed with direct proof. Understanding and formulating valid deductions are important. Organizing thoughts in proof requires special attention; intermediary steps should not be overlooked, and

the flow of the arguments needs to be built logically. Two-column proofs, flow proofs, and pictorial proofs should be seen as valid forms of proof.

According to the findings of the teacher knowledge test on proof, preservice teachers need to develop the ability to ask questions to encourage proof and reasoning. It was observed that preservice teachers do not have the necessary prior knowledge for mathematics tasks to be based on proof and reasoning. It was recognized that non-proof demonstrations were accepted as proof. There are two reasons for this situation: Preservice teachers may see proof as providing simpler justifications at the middle school level and expressing empirical arguments as proofs when necessary. Another explanation is that the preservice teachers may not know the difference between proof and demonstration. Therefore, their misconceptions about what it means to prove at the middle school level should be changed. It was seen that preservice teachers were not aware of the limitations of empirical arguments. In the expert opinions, it was concluded that preservice teachers did not sufficiently understand the need and importance of proof and teaching proof. In the teacher beliefs scale on proof, preservice teachers' scores on beliefs on students' ability to do proof and teaching proof in middle school were low.

The required practices for teaching proof were decided by using the findings of the preliminary studies, and a PCK module was prepared accordingly. Students and prospective teachers should know the need and importance of proof. Therefore, mathematics education standards related to proof and reasoning should be introduced and emphasized. The skills that prepare the ground for proof should be highlighted; thus, it should be shown that proof is not only a product but also a set of process skills. Preservice teachers should be convinced why empirical arguments based on examples are not proofs. It is important to evaluate whether an argument is proof or

not and to categorize student arguments in this respect. Preservice teachers should be taught proof schemes. They should be taught critical pedagogical moves to improve students' proof and reasoning skills, such as asking the right questions, creating proof and reasoning-based tasks, or modifying existing tasks into proof-based ones.

The characteristics of the design study for preservice teachers, which were informed by the findings of the preliminary studies, the literature, and the theoretical framework, were determined from a design perspective. The characteristics compiled in response to the question “What are the characteristics of an intervention program to develop preservice teachers’ teaching proof practices (belief/ knowledge/ attitude) aiming to promote students’ reasoning and proof skills?” are listed below:

- To prepare preservice teachers for proof, their content knowledge and pedagogical content knowledge on proof should be developed. Content knowledge should include knowledge of proof methods, rules of logic, proof forms, and subject matter knowledge. Pedagogical content knowledge should include knowledge of student approaches to teaching proof, knowledge of asking questions and preparing activities for proof and reasoning, and knowledge of practices to develop students' proof skills.
- Rules of logic and proof methods should not be considered only within the framework of mathematics or its branches. Examples from daily life are very influential in developing proof skills.
- In order to develop proof skills, it is necessary to be exposed to proof applications a lot. More repetition and more practice are important for better acquisition of proof skills.

- Proof practices should provide a platform for preservice teachers to discuss their ideas, justify and question all the arguments they put forward and encounter.
- The theoretical knowledge about proof and teaching proof should be supported by practical applications.
- Knowledge of proof should be developed in order for preservice teachers to have positive beliefs about proof.
- In order for preservice teachers to have positive beliefs about students' proof skills, good examples should be presented to them, and what needs to be done to improve students' proof skills should be shown step by step.
- It should be emphasized that proof and reasoning is not a subject but a process skill and should be demonstrated practically throughout the program.

CHAPTER 4

PART 2: A DESIGN STUDY

FOR DEVELOPING PRESERVICE TEACHERS' PROOF SKILLS

This chapter presents details of the methodology, findings, and the discussion of the findings on the design study for improving preservice mathematics teachers' knowledge, beliefs, and teaching practices of proof. In the "Methodology" section, the design of the study, the instruments used for the development and evaluation of the designed program, the participants involved in the study, the data collection procedure, data analysis, validity and reliability of the study, and the role of the researcher are detailed. In the "Findings" section, first, the implementation of the program is elaborated. Then, the impact of the program is presented through the analysis of data collection tools used in the study. Finally, in the "Discussion" section, the findings are evaluated in light of theoretical background and relevant literature. The implications of findings on practice, the limitations of the study, and the suggestions for future research are presented.

4.1 Methodology

4.1.1 Research design

In light of the findings of the preliminary studies presented in the previous chapter, a program (Proof, Reasoning, and Instructional Skills in Mathematics- PRISM) was designed for preservice teachers and implemented in two cycles. The results of these implementations were evaluated in terms of the effectiveness of the PRISM.

This study is a design-based research study that analyzes the current situation to solve a relevant problem for mathematics education, examines the situations that contribute to it, compiles suggestions for the problems encountered and for improving the current situation, designs a learning environment in line with these suggestions, and analyzes how to improve this design. Design-based research is a methodology developed to enhance the impact, transmission, and application of educational research to improve practice. It emphasizes the necessity for theoretical construction and the formulation of design principles that direct, inform, and enhance practice and research within educational settings (Anderson & Shattuck, 2012; McKenny & Reeves, 2018). In design-based research, the design is embedded inside the authentic educational context, and the intervention is initiated through an evaluation of the local learning environment, guided by relevant theories and literature. Usually, several data sources are utilized, and mixed methodologies are employed. The design undergoes numerous iterations to evaluate its effectiveness and identify areas for improvement. Collaboration between researchers and stakeholders is essential for developing design concepts (Anderson & Shattuck, 2012; Barab & Squire, 2004; Wang & Hannafin, 2005). Cobb et al. (2003) asserted that design experiments often involve both the "engineering" of certain learning forms and the systematic examination of those forms within the framework established by the methods used to support them. This created setting is subject to testing and revision, with successive iterations functioning similarly to systematic variation in experimentation.

This design study was conducted in two cycles. In design-based research, devised designs are implemented in various iterations in order to revise, refine, and advance the developed design (Anderson & Shattuk, 2012; Barab & Squire, 2004).

These iterations are specified differently in different research studies, such as phase, cycle, case, and site (Anderson & Shattuck, 2012). The cycles in the study were conducted on a weekly basis, one following the other. The work done in one cycle has guided the planning and implementation of the other cycle; hence, these two cycles affected each other. The diagram of the implemented cycles in the study is shown in Figure 14.

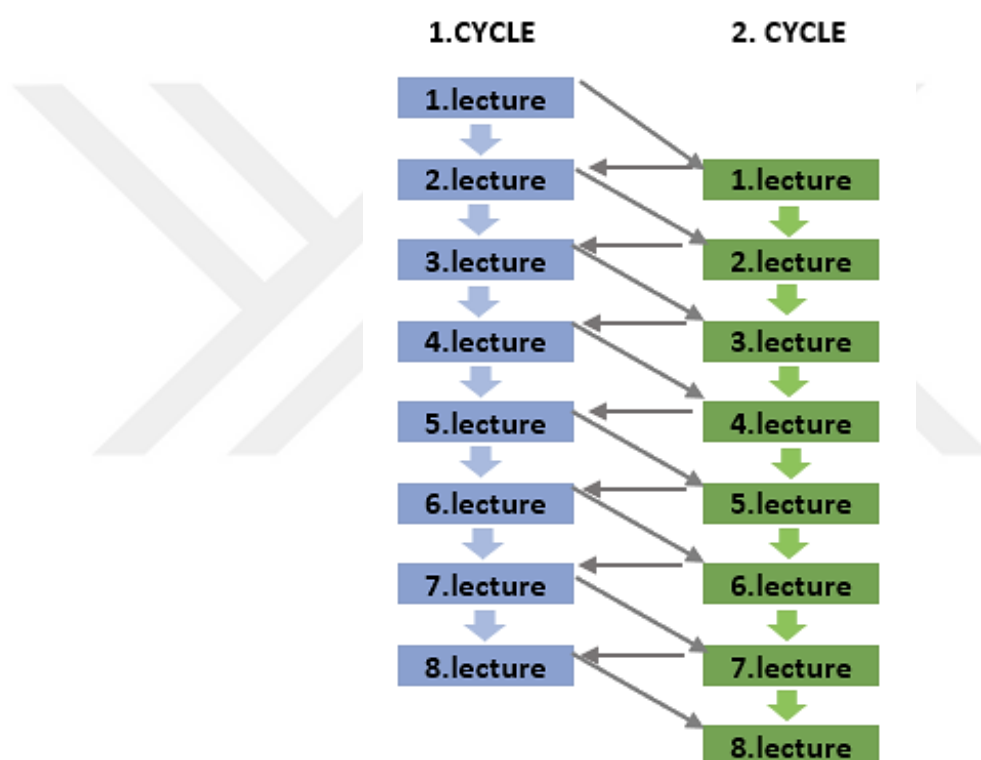


Figure 14. Cycle diagram in the study

As multiple data sources were used to present a complete picture of the studied phenomenon, the details of participants, data instruments, data collection, and data analyses are specified separately. The overall design of this study is summarized in Table 14.

Table 14. The Summary of Design Study

Research Focus	Data Sources	Data Collection Tools	Participants
The impact of the PRISM	The implemented design in 8 weeks	- Observations - Assignments and tasks - Interviews	- 54 preservice teachers (32 in the first and 22 in the second cycle) - 8 interviewees.

4.1.2 Instruments

Data collection for this design study was achieved through instruments as activities in PRISM and assignments and oral and written reports of the participants. The tools for the development of PRISM and the impact of PRISM are separately detailed below.

4.1.2.1 Sources for the designed program-PRISM

The designed program (Proof, Reasoning, and Instructional Skills in Mathematics-PRISM) was developed by using the key ideas in the relevant literature and the findings of the preliminary studies to foster knowledge of proof and develop positive beliefs for proving and teaching proof. PRISM consisted of two modules: (1) The logic and proof methods module and (2) the teaching proof and related skills module. The first module was composed of the skills of “drawing valid conclusions”, “understanding and applying logic rules”, “formulating conditional statements”, “understanding logically equivalent statements”, “sequencing logical arguments”, “understanding and applying direct proof”, “understanding and applying indirect proof methods”, and “understanding and flexibly using different forms of proof”. The second module consisted of activities for convincing why empirical arguments cannot be counted as proof, analyzing and classifying student arguments,

distinguishing proofs from non-proofs, prompting questions to develop proof and reasoning skills, and modifying tasks to promote proof and reasoning skills. For the logic and proof methods module of the program, geometry books that contained chapters on the nature of deductive reasoning, logic rules, and proof were used (Jacobs, 1986; Larson et al., 2007; Long et al., 2015; McGraw-Hill Education, 2005). For the teaching proof module of the program, implementations in the literature and ideas from a book written for preparing teachers to reasoning and proving were used (Arbaugh et al., 2019).

4.1.2.2 Data collection tools of PRISM

The data for the PRISM encompassed activities in the program, participants' written responses to exams, assignments/tasks, quizzes, interviews with several participants of the program and interactions of the participants with each other and with the researcher. The tasks and activities used during the program were also tools for evaluation of participants' learning and progress. While engaging with the tasks, participants' responses and interactions with each other were taken into consideration.

To analyze the impact of PRISM, participants' performances on the written materials (exams, quizzes, etc.) were used. The exams were developed by the instructor of the course; the items related to content of PRISM were used for the evaluation of participants' performances. Quizzes were designed parallel to the learned skills in PRISM to see the participants' progress and to detect problematic parts in order to overcome those. Two major assignments were given to participants: Proof task design assignment and teaching proof assignment. Participants' written reports on those assignments were used as data collection tools. While in the proof

task design assignment only the middle and high school curriculum objectives were used, in the teaching proof assignment, geometry proof tasks to direct middle or high school students were provided to participants from the relevant literature and geometry textbooks. The objectives of the proof task design assignment were chosen from geometry topics in the middle and high school mathematics curricula. A selected geometry objective in the curriculum was assigned to only one participant from each cycle.

The participants presented their work in the last week of the course before reporting the tasks they prepared. It was aimed to have diversity in the objectives and to cover geometry topics as comprehensively as possible. Lastly, semi-structured interviews with eight participants of PRISM were conducted for three purposes: (i) to learn what changed in the participants' knowledge and beliefs in proving and teaching proof from their viewpoints, (ii) to understand which elements of PRISM were useful and which were incomprehensible or complicated, and (iii) to differentiate the impact of PRISM from the impact of the course that the program took place in.

Data for the implementation of the designed program- PRISM were collected through audio records, observation notes, participants' written materials (quizzes, exams, assignments, and reports), and video records of interviews.

4.1.3 Participants

Participants of the design were participants of PRISM, participants of the proof task assignment, participants of the teaching proof practice, and interview participants. As some of the participants of PRISM did not contribute to other data collection instruments, participants in each one separately specified.

4.1.3.1 Participants of PRISM

Participants of PRISM were students who enrolled in the Teaching Geometry course in a public university in the 2023-2024 Spring semester. There were 54 students in two sections/cycles (32 and 22, respectively). The demographic information of participants is shown in Table 15. The course and, hence, the participants were selected based on the purpose of the study. Hence sampling was purposeful (Fraenkel et al., 2011).

Table 15. Demographic Information of the Participants in the Intervention Study

Section 1		Section 2		Total
Department	<i>f</i>	Department	<i>f</i>	<i>f</i>
Teaching elementary school	27	Teaching elementary school	19	46
Teaching high school	5	Teaching high school	3	8
Gender		Gender		
Female	25	Female	11	36
Male	7	Male	11	18
Total	32		22	54

4.1.3.2 Participants of the proof task assignment

All students taking the course and participating in the PRISM were required to prepare a proof and reasoning-based task based on a predetermined curriculum objective. 28 preservice teachers from the first cycle and 20 preservice teachers in the second cycle prepared a report for the proof task assignment.

4.1.3.3 Participants of the teaching proof practice

Students taking the course and participating in the PRISM were required to teach or ask at least one of the predetermined geometry proof tasks to a middle or high school student as part of the course requirement. 31 preservice teachers from the first cycle and 19 preservice teachers in the second cycle were the participants in the teaching proof practice.

4.1.3.4 Interview participants

Interviews at the end of the PRISM were conducted with eight female students who were actively participating in the program and attended all lectures of PRISM. The voluntary participants were chosen purposefully based on their performances during the designed program.

4.1.4 Data collection procedure

Based on the findings of the preliminary studies, a program- PRISM was designed for preservice mathematics teachers. It was planned to be immersed in a Teaching Geometry course during the Spring 2024 semester. Teaching Geometry course was selected for the following reasons:

- The course aims to empower preservice teachers in content and pedagogical content knowledge in geometry by focusing on proofs, justifications, and geometric constructions. In PRISM, content for developing the knowledge of proof was not limited to geometry; daily life examples and other domains of mathematics were also used to underline the proof skills of participants.
- The study was designed to be implemented in two cycles. The fact that the course would be taught in two sections was consistent with the design of the

study. Exposing students who were expected to learn the same topics to similar stimuli related to proof and reasoning was important to interpret the findings of the study in a more meaningful way and to reduce confounding variables. The same content is delivered, but one section follows the other in terms of timing. This situation shaped the design of this study; while two groups were given the same intervention sequentially, the observations and the implementations in the first cycle informed the second one. Similarly, what was learned in the second section was addressed again briefly in the first one.

- The instructor of the course agreed on such an intervention after looking at the plans. As preservice teachers struggle with geometry proofs and transferring proof skills into their future teaching practices, this intervention was considered an extra opportunity for improving preservice teachers' proof knowledge and skills.

The designed program- PRISM began in the third week of the semester and took 8 lectures for each cycle. Each lecture took one class hour per week. The second cycle had the first lecture in the fourth week, and throughout the intervention, the second cycle followed the first one a week later. As the PRISM comprises two parts (content knowledge and pedagogical content knowledge), the instructor of the course requested the completion of the CK part before the midterm in order to provide equal opportunities for the participants of the two sections. Hence, the third and fourth lectures for the second cycle were conducted in the same week. The timeline of the PRISM can be seen in Figure 15.

Date (2024)	Week	Course Content	Cycle 1	Cycle 2
Feb 13-14	1	Introduction to the course, preliminary activities		
Feb 20-21	2	Levels of geometric thinking, Review of basic logic rules		
Feb 27-28	3	Review of proof methods; axioms related to points, lines and planes	1 st lecture	
March 5-6	4	Measurement of distance, angles, measurement and congruency of angles	2 nd lecture	1 st lecture
March 12-13	5	Basic constructions using straightedge and compass	3 rd lecture	2 nd lecture
March 19-20	6	Constructions by folding papers making Constructions using linkage strips	4 th lecture	3 rd & 4 th lecture
March 26-27	7	Congruency in triangles and quadrilaterals Parallel lines and planes	MIDTERM	
April 2-3	8	Quadrilaterals	5 th lecture	
April 16-17	9	Circles, Area	6 th lecture	5 th lecture
April 25	10	Goals of Geometry curricula, NCTM standards; Geometric working spaces model and Paradigms	7 th lecture	6 th lecture
April 30	11	Similarity	8 th lecture	7 th lecture
May 7-8	12	Investigating 3D shapes and spatial reasoning		8 th lecture
May 14-15	13	Task assignment Presentations	END OF THE COURSE/ PRESENTATIONS	

Figure 15. The timeline of the PRISM

After the implementation of the program, the final exams, assignments, and reports of the participants were analyzed. Based on the overall performances of participants, eight preservice teachers were asked to be interviewed about the program and their development in knowledge and skills of proof. The interviews were scheduled based on the interviewees' availability and conducted online. Before the meeting began, participants' oral consent was requested and obtained. The interviews lasted approximately 30 minutes. The timeline of the design study can be seen in Table 16.

Table 16. The Timeline of the Design Study

Timeline	Activity
Oct 2023 - Dec 2023	Compiling the findings from the knowledge test, beliefs scale, policy analyses, and interviews with experts
Dec 2023 - May 2024	Preparation of the Designed Program- PRISM
Feb 2024 - May 2024	Implementation of the PRISM
May 2024 - June 2024	Interviews with participants of the PRISM
June 2024 - Sept 2024	Analyzing the data from the PRISM and writing down the report on findings and their interpretations

4.1.5 Data analysis

Research questions in the study and the multiple data sources necessitated different data analysis methods. The study consists of both quantitative and qualitative data analysis approaches. The audio-recorded interactions during the implementation of PRISM were transcribed and analyzed thematically. Quizzes, exams, and other written materials were analyzed through the developed rubrics and available analytical frameworks (e.g., Balacheff, 1988; Harel & Sowder, 1998). The proof task assignment was analyzed using two methods: Participants' expressions were analyzed through the emerged codes and themes. Also, the "Integrated framework for analyzing ways of reasoning" by Sevinc et al. (2022) was used to analyze the ways of reasoning promoted in the tasks produced by participants. The Integrated framework for analyzing ways of reasoning (Sevinc et al., 2022) was developed to analyze the modes of reasoning in mathematics textbooks. The framework had not been used before to analyze tasks developed by preservice teachers. Analyses for the proof teaching activity were realized through emerging codes and themes.

4.1.6 Trustworthiness

The trustworthiness of a research study is related to the methodologies employed to ensure validity and reliability (Patton, 2002). Specifically, the trustworthiness of a design research study pertains to the degree to which the conclusions and conjectures have been supported and rational (Gravemeijer & Cobb, 2006). Trustworthiness is examined using the four terms: Credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985).

4.1.6.4.1 Credibility

Credibility refers to the authenticity and veracity of findings in research, and it is related to internal validity (Lincoln & Guba, 1985). To ensure the credibility of the study, the researcher was involved in the study for a long time, triangulated the data sources and methods, made persistent observations, and described the study in detail (Lincoln & Guba, 1985; McKenney & Reeves, 2018).

This study used prolonged engagement to establish credibility (Lincoln & Guba, 1985). This technique entailed fostering sustained engagement with the participants, facilitating a more profound comprehension of their behaviors and dialogues (Cresswell & Miller, 2000). Additionally, the researcher's experience as a teaching assistant further contributed to the study's credibility. Data triangulation was also utilized to ensure credibility. The data triangulation procedure entailed cross-checking the findings using many data collection tools, such as observations, field notes, written documents, and recordings. This facilitated a thorough comprehension of the participants' reasoning and proving competencies and their progress throughout the PRISM. Rich descriptions entail giving thorough and contextualized explanations of the participants, procedures and study background (Shenton, 2004). The method section offers a thorough account of the planning,

implementation, and analysis processes. The incorporation of detailed explanations enables readers to comprehend the study's methodologies and the reasoning behind design choices. This study was conducted in two consecutive cycles, which made it possible to test the developed conjectures, which contributed to the improvement of the internal validity of the study (Bakker & van Eerde, 2015).

