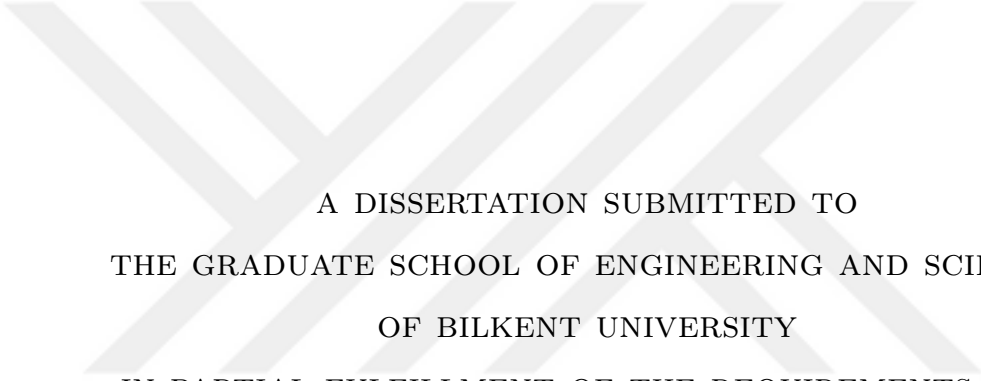


INVESTIGATIONS INTO THE EVOLUTION OF HEATED LIQUID FILMS



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By
Omair A. A. Mohamed
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Luca Biancofiore(Advisor)

Yegân Erdem

A. Kerem Uğuz

Christian Ruyer-Quil

A. Alperen Günay

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

INVESTIGATIONS INTO THE EVOLUTION OF HEATED LIQUID FILMS

Omair A. A. Mohamed
Ph.D. in Mechanical Engineering
Advisor: Luca Biancofiore
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The evolution of the free surface of a heated liquid film is directly tied to the performance and efficiency of various industrial systems. Therefore, we investigate the spatiotemporal evolution of heated liquid films across a range of different settings by formulating distinct hydro-thermal models taking into account the effects of inertia, thermocapillarity, evaporation, gas shear, and thermal radiation, where we direct our modeling efforts in each problem on the most dominant physical phenomena. In liquid flows characterized by relatively low Reynolds numbers belonging to the drag-gravity flow regime, we model the hydrodynamics of the film using the long-wave expansion (LWE) methodology and perform linear stability analyses focused on the thermocapillary and evaporative instabilities, as they have a primary influence on the film's evolution in this flow regime. Consequently, the evaporation process is governed by the competition between thermodynamic disequilibrium and diffusion effects dependent on the interface's curvature. We modify the kinetic-diffusion evaporation model of Sultan et al. [Sultan et al., *J. Fluid Mech.* 543, 183, (2005)] and combine it with long-wave theory to derive a governing equation encapsulating the coupled dynamics. We then utilize linear stability theory to derive the system's dispersion relationship, in which the Marangoni effect has two components. The first results from surface tension gradients driven by the uneven heat flux, while the second arises from surface tension gradients caused by imbalances in vapor diffusion. These two components interact with evaporative mass loss and vapor recoil in a rich and complex manner. Moreover, we identify an evaporation regime where a volatile film is devoid of evaporation instabilities. Furthermore, we investigate the effect of film thinning on its stability at the two opposing limits of the evaporation regime, where we find its impact in the diffusion-limited regime to be dependent on the intensity of evaporative phenomena. Finally, we conduct a spatiotemporal analysis which indicates that vapor diffusion effects are correlated with a shift

towards absolute instability. In the second problem, we study the spatiotemporal evolution of an evaporating liquid film sheared by a gas and consider both the inertial and thermal instability modes, where the shearing gas is modeled by imposing a constant shear stress along the liquid's interface. Interestingly, its inclusion in the problem allows the utilization of a one-sided evaporation model, which is precisely the transfer-rate-limited case of the first system we investigated. Once more long-wave theory is used to derive the an evolution for the liquid film which incorporates the role of the shearing gas. Afterwards, linear stability theory is used to investigate the temporal and spatiotemporal characteristics of the flow, where it is found that the evaporation of the film promotes absolute instabilities and can cause convective/absolute transitions. We also find that counter-flowing shearing gas can suppress the inertial instability affirming similar conclusions found by previous studies for a strongly confined isothermal film. Furthermore, the evolution interface equation was solved numerically to explore the film's nonlinear stability. Moreover, we employ self-similarity analysis to probe the shear stress's effect on the film's rupture mechanics. In the third problem we research, the liquid flow's Reynolds number is relatively high, and hence we utilize the weighted-residual integral boundary layer (WIBL) technique [C. Ruyer-Quil and P. Manneville," Eur. Phys. J. B, **15**, 357, (2000)], and direct our attention at directly simulating the temperature field across the film using reduced models. The WIBL hydrodynamic equations are derived expressions obtained via the boundary layer approximation, while the thermal profile is modeled by employing an asymptotic expansion which produces a hierarchy of models in which enhanced sophistication is offset by higher complexity and computational cost. These models are solved numerically revealing how the temperature field across the film is governed by a balance between the conduction across both the liquid film and the solid surface, and their respective radiative emissions, wherein these two transfer phenomena are linked through two corresponding dimensionless numbers associated with both the liquid film and the solid surface.

Keywords: Liquid films, thermocapillarity, evaporation, radiation.

ÖZET

ISITILAN SIVI FİLMLEİN EVRİMİNE İLİŞKİN ARAŞTIRMALAR

Omair A. A. Mohamed
Makine Mühendisliği, Doktora
Tez Danışmanı: Luca Biancofiore
Ağustos 2024

Isıtılmış sıvı filminin serbest yüzeyinin evrimi, çeşitli endüstriyel sistemlerin performansına ve verimliliğine doğrudan bağlıdır. Bu nedenle, atalet, termokapilare, buharlaşma, gaz kayması ve termal radyasyonun etkilerini hesaba katarak farklı hidro-termal modeller formüle ederek, farklı ortamlarda ısıtılmış sıvı filmlerinin uzaysal-zamansal evrimini araştırıyoruz ve her problemde modelleme çabalarımızı en baskın fiziksel olgulara yönlendiriyoruz. Viskozite-yerçekmi akış rejimine ait nispeten düşük Reynolds sayılarıyla karakterize edilen sıvı akışlarında, filmin hidrodinamiğini uzun dalga genleşme (LWE) metodolojisini kullanarak modelliyoruz ve termokapiler ve buharlaşma kararsızlıklarına odaklanan doğrusal kararlılık analizleri gerçekleştiriyoruz, çünkü bunlar bu akış rejiminde filmin evrimi üzerinde birincil bir etkiye sahiptir. Sonuç olarak, buharlaşma süreci, arayüzün eğriliğine bağlı termodinamik dengesizlik ve difüzyon etkileri arasındaki rekabet tarafından yönetilir. Sultan vd. tarafından yapılan kinetik-difüzyon buharlaşma modelini değiştiriyoruz. [Sultan vd., J. Fluid Mech. 543, 183, (2005)] ve bunu uzun dalga teorisiyle birleştirerek bağlı dinamikleri kapsayan bir yönetici denklem türetiyoruz. Daha sonra, Marangoni etkisinin iki bileşeni olduğu sistemin dispersiyon ilişkisini türetmek için lineer kararlılık teorisini kullanıyoruz. İlki, düzensiz ısı akışı tarafından yönlendirilen yüzey gerilim gradyanlarından kaynaklanırken, ikincisi buhar difüzyonundaki dengesizliklerden kaynaklanan yüzey gerilim gradyanlarından kaynaklanmaktadır. Bu iki bileşen, buharlaşma kütle kaybı ve buhar geri tepmesi ile zengin ve karmaşık bir şekilde etkileşime girer. Dahası, uçucu bir filmin buharlaşma kararsızlıklarından yoksun olduğu bir buharlaşma rejimi belirliyoruz. Ayrıca, film incelmesinin buharlaşma rejiminin iki zıt sınırındaki kararlılığı üzerindeki etkisini araştırıyoruz ve difüzyonla sınırlı rejimdeki etkisinin buharlaşma olaylarının yoğunluğuna bağlı olduğunu buluyoruz. Son olarak, buhar difüzyon etkilerinin mutlak kararsızlığa doğru bir kayma ile ilişkili olduğunu gösteren bir uzaysal-zamansal

analiz yürütüyoruz. İkinci problemde, bir gaz tarafından kesilen buharlaşan bir sıvı filminin uzaysal-zamansal evrimini inceliyoruz ve kesme gazının sıvının arayüzü boyunca sabit bir kesme gerilimi uygulayarak modellendiği hem atalet hem de termal kararsızlık modlarını ele alıyoruz. İlginç bir şekilde, probleme dahil edilmesi, incelediğimiz ilk sistemin aktarım hızı sınırlı durumu olan tek taraflı bir buharlaşma modelinin kullanılmasına izin veriyor. Bir kez daha, kesme gazının rolünü içeren sıvı film için bir evrim türetmek için uzun dalga teorisi kullanılıyor. Daha sonra, doğrusal kararlılık teorisi, akışın zamansal ve uzaysal-zamansal özelliklerini araştırmak için kullanılıyor ve burada filmin buharlaşmasının mutlak kararsızlıkları desteklediği ve konvektif/mutlak geçişlere neden olabileceği bulunuyor. Ayrıca, ters akışlı kesme gazının, önceki çalışmalarda güçlü bir şekilde sınırlandırılmış izotermal film için bulunan benzer sonuçları doğrulayarak eylemsizlik kararsızlığını bastırabileceğini bulduk. Dahası, evrim arayüzü denklemi, filmin doğrusal olmayan kararlılığını keşfetmek için sayısal olarak çözüldü. Dahası, kesme geriliminin filmin kopma mekaniği üzerindeki etkisini araştırmak için öz-benzerlik analizini kullanıyoruz. Araştırdığımız üçüncü problemde, sıvı akışının Reynolds sayısı nispeten yüksektir ve bu nedenle ağırlıklı rezidüel integral sınır tabakası (WIBL) tekniğini [C. Ruyer-Quil ve P. Manneville, "Eur. Phys. J. B, **15**, 357, (2000)] kullanıyoruz ve dikkatimizi indirgenmiş modeller kullanarak film boyunca sıcaklık alanını doğrudan simüle etmeye yönleltiyoruz. WIBL hidrodinamik denklemleri, sınır tabakası yaklaşımı yoluyla elde edilen türetilmiş ifadelerdir, termal profil ise gelişmiş karmaşıklığın daha yüksek karmaşıklık ve hesaplama maliyetiyle dengelendiği bir model hiyerarşisi üreten asimptotik bir genişleme kullanılarak modellenmiştir. Bu modeller sayısal olarak çözülerek, film boyunca sıcaklık alanının hem sıvı film hem de katı yüzey boyunca iletim ve bunların ilgili radyasyon emisyonları arasındaki bir denge tarafından nasıl yönetildiği ortaya çıkarılmıştır, burada bu iki transfer olgusu hem sıvı film hem de katı yüzeyle ilişkili iki karşılık gelen boyutsuz sayı aracılığıyla birbirine bağlanmıştır.

Anahtar sözcükler: Sıvı filmler, termokapilarite, buharlaşma, radyasyon.

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Chapter 1

Introduction

Heated liquid films are utilized in numerous industrial applications, which include, among many others, the on-chip cooling of electronics [1, 2], deposition-based production techniques [3, 4, 5], thermal desalination systems [6, 7, 8], air humidifying devices [9], and absorption cooling cycles [10, 11]. In some particular processes, the liquid film is combined with a shearing gas, such as in distillation columns [12], structured packings [13], gas scrubbers [14], and airblast atomizers [15]. Furthermore, liquid films have also been utilized in emerging sustainable energy technologies such as in the cooling and desoiling of solar panels [16], solar-powered steam generation [17], and carbon dioxide capture [18, 19]. These technologies are highly energy-intensive, and therefore, gaining a better understanding of the dynamics of these liquid films carries the potential for significant environmental and economic gains, a goal that is at the core of the global effort to combat climate change. Especially since various studies have demonstrated that the heat and mass transport across a wavy film can be as much as 100% larger than it for a flat film [20, 21].

In particular, investigating the spatiotemporal stability of these films by can help elucidate the complex interactions between the various physical phenomena governing their evolution. The value of such studies is heightened by the additional insight they can provide, such as the role that evaporation plays in the

liquid film’s transition into the turbulent flow regime, which has been linked to the film’s thermal properties in addition to the Reynolds number [22]. Investigations by means of stability analysis can help better comprehend such transitions, as some finite turbulent structures have been shown to emerge from specific infinitesimal perturbations [23]. The problem of a heated evaporating liquid film is a sophisticated and multifaceted challenge bringing together an array of physical phenomena which include inertia, thermocapillarity, phase change, thermodynamics, mass transfer, and diffusion. This complexity has provoked the development of various empirical and mathematical modeling techniques.

The initial efforts to study the dynamic evolution of thin liquid films go back to the pioneering works of Kapitza [24, 25], who was the first to identify the existence of an inertial instability mode, now known as the Kapitza mode or H -mode instability. The novel efforts of Kapitza on isothermal films were succeeded by studies by Benjamin [26] and Yih [27], who determined the critical Reynolds number beyond which a vertical film is always unstable. A major advancement was later made by Benney [28] who devised a lubrication-inspired approach to the problem which exploited the film’s small vertical scale and its slow variation in space and time, a methodology now referred to as the long-wave expansion (LWE). The applicability of this approach was expanded substantially by Ruyer-Quil & Manneville [29, 30] by combining a boundary layer approximation with a projection of the velocity field onto an appropriate set of test functions, with the resulting equations being averaged using weighted residuals. This procedure, denoted as the weighted-residual integral boundary layer (WIBL) method, frees the flow’s variables from being exclusively slaved to the film thickness, and avoids the unphysical blow up of Benney’s technique at moderate Reynolds numbers [31]. The correct mechanism behind the H -mode instability was identified correctly as inertia by Smith [32], with further clarifications about the role of inertia being made by Ruyer-Quil and Manneville [33], as well as Dietze [34].

Thermocapillary effects were introduced into the stability analysis of a liquid film by Sreenivasan and Lin [35], who considered a film flowing over an inclined substrate held at a fixed temperature, and detected a nonzero critical Marangoni number due to the stabilizing action of gravity. The LWE technique was then

applied to the problem by Kelly *et al.* [36] and yielded two critical Reynolds numbers corresponding to a thermocapillary instability as well as a modified H -mode instability. Goussis and Kelly [37] further classified this thermocapillary instability according to the responsible physical mechanism and the resulting form of motion into (i) the long-wave S -mode instability whose horizontal length is much larger than the film's vertical scale, and (ii) the short wave P -mode instability which produces patterns on the same scale as the film's thickness. Ruyer-Quil *et al.* [38] examined long waves in a heated liquid film by applying the WIBL methodology to arrive at coupled equations for the evolution of the film thickness, the flow rate, and the interfacial temperature. Their system of equations was then analyzed by Scheid *et al.* [39] where good agreement was obtained with the Orr-Sommerfeld system, and the advective transport of heat was found to stabilize small-amplitude waves but destabilize large-amplitude waves in the presence of a recirculating flow region.

The impact of evaporation on the stability of liquid films and its relation to thermocapillarity has been studied extensively, both experimentally and theoretically. Berg *et al.* [40] studied the formation of convection patterns in pools of evaporating liquids where the perceived patterns in many liquids were found to mostly depend on the depth of the liquid pool, as opposed to the properties of the liquid. Phase change can also cause the appearance of structures such as festoon instabilities, which were reported near the contact line of droplets by Redon *et al.* [41], as well as height fluctuations as observed by Kavehpour *et al.* [42] along the droplet's entire surface. Kimball *et al.* [43] investigated the evolution of convective structures in thin liquid films where they observed a sequence of transitions as the film thins from large variable cells, to concentric rings and spirals, to the eventual cessation of Marangoni convection. Their analysis suggests that attaining equilibrium conditions at the liquid's surface unifies its temperature and ultimately causes the elimination of the thermocapillary instability. The phenomenon of evaporation-driven climbing streaks was studied extensively by Fournier and Cazabat [44], Vuilleumier *et al.* [45], and Hosoi and Bush [46]. Moreover, the influence of non-uniform heating was studied by Mildinova *et al.* [47], Kalliadasis *et al.* [48], and Yeo *et al.* [49].

Formulations of the evaporation process which do not account for the dynamics of the vapor phase beyond the interfacial boundary conditions are known as *one-sided* models. They represent scenarios where the liquid is evaporating into a vacuum or subjected to continuous gas flow [50]. These models treat the evaporation process as a departure from thermodynamic equilibrium at the liquid interface, and intrinsically predict the effects of evaporation to be destabilizing, as argued by Prosperetti and Plesset [51]. The imbalance of pressure exerted by the liquid molecules departing the interface produces an effect known as vapor recoil, which was first reported experimentally by Hickman [52] in flat liquid films under high vacuum, who found it to both strongly destabilize the interface and dramatically increase the evaporation rate. This one-sided problem was the subject of repeated theoretical efforts such as that undertaken by Palmer [53], who investigated the coupling between the Marangoni and vapor recoil instabilities for long-wave disturbances. Similar studies followed by Burelbach *et al.* [54] who considered a flat film and included the effects of evaporative mass loss and disjoining pressure, Danov *et al.* [55] who accounted for the presence of nonvolatile surfactants, and Joo *et al.* [56] who analyzed a gravity-driven inclined film and incorporated higher order effects. More recently, Mohamed and Biancofiore [57] circumvented the limitations imposed by long-wave theory on the model of Burelbach *et al.* [54] by deriving the corresponding Orr-Sommerfeld system instead. We follow them herein and adopt the terminology *E*-mode to refer to the evaporative instability. Subsequent investigations into this problem were conducted by Oron *et al.* [58], Margerit *et al.* [59], and Merkt and Bestehorn [60]. Additional efforts were made to further develop one-sided models such as those exerted by Ajaev and Homsy [61, 62, 63] who applied their model to investigate the effect of evaporation on wetting phenomena. Ji and Witelski [64] examined the competition between weak phase change and dewetting in thin liquid films, and observed complex evolution dynamics. Notable pursuits were also made to improve the modeling of the thermodynamic balance at the liquid interface such as that conducted by Shklyaev and Fried [65] who wrote a novel set of boundary conditions to replace the conventional Hertz-Knudsen-Langmuir equation with a formulation based on the balance of configurational momentum, which showed realistic applicability to molten metals.

On the other hand, in the absence of sufficient advection of the liquid's vapor from the vicinity of its surface, vapor diffusion becomes an important effect producing intricate phenomena and therefore must be taken into consideration. Deegan *et al.* [66] explained the origin of coffee stains in the context of the diffusion-limited evaporation of droplets into ambient air, while Cachile *et al.* [67] explained the receding of unpinned droplets under similar assumptions. Poulard *et al.* [68] studied a similar problem where contact line instabilities and the nonmonotonic behavior of the receding contact line were explained. These instabilities inspired an important development in the modeling of evaporating liquid films made by Sultan *et al.* [69], who formulated a general evaporation model which incorporates both the thermodynamically determined molecule departure rate as well their diffusion into the ambient gas phase, under the assumption of the static diffusion of the vapor. This assumption was relaxed by Kanatani [70], who investigated a liquid film evaporation into an inert gas where the convection of the vapor was taken into account. Additional progress was made by Zhao and Nadal [71], who addressed the Stefan flow, the temperature jump across the Knudsen layer, and the change in the interface's chemical potential due to the presence of the inert gas.

Accounting for the presence of a gas layer adjacent to the liquid film greatly increases the complexity of the problem, and has drawn considerable interest. Early experimental works made in this regard include the works of Hanratty and Engen [72] who demonstrated the various wave transitions in air-water flows, and Craik [73] who detected waves which travel faster than the interface and others that travel slower than it in film flows inside horizontal channels. More recently, Ó Náraigh *et al.* [74] formulated a single equation accounting for interfacial turbulence, and found that the wave-induced Reynolds stresses do not play an important role in the the growth of interfacial waves. Dietze Ruyer-Quil [75] formulated a low-dimensional model based on a weighted-residual integral boundary layer approach to study the interaction between the gas and liquid phases for horizontal pressure-driven and gravity-driven liquid films, in both the cases of co-flowing or counter-flowing gas phase. Their model successfully predicts the instability onset, phase speed, and growth rate in the long wave limit.

Lavalle *et al.* [76] found that the effect of the counter-flowing gas can be stabilizing, destabilizing, or cause harmonic changes in the cut-off wave number and temporal growth rate depending on the level of confinement of the liquid film. Notably, they found that the H -mode instability can be completely suppressed by a counter-flowing gas given sufficient confinement. Due to the complexity of the coupled gas-liquid problem, a common approach is to simplify the problem by imposing the gas stress at the liquid-gas interface and solving for the liquid phase only. King *et al.* [77] considered a thin isothermal film flowing down an inclined plane subject to a counter-flowing steam of air. The shear stress applied by the airflow onto the liquid interface was prescribed via a constant drag coefficient, while its normal stress was variable and related to the constant shear stress by thin-airfoil theory. Moreover, Sivapuratharasu *et al.* [78] considered the same problem, and in similar fashion, assumed the airflow's shear stress to be constant along the liquid interface but neglected its contribution to the normal stress balance. This constant-stress modeling approach has been applied in a variety of fluid problems including droplets [79, 80], and rivulets [81, 82, 83, 84].

Another important phenomenon associated with counter-current gas flow over a liquid film is *flooding*, which refers to the appearance of large amplitude standing waves in the fluid film at a critical shearing gas flow rate [85, 86]. Additional increases in the gas flow rate beyond the flooding limit lead to the instabilities traveling in the upstream direction and possibly the complete reversal of the liquid flow direction, which can have severe consequences on the performance of systems such as plate heat exchangers, reflux condensers, and nuclear reactor cooling systems. This sudden onset of flooding is associated with a transition in the flow dynamics from convectively unstable to absolutely unstable [87].

The distinction between convective and absolute instabilities was first made by Briggs [88] in the field of plasma physics and later introduced into fluid stability theory by Huerre & Monkewitz [89]. Under this framework, flows are classified by examining their long-time response to an infinitesimal impulse source. If the disturbance grows but is convected away by the flow then the flow is classified as *convectively unstable*. On the other hand, if it grows at its origin and proceeds to contaminate the entire flow field then the flow is classified as *absolutely unstable*.

The spatiotemporal characterization of a falling liquid film was first performed by Brevdo *et al.* [90] who solved the Orr-Sommerfeld eigenvalue problem and found the instabilities to be of the convective type, a result identical to that found by Joo & Davis [91] who had used a model based on Benney's long-wave approach. Ó Náraigh *et al.* [92] employed Orr-Sommerfeld stability theory along with a combination of linear modal analysis and ray analysis to study the problem of two-phase stratified flow of a laminar liquid phase and a fully developed gas phase which can either be turbulent or laminar. Using this model they identified the regimes that produce absolute instabilities and compared their results to experimental flow-regime maps. In order to investigate the flooding phenomenon, Tseluiko and Kalliadasis [93] treated the liquid interface as a solid wall which permitted obtaining closed form solutions for the pressure and tangential stresses imposed by the gas. The same model was applied by Vellingiri *et al.* [87] to investigate the linear spatio-temporal stability of an isothermal liquid film flowing down an inclined plane and subject to a counter-flowing turbulent gas. They proceeded to identify the relationship between the gas flow rate and inclination angle, and the convective-absolute transitions in the flow's stability character. Additionally, they conceived a generalized methodology for identifying the type of instability when the flow is described by non-local equations.

A notable property of thin liquid films has been their tendency to spontaneously rupture once the film thickness reaches a critical value depending on the different forces competing within the film, such as disjoining pressure and surface tension [94]. Both Burelbach *et al.* [54] and Joo *et al.* [56] predicted the occurrence of film rupture due to long-wave instabilities driven by vapor recoil and thermocapillarity. The occurrence of film rupture was confirmed by the direct numerical simulations of non-volatile films performed by Krishnamoorthy *et al.* [95], whose results showed good qualitative agreement with those of long-wave theory. Boos and Thess [96] demonstrated that the rupture caused by the long-wave Marangoni instability occurs via an infinite cascade of increasingly smaller structures. Shklyaev *et al.* [97] considered a generalized one-dimensional Cahn-Hilliard equation suited for modeling the dynamics of a thin liquid film heated from below, and subject to the Marangoni instability. They constructed

self-similar solutions corresponding to the film dynamics and defined a power law governing their evolution. Wei and Duan [98] considered a liquid film undergoing quasi-equilibrium evaporation and identified a set of self-similar solutions along with the power laws governing their behavior. Later on, Dallaston *et al.* [99] demonstrated that the formation of iterated structures as the film ruptures is due to a bifurcation of the continuous self-similar solution set into a discrete set occurring periodically over a logarithmic time scale. Another recent effort is that of Ji and Witelski [100] who investigated evaporation driven rupture of thin films using an elaborate evaporation model, which they used to examine the balance between conservative restoring forces and the non-conservative flux term. They found several categories of finite-time singularities including self-similar and non-self-similar behavior, which were studied both analytically and numerically.

A recent area of interest in the study of the evolution of heated liquid films is the inclusion of the effects of thermal radiation, especially as technological development becomes increasingly energy-conscious and the integration of solar energy into existing industries becomes progressively widespread. For instance, Krauter [16] demonstrated that the efficiency of photovoltaic panels can be significantly increased by introducing the flow of a thin layer of water over the surface of the panels. Similar findings were also made by Shahverdian *et al.* [101], who leveraged a dynamic multi-objective optimization technique. Another study conducted by Ebaid *et al.* [102] compared the performance increases obtained by using a variety of fluids such as a (TiO₂) nanofluid in a water-polyethylene glycol mixture. Various recent studies have attempted to include the effects of thermal radiation on the evolution of liquid films, such as that carried out by Saeki *et al.* [103] who investigated the thermocapillary instability of irradiated transparent liquid films and demonstrated that periodic variation of the reflected energy between the solid wall and the liquid interface has a significant effect on the film's stability.

Despite these efforts, advanced investigations focusing on irradiated liquid films which are sophisticated enough to allow experimental comparison remain limited in scope. This is due to the complex couplings between the various physical

phenomena. Specifically, direct numerical simulations are prohibitively expensive and thus have been mostly limited to either two dimensional studies [104], or restricted domains [105]. Consequently, significant attention has been paid towards developing reduced hydro-thermal models which are robust enough to reproduce the film’s temperature field, while remaining computationally efficient. These include the studies of Aktershev *et al.* [106] who assumed a fully-linear temperature distribution, and Ruyer-Quil *et al.*, whose attempt was based on the WIBL technique [38]. More recently, Cellier and Ruyer-Quil [107] formulated new asymptotic models that cure some of the deficiencies of the previous attempts and significantly expand the range of applicability of the previous models. However, their study involved a constant wall temperature and did not involve radiative heat transfer. Note that we follow their methodology in this manuscript, albeit we do include the radiative heat fluxes.

In this work, we investigate the dynamics of heated liquid films across several distinct environments involving inertia, thermocapillarity, phase change, gas shear, and thermal radiation. For each problem, we write appropriate reduced formulations that encompass the dominant physical influences dictating the film’s spatiotemporal evolution, where we employ various analytical and numerical methodologies to explore a range of phenomena which have not been thoroughly examined in the past. The research done here has produced two publications in *Physical Review Fluids*: O. A. A. Mohamed, M. C. Dallaston, and L. Biancofiore, *Spatiotemporal evolution of evaporating liquid films sheared by a gas*, Phys. Rev. Fluids, **6**, 114002 (2021), and “*Stability analysis of volatile liquid films in different evaporation regimes*”, by O. A. A. Mohamed and L. Biancofiore. The latter of the two has been accepted but is yet to be published as of this writing.

The structure of this dissertation is as follows. In Chap. 2, we study the two-fluid problem of a volatile liquid film and its vapor in the flow domain characterized by low Reynolds numbers, known as the drag-gravity regime due to the balance between viscosity and gravity, within which we find that the presence of vapor diffusion gives rises to complex dynamics resulting from the interplay between the Marangoni instability and evaporation phenomena such vapor recoil, mass loss, and film thinning.

We first introduce the equations governing both the liquid and vapor domains, as well as the scaling employed, and the resulting set of dimensionless parameters. Subsequently, we present our application of the LWE to produce the film's governing equation, which is perturbed to produce the dispersion relationship representing the system's dynamic behavior. Thereafter, this dispersion relationship is utilized to perform a temporal stability analysis where we probe the influence of the evaporation regime on the film's temporal stability, the mass loss instability, the effects resulting from film thinning, and wave celerity. This temporal analysis is then expanded into the spatiotemporal framework through which we examine the impact of the evaporation regime as well as film thinning on the spatiotemporal stability.

In Chap. 3, we consider again consider an evaporating liquid film in the drag-gravity regime, albeit in this problem the free surface is subjected to a shearing gas flow, which introduces additional complexity but also enables a one-sided evaporation model due to the elimination of the vapor gradient above the liquid's surface. Our analysis indicates that the imparted shear stress has important effects on both the thermal instabilities appearing at relatively lower Reynolds numbers, as well as the inertial H -mode which arises at relatively higher values. Moreover, the shear stress has significant consequences on the self-similar rupture mechanics occurring towards film dry-out.

The chapter begins with the introduction of the governing equations, dimensionless parameters, and the modeling of the shear stress, followed by the derivation of the nonlinear governing equation using the long-wave framework. This is succeeded by the application of linear stability theory to obtain the analytical dispersion relationship governing the liquid film's dynamics, which is utilized to perform a linear temporal stability analysis to study the effects of the shear stress. This in turn is subsequently extended into a linear local spatiotemporal stability analysis. In addition, we explore how the shear stress's effect on the flow's stability is specifically tied to the influence of the film thinning. Thereafter, we study the nonlinear dynamics of the flow by solving the governing equation numerically and compare the nonlinear temporal and spatiotemporal dynamics to the results of the prior linear stability analysis. Lastly, we examine the film

rupture dynamics via a self-similarity analysis and analyze the role of different dimensionless parameters on the rupture mechanism.

Chapter 4 is dedicated to the problem of an irradiated film in the drag-inertia flow regime, which is parametrized by relatively larger Reynolds numbers and deeper films. Here, the liquid free surface's evolution is dictated by inertia instead of thermal phenomena, and hence, we shift our focus and modeling strategies accordingly. We adjust our hydrodynamic modeling methodology and employ the WIBL [29, 30] technique to overcome the non-physical finite-time blow-up [31, 108] of the LWE models. Moreover, through a series of careful assumptions and simplifications, we reduce the highly complex hydro-thermal radiation problem into a hierarchy of systems of partial differential equations of increasing complexity. These models reveal how temperature field across the film is governed by a balance between the conduction across both the liquid film and the solid surface, and their respective radiative emissions, wherein these two transfer phenomena are linked through two corresponding dimensionless numbers associated with both the liquid film and the solid surface.

We first define the hydrodynamic problem by deriving the boundary layer expressions from the dimensionless governing equations and subsequently apply the WIBL methodology to obtain the first order WIBL equations, which are verified by recovering the corresponding isothermal equivalent under the LWE framework. Thereafter, we formulate the liquid film's thermal problem by writing the governing system of equations and applying the chosen scaling which produces the parameters governing the conduction-radiation balance. Next, we solve for the Nusselt temperature solution and asymptotically decompose the temperature field following the method developed in Ref. [107] to obtain the set of models governing the temperature field. Finally, we present the results obtained from the numerical simulation of the hydro-thermal coupled systems of equations.

Lastly, Chap. 5 summarizes our findings, discusses possible avenues for improvement, and suggests topics for future work.

Chapter 2

Kinetic-diffusion evaporation at low Reynolds numbers

In this first chapter, we introduce a coupled two-fluid problem where both the liquid and vapor phases contribute to the liquid interface's evolution. The system we investigate consists of a liquid film flowing over a heated plate and its vapor phase, which diffuses into a passive ambient environment, e.g., an inert gas. The static environment about the liquid interface requires accounting for the effects introduced by the diffusion of liquid's vapor as they are non-negligible in such a system. The liquid flow we focus on here is characterized by low Reynolds numbers and the balance between the streamwise component of the gravitational force and the drag emanating from viscosity, whereby inertia and surface tension are relegated to perturbative roles acting to deviate the flow from its semi-parabolic base velocity profile. Due to the primary roles played by gravity and viscosity, this particular flow domain is known as the drag-gravity regime [109]. Moreover, this flow regime's particular properties are reflected in our modeling approach for the problem's hydrodynamics, as it is amenable to compliant to the simple hydrodynamic models such as a the long-wave expansion (LWE) [28], which we will utilize in this chapter.

On the other hand, the evaporation process we consider incorporates the kinetic interactions driving the departure of molecules from the liquid’s interface as well as the diffusion of vapor into the neighboring domain. Consequently, the evaporation process is necessarily governed by the competition between the thermodynamic disequilibrium tied to the liquid film’s local thickness and the diffusion dynamics dependent on the interface’s curvature. The low Reynolds numbers correlated with the drag-gravity flow regime are associated with the appearance of various thermal instabilities which dominate the evolution of the liquid interface, namely the Marangoni *S*-mode [36] and evaporative *E*-mode [50, 57], and thus these particular instabilities are the focus of our investigation in this chapter. As of this writing, the work presented in this chapter has been accepted for publication in *Physical Review Fluids*, under the title “A stability analysis of volatile liquid films in different evaporation regimes”, by O. A. A. Mohamed and L. Biancofiore.

This chapter is structured as follows. Section 2.1 introduces the dimensional systems of equations governing both the liquid and vapor phases and their ensuing nondimensionalization, while Sec. 2.2 showcases our application of the LWE and the derivation of the differential equation governing the dynamic evolution of the liquid film. Section 2.3 then details the application of linear stability theory to obtain the system’s analytical dispersion relationship, and is followed by Sec. 2.4 where we investigate the temporal stability analysis of the coupled system, and finally, in Sec. 2.5 we present our spatiotemporal stability analysis.

2.1 Mathematical formulation

The two-phase problem is formulated by combining the liquid film model of Joo *et al.* [56] with the modified kinetic-diffusion evaporation model of Sultan *et al.* [69]. The liquid film is incompressible, thin, Newtonian, volatile, with constant physical properties, and driven by gravity down a heated plate held at a constant temperature, which is inclined from the horizontal with an angle β , as depicted in Fig. 2.1.

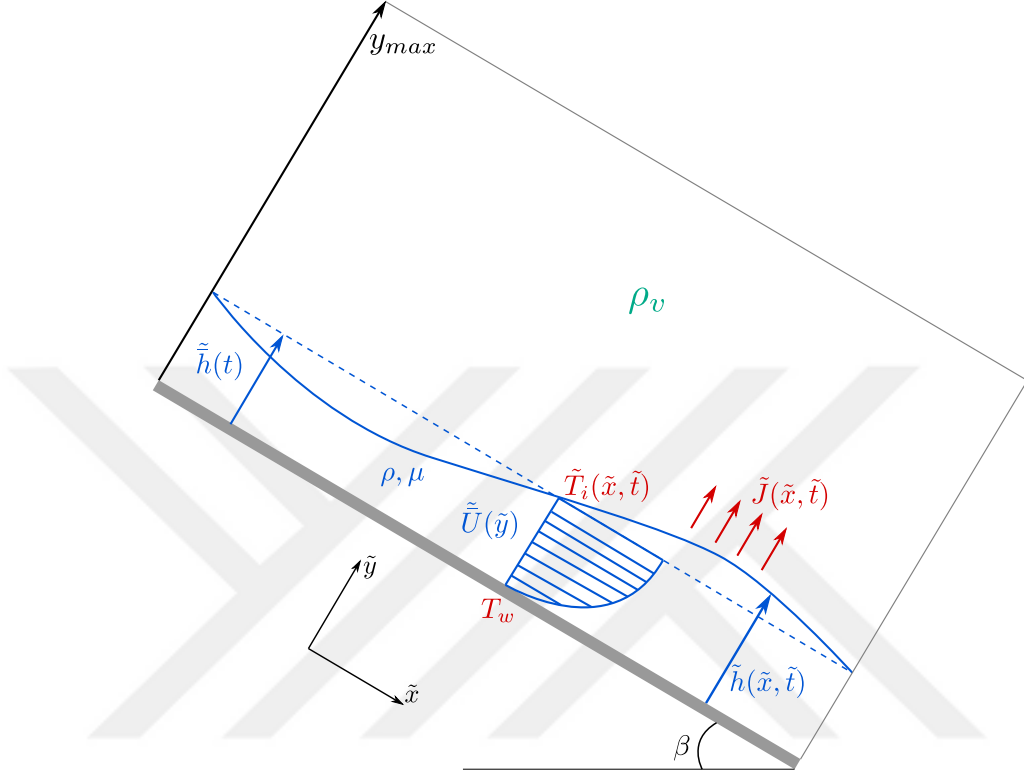


Figure 2.1: Schematic illustration of an evaporating liquid film flowing under the influence of gravity over a heated plate held at a constant temperature. The unperturbed base flow height $\tilde{h}$ is marked by the dashed line, β denotes the inclination angle from the horizontal, and T_w is the constant wall temperature.

Our study is limited to two dimensions where we assign Cartesian $(\tilde{x}, \tilde{y})$ coordinates such that the $\tilde{x}$ coordinate points in the streamwise direction and the $\tilde{y}$ coordinate away from the liquid film into the gas phase. The function $\tilde{h}(\tilde{x}, \tilde{t})$ defines the liquid interface forming the boundary between the two phases, while $\tilde{y}_{max}$ marks the upper limit of the gas domain. The gas phase consists of a static mixture of an inert gas and the liquid's vapor and does not contribute any viscous stresses at the liquid interface. Note that the tilde notation denotes dimensional variables. Our focus here is on the thermal instabilities appearing at relatively low Reynolds numbers, namely the thermocapillary *S*-mode and the evaporative *E*-mode.

2.1.1 Liquid phase governing equations

The equations governing the liquid film are the continuity equation, the Navier-Stokes momentum equations, and the energy equation, which in dimensional vector form read

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad (2.1a)$$

$$\partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} = -\frac{1}{\rho} \nabla \tilde{p} + \nu \nabla^2 \tilde{\mathbf{u}} + \tilde{\mathbf{F}}, \quad (2.1b)$$

$$\partial_t \tilde{T} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} = \alpha \nabla^2 \tilde{T}. \quad (2.1c)$$

$\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ is the liquid velocity vector, $\tilde{p}$ is the liquid's pressure, $\tilde{T}$ is the liquid temperature, $\tilde{\mathbf{F}} = (g \sin \beta, -g \cos \beta)$ is the gravitational body force vector, ρ is the liquid density, ν is the liquid's kinematic viscosity, and $\alpha = \lambda / \rho c_p$ is the liquid's thermal diffusivity involving its thermal conductivity λ and its constant pressure heat capacity c_p .

At the solid wall ($\tilde{y} = 0$), we impose a no slip condition on the velocity and a constant value for the temperature:

$$\tilde{\mathbf{u}} = 0, \quad (2.1d)$$

$$\tilde{T} = T_{wall}. \quad (2.1e)$$

At the interface between the liquid and the gas [$\tilde{y} = \tilde{h}(\tilde{x}, \tilde{t})$], the balances of mass, energy, and stress follow Refs. [54, 56] and respectively read

$$\tilde{J} = \tilde{\rho} (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_i) \cdot \mathbf{n} = \tilde{\rho}_v (\tilde{\mathbf{u}}_v - \tilde{\mathbf{u}}_i) \cdot \mathbf{n}, \quad (2.1f)$$

$$\tilde{J} \left[\mathcal{L} + \frac{1}{2} \left(\frac{\tilde{J}}{\tilde{\rho}_v} \right)^2 \right] = -\lambda \nabla \tilde{T} \cdot \mathbf{n}, \quad (2.1g)$$

$$\frac{\tilde{J}^2}{\tilde{\rho}_v} \mathbf{n} + (\tilde{\mathbf{T}} - \tilde{\mathbf{T}}_g) \cdot \mathbf{n} = 2\tilde{\gamma} \mathcal{K} \mathbf{n} + \nabla_s \tilde{\gamma}. \quad (2.1h)$$

The subscript i denotes quantities which are at the liquid-gas interface, the subscript v denotes quantities associated with the liquid's vapor, and the subscript g denotes those representing the ambient gas phase. $\mathcal{L}$ is the latent heat of vaporization and $\tilde{J}$ is the vapor mass flux. Moreover, $\mathcal{K} = -\frac{1}{2}\nabla_s \cdot \mathbf{n}$ is the mean interface curvature, ∇_s is the surface gradient operator, and $\mathbf{n} = \frac{1}{N}(-\partial_{\tilde{x}}\tilde{h}, 1)$ is the local interface normal unit vector. The stress tensor for the liquid phase is $\tilde{\mathbf{T}} = -\tilde{p}\mathbf{I} + \tilde{\mathbf{P}}$ where $\tilde{\mathbf{P}}$ is the deviatoric stress tensor, while the gas phase's stress tensor is simplified to $\tilde{\mathbf{T}}_g = -\tilde{p}_g\mathbf{I}$, in accordance with our previous assumptions. Furthermore, the liquid's surface tension is defined by a linear equation of state

$$\tilde{\gamma} = \gamma_0 - \left| \frac{d\tilde{\gamma}}{d\tilde{T}} \right| (\tilde{T} - T_{wall}),$$

where γ_0 is the the mean surface tension at T_{wall} and $d\tilde{\gamma}/d\tilde{T}$ is negative for most liquids. Other forces such those resulting from van der Waals intermolecular interactions are not considered.

2.1.2 Vapor phase governing equations

The vapor phase is modeled by modifying Sultan *et al.*'s [69] formulation for the liquid vapor to accommodate the thinning of the film due to evaporative mass loss. Under the assumption of quasi-static diffusion, the Péclet number is very small and the convection-diffusion equation governing the liquid vapor distribution reduces to the Laplace equation:

$$\frac{\partial^2 \tilde{\rho}_v}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\rho}_v}{\partial \tilde{y}^2} = 0. \quad (2.2a)$$

The interface boundary condition is derived by imposing the continuity of the normal vapor flux at liquid interface, thereby equating the kinetic normal vapor flux dictated by the Hertz-Knudsen relationship [51]:

$$\mathbf{J} = v_{th}(\rho_v^{eq}(T_{int}) - \rho_{v|int})\mathbf{n} \quad (2.2b)$$

to that required by Fick's diffusion law:

$$\mathbf{J} = -D\tilde{\rho}_v, \quad (2.2c)$$

where $v_{th} = \alpha_M \sqrt{k_B \tilde{T}_{int} / (2\pi M_w)}$ is the typical thermal kinetic velocity with the accommodation coefficient α_M , the Boltzmann constant k_B , and the molecular weight M_w . D is the diffusion coefficient of vapor into the gas phase, $\tilde{\rho}_v^{eq}$ is the equilibrium vapor density, and the subscript *int* denotes quantities at the liquid interface $\tilde{h}(\tilde{x}, \tilde{t})$. Utilizing a linear equation of state for the vapor's equilibrium density in terms of the liquid's temperature, we obtain at $\tilde{y} = \tilde{h}$:

$$-D \left(1 + v_{th} \frac{d\tilde{\rho}_v^{eq}}{d\tilde{T}} \frac{\mathcal{L}}{\lambda} \tilde{h} \right) (\mathbf{n} \cdot \nabla) \tilde{\rho}_v = v_{th} \left(\tilde{\rho}_v^{eq}(T_{wall}) - \tilde{\rho}_v \right), \quad (2.2d)$$

with the vapor mass flux being

$$\tilde{J} = -D(\mathbf{n} \cdot \nabla) \tilde{\rho}_v|_{int}. \quad (2.2e)$$

Equation (2.2d) couples the vapor problem to the liquid problem through the interface height function $\tilde{h}(\tilde{x}, \tilde{t})$. Further details regarding the derivation of this boundary condition can be found in Ref. [69].

