

SOME NEW BOUNDS ON CLOSED ORDINAL RAMSEY NUMBERS

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ABSTRACT

SOME NEW BOUNDS ON CLOSED ORDINAL RAMSEY NUMBERS

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Partition calculus on ordinals was introduced by Erdős and Rado and is a well-studied subject. Later on, Baumgartner extended this investigation to topological spaces and studied topological partition relations on ordinal spaces.

In recent years, closed and topological partition relations on ordinals have been an active area of research. This thesis is a contribution to the study of closed ordinal Ramsey numbers. More specifically, we provide new lower and upper bounds for closed ordinal Ramsey numbers of pairs of small countable ordinals. These bounds improve the previously known bounds given by Caicedo and Hilton.

Keywords: Ordinals, Ramsey numbers, partition relations, closed ordinal Ramsey numbers

ÖZ

KAPALI ORDİNAL RAMSEY SAYILARI İÇİN BAZI YENİ SINIRLAR

Sağlam, Irmak

Yüksek Lisans, Matematik Bölümü

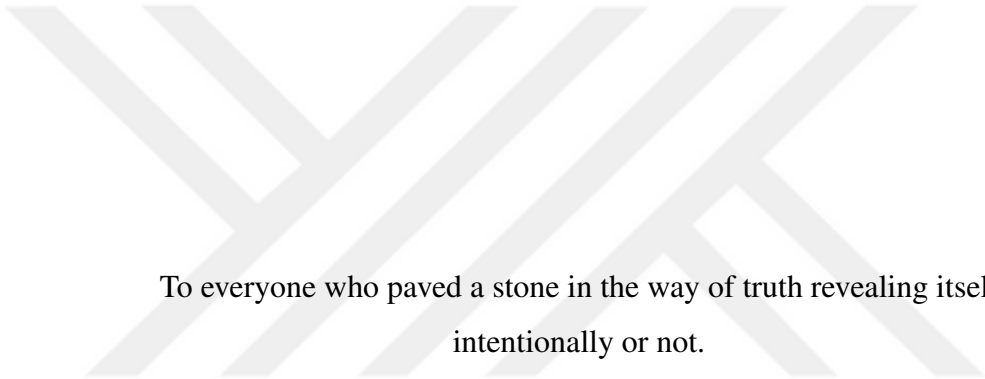
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Ordinaller üzerindeki bölümlleme hesaplamaları ilk olarak Erdős ve Rado tarafından ortaya konmuş ve iyi incelenmiş bir konudur. Daha sonra, Baumgartner bu araştırmaları topolojik uzaylara genişletmiş ve ordinal uzayları üzerindeki topolojik bölümlleme bağıntılarını çalışmıştır.

Son yıllarda, ordinallerin kapalı ve topolojik bölümlleme bağıntıları aktif bir araştırma alanı haline gelmiştir. Bu tez, kapalı ordinal Ramsey sayılarına dair çalışmalara bir katkıdır. Özel olarak, küçük sayılabilir ordinal çiftlerinin kapalı ordinal Ramsey sayılarına yeni alt ve üst sınırlar bulacağız. Bu sınırlar daha önceden Caicedo ve Hilton tarafından verilen sınırları geliştirmektedir.

Anahtar Kelimeler: Ordinaller, Ramsey sayıları, bölümlleme bağıntıları, kapalı ordinal Ramsey sayıları



To everyone who paved a stone in the way of truth revealing itself,
intentionally or not.

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The main results and some parts of this thesis will appear in a joint publication with Burak Kaya [16].

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CHAPTER 1

INTRODUCTION

1.1 Background from Ramsey theory

In broadest terms, Ramsey theory is the study of "necessary occurrences" of different kinds of patterns in (combinatorial) mathematical structures. In particular, it asks the following question: *How big of a structure do we need in order to guarantee the existence of a substructure with a given property?*

Ramsey theory takes its name from Frank Plumpton Ramsey, who observed the following phenomena in 1930: *If we are given two positive integers k and m , we can always find a positive integer r such that, when we color the edges of the complete graph on r vertices to m colors, we can always find a complete subgraph on k vertices that is monochromatic, i.e. all edges of the subgraph have the same color. Similarly, if we color the edges of a complete graph on infinitely many vertices to finitely many colors, then we can always find an infinite complete subgraph that is monochromatic.*