4.1.6.4.2 Transferability

Transferability is related to external validity (Lincoln & Guba, 1985) and generalizability (Bakker & van Eerde, 2015). In a design research study, "what is generalized is a way of interpreting and understanding specific cases that preserve their individual characteristics" (Gravemeijer & Cobb, 2006, p. 47). Hence, the insights obtained from this study can be utilized to analyze and illuminate other pertinent situations related to learning and teaching proof. The objective of a design-based research study is for the established local theory to serve as a reference for others seeking to adapt the design sequences to their own research and contexts (Gravemeijer & Cobb, 2006).

To address transferability, researchers must provide a "thick description" and detailed research context (Bakker & van Eerde, 2015; Gravemeijer & Cobb, 2006; Lincoln & Guba, 1985; McKenney & Reeves, 2018). These descriptions should entail participants' reactions, design processes and "failures and successes, procedures followed, the conceptual framework used, and the reasons for certain choices" (Bakker & van Eerde, 2015, p. 445). In this study, procedures and the progress of the study were detailed.

4.1.6.4.3 Dependability

The dependability is related to the consistency, replicability, and reliability of the findings in qualitative studies (Lincoln & Guba, 1985). It is provided through data triangulation, prolonged engagement in the field, peer review, and audit trials.

Gravemeijer and Cobb (2006) assert that design research studies do not seek replication since it is neither feasible nor desirable (Simon, 1995). They define ecological validity as the objective in design research, where modifications are implemented in different contexts depending on the interpretations of the study's findings (Gravemeijer & Cobb, 2006). One method to ascertain the reliability of a study may involve employing consistent design principles and doing several iterations in design research (McKenney & Reeves, 2018). This study was undertaken in two cycles regarding this matter. The same design principles were adhered to in both cycles.

Different and diverse data sources (observations, written materials, interviews, etc.) were used to address dependability and to examine the concepts and phenomena in the study in depth. In this way, triangulation was achieved. The study took place during a semester where the researcher was both a participant in the context and the instructor of the design. The researcher was a teaching assistant in the Teaching Geometry course, so she was already a natural member of the learning community before the design was implemented. Her observations accumulated and were shaped through a prolonged engagement in the field. Peer review was also utilized to increase the dependability of the study. The researcher consulted academics specialized in mathematics education and the dissertation advisor throughout the study. An audit trail was utilized where the researcher documented

memos throughout the data collection phase to capture the coding process, emphasizing the merged codes and their corresponding themes.

4.1.6.4.4 Confirmability

Confirmability is related to objectivity and neutrality. It addresses the question, "To what extent are the findings influenced by the participants and conditions rather than researcher bias?" (Lincoln & Guba, 1985; McKenney & Reeves, 2018). It can be provided through audio or video recording during data collection. The triangulation of data collection methods can enhance confirmability. Audit trails can also address the conformability of results (Lincoln & Guba, 1985; McKenney & Reeves, 2018). In this study, a variety of data collection tools was used. Design lectures and interviews were audio or video recorded. Transcripts of the interactions or expressions of participants were presented when necessary.

4.1.7 The role of the researcher

The researcher's beliefs and paradigm impact and guide the research (Denzin & Lincoln, 1994). The researcher of this study had several roles. While analyzing the written data or observing the participants' reactions, the researcher took a passive role. On the other hand, in the design part of the study, the researcher was an active participant as she was the designer and implementer of the PRISM while guiding the participants throughout the design process. In design research, researchers should be present in the design environment and immerse themselves in the study to better interpret the events and the outcomes of the design (Gravemeijer & Cobb, 2006). In accordance with the objectives of a design research study, the researcher followed the course from start to finish and worked alongside the instructor of the course. This

enabled her to establish rapport with the participants throughout the program and the course. As participants were students in a teaching geometry course, the researcher tried to support both the students and the lecturer during other course activities. This helped the researcher to become a natural member of the classes where the study took place.

4.2 Findings

This part consists of findings obtained throughout the lectures of PRISM, findings of the proof task design assignment, findings of the proof teaching practice of participants, and findings from the interviews with eight design participants.

4.2.1 Findings from the implementation of the designed program- PRISM

The designed program (Proof, Reasoning, and Instructional Skills in Mathematics- PRISM) consisted of two modules: (1) The logic and proof methods module and (2) the teaching proof and related skills module. The first module was composed of the skills of “drawing valid conclusions,” “understanding and applying logic rules,” “formulating conditional statements,” “understanding logically equivalent statements,” “sequencing logical arguments,” “understanding and applying direct proof,” “understanding and applying indirect proof methods,” and “understanding and flexibly using different forms of proof.” The second module consisted of activities for convincing why empirical arguments cannot be counted as proof, understanding and classifying student arguments, distinguishing proofs from arguments that are not proofs, prompting questions to foster proof and reasoning skills, and designing tasks regarding proof and reasoning skills.

The first module started with an exercise of drawing conclusions based on the given text. Both the literature and the preliminary findings of this study pointed out the necessity to emphasize this part, as preservice teachers sometimes make assumptions beyond the information provided and present unfounded justifications. In this study, preservice teachers were given a short story by J. R. R. Tolkien and presented some statements as conclusions. They were expected to mark these statements as “true,” “false,” or “not certain.” The purpose of the task was to understand and internalize the idea that “we need to use only the given information; we should not suppose ideas that are not present.” In the first cycle, the activity raised many debates among participants. The questions “How do we know that?” and “But can’t we understand/deduce this from the next sentence/paragraph?” were dominant. One problem with the task was the language of the passage. The participants were not very familiar with the world of J. R. R. Tolkien. This was foreseen during the design process, and translations were prepared at the end of the activity sheet. Some statements were discussed more than others. Learning from this experience, a number of changes were made in the second cycle for this task. First, the vocabulary list was revised and expanded. Second, the most controversial statements in the first cycle were identified and presented to the participants of the second cycle (even though they agreed on the correct answer). This was to provide more challenge and understand that the problems in the first cycle were due to the nature of the task or the dynamics of the learning environment. Language issues continued as in the first cycle, but overall performances improved with the help of revision ideas learned from the first cycle. There were intense discussions among the students and with the researcher in both implementations. The participants tried to convince each other, challenge each other, and adopt different perspectives.

In the second lecture, logical statements, their truth tables, and the law of detachment were briefly mentioned. From there, conditional statements were discussed, and the Euler diagram was introduced as a tool to understand conditional statements better. The most important purpose of this lecture was to attain skills for formulating statements in the if-then form since it was observed that preservice teachers struggled with how to start proving when the statement to be proved was not in the conditional form. It was thought that thinking in the if-then form would facilitate the formulation of direct proof, proof by contradiction, and proof by contrapositives. Statements were diversely selected so that different forms of statements could be practiced. Participants in the first cycle had problems converting some statements into the if-then form because they needed to think about them first in English and then in Turkish. Although equivalent statements were the topic of the next lecture, a short introduction was done by differentiating the meaning of statements of p implies q and q implies p (thinking of equivalences of converse statements) as most participants had trouble distinguishing them while formulating conditional statements. In the second cycle, the most problematic statements were questioned to see whether similar problems occur in this group, too. The participants of the second cycle were more successful than those of the first cycle as they discussed the formulation together. For instance, one of the statements in the task was “Susie eats pizza every Thursday.” One participant asked why “If Susie eats pizza, it is Thursday” is incorrect. She stated that when she thinks of it, that sounds very meaningful. One of the participants in the class responded to her in a very rational manner: “Susie can eat pizza any other day too.” This interaction helped them a lot to understand the if-then statements. The researcher repeatedly stated that there should not be any loss of meaning when converting the statement to the

conditional form. The importance of this idea became particularly apparent during the discussion of one example: “Use the stairs instead of the elevator in case of fire.” Participants’ discussions were like, “Do we have to indicate the elevator; wouldn’t it be sufficient to say if there is a fire using the stairs?” “Does the context matter where this conditional statement is used? Because for the people on the ground floor, we don’t need to indicate the elevator”. At the end of the discussions, participants were convinced that the message was not only about using stairs but also about avoiding the use of elevators in case of fire, so they formed the conditional statement without losing any meaning.

At the end of these two lectures, participants were given a quiz to see whether they could construct if-then statements and draw valid conclusions from the given text (see Appendix L). There were three statements to be converted into if-then form and five items based on a given text in the quiz. The average scores of the participants showed that the first group got 1.68 out of 3 points from the if-then statements; the second group got 2.6 points from the same items. Also, for the “drawing valid conclusions” items, the first group attained 3.39 out of 5, and the second group got 3.6 points. The problematic responses on the if-then part were due to keeping the quantifier (always, all, etc.) in the rewritten conditional statement and thinking about the converse instead of the original statement. The last issue came up when the converse of the statement was also meaningful and logical. The problems with “drawing valid conclusions” items were mainly about language problems (understanding an English text, lack of knowledge of some vocabulary) and about unnecessary assumptions. These problematic parts were revisited and discussed after the quiz in both cycles.

The module continued with the exercises of logically equivalent statements. Euler's diagram was used again to clearly show that the converse statements are not logically equivalent. The examples of the module required them to formulate inverse, converse, and contrapositive forms in the exercises. The purpose of these exercises was to help preservice teachers to gain flexibility in the conversion of logically equivalent statements and skills to avoid those that are not logically equivalent. The most frequently observed issue regarding this was the confusion between the original statement and its converse. In this study, in one item, the statement did not begin with "if" and asked for the formulation of the converse of the given statement. The item was: "You cannot comprehend geometry if you do not know how to reason deductively." Most participants in both cycles fell into the trap. Only a small number of them considered the reformulation of the original statement so that the change in the order would be correct in the converse form. Next, definitions were given as biconditional statements. This part was important as they could flexibly use the definitions in their proof constructions. In the following part of the logic module, understanding valid and invalid deductions was the theme. There, a story of a crime and the conditions of the two main suspects were given (see Figure 16). Participants were expected to detect the guilty and the innocent person based on the given information and applying logic rules. While participants are dealing with the problem, Euler's diagram was shown to build a strong connection between this representation and the logic rules: Euler's diagram presents the conditional statements in two sets such that one is the subset of the other.

Suppose that during a trial it is established that **whoever committed the crime is color-blind** and that **Mr. Black and Miss White are two of the main suspects**.

- If the prosecutor produces proof that Mr. Black is colorblind, must the jury conclude that he is guilty?
- If Miss White's lawyer proves that she is not color-blind, does it follow that she must be innocent?

Figure 16. An activity for understanding valid and invalid deductions

Next, the law of syllogism was presented when arguments with two premises were given. This law states that if “p implies q” and “q implies r” are true, then “p implies r” is also true. As this law was implicitly in some of their knowledge repertoires, like transitive property, they did not have difficulty comprehending this rule. The valid and invalid deductions based on the law of syllogism were discussed both in everyday language examples and geometry examples.

The following topic of the module was direct proof. The law of syllogism was reminded, and participants were directed to think about the extended version of this rule. The purpose of this activity was to help preservice teachers to develop the habit of arranging the arguments and their justifications in logical order as they were observed to have difficulties formulating the steps of direct proof in logical order. Daily life examples were asked to foster an understanding of direct proof. In this part, participants were also asked to rearrange given premises to formulate direct proof and reveal “what is to be proved” after concluding the order and following the law of syllogism. After a warm-up exercise, two challenging and multipurpose exercises were given (see Figure 17 and Figure 18). With a given statement, participants were expected to order the steps, but those steps were not directly seen; there were some contrapositives of the required steps. The purpose of this exercise

was to remind them that they may sometimes need the contrapositive of the statements and that it would be much easier to progress with contrapositives in some cases. After they got over the initial reaction of being stunned, participants of both cycles performed well in working with contrapositives and direct proof. Some participants in the second cycle suggested converting the given theorem to the contrapositive form while they were offered to think about switching the steps; their approach was very effective as it required less effort and captured the essence of the exercise.

Theorem.

If this is the last exercise, it is not easy.

Proof.

- If this is arranged in logical order, it is not the last exercise.
- If I can't understand an exercise, I get dizzy when I try to do it.
- If an exercise is easy, it does not give me trouble.
- If I get dizzy while trying to do an exercise, it is giving me trouble.
- If an exercise is not arranged in a logical order, I can't understand it.

Figure 17. The first activity of rearranging premises to reach proof

The next activity was more difficult, as it required both thinking of contrapositives and consisted of many steps. Using letters for conditional statements was suggested to facilitate their task. Although they experienced some difficulties, they comprehended the result and concluded correctly.

A driver makes the following statements about automobiles:

- A front-wheel drive gives a good hold on the road.
- A heavy car must have good brakes.
- Any powerful car is high-priced.
- Light cars do not have a good hold on the road.
- A low-powered car cannot have good brakes.

Is it logical for this driver to buy a cheap front-wheel drive? Show your proof.

Figure 18. Second activity of rearranging premises to reach proof

After the exercises of direct proof, participants were introduced to the story “Lady or Tiger” and given the following directions in Figure 19.

Many puzzles have been invented on this theme. In one of these puzzles, the king puts signs on the doors of the rooms, telling the young man that **one of the signs is true and one is false**. In which room is the lady?

A
IN THIS ROOM THERE
IS A LADY, AND IN
THE OTHER ROOM
THERE IS A TIGER.

B
IN ONE OF THESE ROOMS
THERE IS A LADY, AND
IN ONE OF THESE ROOMS
THERE IS A TIGER.

Figure 19. “Lady or tiger” activity for indirect reasoning

Participants were expected to use indirect reasoning by following only the given information. This activity and the following daily life example of using blood type for paternity tests were informative, and participants displayed good performances in those exercises. After one more geometry example, the indirect proof part of the module was completed.

Before the midterm exam, the logic module was finished with forms of proof (paragraph, flow, two-column), and strategies to successfully complete a proof were

shared. These strategies were planning, finding keywords in the given statement, marking the given information on the figure or drawing the figure if possible, working backward, paying attention to the additional information such as congruent, equal, parallel, etc., being careful not to miss any step, and being sure not to use any information that was not given or derived from definitions, postulates, or previously proved theorems. They followed this part very carefully, not because it was right before the exam but because proof strategies were explicitly and directly offered to them. Some of the responses in this instruction were “Can we use flow proof for our proofs?” and “Does it matter which format we choose?”. One of the aims of this instruction was that the format does not matter, but the logical steps and the completion of proof with the indication of necessary definitions, postulates, or previously proved theorems matter. Participants were assured in that respect, and as a result, some of them preferred two-column proofs in the exam, which was not their approach beforehand.

After the logic module, the midterm exam took place. In the exam, items related to proof constructions and deriving logical conclusions based on the given information were evaluated. The relevant items on the midterm exam are presented in Appendix M.

The participants’ responses showed that they still had issues in presenting rationales for every step. They failed to explain intermediary steps in their proofs. In some cases, they provided reasons without explaining how those reasons were obtained. For instance, some participants claimed to use the ASA (angle-side-angle) theorem in their arguments, but they did not indicate the congruence of which angles and which side. Also, some respondents had the tendency to think the converse of the statement instead of the original ones. Similarly, reaching out what was previously

assumed was observed in some cases. These misconceptions took place many times during the design and the course. Some of the participants successfully used two-column proofs (even though they were not specifically asked to use this form) in their proof attempts during the midterm exam.

After the midterm exam, the “Teaching proof and reasoning” module was initiated for the two groups; one followed the other as in the logic module. This module started with direct instruction on the importance of proof and reasoning and also the place of proof in important documents such as standards by the National Council of Teachers of Mathematics (NCTM) and Common Core State Standards-Mathematics (CCSSM) overview. The preliminary practices of proof, such as pattern-seeking, conjecturing, and writing explanatory proofs, presenting and discussing mathematical arguments, were specifically highlighted in order to prepare preservice teachers for what proof entails: Proof is not only a product; it covers many skills. The importance of transforming mathematics classes to promote reasoning and proof was emphasized. Then, the empirical arguments and thinking through examples were compared, and their contribution to formal proof was discussed.

Later, three activities suggested by Arbaugh et al. (2019) were implemented in turn. The aim of these activities was to be convinced that relying on empirical evidence and patterns from a limited number of cases was not sufficient to construct proofs. Also, it was aimed to foster the idea of why proof matters. In the first activity, “The squares problem”, participants were expected to find a general rule, a pattern, and comment on how they can be certain on their answers (see Figure 20):

Solve this task:

1. How many different 3×3 squares are there in the 4×4 square below?



2. How many different 3×3 squares are there in a 5×5 square?
3. How many different 3×3 squares are there in a 6×6 square?
4. How many different 3×3 squares are there in a 7×7 square?
5. How many different 3×3 squares are there in a 60×60 square?

First work on the problem individually. Then discuss it with your friends.

- Did your methods yield the same result?
- How might you arrive at the totals of the larger squares **without** counting?

Figure 20. The squares problem

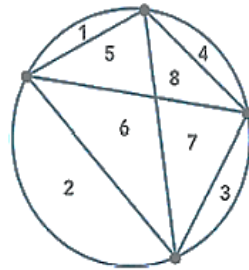
In the second part, “Circle and Spots Problem,” participants were expected to explore a possible relationship between the number of spots and the greatest number of non-overlapping regions into which the circle can be divided. In this task, until the sixth point on the circle, participants might think that there is a pattern of $r = 2^{n-1}$. The purpose of the activity is to be cautious when exploring a pattern through a limited number of observations. The message is clear: We cannot be sure that the rule will apply to the following numbers and examples. At some point, there could be an example that breaks the rule. The activity can be seen in Figure 21.

The idea in the first and second tasks was further emphasized in the third part: The Monstrous Example. There, a passage about whether the expression of $1+1141n^2$ could be a squared number was given. By testing through a computer program, a group of mathematicians found that the expression turns out to be a squared number when $n = 30,693,385,322,765,657,197,397,208$. After giving this passage, many discussions came to the scene. First, we went back to the first task, and the students were directed to questions like: How confident are you in your response? Do you think you found a pattern? How can you be so sure? The responses were formulated around finding a structure. The participants explained that they

found the pattern not only through numbers but also by looking at the structures that remain constant. For instance, an explanation for the method focused on the middle of the squares: They can not be on the side of the original square (the boxes that contain X). So, for a 4x4 square, we can only have four 3x3 squares inside (see Figure 22).

Part 2: Circle and Spots Problem:

- Place different numbers of spots (points) on the circumference of a circle and join each pair of spots by straight lines. Use several drawings of circles and lines to explore a possible relationship between the number of spots and the greatest number of non-overlapping regions into which the circle can be divided.
- The number of non-overlapping regions in a circle with four spots on the circumference:



- Explore the relationship with other numbers of spots and then answer this question:

When there are 15 spots around the circle, is there an easy way to tell for sure what is the greatest number of non-overlapping regions into which the circle can be divided? What pattern did you notice?

Figure 21. Circle and spots problem

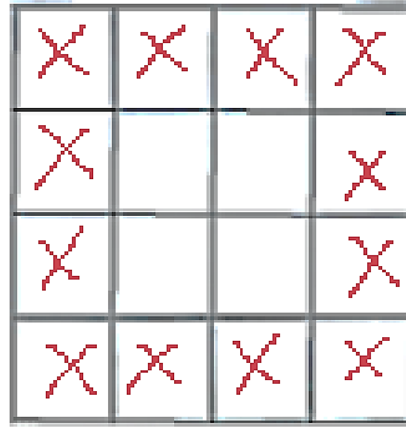


Figure 22. A response to the squares problem

The next instruction in this module was about how we can judge whether an argument is proof or not. Ten student responses (labeled A-J) to the sum of two odd numbers task, “Prove that when you add any two odd numbers, your answer is always even.” were given. Participants were expected to make a judgment as to whether or not the student had proven the conjecture with their rationales. After finishing the activity, they were expected to list criteria for arguments to be counted as proofs.