Moreover, to prevent the saturation of the gas domain with vapor, a vapor outflux is enforced at the upper boundary by imposing the density gradient. We write a boundary condition that is tied to the liquid film's height, which allows the vapor density gradient to react as the film becomes thinner due to the evaporative mass loss. Moreover, the boundary condition is formulated to recover a single-fluid model when the appropriate limit of the two-phase system is taken. In particular, our expression allows the recovery of the exact expressions in Joo *et al.* [56], who considered only the liquid phase in their work. The top boundary condition at $\tilde{y} = y_{max}$ reads:

$$\frac{\partial \tilde{\rho}_v}{\partial \tilde{y}} = - \frac{v_{th} \left(\tilde{\rho}_v^{eq}|_{T_{wall}} - \tilde{\rho}_v^{eq}|_{T_{int}} \right)}{-D(\mathbf{n} \cdot \nabla) \tilde{\rho}_v|_{int} + v_{th} \left(\tilde{\rho}_v^{eq}|_{T_{wall}} - \tilde{\rho}_v^{eq}|_{T_{int}} \right)} \frac{\lambda \Delta T}{\mathcal{L} D} \frac{1}{\tilde{h}}, \quad (2.2f)$$

where the fraction represents the degree of thermodynamic disequilibrium at the interface allowing the recovery of the one-sided model, and $\lambda\Delta T/(\mathcal{L}D)$ is the characteristic vapor density. The formulation of this boundary condition represents an extension of the model of Sultan *et al.* [69] to non-equilibrium evaporation and allows the inclusion of the resultant thinning of the film due to the evaporative mass loss, as their model is restricted to recovering the one-sided case of a film at full height undergoing equilibrium evaporation where net mass loss is zero.

Finally, the system of equations governing the vapor phase is closed by periodic boundary conditions at the horizontal extremities of the domain to facilitate its numerical solution:

$$\tilde{\rho}_v(x_0, \tilde{y}) = \tilde{\rho}_v(x_{max}, \tilde{y}), \quad \left. \frac{\partial \tilde{\rho}_v}{\partial \tilde{x}} \right|_{x_0} = \left. \frac{\partial \tilde{\rho}_v}{\partial \tilde{x}} \right|_{x_{max}}, \quad (2.2g)$$

where x_0 and x_{max} are the horizontal domain limits.

2.1.3 Scaling and dimensionless parameters

The following scales are introduced into the systems of equations (2.1) and (2.2):

$$\begin{aligned} (\tilde{x}, \tilde{y}) &\rightarrow \tilde{h}_0(x^*, y^*), & \tilde{t} &\rightarrow \frac{\tilde{h}_0^2}{\nu} t^*, & \tilde{h} &\rightarrow \tilde{h}_0 h^* \\ (\tilde{u}, \tilde{v}) &\rightarrow \frac{\nu}{\tilde{h}_0}(u^*, v^*, w^*), & \tilde{p} &\rightarrow p_g + \frac{\rho\nu^2}{\tilde{h}_0^2} p^*, & \tilde{J} &\rightarrow \frac{\lambda\Delta T}{\tilde{h}_0\mathcal{L}} J^*, \\ \tilde{T} &\rightarrow T_w + T^*\Delta T, & \tilde{\rho}_v &\rightarrow \rho_v^{eq}(T_{wall}) + \frac{\lambda\Delta T}{\mathcal{L}D} \rho_v^*, \end{aligned} \quad (2.3)$$

where $\Delta T = T_w - T_g$, and $\tilde{h}_0$ is the flat film thickness at the initial time. The star notation denotes dimensionless variables, and is dropped onwards for simplicity, except for some cases where emphasis is required. Introducing these scales into

Eqs. (2.1) and (2.2) yields the liquid film equations as

$$\partial_x u + \partial_y v = 0, \quad (2.4a)$$

$$\partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \partial_{xx} u + \partial_{yy} u + \text{Re} \sin \beta, \quad (2.4b)$$

$$\partial_t v + u \partial_x v + v \partial_y v = -\partial_y p + \partial_{xx} v + \partial_{yy} v - \text{Re} \cos \beta, \quad (2.4c)$$

$$\text{Pr} [\partial_t T + u \partial_x T + v \partial_y T] = \partial_{xx} T + \partial_{yy} T. \quad (2.4d)$$

The scaled wall boundary conditions at ($y = 0$) are

$$u = v = 0, \quad T = 0, \quad (2.4e)$$

while the scaled interface boundary conditions at ($y = h$) become

$$EJ = \frac{1}{N} [-u \partial_x h + v - \partial_t h], \quad (2.4f)$$

$$J + \frac{E^2}{D^2 L} J^3 = \frac{1}{N} [\partial_x h \partial_x T - \partial_y T], \quad (2.4g)$$

$$-\frac{3}{2} \frac{E^2}{D} J^2 - \frac{2}{N^2} [\partial_x u [(\partial_x h)^2 - 1] - \partial_x h (\partial_y u + \partial_x v)] + p \quad (2.4h)$$

$$= -3S(1 - CT) \left[\frac{\partial_{xx} h}{N^3} \right], \quad (2.4i)$$

$$[1 - (\partial_x h)^2] [\partial_y u + \partial_x v] - 4\partial_x h \partial_x u - \tau = -2 \frac{M}{\text{Pr}} [\partial_x T + \partial_x h \partial_y T] N. \quad (2.4j)$$

where $N = \sqrt{1 + (\partial_x h)^2}$ is the modulus of the interface's normal unit vector $\mathbf{n}$.

Similarly, the dimensionless vapor system is

$$\frac{\partial^2 \rho_v}{\partial x^2} + \frac{\partial^2 \rho_v}{\partial y^2} = 0, \quad (2.5a)$$

$$(1 + \chi h)(\mathbf{n} \cdot \nabla) \rho_v = \text{Pe}_k \rho_v \quad \text{at } y = h, \quad (2.5b)$$

$$\frac{\partial \rho_v}{\partial y} = -\frac{\chi}{1 + \chi h} \quad \text{at } y = y_{max}^*, \quad (2.5c)$$

$$\rho_v(x_0^*, y) = \rho_v(x_{max}^*, y), \quad \frac{\partial \rho_v}{\partial x} \Big|_{x_0^*} = \frac{\partial \rho_v}{\partial x} \Big|_{x_{max}^*} \quad (2.5d)$$

with the dimensionless expression for the vapor mass flux obtained from Eq. (2.2e) as

$$J = -(\mathbf{n} \cdot \nabla^*) \rho_{v|int}. \quad (2.5e)$$

Despite the attractiveness of the simplifications offered by considering the diffusion problem in two dimensions, this reduction gives rise to some challenges. In particular, the utilization of techniques utilizing Green's function become rather arduous due to logarithmic growth and the difficulties involved in producing the required far-field behavior. Various methodologies have been implemented to overcome these issues, such as the introduction of an artificial remote boundary [110, 111, 112], or splitting the solution domain and linking the distinct solutions using the method of matched asymptotic expansions [113]. Notably, the resolution of this problem in a finite three-dimensional space is much simpler due to (i) the inherent decay of the Green's function and (ii) the absence of the need to address the solution's behavior at large distances.

The scaled systems of equations (2.4) and (2.2) contain the group of dimensionless variables detailed in Table. 2.1.

Parameter	Expression
Reynolds number (Re)	$g\tilde{h}_0^3/\nu^2$
Prandtl number (Pr)	ν/α
Evaporation number (E)	$\lambda\Delta T/\rho\mathcal{L}\nu$
Density ratio (D)	$3\tilde{\rho}_v/2\tilde{\rho}$
Nondimensional latent heat (L)	$8\tilde{h}_0^2\mathcal{L}/9\nu^2$
Nondimensional surface tension (S)	$\gamma_0\tilde{h}_0/3\tilde{\rho}\nu^2$
Capillary number (C)	$\gamma\Delta T/\gamma_0$
Marangoni number (M)	$ d\tilde{\gamma}/d\tilde{T} \Delta T\tilde{h}_0/2\mu\alpha$
Thermal expansion parameter (χ)	$v_{th}\tilde{h}_0(\mathcal{L}/\lambda)(d\rho_v^{eq}/dT)$
Kinetic Péclet number (Pe_k)	$v_{th}\tilde{h}_0/D$

Table 2.1: The set of dimensionless parameters appearing in the nondimensional liquid and vapor systems of equations, Eqs. (2.4) and (2.5), respectively.

The boundary condition [Eq. (2.5b)] is identical to the expression in Sultan *et al.*, [69] and contains the two parameters determining the evaporation regime: the thermal expansion number χ representing the fluctuations in the equilibrium vapor density with temperature, and the *kinetic* Péclet number Pe_k , which is the ratio of kinetic and diffusion time scales. The ratio $Pe_k/\chi = \lambda/[D(d\rho_v^{eq}/dT)\mathcal{L}]$ which we denote as Γ defines the evaporation regime. Notably, the vapor mass flux can be expressed explicitly in terms of Γ by rearranging Eq. (2.5b) as

$$J = -\frac{\rho_{v|int}}{h + 1/\chi}\Gamma. \quad (2.6)$$

When $\Gamma < 1$ the evaporation rate is dictated to a greater degree by the thermodynamic molecular exchange at the liquid interface, and is thus strongly dependent on the interface's proximity to the heated wall as opposed to its curvature. Consequently, the evaporation rate will be higher (lower) at the film's troughs (crests). The particular case of $\Gamma \rightarrow 0$ represents the exclusion of liquid vapor as a variable in the system, and hence, the complete absence of vapor diffusion phenomena. In this situation, the problem is reduced to a single-fluid (one-sided) system where the evaporation process is slaved to the local film height, since it dictates the energy transfer to the the interface and the resulting local vapor mass flux. This evaporation regime is therefore aptly named the transfer-rate-limited regime, and is the evaporation regime considered by several of the notable works investigating the stability of volatile films [54, 56, 59, 61, 63, 62, 64, 50]. Additionally, under the limit of $\Gamma \rightarrow 0$, the mass flux expression in Eq. (2.6) takes the form

$$J = \frac{1}{h + 1/\chi}, \quad (2.7)$$

which is equivalent to the mass flux expression in the works of Burelbach *et al.* [54] and Joo *et al.* [56] where the quantity $1/\chi$ represents the degree of disequilibrium at the liquid interface and is equivalent to their parameter K . Note that the value of unity that appears in the nominator of Eq. (2.7) because as $\Gamma \rightarrow 0$, $\rho_{v|int} \rightarrow -\infty$, which we interpret as the interfacial region being infinitely unsaturated since the nondimensional value of the saturation vapor density is

zero.

On the other hand, when $\Gamma > 1$, the gas layer directly above the liquid is sufficiently saturated with vapor such that the evaporation rate is largely dependent on the diffusion of vapor into the ambient phase, and kinetic phenomena are relegated to a secondary role. Consequently, the evaporation rate is determined by the interface's curvature since it governs the vapor gradient directly above the liquid film. Under these conditions, the evaporation rate will be higher (lower) at the film's crests (troughs) due to their concave (convex) shape, which allows for higher overall vapor diffusion, in what is known as the sharp-edge effect. For more details please refer to Refs. [66, 69, 114]. Moreover, when $\Gamma \rightarrow \infty$, the layer of gas directly above the liquid surface is completely saturated by vapor and the evaporation process is entirely dependent on the diffusion rate. Thus, the evaporation conditions at this threshold are known as the diffusion-limited evaporation regime [69, 115].

In this work, we investigate both transfer-rate-limited and diffusion-limited evaporation, as well as intermediate cases where both kinetic and diffusion processes compete at the liquid interface.

2.2 Long wave theory and the interface evolution equation

The liquid flow regime of interest is characterized by relatively small Reynolds numbers, and therefore, our utilization of the LWE to model the liquid film is reasonable as this methodology retains its accuracy in comparison to more sophisticated models (e.g., the full Orr-Sommerfeld system of equations) for the flow regime in question [87]. Note that the LWE methodology assumes that surface tension is strong and that the variations in the liquid film's surface are slow in space and time.

We begin our application of the LWE by defining a small parameter $\epsilon = \frac{h}{\ell}$ where ℓ is the flow's horizontal characteristic length and $h \ll \ell$. The liquid system's independent variables are then rescaled using ϵ as

$$x \rightarrow \epsilon x, \quad y \rightarrow y, \quad t \rightarrow \epsilon t. \quad (2.8)$$

Now the system's dependent variables are expanded in terms of ϵ as

$$u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2), \quad (2.9a)$$

$$v = \epsilon(v_0 + \epsilon v_1 + \mathcal{O}(\epsilon^2)), \quad (2.9b)$$

$$p = p_0 + \epsilon p_1 + \mathcal{O}(\epsilon^2), \quad (2.9c)$$

$$T = T_0 + \epsilon T_1 + \mathcal{O}(\epsilon^2). \quad (2.9d)$$

Additionally, the following dimensionless parameters are rescaled:

$$E \rightarrow \epsilon \bar{E}, \quad D \rightarrow \epsilon^2 \bar{D}, \quad S \rightarrow \frac{\bar{S}}{\epsilon^2}. \quad (2.10)$$

Note that the transformations in Eqs. (2.10) can be considered realistic when compared to the data tabulated in Ref. [54] and that since $\mathcal{L}$ is typically quite large, the kinetic energy term in the energy balance at the liquid interface [Eq. (2.4g)] can be neglected [56, 54, 58]. Moreover, we consider the vapor mass flux (J) and the interface height (h) functions to be leading order quantities.

Eqs. (2.8), (2.9), and (2.10) are substituted into the system of equations (2.4) and the ordering procedure allows obtaining the leading temperature solution as

$$T = -J(x, t)y. \quad (2.11)$$

The leading order expression for the temperature in Eq. (2.11) is sufficient to substitute the vapor mass flux derivatives in place of the temperature derivatives in the $\mathcal{O}(\epsilon)$ system since our application of the LWE readily drops terms of order $\mathcal{O}(\epsilon^2)$. The vapor mass flux is retained as an unknown function, and we obtain solutions for the dependent variables (u, v, p) which are then substituted into the

interfacial mass balance [Eq. (2.4f)] to obtain the film evolution equation as

$$\partial_t h + \bar{E}J + \partial_x \left\{ \frac{1}{3} \text{Re} \sin \beta h^3 + \epsilon \left[\frac{2}{15} (\text{Re} \sin \beta)^2 h^6 \partial_x h + \frac{5}{24} \bar{E} (\text{Re} \sin \beta) h^4 J - \frac{1}{3} (\text{Re} \cos \beta) h^3 \partial_x h + \mathcal{M} \partial_x (hJ) h^2 - V_r h^3 J \partial_x J + \bar{S} h^3 \partial_{xxx} h \right] \right\} = 0. \quad (2.12)$$

Equation (2.12) describes the evolution of the liquid interface under the influence of gravity, hydrostatic pressure, thermocapillarity, and evaporation. Note that the exact governing equation obtained by Joo *et al.* [56] can be recovered from this expression by substituting $J = 1/(h+K)$. This expression represents an extension of the model of Sultan *et al.* [69] to include thermocapillary effects at $\mathcal{O}(1)$, as well as convection and evaporation phenomena at $\mathcal{O}(\epsilon)$. Moreover, we have defined $V_r = \bar{E}^2/D$ as the parameter representing the strength of vapor recoil forces, as well as $\mathcal{M} = M/\text{Pr}$, which represents the intensity of the thermocapillary Marangoni effect.

Note that Eq. (2.12) is also treated periodically in the streamwise coordinate x during the numerical simulation of the liquid-vapor system formed by its coupling to Eqs. (2.5). The hydrostatic pressure jump at the boundary is not considered here due to the simple dependence of the leading order pressure solution on the local interface height and surface tension. Nevertheless, the inclination angles involved in our calculations are rather moderate, and therefore, this simplification does not have any significant effect on our findings. For further details, please refer to Sec. 2.6.

2.3 Dispersion relationship derivation

The base states for film height, vapor mass flux, and vapor density can be found by setting $\partial_x = 0$ in Eqs. (2.5) and (2.12) to arrive at the following base state

solutions, respectively:

$$\bar{h}(t) = \sqrt{(\bar{h}_0 + \frac{1}{\chi})^2 - 2\bar{E}t} - \frac{1}{\chi}, \quad \bar{J} = \frac{1}{\bar{h} + \frac{1}{\chi}}, \quad \bar{\rho}_v(t, y) = \bar{J}(\bar{h} - y) - \frac{1}{\Gamma}, \quad (2.13)$$

where $\bar{h}_0$ is the initial dimensionless base flow height at $t = 0$. In contrast to Sultan *et al.*'s [69] formulation in which $\bar{J} = 1$, our formulation results in the base density gradient and the associated base vapor flux depending on $\bar{h}(t)$, and therefore can account for the higher evaporation rate expected as the volatile film's thickness reduces. Note that the expressions for $\bar{h}(t)$ and $\bar{J}$ in Eq. (2.13) are equivalent to those in Burelbach *et al.* [54] and Joo *et al.* [56], with $1/\chi$ taking the place of their interfacial non-equilibrium parameter K .

We now employ the assumption that the perturbation's growth is much faster than the variation in the base flow's height due to evaporation, i.e., that the base flow's height can be considered *frozen* and hence treated as a steady state during perturbation growth [54, 116]. We introduce the following infinitesimal perturbations into the coupled system's variables:

$$h = \bar{h} + \delta h, \quad \rho_v = \bar{\rho}_v + \delta \rho_v, \quad J = \bar{J} + \delta J, \quad (2.14)$$

where $\delta h = H e^{i(kx - \omega t)} + c.c.$, $H \ll 1$, and c.c. stands for the complex conjugate. Due to the applied scaling [Eqs. (2.10)], here $k = k^*/\epsilon$ and $\omega = \omega^*/\epsilon$, where k^* and ω^* are the wavenumber and angular frequency in the nonscaled dimensionless spatial and temporal coordinates (x^*, t^*) . We then utilize Eq. (2.5b) to express $\delta \rho_v$ in terms of δh

$$\delta \rho_v = \bar{J} \frac{\Gamma - 1}{\Gamma + |k| \left(\bar{h} + \frac{1}{\chi} \right)} e^{-|k|(y - \bar{h})} \delta h, \quad (2.15)$$

and subsequently use Eq. (2.5e) to finally express δJ in terms of δh

$$\delta J = \bar{J} |k| \frac{\Gamma - 1}{\Gamma + |k| \left(\bar{h} + \frac{1}{\chi} \right)} \delta h. \quad (2.16)$$

Substituting from Eqs. (2.15) and (2.16) into Eq. (2.12) and linearizing yields an expression for the dispersion relationship governing the evolution dynamics

$$\begin{aligned} \omega = & \left[\text{Re} \sin \beta \bar{h}^2 + \epsilon \frac{5}{6} \text{Re} \sin \beta \bar{E} \bar{J} \bar{h}^3 \right] k \\ & + i\epsilon \left[\frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 - \frac{1}{3} \text{Re} \cos \beta \bar{h}^3 + \mathcal{M} \bar{J} \bar{h}^2 \right] k^2 - i\epsilon \bar{S} \bar{h}^3 k^4 \\ & + \bar{J} \frac{\Gamma - 1}{\Gamma + |k| \left(\bar{h} + \frac{1}{\chi} \right)} \left[\epsilon \frac{5}{24} \text{Re} \sin \beta \bar{E} \bar{h}^4 k |k| - i \bar{E} |k| + i\epsilon (\mathcal{M} - V_r \bar{J}) \bar{h}^3 k^2 |k| \right]. \end{aligned} \quad (2.17)$$

Notably, the evaporation regime parameter Γ appears proportional to the last four terms which in order of appearance in Eq. (2.17) represent wave dispersion, mass loss, the diffusion-dependent component of the Marangoni effect, and vapor recoil.

As in previous literature [54, 56], the Marangoni effect in the volatile film is coupled to the evaporation process, however, under our formulation it has two distinct components. The first is driven by interfacial temperature gradients caused by the unequal heat flux arriving at the interface from the hot wall due to variations in the film's height and is tied to surface curvature, and therefore, is proportional to k^2 . On the other hand, the second term is linked to the evaporation regime and depends on the intensity of the diffusion effects and the resulting vapor density gradient above the liquid interface, while also depending the interface's curvature. Hence, it is proportional to $k^2 |k|$. Interestingly, unlike previous works which deal exclusively with the transfer-rate-limited evaporation regime [50, 54, 56], all of the terms in the dispersion relationship [Eq. (2.17)] are proportional to the wavenumber k , and are therefore considered to be associated with the instability.

For our subsequent analyses, we will assign $\epsilon = 0.05$, $\bar{S} = 0.075$, and $\chi = 10$, unless otherwise stated. Additionally, we will assign the value of $\Gamma = 1000$ to represent the diffusion-limited-regime since it is sufficiently large to approximate the limit of $\Gamma \rightarrow \infty$.

2.4 Temporal stability analysis

For a temporal stability analysis, Eq. (2.17) admits a real wavenumber k and a complex angular frequency ω , and yields expressions for the temporal growth rate and phase speed. The expression for the temporal growth rate allows the investigation of the influence of the evaporation regime on the stability of the film and highlighting some of its special cases. Moreover, it enables us to investigate the impact of the evaporation regime on the mass loss instability as well as the effects arising from film thinning. On the other hand, the expression for the phase speed permits examining the influences of the evaporation regime and film thinning on wave celerity within the film.

2.4.1 The impact of the evaporation regime on the stability of the film

In a temporal analysis, Eq. (2.17) takes the form $\omega(k) = \omega_r + i\omega_i$, where $\omega_r = ck$ is the angular frequency written as

$$\omega_r = \left[\text{Re} \sin \beta \bar{h}^2 + \epsilon \frac{5}{6} \text{Re} \sin \beta \bar{E} \bar{J} \bar{h}^3 + \epsilon \frac{5}{24} \frac{\Gamma - 1}{\Gamma + |k| \left(\bar{h} + \frac{1}{\chi} \right)} \text{Re} \sin \beta \bar{E} \bar{J} \bar{h}^4 |k| \right] k, \quad (2.18)$$

and ω_i is the temporal growth rate reading

$$\omega_i = \epsilon \left[\frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 + \mathcal{M} \bar{J} \bar{h}^2 - \frac{1}{3} \text{Re} \cos \beta \bar{h}^3 - \bar{S} \bar{h}^3 k^2 \right] k^2 + \bar{J} \frac{\Gamma - 1}{\Gamma + |k| \left(\bar{h} + \frac{1}{\chi} \right)} \left[-\bar{E} |k| + \epsilon (\mathcal{M} - V_r \bar{J}) \bar{h}^3 k^2 |k| \right]. \quad (2.19)$$

The evaporation regime parameter Γ appears proportional to three terms in the temporal growth rate which, in order of appearance in Eq. (2.19), represent the contributions of (i) mass loss, (ii) the diffusion-dependent Marangoni effect, and (iii) vapor recoil. Notably, the type of contribution each of these terms serves to the growth rate, whether stabilizing or destabilizing, depends on whether the evaporation regime parameter Γ is smaller or larger than 1, as indicated by the change in their algebraic signs in Eq. (2.19). Moreover, the last two terms in Eq. (2.19) do not appear in Sultan *et al.*'s [69] derivation. Note that the temporal growth rate produced by Eq. (2.19) is validated against a numerically computed counterpart, which is extracted from the numerical simulation of the coupled liquid-vapor system formed by Eqs. (2.5) and Eq. (2.12), for further details refer to Sec. 2.6.

When $\Gamma < 1$, the evaporation process is dictated by thermodynamic disequilibrium at the interface and the roles of the evaporative mass loss and vapor recoil are destabilizing, which is in accordance with what is found in the literature dealing with the transfer-rate-limited regime [54, 56]. However, under this evaporation regime the local evaporation rate across the interface is dictated by the proximity to the hot wall below and is therefore higher at the troughs than at the crests, and hence the local latent cooling is also higher at the troughs. This results in a reduction in the temperature-dependent surface tension gradient between the troughs and the crests and ultimately a tempering of the Marangoni instability. This attenuating effect is evident in Eq. (2.19) in the negative sign of its diffusion-dependent component when $\Gamma < 1$.

On the other hand, when $\Gamma > 1$, the evaporation process is largely dependent on the interface's curvature and the associated diffusion of vapor into the ambient gas phase above the liquid interface. We find in this case that the contributions of mass loss and vapor recoil are stabilizing since the vapor flux is higher at the crests than at the troughs due to the sharp edge effect [66], i.e., their convex curvature, which matches previous findings [66, 69, 115]. Accordingly, the latent cooling in this regime is higher at the crests rather than the troughs which reduces their local temperature, and thus increases the surface tension gradient across the film. Consequently, the diffusion-dependent component of the Marangoni instability

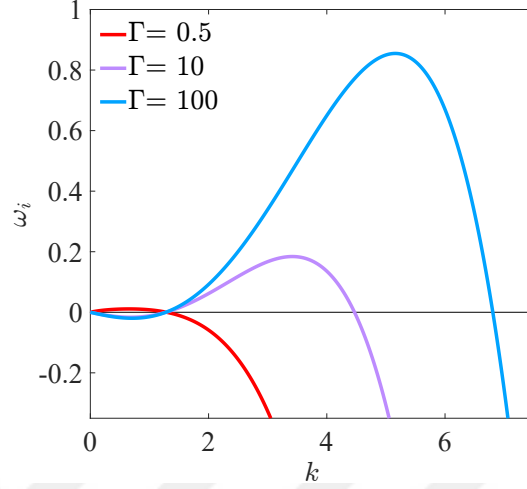


Figure 2.2: Temporal growth rates for $\text{Re} = 2$, $\beta = 25$, $\mathcal{M} = 0.7$, $V_r = 0.1$, $\bar{E} = 0.05$, and increasing values of Γ representing the increasing importance of vapor diffusion.

plays a reinforcing role in this evaporation regime and results in a substantially more unstable liquid film.

This destabilizing influence of vapor diffusion can be seen in Fig. 2.2, where we see that increasing the value of Γ , which represents the heightening of the influence of vapor diffusion on the evaporation process, results in significant increases in the temporal growth rate ω_i and the widening of the range of unstable wave numbers.

Additionally, we find that the Marangoni instability is aided by the vapor recoil and mass loss instabilities when $\Gamma < 1$, as they all contribute positively to the temporal growth rate in Eq. (2.19) under this condition, which also is in accordance with previous results [54, 56]. Conversely, when the evaporation process is dominated by diffusion ($\Gamma > 1$), the Marangoni effect has a strong destabilizing influence that is suppressed by vapor recoil and evaporative mass loss, as shown in Fig. 2.3. When evaporation effects are relatively weak ($V_r = 0.1$, $\bar{E} = 0.05$) as in Fig. 2.3(a), we observe two regions of intense instability in the (k, Γ) space: Firstly, a region of instability for small values of Γ which is caused by the combination of the subdued Marangoni instability, mass loss, and vapor recoil. However, we also see a significantly stronger region of instability at relatively larger values of Γ , which arises mostly from the Marangoni effect as

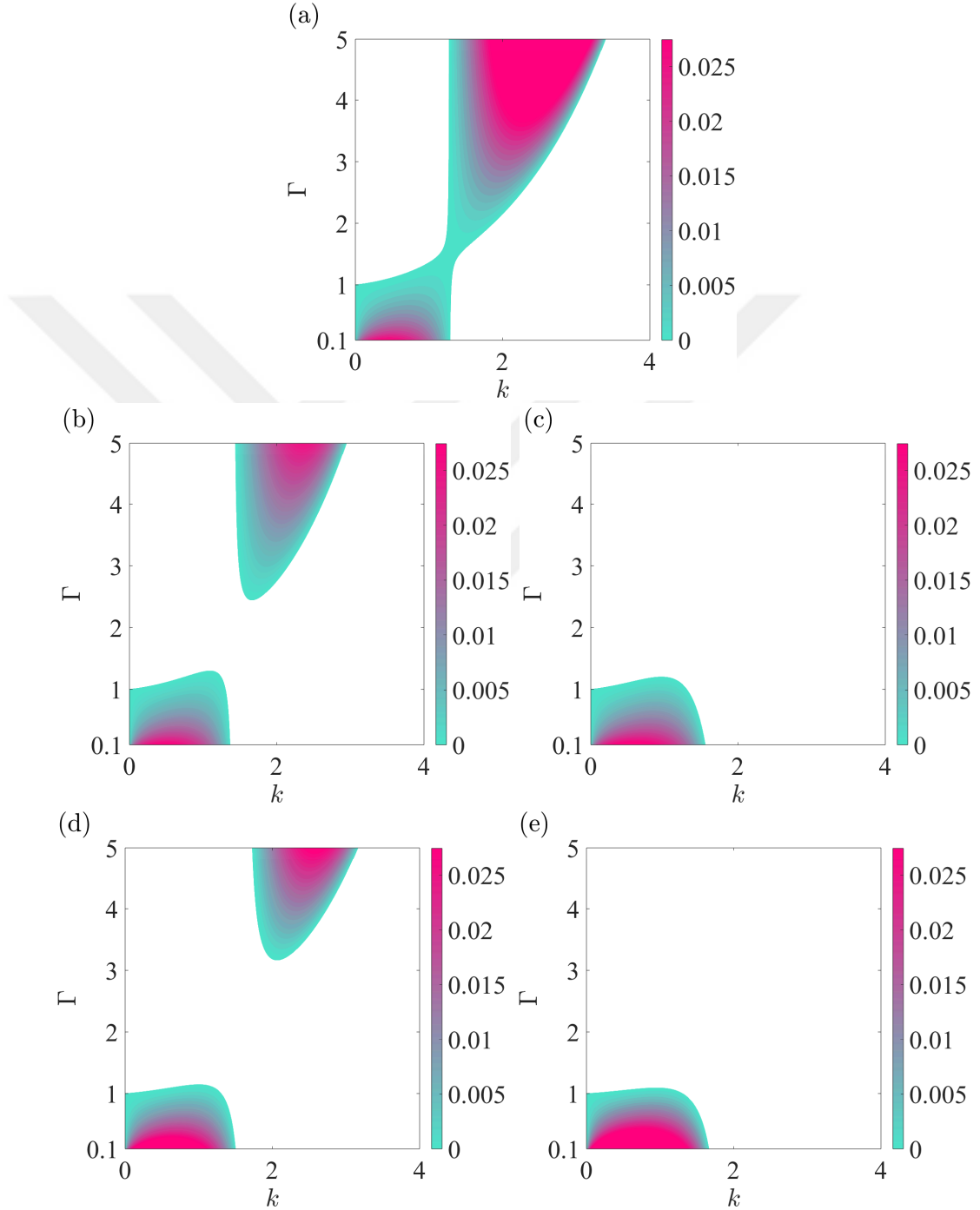


Figure 2.3: Temporal growth rates in the (k, Γ) space as influenced by the strength of evaporation effects for $\text{Re} = 2$, $\beta = 25$, and $\mathcal{M} = 0.7$. (a) Reference case with relatively weak evaporation ($V_r = 0.1$, $\bar{E} = 0.05$). Figs. (b) and (c) depict the effect of stronger vapor recoil, $V_r = 0.2$ and $V_r = 0.4$, respectively. Figs. (d) and (e) depict the effect of stronger mass loss, $\bar{E} = 0.075$ and $\bar{E} = 0.1$, respectively.

amplified by latent cooling. In Fig. 2.3(b) vapor recoil is stronger ($V_r = 0.2$) and has two opposite effects depending on the evaporation regime. At small values of Γ , the stronger vapor recoil strengthens the instability, whereas at relatively larger values of Γ it weakens it. These two opposing trends continue as the value of V_r is further increased to $V_r = 0.4$, as seen in Fig. 2.3(c), where we observe further intensification of the instability in the kinetically-dominated regime, alongside a complete suppression of the perturbation in the diffusion-dominated evaporation regime. Likewise, the effect of evaporative mass loss on the (k, Γ) parameter space is shown in Figs. 2.3(d) and 2.3(e), where again the contrast in its role and the dependence of its influence on the evaporation regime is evident. Figs. 2.3(d) and 2.3(e) show the gradual stabilization (destabilization) that results from raising the value of $\bar{E}$ when Γ is relatively large (small).

2.4.2 Special cases in the evaporation regime

The evaporation regime contains three unique situations represented by three special cases of the temporal growth rate [Eq. (2.19)]. Firstly, the case of $\Gamma \rightarrow 0$ represents the transfer-rate-limited regime where the liquid's vapor does not appear as a variable in the system and the role of vapor diffusion is excluded. This is reflected in the elimination of $|k|$ from Eq. (2.19) since it is associated with the vapor gradient above the liquid interface, and hence in the transfer-rate-limited regime Eq. (2.19) reads

$$\omega_i = \frac{\bar{E}}{\left(\bar{h} + \frac{1}{\chi}\right)^2} + \epsilon k^2 \left[\frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 + \frac{\mathcal{M}}{\chi} \frac{\bar{h}^2}{\left(\bar{h} + \frac{1}{\chi}\right)^2} + \frac{V_r \bar{h}^3}{\left(\bar{h} + \frac{1}{\chi}\right)^3} - \frac{\text{Re}}{3} \cos \beta \bar{h}^3 - \bar{S} \bar{h}^3 k^2 \right], \quad (2.20)$$

in which we have substituted explicitly for the base vapor flux $\bar{J}$. Equation (2.20) recovers the exact expression for the temporal growth rate in Joo *et al.* [56] with the exception of an extraneous higher-order term, where $1/\chi$ has taken the place of their interfacial nonequilibrium parameter K . Remarkably, this limit

naturally allows unifying the two terms representing the Marangoni instability in Eq. (2.19) into a single term, which is compatible with the singular dependence of both drivers of the surface tension gradient on the film thickness, namely the heat flux across the film and the latent cooling due to the vapor flux from the interface. Note that when the limit of $\Gamma \rightarrow 0$ is taken in the original work of Sultan *et al.* [69], only the special case of quasi-equilibrium conditions and $\bar{h} = 1$ is recovered from Burelbach *et al.*'s [54] work.

Secondly, as in the work of Sultan *et al.* [69], the case of $\Gamma \rightarrow \infty$ represents the diffusion-limited regime where the vapor gradient above the film is a primary influence on its stability, which is manifested in the expression for the temporal growth rate under this limit which now reads:

$$\omega_i = -\bar{E}\bar{J}|k| + \epsilon k^2 \left[\frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 + \mathcal{M}\bar{J}(\bar{h}^2 + \bar{h}^3|k|) - V_r \bar{J}^2 \bar{h}^3 |k| - \frac{\text{Re}}{3} \cos \beta \bar{h}^3 - \bar{S} \bar{h}^3 k^2 \right]. \quad (2.21)$$

Note that the term representing the Marangoni instability is maximized in under these conditions since the coefficient associated with its diffusion-dependent component takes its peak value when $\Gamma \rightarrow \infty$, which is consistent with the fact that the augmenting role of latent cooling is maximized when the evaporation process is entirely governed by diffusion.

Lastly, when $\Gamma = 1$, the last three terms in Eq. (2.19) vanish and it reduces to

$$\omega_i = \epsilon \left[\frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 - \frac{\text{Re}}{3} \cos \beta \bar{h}^3 + \mathcal{M}\bar{J}\bar{h}^2 - \bar{S}\bar{h}^3 k^2 \right] k^2, \quad (2.22)$$

which represents a volatile film where the increased evaporation rate at the troughs due to their increased proximity to the hot wall is precisely balanced by the heightened evaporation rate at the crests due to the higher diffusion rate caused by their convex curvature. Interestingly, this special case represents an evaporating film that is naturally devoid of the vapor recoil and mass loss instabilities. As for the Marangoni instability, since the evaporative latent cooling

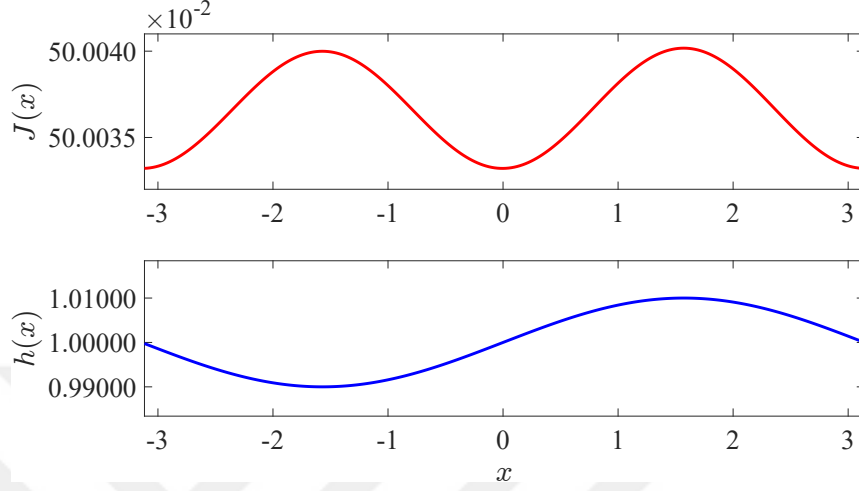


Figure 2.4: Plot of the perturbed liquid interface (bottom) and the corresponding numerically computed vapor flux (top) for $\Gamma = 1$, $k = 1$, and a grid size of $(N_x, N_y) = (256, 64)$. The liquid interface is described by $h(x) = 1 + 0.01 \sin(x)$.

is equalized across the troughs and the crests, it neither diminished nor augmented and is therefore intermediate in strength between the transfer-rate-limited ($\Gamma \rightarrow 0$) and the diffusion-limited ($\Gamma \rightarrow \infty$) cases. We verify that the vapor mass flux is indeed equal at the crests and the troughs of the film when $\Gamma = 1$ by numerically solving the vapor system [Eqs. (2.5)] for a perturbed interface described by $h(x) = 1 + 0.01 \sin(kx)$, as shown in Fig. 2.4. Due to this particular property, we will refer to this special case as the *balanced flux* regime.

2.4.3 The impact of the evaporation regime on the mass loss instability

The role that evaporative mass loss has on the temporal stability of the film is tied to the evaporation regime through several facets. In the complete absence of vapor diffusion effects i.e., the transfer-rate-limited regime, the evaporation process is kinetically driven, and thus, the film's troughs which are closer to the hot wall will decay faster than the crests, since they experience relatively higher evaporation rates. This results in perturbation growth and ultimately a destabilizing influence for evaporative mass loss, as reported in Refs. [54, 56].

Moreover, since the evaporation process here is one-sided, i.e., does not account for a gas phase above the liquid film, there is no vapor gradient restricting the departure of molecules from the interface, and thus, their ability to depart the interface is equal across the film regardless of the local interface curvature, as it is only contingent on the local film height.

These two aspects of the mass loss effect are reflected in the expression for the temporal growth representing this limit [Eq. (2.20)], where we find, similar to previous works dealing with the transfer-rate-limited evaporation regime [54, 56, 69, 50], that the mass loss term (i) contributes positively to the temporal growth rate, and (ii) is independent of the wave number k , which is consistent with the effect being independent of the interface's curvature. Therefore, this term is typically neglected when computing the temporal growth rate [50, 56] in this regime since it is not directly related to the perturbation, and its contribution is restricted to an algebraic shift in the temporal growth rate across all wave numbers. Thus, we keep in accordance with this convention [54, 56] and set $\bar{E}$ to zero when computing the perturbation's temporal growth rates in the transfer-rate-limited regime.

On the other hand, accounting for the liquid's vapor introduces a vapor gradient above the liquid film which varies with its curvature, and hence, the effect of mass loss is not realized equally across the interface since the propensity of the molecules to depart from it depends on the variation in this gradient. Thus, in the presence of vapor diffusion, the role of mass loss is properly tied to the instability through the film height perturbation. Again, these properties are captured in the associated expressions for the temporal growth rate [Eqs. (2.19) and (2.21)], where the mass loss term is proportional to the wave number. Consequently, we retain this term when $\Gamma > 0$.

Ultimately, the influence that mass loss has in the two-phase system, whether stabilizing or destabilizing, is determined by the relative strengths of the kinetic and diffusion phenomena. The effect is destabilizing when kinetic effects dominate ($\Gamma < 1$), as the film's troughs will overall evaporate more intensely due to their proximity to the hot wall, despite the hindrance posed by the unfavorable vapor

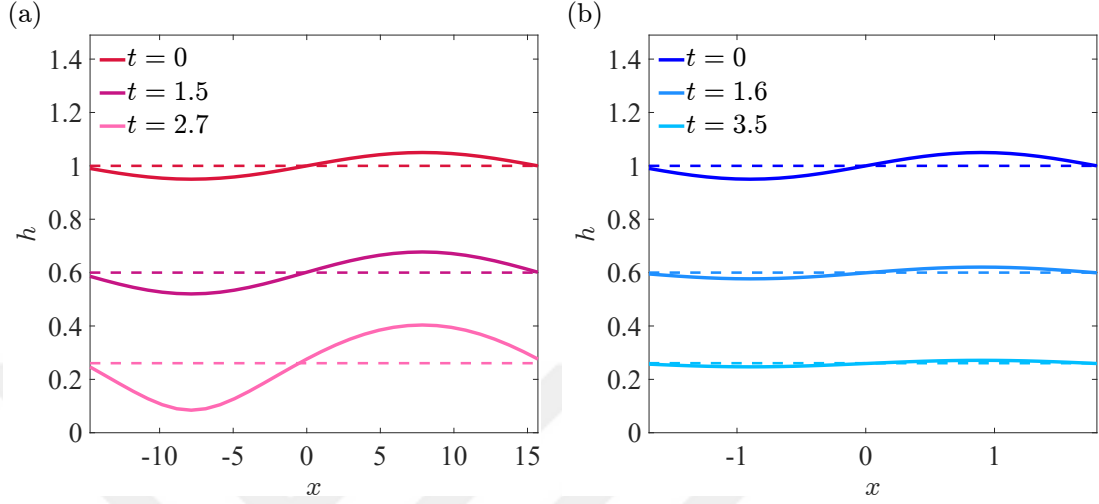


Figure 2.5: Numerical simulations of a thinning evaporating film comparing of the role of evaporative mass loss at the two limits of the evaporation regime, for $\text{Re} = 2$, $\beta = 0$, $\mathcal{M} = 0$, $V_r = 0$, $\bar{E} = 0.1$. (a) Transfer-rate-limited evaporation, $\Gamma = 10^{-6}$, $k = 0.2$, $(N_x, N_y) = (32, 8)$. (b) Diffusion-limited evaporation, $\Gamma = 1000$, $k = 1.75$, $(N_x, N_y) = (32, 1024)$. In both cases the simulation started from an initial interface $h = 1 + 0.05 \sin(kx)$. The dashed lines represent the reducing flat film height at the corresponding time.

gradient. On the contrary, the effect is stabilizing when vapor diffusion effects prevail ($\Gamma > 1$), as the film's crests will evaporate more intensely due to their convex curvature, despite their relative distance from the hot wall.

We demonstrate this contrasting effect of evaporative mass loss in the presence of vapor diffusion by simulating the coupled liquid-vapor system through the numerical solution of Eqs. (2.5) and Eq. (2.12), where we isolate the effect of evaporative mass loss by neglecting vapor recoil and Marangoni phenomena ($V_r = \mathcal{M} = 0$), and keeping $\bar{E} \neq 0$. The results of this simulation are presented in Fig. 2.5 where we chose values of Γ representing the extremities of the evaporation regime to maximize the observable effect. Fig. 2.5(a) depicts an evaporating thinning film at different points in time under transfer-rate-limited conditions ($\Gamma \rightarrow 0$), where the perturbation is observed to grow in time as a result of unequal mass loss across the film. Conversely, Fig. 2.5(b) depicts the film thinning in the diffusion-limited regime ($\Gamma = 1000$), where the evaporative mass loss results in the perturbation's decay and, eventually, the restoration of the flat interface.