Participants were distributed sheets involving these ten student arguments. They first individually decided on whether those arguments could be accounted as proof or not. After they made their decisions, the participants’ labels as “yes” and “no” for each student were marked on the table. Each participant was asked to raise their hand if they believed that the argument was proof. Then, the frequencies were recorded. Ten student responses, their characteristics, and the percentage of correct responses in both cycles are given in Table 17.

Table 17. Participants' Responses to 10 Student Arguments

Student	Characteristic of the Argument	First cycle correct response (%)	Second cycle correct response (%)
A	Begins with a specific example but covers all odd numbers; generic example.	92	100
B	Misunderstanding of the distributive property. No meaningful argument.	100	100
C	Seems like a general argument but deals with only addition of same odd number.	4	8.4
D	Tabulating specific examples; an empirical argument.	100	100
E	No attempt for generalization, example-based argument.	80	67
F	Definition of the addition of two even numbers was used. (use of previously proven statement)	88	100
G	Making general arguments without explaining assumptions and stating the known rules/ theorems/ definitions.	60	8.4
H	A general argument with all terms defined. Every step is justified.	100	100
I	Proof by exhaustion: Every possible case was shown in the table. No option left missing.	40	33
J	Use previously known definitions. A logical and an algebraic argument was provided.	96	100

After collecting the number of responses, the correct responses for each student's argument were shared with participants. Starting with the arguments with the most

incorrect answers, every argument was discussed on the decision to accept it as proof. One such argument was by student C (see Figure 23):

Student C

Any odd number can be written as $2x + 1$. So, let's add two odd numbers.

$$2x + 1 + 2x + 1 = 4x + 2$$

$4x + 2$ is even since 4 and 2 are both even.

Or $2(2x + 1)$ shows that $4x + 2$ is even.

Figure 23. Argument by student C

The problem in this statement was due to the general structure of the argument expressed. However, in the statement, only a generalization about the addition of two identical odd numbers is reached. Only 4% of participants in the first cycle and 8.4% of participants in the second cycle could answer this correctly. With this response by student C, it was discussed that the representations should include all natural numbers (or all cases in the domain).

Another problematic response was the argument by student I. The argument was a proof by exhaustion but participants evaluated it as an empirical argument. As they could not see a generalization (algebraic or not) with the form of $2x+1$ and $2y+1$, they treated this argument as non-proof. 40% of the participants in the first cycle responded correctly, whereas this percentage was 33% for the second cycle. During the discussion, the concept of “proof by exhaustion” came to the scene.

Participants were convinced that the argument covers all cases by systematically listing all possibilities. The argument by student I is shown in Figure 24:

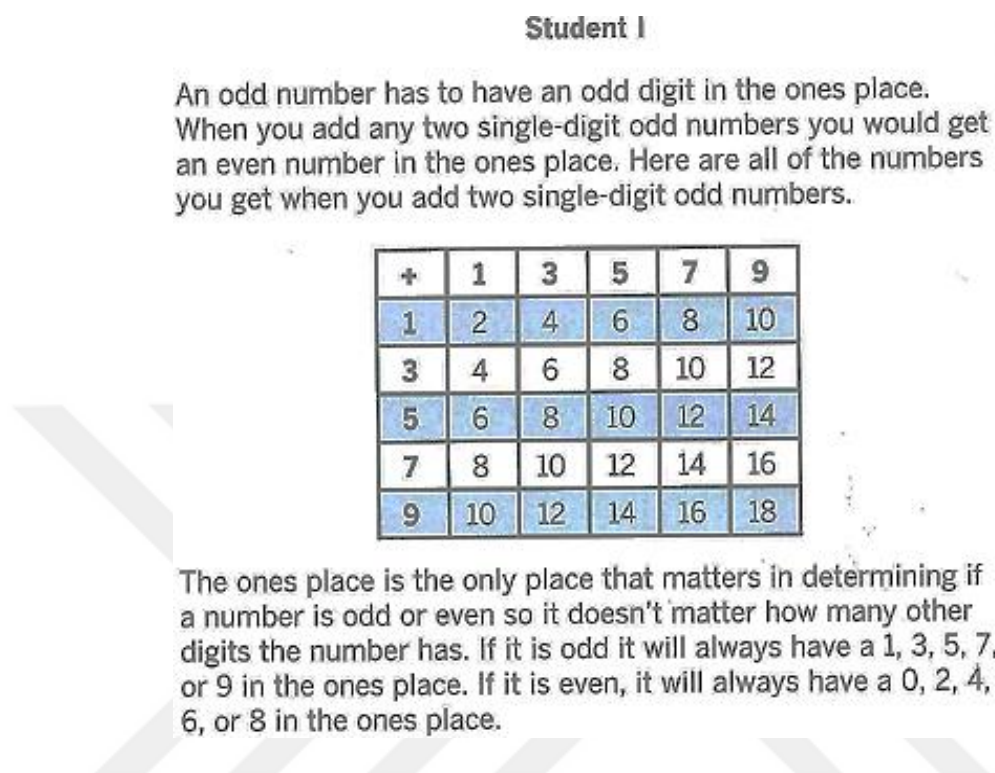


Figure 24. Argument by student I

Several participants stated that the argument by student G was correct and could be counted as proof since it explains the underlying structure and reaches generalization. The problem was that what the argument relied on was missing: There was no definition of odd or even numbers, and the process of circling by twos was not explained. As the argument missed some key elements, the argument could not be accepted as proof. After these discussions and explanations, participants were convinced that it cannot be considered as proof. The argument that led to fruitful discussion was by student G. The argument by student G can be seen in Figure 25.

Student G*

If you add two odd numbers, the two ones left over from the two odd numbers (after circling them by twos) will group together to make an even number.

Figure 25. Argument by student G

The argument by student E (see Figure 26) was similar to the one by student A; it caused participants to investigate whether they were both proof or what makes one proof and the other non-proof. The argument by student E missed the explanation to move from a single example to generalization. Participants who could not find this treated this argument as proof. The opportunity to analyze that argument was important to comprehend generic examples better. After discussions in the classroom, participants could learn the concept of generic examples more meaningfully.

Student E*

If I take the numbers 5 and 11 and organize the counters as shown, you can see the pattern.



You can see that when you put the sets together (add the numbers), the two extra blocks form a pair and the answer is even.

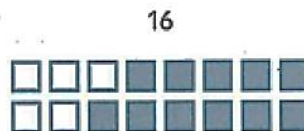


Figure 26. Argument by student E

After investigating and discussing all these student arguments, the criteria for judging whether an argument counts as proof were discussed. The responses of the

participants included “seeking for generalization,” “having a structure,” “use known theorems/definitions,” and “not relying solely on examples”. These criteria list was expanded with other relevant considerations such as: “The statements and definitions that are used in the argument must be ones that are true and accepted by the community because they have been previously justified,” “The conclusion that is reached from the statements must follow logically from the argument made,” and “The mathematics must be correct.” (Arbaugh et al., 2019). Through the end of this part, it was stressed that there might be some personal preferences for the construction of proofs: the type of proof (direct proof, proof by contradiction, etc.), format of the proof (narrative, two-column proof, etc.), used representations (words, symbols, etc.), and how explanatory the proof is.

After evaluating student arguments and deciding on the appropriate criteria for arguments to be proof, the categorization methods of proof attempts were discussed. After hearing their ideas, two commonly used frameworks were introduced: Proof schemes by Harel and Sowder (1998) and Proof schemes by Balacheff (1988). While exploring the categories, 10 student arguments were revisited, and exemplary arguments for each category were investigated. Providing these frameworks and examining arguments by looking at those frameworks improved their pedagogical content knowledge.

The next topic of the teaching proof module was how to help students develop their reasoning and proof capacities. Four main teacher activities were discussed to foster proof skills of students:

1. “Implement tasks that promote reasoning and problem-solving,
2. Use and connect mathematical representations,
3. Pose purposeful questions,

4. Facilitate meaningful discourse.” (Arbaugh et al., 2019, p. 91)

Among the four dimensions, “task selection and implementation” was particularly emphasized. It is highlighted that the classroom tasks should be high-level or cognitively demanding tasks to promote reasoning and proof skills. Also, low-floor high-ceiling tasks were mentioned as “. . . all students can enter and explore at some level, and at the same time they have the potential to engage students in challenging mathematics” (Arbaugh et al., 2019, p.76).

It is very challenging to create tasks that foster reasoning and proof, hence modifying tasks to increase their reasoning-and-proof potential was suggested to participants. To be able to do that, the kind of elements that should be promoted in such tasks were revisited. The elements that served as criteria were “making conjectures, investigating a conjecture, developing a solid argument, and evaluating an argument or a proof activity” (Arbaugh et al., 2019, p. 108). Exemplary tasks from the work of Arbaugh et al. (2019) were presented. Initially, participants were given original tasks and they were asked how they would modify tasks to promote reasoning and proof. Then, the modification ideas were discussed together. One such task is given in Figure 27.

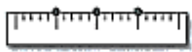
This activity was done for two purposes. When the participants become teachers, they may use the skills they learned; modification of tasks in the absence of curricular materials that target reasoning and proof skills. Also, participants were required to design a task for middle or secondary school students by the end of the term. This activity provides them guidance on how to modify tasks to increase their reasoning and proof skills promotion. The teaching proof and reasoning module ended up with a summary of what had been learned and discussed in the module. The final exam of the course showed that noticeable changes were observed in

participants' proof construction skills (see sample questions in Appendix M). They were more successful in explaining the rationale in every proof step in this task compared to their midterm performances. Most of them were able to determine counterexamples while disproving the given statement. Participants' awareness of the distinction between what is asked and what is to be proved increased.

Task B1

REASONING WITH VISUALS: Explain why this method for drawing a parallelogram works. Name a theorem that supports your answer.

FIGURE 5.4 The Construction Conjectures task as it appeared in a textbook.



1. Use a straightedge to draw a segment. Identify and mark the midpoint.



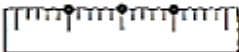
2. Draw a second segment whose midpoint coincides with the first segment's midpoint.



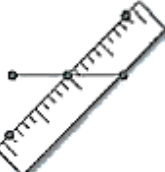
3. Connect the endpoints of the two segments.

Activity 5.3 The Construction Conjectures Task (Task B2)

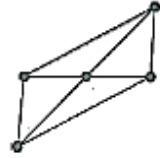
Use a ruler and the three steps outlined below to complete Parts a-d.



1. Use a straightedge to draw a segment. Identify and mark the midpoint.



2. Draw a second segment whose midpoint coincides with the first segment's midpoint.



3. Connect the endpoints of the two segments.

Record your work on the following questions in your notebook or binder.

- Use this construction with a variety of starting segments.
- Write a conjecture about the type of figure your construction produces.
- Write a proof that explains why that figure is produced each time by the construction.

Figure 27. Example for the task modification activity

After the implementation of the program, three different data collection tools were used to obtain information about the effectiveness of the program. First, the participants were asked to prepare a task based on proof and reasoning based on a curriculum objective. Secondly, they were required to ask and/or teach predetermined geometry proof questions to at least one student from the fifth to the tenth-grade level. In this assignment, they were expected to reflect on their own teaching processes and the student's learning and proof processes. Third and lastly, semi-structured interviews were conducted with several participants in the program. In these interviews, the impact of the program on the participants' learning and attitudes was examined, and the participants were asked to evaluate their learning and the program. In addition, the participants were given opportunities for self-reflection. The findings of these studies aiming to demonstrate the effectiveness of the program are presented below in order.

4.2.2 Findings from the task design assignment

Participants of the course were expected to design a task from a given objective in the middle school or high school mathematics curriculum (see Appendix N for the details of the assignment). They were asked to prepare a task that targets the promotion of reasoning and/or proof. Participants presented their ideas in the final week of the semester, but they were required to submit their report a week later on the course LMS system. Forty eight reports were analyzed in two different ways. First, participants' expressions regarding how they promote reasoning or proof in their task design were examined. Second, the developed tasks were analyzed based on the framework by Sevinc et al. (2022).

Participants were required to report how they promoted reasoning or proof through their task. Their responses were coded, and 18 themes emerged based on these codes. These themes and their frequencies are detailed in Table 18.

Table 18. Participants' Expression on Promotion of Reasoning and Proof

Theme	First cycle	Second cycle	Total
justification	25	18	43
group work and discussion	12	9	21
critical thinking	8	5	13
questioning (reason or procedure)	11	1	12
conjecturing	6	4	10
exploration	8	2	10
logic	6	2	8
establish conceptual relationships	5	2	7
use known rules and information	5	2	7
organization	2	4	6
seeking patterns	5	0	5
generalization	5	0	5
drawing conclusions	3	1	4
evidence from measurement	1	2	3
verification	2	0	2
cause-effect relationship	2	0	2
evaluation	2	0	2
mathematical language	1	0	1
Total	109	52	161

These findings showed that the most frequent expression of promotion reasoning and/or proof was coded as “justification.” This code is formulated when the

participants refer to justifying conclusions or computations, providing explanations for the results that they found, and articulating and defending their reasoning throughout the task or at the end of the task. Some exemplary statements were: “Students must use theorems about parallel lines and angles to justify their answers.”, “Students will be expected to provide reasoning for the relationship of $2\pi r$, πr , and circumference.”, and “Students must justify their constructions and the reasoning behind the location of the orthocenter, depending on the type of triangle.”

Another frequent theme was “group work and discussion.” Here, participants mentioned collaborative work, class discussions, and peer evaluations. Communicating the findings was also included in this theme. Statements relevant to this theme were “Students will present their constructions to peers for critique and defense, fostering a classroom environment of collaborative learning and critical evaluation.”, “Creating a discussion about why they draw different objects with the same conditions . . .”, “Through discussion and debate within their groups, students must justify their conclusions and reasoning to their peers. This process fosters critical thinking skills as students evaluate multiple perspectives and refine their own reasoning.” and “Students explain and justify their constructions to their peers, reinforcing their understanding and ability to communicate their reasoning.”

Next, the other findings regarding the themes were evaluated based on the design program. In the design modules, questioning, conjecturing, exploration, logic, use of known rules and information, seeking patterns, generalization, and drawing conclusions were highlighted more than other themes. Their frequencies were observed to be low when considering their emphases in the intervention. For instance, questioning was explicitly mentioned 12 times in the written reports. These questions were in the “why,” “how,” and “how do we know that” forms, asking for

rationales, clarification of problem-solving procedures, and revealing existing knowledge repositories of students. The statements were “The task promotes justification and reasoning by requiring students to explain why a line is perpendicular and how they determined the right angle accurately.” and “If rectangular prism sizes are changed, what changes and why? How?”

Conjecturing was presented as an important step for moving towards proving and emphasized in the intervention, especially in the module for teaching proof. Only ten explicit statements were detected in the participants’ reports. When hypothesizing or predicting was used, it was considered under this theme. Some of the statements were “Third question asks them to test and justify their conjecture. This is important because they will understand why the rule for area of a triangle works, which will save them from rote memorization of the formula without understanding.” and “These activities encourage students to articulate their thought processes, make predictions, and verify outcomes, enhancing their critical thinking skills.”

Exploration, discovery, and observation were also given as important aspects of developing the proof skills of students in the intervention. Ten expressions were related to this theme in the participants’ reports. “Students are encouraged to find out about the number “pi” with their own observations instead of simply memorizing what it is.”, “The student discovers by trial and error which triangle is actually possible to create and which is not possible to create with the rods they use in the direct way. Then, she goes to a mathematical equation with the examination of triangles that she has created and failed to create as a result of various experiments”, and “. . . discover the principle of triangle inequality” were some of the excerpts of participants’ responses.

Participants' designed tasks were also analyzed through the "Integrated framework for analyzing ways of reasoning" by Sevinc et al. (2022). The framework was developed to categorize solved tasks, but it was also used to analyze and categorize unsolved reasoning tasks. Forty eight tasks were analyzed, and the ways of reasoning that the tasks were promoting were summarized in Table 19.

Table 19. Ways of Reasoning in the Designed Tasks

Ways of reasoning	Sub dimensions	<i>f</i>
Appeal to authority	-	4
Simple (1-step) deductive reasoning	-	6
Mathematizing	-	1
Reasoning by analogy	-	1
Reasoning with empirical arguments/ Specific cases	Making claims and generalizing	28
	Justification of a claim	1
Developing conclusions/ justifying/ refuting through deductive reasoning	Generic example	3
	Counterexample	0
	Systematic enumeration	0
	Other	4
Other	Abductive reasoning	1

This analysis showed that most participants prepared tasks that relied on reasoning with empirical arguments or specific cases. In those tasks, students are expected to observe a regularity through measurement, manipulation, or calculator, make a claim (conjecture), and reach a generalization without consulting previously known relevant theorems and rules. These participants tried to build their tasks by letting

students see data-based or measurement-based evidence. Almost none of the tasks asked for justification; exploration and trying to reach a general result (a formula, a principle, a rule etc.) were prevalent in the reports. These findings were aligned with the hesitations of teachers or teacher candidates to ask students to provide formal proof. In the intervention, providing opportunities for proof and reasoning were discussed extensively. They were given lectures on how to modify tasks to increase their reasoning and proof potential. The following elements were suggested to be considered in task design: Making a conjecture, investigating a conjecture, developing an argument, and evaluating an argument. Participants who provided students tasks, including reasoning with empirical arguments or specific cases, tried to ensure all those elements but missed the opportunities (or could not provide due to the objective of the task) for asking for deductive proofs.

Developing conclusions/justifying/refuting through deductive reasoning was observed in 7 tasks; while three were generic examples the rest of the four tasks were directly asking for proofs. The grade level for the tasks in this categorization showed that three tasks were designed for middle school students, and the rest were designed for high schoolers. The important findings for these analyses were that some participants did not perceive the generic examples as proof before the intervention appreciated and internalized generic examples and benefited from that in their task designs. Even though there were only 3 participants who designed a task asking for deductive proof in middle school, it is promising that participants who will become middle school teachers can explore and devise ways to integrate proof into their teaching practices.

In the ways of reasoning for some tasks, “Simple (1-step) deductive reasoning” was observed. This categorization could be considered parallel with

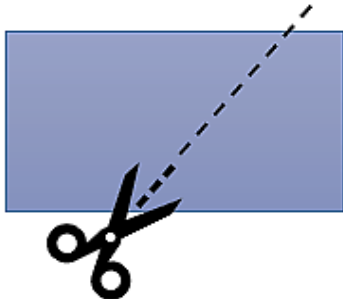
“Developing conclusions/justifying/refuting through deductive reasoning” because the objective of the task and previously learned information on the topic of task limited participants to ask for multi-step formal proofs so they could only give place simple deductive reasoning in their tasks. One example task can be seen in Figure 28.

Work with a partner.

A) 1) Take a rectangular or A4 size paper.

2) Cut the rectangle into two pieces along opposite sides of rectangle. (There will not be a steep cut.)

3) Create a parallelogram from the two parts of the cut rectangle.



- What do you notice about the areas of the rectangle and the parallelogram?
- What do the converging edges of a rectangle form in a parallelogram?

Figure 28. Proof and reasoning task by a participant

In summary, more proof-based activities were expected to be offered by participants. Even though some participants tried to integrate formal proof and deductive reasoning in their tasks, most looked for conjecturing and reaching generalizations from observed regularities. Justifications were mainly empirical and based on experimentation, and participants refrained from asking for proofs and deductive reasoning. These findings pointed to different reasons for this avoidance. First, some participants still find immersing proof ideas in their tasks challenging and/or meaningless, considering student characteristics, grade level, and knowledge

repository. Secondly, the objectives and the grade levels of the task were preselected and randomly assigned to participants. While some offer more chances to include reasoning and proof, in some, establishing conceptual links and exploring through measurements and experimentations become more important due to the characteristics of the objectives. This happened especially in the introduction of main concepts in geometry at lower grade levels.