2.4.4 The impact of the evaporation regime on the effects of film thinning

The effect of the reduction in the base flow height $\bar{h}$ due to evaporation on the thermal instabilities (thermocapillary and evaporative) was studied previously by Mohamed *et al.* [50]. However, their analysis was restricted to the transfer-rate-limited regime corresponding to $\Gamma \rightarrow 0$. In our work, we also investigate the effect of film thinning in the presence of vapor diffusion effects. The competition between the different physical phenomena influencing the stability of the film as it thins is best understood in the limiting cases of transfer-rate-limited evaporation and diffusion-limited evaporation.

The effect of film thinning on the temporal growth rate in the transfer-rate-limited regime ($\Gamma \rightarrow 0$) is shown in Fig. 2.6(a) where we observe a drastic increase in both the temporal growth rate and range of unstable wave numbers as the film thickness is reduced. This results in the stable film corresponding to $\bar{h} = 1$ becoming increasingly unstable as the value of $\bar{h}$ is decreased, repeating the findings of Ref. [50]. On the other hand, Fig. 2.6(b) depicts the temporal growth rate in the case of $\Gamma = 1000$ which represents the diffusion-limited regime, where we observe markedly different behavior as the reduction in $\bar{h}$ causes a reduction in the temporal growth rate indicating a stabilizing effect. Note this is also accompanied by an expansion of the set of unstable wave numbers.

To understand this contrast, we inspect the change in the dominant components of the temporal growth rate with $\bar{h}$ close to the most unstable wave number (denoted k_{max}) in the two disparate regimes, as shown in Figs. 2.6(c) and 2.6(d). We do not include the inertial component in our analysis due to its diminished influence on the thermal instability, or the mass loss component since it is only first order in the wave number while the others are of higher orders. Fig. 2.6(c) corresponds to Fig. 2.6(a) where we plot the two components of the thermal instability in the transfer-rate-limited regime: the Marangoni effect and vapor recoil, along with the two stabilizing phenomena: hydrostatic pressure and surface tension, against the base flow height $\bar{h}$.

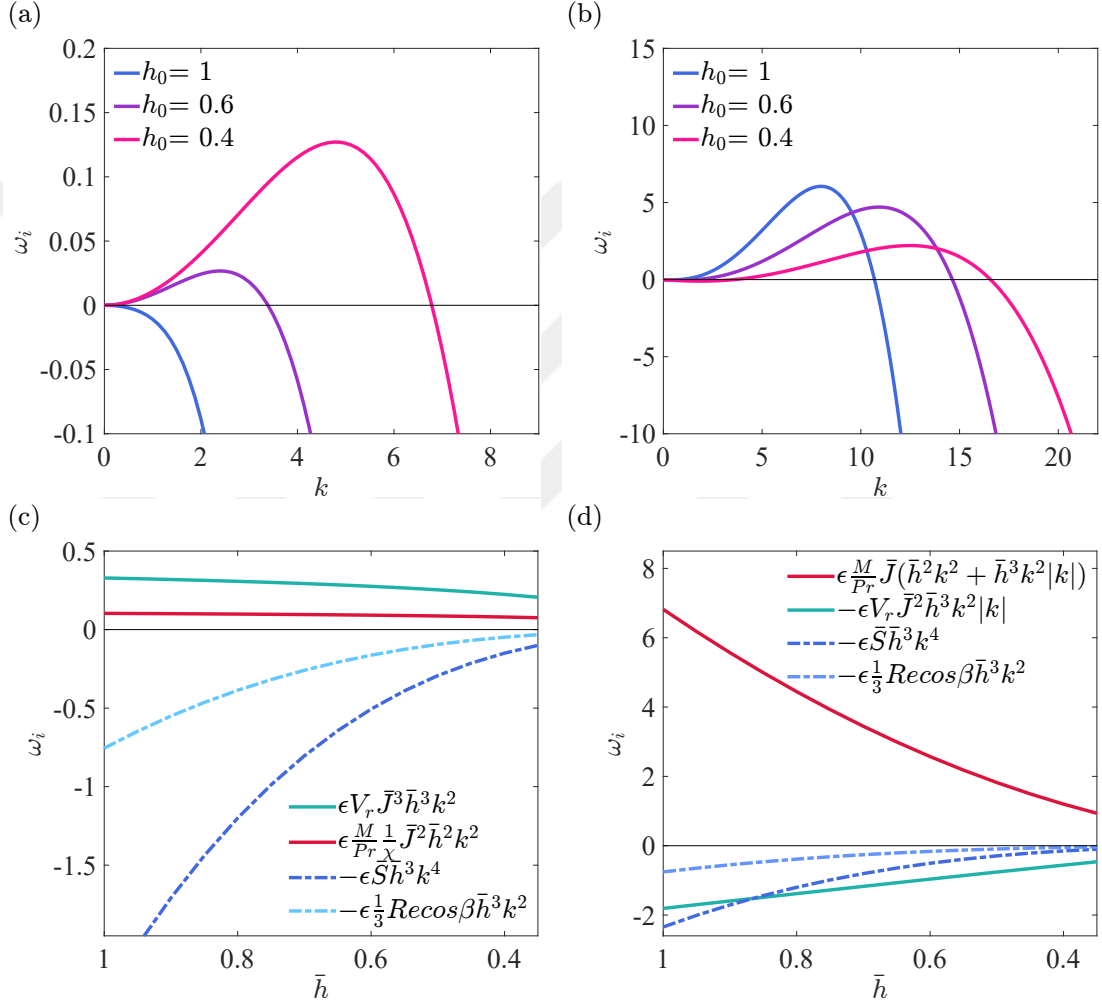


Figure 2.6: Comparison of the effect of film thinning on the temporal stability of the film at the two limits of the evaporation regime for $Re = 2$, $\beta = 25$, $\mathcal{M} = 1.25$, and $V_r = 0.35$. (a) Temporal growth rates in the transfer-rate-limited regime ($\Gamma = 0$, $\bar{E} = 0$). (b) Temporal growth rates in the diffusion-limited regime ($\Gamma = 1000$, $\bar{E} = 0.05$). (c) Competition between the physical effects in the transfer-rate-limited regime ($\Gamma = 0$, $\bar{E} = 0$, $k = 5$). (d) Competition between the physical effects in the diffusion-limited regime ($\Gamma = 1000$, $\bar{E} = 0.05$, $k = 5$).

We find that as the film thins the reduction in the stabilizing effects is significantly larger than the decline in the destabilizing ones, yielding an overall destabilization of the film. This is consistent with the order of these terms in the expression for the temporal growth rate corresponding to this regime [Eq. (2.20)], as it can be seen that the Marangoni term and vapor recoil terms diminish with $\bar{h}$ in $\sim \mathcal{O}(1)$, whereas the hydrostatic pressure and surface tension terms vary with it in $\sim \mathcal{O}(\bar{h}^3)$.

Likewise, Fig. 2.6(d) corresponds to Fig. 2.6(b), where the augmented Marangoni effect remains the sole driver of the thermal instability since vapor recoil becomes a stabilizing factor in the diffusion-limited regime. We notice in this case that the destabilizing Marangoni effect reduces rather fast with $\bar{h}$ in a manner that is comparable to the stabilizing factors it competes against, which now also include vapor recoil, resulting ultimately in an overall stabilization of the film. This behavior again conforms to the order of the terms in the corresponding Eq. (2.21), where the augmented Marangoni term reduces with $\sim \mathcal{O}(\bar{h}^2)$ which is closer to the order of the stabilizing terms with respect to $\bar{h}$.

This heightened sensitivity of the Marangoni term to the base flow height arises in large part due to the reliance of its diffusion-dependent component on the intensity of latent cooling, which loses its efficacy as the film thins in this evaporation regime. This is because the thinner film corresponds to higher interfacial temperatures, yet the evaporation process and hence the latent cooling, remain constrained by the diffusion rate. This is in stark contrast to the transfer-rate-limited case, where the higher interfacial temperatures correspond immediately to higher evaporation rates. Notably, the type of stabilization observed in Fig. 2.6(b) occurs when the stabilizing evaporation effects, namely vapor recoil and mass loss, are strong enough to compensate for the fast reduction in the hydrostatic pressure and surface tension terms as $\bar{h}$ reduces.

In fact, the effect of the reduction in base flow height on the film's stability in the diffusion-limited regime is rather nuanced and is determined by the relative strength of the evaporation phenomena with respect to the Marangoni instability. This is depicted in Fig. 2.7 which plots the perturbation's temporal growth

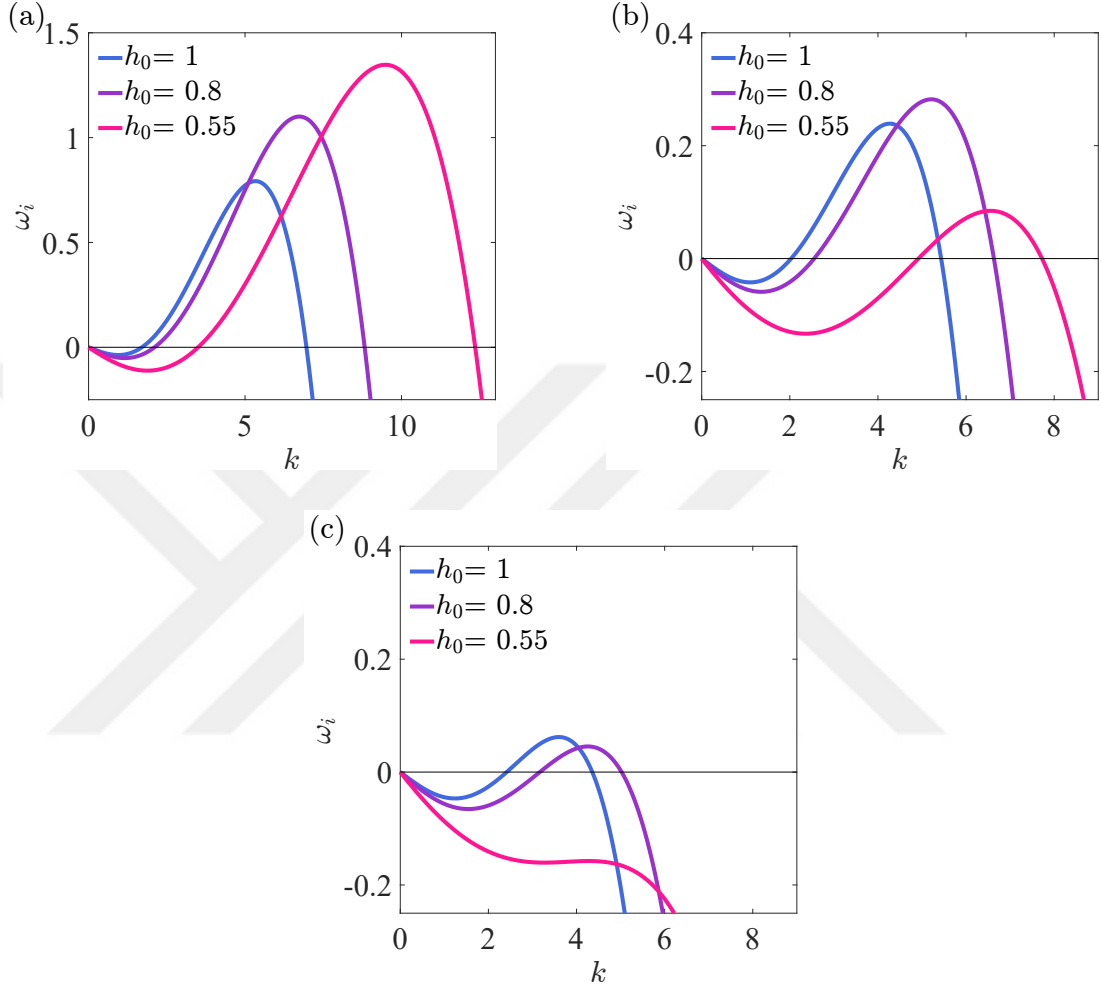


Figure 2.7: Temporal growth rates for a thinning film in the diffusion-limited regime ($\Gamma = 1000$) with varying evaporation intensity. $\text{Re} = 2$, $\beta = 15$, $\mathcal{M} = 0.63$, $\bar{E} = 0.065$, and (a) $V_r = 0.02$, (b) $V_r = 0.14$, and (c) $V_r = 0.21$.

rates in the same thinning film for three different strengths of the vapor recoil parameter. In Fig. 2.7(a), vapor recoil is weak with respect to the Marangoni effect ($V_r = 0.02$). Thus, as the film thins the resulting reduction in the sum of the stabilizing terms is larger than the corresponding decrease in the destabilizing Marangoni term. As a result, the effect of film-thinning is destabilizing.

On the other hand, Fig.2.7(b) depicts the temporal growth rates for the same film where vapor recoil is now significantly stronger ($V_r = 0.14$). In this case, we witness that the initial reduction in base film height to $\bar{h} = 0.8$ has a destabilizing influence as evidenced by the increase in the maximum temporal growth rate,

which is driven by the same destabilizing mechanism observed in Fig. 2.7(a). However, as the film height reduces further, the relatively slow [$\sim \mathcal{O}(\bar{h})$] reduction in the vapor recoil term [see Fig. 2.6(d)] signifies that vapor recoil retains its potency as a stabilizing factor relative to the rapidly decaying Marangoni effect. This ultimately leads to an inversion in the impact of film thinning from destabilizing to stabilizing.

Lastly, when vapor recoil is even stronger ($V_r = 0.21$), as in Fig. 2.7(c), the reduction in base flow height has an immediate stabilizing effect since the decrease in the strength of the Marangoni instability is larger than the decrease in the remaining stabilizing phenomena, which mirrors the case depicted in Fig. 2.6(b). Additionally, we note that due to the relatively weak Marangoni instability, the stabilizing role of mass loss is significant, and consequently, the results in Fig. 2.7 can be reproduced by increasing the mass loss parameter $\bar{E}$ in place of V_r . However, this analogous impact of evaporative mass loss is not depicted here graphically to avoid redundancy.

2.4.5 The impacts of evaporation regime and film thinning on wave celerity

We extend our analysis to investigate the influence of evaporation regime and film thinning on wave propagation within the film by examining the phase speed $c = \frac{\omega_r}{k}$, which is obtained directly from Eq. (2.18) as

$$c = \text{Re} \sin \beta \bar{h}^2 + \epsilon \bar{E} \bar{J} \text{Re} \sin \beta \left(\frac{5}{6} \bar{h}^3 + \frac{\Gamma - 1}{\Gamma + |k| \left(\bar{h} + \frac{1}{\chi} \right)} \frac{5}{24} \bar{h}^4 |k| \right). \quad (2.23)$$

The last two terms in Eq. (2.23) indicate that the phase speed is impacted by evaporative mass loss, mirroring the findings in Refs. [50, 56]. Intriguingly, the third term is also proportional to Γ which implies that the phase speed is affected by the evaporation regime in particular. We demonstrate this dependence in

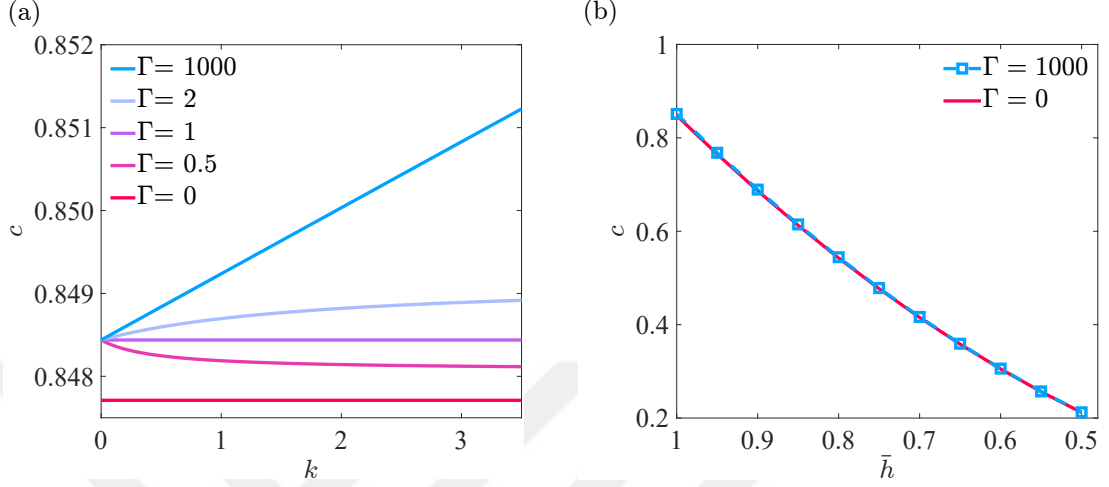


Figure 2.8: Perturbation phase speed versus (a) wave number and (b) base flow height for $\text{Re} = 2$, $\beta = 25$, $\mathcal{M} = 1.25$, $V_r = 0.15$, and $\bar{E} = 0.1$.

Fig. 2.8(a), which shows the variation in wave dispersion across the evaporation regimes, where we find three distinct dispersion relationships.

For the transfer-rate-limited ($\Gamma \rightarrow 0$) and balanced flux ($\Gamma = 1$) cases, the phase speed does not depend on the wave number, and thus, the waves are non-dispersive. On the other hand, for moderate values of Γ representing general evaporation, the waves are nonlinearly dispersive, with this dispersion being positive for $\Gamma < 1$ and negative for $\Gamma > 1$, indicating that the imbalance of vapor diffusion in particular is related to wave dispersion. We note that this positive (negative) dispersion can lead to pulse broadening (compression) when $\Gamma < 1$ ($\Gamma > 1$). However, the effect is negligible since it is only effective for small k .

That being said, when $\Gamma = 1000$ signifying diffusion-limited evaporation, Eq. (2.23) takes the form

$$c = \text{Re} \sin \beta \bar{h}^2 + \epsilon \bar{E} \bar{J} \text{Re} \sin \beta \left(\frac{5}{6} \bar{h}^3 + \frac{5}{24} \bar{h}^4 |k| \right), \quad (2.24)$$

which demonstrates that for diffusion-limited evaporation the waves are linearly negatively dispersive for all k . This suggests that evaporation effects in the diffusion-limited regime can lead to pulse compression. Nevertheless, we find

that the impact of the evaporation regime on phase speed is entirely overshadowed by that of the film thinning. This is depicted in Fig. 2.8(b), where c is almost identical in both evaporation limits but reduces drastically with $\bar{h}$ due to the stronger viscous resistance, reflecting the findings of Ref. [50].

2.5 Spatiotemporal stability analysis

The spatiotemporal dynamics are investigated by determining the flow's long-time response to an infinitesimal impulse source, following the procedure of Huerre and Monkewitz [89]. The nature of the instability is determined by examining the temporal growth rate $\sigma(V)$ associated with the zero-group-velocity ($V = 0$), also known as the *absolute growth rate* denoted typically as $\omega_{i,0}$, ($\omega_{i,0} < 0$ indicates that the flow is convectively unstable, and $\omega_{i,0} > 0$ indicates that it is absolutely unstable). As required by our spatiotemporal analysis, we assume both ω and k to be complex in Eq. (2.17), which is then utilized to obtain the linear impulse response wave packets conveying the nature of the flow.

However, we note that the system's dispersion relationship [Eq. (2.17)] in its general form is not analytic at zero due to the presence of $|k|$, which typically prevents the application of the Briggs pinching criterion [89]. Nevertheless, we can overcome this impediment by restricting our spatiotemporal analysis to wave numbers with positive real part ($|k| \rightarrow +k$), thereby removing the nonanalyticity from the dispersion relationship. This is justified by the fact that the temporal analysis produces a unique maximum temporal growth rate $\omega_{i,max}$ that is also a global maximum for all $\sigma(V)$ [89], and since the expression for the temporal growth rate is symmetric around $k = 0$, we expect to see this unique maximum for $|k| \rightarrow \pm k$. Hence, setting $|k| \rightarrow +k$ in our dispersion relationship produces this dominant mode as a saddle point on the right-hand-side of the (k_r, k_i) plane allowing the classification of the instability. Interestingly, the non-analyticity vanishes from the dispersion relationship [Eq. (2.17)] in the transfer-rate-limited ($\Gamma \rightarrow 0$) and balanced flux ($\Gamma = 1$) cases, where the vapor gradient does not make an effective contribution to the system.

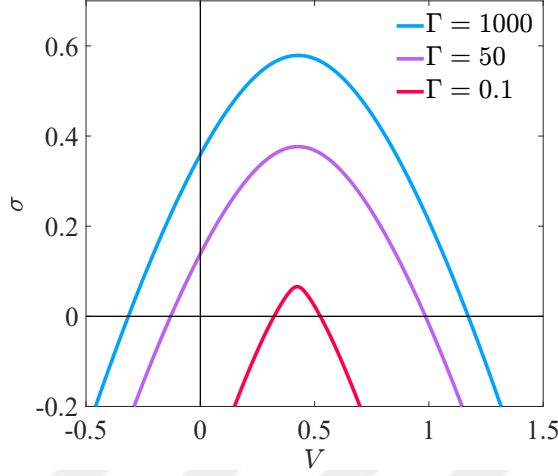


Figure 2.9: Spatiotemporal impulse responses for $\text{Re} = 1$, $\beta = 25$, $\mathcal{M} = 0.7$, $V_r = 0.2$, $\bar{E} = 0.1$, and increasing values of Γ representing the increasing importance of vapor diffusion.

The impact of the evaporation regime on the spatiotemporal stability of the flow can be seen in Fig. 2.9, where increasing the relative importance of vapor diffusion results in increasing the maximum temporal growth rate as well as expanding the range of unstable modes, as evidenced by the widening of the impulse response wave packets as the value of Γ is increased. This is also accompanied by a change in the spatiotemporal nature of the flow from convectively unstable in the $\Gamma = 0.1$ case where vapor diffusion is relatively insignificant, to absolutely unstable for $\Gamma = 50$ which represents a case of evaporation dominated by vapor diffusion. Increasing the value of Γ even further leads to a corresponding increase in the absolute growth rate, as seen in the case of $\Gamma = 1000$ representing the diffusion-limited regime. The spatiotemporal analysis reveals that the evaporation regime has a significant effect on the propagation of the perturbations in the film, with this effect being quite complex, as it cannot be detected by simple observation of the phase speed in the temporal framework [See Fig. 2.8(b)].

Moreover, the evaporation regime also determines the influence of the film thinning on the spatiotemporal stability of the flow, as demonstrated by Figs. 2.10 and 2.11. In the transfer-rate-limited regime ($\Gamma \rightarrow 0$) represented by Fig. 2.10, the reduction in base flow height results in increasing both the maximum temporal growth rate and the absolute growth rate which can lead to transitions

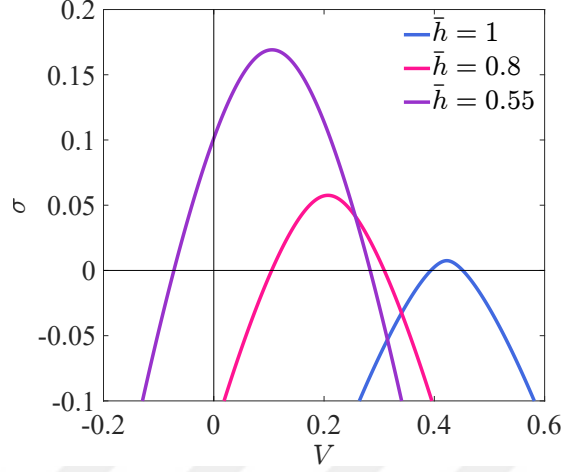


Figure 2.10: Spatiotemporal impulse responses for a thinning film in the transfer-rate-limited regime ($\Gamma = 0$) for $\text{Re} = 2$, $\beta = 15$, $\mathcal{M} = 2.75$, and $V_r = 0.35$, $\bar{E} = 0$.

in the spatiotemporal nature of the instability. Initially the flow is convectively unstable at $\bar{h} = 1$ as both edges of the impulse response are on the same side of the vertical axis. As the film becomes thinner, the maximum temporal growth rate is increased and the convective nature of the instability is weakened, as evidenced by the impulse response shifting closer to the origin resulting in a higher value of $\omega_{i,0}$. Further thinning of the film height leads to additional destabilization as well as causing the left edge of the impulse response to cross the y -axis (i.e., $V = 0$) above the origin, producing a positive value of $\omega_{i,0}$. This signifies that the flow has undergone a transition from convective to absolute instability. Note that this behavior matches the findings reported in Ref. [50] on the effect of film thinning on spatiotemporal stability in the transfer-rate-limited regime.

On the other hand, the impact of film thinning on spatiotemporal stability in the diffusion-limited regime is quite complex and emerges from the rich interactions between the Marangoni and evaporation phenomena, where the intensity of evaporation plays a fundamental role in determining the spatiotemporal nature of the flow. We illustrate this by conducting a spatiotemporal analysis with the same parameter sets used in the temporal analysis (see Fig. 2.7), where the impact of stronger evaporation is again represented by increasing the value of V_r .

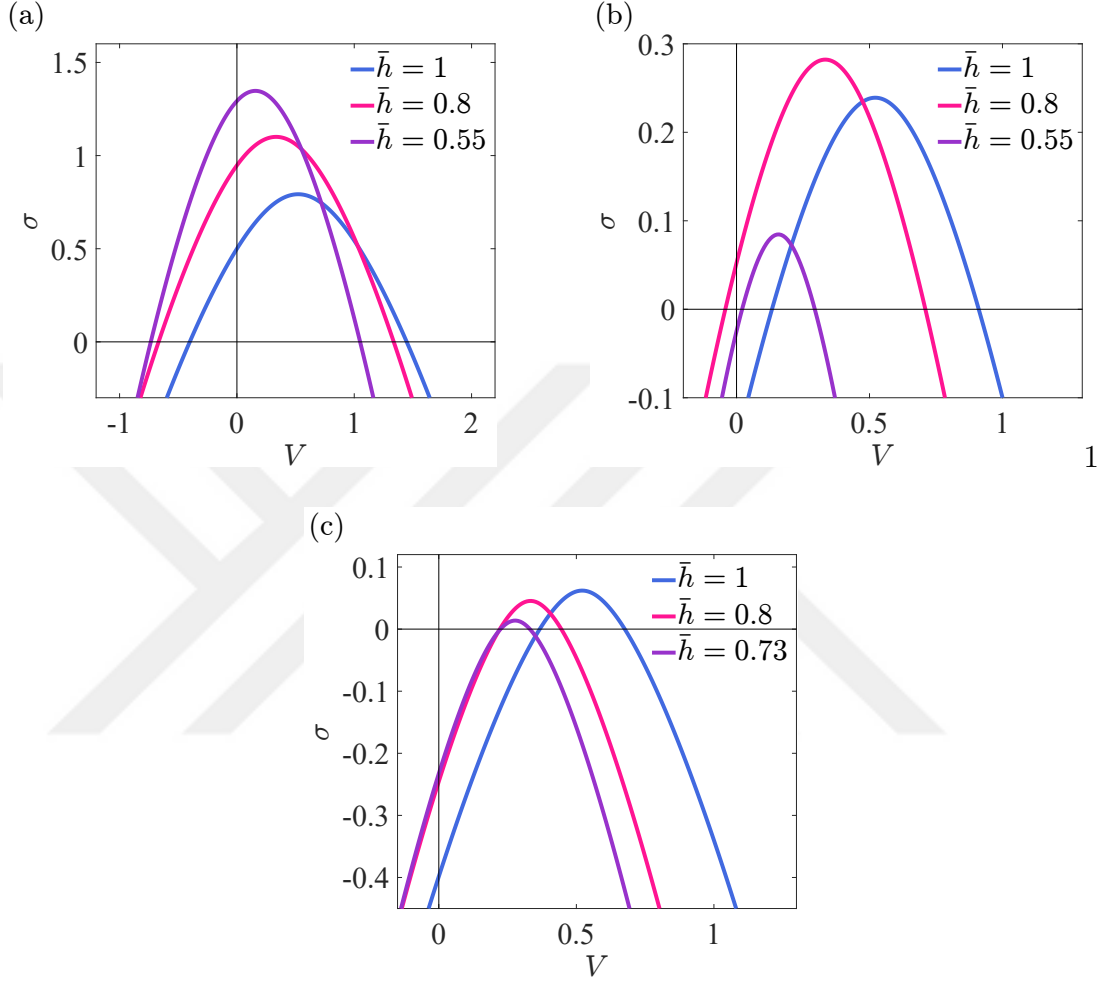


Figure 2.11: Spatiotemporal impulse responses for a thinning film in the diffusion-limited regime ($\Gamma = 1000$) with varying evaporation intensity. $Re = 2$, $\beta = 15$, $\mathcal{M} = 0.63$, $\bar{E} = 0.065$, and (a) $V_r = 0.02$, (b) $V_r = 0.14$, and (c) $V_r = 0.21$.

We observe in Fig. 2.11(a), where vapor recoil is relatively weak ($V_r = 0.02$), that the reduction in film thickness has an effect akin to that in the transfer-limited case depicted in Fig. 2.10. Furthermore, Fig. 2.11(b) corresponds to Fig. 2.7(b), where vapor recoil is significantly stronger, and the influence of film thinning in this intermediate case is again non-monotonous. We notice that the instability is at first of the convective type for $\bar{h} = 1$, and as expected, the reduction in base flow height is initially destabilizing. Notably, this destabilization is accompanied by a transition from convective to absolute instability for $\bar{h} = 0.8$. However, as the base flow height decreases further, film thinning converts into a

stabilizing influence and the instability reverts back to the convective type. Additional increases in vapor recoil intensity restore the monotonous impact of film thinning where this effect is immediately stabilizing, as indicated by the progressive decrease in $\omega_{i,max}$ with $\bar{h}$ in Fig. 2.11(c). Nevertheless, the reduction in base flow height in this case still results in an increase in the absolute growth rate, subduing the flow's convective character.

This propensity of the reduction in film height to shift the impulse response towards the vertical axis is observed in all three cases of Fig. 2.11, and is tied to the inverse relationship between phase speed and film height [see Fig. 2.8(b)], since the center of the impulse response wave packet corresponds to $c(k_{max})$. We note again that this dynamic interplay between film thinning and vapor recoil strength can be replicated by increasing the intensity of the mass loss effect i.e., by raising the value of $\bar{E}$, however, it is not depicted here to avoid redundancy.

2.6 Numerical solution and validation

The liquid-vapor problem is simulated by numerically solving the system formed by the coupling of Eqs. (2.5) and Eq. (2.12) over a domain that is periodic in the streamwise coordinate x . The initial interface is perturbed and is described by $h(x, 0) = \bar{h}_0 + H \sin(kx)$, where $\bar{h}_0$ is the initial flat film height, and H is the infinitesimal initial perturbation amplitude. The vapor density problem described by Eqs. (2.5) is solved by means of a finite difference approach utilizing the iterative successive over-relaxation method [117]. Due to the interface's curvature, the Cartesian (x, y) coordinate system must be transformed into a boundary-fitted coordinate system (ξ, η) using

$$\xi = x, \quad \eta = \frac{y - y_{max}^*}{h(x, t) - y_{max}^*}, \quad (2.25)$$

where the mapping in Eq. (2.25) guarantees a uniform computational plane with constant $\Delta\xi$ and $\Delta\eta$ as required by the finite difference approach. This remapping

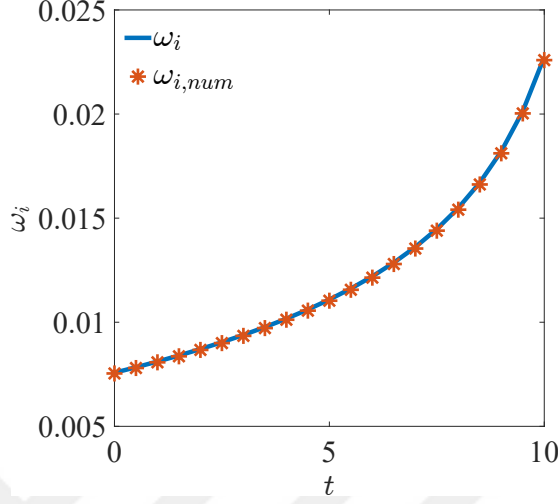


Figure 2.12: Comparison of the analytical and numerical temporal growth rates for $\text{Re} = 2$, $\beta = 25$, $\mathcal{M} = 0.75$, and $V_r = 0.15$, $\bar{E} = 0.05$, $\Gamma = 0.5$, and a grid size of $(N_x, N_y) = (32, 2048)$. The initial interface in the numerical simulation is described by $h(x, 0) = 1 + 0.001 \sin(0.25x)$. The difference between ω_i and $\omega_{i,num}$ remains less than 0.6%.

drastically complicates the spatial derivatives in the system, but results in one-to-one correspondence between the solution points in the computational and physical spaces. For further details on grid transformations, please see Ref. [118].

The evaporative mass flux from the liquid interface can be computed efficiently by rearranging the interfacial boundary condition [Eq. (2.5b)] as:

$$J = -\frac{\rho_{v|int}}{h + 1/\chi} \Gamma. \quad (2.26)$$

J is then substituted into the liquid film's evolution equation [Eq. (2.12)], which is integrated as an initial value problem using MATLAB [119]. The numerical temporal growth rate is extracted by performing the discrete Fourier transform of $h(x, t)$ at each time step as

$$\hat{f}(k, t) = \sum_{j=1}^{n-1} h(j, t) e^{-2\pi i(j-1)(k-1)/(n-1)}, \quad (2.27a)$$

where n is the total number of discrete points in $h(x, t)$. The perturbation amplitude at each time step is then found from

$$\hat{H}(k, t) = 2|\hat{f}(k, t)|, \quad (2.27b)$$

which allows computing the numerical temporal growth rate as

$$\omega_{i,num} = \frac{\dot{\hat{H}}}{\hat{H}}. \quad (2.27c)$$

Our numerical solver and analytical derivation of the temporal growth rate [Eq. (2.19)] are cross-validated by comparing ω_i and $\omega_{i,num}$ as shown in Fig. 2.12, where the difference between the numerical analytical growth rates does not exceed 0.6%.

Note that the number of grid points along the y coordinate is varied according to strength of the vapor density gradient across the vapor domain, as represented by the value of Γ . For very small values of Γ , the vapor density is approximately constant and therefore, a relatively small number of grid points is sufficient to obtain accurate simulation results. On the other hand, relatively large values of Γ are associated with sharp density gradients across the vapor domain requiring a comparatively large number of grid points to accurately resolve. Moreover, in all the simulations presented, we utilized a maximum vapor domain height $y_{max} = 35$, which is sufficiently large to accurately model our system while retaining numerical efficiency.

Chapter 3

Kinetic evaporation at low Reynolds numbers

In this second chapter, we investigate a system where the ambient environment above the liquid film is no longer static, as we introduce a shearing gas which flows continuously above the heated liquid film applying mechanical stress to the interface. The introduction of this mechanically active gas layer into the problem brings with it additional complexity, but also opens the door for some simplifications. Once more, the liquid flow regime of interest is the drag-gravity regime, which allows us again to make use of the LWE [28] to simplify the hydrodynamic problem. Furthermore, for this particular problem, we consider in our investigation a combined thermal instability mode resulting from the coupling of the thermocapillary S -mode [36] and evaporative E -mode [50, 57] instabilities appearing at low Reynolds numbers, as well as the inertial H -mode instability emerging at relatively higher Reynolds numbers. We include this inertial instability mode in our analysis due to the fact that the shearing gas significantly modifies the liquid flow's inertia, in addition to its effects on the thermal instabilities. Notably, the action of the shearing gas allows the reduction of the kinetic-diffusion evaporation process into a purely kinetic one, i.e., permits assuming one-sided evaporation. As for the mechanical stress imparted by the gas onto the liquid interface, our modeling approach allows capturing its effect into

a single nondimensional parameter. The work presented in this chapter has been published in O. A. A. Mohamed, M. C. Dallaston, and L. Biancofiore, *Spatiotemporal evolution of evaporating liquid films sheared by a gas*, Phys. Rev. Fluids, **6**, 114002 (2021).

This chapter is structured as follows: Section 3.1 presents the dimensional system of equations governing the liquid phase, followed by the reduced evaporation model and the modeling methodology for the gas's shear stress. Section 3.2 presents the employment of the LWE to arrive at the differential equation dictating the liquid interface's evolution, and is followed by Sec. 3.3 where the linear stability procedure is utilized to obtain the dispersion relationship encompassing the system's dynamic behavior. Subsequently, Sec. 3.4 presents the results of both our temporal and spatiotemporal analyses. Finally, Sec. 3.5 showcases the results of our nonlinear evolution film evolution study which was conducted by numerically solving the evolution equation. This analysis included nonlinear temporal and spatiotemporal analyses, film rupture analysis, which was extended by utilizing a self-similarity analysis to investigate the rupture mechanics under various conditions.

3.1 Mathematical formulation

The system we formulate is an extension of the study of Joo *et al.* [56] to include the effects of a shearing gas. Again, we consider an incompressible, thin, Newtonian, evaporating liquid film with constant physical properties, driven by gravity down a heated plate held at a constant temperature. The heated plate is inclined from the horizontal with an angle β as depicted in Fig. 3.1. A Cartesian $(\tilde{x}, \tilde{y})$ coordinate system is considered where the $\tilde{x}$ coordinate points in the streamwise direction and the $\tilde{y}$ coordinate in a direction perpendicular to the heated plate, i.e the cross-stream direction. The liquid film's interface is defined by the function $\tilde{y} = \tilde{h}(\tilde{x}, \tilde{t})$. Note that the tilde notation indicates dimensional quantities. The liquid film is sheared at its interface by a turbulent gas, whose

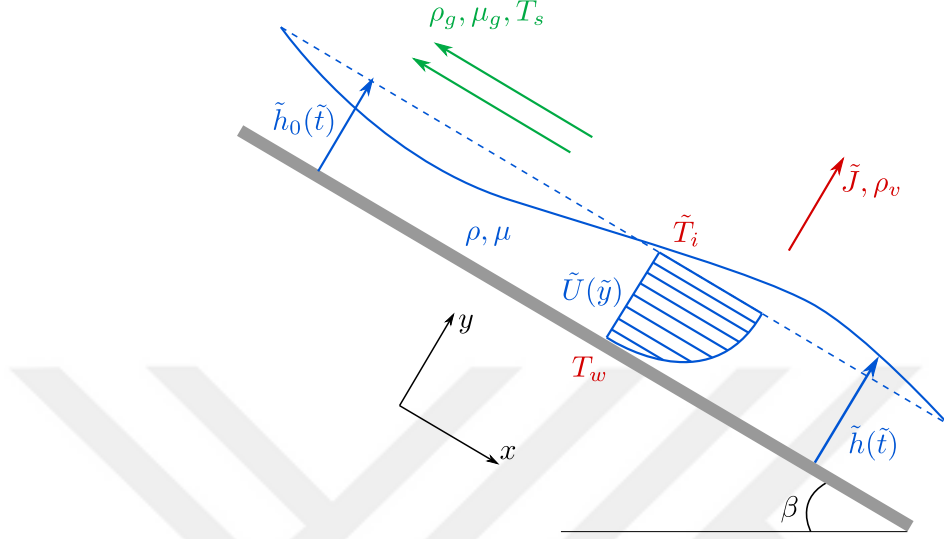


Figure 3.1: Schematic illustration of a heated evaporating liquid film flowing under the influence of gravity, while being sheared by a counter-flowing gas. The dashed line represents the unperturbed base flow thickness $\tilde{h}_0$, β is the inclination angle from the horizontal, and T_w is the constant wall temperature.

effects are accounted for by a constant quantity denoted $\tilde{\tau}$. This variable represents a constant shear stress applied along the entire interface, and is derived in detail in Sec. 3.1.3. For this problem, the liquid film is affected by two instability modes depending on the value of the flow's Reynolds number; for relatively small Reynolds numbers, we consider a thermal instability mode resulting from the combination of the thermocapillary *S*-mode [36] and the evaporative *E*-mode [57] instabilities, while at relatively larger Reynolds numbers, it is destabilized by the *H*-mode instability, as modified by thermocapillarity and evaporation.

3.1.1 Liquid phase governing equations

The liquid film is governed by the continuity equation, Navier-Stokes momentum equations, and the energy equation which in vector form read:

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad (3.1a)$$

$$\partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} = -\frac{1}{\rho} \nabla \tilde{p} + \nu \nabla^2 \tilde{\mathbf{u}} + \tilde{\mathbf{F}}, \quad (3.1b)$$

$$\partial_t \tilde{T} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} = \alpha \nabla^2 \tilde{T}, \quad (3.1c)$$

where $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ is the liquid velocity vector, ρ is the liquid density, $\tilde{p}$ is the liquid's gauge pressure, ν is the liquid's kinematic viscosity, $\tilde{\mathbf{F}} = (g \sin \beta, -g \cos \beta)$ is the gravitational body force vector with g as the gravitational acceleration magnitude, $\tilde{T}$ is the liquid temperature, and $\alpha = \lambda/\rho c_p$ is the liquid's thermal diffusivity such that λ and c_p are the thermal conductivity and constant pressure heat capacity, respectively.

The boundary conditions at the solid wall ($\tilde{y} = 0$) are the no slip, no penetration condition on the velocity, and a constant wall temperature:

$$\tilde{\mathbf{u}} = 0, \quad (3.1d)$$

$$\tilde{T} = T_w. \quad (3.1e)$$

The boundary conditions at the liquid-gas interface $\tilde{y} = \tilde{h}(\tilde{x}, \tilde{t})$ again read

$$\tilde{J} = \rho (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_i) \cdot \mathbf{n} = \rho_v (\tilde{\mathbf{u}}_v - \tilde{\mathbf{u}}_i) \cdot \mathbf{n}, \quad (3.1f)$$

$$\tilde{J} \left[\mathcal{L} + \frac{1}{2} \left(\frac{\tilde{J}}{\rho_v} \right)^2 \right] = -\lambda \nabla \tilde{T} \cdot \mathbf{n}, \quad (3.1g)$$

$$\frac{\tilde{J}^2}{\rho_v} \mathbf{n} + (\tilde{\mathbf{T}} - \tilde{\mathbf{T}}_g) \cdot \mathbf{n} = 2\tilde{\gamma} \kappa \mathbf{n} + \nabla_s \tilde{\gamma}. \quad (3.1h)$$

Quantities with the subscript i are at the liquid-gas interface, those with the subscript v are vapor quantities, and those with the subscript g are associated with the shearing gas. $\tilde{J}$ is the evaporation mass flux, $\mathcal{L}$ is the latent heat of vaporization, ∇_s is the surface gradient operator, $\kappa = -\frac{1}{2} \nabla_s \cdot \mathbf{n}$ is the mean interface curvature, and $\mathbf{n}$ is the local interface normal unit vector with modulus N such that $\mathbf{n} = \frac{1}{N}(-\partial_{\tilde{x}} \tilde{h}, 1)$. Additionally, $\tilde{\gamma} = \gamma_0 - |d\tilde{\gamma}/d\tilde{T}|(\tilde{T}_i - T_s)$ is the

surface tension function with γ_0 as the mean surface tension at the saturation temperature T_s . Moreover, $\tilde{\mathbf{T}} = -\tilde{p}\mathbf{I} + \tilde{\mathbf{P}}$ is the general stress tensor consisting of the sum of the normal stress components and the deviatoric stress tensor $\tilde{\mathbf{P}}$. Other effects such as van der Waals interactions are not considered here.