4.2.3 Findings from the proof teaching practice

Participants were asked to select and ask at least one geometry proof question to a student from fifth to tenth grade (see Appendix O for the directions in detail). In the directions, they were required to give place to the following information:

1. The grade level of the student.
2. Selected task and the reason(s) for selecting it.
3. How the student approached the task and evaluated his/her overall performance (Whether s/he could prove it, did s/he use examples or general statements, did s/he refer to theorems, definitions, or rules, etc.).
4. Directions and guidance that were presented to the students and why they needed those directions/guidance (to clarify the ideas, to challenge the student further, to understand her/his approach, etc.).
5. Reflection on their own performance as a teacher (whether they could promote proof skills, confidence to teach proof skills, etc.).

Participants were asked to collect data and present evidence. The data consisted of 50 preservice teachers' written reports on middle school students or high school students' tasks.

Participants presented diverse tasks to the interviewed students. Their reasons for task selections were mostly about familiarity with the topic, being recently learned, appropriateness of the grade level, and easiness of the task. Some participants perceive the task as an opportunity to introduce proof and indicate the reason as being “fundamental.”. Some of them provided more than one task to see the progress as they expected the first task would provide a learning environment for the follow-up task(s). The most frequently selected task was “Prove that the sum of the interior angles of a triangle is 180 degrees.”

Since the assignment required preservice teachers to assess students’ proof skills and approaches toward proof, participants recorded both cognitive responses and emotional reactions of the participants. In the majority of the responses, preservice teachers reported that students had difficulties in proving; they did not know how to start to prove, they referred to numbers, measurements, or specific examples in order to prove the given statement, and they had issues in differentiating what is given and what is to be proved.

. . . she wanted to start with one triangle, and she preferred assigning number values to angles according to their sum being equal to 180 degrees. I realize that she generally starts by giving numbers to angles because she feels comfortable this way. Afterward, we give a random measure to interior angles like x , y , and z and then calculate the measured exterior angles to generalize the situation.

. . . I thought that (two triangles are congruent) was given by the question [a response by the interviewed student].

. . . My student also had a little difficulty in this sense at the beginning; it took her a little time to understand what she needed to do since she already knew that the conclusion she needed to reach was correct. She made a quick inference and said that the triangle was isosceles. When I asked why, she said that the height is also the median in an isosceles triangle. When I asked if it was the same for all isosceles triangles, she answered yes. In that case, there must be a reason for this. I wanted her to think about the question of why this is so . . . At first, she made the mistake of showing the triangles as congruent

as a reason, but I told her that we did not know that yet, that it was a conclusion we had to reach.

Some students had problems with the geometry knowledge required for proving, hence some preservice teachers struggled with providing that knowledge and their focus diverged a little bit from questioning or providing proof skills. In some cases, preservice teachers indicated that through some hints and directions, they could apply proof ideas and successfully complete their proofs.

. . . With my guidance and the thought-provoking questions I asked him, he was able to reach the results. He made references to the “Z” rule [referring to the congruence of alternate interior angles] and congruent angles, and he was able to do the proof using these rules. He was quite successful for someone who had never done a proof before.

. . . Slowly, she began to understand the basis of proof. I helped her constantly question herself with my questions. She began to investigate the reasons for the rules she uttered that were not valid (rules by heart). As a result, it was slowly approaching the end. She completed the proof . . . After closing the Zoom recording, she said how enjoyable it was to learn the logic and that she could not believe she did it.

Preservice teachers were also asked what kind of directions they provided to students. A few preservice teachers directly explained how to do the proof to the students. Most of them reported that they tried to provide some guidance and hints when students struggled; they tried to establish connections in students’ minds on the supportive theorems, rules, or axioms. Sometimes, they provide this help in each step of the proof, while most of the time they involve students in the proof process with little support. Some of the participants even tried to challenge the students further and move beyond the given task. Participants highlighted the importance of “why” and “how” questions in understanding the rationale of the given geometry statements, in revealing necessary conceptual knowledge, in differentiating the “given” and “reason” of proofs, in exploring whether the concepts, theorems, and rules are understood as it is.

4.2.4 Findings from the interviews with design participants

8 participants were interviewed after the final exams of the semester in order to determine the impact of the PRISM on participants' knowledge of proof and teaching proof and their reflections on the design and implementation of the program. Meetings were arranged to be online, and each lasted approximately 30 minutes. Participants were asked to respond to several questions regarding the overall evaluation of the PRISM, how to reshape it, what they needed throughout the program, the impact of the program on their learning and teaching skills, and reflections on their teaching proof practices (see Appendix P).

The first question of the interview was “When you think about the content of the activities, how did you find their quality? How did you find the difficulty of the activity?”. While participants were responding to that question, they also presented some ideas regarding the following question: “What do you remember most strongly from these applications? Why?”. These two questions were analyzed together.

To reflect on the course, some participants mentioned their first experiences with proving and logic and then compared them with the PRISM activities.

We took Math 111, the Introduction to Logic course at the university, but it wasn't like (what we did) here, and we skipped it. Yours was a little more focused on understanding, understanding logic, understanding evidence, and being able to transfer it. And it was very useful for me. (Yasemin)

It was good that they were related to proof, and to be honest . . . my experience in Math 111 was not very good. (Elif)

Your logic module really increased my self-confidence. For example, no matter how unprejudiced I came to this school, my prejudices about proof increased exponentially because of Math 111, . . . it increased a lot . . . I passed that course. I don't know how I passed, but there was something against proof that stayed with me. Coldness, I might not be able to do it. (Esma)

. . . most of (what we learned in) Math 111 is lost for both me and my semester mates. It is incomplete . . . I felt very comfortable in the class because you continued explaining from the very basics. Because we really started from the very beginning. (Leyla)

Some participants reacted to the effectiveness of the Logic module of the designed course.

I think the logic module was very effective. Because, you know, we use the definitions of Euclid's axioms, the theorems, the definitions of the axioms, we want to prove the theorems, but it is something we should know. But when we got there, we noticed that we were not competent. At some point, I think the logic part doubled the efficiency of the course. That's how I saw it, on my own behalf. Apart from that, you introduced some stages for us to follow, for example, under the name of proof skills. (Esma)

We were doing a logical order. You gave us statements and asked them in the sequences; they were very useful to me. We sorted those statements and formed the result. (Zeynep)

Participants' evaluations focused on some specific activities that were most memorable for them.

One frequent thought in my mind is whether or not these children's proofs are related to the correct proofs or not. There was also a text about Gandalf, I think. That one I remember well. (Elif)

For example, we examined student proofs, you emphasized . . . or something like that, you know, students can go through examples, but here, for example, we see two types of students. One of them makes a generalization by giving examples. One of them stays with examples, or I don't remember the exact terminology there, but this distinction was good. You know, when we see examples in both of them . . . I think it was nice to think like this and to be learning this. (Esma)

. . . It was really surprising for me to expect proof from a student, especially at the middle school level, not knowing as much as we do. We did something like this. We evaluated the students' proofs, whether they were proofs or not. Then we talked about it a lot among ourselves because it was very different. You know, it is proof to us, but actually, it doesn't feel like proof at their level, or rather, they don't know anyway. For example, thinking about these things was very productive for me (Merve).

There was one particular event. “Is this proof or not?” I think this is also very valuable for teaching. What we have learned does not actually correspond to my own experiences as a student. We did not learn in this way, we were not taught in this way. So I don’t know how to evaluate, for example. I don’t know how to teach proof to a student. I am quite sure that I needed to learn this. (Leyla)

In terms of teaching perspective, some participants reacted in the following manner:

What you have shown me makes my job easier for me, so I think we need this after graduation [teaching proof in the future], in general. (Yasemin)

I am frankly more warmed up to proof and to the part that it should be taught in schools. (Elif)

Criticisms were also expressed about the design of the course and about a specific activity as well:

For example, in class, we’re in activity five . . . Then the place we looked at, which is my personal interpretation, was sometimes separate from the subject we were doing at that moment. By the way, for someone else, maybe it may be a relief to go through something different. But I thought, why right now? for some of the activities. But towards the end, I think they all made sense, you know, from the very beginning to the very end. In the last lessons, it started to go very parallel. (In the beginning), it was like connections were not formed. (Nehir)

It was the activity that stayed in my mind in a slightly negative sense. There was something like drawing conclusions in the proof skills section, there was something in the Lord of the Rings context [referring to an activity for drawing proper conclusions]. (Esma)

General problems with the design were related to the timing and duration of the course. Participants found it very limited:

I think the activities were good, but we were trying to do everything quickly . . . If we did the activities in a larger time span, I think it could be better. (Elif)

Participants were directed to the question, “Was there an activity that you had difficulty understanding?” so that the design could be reshaped considering their situation and needs.

That first Gandalf thing we did was a little bit . . . For example, I unconsciously chose “not certain” for most of the questions without thinking much, I have no idea. I had a little bit of difficulty there, filling them in, but

then when we were discussing with our friends it was okay, that text made me a little tired. (Elif)

With sentences that do not contain numerical content . . . sometimes contradict a little bit. Because when we translate those verbal expressions into daily life, there is always a question mark (Nehir)

In the expressions “and, or, if.” Also, we made sentences, I think we rewrote sentences like in the form of converses, contrapositives . . . Yes, they were challenging for me. I had to look up the thing [the meaning of converse, inverse and contrapositive] all the time. Which one we were doing, and what should we change, and so on. (Merve)

Converses, inverses, contrapositives of expressions, etc. (Fatma)

During the design, the idea of using examples from daily life was questioned by the researcher in terms of impact and benefit. Hence, participants were asked the following question: “I tried to present examples from daily life that were not related to mathematics or geometry; do you think this was helpful? Would it be better to emphasize mathematics or geometry more instead?”.

If the course is to be presented as a separate course, I think it is more memorable for me to do this with examples from daily life other than geometry . . . but if it is to be shown within the geometry course, a little more geometry-oriented things can be added. (Yasemin)

In general, I don't think I will use it much in the teaching process. It creates confusion for me, because I don't feel very comfortable with it. It didn't cause any negativity, but it would make me feel better to see the math in those sentences. (Nehir)

I think the ones that seemed irrelevant from the context were also useful . . . especially for me . . . I think it was nice. It was also nice that not everything was through mathematics. For example, we saw things both in there and in geometry. I mean, it was like we didn't skip that either . . . Some of them can be more relatable because they are direct . . . I think there was no problem in starting like that and then going to geometry. I mean, it was good. (Esma)

I think it was positive because it was easier for me to apply it to mathematics when it was sitting in my mind there. (Merve)

It was definitely very useful, but our time with you was a bit limited. Now, if this had been a (different) lecture, it would have been very logical to start with this at the beginning and then increase the math intensity. But considering the time constraints, if you come with pure math, I think it might be less interesting for some people because it can be tiring. Now, for myself, for example, I would be happy if it was math because it really interests me. I would really like to do it, but on the other hand, I think it can be very tiring for some people. So I think it makes a lot of sense to use examples from daily life. It is a very nice option. I wonder if it would be possible for you to provide something like that in addition for those who want it? Someone who wants to do more math. Maybe individually. (Leyla)

I think daily life was good. I think it was balanced, I mean, the things you did were intertwined with mathematics, the activities you did were intertwined with mathematics. I mean, you were emphasizing mathematics. We also better understood the place of mathematics in daily life. (Fatma)

I think it was useful to be from daily life. I think it was useful not to start directly with mathematics in order to put it into a certain logic, but to make use of more daily things or verbal things, but if you think about it for the lesson, for example, I was opening your notes and looking at them. I wonder which methods I can use . . . In the parts where we look at proof methods, it would have been more useful for me to have more mathematical examples. (Zeynep)

In terms of understanding the participants' needs from a broader perspective, they were asked the following questions: "Would it be meaningful to support students taking this course next semester in the same subject? Do you have any suggestions for changes or adjustments? Are there any points during the semester where you can think that you need more support in these areas?". There were answers regarding the duration and the timing of the designed course, the addition of extra reading materials, the implementation of the design as more intertwined with the main course, increasing the amount of the mathematical proof or having an opportunity for PS.

I think there was not enough time. In general, there were parts where it felt like we were trying to accomplish something in an hour. You know, there could have been parts where we thought and discussed more, but it's not about you, of course. That's why, I think it would be more useful to include

more mathematics in the course and . . . I think it can be discussed more in the lessons, but in general, I think it was good. (Zeynep)

I think the time allocated was also short, and it was always at such a fast pace, but did it affect me a lot? No, it didn't. I think we learned the main idea that we need to learn. I think the time allocated for this lesson can be increased because it is important. I think its weight could be increased. (Esma)

I can suggest this: The part that I get the most was the part of the students' proofs. For example, we covered the circle (as a topic in the main course), seeing 1-2 examples of the student, seeing 2 examples only on the circle would be nice. (Nehir)

It is difficult to create an extra lesson, it is difficult to create an extra hour. It can be given at the beginning of the lesson instead of the end, and in addition to that, we can make such activities more focused on the class with quizzes, for example. Because the focus is lost, there can be additional quizzes. (Yasemin)

After the questions about the design, participants were asked questions related to the impact of the design. There, the questions "How do you feel when you think of yourself as someone who is learning how to prove yourself? Can you tell us about your self-confidence, competence, and feelings?" were directed to the participants. All of them emphasized the improvement in their proof knowledge and skills and they specified on what aspects they had troubles with and how they overcame these issues due to the main course and this design.

I definitely think that I have made a positive meaningful development . . . I am much more confident in proving at the end of the semester than at the beginning. I mean, as I said, maybe I can't do everything I come across because of a math deficiency, but I am sure that I am at a much better level. (Leyla)

I think I'm much better. There is a real difference between Fatma before the semester and Fatma now. First of all, I know what the proof is, of course, I knew what it was, but I didn't know how to do it. I didn't know where to start . . . I mean, I couldn't put into practice what I knew, but thanks to you . . . step by step I saw that everything is actually much more straightforward (Fatma)

Especially until the middle of the semester, until the midterm part, I was presenting either too much information or using unnecessary information . . . You know, I was trying to use everything I had, I was either giving too little information or too much information. Towards the end, I started to realize how I could make this proof more concise. So I can honestly say that I am more confident. (Nehir)

To understand the impact of the design on participants' teaching knowledge, "How do you feel when you think of yourself as someone who will teach proof? Can you tell us about your self-confidence, competence, and emotional state?" asked. They were also directed to give details on the process and results of their proof teaching experience.

Honestly, I feel like, okay, we have a big responsibility. Not everyone will be able to achieve this, but I would be happy to be able to do something, even in the mind of a student at the stage of learning what he did and, why he did it, where it came from. Right now, maybe I am not afraid because I took this course and had these experiences. (Elif)

You had us do our teaching proof practice, which I found really nice, frankly, both to measure what the students think and to take their pulse. On the one hand, it was also nice to do such a study. I don't know, it was nice to be warming them up, I warmed up too, frankly, while doing those things. They are more moderate towards proof, not full of fear (Elif)

You know, the part of making a student do it is still very scary for me . . . the questions I ask, what kind of questions I ask, I shouldn't give the answer, but I need to give a clue. That part was very challenging for me. (Merve)

Even if he doesn't give the right answer, I try to ask him questions that can lead to the right answer. This lesson really added something to me. (Fatma)

I had to go step by step (while asking and teaching a geometry proof to a student). I think it was a good opportunity. It was very exciting for me. It was also very exciting for my student, but when I really started and especially when I got into a certain point of the problem like this, I felt a great relief . . . I was more bent over and pointing at things because the student had a hard time applying what he knew. Actually, he knows, but he had difficulty putting it together. Toward the end, I continued to be a little more relaxed and more confident. I liked that too. (Nehir)

I did it with my brother. My brother was in the ninth grade . . . I shouldn't say he was scared . . . his knowledge was incomplete . . . I looked at the curriculum before I studied with him, to see how much he knew, where he could get it from. How was it to guide him? Actually, you know, it requires a skill to prompt them to think by asking questions without saying anything. So I think we need to develop that. Because those questions are important in the process of teaching, in the process of making him learn . . . For example, what can I ask if he can't do this or that, of course, since it is spontaneous at that moment, nothing much happens. He assumes something in a very irrelevant way . . . Asking questions is important, that's what I realized. I mean, I need to work on that a little bit. (Esma)

He sweated a lot, he was nervous, he didn't know what to do . . . I observed that he was trying to establish the congruence of two triangles using the angle-angle-angle principle. But we do not have congruence in the case of angle-angle-angle. But he said that three angles are congruent and two triangles are congruent because they are congruent. For example, I think this was something very specific and wrong . . . I didn't give much guidance . . . but at the end, I felt much more professional. (Fatma)

Finally, their scores on the knowledge test and beliefs scale were shown individually and they were requested to analyze their scores and reflect on their situation in terms of whether there exists progression on those scores. The knowledge test and beliefs scale were administered at the beginning of the semester to all participants of the PRISM and no post-test for these measures was conducted. Participants of the interviews were asked to self evaluate their knowledge of proof and beliefs towards proof and compare them with their previous scores. The scores shown to the participants are summarized in Table 20 and Table 21.

(On her belief that she has the ability to do proof) Here, I think this was something I gave due to our experience in Math 111 and my own struggles there . . . When I compare myself with the beginning, I think there is a (positive) difference. (Elif)

There is definitely a change. I also remember the questions on the scale. My scores were lower, you know, my belief in myself . . . This course affected me, I mean, I think my belief in proof increased on my own behalf . . . I still don't feel very successful in that respect. But there is an improvement compared to the beginning. (Esma)

I think I am progressing, but I still have some weaknesses, I mean . . . I am a little more confident and I think I have learned more and I want to do something like this in the future, in the summer or something like that. (Merve)

At the beginning of the semester, when I think of proof, I think that maybe I actually chose it that way because I believe that there are very formal steps, if this is not done, it is not considered proof. I believe that there are strict rules . . . if I don't write that, the teacher will not count it as proof, it will not be accepted. But as I said, I think the task you did was very useful. As I said, after seeing the answers of those students, some of them were considered proofs even though they were not for me. For example, after seeing them, I realize now that there are different representations of proof. That's why its place in middle school is more acceptable for me now. (Fatma)

My belief in myself in ability to prove is not that anymore, it is definitely more than (that). I mean, I don't believe that I'm perfect. After all, I'm still a student, but I think I'm better than what I was before. I think I've made very good progress from the beginning of the semester to the current Fatma. (Fatma)

I think the reason why the pedagogical part is low is that the lack of belief in myself may have also affected this. Because I think the question of how I can teach the student when I can't even do it myself is a huge issue. Because I don't know (prove) myself, how can I teach students something I don't know myself? (Fatma)

Of course my belief strengthened as I was able to do it. That's why, you know, it's something that can happen as you work. I feel better about that too. (Zeynep)

While tutoring students, for example, I try to use reasoning most of the time, especially the instructions you mentioned. I don't want to say (things/rules) directly. I'm trying to ensure that the student conducts that reasoning himself and reaches the result by making a generalization or by using his previous knowledge. I apply them. I feel better in that regard. (Zeynep)

Table 20. Performances of Interviewed Participants on Beliefs Scale on Proof

Pseudonyms	Role and Importance of Proof	Place of Proof in Middle School	Mastery in Proof Related Skills	The Need for Proof	Ability to Do Proof
Yasemin	4.28	3.28	4	4	2.4
Elif	4.37	2.1	3	3.67	1
Nehir	5	4.55	4	5	2.2
Esma	3.97	3.73	3.34	4.34	2.6
Merve	5	4.55	3	4.67	2
Leyla	4.55	3.73	3	4	2.6
Fatma	3.73	3.27	4	4	2
Zeynep	4.73	3.64	3	4.34	2.2

Note. Maximum score for each dimension could be 5.

Table 21. Performances of Interviewed Participants on Knowledge Test on Proof

Pseudonyms	CK	PCK
Yasemin	6	4
Elif	9	2
Nehir	6	4
Esma	7	2
Merve	4	2
Leyla	9	5
Fatma	8	3
Zeynep	8	5

Note. Maximum score for each component could be 10.

Overall, the interviewed participants' self-reflections pointed out the positive impact of the designed module. Analyzing the difficulties they experienced and trying to understand the reasons for this was important in terms of shedding light on future studies and designs. Similarly, identifying the interventions that worked was of great importance in terms of preparing the ground for the design of the next studies. It is also possible to consider the results obtained here for the participants who were not interviewed because various observations were made throughout the implementation and some verbal feedback was also received from the participants. It was important for the aims of the study that the participants, who had no previous experience in teaching proof, understood that different skills were needed to teach proof, and that they gained positive outcomes from their first experience.

4.3 Discussion of the design study

4.3.1 Overview of the research findings

After analyzing the current situation through analysis of policy texts and discourses, examining expert opinions on the issues related to learning and teaching proof, investigations of knowledge and beliefs of preservice teachers on proof, it was thought that the first step to solve the problems related to the teaching proof and proof skills, which are frequently encountered in the literature, was to develop and implement a teacher training program. In order to prepare teachers for proof and proof teaching, a program was prepared for preservice teachers who have similar characteristics to teachers and are more accessible. It was meaningful to work on developing these skills before starting to teach.

The program, which was implemented with preservice teachers for 8 weeks, focused on the methods of proof and the rules of logic, as well as the pedagogical

knowledge required for the teaching of proof. In the proof methods and logic module of the program, it was aimed to provide the important skills necessary for proof and to prevent misconceptions and negative beliefs that were already held. The emphasized skills were deducing only from existing knowledge, justifying each step, knowing and applying the rules of logic, expressing statements as conditional statements, understanding and constructing logically equivalent statements, knowing that a statement and its opposite are not the same and may not have the same truth value, knowing that a statement and its contrapositive are the same and have the same truth value, distinguishing valid and invalid inferences, knowing and applying the direct proof method, knowing and applying indirect proof methods, mastering different forms of proof and using them flexibly.

In the evaluations carried out as a result of the studies conducted in this section, the most frequently observed difficulties that participants had were to distinguish between a statement and its converse while proving, to distinguish the given information from the information to be obtained, and to present the given statement in the if-then form. More time was allocated to solving these problems during this module of the program. The evaluations at the end of the module and the program and the participants' self-evaluations revealed that the participants made progress in these areas and performed better than before. These findings suggest that domain knowledge about proof can be improved with sufficient intervention (Lin et al., 2012).

As participants acquired new knowledge and skills related to proof and the teaching of proof, they simultaneously reassessed and abandoned certain long-standing ideas, beliefs, and practices associated with proving and teaching proof at the middle school level. This abandonment is sometimes referred to as “unlearning”

(Cochran-Smith, 2003). Unlearning is about questioning oneself on what she learned and relearning it. It is related to a reflective experience of one's own learning. An analogy developed by a participant in a study (Nemoto, 2007) is helpful in conceptualizing unlearning: "I imagined I had knit a sweater in a conventional manner and then unraveled it into the yarn that comprised it and knitted it again while holding it up to my own body" (p. 36). The previously learned knowledge is important for developing new insights even if that knowledge was formulated in an unproductive manner.

Another type of knowledge required for teaching proof is pedagogical content knowledge (Lesseig, 2016; Stylianides, 2011; Stylianides & Ball, 2008). This includes what is called in the literature specialized content knowledge (SCK), knowledge of students and content (KSC), and knowledge of teaching and content (KTC) (Ball et al., 2005). Accordingly, the knowledge that teachers need to know in order to teach proof was compiled and included in the proof teaching module of the implemented program. The knowledge targeted in this module included the following: The place of proof in mathematics education, activities and practices that prepare students for proof, understanding the difference between empirical arguments and thinking through examples, the inadequacy of constructing arguments based only on examples and that it cannot be considered as proof, the importance of basing the identified patterns on a rule or a structure, examining the cases where student arguments are considered as proofs, categorizing student proofs, preparing and presenting tasks to prepare students for proof, transforming existing tasks into proof-based tasks, preparing students for proof through effective questioning practices.

One of the results obtained as a result of those activities and discussions in the teaching of proof module of the program was the difficulties that preservice teachers had in presenting skills related to proof. At first, the participants' attitudes were not very positive towards different approaches to proof, but over time, they were able to better distinguish the basic elements of proof and better evaluate student arguments. At the end of the module, it became clear why "generic example" could be considered as proof when it was a concept that was not in the participants' repertoire of knowledge and was not initially accepted by the participants. While the knowledge of algebra was a sine qua non for proofs before, it was understood that proofs can be constructed without the using algebra. This understanding brought the participants closer to the idea that proof can be taught before learning algebra. This reinforced the need for students to be introduced to proof-related activities at an early age. At the end of the module, the participants, who thought that proof should only be expressed verbally, learned that proof could also be created with visuals, which is called "pictorial proof." One of the most important outcomes of this module was the discovery of the importance of asking questions while teaching proof and proof-related skills. Questions such as the ones asking not only to find the correct answer but also how that answer was reached, what assumptions were used, what can be achieved when going beyond the given situation, and what justifications it is based on have become more prominent with this program. Since creating a proof task from scratch would be a very laborious task, the idea of how existing tasks could be organized in this direction was beneficial for the participants. Accordingly, at the end of the module, participants were expected to create proof-oriented activities and tasks by taking into account a learning outcome in the curriculum. While justification group work and discussions were the prominent emphases in the resulting products,

the skills emphasized in the program (questioning, conjecturing, exploration, logic, use of known rules and information, seeking patterns, generalization, and drawing conclusions) were not included in the works prepared by the participants as much as expected. This is actually a realistic finding because the participants had no prior experience in task organization. In addition, each of the objectives may not be sufficiently suitable for preparing a task for proof. Nevertheless, the tasks produced can be considered very suitable to be improved over time and to be proof-oriented.

When participants' tasks were analyzed according to the "Integrated framework for analyzing ways of reasoning" prepared by Sevinc et al. (2022), it was revealed that the parts involving reasoning were mostly based on empirical arguments and focused more on assertions and generalizations. Very few of the prepared tasks were on deductive reasoning and justification. This can be explained in two ways: The structure of some curriculum objectives may not be appropriate to focus on a proof. Secondly, the participants may still not have reached a sufficient level to be able to integrate proof into their lessons or may have negative beliefs about it. In informal discussions with the participants, it was observed that they were still hesitant to make proof an element of their lessons. The efforts of some participants, albeit a small number, to conduct proof-oriented studies were promising. The observation of "generic examples," especially at the secondary school level, shows that some participants benefited from the content of the program.

At the end of the program, the participants were asked to teach proof to at least one student through predetermined tasks and to share what they experienced in their teaching process. In these studies, participants were expected to reflect on their cognitive and emotional states for both themselves and the learners. In the written feedback obtained from the participants, it was understood that behaviors such as not

knowing how to start and not knowing which information to use were frequently experienced when students were asked to do proofs (Moore, 1994). In the responses of many preservice teachers, it was stated that directives and hints were very useful (Stylianides, 2011). Proof attempts made by students through examples were questioned by the preservice teachers and it was emphasized that examples cannot be considered as proofs (Stylianides, 2008). A problem that is frequently encountered in the literature is the confusion of the statement given and the statement to be proved (Moore, 1994). When the students who were taught proof used the statement to be proved in the proof process, the preservice teachers objected and intervened. When the preservice teachers could not get a response from the students about the proof, they taught it directly, and this mostly resulted in the understanding of the proof by the students. This was one of the indicators that students could understand the proof when they were introduced. This can be considered as a situation that supports the idea of exposing students to the proof more, which we have encountered in the literature and expert opinions in the preliminary studies (Bieda, 2010; Stylianides & Ball, 2008).

Students' emotional reactions were consistent with the findings in the literature on proof skills. Accordingly, negative emotions, such as anxiety when they do not know how to start a proof or how to proceed, and they show this with their behaviors and expressions, fear of failure, and timid behavior during the proving process, and also positive emotions, such as feeling of success and satisfaction, increased motivation to solve another proof question, and enjoyment were reported.

In this study, in which preservice teachers experienced teaching proof for the first time, they either made students experience the process of proof by asking questions and giving directions, or they directly presented the idea of proof for

students who had difficulties, depending on the student profile and approach. In both cases, it was challenging for the preservice teachers to observe whether the students followed the process cognitively and emotionally. Nevertheless, they found these experiences instructive and inspiring for their future teaching experiences.

Interviews were conducted with eight participants at the end of the program in order to see what the program contributed to the participants, to differentiate the effect of the program from the Geometry teaching course they had taken, to get feedback about the positive and negative aspects of the program and thus to collect data that will prepare the ground for future studies and to compile data that will support other findings of this study and provide different perspectives. From the participants' responses, we learned that their past experiences in learning proofs caused them to develop prejudices and a negative attitude from the beginning. For this reason, they stated that the logic and proof methods module helped with their prejudices and increased their self-confidence.

The reactions to the module on teaching proof were similarly very positive. The reason for this was that although the participants had prior experience with proof, they had not taken any course on teaching proof and had not practiced it. It was a very important achievement for this study that they considered including proof in their future teaching practice and saw proof as an important part of mathematics education.

The feedback about the positive aspects of the program, as well as the aspects that did not go well or were not effective enough, was very valuable. Participants pointed out the difficulty in the structure or language of some of the activities, and some participants suggested changes in the time and duration of the activities. There were also participants who said that they needed more time to be allocated to some

activities in the section on proof methods and logic. These views were invaluable from a design-based research perspective because hearing the reflections and experiences of learners about the learning process is a necessary move to improve current practices. Learners are at the center of the design, so their self-evaluation and feedback on learning processes and learning conditions, along with other assessment tools, can have the power to improve or change learning designs. When we evaluate the participants' reflections on their own development and their feelings about the proof, we understand that they now feel more confident and competent about the proof, both as learners and as teachers. Seeing themselves as more successful in proof has also changed their belief that they can teach proof in a positive way. Nevertheless, by benefiting from the experience of teaching proofs requested in this program, it was observed that they were aware that it was not an easy practice, but they had a strong belief that they could get better. The participants were very pleased with what the program brought to them and wished that other preservice teachers would benefit from it in the future. All these findings were very encouraging for shaping the next cycle of the design and continuing the study.

In this study, a local instructional theory was developed on the process of learning proof, how to teach proving, and the ways to support that learning. The designed program and its characteristics, and the overall findings of the study can suggest teacher educators or researchers who conduct professional development courses, a series of carefully structured, exemplary instructional activities designed to aid teachers or preservice teachers in developing their skills for understanding and teaching mathematical proof. A local instructional theory informs researchers how a new approach works so that they might adapt it to their own learning context (Gravemeijer, 2004). The objectives, assumptions, rationale, and related instructional

activities, supplemented by examples from the PRISM, were clearly outlined. Through this approach, it contributed to the development of the concept of local instructional theory.

4.3.2 Implications for practice

The findings of this study provide important insights for improving preservice teachers' knowledge about proving and proof-related beliefs that will be of interest to many in the field of mathematics education, including middle and high school mathematics teachers, elementary school teachers, preservice teachers, teacher educators, mathematicians, mathematics education researchers, educational policymakers, and other educational stakeholders.

The most important finding of this study is the observation that preservice teachers' knowledge of proof and their attitudes and beliefs towards proof changed positively when they were provided with comprehensive, knowledge- and practice-based training to improve their knowledge and beliefs about proof and teaching proof. This finding is crucial to see that the education offered to preservice teachers in universities has the potential to bring preservice teachers and future students to the desired level in terms of proof skills and attitudes towards proof when implemented within the framework of the dimensions of the program implemented in this study. For this reason, teacher educators may consider using the ideas in the program developed for this study for the applications in their own courses. Since a similar application can be applied to teachers in the context of in-service training and professional development, there may be many advantages that the Ministry of National Education, in other words, education policymakers, can consider. A better-

equipped teacher with a positive attitude points to a better-educated student group. This may lead to more successful results in mathematics education.

4.3.3 Limitations of the study

Although the PRISM was tried to be as comprehensive as possible in terms of content, some limitations existed considering the implementation period. In a wider time period, the activities considered can be presented more effectively.

The PRISM program was implemented in a Teaching Geometry course. The fact that the aims of the design of the course and the skills to be developed in the course were parallel formed an important basis for the implementation of the program in this course. In the Teaching Geometry course, proof skills have an important place in learning Euclidean geometry. Since the proof skills targeted in the program are also included in this course, it can be thought that the information learned in the course and the experiences gained have an effect on the data obtained in the program. The effect of the course was not ignored at every stage of the study and the participants were asked to give feedback about the program based on the questions asked and observations made. Especially in written data sources and interviews, the evaluations made about the course were separated. In the questions directed to the participants during the interviews, it was emphasized that they should put aside the contribution of the course and make evaluations only according to the PRISM. Although there are parts related to geometry in the program, examples and activities involving daily life examples and other areas of mathematics separated the program from the course, even if only slightly, so that some parts of the program were distinctive from the course in the evaluations of the participants.

At the end of the program, the interviewees were selected on the basis of convenience sampling as a result of observations and evaluations in the course and the program. Accordingly, preservice teachers who were interested in their own learning and development and who were not afraid to provide feedback on the course and the program, who were slightly more motivated, and who had concerns and efforts about learning and teaching were considered for the interviews. These people may not represent all participants in the PRISM. However, in terms of their success in the proof, the participants were determined to represent the participants in general and not only one success group.

The implementation of the program following one hour of the lessons of the regular geometry teaching course may have affected the performance of the participants. Especially the participants who took a lot of courses during the day and came to this course showed signs of fatigue from time to time during the program application at the end of the course, and a slight decrease in the number of active participation was observed.

4.3.4 Suggestions for future research

The given activities can be revised according to the needs of the preservice teachers, their current situation, and the places where they have difficulties. In the study, some activities were done with expressions from daily life. There were participants who found this useful, as well as participants who thought that the number of mathematical/geometry examples should be increased. In light of these views, although it is useful to work with examples that can be encountered in daily life or entirely fictional examples in activities related to logic rules and proof methods, the number of mathematics and geometry-based examples or activities should be

increased. In this way, the interconnectedness of the rules and the procedures to be followed can be better observed.

The task design assignment provided an opportunity for the participants to apply the proof knowledge they had learned. Repeating this study several times and preparing a learning environment where participants can receive more feedback and brainstorm with other participants may be aspects to be considered in future studies. In addition, participants can be expected to apply the reasoning and proof-based tasks they designed and revise the tasks they prepared as a result of the evaluation of these applications. In this way, participants who gain more experience in teaching proof from the planning phase to the teaching phase can reinforce what they have learned. Similarly, the practice of proof with a middle school or high school student can be practiced several times to increase the preservice teachers' experience with the practice of proof. After developing a proof teaching series consisting of various interrelated proof tasks, participants can be asked to apply these tasks to a student one at a time according to the specified order. With this study, participants can both monitor and support the development of students' proof skills and improve their own teaching skills over time. Feedback can be provided to the participants between the different teaching practices and they can be expected to make reflective evaluations afterwards.

The implementation period of the program can be extended and the timing of the program can be considered in a time period where the participants can be more productive. Acquiring proof skills and learning the knowledge and rules related to proof require a cognitively intensive effort. The program was implemented in the geometry teaching course and the participants benefited, but a separate program or a separate course for preservice teachers could be considered for more practice and

activities. Another application with preservice teachers could be to integrate this program into a different area of mathematics. The data obtained as a result of the program in a course such as algebra teaching can be compared with the data of this study and the common and differentiating findings and their reasons can be discussed.

This program for preservice teachers can also be implemented with teachers as an in-service training or professional development course. Teachers who exhibit similar characteristics to preservice teachers may also have difficulties in proof knowledge and proof teaching knowledge. A program designed first to analyze their knowledge and beliefs and then to develop their knowledge and improve their beliefs, as was done with the preservice teachers, can make mathematics teachers better equipped to teach proof and positively affect their teaching practices. In addition, teachers can be asked to apply the skills they have gained from this program in their own classrooms and make a reflective evaluation of the results of the application. In this way, a ground would be created for students in real classroom environments to be introduced to proof.

APPENDIX A

INTERVIEW QUESTIONS WITH EXPERTS

Semi-Structured Interview Questions with Mathematicians

1. You include proof in various courses in your department. Can you evaluate student competencies at the beginning and end of these courses (especially introductory courses like Math 111) in terms of proof skills?
2. How does a good student prove and what does he need to know? [How should the proof be done and what skills are important/required for this?]
3. Why do you think unsuccessful students can't prove? What factors do you think are effective in this?
4. What would you recommend teaching and gaining skills for high school students or previous graders to be ready to proving?
5. What is the place of proof in your “doing mathematics” process?
6. Do actions such as trial and error, and working on examples contribute to your proof process? Do you include them?

Semi-Structured Interview Questions with Mathematics Teachers

1. What do you think is the place of proof in mathematics?
2. What do you think is the place of proof in school mathematics? In middle school math?
3. Is it difficult to prove? What are the difficulties, if any? What are the reasons?
4. Do you think proof should be taught in schools?

If so, at what level should one start?

How should it be taught?

5. Let's say you are in one of the decision-making positions in the curriculum. It was decided to include proof in the mathematics education curriculum. What grade level would you recommend starting with?
6. How do you see yourself in proving? What's it like to prove for you?
7. When did you first encounter proof?
8. As a teacher, what do you think about proving and teaching proving?
9. What do you think is necessary for students to prove or be ready to prove?
10. Let's say you learned that a new curriculum is being prepared at the end of this month, and the middle school curriculum includes skills related to logic, introduction to proof, reasoning.

How would you feel? (emotional, teacher competencies, etc.)

Would you take action? (What would you do?)

11. Some standards and principles (related to reasoning and proof) determined in mathematics education are emphasized by NCTM. To what extent do you think these principles are applicable at the middle school level?

- ☐ Applicable
- ☐ Reasonable
- ☐ Not Applicable

Semi-Structured Interview Questions with Mathematics Teacher Educators

First of all, I would like to start with teachers and teacher candidates.

1. Based on your observations, how would you evaluate the proof proficiency of teachers/preservice teachers?
2. What do you think are the main reasons for the difficulty in proving?
3. Do you include proof skills and teaching these skills in your undergraduate courses?
4. What should we pay attention to if we want to offer in-service proof training to teachers?
5. What should be offered to prospective teachers for preservice training?
(Should it be?)

Let's continue with the students.

1. Proof proficiency of high school/secondary school students does not show much variation in the literature and a negative picture is drawn. Is it possible to modify this table? Why/how?
2. What do you think are the most critical reasons for this situation?
3. If you had the opportunity to make a big change in the curriculum, would you include proof skills in the curriculum?

Which level would it be more appropriate to start? High school/middle school or else?

What knowledge/skills would you focus on?

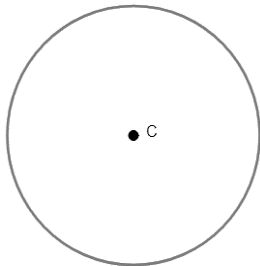
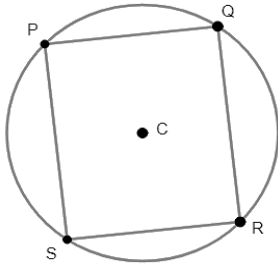
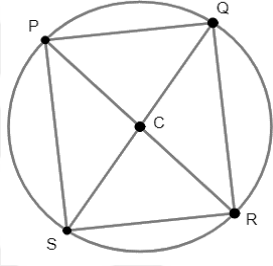
APPENDIX B

TEACHER KNOWLEDGE TEST ON PROOF AND ON TEACHING PROOF (ENGLISH)

1. PART

It is expected you to respond proof items in this part. Please try to answer each item.

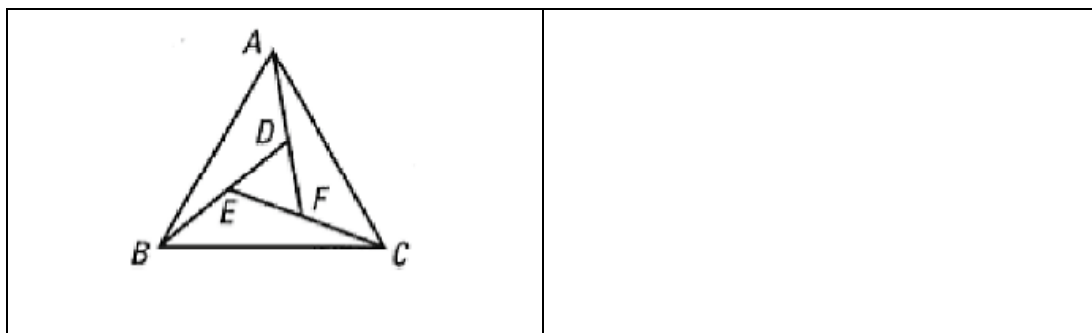
1. Item

Deniz sketches a circle, and he calls the center C.	He then draws a quadrilateral PQRS, whose corners lie on the circle.	He then draws the diagonals of the quadrilateral.
		