3.1.2 Reduced evaporation model

The constant flow of gas leads to the continuous removal of vapor molecules from the region above the liquid interface which prevents the formation of a vapor gradient above the liquid. Therefore, it becomes reasonable to exclude vapor diffusion effects and treat the evaporation process as purely one-sided, where the evaporation rate is dictated by the departure from thermodynamic equilibrium at the liquid interface, i.e., the evaporation is considered transfer-rate-limited. The particular formulation for the vapor mass flux we utilize is that of Burelbach *et al.* [54], which employs a constitutive relationship based on kinetic theory [53] and reads

$$\tilde{J} = \left(\frac{\alpha \rho_v \mathcal{L}}{T_s^{\frac{3}{2}}} \right) \left(\frac{M_w}{2\pi R_g} \right)^{\frac{1}{2}} (\tilde{T}_i - T_s). \quad (3.2)$$

M_w is the liquid's molecular weight, α is the accommodation coefficient, and R_g is the universal gas constant. Equation (3.2) relates the evaporative mass flux to the interfacial temperature, which in turn depends on the liquid interface's height, thereby completing the system formed by Eqs. (3.1) as a self-contained one-sided model.

As for the effect of the gas flow on the interfacial temperature, previous studies have demonstrated that the heat transfer from an evaporating liquid film subjected to laminar or turbulent gas flow is dominated by the latent heat transport associated with the evaporation of the film, as opposed to sensible heat transfer, particularly heat convection due to the gas motion [120, 121]. Therefore, we consider the effect of sensible heat convection on the evaporation process to be negligible.

3.1.3 Shear stress modeling

We account for the gas's shear stress at the interface by first expanding $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{T}}_g$ in Eq. (3.1h) to get

$$\frac{\tilde{J}^2}{\rho_v} \mathbf{n} + \tilde{\mathbf{P}} \cdot \mathbf{n} - \tilde{p} \mathbf{n} - \tilde{\mathbf{P}}_g \cdot \mathbf{n} + \tilde{p}_g \mathbf{n} = 2\sigma \kappa \mathbf{n} + \nabla_s \sigma. \quad (3.3a)$$

Taking the dot product with $\mathbf{n}$ for the normal stress balance

$$\frac{\tilde{J}^2}{\rho_v} + (\tilde{\mathbf{P}} \cdot \mathbf{n}) \cdot \mathbf{n} - \tilde{p} - (\tilde{\mathbf{P}}_g \cdot \mathbf{n}) \cdot \mathbf{n} + \tilde{p}_g = 2\sigma \kappa, \quad (3.3b)$$

and taking the dot product with the tangent unit vector $\mathbf{t} = \frac{1}{N}(1, \partial_{\tilde{x}} \tilde{h})$ for the shear stress balance in the $(\tilde{x}, \tilde{y})$ plane

$$(\tilde{\mathbf{P}} \cdot \mathbf{n}) \cdot \mathbf{t} - (\tilde{\mathbf{P}}_g \cdot \mathbf{n}) \cdot \mathbf{t} = \nabla_s \sigma \cdot \mathbf{t}. \quad (3.3c)$$

The quantity $-(\tilde{\mathbf{P}}_g \cdot \mathbf{n}) \cdot \mathbf{n} + \tilde{p}_g$ in Eq. (3.3b) is associated with the gas's normal stress and is neglected, while the quantity $(\tilde{\mathbf{P}}_g \cdot \mathbf{n}) \cdot \mathbf{t}$ in Eq. (3.3c) is replaced with a constant quantity denoted $\tilde{\tau}$, whose magnitude represents the strength of the shear gas flowing against the interface. This is justified by the fact that under the LWE, the fluid layer's disruption of the gas flow is assumed to be minimal due to the fluid interface's slow variation in space, and therefore, when the normal and shear stresses imparted by the gas on the interface are subjected to small perturbations, the leading order normal stress is insignificant, while the leading order shear stress effect is found to be quite substantial [77]. Under these assumptions, the role of the shearing gas is confined to modifying the fluid base flow. This modeling approach has been previously applied in a variety of fluid problems including droplets [79, 80], rivulets [81, 82, 83, 84], and thin liquid films [77, 78]. Positive values of $\tilde{\tau}$ are indicative of a gas which is adding kinetic energy to the liquid interface by flowing along it in the same direction with higher momentum, while negative values of $\tilde{\tau}$ represent either a counter-flowing gas, or a co-flowing gas which is taking away energy from the liquid flow by flowing along it with lower momentum.

3.1.4 Scaling and dimensionless parameters

The following scales are introduced into the system of equations (3.1):

$$\begin{aligned}
 (\tilde{x}, \tilde{y}) &\rightarrow \tilde{h}_0(x^*, y^*), & \tilde{t} &\rightarrow \frac{\tilde{h}_0^2}{\nu} t^*, & \tilde{h} &\rightarrow \tilde{h}_0 h^* \\
 (\tilde{u}, \tilde{v}) &\rightarrow \frac{\nu}{\tilde{h}_0}(u^*, v^*, w^*), & \tilde{p} &\rightarrow p_g + \frac{\rho \nu^2}{\tilde{h}_0^2} p^*, & \tilde{J} &\rightarrow \frac{\lambda \Delta T}{\tilde{h}_0 \mathcal{L}} J^*, \\
 & & \tilde{T} &\rightarrow T_s + T^* \Delta T,
 \end{aligned} \tag{3.4}$$

where $\Delta T = T_w - T_s$, $\tilde{h}_0$ is the mean film thickness at the initial time, and the star notation denotes non-dimensional variables. Introducing these scales into Eqs. (3.1) and dropping the star notation, in component form they read

$$\partial_x u + \partial_y v = 0, \tag{3.5a}$$

$$\partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \partial_{xx} u + \partial_{yy} u + \text{Re} \sin \beta, \tag{3.5b}$$

$$\partial_t v + u \partial_x v + v \partial_y v = -\partial_y p + \partial_{xx} v + \partial_{yy} v - \text{Re} \cos \beta, \tag{3.5c}$$

$$\text{Pr} [\partial_t T + u \partial_x T + v \partial_y T] = \partial_{xx} T + \partial_{yy} T. \tag{3.5d}$$

The scaled wall boundary conditions at ($y = 0$) are

$$u = v = 0, \quad T = 1, \tag{3.5e}$$

while the scaled interface boundary conditions at ($y = h$) become

$$EJ = \frac{1}{N} [-u \partial_x h + v - \partial_t h], \tag{3.5f}$$

$$J + \frac{E^2}{D^2 L} J^3 = \frac{1}{N} [\partial_x h \partial_x T - \partial_y T], \tag{3.5g}$$

$$\begin{aligned}
 -\frac{3}{2} \frac{E^2}{D} J^2 - \frac{2}{N^2} [\partial_x u [(\partial_x h)^2 - 1] - \partial_x h (\partial_y u + \partial_x v)] + p \\
 = -3S(1 - CT) \left[\frac{\partial_{xx} h}{N^3} \right],
 \end{aligned} \tag{3.5h}$$

$$[1 - (\partial_x h)^2] [\partial_y u + \partial_x v] - 4\partial_x h \partial_x u - \tau = -2 \frac{M}{P} [\partial_x T + \partial_x h \partial_y T] N, \tag{3.5i}$$

$$KJ = T. \tag{3.5j}$$

Parameter	Expression
Reynolds number (Re)	$g\tilde{h}_0^3/\nu^2$
Prandtl number (Pr)	ν/α
Evaporation number (E)	$\lambda\Delta T/\rho\mathcal{L}\nu$
Density ratio (D)	$3\tilde{\rho}_v/2\tilde{\rho}$
Nondimensional latent heat (L)	$8\tilde{h}_0^2\mathcal{L}/9\nu^2$
Nondimensional surface tension (S)	$\gamma_0\tilde{h}_0/3\tilde{\rho}\nu^2$
Capillary number (C)	$\gamma\Delta T/\gamma_0$
Marangoni number (M)	$ d\tilde{\gamma}/d\tilde{T} \Delta T\tilde{h}_0/2\mu\alpha$
Dimensionless shear stress (τ^*)	$(N\tilde{h}_0^2/\mu\nu)\tilde{\tau}$
Equilibrium parameter (K)	$\frac{\lambda T_s^{3/2}\sqrt{2\pi R_g}}{\alpha_M\tilde{h}_0\rho_v\mathcal{L}^2\sqrt{M_w}}$

Table 3.1: The set of dimensionless parameters appearing in the nondimensional one-sided systems of equations, Eqs. (3.5).

The scaling (3.4) upon insertion into Eqs. (3.5) produces the set of dimensionless parameters listed in Table. 3.1. Note that this is the same set of dimensionless parameters appearing in Table. 2.1, except for the last two parameters which are specifically relevant to the problem at hand.

3.2 Long wave theory and the interface evolution equation

Since the liquid flow domain of interest is the drag-gravity regime, we again employ the LWE to model the liquid's hydrodynamics. We introduce a small parameter $\epsilon = \frac{h}{\ell}$ where ℓ is the flow's horizontal characteristic length, such that $h \ll \ell$. The parameter ϵ is used to rescale the system's independent variables as

$$x^* \rightarrow \epsilon x, \quad y^* \rightarrow y, \quad t^* \rightarrow \epsilon t. \quad (3.6)$$

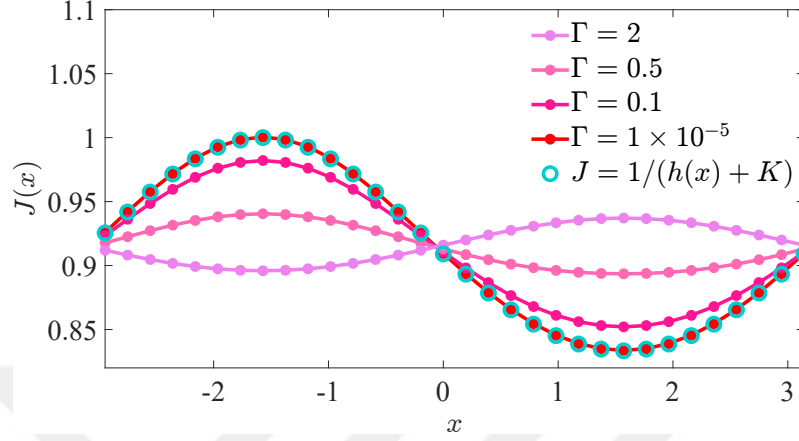


Figure 3.2: Recovery of the one-sided interfacial evaporative mass flux from the general evaporation model for $1/\chi = K = 0.1$ by computing the vapor density gradient using Eqs. (2.5) followed by the utilization of Eq. (2.6). The liquid interface is described by $h(x) = 1 + 0.1(x)$ and $(N_x, N_y) = (32, 32)$.

We also expand the system's dependent variables in terms of ϵ as

$$u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2), \quad (3.7a)$$

$$v = \epsilon(v_0 + \epsilon v_1 + \mathcal{O}(\epsilon^2)), \quad (3.7b)$$

$$p = p_0 + \epsilon p_1 + \mathcal{O}(\epsilon^2), \quad (3.7c)$$

$$T = T_0 + \epsilon T_1 + \mathcal{O}(\epsilon^2), \quad (3.7d)$$

$$J = J_0 + \epsilon J_1 + \mathcal{O}(\epsilon^2). \quad (3.7e)$$

Note here that availability of a complete system of equations under this one-sided formulation allows the inclusion of the evaporative mass flux function in the asymptotic expansion where the resulting solution reads at leading order as

$$J(x, t) = \frac{1}{h(x, t) + K} \quad (3.8)$$

Notably, this exact expression can be recovered from our general kinetic-diffusion evaporation model presented in Sec. 2.1.2 when the limit of no vapor diffusion is considered ($\Gamma \rightarrow 0$), where $1/\chi$ takes the place of K [see Eq. 2.7]. This is demonstrated in Fig. 3.2 which shows the recovery of the one-sided evaporation flux [$J = 1/(h(x) + K)$] as the value of Γ is gradually reduced towards

zero. The vapor mass flux is computed numerically from the vapor density system [Eqs. (2.5)] followed by the utilization of Eq. (2.6).

Furthermore, the following dimensionless parameters are also rescaled with ϵ :

$$E \rightarrow \epsilon \bar{E}, \quad D \rightarrow \epsilon^2 \bar{D}, \quad S \rightarrow \frac{\bar{S}}{\epsilon^2}. \quad (3.9)$$

We consider the shear stress τ and the interface height function $h(x, t)$, to be leading order quantities. Note that the transformations in (3.9) can be considered realistic when compared to the data tabulated in Ref. [54].

Substituting from Eqs. (3.6), (3.7), and (3.9) into the system of equations (3.5), we obtain solutions for the dependent variables (u, v, p, T, J) up to $\mathcal{O}(\epsilon)$ which are then substituted into Eq. (4.1e) to obtain

$$\begin{aligned} \partial_t h + \partial_x \left[\frac{\text{Re} \sin \beta}{3} h^3 + \frac{\tau}{2} h^2 \right] + \frac{\bar{E}}{h + K} \\ + \epsilon \partial_x \left[\frac{2}{15} (\text{Re} \sin \beta)^2 h^6 \partial_x h + \mathcal{M} \frac{h^2}{(h + K)^2} \partial_x h + V_r \frac{h^3}{(h + K)^3} \partial_x h \right. \\ \left. - \frac{1}{3} \text{Re} \cos \beta h^3 \partial_x h + \bar{S} h^3 \partial_{xxx} h \right] + \epsilon \partial_x \left[\frac{5\bar{E}}{24} \text{Re} \sin \beta \frac{h^4}{h + K} \right] \\ + \epsilon \bar{E} \text{Pr} \frac{h^3}{(h + K)^3} \left[\frac{\bar{E}}{3(h + K)} + \frac{1}{120} \text{Re} \sin \beta (7h - 15K) h \partial_x h \right] \\ + \epsilon \tau \left[\frac{1}{12} \bar{E} P \frac{h^4}{(h + K)^3} \partial_x h + \frac{2}{15} \text{Re} \sin \beta \partial_x (h^5 \partial_x h) \right] = 0, \end{aligned} \quad (3.10)$$

which describes the evolution of the liquid interface under the influence of gravity, hydrostatic pressure, thermocapillarity, evaporation, and gas shear as modeled by τ . Note that this is the same equation obtained by Joo *et al.* [56] except for the additional terms proportional to τ . We define $V_r = \bar{E}^2/D$, which is a parameter representing the strength of the vapor recoil forces, and $\mathcal{M} = KM/P$, which represents the strength of the Marangoni forces as modified by evaporation effects. Interestingly, Eq. (3.10) can also be obtained by substituting the particular form of the evaporative mass flux function employed in this chapter [Eq. (3.8)] into the film evolution equation under our kinetic-diffusion formulation [Eq. (2.12)].

Moreover, by making various assumptions within Eq. (3.10), several previously derived models can be recovered. When $\tau = 0$, the exact equation derived by Joo *et al.* [56] is acquired. If then we set $\text{Re} = 0$, a generalization of the equation of Burelbach *et al.* [54] with first order evaporation and convection terms is obtained. If we retain Re and set $\bar{E} = V_r = 0$, the evaporative mass flux naturally goes to zero, K takes the role of the Biot number, and Eq. (3.10) then describes a nonvolatile film evolving under the influence of gravity, thermocapillary forces, and gas shear. If further $\beta = \tau = 0$, the equation derived by Davis [122] is retrieved albeit without accounting for van der Waals forces. Finally, if $\tau = 0$, the film is isothermal ($\bar{E} = V_r = \mathcal{M} = 0$), but $\beta \neq 0$, Eq. (3.10) resembles Benney's [28] original equation for falling liquid films.

3.3 Dispersion relationship derivation

We now follow Joo *et al.* [56] and derive the expressions for a base flow that is spatially uniform but temporally evolving. We set $\partial_x = 0$ in equation (3.10) to get:

$$\partial_t \bar{h} + \frac{\bar{E}}{\bar{h} + K} + \epsilon \bar{E}^2 \text{Pr} \frac{\bar{h}^3}{3(\bar{h} + K)^4} = 0. \quad (3.11)$$

Solving equation (3.11) we obtain:

$$\begin{aligned} \bar{h}(t) = \Omega - K + \epsilon \frac{\bar{E} \text{Pr}}{3\Omega} \left(\frac{K(2K^2 + 6K + 3)}{1 + K} \right. \\ \left. + \frac{3K^2}{2} \ln \frac{\Omega}{1 + K} + \frac{K^3}{\Omega} - 3K\Omega - \bar{E}t \right) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (3.12)$$

where $\Omega = (K^2 + 2K + 1 - 2\bar{E}t)^{1/2}$. Moreover, emulating Burelbach *et al.* [54], we employ the *frozen interface* approximation [54, 116] which assumes that the time scale associated with the variation in base flow height is much larger than that associated with the perturbation's growth. In other words, we examine the stability of a base flow that is frozen in time. The amount of time it takes the liquid film governed by a particular parameter set to completely evaporate is

called the *dry-out time*, denoted t_D . An analytical expression for t_D can be found by setting $\bar{h}(t) = 0$ and solving for t in equation (3.12) to get

$$t_D = \frac{1 + 2K}{2\bar{E}}, \quad (3.13)$$

which is identical to the expressions obtained in Refs. [54] and [56]. We define $t_f = t/t_D$ as the fraction of time until complete film evaporation, which in the subsequent stability analysis determines how much remains of the evaporating liquid film. We now assume that the interface height function $h(t)$ is perturbed by an infinitesimal harmonic perturbation such that:

$$h(t) = \bar{h}(t) + \delta H e^{i(kx - \omega t)}, \quad (3.14)$$

where H is the initial disturbance amplitude and δ is a small linearization parameter ($\delta \ll 1$). Note that due to the applied scaling (Eqs. (3.9)), here $k = k^*/\epsilon$ and $\omega = \omega^*/\epsilon$, where k^* and ω^* are the wavenumber and angular frequency in the non-scaled dimensionless spatial and temporal coordinates (x^*, t^*) . Subtracting Eq. (3.11) from Eq. (3.10) and substituting Eq. (3.14) we obtain:

$$\begin{aligned} \omega(k) = & -i\epsilon \bar{S} \bar{h}^3 k^4 + i\epsilon \left[\frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 + V_r \frac{\bar{h}^3}{(\bar{h} + K)^3} + \mathcal{M} \frac{\bar{h}^2}{(\bar{h} + K)^2} \right. \\ & - \frac{1}{3} \text{Re} \cos \beta \bar{h}^3 + \frac{2}{15} \text{Re} \sin \beta \bar{h}^5 \tau \left. \right] k^2 + \left[\text{Re} \sin \beta \bar{h}^2 + \bar{h} \tau \right. \\ & + \epsilon \left(\frac{5}{6} \text{Re} \sin \beta \bar{E} \frac{\bar{h}^3}{\bar{h} + K} + \frac{7}{120} \bar{E} \text{Re} \sin \beta \text{Pr} \frac{\bar{h}^5}{(\bar{h} + K)^3} \right. \\ & - \frac{5}{24} \text{Re} \sin \beta \bar{E} \frac{\bar{h}^4}{(\bar{h} + K)^2} - \frac{1}{8} \text{Re} \sin \beta \bar{E} \text{Pr} \frac{\bar{h}^4}{(\bar{h} + K)^3} K \\ & \left. \left. + \frac{1}{12} \bar{E} \text{Pr} \frac{\bar{h}^4}{(\bar{h} + K)^3} \tau \right) \right] k + i \frac{\bar{E}}{(\bar{h} + K)^2} - i\epsilon \text{Pr} \bar{E}^2 \frac{\bar{h}^2 (3K - \bar{h})}{3(\bar{h} + K)^5}. \end{aligned} \quad (3.15)$$

Eq. (3.15) is a dispersion relationship of the form $\omega(k)$ governing the linear dynamics of the liquid film. Note that the last two terms in Eq. (3.15) are not proportional to k , but are an algebraic addition to the imaginary part of the angular frequency ω as a function of the base state height $\bar{h}$ [54, 56], which is what expected from the mass loss instability in the transfer-rate-limited

evaporation regime. As expected, this quantity causes an algebraic shift in the perturbation's growth rate not properly related to the instability itself, but instead to the reduction in base state height due to mass loss. Hence, the contribution of these two terms will be neglected when computing the temporal growth rates using Eq. (3.15). For further details, please refer to Sec. 2.4.3. Note that in the ensuing analyses, the following parameter values are fixed: $\epsilon = 0.15$, $\bar{S} = 0.15$, $K = 0.1$, $\text{Pr} = 1$ unless otherwise stated.

3.4 Linear Stability

3.4.1 Temporal stability analysis

For the purposes of temporal stability analysis it is assumed that the wavenumber k is real and the angular frequency ω is complex. Under these assumptions, Eq. (3.15) takes the form

$$\omega(k) = \omega_r + i\omega_i, \quad (3.16)$$

where $\omega_r = kc$ is the the real angular frequency

$$\begin{aligned} \omega_r = k \Big[& \text{Re} \sin \beta \bar{h}^2 + \bar{h} \tau + \epsilon \bar{E} \left(\frac{5}{6} \text{Re} \sin \beta \frac{\bar{h}^3}{\bar{h} + K} \right. \\ & + \frac{7}{120} \text{Re} \sin \beta \text{Pr} \frac{\bar{h}^5}{(\bar{h} + K)^3} - \frac{5}{24} \text{Re} \sin \beta \frac{\bar{h}^4}{(\bar{h} + K)^2} \\ & \left. - \frac{1}{8} \text{Re} \sin \beta \text{Pr} K \frac{\bar{h}^4}{(\bar{h} + K)^3} + \frac{1}{12} \text{Pr} \frac{\bar{h}^4}{(\bar{h} + K)^3} \tau \right) \Big], \end{aligned} \quad (3.17a)$$

and ω_i is the temporal growth rate

$$\begin{aligned} \omega_i = \epsilon \Big[& \frac{2}{15} (\text{Re} \sin \beta)^2 \bar{h}^6 + \frac{2}{15} \text{Re} \sin \beta \bar{h}^5 \tau + \mathcal{M} \frac{\bar{h}^2}{(\bar{h} + K)^2} \\ & + V_r \frac{\bar{h}^3}{(\bar{h} + K)^3} - \frac{1}{3} \text{Re} \cos \beta \bar{h}^3 - \bar{S} \bar{h}^3 k^2 \Big] k^2. \end{aligned} \quad (3.17b)$$

Eqs. (3.17) show that the effect of τ depends on whether the liquid film is moving or stationary. Generally, for a moving film ($\beta \neq 0$) positive values of τ increase the temporal growth rates corresponding to the band of unstable wavenumbers. By contrast, the opposite is true for negative values of τ . Furthermore, τ 's influence on the H -mode is much more pronounced than it on the thermal mode. This is due to τ 's role as a multiplier of the flow's existing inertia as it appears as an algebraic variation to the film's velocity. For a stationary film ($\beta = 0$), τ does not change the temporal growth rate since there is no base flow. As for the perturbation phase speed, τ will always change its value regardless of whether the film is flowing or not, whereby this change is contingent on the sign of τ .

Fig. 3.3(a) shows a plot of the maximum temporal growth rate across the range of unstable wave numbers versus the flow's Reynolds number. The figure shows the existence of sets of Reynolds number values to which the corresponding maximum temporal growth rates are positive. Moving along the Re-axis, the first unstable set of Reynolds numbers corresponds to the thermal instability mode resulting from a combination of the Marangoni S -mode and the evaporative E -mode, whereas the second unstable set of values corresponds to the inertial H -mode instability, albeit modified by Marangoni and evaporation effects. The interconnection between the shear stress's role and the flow's inertia is highlighted by the different colored lines representing different values of τ . At low Reynolds numbers, where the flow's inertia is weak, the effect of the shear stress is significantly weaker than its effect at higher Reynolds number where the flow is dominated by inertia.

The effect of the film thinning can be observed by comparing Fig. 3.3(a) to Fig. 3.3(b), which also shows a plot of the maximum temporal growth rate versus Reynolds number, however at reduced base flow thickness ($t_f = 0.35$). We find that in general the evaporation of the film, and the consequent reduction in its base flow thickness, lead to the strengthening of the thermal instability mode, as indicated by the expansion of the set of unstable values of Re and the increase in their corresponding maximum temporal growth rates. This occurs despite the thinner film being dominated by viscosity since the strength of the Marangoni and vapor recoil effects are significantly increased. These effects are proportional

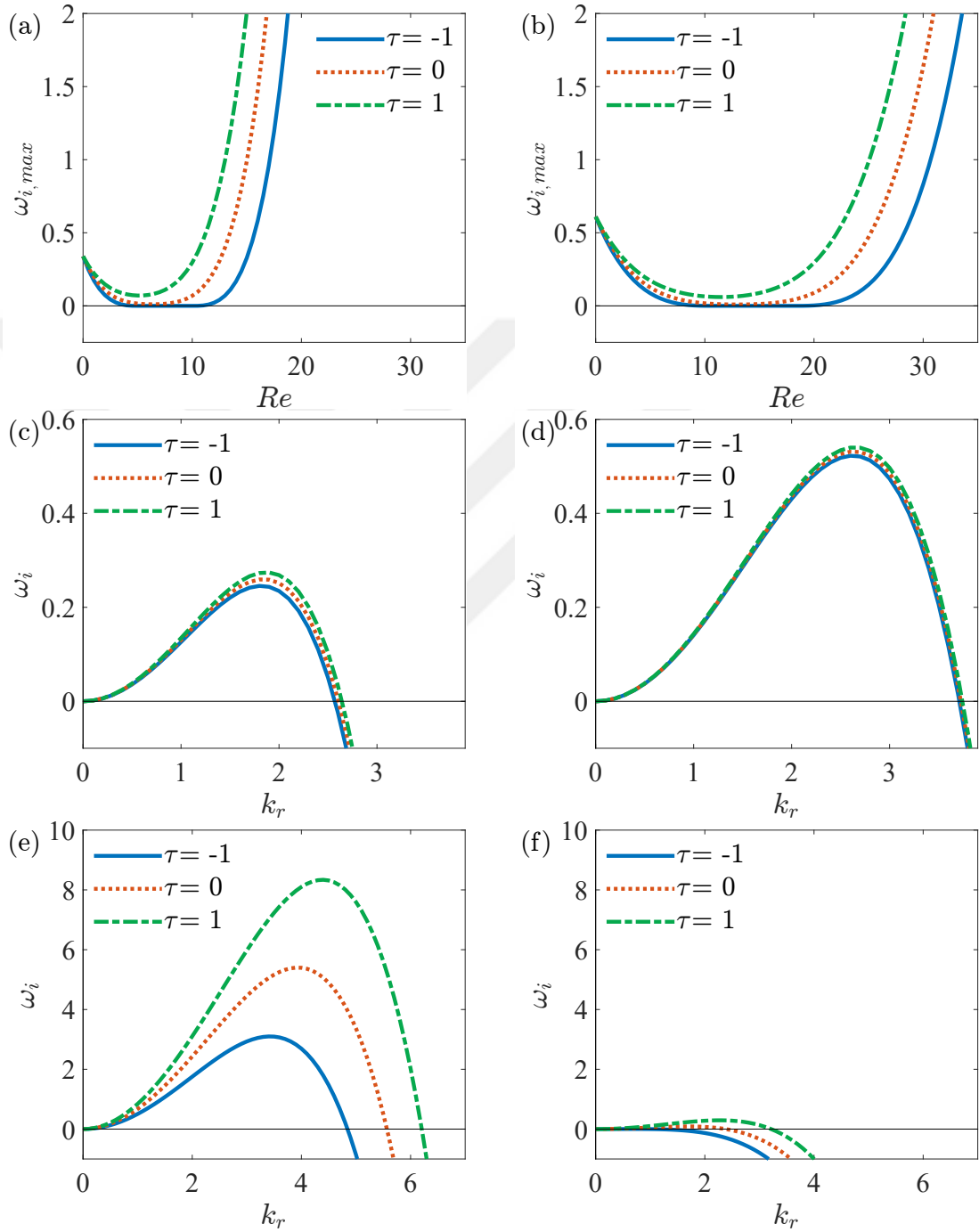


Figure 3.3: Temporal growth rates for $\beta = 25$, $\mathcal{M} = 0.5$, $\bar{E} = 0.1$, $V_r = 1$. (a) $\omega_{i,max}$ versus Re at $t_f = 0$, (b) $\omega_{i,max}$ versus Re at $t_f = 0.35$, (c) ω_i versus k for $Re = 0.5$ and $t_f = 0$, (d) ω_i versus k for $Re = 0.5$ and $t_f = 0.35$, (e) ω_i versus k for $Re = 20$ and $t_f = 0$, and (f) ω_i versus k for $Re = 20$ and $t_f = 0.35$. Note that the parameters in (c) and (d) are in the range of the thermal instability mode, while those of (e) and (f) are in the range of the H -mode instability.

to the temperature gradient across the film which becomes sharper as the film becomes thinner. In contrast, the thinning of the film results in the dampening of the H -mode instability due to the increased prominence of viscous forces. This appears in the plot as the contraction of the second set of unstable Reynolds numbers along with a decrease in their corresponding maximum temporal growth rates.

Fig. 3.3(c) depicts a plot of the temporal growth rate versus wavenumber for different values of τ at a value of Reynolds number belonging to the thermal instability mode and at full film height. It can be seen that the shear stress's effect on the thermal instability's growth rate is rather small since its effect is proportional to the film's Reynolds number. On the other hand, Fig. 3.3(d) shows the temporal growth rates for the same film, but at a smaller thickness ($t_f = 0.35$), where the aforementioned strengthening of the thermal instability mode, and the reduced impact of gas shear, are clearly apparent.

The effects of the shear stress and film thinning on the H -mode instability are presented Figs. 3.3(e) and 3.3(f). Fig. 3.3(e) plots the temporal growth rate versus wavenumber for different values of τ at $Re = 20$. The plots evidently show the significant impact of the shear stress on the H -mode instability, as seen in the large shifts in the temporal growth rate curves corresponding to the different values of τ . Again, these effects are investigated at lower film thickness where Fig. 3.3(f) shows the strong suppressive nature of the negative values of τ on the lva -mode instability. This suppressive behavior can be qualitatively linked to that which appears in the work of Lavalley *et al.* [76] such that a negative value of τ is equivalent to a counter-current shearing gas flowing alongside the liquid interface in a highly confined setting.

The contrast between the shear stress's effect on the thermal instability and that on the H -mode instability is highlighted in Fig. 3.4, where the maximum temporal growth rate and phase speed are depicted versus τ for (a) $Re = 0.5$ which belongs to the range of the thermal instability mode, and (b) $Re = 20$ which is in the range of the H -mode instability. In Fig. 3.4(a) it can be seen that for a range of negative values of τ the perturbation has a maximum temporal growth

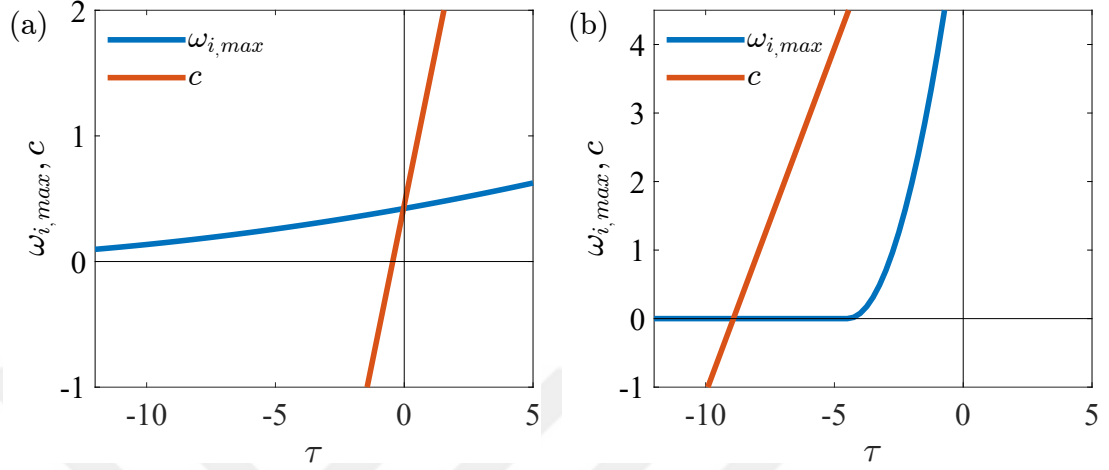


Figure 3.4: Maximum temporal growth rate and perturbation phase speed versus τ for $\beta = 25$, $\mathcal{M} = 1$, $\bar{E} = 0.1$, $V_r = 1$, $t_f = 0$, and (a) $\text{Re} = 1$ (thermal mode instability), (b) $\text{Re} = 20$ (H -mode instability).

rate that is positive along with a negative phase speed, signifying an upstream-traveling perturbation. Contrarily, Fig. 3.4(b) shows that the maximum temporal growth rate is zero for perturbations with negative phase speed, so no upstream convection of H -mode instabilities occurs.

The reduction in film thickness due to evaporation results in it becoming increasingly dominated by viscosity, which not only resists the growth of the perturbations, but also their convection in the film. The thinner film's increased resistance to the convection of the perturbations is demonstrated in Fig. 3.5 which plots perturbation phase speed versus (a) inclination angle, and (b) τ for an increasingly thinner film. In Figs. 3.5(a) and 3.5(b) we see a significant reduction in the slope of the plots indicating a weakening of the convective nature of the flow with larger values of t_f . This demonstrates that the thinning of the film results in a significant reduction in the efficacy of convective drivers such as inclination angle and gas shear.

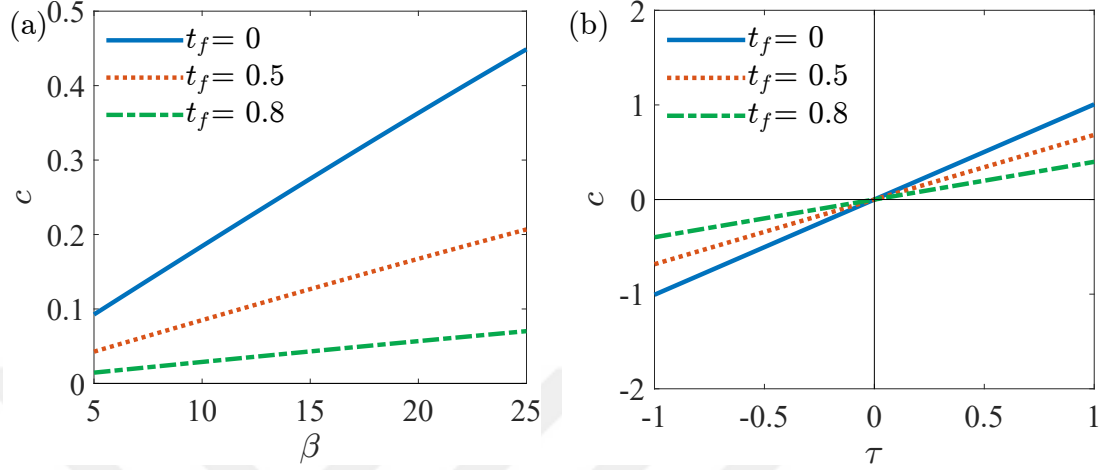


Figure 3.5: Perturbation phase speed versus (a) inclination angle for $\tau = 0$, and (b) τ for $\beta = 0$, $\text{Re} = 1$, $\mathcal{M} = 0.5$, $\bar{E} = 0.1$, $V_r = 1$, and increasing values of t_f .

3.4.2 Spatiotemporal stability analysis

We investigate the spatiotemporal evolution of the film following a procedure based on the seminal work of Huerre & Monkewitz [89], where the spatiotemporal nature of the flow is determined by examining its long-time response to an infinitesimal impulse source. The distinction between the two types of instability is made by inspecting the temporal growth rate (denoted σ) corresponding to the zero group velocity solution ($d\omega/dk \equiv V = 0$), known as the *absolute growth rate* and denoted by $\omega_{i,0}$, where $\omega_{i,0} < 0$ indicates that the instability is convective, and $\omega_{i,0} > 0$ indicates that it is absolute. Examples of impulse response analyses can be found in literature for different flow systems, see e.g. [123, 124]. Eq. (3.15) is used to obtain the full linear impulse response wavepackets for different parameter sets, as shown in Fig. 3.6 where the effect of film evaporation on both the thermal and H -mode instabilities is investigated. Fig. 3.6(a) depicts the impulse response wavepackets for a film at full thickness and $\text{Re} = 1$, i.e. where the film is subject to the thermal instability mode. The plot shows three distinct wavepackets corresponding to three values of τ , where the wavepacket associated with $\tau = 1$ has the highest peak due to the shear stress's contribution to the growth rate. The central wavepacket corresponding to $\tau = 0$ has its edges on opposite sides of the $V = 0$ axis, resulting in $\omega_{i,0} > 0$ and thus represents a perturbation which is absolute in nature. On the other hand, the two outermost

wavepackets corresponding to $\tau = -1$ and $\tau = 1$ have both fronts on the same side of the $V = 0$ axis, resulting in $\omega_{i,0} < 0$ and indicating that they represent perturbations which are convected by the flow upstream and downstream, respectively.

The effect of film thinning on the thermal instability's impulse response is demonstrated by Fig. 3.6(b) where $t_f = 0.8$. The aforementioned amplification of the thermal instability mode is reproduced by our spatiotemporal analysis as all the wavepackets are noticeably more prominent. Furthermore, at this reduced film thickness all three wavepackets include positive values of $\omega_{i,0}$ and, therefore, correspond to absolute instabilities. This indicates that the thinning of the film caused a convective-absolute transition of the thermal instability mode in the cases of $\tau = \pm 1$. This convective-absolute transition due to film thinning seen in Figs. 3.6(a) and 3.6(b) is analogous to that observed under the special transfer-rate-limited case of the kinetic-diffusion evaporation model in Chap. 2 (see Fig. 2.10).

Additionally, noting that the group velocity at the center of the impulse response wavepackets in Fig. 3.6 corresponds to the perturbation phase speed c from the temporal stability analysis [125], we notice again here that the thinning of the film decreases the phase speed of the perturbations, as observed in Fig. 3.5. However, comparing the wavepackets in Figs. 3.6(a) and 3.6(b) we observe that the edges of the wavepackets corresponding to $\tau = 0$ and $\tau = 1$ have significantly larger velocities, which implies that the overall progression of the thermal instability mode is enhanced by the reduction in film thickness. In other words, the thinning of the film hinders the convection of the thermal instability mode by the shearing gas or the flow's inertia, but actually increases the rate at which it spreads via its inherent instability mechanism.

As for the H -mode instability, Fig. 3.6(c) presents the impulse response wavepackets for a film at full thickness and $\text{Re} = 20$, which is sufficiently high to induce it. The strong effect that the shear stress has on the temporal growth rate of the H -mode instability is again clearly demonstrated by the significant change in the curves as the value of τ is varied. The considerable suppressive effect of the

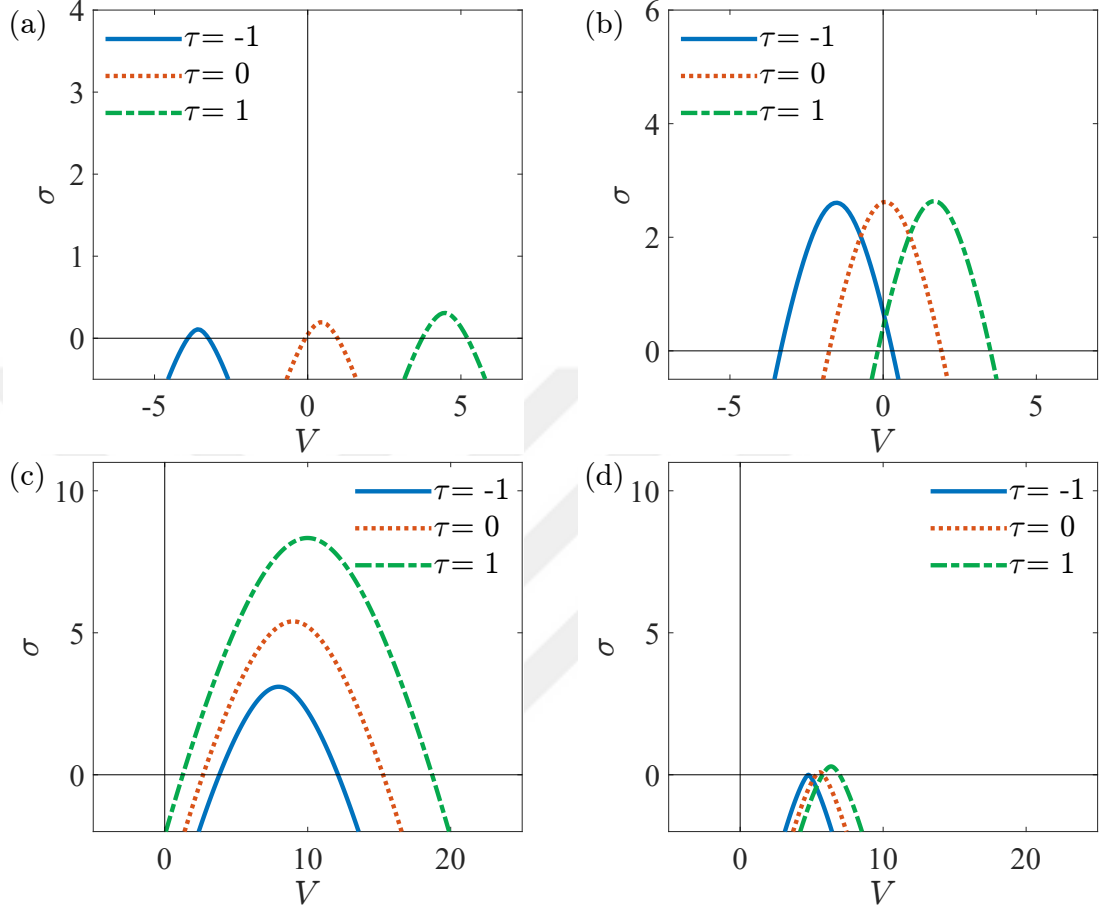


Figure 3.6: Impulse response wavepackets for $\beta = 25$, $\mathcal{M} = 0.5$, $\bar{E} = 0.1$, $V_r = 1$ and (a) $\text{Re} = 1$ and $t_f = 0$, (b) $\text{Re} = 1$ and $t_f = 0.8$, (c) $\text{Re} = 20$ and $t_f = 0$, and (d) $\text{Re} = 20$ and $t_f = 0.35$. Note that the parameters in (a) and (b) are in the range of the thermal instability mode, while those in (c) and (d) are in the range of the H -mode instability.

film thinning on the H -mode instability becomes apparent by comparing Figs. 3.6(c) and 3.6(d), that differ only by the value of t_f : (c) $t_f = 0$ and (d) $t_f = 0.35$.