Deniz says: “Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross at the center of the circle”.		
Is Deniz right?	YES / NO	
Explain your answer:		

2. Item

Prove the statement below.

Given: $\triangle ABC$ is equilateral, $\angle CAD \cong \angle ABE \cong \angle BCF$	
Prove: $\triangle DEF$ is equilateral.	



3. Item

Here are two statements.

- I. If a figure is a rectangle, then its diagonals bisect each other.
- II. If the diagonals of a figure bisect each other, the figure is a rectangle.

Which is correct?

- A) To prove I is true, it is enough to prove that II is true.
- B) To prove II is true, it is enough to prove that I is true.
- C) To prove II is true, it is enough to find one rectangle whose diagonals bisect each other.
- Ç) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.
- D) None of A-D is correct.

2. PART

You are expected to evaluate some students' responses in proof-related tasks in this part. Please try not to leave any items empty.

4. Item

QUESTION: "Nalan says that if a figure has four sides and the opposite sides are parallel, it must be a square. Do you agree or disagree with Nalan? You can use geometrical shapes in your answer" (adapted from NCTM, 2005, p.78)

The response given by Cem for this question was: "I do not agree with Nalan because a square has 4 equal sides and 2 pairs of opposite sides that are parallel."

What kind of questions should a teacher ask to Cem based on his response?

5. Item:

Evaluate the student arguments for proving the following expression: “The sum of two odd numbers is an even number”

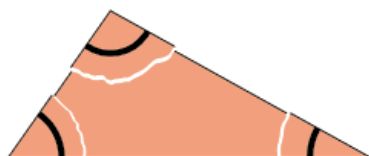
A. Ayşe	B. Burak
x is an odd number. y is an odd number. $2x+1$ ve $2y+1$ are odd numbers. $2x+1+2y+1=2(x+y+1)$ is an even number.	When we group odd numbers by 2, we have 1 left. When we add two odd numbers, leftover ones will form a new group. That is why our sum will be an even number.
C. Ceren	D. Derya
$3+3=6$ $3+5=8$ $3+7=10$ $5+5=10$	All odd numbers can be written as $2n+1$ for every integer n. When we add two odd numbers, we get: $(2k+1)+(2m+1)=(2k+2m)+(1+1)$ $=2k+2m+2=2(k+m+1)$. This is an even number.
E. Emre	F. Fatma
Odd #: ○○ ○○ ○ Odd #: ●● ●● ● ○ When we add, we have an even #.	$1+3=4$ $19+91=110$ $1011+509=1620$ $1000001+7=1000008$

- Which student(s)' response(s) would you accept as correct?
- If this was an exam question, how would you order students' responses from the highest grade to the lowest grade?
- How would you give feedback to students' responses?

6. Item

Identify the suitable method(s) for middle school students from the proofs developed for “The sum of the interior angles of a triangle is 180° ”. Explain the pros and cons of these methods.

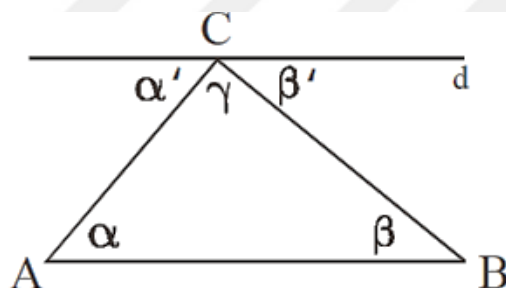
1. Method:



Cutting the angles of a triangle from the paper triangle, assembling the angles to form a straight angle and having students to explore that the sum of the interior angles of a triangle is 180° .

2. Method:

ABC is a triangle and α , β , and γ are the interior angles of this triangle. Hence, we will try to prove that $\alpha + \beta + \gamma = 180^\circ$.



Draw the line d, that is parallel to AB and passes from point C. Name the angles α' and β' as in the drawing. α and α' and also β and β' are congruent since they are alternate interior angles. Hence, α can be substituted for α' and β can be substituted for β' . Since d is a straight line, we get $\alpha + \beta + \gamma = 180^\circ$.

APPENDIX C

TEACHER KNOWLEDGE TEST ON PROOF AND ON TEACHING PROOF

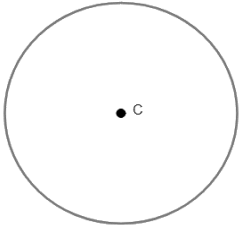
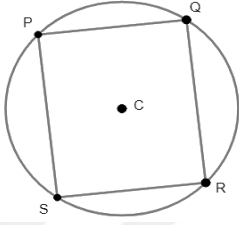
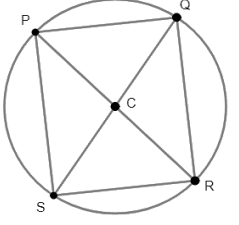
(TURKISH)

MATEMATİK ÖĞRETMENLERİNİN İSPAT VE İSPAT ÖĞRETİMİNE DAİR BİLGİLERİ ÖLÇEĞİ

1.KISIM

Bu kısımda yer alan ispat sorularını yanıtlamanız beklenmektedir. Lütfen her soruyu olabildiğince yanıtlayıp boş bırakmamaya gayret ediniz.

1. Madde

Deniz bir çember çizer ve çemberin merkezini C olarak isimlendirir.	Ardından köşeleri bu çemberin üzerinde olan PQRS dörtgeni çizer.	Son olarak bu dörtgenin köşegenlerini çizer.
		
Deniz'in bu çizimleri ardından oluşan fikri şöyledir: "Köşeleri çemberin üzerinde olan hangi dörtgeni çizersem çizeyim köşegenler her zaman çemberin merkezinde kesişecektir"		
Deniz haklı mıdır?	EVET / HAYIR	
Cevabınızı açıklayınız:		

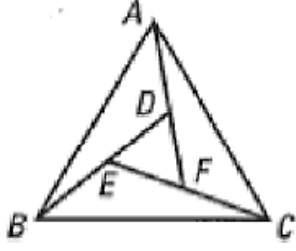
2. Madde

Aşağıdaki ifadeyi ispatlayınız.

Verilenler: $\triangle ABC$ eşkenardır ve
 $\angle CAD \cong \angle ABE \cong \angle BCF$

İspat:

İspatlanacak: $\triangle DEF$ eşkenardır.



3. Madde

Aşağıda iki ifade bulunmaktadır.

- I. Bir şekil dikdörtgense köşegenleri birbirini ortalar.
- II. Bir şeklin köşegenleri birbirini ortalarsa, bu şekil dikdörtgendir.

Buna göre hangisi doğrudur?

- A) I'in doğruluğunu ispatlamak için II'nin doğruluğunu ispatlamak yeterlidir.
- B) II'in doğruluğunu ispatlamak için I'nin doğruluğunu ispatlamak yeterlidir.
- C) II'in doğruluğunu ispatlamak için köşegenleri birbirini ortalamayan bir dikdörtgen bulmak yeterlidir.
- Ç) II'in yanlış olduğunu göstermek için köşegenleri birbirini ortalamayan ve dikdörtgen olmayan bir şekil bulmak yeterlidir.
- D) Yukarıdaki seçeneklerin hiçbiri doğru değildir.

2. KISIM

Bu kısımda yer alan maddelerde bazı öğrencilerin ispat ile ilintili sorulara verdikleri yanıtları değerlendirmeniz beklenmektedir. Lütfen hiçbir maddeyi boş bırakmayınız.

4. Madde

SORU: “Nalan’a göre eğer bir şekil dört kenara sahipse ve bu şeklin karşılıklı kenarları paralelse bu şekil kare olmalıdır. Nalan’ın fikrine katılıyor musun? Cevabında geometrik şekiller kullanabilirsin.” (NCTM, 2005, s.78’den adapte edilip Türkçeleştirildi.)

Bu soruya Cem’in verdiği yanıt şöyle olmuştur: “Nalan’a katılmıyorum çünkü karenin 4 eş kenarı 2 çift karşılıklı paralel kenarı vardır.”

Cem’in ifadesinden yola çıkarak bu konu hakkındaki bildiklerini açığa çıkarmak için öğretmeni ne tür sorular sormalıdır?

5. Madde:

“İki tek sayının toplamı bir çift sayıdır” ifadesini ispatlamak için öne sunulan öğrenci argümanlarını değerlendiriniz.

A. Ayşe	B. Burak
x bir tek sayıdır. y bir tek sayıdır. $2x+1$ ve $2y+1$ tek sayılardır. $2x+1 + 2y+1 = 2(x+y+1)$ çift sayıdır.	Tek sayıları ikili ikili grupladığımızda geriye 1 kalır. İki tek sayıyı topladığımızda geriye kalan 1'ler kendi arasında grup oluşturur. Bu yüzden toplamımızın sonucu çift sayı olur.
C. Ceren	D. Derya
$3+3=6$ $3+5=8$ $3+7=10$ $5+5=10$ Bu yüzden iki tek sayının toplamı hep çift sayı olur.	Tüm tek sayılar n bir tam sayı için $2n+1$ olarak yazılır. İki tek sayıyı topladığımızda: $(2k+1) + (2m+1) = (2k+2m) + (1+1)$ $= 2k+2m+2$ yani $2(k+m+1)$ elde ederiz. Bu da bir çift sayıdır.
E. Emre	F. Fatma
Tek sayı: ●● ●● ●● Tek sayı: ●● ●● ●● =  ●● ●● ●● Topladığımızda sonuç çift sayı	$1+3=4$ $19+91=110$ $1011+509=1620$ $1000001+7=1000008$ Buna göre iki tek sayının toplamı hep çift sayı olur.

a) Hangi öğrenci cevabını/cevaplarını doğru olarak kabul ederdiniz?

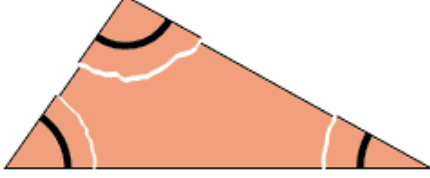
b) Bu bir sınav sorusu olsaydı en yüksek değerlendirmeden en düşük değerlendirmeye öğrenci cevaplarını nasıl sıralardınız?

c) Öğrencilerin cevaplarına nasıl dönütler verirdiniz?

6. Madde

“Üçgenin iç açılarının ölçüleri toplamı 180 derecedir” ifadesini ispatlamak için yapılan sınıf içi uygulamalardan ortaokul öğrencileri için uygun olanları belirleyiniz. Bu uygulamaların güçlü ve zayıf yönlerini açıklayınız.

1. Yöntem:

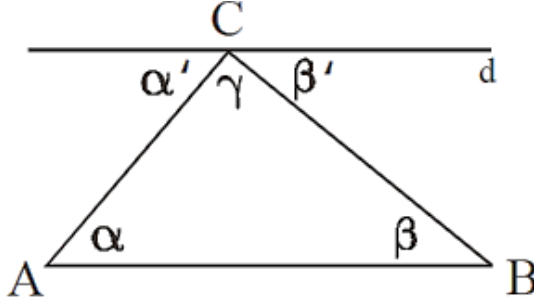


Üçgeni oluşturan açılar kağıt üzerinde kestikten sonra bu açılar birleştirilip doğrusal açı oluşturularak üçgenin iç açıları ölçüler toplamının 180 derece olduğunu keşfettirmek.



2. Yöntem:

ABC bir üçgen ve α, β , ve γ bu üçgenin iç açıları olsun. Öyleyse ispatlamaya çalıştığımız ifade $\alpha + \beta + \gamma = 180^\circ$ olur.



ABC üçgenin AB kenarına paralel ve C noktasından geçecek şekilde d doğrusu çizilir. Çizimde görüleceği gibi bilinmeyen açılar α' ve β' olarak adlandırılır. α ve α' ile β ve β' iç ters açı oldukları için ölçüleri eşittir. Bu yüzden d doğrusu üzerindeki α' yerine α , β' yerine ise β yazılabilir. d doğrusu doğrusal olduğundan $\alpha + \beta + \gamma = 180^\circ$ olur.

APPENDIX D

EXPERT OPINION FORM FOR BELIEFS SCALE

Uzman Görüşü-Öğretmen İnançları

Ortaokul ve lise matematik öğretmenlerinin ispata ve ispat öğretimine dair inançlarını ölçmek için bir ölçek hazırladık. Bu ölçekteki maddelerin karşılık geldiği kategoriye belirleyebilmek adına yardımınıza ihtiyacımız var.

Altı bölümden oluşan bu formda, her bölümde ayrı kategorilere denk geleceğini düşündüğümüz maddeleri listeledik. Sizden istediğimiz bu maddelerin belirtilen kategoriye uygunluk derecesini işaretlemeniz.

Maddeleri belirtilenin dışında bir kategorinin altında görüyorsanız, maddelerin dili ve ifade biçimiyle ilgili düzeltilmesi gereken yerler tespit ederseniz veya başka notlarınız varsa düşüncelerinizi her bölümün altındaki notlar kısmına işlemenizden memnuniyet duyarız.

Desteyiniz için çok teşekkür ederiz.

Melek Pesen & Prof. Dr. Emine Erkin
Boğaziçi Üniversitesi Öğrenme Bilimleri Doktora Programı

Bu bölümde "İspatın doğası ve matematikteki rolü ile ilgili inançlar" kategorisinde olabileceğini düşündüğümüz maddeleri listeledik. Lütfen bu maddelerin uygunluk derecelerini işaretleyiniz.

İspatın doğası ve matematikteki rolü ile ilgili inançlar

	Uygun değil	Olabilir	Uygun	Çok uygun
Matematiksel ifadelerin doğruluğunu göstermek için ispat gerekir.	☐	☐	☐	☐
İspat matematikçiler için en temel çalışmadır.	☐	☐	☐	☐
Matematiksel ispatın temel işlevi ifadeleri sistematik bir şekilde sunarak doğrulamaktır.	☐	☐	☐	☐
Matematiksel ispatın en önemli işlevi matematiksel anlamayı sağlamaktır.	☐	☐	☐	☐
Doğruluğu kesin olan bir ifadeyi yeniden ispatlamak gereksizdir.	☐	☐	☐	☐

Doğruluğu kesin
olan bir ifadeyi
yeniden
ispatlamak
gereksizdir.

☐☐☐☐

Daha önceden
matematikçiler
tarafından
ispatlanan
ifadeleri tekrar
ispatlamak
gereksiz bir iştir.

☐☐☐☐

İspat
matematiğin
vazgeçilmez bir
unsurudur.

☐☐☐☐

İspatın önemine
yapılan vurguyu
anlayamıyorum.

☐☐☐☐

Eklemek istedikleriniz/ notlarınız:

Yanıtınız

Bu bölümde "İspatın okul matematiğindeki rolüne dair inançlar" kategorisinde olabileceğini düşündüğümüz maddeleri listeledik. Lütfen bu maddelerin uygunluk derecelerini işaretleyiniz.

İspatın okul matematiğindeki rolüne dair inançlar

	Uygun değil	Olabilir	Uygun	Çok uygun
İspat matematik dersinde öncelikli bir alan değildir.	☐	☐	☐	☐
İspat soruları öğrencilerin düşünme biçimlerini göstermek adına önemlidir.	☐	☐	☐	☐
İspat öğrencilerin matematiksel kavramları daha iyi öğrenmesine olanak sağlar.	☐	☐	☐	☐
Matematik dersinde yapılan ispatların öğrencilerin düşünme becerilerine faydası olmaz.	☐	☐	☐	☐
İspat teorik matematik için gereklidir, oysaki okul matematiğinde pratik uygulamalar daha önemlidir.	☐	☐	☐	☐

İspat çalışmaları
problem çözme
becerisini
geliştirir.

☐☐☐☐

Matematiksel
dilini anlaşılması
ve ispat
uygulamaları
birbirlerini
destekler.

☐☐☐☐

İspat mantıksal
düşünmeyi
geliştirmek için
gereklidir.

☐☐☐☐

Ekleme istedikleriniz/ notlarınız:

Yanıtınız

Bu bölümde "İspatın okulda öğretimi ile ilgili inançlar" kategorisinde olabileceğini düşündüğümüz maddeleri listeledik. Lütfen bu maddelerin uygunluk derecelerini işaretleyiniz.

İspatın okulda öğretimi ile ilgili inançlar

	Uygun değil	Olabilir	Uygun	Çok uygun
İspat ortaokulda esnek kurallarla sunulmalıdır.	☐	☐	☐	☐
İspat lise düzeyinde esnek kurallarla sunulmalıdır.	☐	☐	☐	☐
Teoremleri ezberletmektense matematiği ispat ile öğretmek daha anlamlıdır.	☐	☐	☐	☐
Formel ispattansa esnek açıklamalar ve gerekçeler benim dersimde yeterli olur.	☐	☐	☐	☐
Öğrencilere teoremlerin ispatı sunulmalı ama öğrencilerden ispat yapmaları beklenmemelidir.	☐	☐	☐	☐

Normalde ispat sayılamayacak argümanlar ortaokul seviyesinde ispat olarak kabul edilebilir.

☐☐☐☐

Matematik ve geometri dersinde kullanılan ifade ve teoremlerin çoğu ispatlanmalıdır.

☐☐☐☐

İspat ile ilgili olan "Nereden biliyorsun?" sorusu öğrencilerime lise/ üniversite sınavlarında yardımcı olur.

☐☐☐☐

Ortaokul düzeyinde bir argümanın matematiksel ifadeler içermese de doğru ve anlaşılır olması yeterlidir.

☐☐☐☐

İspat öğretimine ortaokulda geometri konularından itibaren başlanabilir.

☐☐☐☐

İspat olmayacak argümanları ispat gibi kabul etmek ileriki yıllarda öğrencilerin ispat becerilerine olumsuz olarak yansır.

☐☐☐☐

Öğrenciler için bir teoremin anlamını bilmek ispatını yapmaktan daha önemlidir.

☐☐☐☐

Ortaokul derslerinde ispatların üzerinde durulması öğrencilere ileride faydalı olur.

☐☐☐☐

Örüntüleri keşfetme ve varsayımda bulunma gibi çalışmalar ortaokul seviyesinde öğrencileri ispata hazırlar.

☐☐☐☐

Matematiksel ifadeleri gerekçelendirmede illa cebir bilmeye gerek yoktur.

☐☐☐☐

Ortaokulda öğrencilerden ispat yapması beklenmemelidir.

☐☐☐☐

İspatı ayrı bir konu olarak görmektense bir beceri olarak ele alırsak hemen hemen birçok matematik dersinde ispata yer verebiliriz.

☐☐☐☐

İspata ortaokul matematik müfredatında yer vermeye gerek yoktur.

☐☐☐☐

Matematiksel ispata en erken lisede yer verilmelidir.

☐☐☐☐

Matematiksel ispat en erken üniversite döneminde öğretilmelidir.

☐☐☐☐

Matematiksel ispatı sadece matematik alanında eğitim alabilecek öğrencilere öğretmek gerekir.

☐☐☐☐

Öğrencilerin seviyelerine göre ispat çalışmaları yapmak mümkündür.

☐☐☐☐

Ekleme istedikleriniz/ notlarınız:

Yanıtınız

Bu bölümde "Öğrencilerin ispat yeterlilikleri ile ilgili inançlar" kategorisinde olabileceğini düşündüğümüz maddeleri listeledik. Lütfen bu maddelerin uygunluk derecelerini işaretleyiniz.

Öğrencilerin ispat yeterlilikleri ile ilgili inançlar

	Uygun değil	Olabilir	Uygun	Çok uygun
Ortaokul seviyesinde ispat sadece üstün yetenekli öğrencilere öğretilmelidir.	☐	☐	☐	☐
Ortaokul öğrencileri yapılan ispatları anlayıp ispat oluşturabilecek yetkinliktedir.	☐	☐	☐	☐
Cebirsel ifadeleri kullanamamak ortaokul öğrencilerinin ispat oluşturmalarına engel teşkil eder.	☐	☐	☐	☐
Öğrenciler ispatın önemini ve gereğini kavrayabilirler.	☐	☐	☐	☐
Öğrenciler matematiksel ispat sürecini matematikçilerle benzer şekilde deneyimleyebilir.	☐	☐	☐	☐
Öğrencilerin çoğunluğu ispatlamaları gereken ifadeleri ancak örneklerle gösterebilirler.	☐	☐	☐	☐

Birçok öğrenci matematiksel ispat sırasında verilen ifadeleri sorgulamadan kabul eder.

☐☐☐☐

Bir ifadeyi ispat etmek gelişimsel bir süreç gerektirir. Öğrencilerimizin büyük çoğunluğu gereken seviyeye erişememiştir.

☐☐☐☐

Benim sınıfımda birçok kuralın ve teoremin ispatı öğrenciler tarafından gerçekleştirilmektedir.

☐☐☐☐

Öğrencilerimin bilişsel gelişimleri açısından ispat yapmaya hazır olmadıklarını düşünüyorum.

☐☐☐☐

İspat öğretimi ortaokul seviyesinde ezberden öteye geçemez. Öğrenciler ispatın mantığını bu yaşlarda içselleştiremez.

☐☐☐☐

Ortaokul öğrencilerinin yapabilecekleri ispatlar vardır.

☐☐☐☐

İspatı öğrenmek zordur.

☐☐☐☐

Ekleme istedikleriniz/ notlarınız:

Yanıtınız

Bu bölümde "Öğretmenlerin kendi ispat yeterlilikleri ile ilgili inançları" kategorisinde olabileceğini düşündüğümüz maddeleri listeledik. Lütfen bu maddelerin uygunluk derecelerini işaretleyiniz.

Öğretmenlerin kendi ispat yeterlilikleri ile ilgili inançları

	Uygun değil	Olabilir	Uygun	Çok uygun
Matematiksel ispatlar için gerekli olan teorem, aksiyom ve tanımları kullanabilirim.	☐	☐	☐	☐
Matematiksel ispat için gerekli olan mantık kurallarına hakimim.	☐	☐	☐	☐
Matematiksel ispatları sözlü, sembolik veya gösterimsel olarak ifade edebilirim.	☐	☐	☐	☐
Matematiksel ispat yöntemlerini kolaylıkla kullanabilirim.	☐	☐	☐	☐
Matematiksel ifadeleri koşullu olarak anlamlandırıp ispat edebilirim.	☐	☐	☐	☐
Matematiksel ispat benim için zordur.	☐	☐	☐	☐

Matematiksel ifadelere yapılan ispatları anlamakta güçlük çekmem.	☐	☐	☐	☐
İspat yöntemlerinin hangisini ne zaman kullanmam gerektiği konusunda kafa karışıklıkları yaşayabiliyorum.	☐	☐	☐	☐
İspata sınıf uygulamalarında yer verebilmek ve ispat bilgimi geliştirmek için desteğe/eğitime ihtiyacım var.	☐	☐	☐	☐
İspat yapmaya nasıl başlamak gerektiği konusunda zorluk çekebiliyorum.	☐	☐	☐	☐
İspatları anlamada genellikle zorlanırım.	☐	☐	☐	☐
İspatı öğretmesi zordur.	☐	☐	☐	☐
Öğrencilere ispatı öğretecek ve derslerimde ispat uygulamaları yapacak kadar donanımlı değilim.	☐	☐	☐	☐
İspat yaparken uygulamam gereken stratejileri bazen kestiremiyorum.	☐	☐	☐	☐

Bu bölümde "Çevresel faktörlerin etkisine dair inançlar" kategorisinde olabileceğini düşündüğümüz maddeleri listeledik. Lütfen bu maddelerin uygunluk derecelerini işaretleyiniz.

Çevresel faktörlerin etkisine dair inançlar

	Uygun değil	Olabilir	Uygun	Çok uygun
Müfredat matematik dersinde ispata yer vermeye engel oluyor.	☐	☐	☐	☐
Merkezi sınavlar nedeniyle ispata yer verilemiyor.	☐	☐	☐	☐
Yeni nesil sorular matematik dersinde ispata öncelik vermeyi sağlıyor.	☐	☐	☐	☐
Müfredattaki konuların yoğunluğu derslerde ispata yer vermeyi zorlaştırır.	☐	☐	☐	☐
Okul idareleri derslerde ispata yer verip vermeme konusunda inisiyatifini öğretmenlere bırakır.	☐	☐	☐	☐

Ekleme istedikleriniz/ notlarınız:

Yanıtınız

APPENDIX E

FACTOR LOADINGS OF THE BELIEFS SCALE ON PROOF

	Factor					Uniqueness
	1	2	3	4	5	
Item 23	0.696					0.471
Item 21	0.639					0.393
Item 39	0.595					0.573
Item 7	0.579					0.629
Item 22	0.558					0.548
Item 25	0.524					0.659
Item 2	0.524					0.716
Item 12	0.512					0.680
Rev 20	0.465					0.617
Item 8	0.446					0.722
Item 40	0.430					0.760
Rev 37		0.630				0.478
Rev 31		0.600				0.557
Rev 29		0.597				0.581
Item 9		0.572				0.506
Rev 24		0.493				0.426
Item 18		0.449				0.588
Rev 27		0.434				0.752
Rev 17		0.377				0.744
Rev 35		0.358				0.831
Rev 3		0.331				0.799
Rev 34		0.327				0.841
Item 10			0.653			0.375

	Factor					Uniqueness
	1	2	3	4	5	
Item 5			0.653			0.489
Item 15			0.480			0.591
Rev 1				0.601		0.540
Rev 19				0.566		0.551
Rev 4				0.361		0.747
Rev 33					0.756	0.404
Rev 26					0.724	0.458
Rev 30					0.715	0.400
Rev 36					0.650	0.567
Rev 38					0.413	0.752

Note. 'Principal axis factoring' extraction method was used in combination with a 'varimax' rotation.

APPENDIX F

TEACHER BELIEFS SCALE (ENGLISH)

	Strongly disagree	Disagree	Neutral	Agree	Strongly Agree
It is unnecessary to prove already proven statements.					
Proof enables students to learn mathematical concepts better.					
Proof should be taught only to students who can study mathematics in the future.					
At the middle school level, proof should be taught only to gifted students.					
I can use required theorems, axioms and definitions for proof.					
Proof is not allowed for high-stakes exams.					
Proof is an indispensable component of mathematics.					
It is possible to do proof studies according to the level of students.					
Middle school students can understand proofs and construct proofs.					
I know the logic rules necessary for proof.					
Proof has gained importance due to "new generation questions" in high-stakes exams.					
With the proof questions, we can observe students' thinking.					
Arguments that would not normally be considered proof can be accepted as proof at the middle school level.					
Not being able to use algebraic expressions prevents middle school students from constructing proofs.					
I can express proofs verbally, symbolically or with representations.					

	Strongly disagree	Disagree	Neutral	Agree	Strongly Agree
The density of the subjects in the curriculum makes it difficult to include proof in the lessons.					
Proof should not be a priority area in mathematics courses.					
Teaching of proof can be started from middle school.					
It is unnecessary to prove obviously true statements.					
Proofs made in mathematics classes do not help students' thinking skills.					
Focusing on proofs in middle school lessons will be beneficial for students in the future.					
Students can comprehend the importance and necessity of proof.					
Proof is necessary to develop logical thinking.					
There is no need to include proof in the middle school curriculum.					
Students can experience a real mathematical proof process.					
I am having difficulty with how to start proof.					
In school mathematics, problem-solving is more important than proof.					
Proof should be presented in middle school with flexible rules.					
I think that students are not ready to prove in terms of their cognitive development.					
Mathematical proof is difficult for me.					
In middle school, students should not be expected to prove.					
Many students accept the proof statements without questioning.					

	Strongly disagree	Disagree	Neutral	Agree	Strongly Agree
I can be confused about when to use which proof methods.					
For students, understanding a theorem is more important than proving it.					
Students can only give empirical examples when they are asked to prove.					
I am having trouble understanding some of the proofs.					
The teaching of proof cannot go beyond memorization at the middle school level.					
Proof is difficult to teach.					
We can include proof as a skill in the mathematics lesson.					
If there is a professional development opportunity about teaching proof, I would like to take advantage of it / I would participate.					

APPENDIX G

TEACHER BELIEFS SCALE (TURKISH)

İSPATA VE ÖĞRETİMİNE YÖNELİK ÖĞRETMEN İNANÇLARI ÖLÇEĞİ

Bu ölçek öğretmenlerin ve öğretmen adaylarının ispata ve ispat öğretimine yönelik inançlarını tespit etmek için hazırlanmıştır. Lütfen her maddeyi kendinizi ve öğretmenlik uygulamalarınızı (veya staj deneyimlerinizi) düşünerek değerlendiriniz.

5= Kesinlikle katılıyorum;

4= Katılıyorum;

3= Ne katılıyorum ne katılmıyorum;

2= Katılmıyorum;

1= Kesinlikle katılmıyorum.

	5	4	3	2	1
Daha önceden matematikçiler tarafından ispatlanan ifadeleri tekrar ispatlamak gereksiz bir iştir.					
İspat öğrencilerin matematiksel kavramları daha iyi öğrenmesine olanak sağlar.					
Matematiksel ispatı sadece matematik alanında eğitim alabilecek öğrencilere öğretmek gerekir.					
Ortaokul seviyesinde ispat sadece üstün yetenekli öğrencilere öğretilmelidir.					
Matematiksel ispatlar için gerekli olan teorem, aksiyom ve tanımları kullanabilirim.					
Merkezi sınavlar nedeniyle ispata yer verilemiyor.					
İspat matematiğin vazgeçilmez bir unsurudur.					
İspat çalışmaları problem çözme becerisini geliştirir.					
Öğrencilerin seviyelerine göre ispat çalışmaları yapmak mümkündür.					
Ortaokul öğrencileri yapılan ispatları anlayıp ispat oluşturabilecek yetkinliktedir.					
Matematiksel ispat için gerekli olan mantık kurallarına hakimim.					
Yeni nesil sorular matematik dersinde ispata öncelik vermeyi sağlıyor.					

	5	4	3	2	1
Matematiksel ispatın temel işlevi ifadeleri sistematik bir şekilde sunarak doğrulamaktır.					
İspat soruları öğrencilerin düşünme biçimlerini göstermek adına önemlidir.					
Normalde ispat sayılamayacak argümanlar ortaokul seviyesinde ispat olarak kabul edilebilir.					
Cebirsel ifadeleri kullanamamak ortaokul öğrencilerinin ispat oluşturmalarına engel teşkil eder.					
Matematiksel ispatları sözlü, sembolik veya gösterimsel olarak ifade edebilirim.					
Müfredattaki konuların yoğunluğu derslerde ispata yer vermeyi zorlaştırır.					
Matematiksel ifadelerin doğruluğunu göstermek için ispat gerekir.					
İspat matematik dersinde öncelikli bir alan <u>değildir</u> .					
İspat öğretimine ortaokulda geometri konularından itibaren başlanabilir.					
Ortaokul öğrencilerinin yapabilecekleri ispatlar vardır.					
Matematiksel ispat yöntemlerini kolaylıkla kullanabilirim.					
Doğruluğu kesin olan bir ifadeyi yeniden ispatlamak gereksizdir.					
Matematik dersinde yapılan ispatların öğrencilerin düşünme becerilerine faydası <u>olmaz</u> .					
Ortaokul derslerinde ispatların üzerinde durulması öğrencilere ileride faydalı olur.					
Öğrenciler ispatın önemini ve gereğini kavrayabilirler.					
Matematiksel ifadeleri koşullu olarak anlamlandırıp ispat edebilirim.					
Matematiksel ispatın en önemli işlevlerinden biri matematiksel anlamayı sağlamaktır.					
İspat mantıksal düşünmeyi geliştirmek için gereklidir.					
İspata ortaokul matematik müfredatında yer vermeye gerek yoktur.					

	5	4	3	2	1
Öğrenciler matematiksel ispat sürecini matematikçilerle benzer şekilde deneyimleyebilir.					
İspat yapmaya nasıl başlamak gerektiği konusunda zorluk çekebiliyorum.					
Okul matematiğinde problem çözme ispattan daha önemlidir.					
İspat ortaokulda esnek kurallarla sunulmalıdır.					
Öğrencilerin bilişsel gelişimleri açısından ispat yapmaya hazır olmadıklarını düşünüyorum.					
Matematiksel ispat benim için zordur.					
Ortaokulda öğrencilerden ispat yapması <u>beklenmemelidir.</u>					
Birçok öğrenci matematiksel ispat sırasında verilen ifadeleri sorgulamadan kabul eder.					
İspat yöntemlerinin hangisini ne zaman kullanmam gerektiği konusunda kafa karışıklıkları yaşayabiliyorum.					
Öğrenciler için bir teoremin anlamını bilmek ispatını yapmaktan daha önemlidir.					
Öğrenciler ispatlamaları gereken ifadeleri ancak örneklerle gösterebilirler.					
İspatları anlamada genellikle zorlanırım.					
Örüntüleri keşfetme ve varsayımda bulunma gibi çalışmalar ortaokul seviyesinde öğrencileri ispata hazırlar.					
İspat öğretimi ortaokul seviyesinde ezberden öteye geçemez.					
İspatı öğretmesi zordur.					
İspat bir beceri olarak ele alınırsa birçok matematik dersinde ispata yer verebiliriz.					
Öğrencilerimizin büyük çoğunluğu ispat yapmak için gereken gelişimsel seviyeye erişememiştir.					
İspat öğretimi konusunda kendimi geliştirmek için desteğe/eğitime ihtiyacım var.					
İspat olmayacak argümanları ispat gibi kabul etmek ileriki yıllarda öğrencilerin ispat becerilerine olumsuz olarak yansır.					

APPENDIX H

ETHICAL APPROVALS

Evrak Tarih ve Sayısı: 19.12.2021-43509

T.C.
BOĞAZİÇİ ÜNİVERSİTESİ
SOSYAL VE BEŞERİ BİLİMLER YÜKSEK LİSANS VE DOKTORA TEZLERİ ETİK İNCELEME
KOMİSYONU
TOPLANTI KARAR TUTANAĞI

Toplantı Sayısı : 25
Toplantı Tarihi : 15.12.2021
Toplantı Saati : 15:00
Toplantı Yeri : Zoom Sanal Toplantı
Bulunanlar : Prof. Dr. Ebru Kaya, Prof. Dr. Fatma Nevra Seggie, Dr. Öğr. Üyesi Yasemin Sohtorik İlkmen
Bulunmayanlar :

Melek Pesen
Öğrenme Bilimleri

Sayın Araştırmacı,
"Ortaokul öğrencileri için ispat temelli geometri dersleri geliştirmek" başlıklı projeniz ile ilgili olarak yaptığımız SBB-EAK 2021/84 sayılı başvuru komisyonumuz tarafından 15 Aralık 2021 tarihli toplantıda incelenmiş ve uygun bulunmuştur.

Bu karar tüm üyelerin toplantıya çevrimiçi olarak katılımı ve oybirliği ile alınmıştır. COVID-19 önlemleri kapsamında kurul üyelerinden ıslak imza alınmadığı için bu onay mektubu üye ve raportör olarak Fatma Nevra Seggie tarafından bütün üyeler adına e-imzalanmıştır.

Saygılarımızla, bilgilerinizi rica ederiz.

Prof. Dr. Fatma Nevra SEGGIE
ÜYE

e-imzalıdır
Prof. Dr. Fatma Nevra SEGGIE
Raportör



T.C.
İSTANBUL VALİLİĞİ
İl Millî Eğitim Müdürlüğü

Sayı : E-59090411-44-74641910
Konu : Anket ve Araştırma İzni (Melek PESEN)

18.04.2023

BOĞAZİÇİ ÜNİVERSİTESİ REKTÖRLÜĞÜNE
(Sosyal Bilimler Enstitüsü Müdürlüğü)

İlgi : a) Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 21.01.2020 tarihli ve 2020/2 sayılı genelgesi.
b) Valilik Makamının 17.04.2023 tarihli ve E-59090411-20-74544488 sayılı oluru.

Valilik Makamının Anket ve Araştırma İzni konulu ilgi (b) oluru ve kullanılması uygun görülen ölçme araçlarının Müdürlüğümüzce mühürlenmiş örnekleri ekte gönderilmiştir.

İlgi (a) genelgenin 28. maddesinde; "Araştırma uygulama izni alan kamu kurum ve kuruluşları, uluslararası kuruluşlar, üniversiteler, sivil toplum kuruluşları ve araştırmacılar tamamladıkları bilimsel araştırma ile ilgili sonuç raporlarını, izni aldıkları ilgili birime çalışma bitiminden itibaren 30 gün içerisinde göndereceklerdir." ifadesi yer almaktadır.

Olur gereğince işlem yapılması ve araştırma sonuç raporunun ekte sunulan örneğe göre Müdürlüğümüz Strateji Geliştirme Şubesine gönderilmesi hususlarında gereğini arz ederim.

Mustafa ERŞAHİN
İl Millî Eğitim Müdürü a.
Şube Müdürü

Ek:
1- Valilik Oluru (1 Sayfa)
2- Rapor Örneği
3- Ölçekler

APPENDIX I

RUBRIC FOR THE KNOWLEDGE TEST

Item 1 (max 4 pt.)

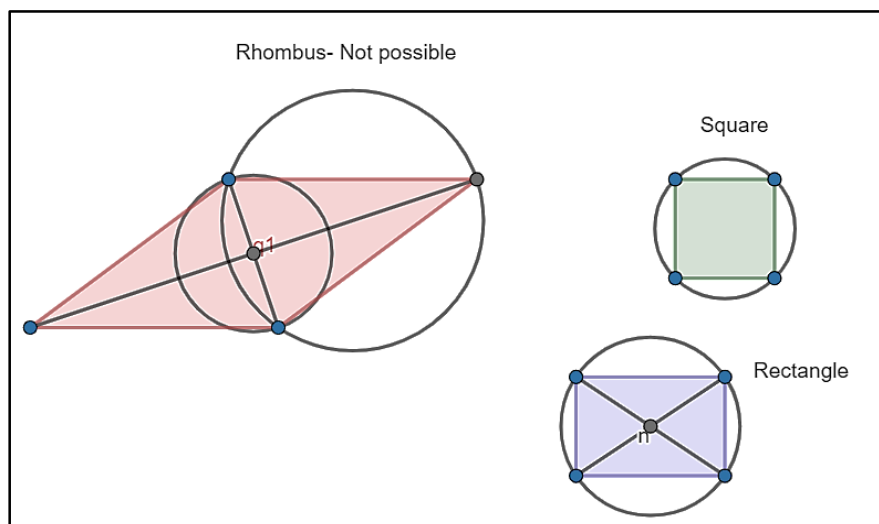
a) The response is correct (1 pt.) / The response is false (0 pt.)

b) *(If it is correct)*

- The explanation is false (0 pt.) /
- The explanation is correct but relies only on repetition of an authority (teacher/book/a knowledgeable peer) source (1pt.)
- The explanation is correct and based on empirical justifications (2 pt.) /
- The explanation is correct and based on formal justifications (3 pt.)

HOW TO INTERPRET IT:

Deniz is not right. A counter-example can be given (most usually think of a trapezoid with all the vertices halfway around the circle). The answer we want as a formal justification could be as follows: For the diagonals to be diameters, the angles seeing the diameters must be 90 degrees. This is only possible for rectangles and squares. The most common justification given by students is that this statement is not valid for non-regular quadrilaterals. However, a rectangle is not a regular quadrilateral either. Therefore, the reasoning that this is only possible for regular quadrilaterals is incorrect (because this cannot be true for a rhombus).



Item 2 (max 5pt.)