The interaction between the shear stress and the effects of film evaporation can also be visualized by plotting the absolute growth rate in the (Re, V_r) space as shown in Fig. 3.7 where the blue, red, and green regions correspond to convectively unstable flow, absolutely unstable flow, and stable flow, respectively. Figs. 3.7(a), 3.7(b), and 3.7(c) plot the absolute growth rate in a film being sheared by a counter-flowing gas ($\tau = -1$) for different values of t_f . Note the existence of two

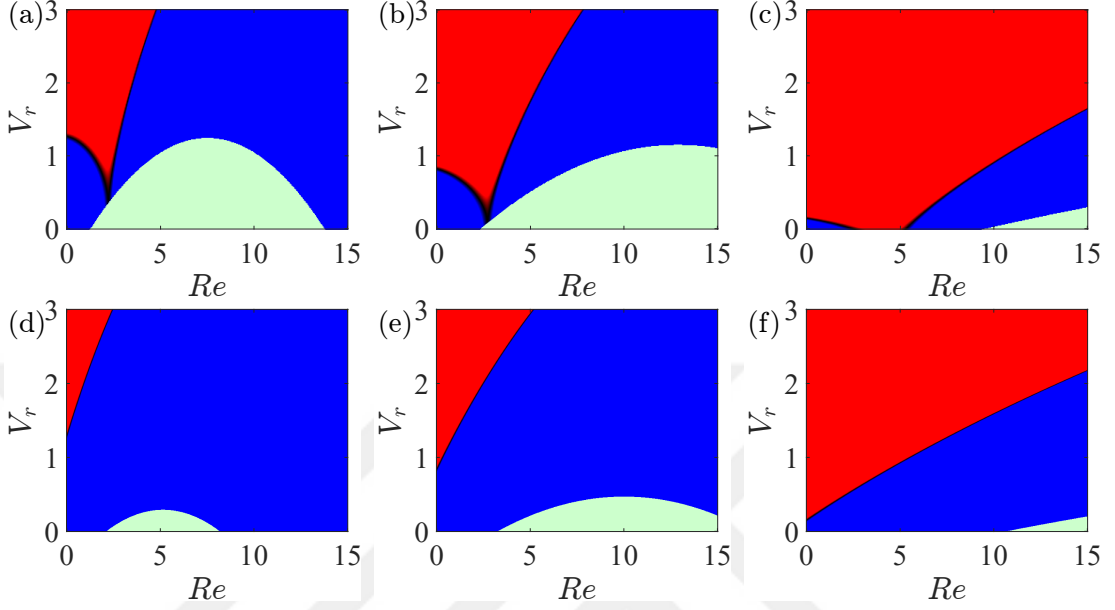


Figure 3.7: Absolute growth rate in the (Re, V_r) space for $\beta = 25$, $\mathcal{M} = 0.5$, $\bar{E} = 0.1$. (a,b,c) $\tau = -1$ and $t_f = 0$, $t_f = 0.3$, and $t_f = 0.7$, respectively. (d,e,f) $\tau = 1$ and $t_f = 0$, $t_f = 0.3$, and $t_f = 0.7$, respectively. The red, blue, and green regions indicate absolutely unstable, convectively unstable, and stable flow, respectively.

regions corresponding to convectively unstable flow: (i) the region on the left contains (Re, V_r) values belonging to the thermal instability mode, and (ii) the region on the right contains (Re, V_r) values belonging to the H -mode instability. In the former (latter) region the perturbation is convected upstream (downstream) by the shear stress (flow). In between there is a parameter set resulting in absolutely unstable flow where the downstream convection by the flow's inertia is balanced by the negative τ value representing a counter-flowing shearing gas. As the value of t_f is increased, the parameter set corresponding to the absolutely unstable flow expands at the expense of those associated with convective instability, while that resulting in stable flow enlarges raising the critical Re associated with the onset of the H -mode instability outside the parameter set we considered. In Fig. 3.7(c) where $t_f = 0.7$, the amplified convective thermal instability mode persists up to Reynolds numbers of $\mathcal{O}(10)$.

On the other hand, Figs. 3.7(d), (e), and (f) plot the absolute growth rate in a film being sheared by a co-flowing gas ($\tau = 1$) for different values of t_f .

Due to the positive value of τ , the parameter space is dominated by convectively unstable flow as the parameter set resulting in absolutely unstable flow is limited to the upper left corner of the parameter space. As the film thins the parameter sets corresponding to stable and absolutely unstable flow grow into regions of the parameter space previously occupied by convectively unstable flows.

3.5 Nonlinear evolution

The linear stability analysis in Sec. 3.4 is only valid when the perturbation amplitude is small. To relax this constraint we solve Eq. (3.10) numerically as an initial value problem using MATLAB [126]. A perturbation in the form of a Gaussian wavepacket localized about $x = 0$ is introduced, such that the initial interface is described by $h(x, 0) = \bar{h}_0 + A \exp(-10x^2)$ where $\bar{h}_0$ is the initial flat film height and A is the Gaussian wavepacket's maximum amplitude. For the numerical simulations we keep $\epsilon = 0.15$, $\bar{S} = 0.15$, $K = 0.1$, $\text{Pr} = 1$.

3.5.1 Solver validation

The solver is validated using the same methodology described in Sec. 2.6. We compare the temporal growth rates obtained analytically from Eq. (3.17b) to those computed from numerical data resulting from solving Eq. 3.10 for an interface disturbed initially by a small-amplitude perturbation. For a particular mode within the Gaussian wavepacket with an amplitude $\hat{h}(k, t)$, the numerical and analytical temporal growth rates are related by

$$\frac{\dot{\hat{h}}}{\hat{h}} = \omega_i. \quad (3.18)$$

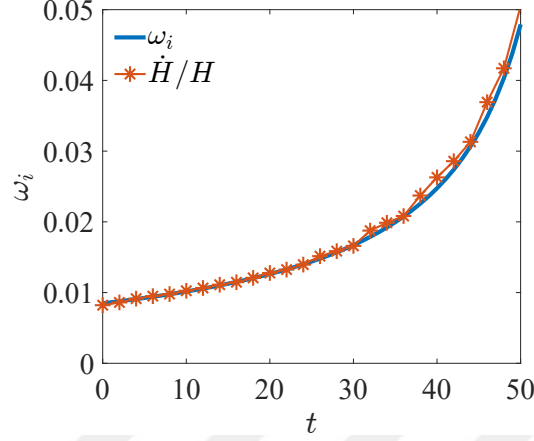


Figure 3.8: Comparison of the analytical and numerical temporal growth rates for $\beta = 5$, $\text{Re} = 1$, $\mathcal{M} = 0.2$, $\bar{E} = 0.01$, $V_r = 0.2$, and $t_f = 0$. The initial interface in the numerical simulation is described by $h(x, 0) = \bar{h}_0 + A \exp(-10x^2)$, where $A = 0.005$.

To compute the left-hand side of Eq. (3.18) we find $\hat{h}(k, t)$ via the discrete Fourier transform of $h(x, t)$ as

$$\hat{h}(k, t) = 2|\hat{f}(k, t)|, \quad \text{where} \quad \hat{f}(k, t) = \sum_{j=1}^{n-1} h(j, t) e^{-2\pi i(j-1)(k-1)/(n-1)}, \quad (3.19)$$

and n is the total number of discrete points in $h(x, t)$. We subsequently take its derivative numerically to obtain $\hat{\dot{h}}$. The right-hand side of Eq. (3.18) is computed from Eq. (3.17b).

An example of this validation is shown in Fig. 3.8, where the error between the numerical and analytical values was found to be $\approx 1\%$ during linear perturbation growth. Further validation in the nonlinear regime was conducted by comparing the our simulation results to those in Ref. [56], where excellent agreement was obtained.

3.5.2 Interface evolution

Fig. 3.9 shows the results of simulating an evaporating film flowing down an inclined plane while being assisted by a co-flowing shearing gas, represented by a

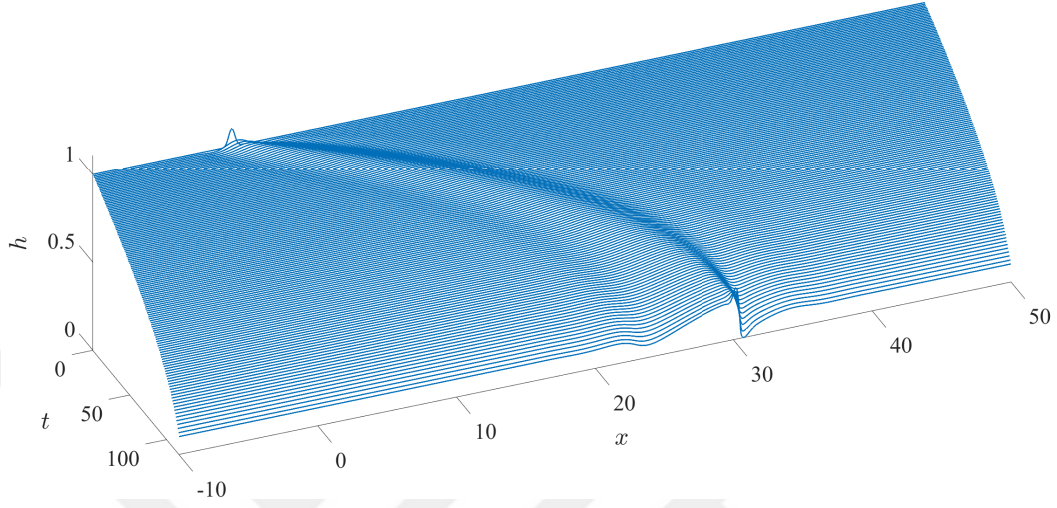


Figure 3.9: Evolution of the fluid interface for $A = 0.1$, $\beta = 5$, $\text{Re} = 1$, $\mathcal{M} = 0.1$, $\bar{E} = 0.005$, $V_r = 0.2$, $\tau = 0.3$ and $t_f = 0$.

positive value of τ , and with $A = 0.1$. The parameter set was chosen specifically such that the evaporation of the film and the growth of the perturbation are sufficiently slow to allow observing several stability regimes. We observe that the initial perturbation amplitude decays as it travels downstream, however, as the film's thickness reduces the thermal instability mode arises and the perturbation begins to grow. At this stage the instability is convective in nature and is carried downstream by the flow. Further thinning of the film causes the perturbation to undergo a convective-absolute transition, and the instability can be seen to start spreading in the upstream direction as well. Notably, the reduction in the perturbation's phase speed is clearly demonstrated by the reduction in the rate of progression of the wavepacket's center when observed in the (x, t) plane.

3.5.2.1 Temporal evolution

The dynamics exhibited in Fig. 3.9 are investigated, first, by a temporal stability analysis. The temporal evolution of the perturbation is examined in the linear regime by computing the maximum temporal growth rate using Eq. (3.17b), and in the nonlinear regime by tracking the maximum perturbation height, denoted h'_{max} , which represents the maximum deviation of the film's interface from the flat

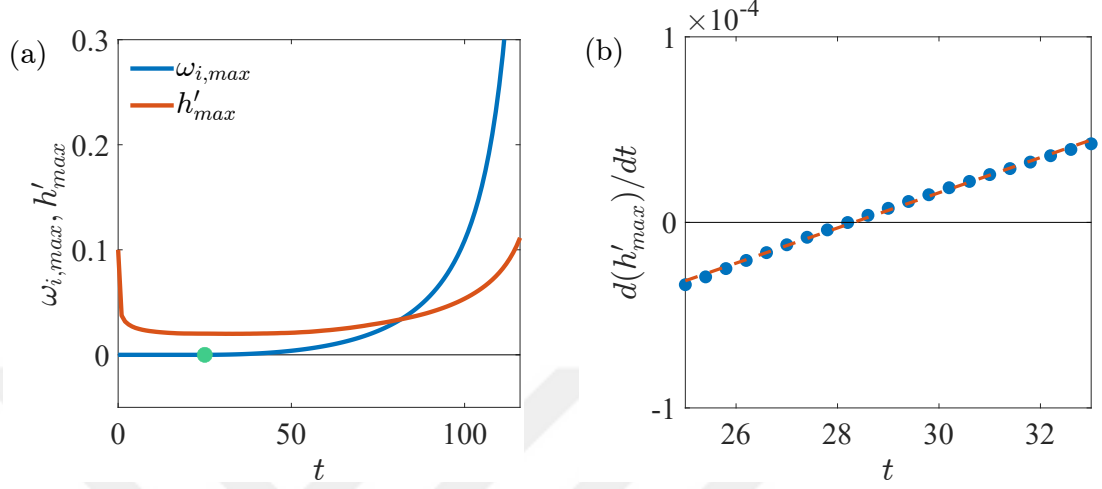


Figure 3.10: Temporal stability analysis for $\beta = 5$, $\text{Re} = 1$, $\mathcal{M} = 0.1$, $\bar{E} = 0.005$, $V_r = 0.2$, $\tau = 0.3$ and $t_f = 0$. (a) $\omega_{i,max}$ and h'_{max} versus time, and (b) dh'_{max}/dt versus time.

base flow, such that $h'_{max}(t) = |h(x, t) - \bar{h}(t)|$. Both these quantities are plotted versus time in Fig. 3.10(a), where $\omega_{i,max}$ is found to be zero up to $t = 24.5$ (marked by the green circle), indicating that the fluid film is indeed initially linearly stable. The thinning of the film beyond this point results in the flow becoming temporally unstable due to the onset of the thermal instability mode.

Furthermore, the same qualitative behavior predicted by linear stability analysis is observed in the nonlinear regime, as h'_{max} initially decreases then starts to increase. We attempt to determine this inflection point, i.e the point in time at which h'_{max} begins to grow, by numerically computing $d(h'_{max})/dt$ and fitting the data around the transition point linearly. We consider the intercept of the linear fitting function with the t axis to be the transition point from stable to unstable flow, which occurs at $t = 28.3$, 15.5% higher than the value predicted by linear stability theory. This difference is attributed to nonlinear effects.

3.5.2.2 Spatiotemporal evolution

The wavepacket's spatiotemporal evolution within the flow domain can be observed by tracking the perturbation's edges, which are defined by the two points

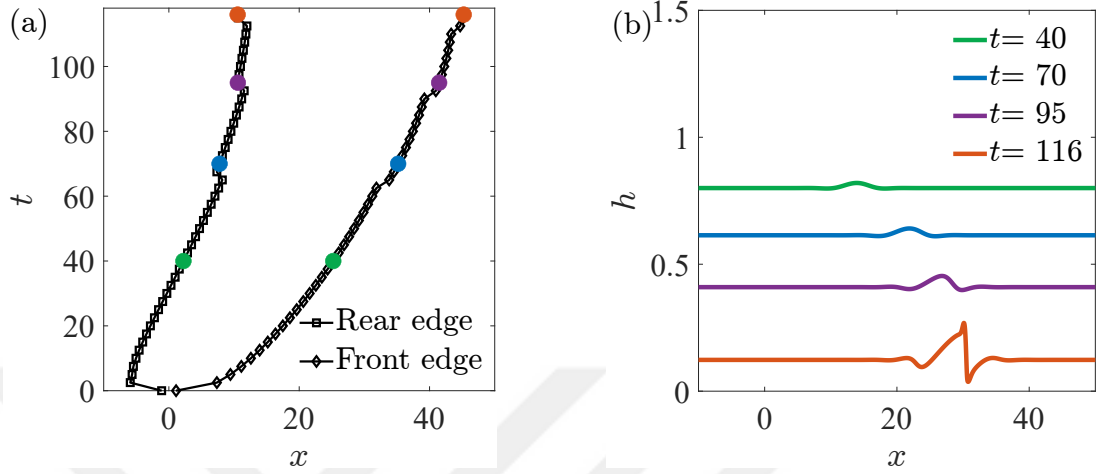


Figure 3.11: Visualization of wavepacket spreading in the flow domain for $\beta = 5$, $\text{Re} = 1$, $\mathcal{M} = 0.1$, $\bar{E} = 0.005$, $V_r = 0.2$, $\tau = 0.3$ and $t_f = 0$. (a) Front and rear wavepacket edge positions in time. (b) The evolution of the interface profile through the convective-absolute transition.

on the interface where $|\partial_x h|$ increases beyond a small tolerance, indicating the existence of a perturbed interface deviating from the flat base flow. The loci of these points are plotted against time in Fig. 3.11(a), which shows that the wavepacket's edges both initially travel downstream forming a convective instability. Subsequently, the rear edge begins to slow down relative to the front edge, eventually reversing direction and traveling upstream, indicating the transition to absolutely unstable flow. The colored circles mark the wavepacket's edges at three points in time corresponding to the interface profiles in Fig. 3.11(b). Comparing the three interface profiles, the transition from convective to absolute instability is clearly apparent.

The spatiotemporal dynamics in the nonlinear regime are further examined by generating the complete impulse response wavepackets by numerically computing the temporal growth rate along each ray ($V = x/t$). We consider the perturbation amplitude at any point along the streamwise coordinate as the maximum perturbation height $h'_{max}(x, t)$, a quantity which is shown to behave asymptotically [127] as

$$h'_{max} \propto t^{-1/2} e^{\sigma(V)t}, \quad V = x/t = \text{constant}, \quad \text{for } t \rightarrow \infty, \quad (3.20)$$

where again σ is the temporal growth rate along a given ray $V = x/t = \text{constant}$, and which can be found analytically as

$$\sigma(V) \sim \frac{d}{dt} \left(\ln \left[t^{1/2} h'_{max}(Vt, t) \right] \right), \quad \text{for } t \rightarrow \infty. \quad (3.21)$$

This expression can be discretized in a form suitable for numerical computation as

$$\sigma(V) \approx \frac{\ln \left[h'_{max}(Vt_2, t_2) / h'_{max}(Vt_1, t_1) \right]}{t_2 - t_1} + \sigma_0(t_2, t_1), \quad (3.22)$$

where $\sigma_0(t_1, t_2) = \ln(t_2/t_1) / [2(t_2 - t_1)]$ is a finite-time correction term. For further details, see Ref. [128].

Note that the analytical absolute growth rate is obtained under the notion that the base flow height does not vary the growth of the perturbation, i.e under the *frozen interface* assumption [54, 116]. In order to allow comparison with linear theory, we enforce this assumption when numerically computing the nonlinear absolute growth rate by simulating the interface evolution at a constant base flow height ($\bar{E} = 0$). Furthermore, the temporal growth rate was monitored over time to ensure that the perturbations were growing nonlinearly at the time when the nonlinear impulse response was acquired. Additionally, the resulting impulse responses were obtained at a value of t_2 large enough to ensure the saturation of the computed values of σ , see Eq. (3.21). The value of $\omega_{i,0}$ was approximated by linearly interpolating between the two peaks closest to the $V = 0$ axis, which is sufficiently accurate for our qualitative analysis.

Fig. 3.12(a) plots the linear and nonlinear absolute growth rates versus film thickness for the range of values of $\bar{h}$ where the flow is temporally unstable. We observe that initially when the film is relatively thick the nonlinear absolute growth rate is higher than its linear counterpart, indicating that finite instabilities are less convective than infinitesimal ones for this set of parameters. However, as the film height reduces, the linear and nonlinear absolute growth rates converge. Note that the two growth rates appear to slightly diverge again towards the end of the range of $\bar{h}$ values considered. However this is due to the film rupturing at

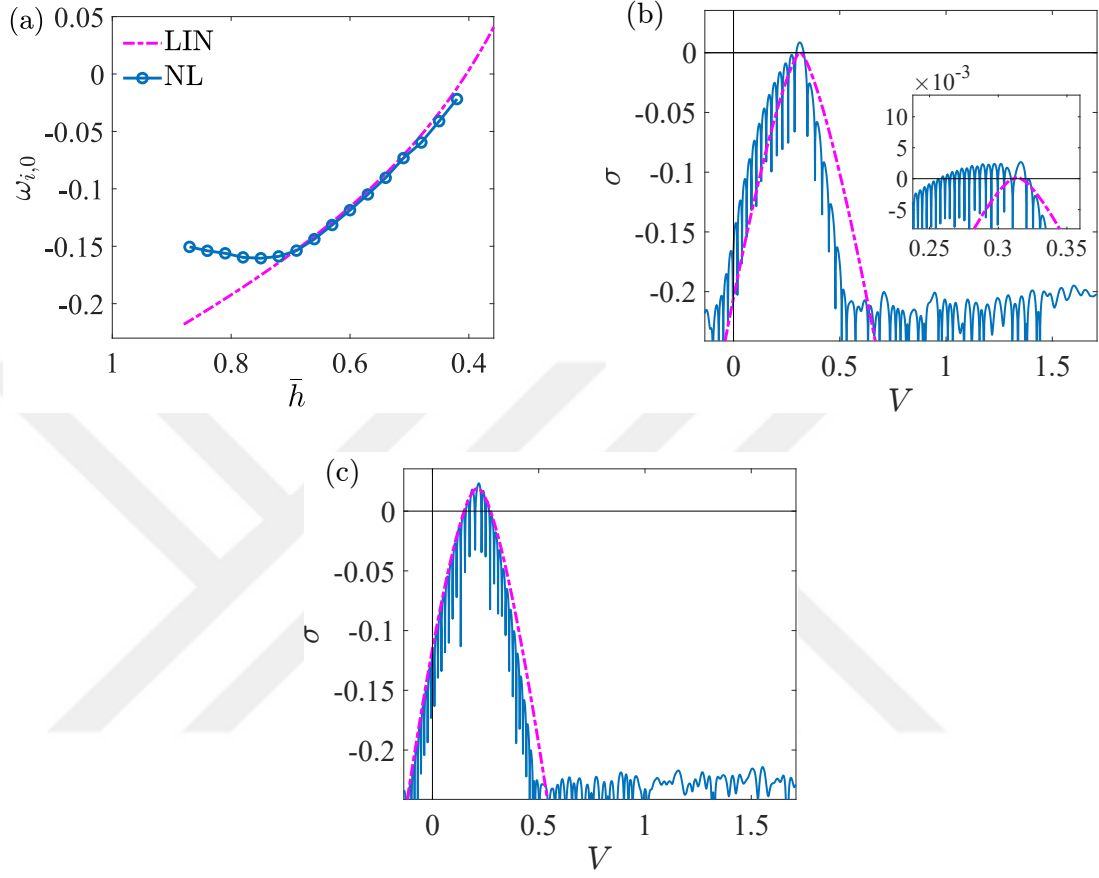


Figure 3.12: Analytical (linear) and numerical (nonlinear) impulse responses for $\beta = 5$, $\text{Re} = 1$, $V_r = 0.2$, $\mathcal{M} = 0.1$, $\bar{E} = 0.005$, $\tau = 0.3$ and $t_f = 0$. (a) Absolute growth rate versus $\bar{h}$. (b) Linear and nonlinear impulse response wavepackets for $\bar{h} = 0.84$. (c) Linear and nonlinear impulse response wavepackets for $\bar{h} = 0.6$. Note that *LIN* (magenta dashed lines) and *NL* (blue continuous lines) indicate linear and nonlinear data, respectively.

a time which is too early to allow accurate estimation of the growth rates.

The impulse response wavepacket corresponding to $\bar{h} = 0.84$ is depicted in Fig. 3.12(b), where the large disparity between the linear and nonlinear responses behind the difference in absolute growth rate is seen. The inset of Fig.3.12(b) details the wavepackets' peaks and demonstrates the significant difference in the upstream and downstream front velocities of the linear and nonlinear responses. According to the Dee & Langer criterion [129], this implies that in the case of finite perturbations, both front velocities are nonlinearly selected as the nonlinear wavepacket's saturation produced wave fronts with velocities higher than those

which would result from an infinitesimal perturbation. On the other hand, the impulse response wavepacket corresponding to $\bar{h} = 0.6$ is shown in Fig. 3.12(c), where we observe that the saturated nonlinear wavepacket's fronts match their linear counterparts indicating that the front velocities are linearly selected, which suggests that for this thinner film the front dynamics are dictated by infinitesimal modes governed by the linear dispersion relationship Eq. (3.15). The convergence of the nonlinear front velocities towards their linear equivalent as the film thins suggests that this effect is related to the strength of the instability, since the thinning of the film results in a significantly stronger thermal instability mode.

This is investigated by generating both the linear and nonlinear impulse responses for a constant film thickness of $\bar{h} = 1$ and increasingly stronger instabilities (higher values of $\mathcal{M}$ and V_r), see Fig. 3.13. Both the linear and nonlinear absolute growth rates are plotted versus increasing values of $\mathcal{M}$ in Fig. 3.13(a). The same trend observed in Fig. 3.12 appears here as the nonlinear value of $\omega_{i,0}$ is initially larger than its linear counterpart. However, again this deviation diminishes with larger values of $\mathcal{M}$ until the eventual convergence of the linear and nonlinear absolute growth rates. This is confirmed by plotting the linear and nonlinear impulse response wavepackets for $\mathcal{M} = 0.32$, as shown in Fig. 3.13(b), where the disparity between the linear and nonlinear responses is apparent, and those corresponding to $\mathcal{M} = 0.64$, as shown in Fig. 3.13(c), where their concurrence is demonstrated.

Moreover, Fig. 3.13(d) shows the linear and nonlinear absolute growth rates for a film with a constant thickness $\bar{h} = 1$, and increasing values of V_r . Once more we observe initially higher values of the nonlinear absolute growth rate and the subsequent convergence to the linear value as V_r is increased. This is confirmed by plotting the linear and nonlinear impulse response wavepackets in Figs. 3.13(e) and 3.13(f), corresponding to $V_r = 0.42$ and $V_r = 0.86$, respectively.

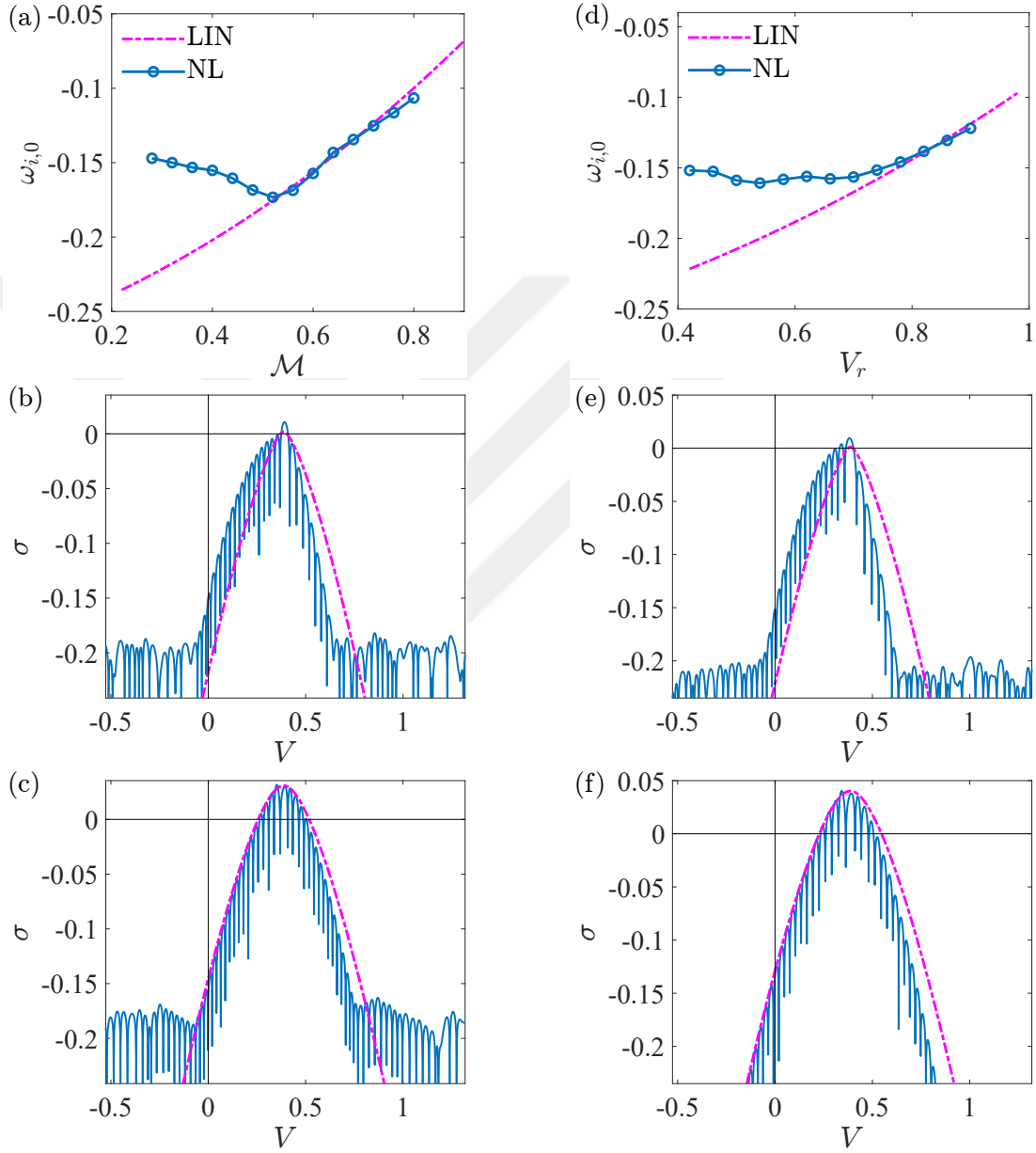


Figure 3.13: Analytical (linear) and numerical (nonlinear) wavepacket impulse responses for $\beta = 5$, $\text{Re} = 1$, $\bar{E} = 0.005$, $\tau = 0.3$ and $t_f = 0$. (a) Absolute growth rate versus $\mathcal{M}$ for $V_r = 0.2$. Linear and nonlinear impulse response wavepackets for (b) $\mathcal{M} = 0.32$ and (c) $\mathcal{M} = 0.64$. (d) Absolute growth rate versus V_r for $\mathcal{M} = 0.1$. Linear and nonlinear impulse response wavepackets for (e) $V_r = 0.42$ and (f) $V_r = 0.86$. Note that *LIN* (magenta dashed lines) and *NL* (blue continuous lines) indicate linear and nonlinear data, respectively.

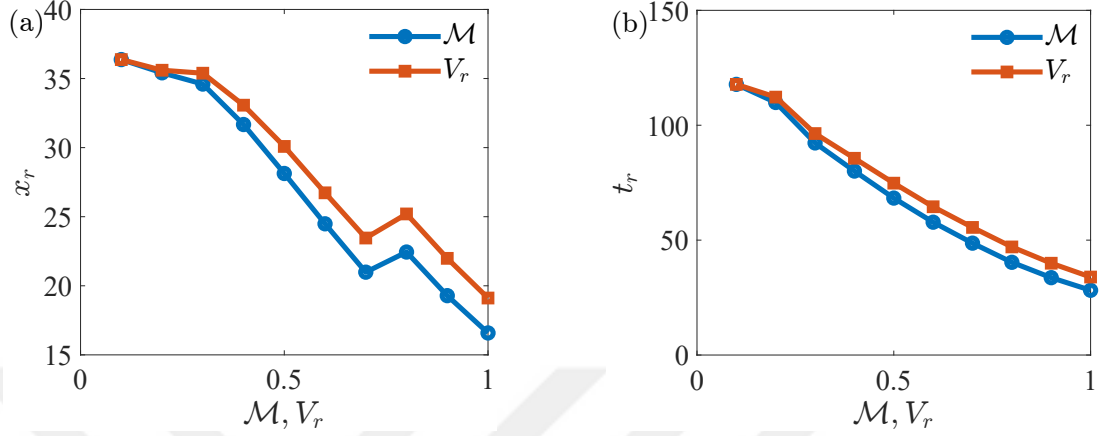


Figure 3.14: Rupture (a) location, and (b) time versus $\mathcal{M}$ and V_r for $\beta = 5$, $\text{Re} = 1$, $\bar{E} = 0.005$, $\tau = 0.5$ and $t_f = 0$.

3.5.3 Rupture analysis

Thermal instabilities can lead to the rupture of the fluid film, which can have significant negative effects on applications sensitive to the film's integrity. Therefore we examine the effects of the shear stress on rupture location (denoted x_r) and rupture time (denoted t_r). We expect a shearing gas to play a role in mitigating these issues by either preventing rupture altogether, or altering the evolution of the interface such that film rupture is delayed until the disturbance exits the region in which it would be detrimental to the system. Note that in the presence of more than one simultaneous effective rupture location, the one closest to the origin of the perturbation ($x = 0$) is considered to be the rupture point. In Fig. 3.14(a) rupture location x_r is plotted against increasing values of both $\mathcal{M}$ and V_r . We find that increasing either parameter causes the rupture location to shift closer towards the origin. This occurs since an increase in $\mathcal{M}$ or V_r enhances the perturbation's temporal growth rate, allowing less time for it to travel before film rupture. However, this trend does not continue across the range of values of $\mathcal{M}$ and V_r we considered, as at a certain value of either parameter the rupture location is abruptly shifted downstream before again resuming its upstream trend. On the other hand, rupture time does not display this harmonic behavior as t_r decreases monotonically across the considered range of $\mathcal{M}$ and V_r values, see Fig. 3.14(b).

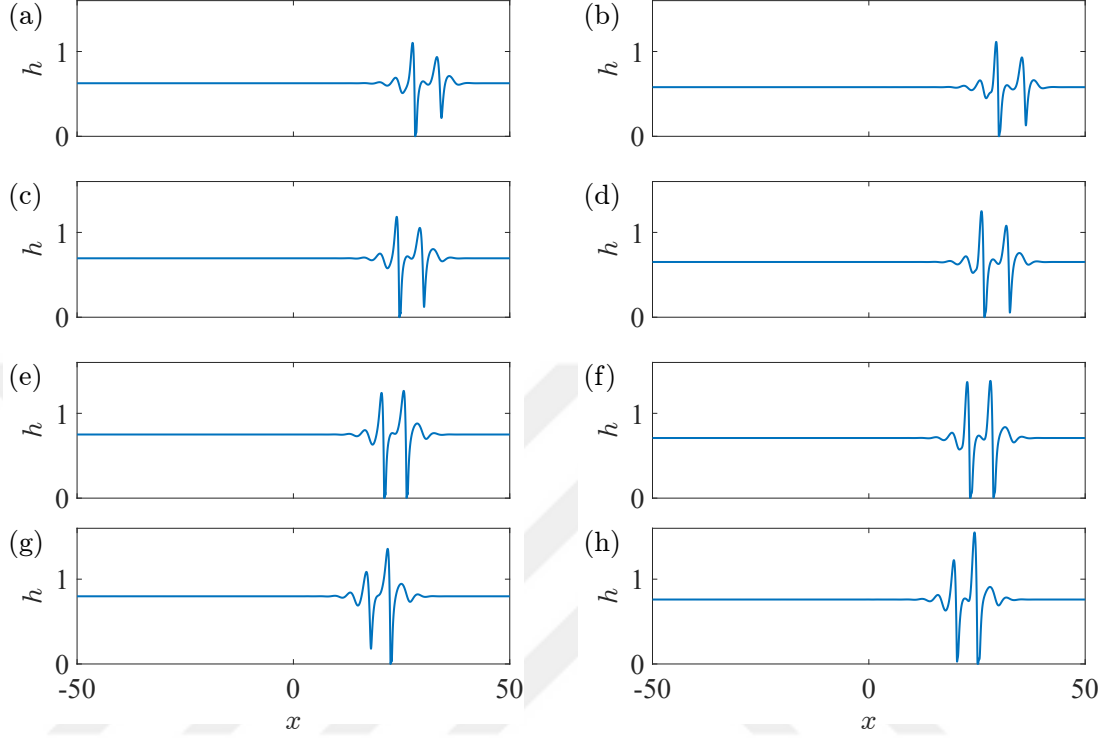


Figure 3.15: Interface profiles at film rupture for $\beta = 5$, $\text{Re} = 1$, $\bar{E} = 0.005$, $\tau = 0.5$, $t_f = 0$, (a,c,e,g) $V_r = 0.1$ and $\mathcal{M} = 0.6$, $\mathcal{M} = 0.7$, $\mathcal{M} = 0.8$, and $\mathcal{M} = 0.9$, respectively and (b,d,f,h) $\mathcal{M} = 0.1$ and $V_r = 0.6$, $V_r = 0.7$, $V_r = 0.8$, and $V_r = 0.9$, respectively.

We investigate this further by plotting the interface profiles at rupture for different values of $\mathcal{M}$ and V_r , as shown in Fig. 3.15. In particular in Figs. 3.15(a), (c), (e) and (g) we keep $V_r = 0.1$ but vary the value of $\mathcal{M}$, while in Figs. 3.15(b), (d), (f) and (h) the value of $\mathcal{M} = 0.1$ is retained while the value of V_r is increased. The rupture profile corresponding to $\mathcal{M} = 0.6$ is presented in Fig. 3.15(a) and shows that film rupture occurs towards rear edge of the perturbation, while the front edge remains far from rupture. When the value of $\mathcal{M}$ is increased to 0.7, the same dynamic is preserved, however the front edge of the perturbation is significantly closer to rupture as it occurs, as seen in Fig. 3.15(c). Further increases in the value of $\mathcal{M}$ show that the rupture location is changed from the rear of the perturbation towards to the front as the value of $\mathcal{M}$ crosses a critical threshold, see Figs. 3.15(e) and 3.15(g). Additional increases in the value of $\mathcal{M}$ restore the original trend of moving closer towards the origin of the disturbance. Interestingly, the same dynamic is reproduced when the value

of $\mathcal{M}$ is kept constant, while the value of V_r is increased. Here again we observe a shift in the evolution dynamics as the strength of the thermal instability mode is increased, akin to that observed in Sec. 3.5.2.2.

3.5.4 Self-similarity analysis

Self-similar analysis of fourth-order thin-film type equations has been performed extensively in the past; see Refs. [97, 99, 100, 130, 131], and is a useful framework within which to understand and predict the rupture of thin films. The solutions of Eq. (3.10) rupture in finite or infinite time, or exhibit other long-time behavior, depending on the presence or absence of thermal disequilibrium (K), evaporative mass loss ($\bar{E}$), and shear (τ). In the absence of mass loss ($\bar{E} = 0$), a possible fate of the solutions is to evolve to long-time traveling waves, or to exhibit permanent chaotic behavior. Indeed, a weakly nonlinear analysis of Eq. (3.10) with $\bar{E} = 0$ will result in the Kuramoto-Sivashinsky equation [132] which is a paradigm equation that exhibits chaos. As typical with Benney-type equations which include an inertial term proportional to $\partial_x(h^6\partial_x h)$, the solution may also blow up in finite time [31, 108]. This blow-up is due to a breakdown in the assumptions leading to the Benney equation and is not physical. In simulations the Reynolds number is kept sufficiently small to avoid this blow-up. In this section we investigate the fates of the solutions of Eq. (3.10) depending on the values of the dimensionless parameters.

3.5.4.1 Finite-time rupture: $K = 0$

When $K = 0$ we observe that the generic behavior of solutions to Eq. (3.10) is to rupture ($h \rightarrow 0$) at a finite point x_r and finite time t_r . This rupture will correspond to a singularity of the solution, as the time and space derivatives will become infinite at the rupture point. The behavior close to rupture can be found

by performing the similarity transformation:

$$h(x, t) = (t_r - t)^\alpha f(\xi, s), \quad \xi = \frac{x - x_r}{(t_r - t)^\beta}, \quad \text{and} \quad s = -\ln(t_r - t). \quad (3.23)$$

Under this transformation we obtain $\partial_t h = (t_r - t)^{\alpha-1}(\partial_s f - \alpha f + \beta \xi \partial_\xi f)$, and each term in Eq. (3.10) is scaled by a power of $(t_r - t)$. The exponents α and β are determined by balancing the time derivative term with surface tension ($\propto (t_r - t)^{4\alpha-4\beta}$) and Marangoni/vapor recoil terms ($\propto (t_r - t)^{\alpha-2\beta}$). Thus, we can write

$$\alpha - 1 = 4\alpha - 4\beta = \alpha - 2\beta \quad \Rightarrow \quad \alpha = \frac{1}{3}, \quad \beta = \frac{1}{2}.$$

Under this scaling, it can be easily checked that every other term in Eq. (3.10) is multiplied by a higher power of $(t_r - t)$, and is thus negligible in the limit that rupture is reached, $t \rightarrow t_r$. In particular, this rupture will occur independent of the presence of evaporative mass loss ($\bar{E} > 0$) and shear ($\tau > 0$).

Close to rupture the scaled profile satisfies the following,

$$\partial_s f - \frac{1}{3}f + \frac{1}{2}\xi \partial_\xi f + \epsilon \partial_\xi [(\mathcal{M} + V_r) \partial_\xi f + \bar{S} f^3 \partial_{\xi\xi} f] = 0, \quad (3.24)$$

where the neglected terms go to zero as $s \rightarrow \infty$ ($t \rightarrow t_r$). For instance, the evaporative mass loss term will be proportional to $(t_r - t)^{1-2\alpha} = (t_r - t)^{1/3} \rightarrow 0$, while the shear stress term will be proportional to $(t_r - t)^{1+\alpha-\beta} = (t_r - t)^{7/6} \rightarrow 0$. The parameters $\epsilon, \mathcal{M}, V_r$ and $\bar{S}$ may be scaled out by taking

$$\xi = \epsilon^{1/2}(\mathcal{M} + V_r)^{1/2} X, \quad f = \epsilon^{1/3}(\mathcal{M} + V_r)^{2/3} \bar{S}^{-1/3} F, \quad (3.25)$$

so that we obtain

$$\partial_s F - \frac{1}{2}F + \frac{1}{3}X \partial_X F + \partial_X [(\partial_X F + F^3 \partial_{XX} F)] = 0. \quad (3.26)$$

The thin film equation corresponding to Eq. (3.26) is a special case of the family of thin-film type equations with similarity solutions considered in Dallaston *et al.* [99]. In their work it was shown that Eq. (3.26) has an infinite number of

steady states ($\partial_s F = 0$), corresponding to similarity solutions, the first of which is linearly stable. This situation is qualitatively similar to the rupture due to van der Waals attraction [130, 131], and the connection between that problem and Eq. (3.26) was made in Ref. [99] via numerical continuation. The existence of a stable similarity solution implies that the rupture behavior of Eq. (3.10) is universal in the $K = 0$ case.

We demonstrate the convergence to the universal rupture profile by solving Eq. (3.10) numerically until rupture and rescaling the interface data and spatial coordinate at each point in time using the transformations represented by Eqs. (3.23) and (3.25). Fig. 3.16(a) shows the rescaled numerical interface of a film free of any gas shear ($\tau = 0$) at $t'_r = t_r - t = 2.596$, in comparison to the analytical self-similar solution to Eq. (3.26). The numerical and analytical profiles are divergent from each other which is attributed to the fact that at this point in time the interface is still not close enough to rupture to satisfy the assumptions upon which Eq. (3.26) is based. However, as the value of t'_r decreases and $h \rightarrow 0$, the scaled numerical profile approaches and eventually agrees with the analytical one, as seen in Figs. 3.16(b) and 3.16(c), respectively.

Figs. 3.16(d), (e), and (f) compare the numerical and analytical profiles at different values of t'_r for the same parameter set, but with an applied gas shear ($\tau = 0.3$), where the same dynamic is observed. Note that when the film is not close to rupture, the scaled numerical profiles corresponding to the cases with and without gas shear are dissimilar due to the effect of the non-zero shear stress being significant at these film thicknesses. However, as the film approaches rupture the role of the shear stress is severely diminished to the extent where it becomes negligible, see Figs. 3.16(c) and (f). This ultimately goes to show that the rupture mechanism for $K = 0$ is indeed universal.

Note that if K is small but non-zero, the solution to Eq. (3.10) will initially tend to the stable similarity solution of Eq. (3.24) until h and K are comparable in size, after which a new dynamic regime appears. We demonstrate this numerically by computing the scaled numerical interface as the film approaches rupture for $K = 0.03$, see Fig. 3.17. As $h \rightarrow 0$, the scaled numerical solution approaches the

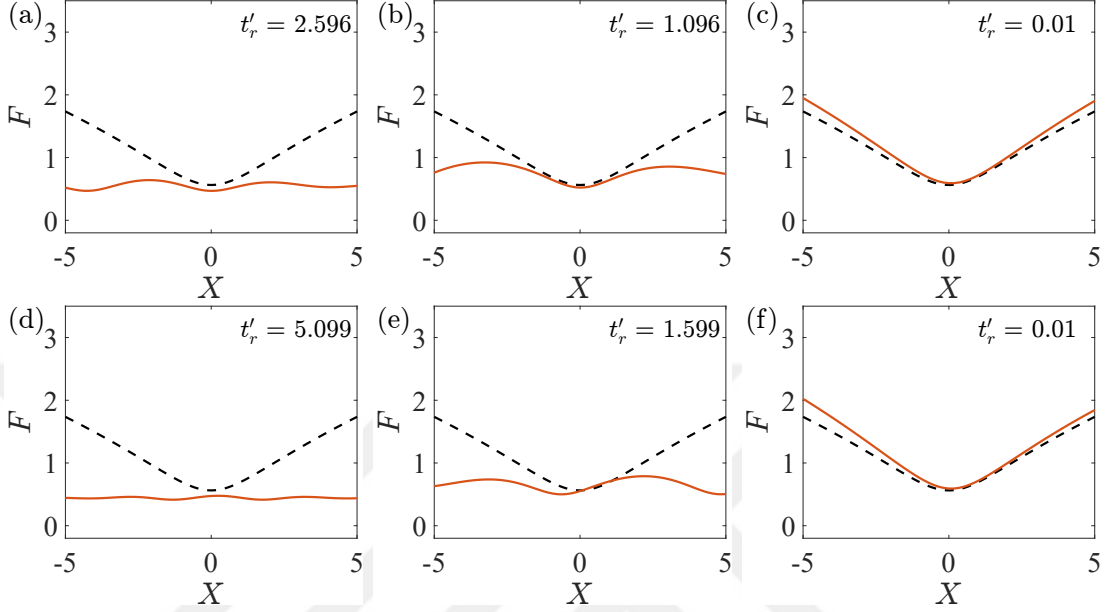


Figure 3.16: The scaled numerical interface compared to the analytical self-similar solution of Eq. (3.26) at different times towards rupture for $K = 0$, $\beta = 0$, $\text{Re} = 1$, $\mathcal{M} = 0.5$, $\bar{E} = 0$, $V_r = 1$, $t_f = 0$. (a,b,c) $\tau = 0$, and (d,e,f) $\tau = 0.3$. Note that $t'_r = t_r - t$.

analytical solution, as seen in Figs. 3.17(a), (b), and (c). However, as h becomes smaller than K , the numerical solution departs from the analytical profile and exhibits new dynamic behavior, see Figs. 3.17(d), (e), and (f).

3.5.4.2 Non-singular rupture: $K > 0$, $\bar{E} > 0$

The similarity solution discussed in Sec. 3.5.4.1 is valid if $K = 0$, or in the intermediate time when $\min(h) \gg K$. However, when $h \ll K$, the dynamics will be different and the behavior depends on whether the evaporative mass loss terms are present ($\bar{E} > 0$) or absent ($\bar{E} = 0$). When $\bar{E} > 0$, the evaporative mass loss term cannot be neglected in any self-similar analysis. However, this term is not a destabilizing term, and we do not see the universal self-similarity that results from the competition between surface tension and Marangoni/vapor recoil forces. Instead, the thickness goes to zero at a point with finite speed ($\partial_t h \sim -\bar{E}/K$) without the appearance of any singularities in the solutions. This conclusion is corroborated numerically in Fig. 3.18 where it is seen that the scaled numerical

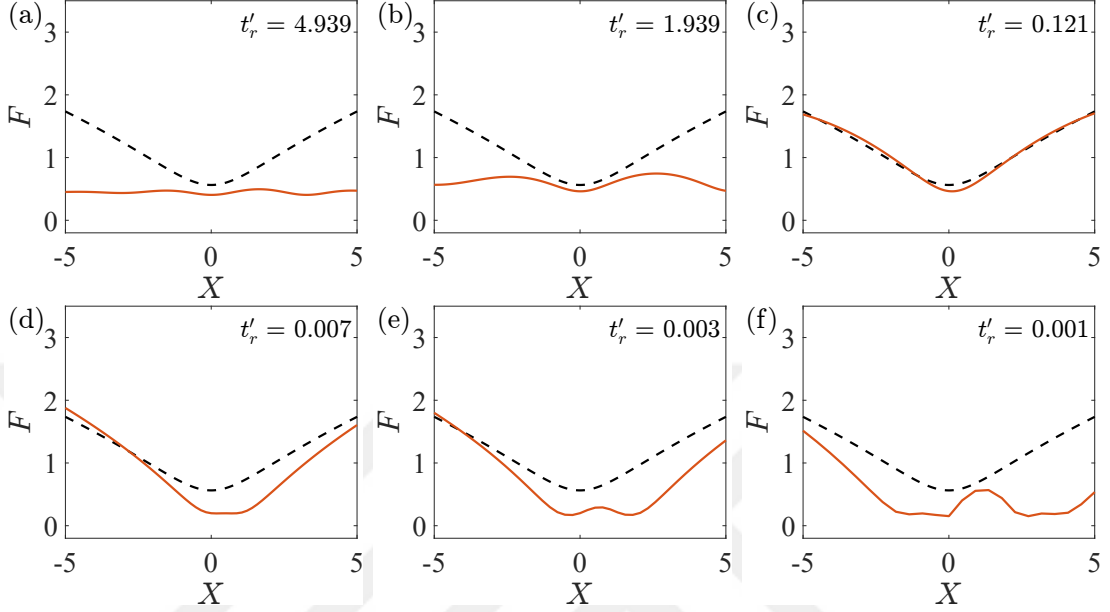


Figure 3.17: The scaled numerical interface compared to the analytical self-similar solution at different times towards rupture for $K = 0.03$, $\beta = 0$, $\text{Re} = 1$, $\mathcal{M} = 0.5$, $\bar{E} = 0$, $V_r = 1$, $\tau = 0$, and $t_f = 0$. Observe the deviation from the analytical self-similar solution in plots (d), (e), and (f) and the appearance of the new dynamic behavior. Note that $t'_r = t_r - t$.

interface does not go towards the self-similar solution but instead flattens out indicating non-singular rupture.

3.5.4.3 Infinite-time rupture: $K > 0$, $\bar{E} = 0$, $\tau \geq 0$

If $K > 0$ and $\bar{E} = \tau = 0$, we find the most important terms again come from balancing the time derivative with the Marangoni force and the surface tension. However, due to the effect of K on the Marangoni term, we obtain the approximate behavior

$$\partial h_t + \epsilon \partial_x \left[\frac{\mathcal{M}}{K^2} h^2 \partial_x h + \bar{S} h^3 \partial_{xxx} h \right]. \quad (3.27)$$

For this equation an infinite-time, rather than finite-time similarity ansatz is appropriate:

$$h(x, t) = t^{-\alpha} f(\xi, s), \quad \xi = t^\beta (x - x_0), \quad \text{and} \quad s = \ln t. \quad (3.28)$$

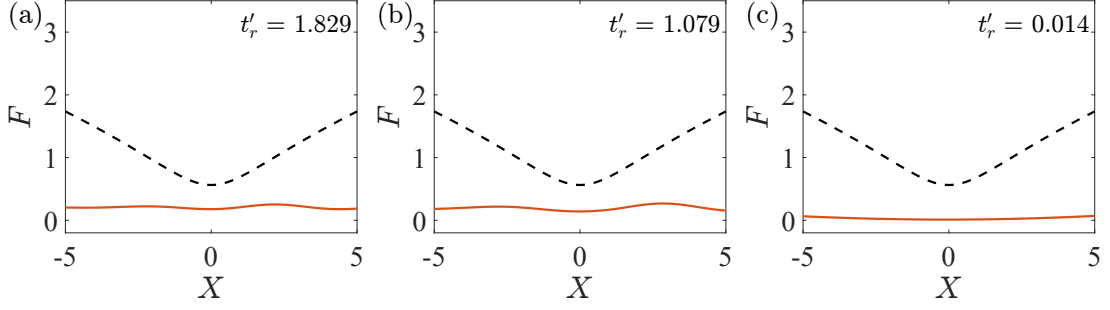


Figure 3.18: The scaled numerical interface compared to the analytical self-similar solution at different times towards rupture for $K = 0.2$, $\beta = 0$, $\text{Re} = 1$, $\mathcal{M} = 0.5$, $\bar{E} = 0.05$, $V_r = 1$, $\tau = 0$, and $t_f = 0$. Note that $t'_r = t_r - t$.

Balancing the three terms gives $\alpha = 1$, $\beta = 1/2$, resulting in the equation

$$\partial_s f - f + \frac{1}{2}\xi\partial_\xi f + \epsilon\partial_\xi \left[\frac{\mathcal{M}}{K^2} f^2 \partial_\xi f + \bar{S} f^3 \partial_{\xi\xi\xi} f \right] = 0. \quad (3.29)$$

All other terms in Eq. (3.10) are of lower power in t , and therefore negligible in the limit $t \rightarrow \infty$. As previously observed in Refs. [96] and [97], the behavior of Eq. (3.27), as time becomes large, is to form an infinite cascade of geometrically shrinking oscillations as $h \rightarrow 0$. This likely corresponds to the existence of a periodic (in s) solution to Eq. (3.29) for this phenomenon observed in finite-time rupture [99].

On the other hand, if $K > 0$, $\bar{E} = 0$, but $\tau > 0$, the non-zero shear stress has the effect of removing the infinite-time rupture solution. If we carry out the similarity transformation as in Eq. (3.29), but have non-zero shear, we obtain

$$\partial_s f - f + \frac{1}{2}\xi\partial_\xi f + \epsilon\partial_\xi \left[\frac{\mathcal{M}}{K^2} f^2 \partial_\xi f + \bar{S} f^3 \partial_{\xi\xi\xi} f \right] + t^{1/2} \partial_\xi \left[\frac{\tau}{2} f^2 \right] = 0. \quad (3.30)$$

The positive power of t on the shear term implies that the cascading similarity solution cannot be reached. In this case, it is observed that the solutions evolve chaotically as the film becomes very thin.

Fig. 3.19(a) shows the scaled numerical interface at different times towards rupture for $\tau = 0$. As $h \rightarrow 0$, the scaled numerical interface's oscillations form

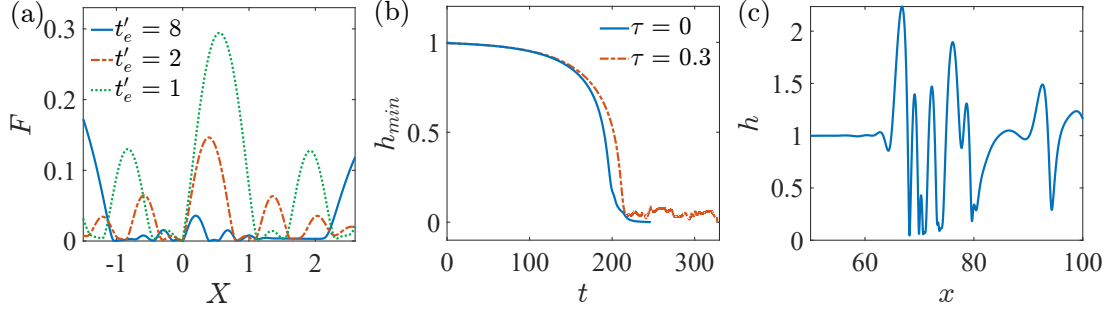


Figure 3.19: Film evolution close to rupture for $K = 0.4$, $\beta = 0$, $\text{Re} = 1$, $\mathcal{M} = 0.5$, $\bar{E} = 0$, $V_r = 1$, and $t_f = 0$. (a) The scaled numerical interface for $\tau = 0$ at different values of $t'_e = t_{end} - t$. (b) Minimum interface height versus time for $\tau = 0$ and $\tau = 0.3$. (c) Chaotic interface evolution for $\tau = 0.3$ at $t = 286.1$.

a geometric hierarchy which repeats for a vanishing spatial coordinate. Since the film in this case ruptures in infinite-time, note the usage of $t'_e = t_{end} - t$ instead of t'_r where t_{end} is the value of time at which the numerical solution was stopped. Fig. 3.19(b) plots the minimum interface height versus time for $\tau = 0$ and $\tau = 0.3$. It is observed that in the case of $\tau = 0$, h_{min} approaches zero asymptotically which is consistent with infinite-time rupture. However, for non-zero shear ($\tau = 0.3$), h_{min} behaves erratically when it becomes small. This chaotic interface evolution is further highlighted by plotting the fluid interface in this regime as shown in Fig. 3.19(c).

Chapter 4

Irradiated films at moderate Reynolds numbers

In this fourth chapter, we study a heated liquid film for which the flow is characterized by relatively larger velocities and film depth, and where the temperature gradient across the film is driven by an incoming radiation flux that passes unimpeded through the liquid surface, is absorbed by and reflected off of the solid wall, and then subsequently exchanged in the infrared spectrum between wall and the free surface. This radiative flux along with the liquid flow's specific parameters play the primary role in directing our modeling strategies as well as the scope of physical phenomena we include in our formulations.

The relatively higher Reynolds numbers and film thickness produce a flow regime that is governed by the balance between viscous damping forces and inertia, and is therefore fittingly named the drag-inertia flow regime [109]. This flow domain is not amenable to the simplified hydrodynamic model we had employed in the previous chapters (the LWE [28]), as it is prone to non-physical finite-time blow-up even at relatively low Reynolds numbers, which is caused by strong nonlinearities leading to the loss of the traveling-wave solutions [31, 108]. Consequently, we employ here a more advanced modeling methodology which unites a gradient expansion with weighted residual techniques using polynomials

as test functions to formulate a model where the evolution of the film is captured by the interface height function as well as the flow rate. This formulation is named the Weighted Integral Boundary Layer model (WIBL) [29, 30], and has been demonstrated to be accurate up to moderate Reynolds numbers ($Re \approx 50$). For further details, please refer to Refs. [108, 109].

In this type of system we are trying to model, the hydrodynamic development of the film has a primary influence on the temperature profile across it. Therefore, in this chapter the thermocapillary *S*-mode [36] and evaporative *E*-mode [57, 50] instabilities which we previously investigated in Chaps. 2 and 3 will not be considered due to their diminished influence in the drag-inertia flow regime. Notably, the role of phase change is mostly relegated to modifying the heat transfer properties from the liquid's surface, specifically, it is possible to incorporate its effect as a modification to the liquid interface's Biot number, yielding an *evaporative* Biot number which takes into account the latent heat loss. For additional details, please refer to Refs. [60, 133, 134, 135, 136]. Nevertheless, our main focus in this chapter is on the primary hydrodynamic and thermal modeling of problem, and we thus postpone this aspect to future work and direct our efforts at the development of an efficient model for the coupled hydrodynamic-thermal radiation problem, where we decompose the temperature variable following an asymptotic strategy similar to that developed by Cellier and Ruyer-Quil in Ref. [107]. Our effort results in a hierarchy of models formed by one or more coupled evolution equations by which the temperature profile across the film can be studied.

This chapter consists of the following: Sec. 4.1 presents the definition of the hydrodynamic problem by first deriving the boundary layer equations from the dimensionless governing equations and subsequently applying the WIBL methodology to obtain the first order WIBL equations. Thereafter, in Sec. 4.2 we formulate the liquid film's thermal problem by writing the governing system of equations, applying the chosen scaling, solving for the Nusselt temperature solution, and then decomposing the temperature field to obtain the set of models governing the temperature field. Finally, Sec. 4.3 presents the results obtained from the numerical simulation of the hydro-thermal coupled systems of equations.

4.1.1 Boundary layer equations

Following the procedure detailed in Ref. [109], we derive the x -momentum boundary layer equation for the liquid film as required by the WIBL method. We start from the same scaled two-dimensional expressions for continuity and the Navier-Stokes momentum equations that appeared in Eqs. (2.4) and Eqs. (3.5), which are restated here for clarity:

$$\partial_x u + \partial_y v = 0, \quad (4.1a)$$

$$\partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \partial_{xx} u + \partial_{yy} u + \text{Re} \sin \beta, \quad (4.1b)$$

$$\partial_t v + u \partial_x v + v \partial_y v = -\partial_y p + \partial_{xx} v + \partial_{yy} v - \text{Re} \cos \beta. \quad (4.1c)$$

The scaled wall boundary conditions at ($y = 0$) are

$$u = v = 0, \quad (4.1d)$$

while the scaled interface boundary conditions at ($y = h$) read

$$v = \partial_t h + u \partial_x h, \quad (4.1e)$$

$$p = \frac{2}{N^2} [\partial_x u [(\partial_x h)^2 - 1] - \partial_x h (\partial_y u + \partial_x v)] + 3S(1 - CT) \left[\frac{\partial_{xx} h}{N^3} \right]. \quad (4.1f)$$

We now define a small ordering parameter ϵ , which is used in the following transformations:

$$(\partial_x, \partial_t) \rightarrow \epsilon(\partial_x, \partial_t), \quad \partial_{xx} \rightarrow \epsilon^2 \partial_{xx}. \quad (4.2)$$

Moreover, we assume that all dimensionless parameters are $\mathcal{O}(1)$ except for the dimensionless surface tension S which we assume is $\mathcal{O}(\epsilon^{-2})$. Meanwhile, all terms $\mathcal{O}(\epsilon^3)$ and higher are neglected. Consequently, the transformations in Eq. (4.2) rewrite Eqs. (4.1) into

$$\partial_x u + \partial_y v = 0, \quad (4.3a)$$

$$\epsilon (\partial_t u + u \partial_x u + v \partial_y u) = -\epsilon \partial_x p + \epsilon^2 \partial_{xx} u + \partial_{yy} u + \text{Re} \sin \beta, \quad (4.3b)$$

$$\epsilon^2 (\partial_t v + u \partial_x v + v \partial_y v) = -\partial_y p + \epsilon \partial_{yy} v - \text{Re} \cos \beta, \quad (4.3c)$$

with the wall boundary conditions at ($y = 0$) as

$$u = v = 0, \quad (4.3d)$$

and the transformed interface boundary conditions at ($y = h$) as

$$v = \partial_t h + u \partial_x h, \quad (4.3e)$$

$$p = 2\epsilon [\partial_y v - \partial_x h \partial_y u] - \epsilon^2 3S(1 - CT) \partial_{xx} h. \quad (4.3f)$$

To advance, we eliminate the pressure gradient from the x -momentum relation. We first solve for the pressure by neglecting $\mathcal{O}(\epsilon^2)$ terms in the y -momentum expression [Eq. (4.3c)], integrating with respect to y , and then utilizing a truncated form [at $\mathcal{O}(\epsilon)$] of the pressure relation Eq. [(4.3f)] to get

$$p(y) = \text{Re} \cos \beta (h - y) - \epsilon^2 3S \partial_{xx} h + \epsilon (\partial_y v + \partial_y v|_{y=h}) + \mathcal{O}(\epsilon^2) \quad (4.4)$$

Substituting the pressure from Eq. (4.4) into the transformed x -momentum expression [Eq. (4.3b)], we finally obtain the second order x -momentum boundary layer equation we require, which reads

$$\begin{aligned} \epsilon [\partial_t u + \partial_x (u^2) + \partial_y (uv)] &= -\epsilon \text{Re} \cos \beta \partial_x h \\ &+ \epsilon^3 3S \partial_{xxx} h + \epsilon^2 [\partial_{xx} u - \partial_x (\partial_y v|_{y=h})] \\ &+ \epsilon^2 \partial_{xx} u + \partial_{yy} u + \text{Re} \sin \beta. \end{aligned} \quad (4.5)$$

4.1.2 Weighted residual integral boundary layer equations

The weighted residual integral boundary layer (WIBL) improves over the LWE by releasing the flow's dynamics from being completely slaved solely to the film thickness by introducing additional variables into the modeling methodology, such as the flow rate or inertial correction terms. The introduction of these additional variables is necessary in the drag-inertia flow domain to correctly model the flow since various flow properties in this regime are true degrees of freedom which

require their own evolution. The introduction of these additional variables is facilitated by an averaging procedure which allows the description of the fluid motion by a column of fluid between the solid surface and the interface, in place of a single particle on the liquid surface.

The WIBL methodology begins by projecting the velocity field onto a set of polynomial test functions and then substituting this expansion into the boundary layer approximation [Eq. (4.5)]. This is followed by careful ordering arguments in order to retain only the number of independent fields that is required to approximate the velocity field in a consistent manner at the chosen order of ϵ . Subsequently, the resulting equations are averaged using the method of weighted residuals using the Galerkin method where the weights are the test functions themselves. This method allows generating several models which increase in complexity with higher orders of approximation, where the simplest makes use of the first order boundary layer expression and produces two coupled equations consistent at order $\mathcal{O}(\epsilon)$, namely for the the film thickness and the flow rate. Due to the complexity and tediousness of the WIBL methodology, we will only present the main steps and equations in this discussion. For a detailed overview of the steps taken under the WIBL method, please see Ref. [109].

4.1.2.1 First order WIBL model

The term first order model refers to a hydrodynamic model that is consistent at order $\mathcal{O}(\epsilon)$, formed by two-coupled equations for the film thickness and the flow rate. The formulation begins from the first order boundary layer approximation Eq. [(4.5)] which reads:

$$\epsilon(\partial_t u + u\partial_x u + v\partial_y u) = -\epsilon \text{Re} \cos \beta \partial_x h + \epsilon^3 3S \partial_{xxx} h + \partial_{yy} u + \text{Re} \sin \beta. \quad (4.6)$$

The streamwise velocity is expanded as

$$u(x, y, t) = \sum_{j=0}^{j_{\max}} a_j(x, t) f_j \left[\frac{y}{h(x, t)} \right], \quad (4.7)$$

where the ordering arguments for the first order model indicate that we only require the first term for consistency at $\mathcal{O}(\epsilon)$, for which we find

$$a_0 = 3\frac{q}{h}, \quad (4.8a)$$

$$f_0 = \frac{y}{h} - \frac{1}{2} \left(\frac{y}{h}\right)^2. \quad (4.8b)$$

Moreover, from continuity we have

$$v = - \int_0^y \partial_x u dy. \quad (4.9)$$

Now, we can write the first residual $\mathcal{R}_0$ from Eq. (4.6) as

$$\begin{aligned} \mathcal{R}_0 = \int_0^h f_0 [\epsilon (\partial_t u + u \partial_x u + v \partial_y u)] dy \\ + \frac{q}{h^2} - \frac{h}{3} (\text{Re} \sin \beta - \epsilon \text{Re} \cos \beta \partial_x h + \epsilon^3 3S \partial_{xxx} h) \end{aligned} \quad (4.10)$$

Evaluating the expression in Eq. (4.10) yields an expression governing the time-evolution of the flow rate denoted $q(x, t)$. The second equation is obtained by integrating the continuity equation [Eq. (4.3a)] across the film and making use of the boundary conditions in Eqs. (4.3d) and (4.3e) in conjunction with Leibniz's integral rule to obtain an expression linking the flow rate and film's thickness function $h(x, t)$. Together, these two equations form the first order WIBL model, which reads:

$$\frac{\partial h}{\partial t} = -\partial_x q \quad (4.11a)$$

$$\begin{aligned} \epsilon \frac{\partial q}{\partial t} = \frac{5}{6} \text{Re} \sin \beta h - \frac{5}{2} \frac{q}{h^2} - \epsilon \frac{17}{7} \frac{q \partial_x q}{h} \\ + \epsilon \frac{9}{7} \frac{q^2 \partial_x h}{h^2} - \epsilon \frac{5}{6} \text{Re} \cos \beta h \partial_x h + \epsilon^3 S h \partial_{xxx} h, \end{aligned} \quad (4.11b)$$

and which will be used to model the film's hydrodynamics in the coupled hydro-thermal problem.

We can verify the correctness of Eqs. (4.11) by comparison to the equivalent isothermal nonvolatile LWE expression from Chaps. 2 and 3. We accomplish this

by expanding the flow rate as

$$q = q^{(0)} + \epsilon q^{(1)} \quad (4.12)$$

and substituting back into Eqs. (4.11), where we are able to obtain:

$$q^{(0)} = \frac{1}{3}h^3 Re \sin \beta, \quad (4.13a)$$

$$q^{(1)} = \frac{2}{15}(\text{Re} \sin \beta)^2 h^6 \partial_x h - \frac{1}{3}\text{Re} \cos \beta h^3 \partial_x h + \epsilon^2 S h^3 \partial_{xxx} h. \quad (4.13b)$$

Inserting the expressions from Eqs. (4.12) and (4.13) into Eq. (4.11a), we recover

$$\begin{aligned} \partial_t h + \text{Re} \sin \beta h^2 \partial_x h + \epsilon \left[\frac{4}{5} (\text{Re} \sin \beta)^2 h^5 (\partial_x h)^2 \right. \\ \left. + \frac{2}{15} (\text{Re} \sin \beta)^2 h^6 \partial_{xx} h - \text{Re} \cos \beta h^2 (\partial_x h)^2 - \frac{1}{3} \text{Re} \cos \beta h^3 \partial_{xx} h \right. \\ \left. + \epsilon^2 3 S h^2 \partial_x h \partial_{xxx} h + \epsilon^2 S h^3 \partial_{xxxx} h \right], \quad (4.14) \end{aligned}$$

which is equivalent to Eqs. (2.12) and (3.10) when the terms related to thermo-capillarity and evaporation are removed. Notice the significantly lower degree of nonlinearity of Eqs. (4.11) when compared to Eq. (4.14). In particular, the term $\partial_x(h^6 \partial_x h)$ is absent, which has been identified to be the source of the unphysical blow-up of the LWE equations [31, 108]. For our comparison between the numerical performance of Eqs. (4.11) and Eq. (4.14), see Sec. 4.3.1.

4.2 Thermal formulation

In order to develop the film's thermal formulation, we must first define the radiation problem by making the following assumptions. We presume the bulk of the liquid to be transparent to optical and infrared radiation, and thus the radiation heat transfer is governed by the infrared exchange between the liquid interface and the solid wall. In addition, the liquid interface is fully transparent to visible light, but behaves as a black body in the infrared spectrum. Finally, we assume the solid wall to be an absorbing, reflecting, and non-transmitting gray surface.

4.2.1 Solid wall temperature problem

We begin our formulation by obtaining a solution for the temperature distribution within the solid wall, which is required for the development of the boundary condition at the bottom of the liquid layer to include the convective heat loss from the solid wall to the ambient environment. In the following, T_w will be used to denote the temperatures associated with the solid wall, while T_l will be purposed for those associated the liquid phase. The dimensional energy equation governing the solid domain reads

$$\partial_t \tilde{T}_w = \frac{\lambda_w}{\rho_w C_{p,w}} \left(\partial_{\tilde{x}\tilde{x}} \tilde{T}_w + \partial_{\tilde{y}\tilde{y}} \tilde{T}_w \right), \quad (4.15a)$$

where $\tilde{T}_w$ is the temperature function within the solid wall, λ_w is its thermal conductivity, ρ_w is its mass density, and $C_{p,w}$ is its specific heat capacity. The ratio $k_w/(\rho_w C_{p,w})$ forms the wall's thermal diffusivity.

Assuming that the wall temperature varies slowly in time and neglecting the axial conduction term within the wall, Eq. (4.15a) is simplifies into

$$\partial_{\tilde{y}\tilde{y}} \tilde{T}_w = 0. \quad (4.15b)$$

The boundary condition at the top of the solid wall ($\tilde{y} = 0$) is written by summing all the heat fluxes at the boundary and takes the form

$$\alpha_s \dot{q}_s + \alpha_w \sigma \tilde{T}_{l,fs}^4 - \epsilon_w \sigma \tilde{T}_w|_{\tilde{y}=0} - \lambda_w \partial_{\tilde{y}} \tilde{T}_w|_{\tilde{y}=0} - h^{wl} (\tilde{T}_w|_{\tilde{y}=0} - \tilde{T}_{l,\infty}) = 0, \quad (4.16a)$$

where α_s is the absorptivity of the solid surface to the incoming solar radiation flux $\dot{q}_s$, α_w is the absorptivity of the solid surface to infrared radiation, ϵ_w is the emissivity of the solid surface, $\tilde{T}_{l,fs}$ is the temperature of the liquid interface at $\tilde{y} = \tilde{h}$, σ is the Stefan-Boltzman constant, h^{wl} is the heat transfer coefficient between the solid plate and the liquid, and $\tilde{T}_{l,\infty}$ is the temperature of the liquid bulk. The terms in Eq. (4.16a) represent in order the incoming solar radiation flux, the incoming infrared radiation from the liquid's surface, the infrared radiation lost to the liquid interface, the conduction of heat into the bulk of the solid, and

lastly the convective heat lost to liquid layer directly above the solid surface.

Eq. (4.16a) can be simplified by neglecting the thermal resistance layer at the solid-liquid interface [109], allowing the replacement of heat convection with heat conduction. Moreover, we suppose that the temperature difference between the solid surface and the liquid interface is small enough to warrant the application of Kirchoff's law of radiation ($\alpha_w = \epsilon_w$). Now, Eq. (4.16a) reads

$$\alpha_s \dot{q}_s + \epsilon_w \sigma \left[\tilde{T}_{l,fs}^4 - \tilde{T}_{w|\tilde{y}=0}^4 \right] - \lambda_w \partial_{\tilde{y}} \tilde{T}_{w|\tilde{y}=0} + \lambda_l \partial_{\tilde{y}} \tilde{T}_{l|\tilde{y}=0} = 0 \quad (4.16b)$$

where λ_l is the liquid's thermal conductivity. This completes the formulation of the boundary condition at the solid-liquid interface.

Similarly, the boundary condition at the wall's lower boundary is written by summing the heat fluxes at $\tilde{y} = -d$:

$$\lambda_w \partial_{\tilde{y}} \tilde{T}_{w|\tilde{y}=-d} - h^{w\infty} (\tilde{T}_{w|\tilde{y}=-d} - \tilde{T}_{\infty}) = 0, \quad (4.17)$$

where $h^{w\infty}$ is the heat transfer coefficient between the wall and the ambient gas. Note that Eq. 4.17 is simply Newton's law of cooling.

Equations (4.15b), (4.16b), and (4.17) constitute the detailed description of the wall's temperature problem. However, the fourth order temperature terms in the solid-liquid boundary condition [Eq. (4.16b)] are quite challenging to deal with at this point, and therefore, this complex boundary condition is replaced with a temperature continuity condition ($\tilde{T}_{w|\tilde{y}=0} = \tilde{T}_{l|\tilde{y}=0}$) in the solid wall's temperature problem, whereas the explicit expression for the heat fluxes at $\tilde{y} = 0$ is deferred to the liquid temperature problem. Hence, the temperature problem for the solid wall can be rewritten as

$$\partial_{\tilde{y}\tilde{y}} \tilde{T}_w = 0, \quad (4.18a)$$

$$\tilde{T}_w = \tilde{T}_l, \quad \text{at } \tilde{y} = 0, \quad (4.18b)$$

$$\lambda_w \partial_{\tilde{y}} \tilde{T}_{w|\tilde{y}=-d} - h^{w\infty} (\tilde{T}_{w|\tilde{y}=-d} - T_{\infty}) = 0, \quad \text{at } \tilde{y} = \tilde{h}. \quad (4.18c)$$

Equations (4.18a) can be readily solved to obtain the following solution for the

wall's temperature distribution:

$$\tilde{T}_w = \frac{h^{w\infty}(\tilde{T}_{l|\tilde{y}=0} - T_\infty)}{\lambda_w + h^{w\infty}d} \tilde{y} + \tilde{T}_{l|\tilde{y}=0} \quad (4.19)$$

Equation (4.19) will be used for the substitute for the wall temperature in the liquid layer's temperature problem. For further details on the aforementioned assumptions and derivations, please refer to Ref. [109].

4.2.2 Liquid layer temperature problem

From this point onward, the subscript l attached to the liquid's variables is dropped, while the subscript w assigned to those associated with the solid wall is retained for clarity. The liquid temperature problem is governed by the energy equation which states:

$$\rho C_p \frac{D\tilde{T}}{Dt} = \lambda \nabla^2 \tilde{T} + \mu \tilde{\Phi}, \quad (4.20)$$

where μ is the liquid's dynamic viscosity and Φ represents viscous dissipation, which for the typical flow velocity and viscosity encountered in our system is negligible [137]. Hence, Eq. (4.20) simplifies to

$$\partial_t \tilde{T} + \tilde{u} \partial_{\tilde{x}} \tilde{T} + \tilde{v} \partial_{\tilde{y}} \tilde{T} = \alpha (\partial_{\tilde{x}\tilde{x}} \tilde{T} + \partial_{\tilde{y}\tilde{y}} \tilde{T}), \quad (4.21a)$$

where $\alpha = k/(\rho C_p)$ is the liquid's thermal diffusivity.

At the liquid's bottom boundary ($\tilde{y} = 0$), we restate flux balance which we had already written when developing the solid wall problem [Eq. (4.16b)], albeit we substitute for the temperature of the solid wall using Eq. (4.19). This yields an expression for the boundary condition only in terms of the liquid temperature and its derivatives:

$$\alpha_s \dot{q}_s + \epsilon_w \sigma \left[\tilde{T}_{fs}^4 - \tilde{T}_{|y=0}^4 \right] - \lambda_w \frac{h^{w\infty}(\tilde{T}_{|\tilde{y}=0} - T_\infty)}{\lambda_w + h^{w\infty}d} + k \partial_{\tilde{y}} \tilde{T}_{|\tilde{y}=0} = 0. \quad (4.21b)$$

At the liquid's free interface ($\tilde{y} = \tilde{h}$) the balance of conductive, convective, and radiative fluxes can be expressed as

$$\epsilon_w \sigma \left[\tilde{T}_{|\tilde{y}=0}^4 - \tilde{T}_{fs}^4 \right] + \lambda \partial_{\tilde{x}} \tilde{T}_{fs} \partial_{\tilde{x}} \tilde{h} - \lambda \partial_{\tilde{y}} \tilde{T}_{fs} - h^{fs} (\tilde{T}_{fs} - T_\infty) = 0, \quad (4.21c)$$

where h^{fs} is the heat transfer coefficient at the liquid's free surface. The terms represent in order the radiative exchange between the liquid's surface and the wall, the conductive heat flux arriving from the liquid bulk, and the convective heat loss to the ambient environment. Equations (4.21a), (4.21b), and (4.21c) form the system of equations governing the liquid temperature problem.

4.2.3 Scaling and dimensionless parameters

We use the same scales we utilized in 2.3 and 3.4 except for the temperature variable, which we rescale using a characteristic wall temperature T_c here to avoid difficulties with the fourth order temperature terms in Eqs. (4.21b) and (4.21c).

$$\begin{aligned} (\tilde{x}, \tilde{y}) &\rightarrow \tilde{h}(x^*, y^*), & \tilde{t} &\rightarrow \frac{\tilde{h}^2}{\nu} t^*, & \tilde{h} &\rightarrow \tilde{h} h^* \\ (\tilde{u}, \tilde{v}) &\rightarrow \frac{\nu}{\tilde{h}}(u^*, v^*, w^*), & \tilde{T} &\rightarrow T_c T^* \end{aligned} \quad (4.22)$$

Note that the star notation will be dropped onwards for simplicity. Applying these scales results in the following nondimensional system of equations for the liquid temperature problem:

$$\text{Pr} [\partial_t T + u \partial_x T + v \partial_y T] = \partial_{xx} T + \partial_{yy} T, \quad (4.23a)$$

$$y = 0 : Q_s + [T_{fs}^4 - T_{|y=0}^4] - N_{cr,w} Bi_w (T_{|y=0} - \mathcal{T}) + N_{cr,l} \partial_y T_{|y=0} = 0, \quad (4.23b)$$

$$y = h : [T_{fs}^4 - T_{|y=0}^4] + N_{cr,l} [\partial_x T_{fs} \partial_x h - \partial_y T_{fs} - Bi_{fs} (T_{fs} - \mathcal{T})] = 0. \quad (4.23c)$$

System of Eqs. (4.23) contains the following dimensionless parameters: Pr is the Prandtl number, Q_s is the incident solar flux parameter, $N_{cr,w}$ is the wall

conduction-radiation parameter, $N_{cr,l}$ is the liquid conduction-radiation parameter, Bi_w is the wall Biot number, Bi_{fs} is the liquid interface Biot number, and $\mathcal{T}$ is the dimensionless ambient temperature. The mathematical forms and interpretations of these numbers are listed in Table 4.1. Of particular interest are the parameters $N_{cr,w}$ and $N_{cr,l}$, which are intimately tied to the radiation fluxes within the system.

Parameter	Expression	Physical significance
Pr	ν/χ	Momentum diffusivity to thermal diffusivity.
Q_s	$\frac{\alpha_s \dot{q}_s}{(\epsilon_w \sigma T_c^4)}$	Absorbed solar flux to infrared radiation emitted from the wall.
$N_{cr,w}$	$\frac{k_w}{(\epsilon_w \sigma T_c^3 \tilde{h})}$	Wall conductive heat transfer to infrared radiation emitted from the wall.
$N_{cr,l}$	$\frac{k_l}{(\epsilon_w \sigma T_c^3 \tilde{h})}$	Liquid conductive heat transfer to infrared radiation emitted from the wall.
Bi_w	$\frac{h^{w\infty} \bar{h}}{(k_w + h^{w\infty} d)}$	Wall internal conductive resistance versus convective resistance at the surface.
Bi_{fs}	$\frac{h^{fs} \tilde{h}}{k_l}$	Liquid internal conductive resistance versus convective resistance at the interface.
$\mathcal{T}$	T_∞/T_c	Ratio of ambient and wall temperatures.

Table 4.1: The set of dimensionless parameters appearing in the scaled liquid temperature problem [Eqs. (4.23)].

4.2.4 Order of magnitude analysis

Due to the complexity of the expressions for the boundary conditions at the solid wall and the liquid interface, we compute typical values for the dimensionless parameters in Table 4.1 in an attempt to identify any differences in order of magnitude large enough to allow for some simplifications. We compute the values of the dimensionless parameters for a typical setting (water at 25° flowing over a

generic photovoltaic surface) to obtain an approximation of their relative magnitudes. The values we use are acquired from common literature, and are listed in Table 4.2.

Physical property	Symbol	Value	Units
Solar absorptivity	α_s	0.8	Nondimensional
Emissivity	ϵ_w	0.75	Nondimensional
Stefan–Boltzmann constant	σ	5.6×10^{-8}	W/(m ² K ⁴)
Incident solar flux	q_s	900	W/m ²
Wall thermal conductivity	k_w	1000	W/(mK)
Liquid thermal conductivity	k_l	0.6	W/(mK)
Typical liquid film thickness	$\tilde{h}$	0.01	m
Typical wall temperature	T_c	308	K
Typical ambient temperature	$\tilde{T}_\infty$	293	K

Table 4.2: Physical properties for water at 25° used in the computation of the system’s dimensionless parameters.

Based on the properties in Table 4.2, we compute the typical values of the dimensionless parameters as

$$\begin{aligned} \text{Pr} &\approx 6.2, & Q_s &\approx 1.9, & N_{cr,w} &\approx 1.65 \times 10^4, & N_{cr,l} &\approx 4.9, \\ Bi_w &\approx 5 \times 10^{-2}, & Bi_{fs} &\approx 1 \times 10^{-2}, & \mathcal{T} &\approx 0.98. \end{aligned}$$

When these values are substituted into the system of equations (4.23), we find that the terms in the equation are sufficiently close in magnitude that no simplification is possible. Therefore, all of the terms in Eqs. (4.23) are retained.

4.2.5 Nusselt temperature solution

In order to be able to fully decompose the temperature variable, we first find the Nusselt temperature solution corresponding to steady state condititions and

a flat liquid interface. Setting $\partial_x = \partial_t = 0$ in Eqs. (4.23) we get

$$\partial_{yy} T_{Nu} = 0, \quad (4.24a)$$

$$y = 0 : Q_s + [T_{fs}^4 - T_{Nu|y=0}^4] - N_{cr,w} Bi_w (T_{Nu|y=0} - \mathcal{T}) + N_{cr,l} \partial_y T_{Nu|y=0} = 0, \quad (4.24b)$$

$$y = h : -[T_{fs}^4 - T_{Nu|y=h}^4] + N_{cr,l} [-\partial_y T_{Nu,f} - Bi_{fs} (T_{Nu,f} - \mathcal{T})] = 0. \quad (4.24c)$$

The fourth order temperature terms in the boundary conditions [Eqs. (4.24b) and (4.24c)] prevent a direct solution for T_{Nu} since obtaining a general closed form solution is not possible for undetermined coefficients, i.e., without assigning numerical values to the parameters. In order to proceed further, we split the Nusselt temperature solution T_{Nu} into a reference temperature field $\bar{T}_{Nu,0}$ and a small temperature correction Θ_{Nu}

$$T_{Nu} = \bar{T}_{Nu,0} + \Theta_{Nu}, \quad \Theta_{Nu} \ll 1. \quad (4.25)$$

$\bar{T}_{Nu,0}$ represents a reference temperature field established by conduction and convection, while $\bar{\Theta}$ is a small correction accounting for radiation the radiation exchange and the influence of interface perturbations on the temperature profile across the film.

Substituting from Eq. (4.25) into Eqs. (4.24) and linearizing in Θ_{Nu} , we get the following system of equations for $\bar{T}_{Nu,0}$:

$$\partial_{yy} \bar{T}_{Nu,0} = 0, \quad (4.26a)$$

$$y = 0 : Q_s - N_{cr,w} Bi_w (\bar{T}_{Nu,0|y=0} - \mathcal{T}) + N_{cr,l} \partial_y \bar{T}_{Nu,0|y=0} = 0, \quad (4.26b)$$

$$y = 1 : N_{cr,l} [-\partial_y \bar{T}_{Nu,0|y=1} - Bi_{fs} (\bar{T}_{Nu,0|y=1} - \mathcal{T})] = 0. \quad (4.26c)$$

Note that the interfacial boundary condition is written at $y = 1$ in Eq. (4.26c) as $\bar{T}_{Nu,0}$ represents the temperature field corresponding to a flat film and does not

admit liquid interface perturbations. Solving Eqs. (4.26) gives

$$\bar{T}_{Nu,0} = \frac{1}{N_{cr,w}Bi_w(1 + Bi_{fs}\bar{h})} \left[Q_s Bi_{fs}(1 - y) + Q_s \right] + \mathcal{T}. \quad (4.27)$$

On the other hand, the system of equations for Θ_{Nu} reads

$$\partial_{yy}\Theta_{Nu} = 0, \quad (4.28a)$$

$$y = 0 : \quad \left[4\bar{T}_{Nu,0|y=1}^3 \Theta_{Nu|y=h} - 4\bar{T}_{Nu,0|y=0}^3 \Theta_{Nu|y=0} \right] - N_{cr,w}Bi_w \Theta_{Nu|y=0} + N_{cr,l} \partial_y \Theta_{Nu|y=0} = 0, \quad (4.28b)$$

$$y = h : \quad - \left[4\bar{T}_{Nu,0|y=1}^3 \bar{\Theta}_{Nu|y=h} - 4\bar{T}_{Nu,0|y=0}^3 \bar{\Theta}_{Nu|y=0} \right] + N_{cr,l} \left[- \partial_y \bar{\Theta}_{Nu|y=h} - Bi_{fs} \mathcal{T} \right] = 0, \quad (4.28c)$$

which can be solved to obtain

$$\Theta_{Nu} = - \frac{N_{cr,l}Bi_{fs}\mathcal{T}}{N_{cr,w}Bi_w} \left(\frac{4\bar{T}_{Nu,0|y=0} - 4\bar{T}_{Nu,0|y=1} + N_{cr,w}Bi_w}{\bar{T}_{Nu,0|y=1}h + N_{cr,l}} y + 1 \right). \quad (4.29)$$

From Eqs. (4.27) and (4.29), the Nusselt temperature solution can be written as

$$T_{Nu} = \frac{1}{N_{cr,w}Bi_w(1 + Bi_{fs}\bar{h})} \left[Q_s Bi_{fs}(1 - y) + Q_s \right] + \mathcal{T} - \frac{N_{cr,l}Bi_{fs}\mathcal{T}}{N_{cr,w}Bi_w} \left(\frac{4\bar{T}_{Nu,0|y=0} - 4\bar{T}_{Nu,0|y=1} + N_{cr,w}Bi_w}{\bar{T}_{Nu,0|y=1}h + N_{cr,l}} y + 1 \right). \quad (4.30)$$

It is important to keep in mind that $y \in [0, 1]$ in $\bar{T}_{Nu,0}$, while $y \in [0, h(x, t)]$ in Θ_{Nu} . This distinction is always implied when referring to the Nusselt solution at the free surface denoted $T_{Nu|fs}$.