Justifications	
Available	2
Not available	0
Partial	1
Is the proof completed?	
Yes	1
No	0
Are the hypotheses correct and formal?	
Yes	1
No	0
Are the concepts and theorems used correctly?	
Yes	1
No	0

HOW TO INTERPRET:

ABC is an equilateral triangle and the angles CAD, ABE, and BEF are congruent (say α). Using this information, the prospective teacher should state that the angles

BAD, CBE, and ACF are congruent and have measure $(60-a)$. From here on, it can be proceeded in two ways:

The measures of the angles ADB, CFA, and BEC are 120 degrees. Because the sum of the measures of the interior angles in triangles is 180 degrees. Since the sum of the measures a and $60-a$ gives 60 degrees, the third angle with unknown measure must be 120 degrees. The measures of the angles FDB, EFA, and DEC must be 60 degrees because these angles form a linear angle with ADB, CFA, and BEC (linear pair in English). Since all the angles of the DEF triangle are 60 degrees, it is an equilateral triangle.

The measures of the angles FDB, EFA, and DEC must be 60 degrees because the sum of the measures of the two interior angles equals the exterior angle of the other angle ($a+(60-a)=60$). Since all the angles of the triangle DEF are 60 degrees, this triangle is an equilateral triangle.

Item 3 (max 1 point):

The correct response is option D (1 point). If another option is selected the respondent gets 0 points. If multiple choices were selected (even if one of them is option D) the respondent gets 0 points. If it left unanswered, the participant gets 0 points.

Item 4 (max 2 pt.)

- The questions are based on remembering rules and facts only (memorization) and do not promote student reasoning and justifications (0 pt.)
- The questions emphasize the relationship between the properties of polygons but do not promote student reasoning and justifications (1 pt.)

- The questions are built around the relationship between the properties of polygons, and they promote student reasoning and justifications (2 pt.)

HOW TO INTERPRET:

The purpose of this question is to examine the student's knowledge and ability to ask questions to reveal misconceptions. According to Nalan's explanation, there are 4 shapes that have four sides and whose opposite sides are parallel: Parallelogram, rectangle, square, and rhombus. Nalan only mentioned the square. Cem's opposition to this statement was based on a new assumption: That the shape also has equal sides.

The questions to be asked to reveal what Cem knows can be categorized as follows:

Response	Evaluation
I disagree with Nalan because a parallelogram also has four sides and the opposite sides are parallel, we cannot say that it must be a square.	Here, no question was asked to reveal Cem's thoughts. Nalan's statement was evaluated directly. Therefore, this kind of answer should be given 0 points.
Do the properties of the rectangle match the properties I have given? Do the properties of the parallel side match the properties I have given?	The existence of other quadrilaterals is pointed out here, but no question is asked about Cem's reasoning.
• Does having 4 equal sides make a difference?	Here, questions were asked about the incorrect reasoning in the student's argument. At the same time, attention was

- Does it matter if there are congruent or parallel sides? - Are there only square quadrilaterals with parallel sides?	drawn to quadrilaterals with similar properties.
---	--

Item 5a (max 3pt.)

- Only informal options are accepted/ No answer----- 0pt.
- Some informal and formal options are accepted / All options are accepted----- 1pt.
- Some formal options are accepted [In this case, Ayşe's response might be included]----- 2 pt.
- All formal options are accepted (Derya, Burak, Emre) [In this case, Ayşe's response should not be included]----- 3 pt.

<i>Non-Proofs</i>	<i>Proofs</i>
Fatma	Burak
Ceren	Derya
Ayşe*	Emre
Ayşe's argument might seem like proof. However, her generalizations apply only to the addition of odd numbers obtained from odd numbers.	

HOW TO INTERPRET:

Answers in different combinations were observed. Although Ayşe's answer can be considered as a proof, she used an odd number to express the sum of two odd numbers to obtain an odd number. Therefore, not every situation is covered. Some of

the preservice teachers may tend to accept this answer as a proof considering the formal structure of the argument.

Sample scoring can be done as follows:

- I would accept all of them (1 point)
- Derya, Fatma, Emre (1 point)
- Ayşe, Derya, Burak, Emre (2 points)
- Ayşe, Derya (2 points)
- Burak, Derya, Emre (3 points)

Item 5b (max 2 pt.)

- In the ordering, all non-proof arguments are given higher grades/ Not all arguments evaluated (0 pt.)
- In the ordering, some non-proof arguments are given higher grades (1 pt.)
- In the ordering, all formal arguments are given higher grades (2 pt.)

HOW TO INTERPRET:

For 2 points, the expected answer is: $D > B > E > A > F > C$ (D, B, E can be greater than or equal to each other, but must be greater than A, and A must be greater than C and F (C and F can be greater than or equal to each other, but the best assessment is that F is greater than C). The question asks everyone to make an assessment and rank them accordingly. Answers that do not address all options should be awarded 0 points.

- $D > A > B > E > F > C$ (1 mark, because an answer without formal proof is evaluated high)
- Emre, Burak, Derya, Ayşe, Ceren = Fatma (2 points)

- I would give them all full marks (1 point)
- Derya highest, Burak lowest (0 points)
- Ceren > Ayşe > Emre (0 points)

Item 6 (max 3 pt.)

- No answer/ A method was stated without a rationale/ No option was selected; only pros and cons are listed. (0 pt.)
- Informal argument is favored without considering proof skills. (1 pt.)
- Formal argument is favored (considering factors other than proof skills) (2 pt.)
- Formal argument is favored (considering proof skills) (3 pt.)

HOW TO INTERPRET:

Method 1 cannot be considered as a proof, it can only be considered as a demonstration. It will be common to find student teachers who think that this can be considered as a proof for students with lower grade levels/middle school students. In this case, they should be evaluated as preferring informal argumentation (1 or 2 points depending on the situation).

APPENDIX J

DIMENSIONS OF THE BELIEFS SCALE AND ITEMS IN EACH DIMENSION

(ENGLISH & TURKISH)

Role and Importance of Proof (Factor 1)	
Item 23	Proof is necessary to develop logical thinking. (İspat mantıksal düşünmeyi geliştirmek için gereklidir.)
Item 21	Focusing on proofs in middle school lessons will be beneficial for students in the future. (Ortaokul derslerinde ispatların üzerinde durulması öğrencilere ileride faydalı olur.)
Item 39	We can include proof as a skill in the mathematics lesson. (Matematik dersinde bir beceri olarak ispata yer verebiliriz.)
Item 7	Proof is an indispensable component of mathematics. (İspat matematiğin vazgeçilmez bir unsurudur.)
Item 22	Students can comprehend the importance and necessity of proof. (Öğrenciler ispatın önemini ve gereğini kavrayabilirler.)
Item 25	Students can experience a real mathematical proof process. (Öğrenciler gerçek bir matematiksel ispat sürecini deneyimleyebilirler.)
Item 2	Proof enables students to learn mathematical concepts better. (İspat öğrencilerin matematik kavramlarını daha iyi öğrenmesini sağlar.)
Item 12	With the proof questions, we can observe students' thinking. (İspat soruları ile öğrencilerin düşünme biçimlerini gözlemleyebiliriz.)
Item 20	Proofs made in mathematics classes do not help students' thinking skills. (Matematik dersinde yapılan ispatların öğrencilerin düşünme becerilerine faydası olmaz.)

Item 8	It is possible to do proof studies according to the level of students. (<i>Öğrencilerin seviyelerine göre ispat çalışmaları yapmak mümkündür.</i>)
Item 40	İspat öğretimi konusunda bir mesleki gelişim fırsatı olursa değerlendirmek isterim/katılıyorum. (<i>İspat öğretimi konusunda bir mesleki gelişim fırsatı olursa değerlendirmek isterim/katılıyorum.</i>)
Place of Proof in Middle School (Factor 2)	
Item 17	Proof should not be a priority area in mathematics courses. (<i>İspat matematik dersinde öncelikli bir alan olmamalıdır.</i>)
Item 37	The teaching of proof cannot go beyond memorization at the middle school level. (<i>İspat öğretimi ortaokul seviyesinde ezberden öteye geçemez.</i>)
Item 31	In middle school, students should not be expected to prove. (<i>Ortaokulda öğrencilerden ispat yapması beklenmemelidir.</i>)
Item 29	I think that students are not ready to prove in terms of their cognitive development. (<i>Öğrencilerin bilişsel gelişimleri açısından ispat yapmaya hazır olmadıklarını düşünüyorum.</i>)
Item 9	Middle school students can understand proofs and construct proofs. (<i>Ortaokul öğrencileri ispatları anlayıp ispat oluşturabilir.</i>)
Item 24	There is no need to include proof in the middle school curriculum. (<i>Ortaokul müfredatında ispata yer vermeye gerek yoktur.</i>)
Item 18	Teaching of proof can be started from middle school. (<i>İspat öğretimine ortaokuldan itibaren başlanabilir.</i>)
Item 27	In school mathematics, problem solving is more important than proof. (<i>Okul matematiğinde problem çözme ispattan daha önemlidir.</i>)
Item 35	Students can only give empirical examples when they are asked to prove.

	(Öğrenciler ispatlamaları gereken ifadeleri ancak örneklerle gösterebilirler.)
Item 3	Proof should be taught only to students who can study mathematics in the future. (İspatı sadece matematik alanında eğitim alabilecek öğrencilere öğretmek gerekir.)
Item 34	For students, understanding a theorem is more important than proving it. (Öğrenciler için bir teoremi kavramak ispatını yapmaktan daha önemlidir.)
Mastery in Proof-Related Skills (Factor 3)	
Item 5	I can use required theorems, axioms and definitions for proof. (İspat için gerekli teorem, aksiyom ve tanımları kullanabilirim.)
Item 10	I know the logic rules necessary for proof. (İspat için gerekli mantık kurallarını bilirim.)
Item 15	I can express proofs verbally, symbolically or with representations. (İspatları sözlü, sembolik veya gösterimsel olarak ifade edebilirim.)
The Need for proof (Factor 4)	
Item 1	It is unnecessary to prove already proven statements. (Daha önceden ispatlanan ifadeleri tekrar ispatlamak gerekmez.)
Item 19	It is unnecessary to prove obviously true statements. (Doğruluğu kesin olan bir ifadeyi yeniden ispatlamak gereksizdir.)
Item 4	At the middle school level, proof should be taught only to gifted students. (Ortaokul seviyesinde ispat sadece üstün yetenekli öğrencilere öğretilmelidir.)
Ability to do Proof (Factor 5)	
Item 26	I am having difficulty with how to start proof.

	(İspat yapmaya nasıl başlamak gerektiği konusunda zorluk çekebiliyorum.)
Item 33	I can be confused about when to use which proof methods. (İspat yöntemlerinin hangisini ne zaman kullanmam gerektiği konusunda kafa karışıklıkları yaşayabiliyorum.)
Item 30	Mathematical proof is difficult for me. (Matematiksel ispat benim için zordur.)
Item 36	I am having trouble understanding some of the proofs. (Bazı ispatları anlamakta güçlük çekiyorum.)
Item 38	Proof is difficult to teach. (İspatı öğretmesi zordur.)

APPENDIX K

CORRELATION MATRIX FOR THE KNOWLEDGE AND BELIEFS SCORES

		Total know.	CK	PCK	Total belief	F1	F2	F3	F4
CK	r	0.900** *	-						
	p	< .001	-						
PCK	r	0.716** *	0.341** *	-					
	p	< .001	< .001	-					
Total belief	r	0.347** *	0.311** *	0.251** *	-				
	p	< .001	< .001	< .001	-				
F1	r	0.371** *	0.317** *	0.293** *	0.819** *	-			
	p	< .001	< .001	< .001	< .001	-			
F2	r	0.307** *	0.272** *	0.227** *	0.856** *	0.576** *	-		
	p	< .001	< .001	< .001	< .001	< .001	-		
F3	r	0.245** *	0.222** *	0.173** *	0.501** *	0.414** *	0.213** *	-	
	p	< .001	< .001	0.02	< .001	< .001	< .001	-	
F4	r	0.248** *	0.252** *	0.132*	0.553** *	0.392** *	0.436** *	0.154**	-
	p	< .001	< .001	0.021	< .001	< .001	< .001	0.007	-
F5	r	-0.068	-0.052	-0.064	0.470** *	0.120*	0.266** *	0.298** *	0.029
	p	0.235	0.367	0.263	< .001	0.036	< .001	< .001	0.613

Note 1. * $p < .05$, ** $p < .01$, *** $p < .001$

Note 2. "Total know." represents total knowledge score on knowledge test on proof.

"CK" represents content knowledge score on the same test. "PCK" represents pedagogical content knowledge score on the same test.

Note 3. "Total belief" represents total belief score on beliefs scale on proof.

Note 4. "F1" is the average score of Factor 1: Role and importance of proof. "F2" is the average score of Factor 2: Place of proof in middle school. "F3" is the average score of Factor 3: mastery in proof-related skills. "F4" is the average score of Factor 4: The need for proof. "F5" is the average score of Factor 5: Ability to do proof.

APPENDIX L

SAMPLE QUIZ IN DESIGN STUDY

Rewrite the statements in the if-then form:

1. Rachel always completes her homework.
2. Alex goes to school every weekday.
3. All congruent segments have the same length.

Read the text first, and mark each conclusion as "true (T)," "false (F)," or "not certain (NC)."

“In answer to your question what we got out of English so far I am answering that so far I got without a doubt nothing out of English. Teachers were sourcastic sourpuses or nervous wrecks. Half the time they were from other subjects or only subs.”

“Also no place to learn. Last term we had no desks to write only wet slabs from the fawcets because English was in the Science Lab and before that we had no chairs because of being held in gym where we had to squatt.”

“Even the regulars Mrs. Lewis made it so boering I wore myself out yawning and Mr. Loomis (a Math) hated teaching and us.”

BEL KAUFMAN, *Up the Down Staircase*

The student writing this essay dislikes English. _____

There are several spelling mistakes in this essay. _____

The school is overcrowded. _____

Mrs. Lewis was not one of the substitute teachers. _____

Mr. Loomis would rather teach mathematics than English. _____

APPENDIX M

SAMPLE MIDTERM AND FINAL QUESTIONS

Midterm Questions

Q1. Prove the following statements. State the proof method you use.

- a) If two angles form a linear pair, then bisectors of these angles form a right angle.
- b) If segments drawn from the endpoints of the base of an isosceles triangle are perpendicular to the opposite sides, then those segments are congruent.

Q2. For each statement provided below, determine whether the conclusions presented afterward can be logically derived from the given statement. Mark your answers as “True” if the conclusion can be reached, “False” if it cannot. You can consult theorems/axioms/definitions that you have learned and proved so far.

- a) *Statement:* The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles.

Conclusion: The sum of the measures of the exterior angles of a triangle is 360° .

- b) *Statement:* If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

Conclusion: If two sides of a triangle are congruent, then the sides opposite sides are congruent.

- c) *Statement:* If a quadrilateral has two pairs of parallel sides, then the quadrilateral is a parallelogram.

Conclusion: If one pair of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Final Questions

Q1. Decide whether each of the following statements is true in ALL cases, true in SOME cases and false in others, or true in NO case. Please write ALL, SOME, or NO as your decision and justify your response as follows:

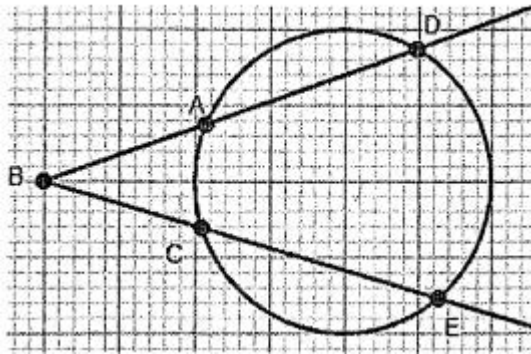
If it is true in ALL cases, prove it by indicating which theorems, axioms, rules and definitions you use.

If it is true in NO case, disprove it by indicating which theorems, axioms, and definitions you use.

If it is true in SOME cases, give an example of a case which makes the statement FALSE.

- a) A set of parallel lines intercepts congruent segments on any transversal.
- b) If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.
- c) Given a correspondence A-D, B-E, C-F between two triangles, if $m(\angle C) = m(\angle F) = 90^\circ$ and $|AB| = |DE|$, then these triangles are similar.
- ç) In a circle, if two chords are equidistant from the center, then they are congruent.

Q3. In the figure below, a circle and two secant lines are shown. Let $m(\text{arc } AC) = x$ and $m(\text{arc } DE) = y$. Find the measure of $\angle DBE$ in terms of x and y . Justify your result.



APPENDIX N

INTERVENTION- TASK DESIGN ASSIGNMENT IN THE PROGRAM

In this assignment, you are expected to design a task related to the MEB objectives assigned to you. The task should satisfy the following criteria:

- It should emphasize justification and reasoning.
- ...

Please explain

In what ways does your task promote justification and reasoning?



APPENDIX O

FINAL EXAM TAKE-HOME PART

Directions:

- In this assignment, you are expected to ask a geometry proof question to a middle school or lower-secondary school (ninth/tenth grade) student.
- Choose at least **one of the tasks** listed below (You may also prefer another task; if you do so, please provide its details and state the reason for your task selection).
- While the student is working on the task, write down your observations on the responses and reactions of the student. You can mention emotional/affective reactions (worry, fear, etc.), too.
- If the student struggles with the task, explain how to proceed and direct questions such as the following: “Can you explain how we proceed to produce a proof?”, “What do you think we should indicate to be able to prove a statement?”, “If you encounter a similar geometry proof task can you prove it? How confident are you?”

In your report, give details on:

1. The grade level of the student.
2. Selected task and the reason(s) for selecting it.
3. How the student approached the task and evaluation of the student’s overall performance (Whether s/he could prove it, did s/he use examples or general statements, did s/he refer to theorems, definitions, or rules, etc.).
4. Directions and guidance that you presented to the students and why you needed those directions/guidance (to clarify the ideas, to further challenge the student, to understand her/his approach, etc.).
5. Reflection about your own performance as a teacher (whether you could promote proof skills, confidence to teach proof skills etc.).

TASKS

1. Prove that the sum of the interior angles of a triangle is 180 degrees.
2. Prove that the sum of the exterior angles of a triangle (or any polygon) is 360 degrees.
3. Prove that the measure of any exterior angle is equal to the sum of the measure of two remote interior angles.
4. Prove that the measures of alternate interior angles are equal when two parallel lines are cut by a transversal.
- 5.

Write a proof.

Given: $\triangle ABC$ with perpendicular bisector $\overline{BD}$

Prove: $\triangle ABD \cong \triangle CBD$ and
 $\angle A \cong \angle C$

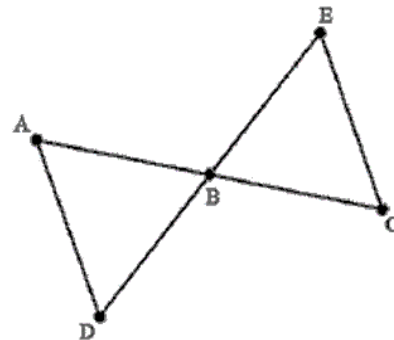


6.

Write a proof.

Given: $\overline{AC}$ and $\overline{DE}$ bisect each other at B

Prove: $\overline{AD} \cong \overline{EC}$



APPENDIX P

INTERVIEW QUESTIONS AFTER AT THE END OF THE DESIGNED
PROGRAM

1. First of all, when you think about the content of the activities, how did you find their quality? How did you find their intensity?
2. When you take a look at your memory, what did you remember the most from these applications? Why?
3. Was there an activity that you had difficulty or had difficulty understanding?
4. I tried to present examples from daily life that were not related to mathematics or geometry; do you think this was helpful? Would it have been better to emphasize mathematics or geometry more instead?
5. Would it be meaningful to support students taking this course next semester in the same subject? Do you have any suggestions for changes or adjustments? Are there any points during the semester where you can think that you need more support in these areas?
6. How do you feel when you think of yourself as someone who is learning how to prove yourself? Can you tell us about your self-confidence, competence, and feelings?
7. How do you feel when you think of yourself as someone who will teach proof? Can you tell us about your self-confidence, competence, and emotional state?
8. How was the take-home part?

9. Let's see how you did on the scales I administered at the beginning of the semester; how do you think of yourself now?



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