4.2.6 Temperature decomposition

Assuming that the temperature distribution doesn't depart significantly from the Nusslet solution in Eq. (4.30), we expand it asymptotically as

$$T = T_{Nu} + T^{(0)} + \epsilon T^{(1)}. \quad (4.31)$$

$T^{(0)}$ and $T^{(1)}$ are corrections to the temperature field such that the amplitudes associated with these corrections reflect the spatiotemporal variations in the temperature distribution within the film. Specifically, an appropriate ansatz is proposed for $T^{(0)}$ and is projected across the film using polynomial functions which are chosen specifically to ensure that the ansatz is satisfied at the boundaries. Moreover, the decomposition is made uniquely determined by utilizing $T^{(1)}$ to write suitable gauge conditions.

Once, the complete temperature decomposition is substituted into system (4.23), such that we retain terms only up to $\mathcal{O}(\epsilon)$, except for the axial conduction terms in the governing equation which are kept despite being $\mathcal{O}(\epsilon^2)$ in order to resolve the hyperbolic stagnation point in large amplitude waves and prevent the appearance of singularities in the temperature field [138]. In addition, this aligns with the *least-degeneracy principle*, where all relevant physical effects are accounted for at leading order. Hence, after inserting the temperature decomposition into Eq. (4.23a) and rearranging for $\partial_{yy}T^{(1)}$ it reads

$$\begin{aligned} \partial_{yy}T^{(1)} = \text{Pr} \left[\partial_t T_{Nu} + \partial_t T^{(0)} + u \left(\partial_x T_{Nu} + \partial_x T^{(0)} \right) \right. \\ \left. + v \left(\partial_y T_{Nu} + \partial_y T^{(0)} \right) \right] - \partial_{xx} T_{Nu} - \partial_{xx} T^{(0)}. \end{aligned} \quad (4.32)$$

Equation (4.32) can be integrated twice to obtain a polynomial solution for $T^{(1)}$ which produces the governing equations once the integration constants are found that the gauge conditions are applied. As for the velocities, we assume that the streamwise velocity doesn't deviate significantly from the parabolic Nuesselt velocity profile such that $u = 3\frac{q}{h} \left[\frac{y}{h} - \frac{1}{2} \left(\frac{y}{h} \right)^2 \right]$ and $v = -\int_0^y \partial_x u dy$, which is consistent with the approximation proposed for u in the derivation of the first

order WIBL model in Sec. 4.1.2.1. The decomposed boundary conditions are not shown here for conciseness.

The ansatz chosen for $T^{(0)}$ consists of variables which capture the deviation of the temperature field from the Nusselt solution close to the important regions of the liquid film. In our case, we introduce the variables which monitor the temperatures at the liquid interface and the solid wall. This framework can be used to generate a hierarchy of models of varying sophistication, where the degree of complexity depends on the number of variables embedded in the ansatz proposed for $T^{(0)}$. Naturally, the models with fewer variables are simpler to handle analytically and implement numerically, at the cost of being less robust.

4.2.7 Thermal evolution equations

In the following we present three models in which the ansatz for $T^{(0)}$ consists of one, two, and four variables where we set the polynomial functions and gauge conditions appropriately for each model. Due to the complexity of the problem at hand, particularly the boundary conditions containing the fourth order temperatures which become extremely unwieldy once decomposed, performing the aforementioned steps is not straightforward. Therefore, a set of *MATLAB* [119] scripts are written which carefully manage and manipulate the expressions until the final equations are generated symbolically, which are then passed on to another set of scripts which automatically discretize and rewrite the equations in a manner that is suitable for numerical computation.

4.2.7.1 Single-variable model

In this model, we introduce a single variable $\theta_1 = T_{fs}$ to monitor the temperature of the liquid's surface. Therefore, we put forth the following ansatz for $T^{(0)}$:

$$T^{(0)} = (\theta_1 - T_{Nu|fs}) P_1(y), \quad (4.33a)$$

where $P_1 = \frac{y}{h}$. Thus, the complete temperature decomposition becomes

$$T = T_{Nu} + (\theta_1 - T_{Nu|fs}) \left(\frac{y}{h} \right) + \epsilon T^{(1)}. \quad (4.33b)$$

The appropriate gauge condition for this model is

$$T_{fs}^{(1)} = 0. \quad (4.33c)$$

We substitute the decomposition Eq. (4.33b) into Eq. (4.32), integrate twice in the y coordinate, find the integration constants using the boundary conditions [Eqs. (4.23b) and (4.23c)], and finally utilize the gauge condition (4.33c) to arrive at the following equation for θ_1

$$\begin{aligned} \partial_t \theta_1 = & \frac{1}{\mathcal{F}_0} \left\{ -h^5 \left(\frac{3 A_1 \text{Pr} \partial_x q}{40 h^4} - \frac{3 A_2 \text{Pr} \partial_x q}{40 h^3} + \frac{3 A_2 \text{Pr} \partial_x q}{40 h^4} + \frac{3 B_1 \text{Pr} \partial_x q}{40 h^4} \right. \right. \\ & - \frac{3 \text{Pr} \theta_1 \partial_x q}{40 h^4} + \frac{9 \text{Pr} q \partial_x \theta_1}{40 h^4} + \frac{9 A_2 \text{Pr} q \partial_x h}{40 h^4} \left. \right) - h^4 \left(\frac{3 A_1 \text{Pr} \partial_x q}{8 h^3} \right. \\ & - \frac{3 A_2 \text{Pr} \partial_x q}{8 h^2} + \frac{3 A_2 \text{Pr} \partial_x q}{8 h^3} + \frac{3 B_1 \text{Pr} \partial_x q}{8 h^3} - \frac{3 \text{Pr} \theta_1 \partial_x q}{8 h^3} + \frac{3 \text{Pr} q \partial_x \theta_1}{4 h^3} \\ & \left. + \frac{3 A_2 \text{Pr} q \partial_x h}{4 h^3} \right) + \mathcal{F}_1 - h^3 \left(-\frac{\partial_{xx} \theta_1}{6 h} - \frac{(\partial_x h)^2 \theta_1}{3 h^3} + \frac{\theta_1 \partial_{xx} h}{6 h^2} + \frac{\partial_x \theta_1 \partial_x h}{3 h^2} \right. \\ & + \frac{A_1 (\partial_x h)^2}{3 h^3} + \frac{A_2 (\partial_x h)^2}{3 h^3} + \frac{B_1 (\partial_x h)^2}{3 h^3} - \frac{A_1 \partial_{xx} h}{6 h^2} - \frac{A_2 \partial_{xx} h}{6 h^2} - \frac{B_1 \partial_{xx} h}{6 h^2} \\ & + \frac{A_1 \text{Pr} \partial_t h}{2 h^2} + \frac{A_2 \text{Pr} \partial_t h}{2 h^2} + \frac{B_1 \text{Pr} \partial_t h}{2 h^2} - \frac{\text{Pr} \theta_1 \partial_t h}{2 h^2} \left. \right) + \frac{h}{D1} \left[-4 A_1^3 B_1 \right. \\ & - A_1^4 + N_{cr,l} \left(A_2 - \text{Bi}_{fs} (\mathcal{T} - \theta_1) - \frac{A_1 \partial_{xx} h}{2} - \frac{A_2 \partial_{xx} h}{2} - \frac{B_1 \partial_{xx} h}{2} \right. \\ & - \frac{A_1}{h} - \frac{A_2}{h} - \frac{B_1}{h} - \frac{h \partial_{xx} \theta_1}{2} + \frac{\theta_1 \partial_{xx} h}{2} + \frac{\theta_1}{h} + \partial_x \theta_1 \partial_x h - \frac{(\partial_x h)^2 \theta_1}{h} \\ & + \frac{3 A_1 \text{Pr} \partial_t h}{2} + \frac{3 A_2 \text{Pr} \partial_t h}{2} + \frac{3 B_1 \text{Pr} \partial_t h}{2} + \frac{9 A_1 \text{Pr} \partial_x q}{8} + \frac{9 A_2 \text{Pr} \partial_x q}{8} \\ & + \frac{9 B_1 \text{Pr} \partial_x q}{8} - \frac{3 \text{Pr} \theta_1 \partial_t h}{2} - \frac{9 \text{Pr} \theta_1 \partial_x q}{8} + \frac{15 \text{Pr} q \partial_x \theta_1}{8} + \frac{A_1 (\partial_x h)^2}{h} \\ & + \frac{A_2 (\partial_x h)^2}{h} + \frac{B_1 (\partial_x h)^2}{h} + \frac{15 A_2 \text{Pr} q \partial_x h}{8} - \frac{9 A_2 \text{Pr} h \partial_x q}{8} \left. \right) + \theta_1^4 \\ & \left. + \frac{4 A_1^3}{4 A_1^3 + \text{Bi}_w N_{cr,w}} \mathcal{F}_1 \right] + \frac{N_{cr,l}}{D2} \left[-4 A_1^3 B_1 - A_1^4 + N_{cr,l} \left(A_2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \text{Bi}_{\text{fs}} (\mathcal{T} - \theta_1) - \frac{A_1 \partial_{xx} h}{2} - \frac{A_2 \partial_{xx} h}{2} - \frac{B_1 \partial_{xx} h}{2} - \frac{A_1}{h} - \frac{A_2}{h} - \frac{B_1}{h} \\
& - \frac{h \partial_{xx} \theta_1}{2} + \frac{\theta_1 \partial_{xx} h}{2} + \frac{\theta_1}{h} + \partial_x \theta_1 \partial_x h - \frac{(\partial_x h)^2 \theta_1}{h} + \frac{3 A_1 \text{Pr} \partial_t h}{2} \\
& + \frac{3 A_2 \text{Pr} \partial_t h}{2} + \frac{3 B_1 \text{Pr} \partial_t h}{2} + \frac{9 A_1 \text{Pr} \partial_x q}{8} + \frac{9 A_2 \text{Pr} \partial_x q}{8} + \frac{9 B_1 \text{Pr} \partial_x q}{8} \\
& - \frac{3 \text{Pr} \theta_1 \partial_t h}{2} - \frac{9 \text{Pr} \theta_1 \partial_x q}{8} + \frac{15 \text{Pr} q \partial_x \theta_1}{8} + \frac{A_1 (\partial_x h)^2}{h} + \frac{A_2 (\partial_x h)^2}{h} \\
& + \frac{B_1 (\partial_x h)^2}{h} + \frac{15 A_2 \text{Pr} q \partial_x h}{8} - \frac{9 A_2 \text{Pr} h \partial_x q}{8} \Big) + \theta_1^4 \\
& \left. + \frac{4 A_1^3 \mathcal{F}_1}{4 A_1^3 + \text{Bi}_w N_{cr,w}} \right] \Big\}, \tag{4.34}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_0 = & \frac{3 N_{cr,l} \text{Pr} h^2}{2 \left(N_{cr,l} - \frac{4 A_1^3 N_{cr,l}}{4 A_1^3 + \text{Bi}_w N_{cr,w}} \right)} - \frac{\text{Pr} h^2}{2} \\
& + \frac{3 N_{cr,l}^2 \text{Pr} h}{2 \left(4 A_1^3 + \text{Bi}_w N_{cr,w} \right) \left(N_{cr,l} - \frac{4 A_1^3 N_{cr,l}}{4 A_1^3 + \text{Bi}_w N_{cr,w}} \right)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_1 = & \frac{4 A_1^3 B_1 - Q_s + A_1^4 - \theta_1^4 - A_2 N_{cr,l} + A_1 \text{Bi}_w N_{cr,w}}{4 A_1^3 + \text{Bi}_w N_{cr,w}} \\
& + \frac{B_1 \text{Bi}_w N_{cr,w} - \text{Bi}_w N_{cr,w} \mathcal{T} + \frac{A_1 N_{cr,l}}{h} + \frac{A_2 N_{cr,l}}{h} + \frac{B_1 N_{cr,l}}{h} - \frac{N_{cr,l} \theta_1}{h}}{4 A_1^3 + \text{Bi}_w N_{cr,w}},
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{Q_s(1 + \text{Bi}_{fs})}{N_{cr,w} \text{Bi}_w(1 + \text{Bi}_{fs}) + N_{cr,l} \text{Bi}_{fs}} + \mathcal{T}, \\
A_2 &= - \frac{Q_s \text{Bi}_{fs}}{N_{cr,w} \text{Bi}_w(1 + \text{Bi}_{fs}) + N_{cr,l} \text{Bi}_{fs}}, \\
B_1 &= - \frac{N_{cr,l} \text{Bi}_{fs} \mathcal{T}}{N_{cr,w} \text{Bi}_w}, \\
B_2 &= - \frac{N_{cr,l} \text{Bi}_{fs} \mathcal{T}}{N_{cr,w} \text{Bi}_w} \frac{4 \bar{T}_{Nu,0|y=0} - 4 \bar{T}_{Nu,0|y=1} + N_{cr,w} \text{Bi}_w}{4 \bar{T}_{Nu,0|y=1} h(x, t) + N_{cr,l}}, \\
D1 &= N_{cr,l} - \frac{4 A_1^3 N_{cr,l}}{4 A_1^3 + \text{Bi}_w N_{cr,w}},
\end{aligned}$$

$$D2 = (4 A_1^3 + Bi_w N_{cr,w}) \left(N_{cr,l} - \frac{4 A_1^3 N_{cr,l}}{4 A_1^3 + Bi_w N_{cr,w}} \right).$$

4.2.7.2 Two-variable model

Here, we propose a variable θ_0 which represents the liquid's temperature at the solid wall such that $\theta_0 = T|_{y=0}$, which is coupled to a second variable representing temperature at the liquid interface, which is denoted $\theta_1 = T_{fs}$. Consequently, the ansatz for T_0 is written as

$$T^{(0)} = T_{Nu} + (\theta_0 - T_{Nu|y=0}) P_0(y) + (\theta_1 - T_{Nu|fs}) P_1(y). \quad (4.35a)$$

The polynomials required to make this ansatz consistent are

$$P_0 = -\frac{y}{h} + 1, \quad P_1 = \frac{y}{h}. \quad (4.35b)$$

Moreover, the following two gauge are required to close the system:

$$T_{|y=0}^{(1)} = 0, \quad T_{|fs}^{(1)} = 0. \quad (4.35c)$$

The coupling which appears in the boundary conditions between the liquid temperature at the solid wall and the liquid interface through the radiative exchange is reflected in this here, as the two gauge conditions in Eqs. (4.35c) produce two coupled expressions. The evolution equation for the interfacial temperature variable θ_1 reads

$$\begin{aligned} \partial_t \theta_1 = & \frac{A_2}{D1} + \frac{\mathcal{T}}{D1} - \frac{\theta_1}{D1} + \frac{Q_s}{D2} - \frac{2(\partial_x h)^2 \theta_0}{D3} + \frac{2(\partial_x h)^2 \theta_1}{D3} \\ & + \frac{h^2 \partial_{xx} \theta_1}{D3} + \frac{h^2 \partial_{xx} \theta_0}{D4} - \frac{A_2 h}{D1} + \frac{h \theta_0^4}{D7} - \frac{h \theta_1^4}{D7} - \frac{2 A_2 (\partial_x h)^2}{D3} \\ & - \frac{N_{cr,l} (\partial_x h)^2 \theta_0}{D6} + \frac{N_{cr,l} (\partial_x h)^2 \theta_1}{D6} + \frac{Bi_{fs} \mathcal{T} h}{D1} - \frac{Bi_{fs} h \theta_1}{D1} \\ & + \frac{N_{cr,l} \partial_x \theta_0 \partial_x h}{D2} - \frac{N_{cr,l} \partial_x \theta_1 \partial_x h}{D2} + \frac{Bi_{fs} N_{cr,l} \mathcal{T}}{D2} - \frac{Bi_{fs} N_{cr,l} \theta_1}{D2} \\ & - \frac{Pr h^2 \partial_t \theta_0}{D8} + \frac{A_2 h \partial_{xx} h}{D3} + \frac{h \theta_0 \partial_{xx} h}{D3} - \frac{h \theta_1 \partial_{xx} h}{D3} + \frac{A_2 N_{cr,l} \partial_{xx} h}{D5} \end{aligned}$$

$$\begin{aligned}
& + \frac{N_{cr,l} h \partial_{xx} \theta_0}{D5} + \frac{N_{cr,l} \theta_0 \partial_{xx} h}{D5} + \frac{N_{cr,l} h \partial_{xx} \theta_1}{D5} - \frac{N_{cr,l} \theta_1 \partial_{xx} h}{D5} \\
& - \frac{A_2 N_{cr,l} (\partial_x h)^2}{D6} + \frac{2 \partial_x \theta_0 h \partial_x h}{D3} - \frac{2 \partial_x \theta_1 h \partial_x h}{D3} - \frac{A_2 \text{Pr} h \partial_t h}{D1} \\
& - \frac{33 A_2 \text{Pr} h \partial_x q}{D9} - \frac{\text{Pr} h \theta_0 \partial_t h}{D1} + \frac{\text{Pr} h \theta_1 \partial_t h}{D1} - \frac{27 \text{Pr} h q \partial_x \theta_1}{D10} \\
& - \frac{33 \text{Pr} h \theta_0 \partial_x q}{D9} - \frac{21 \text{Pr} h q \partial_x \theta_0}{D9} + \frac{33 \text{Pr} h \theta_1 \partial_x q}{D9} \\
& - \frac{3 A_2 N_{cr,l} \text{Pr} \partial_t h}{D5} - \frac{9 A_2 N_{cr,l} \text{Pr} \partial_x q}{D11} - \frac{3 N_{cr,l} \text{Pr} \theta_0 \partial_t h}{D5} \\
& - \frac{3 N_{cr,l} \text{Pr} h \partial_t \theta_0}{D5} + \frac{3 N_{cr,l} \text{Pr} \theta_1 \partial_t h}{D5} - \frac{9 N_{cr,l} \text{Pr} \theta_0 \partial_x q}{D11} \\
& - \frac{9 N_{cr,l} \text{Pr} q \partial_x \theta_0}{D11} + \frac{9 N_{cr,l} \text{Pr} \theta_1 \partial_x q}{D11} - \frac{15 N_{cr,l} \text{Pr} q \partial_x \theta_1}{D11} \\
& + \frac{33 A_2 \text{Pr} h^2 \partial_x q}{D9} - \frac{27 A_2 \text{Pr} h q \partial_x h}{D10} - \frac{15 A_2 N_{cr,l} \text{Pr} q \partial_x h}{D11} \\
& + \frac{9 A_2 N_{cr,l} \text{Pr} h \partial_x q}{D11}, \quad (4.36a)
\end{aligned}$$

where

$$\begin{aligned}
D1 &= \text{Pr} h^2 + \frac{3 N_{cr,l} \text{Pr} h}{2 \text{Bi}_w N_{cr,w}}, & D2 &= \text{Bi}_w N_{cr,w} \text{Pr} h^2 + \frac{3 N_{cr,l} \text{Pr} h}{2}, \\
D3 &= 3 \text{Pr} h^2 + \frac{9 N_{cr,l} \text{Pr} h}{2 \text{Bi}_w N_{cr,w}}, & D4 &= 6 \text{Pr} h^2 + \frac{9 N_{cr,l} \text{Pr} h}{\text{Bi}_w N_{cr,w}}, \\
D5 &= 2 \text{Bi}_w N_{cr,w} \text{Pr} h^2 + 3 N_{cr,l} \text{Pr} h, & D6 &= \text{Bi}_w N_{cr,w} \text{Pr} h^3 + \frac{3 N_{cr,l} \text{Pr} h^2}{2}, \\
D7 &= N_{cr,l} \text{Pr} h^2 + \frac{3 N_{cr,l}^2 \text{Pr} h}{2 \text{Bi}_w N_{cr,w}}, & D8 &= 2 \text{Pr} h^2 + \frac{3 N_{cr,l} \text{Pr} h}{\text{Bi}_w N_{cr,w}}, \\
D9 &= 40 \text{Pr} h^2 + \frac{60 N_{cr,l} \text{Pr} h}{\text{Bi}_w N_{cr,w}}, & D10 &= 20 \text{Pr} h^2 + \frac{30 N_{cr,l} \text{Pr} h}{\text{Bi}_w N_{cr,w}}, \\
D11 &= 8 \text{Bi}_w N_{cr,w} \text{Pr} h^2 + 12 N_{cr,l} \text{Pr} h.
\end{aligned}$$

Meanwhile, the equation governing the liquid temperature at the wall θ_0 is

$$\begin{aligned}
\partial_t \theta_0 &= \frac{\partial_{xx} \theta_0}{3 \text{Pr}} - \partial_t \theta_1 + \frac{\partial_{xx} \theta_1}{3 \text{Pr}} + \frac{3 A_2 \partial_x q}{4} - \frac{A_2 \partial_t h}{h} - \frac{3 A_2 \partial_x q}{4 h} - \frac{\theta_0 \partial_t h}{h} + \frac{\theta_1 \partial_t h}{h} \\
& - \frac{3 \theta_0 \partial_x q}{4 h} - \frac{3 q \partial_x \theta_0}{4 h} + \frac{3 \theta_1 \partial_x q}{4 h} - \frac{5 q \partial_x \theta_1}{4 h} + \frac{2 \partial_x \theta_0 \partial_x h}{3 \text{Pr} h} - \frac{2 \partial_x \theta_1 \partial_x h}{3 \text{Pr} h}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2 \text{Bi}_{\text{fs}} \mathcal{T}}{3 \text{Pr} h} - \frac{2 \text{Bi}_{\text{fs}} \theta_1}{3 \text{Pr} h} - \frac{2 A_2 (\partial_x h)^2}{3 \text{Pr} h^2} + \frac{A_2 \partial_{xx} h}{3 \text{Pr} h} - \frac{2 (\partial_x h)^2 \theta_0}{3 \text{Pr} h^2} \\
& + \frac{2 (\partial_x h)^2 \theta_1}{3 \text{Pr} h^2} + \frac{2 (\partial_x h)^2 \theta_1}{3 \text{Pr} h^2} + \frac{2 Q_s}{3 N_{cr,l} \text{Pr} h} + \frac{\theta_0 \partial_{xx} h}{3 \text{Pr} h} \\
& - \frac{\theta_1 \partial_{xx} h}{3 \text{Pr} h} - \frac{5 A_2 q \partial_x h}{4 h} + \frac{2 \text{Bi}_w N_{cr,w} \mathcal{T}}{3 N_{cr,l} \text{Pr} h} - \frac{2 \text{Bi}_w N_{cr,w} \theta_0}{3 N_{cr,l} \text{Pr} h}
\end{aligned} \tag{4.36b}$$

4.2.7.3 Four-variable model

Under this formulation we propose two supplementary variables derived from the liquid's temperatures at the solid wall and free interface, bringing the total number of parameters introduced to four. The ansatz for $T^{(0)}$ now becomes

$$\begin{aligned}
T^{(0)} = T_{Nu} + (\theta_0 - T_{Nu|y=0}) P_0(y) + (\theta_1 - T_{Nu|fs}) P_1(y) \\
+ \phi_0 P_2(y) + \phi_1 P_3(y), \tag{4.37a}
\end{aligned}$$

where $\phi_0 = \frac{1}{h^2} \partial_{yy} T|_{y=0}$ and $\phi_1 = \frac{1}{h^2} \partial_{yy} T|_{y=h}$. The factor $1/h^2$ is added to ensure the consistency of the ansatz's units. These two additional variables incorporate into the modeling additional information about the diffusion of heat in the regions close to the free surface and the solid wall, and help capture higher order effects. The polynomials that make Eq. (4.37a) consistent are

$$\begin{aligned}
P_0 &= -\frac{y}{h} + 1, & P_1 &= \frac{y}{h}, \\
P_2 &= -\frac{1}{6} \left(\frac{y}{h}\right)^3 + \frac{1}{2} \left(\frac{y}{h}\right)^2 - \frac{1}{3} \left(\frac{y}{h}\right), & P_3 &= \frac{1}{6} \left(\frac{y}{h}\right)^3 + \frac{1}{5} \left(\frac{y}{h}\right).
\end{aligned} \tag{4.37b}$$

Meanwhile the required gauge conditions are:

$$T_{|y=0}^{(1)} = 0, \quad T_{|fs}^{(1)} = 0, \quad \partial_{yy} T_{|y=0}^{(1)} = 0, \quad \partial_{yy} T_{|fs}^{(1)} = 0. \tag{4.37c}$$

The system of equations (4.37) yields the following four governing equations:

$$\partial_t \theta_0 = \frac{\phi_0}{3 \text{Pr} h^2} + \frac{\partial_{xx} \theta_0}{3 \text{Pr}} \tag{4.38a}$$

$$\begin{aligned}
\partial_t \theta_1 = & \frac{\partial_{xx} \theta_1}{3 \text{Pr}} + A_2 \partial_x q + \frac{\phi_1}{3 \text{Pr} h^2} - \frac{A_2 \partial_t h}{h} - \frac{A_2 \partial_x q}{h} + \frac{\phi_0 \partial_t h}{6 h} + \frac{\phi_1 \partial_t h}{3 h} - \frac{\theta_0 \partial_t h}{h} \\
& + \frac{\theta_1 \partial_t h}{h} + \frac{\phi_0 \partial_x q}{6 h} + \frac{\phi_1 \partial_x q}{3 h} - \frac{\theta_0 \partial_x q}{h} + \frac{\theta_1 \partial_x q}{h} - \frac{3 q \partial_x \theta_1}{2 h} - \frac{\partial_x \phi_0 \partial_x h}{9 \text{Pr} h} \\
& - \frac{2 \partial_x \phi_1 \partial_x h}{9 \text{Pr} h} + \frac{2 \partial_x \theta_0 \partial_x h}{3 \text{Pr} h} - \frac{2 \partial_x \theta_1 \partial_x h}{3 \text{Pr} h} - \frac{2 A_2 (\partial_x h)^2}{3 \text{Pr} h^2} + \frac{A_2 \partial_{xx} h}{3 \text{Pr} h} \\
& + \frac{(\partial_x h)^2 \phi_0}{9 \text{Pr} h^2} + \frac{5 (\partial_x h)^2 \phi_1}{9 \text{Pr} h^2} - \frac{2 (\partial_x h)^2 \theta_0}{3 \text{Pr} h^2} + \frac{2 (\partial_x h)^2 \theta_1}{3 \text{Pr} h^2} - \frac{\phi_0 \partial_{xx} h}{18 \text{Pr} h} \\
& - \frac{\phi_1 \partial_{xx} h}{9 \text{Pr} h} + \frac{\theta_0 \partial_{xx} h}{3 \text{Pr} h} - \frac{\theta_1 \partial_{xx} h}{3 \text{Pr} h} - \frac{3 A_2 q \partial_x h}{2 h}
\end{aligned} \tag{4.38b}$$

$$\begin{aligned}
\partial_t \phi_0 = & 12 \partial_t \theta_0 - \partial_t \phi_1 + 12 \partial_t \theta_1 + \frac{\partial_{xx} \phi_0}{3 \text{Pr}} + \frac{\partial_{xx} \phi_1}{3 \text{Pr}} - \frac{4 \partial_{xx} \theta_0}{\text{Pr}} - \frac{4 \partial_{xx} \theta_1}{\text{Pr}} \\
& - 9 A_2 \partial_x q + \frac{12 A_2 \partial_t h}{h} + \frac{9 A_2 \partial_x q}{h} - \frac{\phi_0 \partial_t h}{h} - \frac{\phi_1 \partial_t h}{h} + \frac{12 \theta_0 \partial_t h}{h} \\
& - \frac{12 \theta_1 \partial_t h}{h} - \frac{q \partial_x \phi_0}{h} - \frac{\phi_0 \partial_x q}{h} - \frac{11 q \partial_x \phi_1}{10 h} - \frac{11 \phi_1 \partial_x q}{10 h} + \frac{9 \theta_0 \partial_x q}{h} \\
& + \frac{9 q \partial_x \theta_0}{h} - \frac{9 \theta_1 \partial_x q}{h} + \frac{15 q \partial_x \theta_1}{h} + \frac{2 \partial_x \phi_0 \partial_x h}{3 \text{Pr} h} + \frac{2 \partial_x \phi_1 \partial_x h}{3 \text{Pr} h} \\
& - \frac{8 \partial_x \theta_0 \partial_x h}{\text{Pr} h} + \frac{8 \partial_x \theta_1 \partial_x h}{\text{Pr} h} - \frac{8 \text{Bi}_{fs} \mathcal{T}}{\text{Pr} h} + \frac{8 \text{Bi}_{fs} \theta_1}{\text{Pr} h} + \frac{8 A_2 (\partial_x h)^2}{\text{Pr} h^2} \\
& - \frac{4 A_2 \partial_{xx} h}{\text{Pr} h} - \frac{4 (\partial_x h)^2 \phi_0}{3 \text{Pr} h^2} - \frac{8 (\partial_x h)^2 \phi_1}{3 \text{Pr} h^2} + \frac{8 (\partial_x h)^2 \theta_0}{\text{Pr} h^2} - \frac{8 (\partial_x h)^2 \theta_1}{\text{Pr} h^2} \\
& - \frac{8 Q_s}{N_{cr,l} \text{Pr} h} + \frac{\phi_0 \partial_{xx} h}{3 \text{Pr} h} + \frac{\phi_1 \partial_{xx} h}{3 \text{Pr} h} - \frac{4 \theta_0 \partial_{xx} h}{\text{Pr} h} + \frac{4 \theta_1 \partial_{xx} h}{\text{Pr} h} + \frac{15 A_2 q \partial_x h}{h} \\
& - \frac{8 \text{Bi}_w N_{cr,w} \mathcal{T}}{N_{cr,l} \text{Pr} h} + \frac{8 \text{Bi}_w N_{cr,w} \theta_0}{N_{cr,l} \text{Pr} h}
\end{aligned} \tag{4.38c}$$

$$\begin{aligned}
\partial_t \phi_1 = & -\frac{1}{D1} \left\{ -\frac{A_2}{D1} - \frac{\mathcal{T}}{D1} + \frac{\theta_1}{D1} - \frac{Q_s}{D2} - \frac{13 (\partial_x h)^2 \phi_1(x, t)}{D3} - \frac{23 (\partial_x h)^2 \phi_0}{D4} \right. \\
& + \frac{2 (\partial_x h)^2 \theta_0}{D5} - \frac{2 (\partial_x h)^2 \theta_1}{D5} + \frac{h^2 \partial_{xx} \phi_1(x, t)}{D3} + \frac{7 h^2 \partial_{xx} \phi_0}{D6} - \frac{h^2 \partial_{xx} \theta_0}{D7} \\
& - \frac{h^2 \partial_{xx} \theta_1}{D5} + \frac{A_2 h}{D1} - \frac{h \theta_0^4}{D8} + \frac{h \theta_1^4}{D8} + \frac{2 A_2 (\partial_x h)^2}{D5} + \frac{N_{cr,l} (\partial_x h)^2 \theta_0}{D9} \\
& - \frac{N_{cr,l} (\partial_x h)^2 \theta_1}{D9} - \frac{\text{Bi}_{fs} \mathcal{T} h}{D1} + \frac{\text{Bi}_{fs} h \theta_1}{D1} + \frac{N_{cr,l} \partial_x h \partial_x \phi_0}{D10} \\
& \left. + \frac{N_{cr,l} \partial_x h \partial_x \phi_1(x, t)}{D10} - \frac{N_{cr,l} \partial_x h \partial_x \theta_0}{D2} + \frac{N_{cr,l} \partial_x h \partial_x \theta_1}{D2} \right\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{Bi_{fs} N_{cr,l} \mathcal{T}}{D2} + \frac{Bi_{fs} N_{cr,l} \theta_1}{D2} + \frac{\text{Pr} \partial_t \theta_0 h^2}{D11} + \frac{\text{Pr} \partial_t \theta_1 h^2}{D1} - \frac{A_2 h \partial_{xx} h}{D5} \\
& - \left\{ \frac{7 \text{Pr} h^2}{D12} \left(12 \partial_t \theta_0 + 12 \partial_t \theta_1 + \frac{\partial_{xx} \phi_0}{3 \text{Pr}} + \frac{\partial_{xx} \phi_1(x, t)}{3 \text{Pr}} - \frac{4 \partial_{xx} \theta_0}{\text{Pr}} \right. \right. \\
& - \frac{4 \partial_{xx} \theta_1}{\text{Pr}} - 9 A_2 \partial_x q + \frac{12 A_2 \partial_t h}{h} + \frac{9 A_2 \partial_x q}{h} - \frac{\phi_0 \partial_t h}{h} - \frac{\phi_1(x, t) \partial_t h}{h} \\
& + \frac{12 \theta_0 \partial_t h}{h} - \frac{12 \theta_1 \partial_t h}{h} - \frac{q \partial_x \phi_0}{h} - \frac{\phi_0 \partial_x q}{h} - \frac{11 q \partial_x \phi_1(x, t)}{10 h} \\
& - \frac{11 \phi_1(x, t) \partial_x q}{10 h} + \frac{9 \theta_0 \partial_x q}{h} + \frac{9 q \partial_x \theta_0}{h} - \frac{9 \theta_1 \partial_x q}{h} + \frac{15 q \partial_x \theta_1}{h} \\
& + \frac{2 \partial_x \phi_0 \partial_x h}{3 \text{Pr} h} + \frac{2 \partial_x \phi_1 \partial_x h}{3 \text{Pr} h} - \frac{8 \partial_x \theta_0 \partial_x h}{\text{Pr} h} + \frac{8 Bi_{fs} \theta_1}{\text{Pr} h} + \frac{8 A_2 (\partial_x h)^2}{\text{Pr} h^2} \\
& - \frac{4 A_2 \partial_{xx} h}{\text{Pr} h} - \frac{4 (\partial_x h)^2 \phi_0}{3 \text{Pr} h^2} - \frac{8 (\partial_x h)^2 \phi_1(x, t)}{3 \text{Pr} h^2} + \frac{8 (\partial_x h)^2 \theta_0}{\text{Pr} h^2} \\
& - \frac{8 (\partial_x h)^2 \theta_1}{\text{Pr} h^2} - \frac{8 Q_s}{N_{cr,l} \text{Pr} h} + \frac{\phi_0 \partial_{xx} h}{3 \text{Pr} h} + \frac{\phi_1(x, t) \partial_{xx} h}{3 \text{Pr} h} - \frac{4 \theta_0 \partial_{xx} h}{\text{Pr} h} \\
& + \frac{4 \theta_1 \partial_{xx} h}{\text{Pr} h} + \frac{15 A_2 q \partial_x h}{h} - \frac{8 Bi_w N_{cr,w} \mathcal{T}}{N_{cr,l} \text{Pr} h} + \frac{8 Bi_w N_{cr,w} \theta_0}{N_{cr,l} \text{Pr} h} \left. \right) \Bigg\} \\
& + \frac{2 h \phi_1(x, t) \partial_{xx} h}{D3} + \frac{7 h \phi_0 \partial_{xx} h}{D4} - \frac{h \theta_0 \partial_{xx} h}{D5} + \frac{h \theta_1 \partial_{xx} h}{D5} \\
& - \frac{A_2 N_{cr,l} \partial_{xx} h}{D13} + \frac{N_{cr,l} h \partial_{xx} \phi_0}{D14} + \frac{N_{cr,l} \phi_0 \partial_{xx} h}{D14} + \frac{N_{cr,l} h \partial_{xx} \phi_1(x, t)}{D14} \\
& + \frac{N_{cr,l} \phi_1(x, t) \partial_{xx} h}{D14} - \frac{N_{cr,l} h \partial_{xx} \theta_0}{D13} - \frac{N_{cr,l} \theta_0 \partial_{xx} h}{D13} - \frac{N_{cr,l} h \partial_{xx} \theta_1}{D13} \\
& + \frac{N_{cr,l} \theta_1 \partial_{xx} h}{D13} + \frac{7 \partial_x h \partial_x \phi_0 h}{D15} + \frac{4 \partial_x h \partial_x \phi_1(x, t) h}{D3} + \frac{A_2 N_{cr,l} (\partial_x h)^2}{D9} \\
& - \frac{2 \partial_x h \partial_x \theta_0 h}{D5} + \frac{2 \partial_x h \partial_x \theta_1 h}{D5} - \frac{N_{cr,l} (\partial_x h)^2 \phi_0}{D16} - \frac{N_{cr,l} (\partial_x h)^2 \phi_1}{D17} \\
& + \frac{A_2 \text{Pr} \partial_t h h}{D1} + \frac{33 A_2 \text{Pr} \partial_x q h}{D18} - \frac{7 \text{Pr} \partial_t h h \phi_0}{D19} - \frac{2 \text{Pr} \partial_t h h \phi_1}{D20} \\
& + \frac{\text{Pr} \partial_t h h \theta_0}{D1} - \frac{\text{Pr} \partial_t h h \theta_1}{D1} - \frac{3 \text{Pr} \partial_x q h \phi_0}{D21} - \frac{19 \text{Pr} \partial_x \phi_0 h q}{D22} \\
& - \frac{23 \text{Pr} \partial_x \phi_1(x, t) h q}{D22} - \frac{73 \text{Pr} \partial_x q h \phi_1(x, t)}{D23} + \frac{27 \text{Pr} \partial_x \theta_1 h q}{D24} \\
& + \frac{33 \text{Pr} \partial_x q h \theta_0}{D18} + \frac{21 \text{Pr} \partial_x \theta_0 h q}{D18} - \frac{33 \text{Pr} \partial_x q h \theta_1}{D18} + \frac{3 A_2 N_{cr,l} \text{Pr} \partial_t h}{D13} \\
& + \frac{9 A_2 N_{cr,l} \text{Pr} \partial_x q}{D25} - \frac{N_{cr,l} \text{Pr} \partial_t h \phi_0}{D25} - \frac{N_{cr,l} \text{Pr} \partial_t h \phi_1(x, t)}{D25} \\
& + \frac{3 N_{cr,l} \text{Pr} \partial_t h \theta_0}{D13} + \frac{3 N_{cr,l} \text{Pr} \partial_t \theta_0 h}{D13} - \frac{3 N_{cr,l} \text{Pr} \partial_t h \theta_1}{D13}
\end{aligned}$$

$$\begin{aligned}
& + \frac{3 N_{cr,l} \text{Pr} \partial_t \theta_1 h}{D13} - \frac{N_{cr,l} \text{Pr} \partial_x \phi_0 q}{D25} - \frac{N_{cr,l} \text{Pr} \partial_x q \phi_0}{D25} \\
& - \frac{11 N_{cr,l} \text{Pr} \partial_x \phi_1(x, t) q}{D26} - \frac{11 N_{cr,l} \text{Pr} \partial_x q \phi_1(x, t)}{D26} + \frac{9 N_{cr,l} \text{Pr} \partial_x q \theta_0}{D25} \\
& + \frac{9 N_{cr,l} \text{Pr} \partial_x \theta_0 q}{D25} - \frac{9 N_{cr,l} \text{Pr} \partial_x q \theta_1}{D25} + \frac{15 N_{cr,l} \text{Pr} \partial_x \theta_1 q}{D25} \\
& - \frac{N_{cr,l}}{D25} \text{Pr} h \left(12 \partial_t \theta_0 + 12 \partial_t \theta_1 + \frac{\partial_{xx} \phi_0}{3 \text{Pr}} + \frac{\partial_{xx} \phi_1(x, t)}{3 \text{Pr}} - \frac{4 \partial_{xx} \theta_0}{\text{Pr}} \right. \\
& - \frac{4 \partial_{xx} \theta_1}{\text{Pr}} - 9 A_2 \partial_x q + \frac{12 A_2 \partial_t h}{h} + \frac{9 A_2 \partial_x q}{h} - \frac{\phi_0 \partial_t h}{h} - \frac{\phi_1(x, t) \partial_t h}{h} \\
& + \frac{12 \theta_0 \partial_t h}{h} - \frac{12 \theta_1 \partial_t h}{h} - \frac{q \partial_x \phi_0}{h} - \frac{\phi_0 \partial_x q}{h} - \frac{11 q \partial_x \phi_1(x, t)}{10 h} \\
& - \frac{11 \phi_1(x, t) \partial_x q}{10 h} + \frac{9 \theta_0 \partial_x q}{h} + \frac{9 q \partial_x \theta_0}{h} - \frac{9 \theta_1 \partial_x q}{h} + \frac{15 q \partial_x \theta_1}{h} \\
& + \frac{2 \partial_x \phi_0 \partial_x h}{3 \text{Pr} h} + \frac{2 \partial_x \phi_1(x, t) \partial_x h}{3 \text{Pr} h} - \frac{8 \partial_x \theta_0 \partial_x h}{\text{Pr} h} + \frac{8 \partial_x \theta_1 \partial_x h}{\text{Pr} h} - \frac{8 Bi_{fs} \mathcal{T}}{\text{Pr} h} \\
& + \frac{8 Bi_{fs} \theta_1}{\text{Pr} h} + \frac{8 A_2 (\partial_x h)^2}{\text{Pr} h^2} - \frac{4 A_2 \partial_{xx} h}{\text{Pr} h} - \frac{4 (\partial_x h)^2 \phi_0}{3 \text{Pr} h^2} - \frac{8 (\partial_x h)^2 \phi_1(x, t)}{3 \text{Pr} h^2} \\
& + \frac{8 (\partial_x h)^2 \theta_0}{\text{Pr} h^2} - \frac{8 (\partial_x h)^2 \theta_1}{\text{Pr} h^2} - \frac{8 Q_s}{N_{cr,l} \text{Pr} h} + \frac{\phi_0 \partial_{xx} h}{3 \text{Pr} h} + \frac{\phi_1(x, t) \partial_{xx} h}{3 \text{Pr} h} \\
& - \frac{4 \theta_0 \partial_{xx} h}{\text{Pr} h} + \frac{4 \theta_1 \partial_{xx} h}{\text{Pr} h} + \frac{15 A_2 q \partial_x h}{h} - \frac{8 Bi_w N_{cr,w} \mathcal{T}}{N_{cr,l} \text{Pr} h} \\
& + \frac{8 Bi_w N_{cr,w} \theta_0}{N_{cr,l} \text{Pr} h} \left. \right) - \frac{33 A_2 \text{Pr} \partial_x q h^2}{D18} + \frac{27 A_2 \text{Pr} \partial_x h h q}{D24} \\
& + \frac{15 A_2 N_{cr,l} \text{Pr} \partial_x h q}{D25} - \frac{9 A_2 N_{cr,l} \text{Pr} \partial_x q h}{D25} \Bigg\} \tag{4.38d}
\end{aligned}$$

With

$$\begin{aligned}
D1 &= \frac{\text{Pr} h^2}{15} + \frac{N_{cr,l} \text{Pr} h}{8 Bi_w N_{cr,w}}, & D2 &= \frac{Bi_w N_{cr,w} \text{Pr} h^2}{15} + \frac{N_{cr,l} \text{Pr} h}{8}, \\
D3 &= 3 \text{Pr} h^2 + \frac{45 N_{cr,l} \text{Pr} h}{8 Bi_w N_{cr,w}}, & D4 &= 12 \text{Pr} h^2 + \frac{45 N_{cr,l} \text{Pr} h}{2 Bi_w N_{cr,w}}, \\
D5 &= \frac{\text{Pr} h^2}{5} + \frac{3 N_{cr,l} \text{Pr} h}{8 Bi_w N_{cr,w}}, & D6 &= 24 \text{Pr} h^2 + \frac{45 N_{cr,l} \text{Pr} h}{Bi_w N_{cr,w}}, \\
D7 &= \frac{2 \text{Pr} h^2}{5} + \frac{3 N_{cr,l} \text{Pr} h}{4 Bi_w N_{cr,w}}, & D8 &= \frac{N_{cr,l} \text{Pr} h^2}{15} + \frac{N_{cr,l}^2 \text{Pr} h}{8 Bi_w N_{cr,w}}, \\
D9 &= \frac{Bi_w N_{cr,w} \text{Pr} h^3}{15} + \frac{N_{cr,l} \text{Pr} h^2}{8}, & D10 &= \frac{4 Bi_w N_{cr,w} \text{Pr} h^2}{5} + \frac{3 N_{cr,l} \text{Pr} h}{2},
\end{aligned}$$

$$\begin{aligned}
D11 &= \frac{2 \Pr h^2}{15} + \frac{N_{cr,l} \Pr h}{4 Bi_w N_{cr,w}}, & D12 &= 8 \Pr h^2 + \frac{15 N_{cr,l} \Pr h}{Bi_w N_{cr,w}}, \\
D13 &= \frac{2 Bi_w N_{cr,w} \Pr h^2}{15} + \frac{N_{cr,l} \Pr h}{4}, & D14 &= \frac{8 Bi_w N_{cr,w} \Pr h^2}{5} + 3 N_{cr,l} \Pr h, \\
D15 &= 6 \Pr h^2 + \frac{45 N_{cr,l} \Pr h}{4 Bi_w N_{cr,w}}, & D16 &= \frac{2 Bi_w N_{cr,w} \Pr h^3}{5} + \frac{3 N_{cr,l} \Pr h^2}{4}, \\
D17 &= \frac{Bi_w N_{cr,w} \Pr h^3}{5} + \frac{3 N_{cr,l} \Pr h^2}{8}, & D18 &= \frac{8 \Pr h^2}{3} + \frac{5 N_{cr,l} \Pr h}{Bi_w N_{cr,w}}, \\
D19 &= 4 \Pr h^2 + \frac{15 N_{cr,l} \Pr h}{2 Bi_w N_{cr,w}}, & D20 &= \Pr h^2 + \frac{15 N_{cr,l} \Pr h}{8 Bi_w N_{cr,w}}, \\
D21 &= \frac{28 \Pr h^2}{15} + \frac{7 N_{cr,l} \Pr h}{2 Bi_w N_{cr,w}}, & D22 &= \frac{56 \Pr h^2}{3} + \frac{35 N_{cr,l} \Pr h}{Bi_w N_{cr,w}}, \\
D23 &= \frac{112 \Pr h^2}{3} + \frac{70 N_{cr,l} \Pr h}{Bi_w N_{cr,w}}, & D24 &= \frac{4 \Pr h^2}{3} + \frac{5 N_{cr,l} \Pr h}{2 Bi_w N_{cr,w}}, \\
D25 &= \frac{8 Bi_w N_{cr,w} \Pr h^2}{15} + N_{cr,l} \Pr h, & D26 &= \frac{16 Bi_w N_{cr,w} \Pr h^2}{3} + 10 N_{cr,l} \Pr h.
\end{aligned}$$

4.3 Numerical simulations

4.3.1 Hydrodynamic evolution

The isothermal hydrodynamic problem represented by Eqs. (4.11) is solved by finite difference using *MATLAB* [119] using forward time-stepping such that

$$h^{j+1} = f(q)^j, \quad (4.39a)$$

$$q^{j+1} = f(q, \partial_x q, h, \partial_x h)^j, \quad (4.39b)$$

where j denotes the current time step. The verification of this numerical solver is demonstrated in Fig. 4.2 where we present its comparison to the LWE numerical algorithm employed in Chaps. 2 and 3. We find that our LWE result validates the WIBL solution as there is no discernible difference between their results during early film evolution. Moreover, we observe the expected finite-time blow-up behavior typical of the LWE in the drag-inertia flow regime, see. Figs. (4.2)(c) and (4.2)(d). The combination of this numerical verification with the analytical

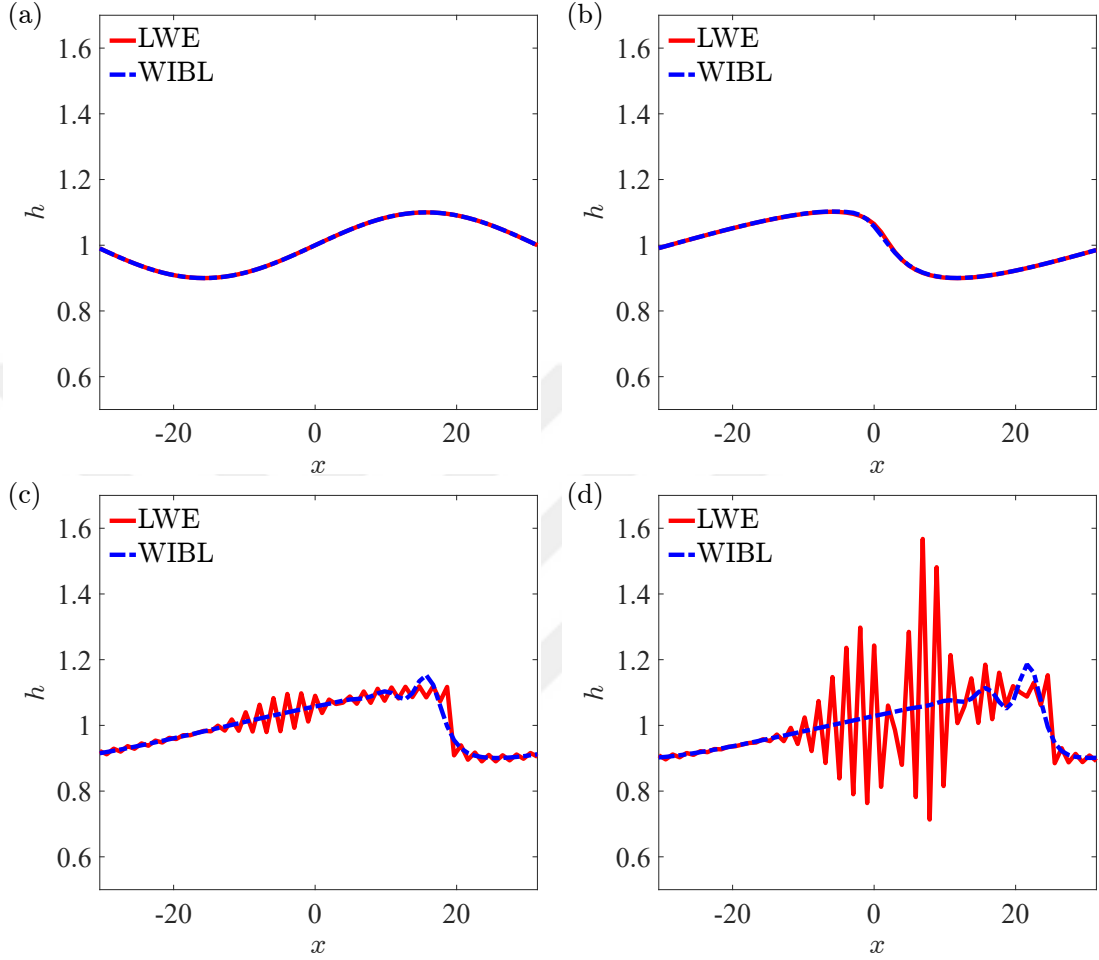


Figure 4.2: Comparison of the film evolution produced by the LWE and WIBL solvers for an initial film defined as $h_0 = 1 + 0.1 \sin(kx)$ for $\epsilon = 0.05$, $k = 0.1$, $S = 30$, $\text{Re} = 20$, $\beta = 25$, $N_x = 64$, and $\Delta t = 1 \times 10^{-4}$. (a) $t = 0$, (b) $t = 4$, (c) $t = 6$, and (d) $t = 6.7$.

corroboration discussed in Sec. 4.1.2.1 establishes our confidence in the hydrodynamic numerical algorithm prior to its coupling to the thermal models developed in Sec. 4.2.7.

4.3.2 Hydro-thermal evolution

Here, we present the results of our numerical solution of the coupled hydro-thermal problem we have developed in Secs. 4.1 and 4.2. Note that we use an

explicit numerical scheme for the temperatures' governing equations, similar to that described by Eqs. (4.39). Specifically, we solve Eq. (4.11) coupled to both the single-variable model [Eq. (4.34)] as well as the two-variable model [Eqs. (4.36)]. The solution of the four-variable model represented by Eqs. (4.38) is deferred to future work.

In order to produce meaningful simulations, we first determine the parameter space in which our approximate Nusselt solution is valid, namely, where the temperature correction Θ_{Nu} remains small and doesn't result in a negative value for T_{Nu} . In general, this occurs when the coefficient

$$\frac{N_{cr,l}Bi_{fs}\mathcal{T}}{N_{cr,w}Bi_w}$$

appearing in Θ_{Nu} becomes large [see Eq. (4.30)], which can occur either when $Bi_{fs} \gg 1$ or $Bi_w \ll 1$. Interestingly, the typical values computed for a water film in Sec. 4.2.4 show that $N_{cr,l}Bi_{fs}\mathcal{T} \ll N_{cr,w}Bi_w$, and therefore this non-physical situation can be generally avoided.

Fig. 4.3 shows the variation in the liquid's Nusselt temperatures at the wall and the liquid interface with various parameters. Fig. 4.3(a) depicts the variation in the Nusselt temperature solution at the solid wall and the free surface with $N_{cr,w}$, where we observe that both temperatures initially decrease with higher values of $N_{cr,w}$, but only up to a certain point, beyond which they both approach constant values. On the other hand, Fig. 4.3(b), depicts the variations in these temperatures with $N_{cr,l}$, where we observe drastically different behavior. Both temperatures decrease with higher values of $N_{cr,l}$, where this variation is initially nonlinear for the interface temperature $T_{Nu|y=1}$ before becoming linear. Notably, the decrease in the wall temperature with $N_{cr,l}$ is always linear for the parameter range we included. Note that further increases in $N_{cr,l}$ will lead to negative values of T_{Nu} due to the temperature correction Θ_{Nu} becoming too large.

In similar fashion, Fig.4.3(c) presents the change in T_{Nu} both at the wall and the free surface with the interface's Biot number Bi_{fs} . When $Bi_{fs} \ll 1$, we see that the Nusselt temperature values at the wall and free surface converge due

to the elimination of the temperature gradient across the film, which is what is expected. However, we also observe that both temperatures decrease with higher values of Bi_{fs} , which can again lead to negative temperatures beyond a certain limit. Fig. 4.3(d), displays similar behavior to Fig. 4.3(a), albeit with Bi_w , where we see an initial decrease in both $T_{Nu|y=0}$ and $T_{Nu|y=1}$ as the value of Bi_w increases, followed by an approach towards constant values.

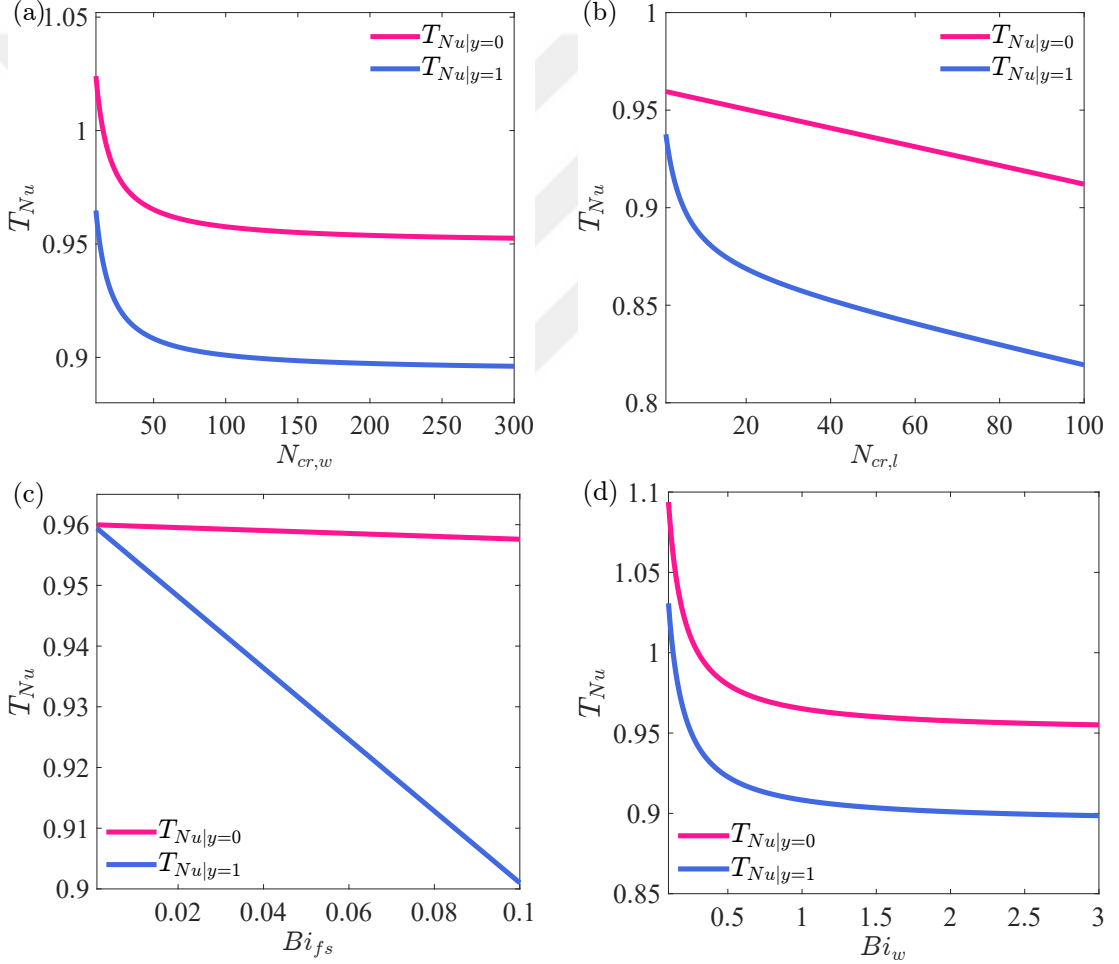


Figure 4.3: Comparison of the variation in the Nusselt temperature solution T_{Nu} with various parameters for $Q_s = 2$, $\mathcal{T} = 0.95$, and (a) $Bi_{fs} = 0.1$, $Bi_w = 2$, $N_{cr,l} = 5$, and different values of $N_{cr,w}$, (b) $Bi_{fs} = 0.1$, $Bi_w = 2$, $N_{cr,w} = 100$, and different values of $N_{cr,l}$, (c) $Bi_w = 0.1$, $N_{cr,l} = 5$, $N_{cr,w} = 100$, and different values of Bi_{fs} , (d) $Bi_w = 0.1$, $N_{cr,l} = 5$, $N_{cr,w} = 100$, and different values of Bi_w .

Now that the admissible parameter range has been established, we can proceed with the numerical simulations. We set our parametric study within the parameter ranges where we observed significant variations in the Nusselt temperature solution in Fig. 4.3.

We first study the effects of the radiative-conductive balance in the heat transfer process by inspecting the temperature field in a developing flow at two distinct values of the wall's conduction-radiation parameter $N_{cr,w}$. Figs. 4.4(a), (b), and (c) present snapshots (at $t = 10, 15$, and 20 , respectively) taken from the temporal evolution of a film where $N_{cr,w} = 5$, signifying that the radiative flux emitted from the solid wall is comparable to the heat flux conducted across it and away from the film. Consequently, we find that the values of θ_1 and θ_0 are relatively high, and that these temperatures react to the interfacial perturbations, which is reflection of the radiative exchange between the solid wall and the liquid interface. In addition, we find a noticeable difference between θ_0 and θ_1 . Contrastingly, Figs. 4.4(d), (e), and (f) depict a film with the same parameters and at the same points in time, albeit with $N_{cr,w} = 50$, indicating a strong propensity of the solid wall to conduct heat away from the liquid film, instead of radiating it towards the free surface. This leads to relatively lower and closer wall and free surface temperatures, which are also quite flat, suggesting that the heat flux across the film is quite weak. Moreover, we observe that the interface temperature computed by the single-variable model (θ_1^*) fares quite well as its profile follows that of its equivalent computed using the two-variable model (θ_1).

Meanwhile, and in similar fashion, Figs. 4.5 (a), (b), and (c) present temporal snapshots of an evolving film taken at $t = 10, 15$, and 20 , where we set $N_{cr,l} = 2$. This represents a liquid film in which a significant part of the heat absorbed at the solid wall is radiated immediately across to the liquid surface, with the conductive exchange being relegated to a secondary role. Consequently, we find that θ_0 and θ_1 settle relatively quickly towards each other where their profiles become relatively flat. In comparison, the frames in Figs. 4.5 (d), (e), and (f) are taken again at the same points in time for the same liquid film, albeit with $N_{cr,l} = 20$. This value of the conduction-radiation parameter signifies a strong degree of heat conduction through the film, which is highly dependent on the

liquid interface's shape. Hence, the temperatures at the wall and the interface depart from one another significantly, and do not show any signs of settling due to the free surface's large deformations. Interestingly, we witness here that the single-variable model does not remain close to the two-variable model.

The analytical models numerical programs developed here still have ample room for a much wider parametric study, however, this is deferred to future work, as the main goal of this effort was the analytical derivation of the models, and their successful programmatic implementation.

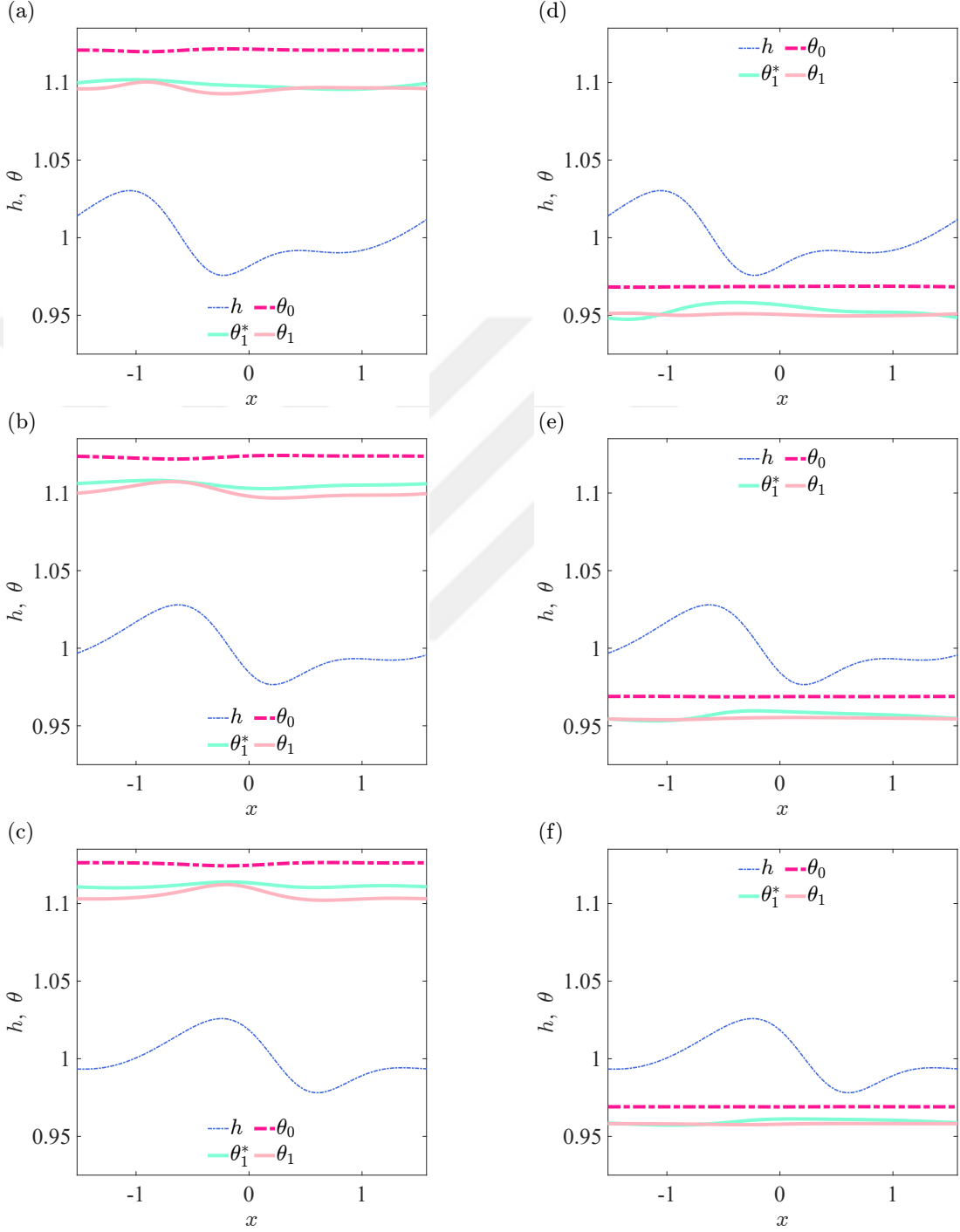


Figure 4.4: Numerical simulations of the liquid film's interface height and temperature field for $\epsilon = 0.05$, $k = 2$, $S = 30$, $\text{Re} = 15$, $\beta = 25$, $\text{Pr} = 50$, $Q_s = 2$, $Bi_{fs} = 0.1$, $Bi_w = 2$, $N_{cr,l} = 2$, $\mathcal{T} = 0.95$. For (a), (b), and (c) $N_{cr,w} = 5$, and $t = 10, 15$, and 20 . For (d), (e), and (f) $N_{cr,w} = 50$, and $t = 10, 15$, and 20 . θ_1^* is the interface temperature obtained using the single-variable model, while θ_0 and θ_1 are the liquid's wall and interface temperatures from the two-variable model, respectively.

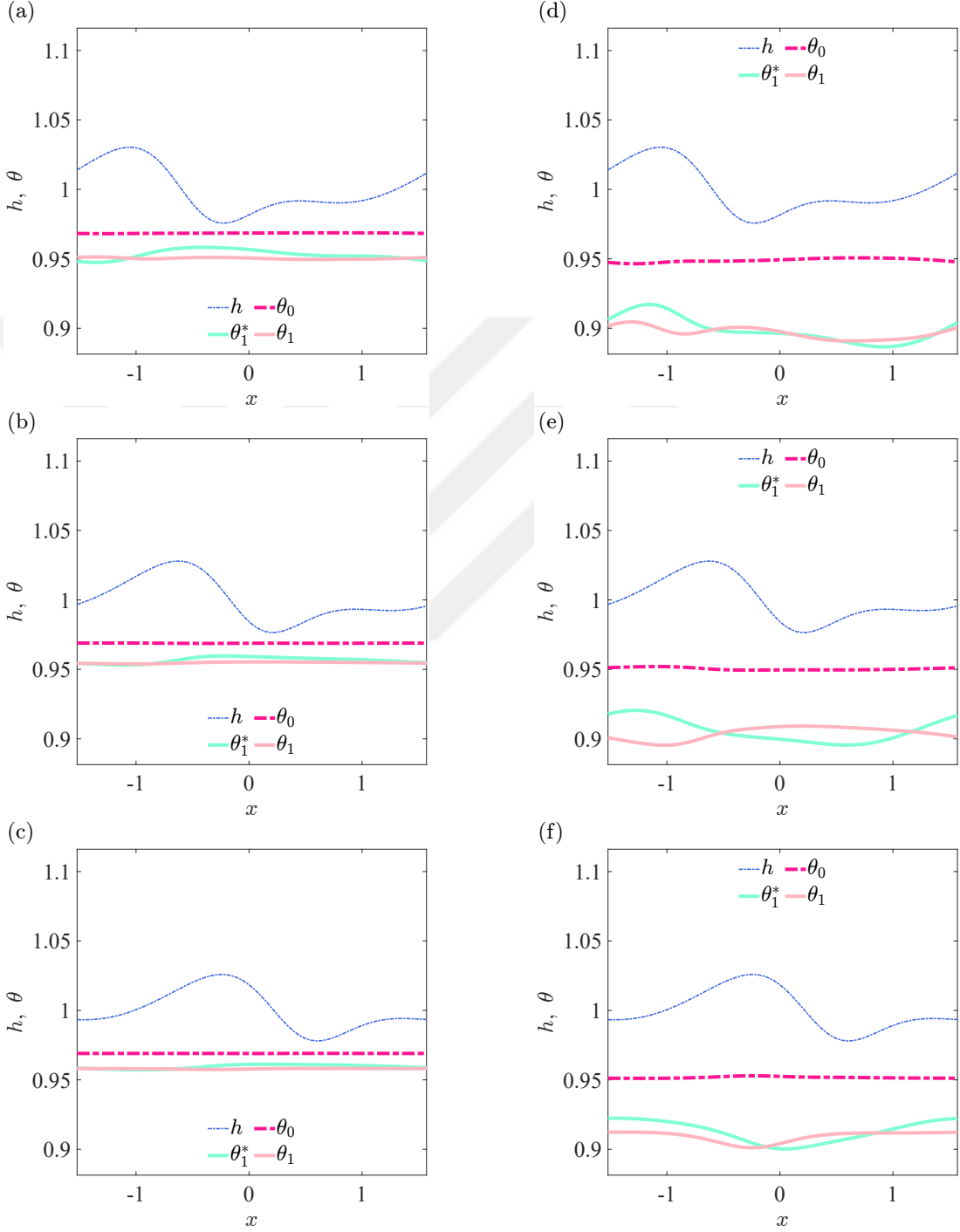


Figure 4.5: Numerical simulations of the film's interface and temperature field for $\epsilon = 0.05$, $k = 2$, $S = 30$, $\text{Re} = 15$, $\beta = 25$, $\text{Pr} = 50$, $Q_s = 2$, $Bi_{fs} = 0.1$, $Bi_w = 2$, $N_{cr,w} = 50$, $\mathcal{T} = 0.95$. For (a), (b), and (c) $N_{cr,l} = 2$, and $t = 10, 15$, and 20 . For (d), (e), and (f) $N_{cr,l} = 20$, and $t = 10, 15$, and 20 . θ_1^* is the interface temperature obtained using the single-variable model, while θ_0 and θ_1 are the liquid's wall and interface temperatures from the two-variable model, respectively.

Chapter 5

Conclusions and future work

5.1 Conclusions

We have investigated the evolution of heated liquid films across a range of systems by formulating different hydro-thermal models taking into account the effects of inertia, thermocapillarity, evaporation, gas shear, and thermal radiation, where we focused our modeling efforts on the most dominant physical phenomena in each problem. In Chaps. 2 and 3, we studied heated volatile films in the drag-gravity flow regime where inertial forces are relatively weak, and thermal effects such as thermocapillarity and evaporation dominate the film's development, and ultimately its effectiveness. In this flow domain, our modeling strategy for the hydrodynamics was based on the LWE due to the relatively low Reynolds numbers expected, where this methodology is sufficiently accurate and adequately robust for our qualitative assessment. The primary focus of our research in this flow regime was directed at the thermocapillary S [36] and evaporative E [50, 57] instability modes owing to their primary influence over the interface's evolution in such conditions. Meanwhile, our emphasis in Chap. 4 was on irradiated films in the drag-inertia flow regime, which is parametrized by relatively larger Reynolds numbers and thicker films. Here, the film's evolution is dictated by inertia and its performance is tied to the development of the temperature profile across

it, and thus, we adapt our attention and our modeling strategies accordingly by employing the WIBL [29, 30] technique for the hydrodynamics, and a new asymptotic model for the irradiated temperature field.

In Chap. 2 we analyzed the general problem of a heated volatile film in the drag-gravity flow domain, where the accumulation of the liquid’s vapor above the free surface and its subsequent diffusion into the neighboring phase is not negligible and must be treated as a fundamental part of the physical model. Therefore, we utilized a two-fluid model of a heated liquid film and its diffusing vapor where the specific role played by diffusion is determined by the kinetic-diffusive balance at the liquid interface. In particular, we investigated the linear stability of an evaporating liquid film flowing down a inclined plane across a wide range of evaporation regimes by theoretical analysis and numerical simulation. We modified the general evaporation model of Sultan *et al.* [69] to allow for thermodynamic disequilibrium and the resultant mass loss, and combined it with the film model of Joo *et al.* [56] to obtain a coupled liquid-vapor system. Subsequently, we performed linear stability analyses in both the temporal and spatiotemporal frameworks. The main focus of our investigation here was the impact of the evaporation regime on the Marangoni instability and its complex coupling to the evaporation effects, namely, vapor recoil, mass loss, and film thinning. These rich interactions were yet to be reviewed comprehensively in the presence of vapor diffusion effects; which was the inspiration behind this effort.

Under our formulation, the Marangoni effect has two components: the first of which is linked to the heat flux rising through the bulk of the liquid film, is dependent only on the liquid interface’s curvature, and is always destabilizing. The second component on the other hand, is caused by the imbalance in vapor diffusion above the interface, and is influenced by both the liquid interface’s curvature and the vapor gradient above it. The specific contribution of this component depends on the evaporation regime: it is stabilizing and causes an attenuation of the Marangoni instability when evaporation is dictated by kinetics, while it is highly destabilizing and significantly strengthens the Marangoni instability when vapor diffusion is the predominant effect. This dependence on the evaporation regime is also shared by the mass loss and vapor recoil effects, which are destabilizing

when evaporation is dominated by kinetics, and stabilizing when it is governed by diffusion.

The presence of the diffusion-dependent component of the Marangoni instability significantly destabilizes the film due to the heightened latent cooling imbalance introduced by the vapor density gradient, with this effect being maximized under diffusion-limited evaporation. Nevertheless, this destabilizing mechanism competes against vapor recoil and evaporative mass loss, which play a stabilizing role in diffusion-dominated evaporation. These evaporative effects when sufficiently strong can completely subdue the Marangoni instability and restore the stable film. Interestingly, when the influences of kinetic and diffusion phenomena on the evaporation process are perfectly balanced, the vapor mass flux is equalized across the film which negates the vapor recoil and mass loss instabilities. We refer to this specific scenario in the evaporation regime as the *balanced flux* case.

Moreover, we investigated the impact of the evaporation regime on the mass loss instability and elucidated its relationship to the wave number. We attributed the independence of the mass loss's contribution to the temporal growth rate from the wave number in the transfer-rate-limited regime to the absence of a vapor density gradient above the liquid film, which allows the vapor molecules to freely depart the liquid interface, and leaves the local film height as the sole variable dictating the instability. On the other hand, accounting for the liquid's vapor introduces a variable density gradient reliant on the interface's shape, which regulates the molecules' propensity to depart from it and connects the instability to the wave number.

The influence of film thinning on the film's stability was also investigated at the limits of the evaporation regime. In the transfer-rate-limited regime, film thinning destabilizes the film since the decrease in the strength of the stabilizing forces with the film's height far outpaces the weakening of the destabilizing phenomena. Conversely, the influence of film thinning in the diffusion-limited regime is quite nuanced and depends on the relative strength of the evaporation phenomena to the Marangoni instability. In fact, the effect varies between being destabilizing when evaporation effects are relatively weak, being nonmonotonic

when for moderately strong evaporation, and lastly, stabilizing when evaporation effects are on par in strength with the Marangoni instability.

In addition, we studied the influence of the evaporation regime on the perturbation phase speed, where we found the waves to be nondispersive in the cases where vapor diffusion is nonexistent or balanced across the interface. On the other hand, the imbalance in vapor diffusion gives rise to nonlinear dispersion at small wave numbers, which is positive (negative) in kinetically-dominated (diffusion-dominated) evaporation. This dispersion becomes linearly negative for all wavenumbers in the diffusion-limited regime. Nevertheless, the changes in the phase speed with the evaporation regime are very small in comparison its drastic reduction with film thickness.

Finally, our spatiotemporal analysis revealed that the increase in the temporal growth rate with the rising importance of vapor diffusion is accompanied by an increase in the absolute growth rate, representing a shift towards absolute instability and leading to convective-absolute transitions. We also investigated the impact of film thinning on the spatiotemporal stability, where we witnessed a progression towards absolute instability as the film became thinner in the transfer-rate-limited regime, and an intricate relationship between evaporation effects and the absolute growth rate in the diffusion-limited regime. Under relatively weak evaporation, the decrease in film height increases the absolute growth rate and brings forth convective absolute transitions, whereas moderate evaporation brings about nonmonotonic changes in the absolute growth rate leading to convective-absolute-convective transitions as the film thins. Nevertheless, relatively strong evaporation restores the monotonic increase in the absolute growth rate as the film thins.

In Chap. 3, we considered the case of a dynamic environment above the liquid film where a mechanically-active gas flow continuously shears the liquid's surface. The presence of this gas layer introduces additional complexity into the problem as we must include the stress imparts into our formulation, but also justifies the utilization of a simpler one-sided evaporation model due to the constant advection of the vapor's molecules away from the liquid's surface by the gas flow, preventing

the build-up of an effective vapor gradient. This permits the assumption that the evaporation process is transfer-rate-limited where the evaporative mass flux is tied to the interfacial temperature, which ultimately allows modeling the problem with a closed system of equations representing a single-fluid. Notably, our formulation of this particular system is precisely the special transfer-rate-limited case of the problem formulated in Chap. 2.

Once again, we focused on the thermal S and E modes of instability and investigated the effects of a shearing gas on the evolution of an evaporating liquid film via linear stability theory, numerical simulation, and self-similarity analysis. The one-sided model of Burelbach *et al.* [54] was used to account for the evaporation of the fluid film, while the shear stress imparted by the gas on the free interface was prescribed as a constant quantity, denoted τ , whose role was confined to modifying the base flow. The main motivation behind this chapter was investigating the effects of the shearing gas on the temporal and spatiotemporal development of the perturbations arising from thermal effects, namely thermocapillary and evaporation phenomena, which had not yet been thoroughly examined.

Through a linear stability analysis we found that (i) the shear gas's influence is much more pronounced on the temporal growth rate of the inertial H-mode instability and (ii) its influence depends on whether the fluid film under consideration is moving ($\beta \neq 0$) or stationary ($\beta = 0$). In a moving film the shear stress modifies the flow's existing inertia and therefore affects both the temporal growth rate and the perturbation phase speed, while in a stationary film it only changes the perturbation phase speed. We found that for a sufficiently negative value of τ the H-mode instability is completely suppressed. This result qualitatively matched that found by Lavalle *et al.* [76] who used a more complex model (limited to isothermal films), combined with experimental validation.

We also investigated how the evaporation and consequent thinning of the film affects the perturbations and their relationship with the shearing gas. We found that the thinning of the film in general suppresses the H-mode instability, but enhances the thermal instability mode. The influence of the shearing gas on both these instability modes is diminished as the film thins due to the increasing

dominance of viscous forces. Additionally, the thinning of the film results in a reduction in perturbation phase speed, indicating a resistance for the convection of the perturbations and possibly a transition into the absolute instability regime. Our subsequent spatiotemporal analysis confirms this result showing that the ability of the base flow to advect the perturbations away from their origin is diminished as the film thins. However, we find that the thinning of the film results in the activation of new modes with higher group velocities, resulting in an overall faster spread of the perturbations via their own mechanism.

Additionally, the interface evolution equation was solved numerically to study the film's evolution when subject to finite perturbations. A set of dimensionless parameters was chosen to show the evolution of the flow dynamics of an evaporating film from stable, to convectively unstable, and ultimately to absolutely unstable. A temporal analysis showed that the dynamic behavior observed in the linear regime persists qualitatively for finite perturbations. Moreover, a subsequent nonlinear spatiotemporal investigation demonstrated the effects of the film thinning due to evaporation on its evolution, where it was shown to cause a transition in the type of instability from convective to absolute. Additionally, the impulse response wavepackets were computed numerically for finite perturbations where we found that, in the case of weaker thermal instabilities, the linear and nonlinear responses are different. Particularly, the nonlinear absolute growth rate is higher than its linear counterpart, indicating that finite perturbations are relatively less convective than infinitesimal disturbances. Consequently, according to the Dee & Langer [129] criterion, the front velocities of nonlinear instabilities are nonlinearly selected as they are imposed by the nonlinearly saturated wavepacket. On the other hand, when the thermal mode instability is relatively strong the linear and nonlinear wave fronts converge suggesting that the wave front dynamics are dictated by infinitesimal modes governed by the linear dispersion relationship.

Furthermore, we investigated the role of the dimensionless shear stress, among other dimensionless parameters, on the rupture mechanism through the means of a self-similarity analysis, which was compared to the scaled numerical simulation data. Our results showed that the dominant parameters in determining the fates of the solutions to the evolution equation are the equilibrium parameter K , the

evaporation number $\bar{E}$, and the constant shear stress τ . In the case of $K = 0$ the film ruptures in finite-time where this rupture mechanism is universal. When K is non-zero but small the solutions to the evolution equation will approach the stable self-similarity solution until h and K are comparable in size, beyond which a new dynamic behavior is detected. On the other hand when $K > 0$ and $\bar{E} > 0$ the film thickness goes to zero at a point with finite speed $\partial_t h \sim -\bar{E}/K$ without the appearance of any singularities in the solutions. Again we find this rupture mechanism to be universal regardless of the value of τ . Lastly, in the case of $K > 0$ and $\bar{E} = 0$, the film ruptures in infinite-time where our analysis identified the appearance of an infinite cascade of geometrically shrinking oscillations when $\tau = 0$, and chaotic evolution when $\tau > 0$.

In Chap. 4 we considered an irradiated liquid film that is characterized by relatively higher Reynolds numbers and film thicknesses, and whose evolution is thus dictated primarily by inertial effects. The liquid film is subjected to solar radiation where this flux passes through the liquid surface, is absorbed by and reflected off of the the solid wall, and then exchanged in the infrared spectrum between the wall and the liquid interface. This thermal radiation problem drives the temperature profile across the liquid film. Moreover, in the drag-inertia flow domain, the influence of the thermal instabilities, i.e., the thermocapillary S -mode and evaporative E -mode, is diminished and is therefore not considered. Moreover, the role of phase change at the liquid interface can be modeled as a modification to it's Biot number. This latter adjustment is deferred to future work.

The WIBL [29, 30] hydrodynamic equations are derived from the boundary layer approximation of the governing system of equations and are then verified by expanding the flow rate and recovering the exact corresponding isothermal evolution equation resulting from the LWE, which appeared in Chaps. 2 and 3. Subsequently, the liquid film's thermal problem is developed by first solving for the temperature profile across the solid wall and in turn inserting this solution into the liquid problem, which accounted for the convective heat transfer to the ambient environment occurring at the bottom of the solid surface. Due to the complexity of the boundary conditions, a closed form solution for the Nusselt

temperature does not exist. Nevertheless, an approximate Nusselt temperature solution was found by decomposing the flat-film temperature into a reference field established by conduction across the film and convection heat losses from the liquid’s surface and the bottom of the solid plate, in addition to a small temperature correction accounting for the radiation exchange across the film and the influence of the fluctuations in the film’s thickness on the temperature field.

The thermal profile within the film was modeled by asymptotically decomposing the temperature variable and truncating the system of equations consistently in line with the *least-degeneracy principle*, such that all physical effects are retained at leading order. We varied the complexity of this decomposition to produce a hierarchy of models of increasing sophistication, each facilitated by an appropriate set of gauge conditions. These models allowed inspecting the influence of the different physical effects on the film’s thermal profile by tracking the evolution of the temperature at the free surface and the solid wall. The most complex model involved additional quantities based on the second derivatives of the temperatures which were introduced to capture information about the diffusion of the heat as well as other higher order effects. Specifically, we formulated thermal evolution models consisting of one, two, and four variables, where the enhanced sophistication of the model is offset by higher complexity and computational cost. Notably, the manual derivation and of the models is very arduous and impractical, and therefore, a set of *MATLAB* [119] scripts was crafted that obtains the equations symbolically and then automatically discretizes them in a form suitable for numerical evaluation. Subsequently, the evolution equations were solved where we focused on investigating the effects of the conduction-radiative parameters, where various of evolution dynamics were observed.

5.2 Future work

Accounting for vapor diffusion in the stability analysis of an evaporating liquid film gives rise to several complex and interesting dynamics resulting from the

competition between the various physical phenomena at the liquid interface. Nevertheless, this problem still offers plenty of opportunities for further study beyond our investigation. The limitations imposed by the long wave expansion can be relaxed by formulating the Orr-Sommerfeld eigenvalue problem while including the kinetic-diffusion balance at the liquid interface, which would allow reliably investigating perturbations at higher wave numbers. The results obtained from such models could then be compared to those obtained from numerical simulations of the two-phase problem [139]. Moreover, more advanced evaporation models that capture additional effects could be used, such as that of Zhao and Nadal [71], which accounts for the Stefan flow, the temperature jump across the Knudsen layer, and the change in the interface's chemical potential due to the presence of the inert gas. Furthermore, the effect of vapor diffusion on other instabilities such as the transverse Marangoni P -mode [37] could be studied. Additionally, three-dimensional effects such as boundary confinement can be included as they have been shown to significantly alter the stability of liquid films [87, 140].

Likewise, the introduction of a shearing gas into the problem of an evaporating thin liquid film adds a lot of complexity and richness to an already interesting problem, which has ample room for additional analysis. Due to the limitations of the LWE approach, the effect of the shearing gas on the H -mode instability could be studied at the moderate Reynolds numbers characterizing the drag-inertia flow regime by utilizing the WIBL technique. Additionally, the small wavenumber limit could be overcome by introducing the shearing gas effects into the Orr-Sommerfeld eigenvalue problem for an evaporating liquid film, as done by Mohamed and Biancofiore [57]. The results obtained from these improved models could be then compared to those obtained via the direct numerical simulation of the two-phase problem, via ad-hoc algorithms developed for evaporating flows [141].

Moreover, the limitations of our shearing gas model could be overcome by considering the full interaction between the shearing gas and the liquid film using more advanced models such as those formulated by Ó Náráigh *et al.* [92] and Vellingiri *et al.* [87]. These formulations would allow including the role of the

normal stress imparted by the gas on the film's surface. In addition it would be possible also to focus specifically on the interaction between the gas and the thermal instabilities. For instance the three-dimensional effects of the turbulent gas could have a significant effect on other instability modes such as the transverse P-mode [37].

Lastly, our attempt at modeling an irradiated film in the drag-inertia flow regime can be expanded further by accounting for thermocapillary effects by accounting for the Marangoni stresses at the liquid interface, as well as introducing the effects of phase change by formulating an appropriate *evaporative* Biot number which takes into account the latent heat losses due to phase change. Moreover, the two-dimensional formulation can be expanded to include the third span-wise coordinate allowing the inclusion of three-dimensional effects, thus increasing the utility of the model by making experimental comparisons viable. In addition, improvements can also be made in the hydrodynamic modeling aspect by making use of higher order WIBL formulations. Additionally, a wider parameteric study can be conducted using the analytical models and numerical algorithms developed herein.

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