These results are known as the finite Ramsey's theorem and the infinite Ramsey's theorem respectively. As an example, consider the case $k = 3$ and $m = 2$. We claim that we can choose $r = 6$, that is, when we color the edges of a complete graph K_6 on 6 vertices to 2 colors, we can always find a monochromatic complete subgraph K_3 on 3 vertices. To see this, assume that we are coloring the edges of K_6 on the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ to red and blue. Let us look at the edges incident to v_1 . There are 5 such edges and so, by the pigeonhole principle, at least three of these must be of the same color. Without loss of generality, we may assume that these edges are $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$ and their color is red. Now let us look at the colors of the edges $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_2, v_4\}$. If all of these edges are colored to blue, then

we have a blue K_3 on vertices $\{v_2, v_3, v_4\}$ and we are done. Otherwise, at least one of these edges is red, say, $\{v_2, v_3\}$ is red. Then we have a red K_3 on the vertices $\{v_1, v_2, v_3\}$ and we are done. See Figure 1.1 for an illustration of the two possible scenarios of this argument.

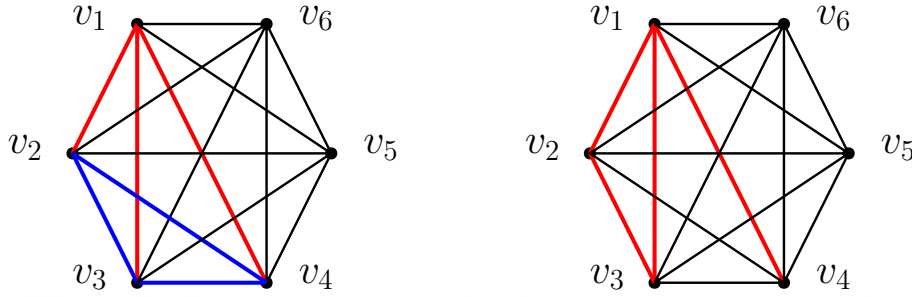


Figure 1.1: Two possible scenarios for K_6

While there were similar results of this type before Ramsey, for example, Hilbert's cube lemma in 1892, Schur's theorem from 1916 and van der Waerden's theorems from 1927, one may consider Ramsey's theorems as having given birth to the Ramsey theory. For a detailed historical background on Ramsey-like results before Ramsey, see [25].

Let us introduce some definitions and Rado's arrow notation to formalize the relations that are needed in Ramsey theory. For the set-theoretic background needed for the rest of this thesis, we refer the reader to [14]. Let X be a set, μ be a cardinal and $n \in \mathbb{N}^+$. We write

$$[X]^n = \{Y \subseteq X : |Y| = n\}$$

to denote the set of all n -subsets of X . A μ -coloring of (n -subsets of) X is a function $c : [X]^n \rightarrow \mu$. For a μ -coloring $c : [X]^n \rightarrow \mu$ and $i \in \mu$, a subset $Y \subseteq X$ is called i -homogeneous (or homogeneous of color i) if we have $[Y]^n \subseteq c^{-1}(\{i\})$.

Definition 1 (Rado's arrow notation). *Let κ, λ, μ be cardinals and $n \in \mathbb{N}^+$. Then we will write*

$$\kappa \rightarrow (\lambda)_\mu^n$$

if for every μ -coloring $c : [\kappa]^n \rightarrow \mu$, there exist $i \in \mu$ and an i -homogeneous subset $X \subseteq \kappa$ such that $|X| = \lambda$.

Relations such as $\kappa \rightarrow (\lambda)_\mu^n$ between cardinals are often called *partition relations*. Note that in the verbal description of Ramsey's theorems, we were talking about the colorings of the edges of complete graphs, whereas, in Rado's arrow notation, our colorings are on the set $[\kappa]^n$. On the other hand, there is a natural correspondence between the set of m -colorings of $[\kappa]^n$ and the colorings of the edges of the complete n -uniform hypergraph on κ vertices to m colors. Consequently, many results in Ramsey theory can be interpreted as statements regarding suitably chosen hypergraphs. For more details regarding this hypergraph interpretation, see [23].

Using Rado's arrow notation, the finite Ramsey's theorem can be expressed as follows,

$$\text{For all } k, n, m \in \mathbb{N}^+, \text{ there exists } r \in \mathbb{N}^+ \text{ such that } r \rightarrow (k)_m^n. \quad (1.1)$$

Similarly, the fact that every coloring of the edges of K_6 to 2 colors contains a monochromatic K_3 , which we proved earlier, can be expressed as $6 \rightarrow (3)_2^2$. On the other hand, we have $5 \not\rightarrow (3)_2^2$, that is, there exists a coloring of the edges of K_5 to 2 colors so that it contains no monochromatic K_3 . Such a coloring is depicted in Figure 1.2.

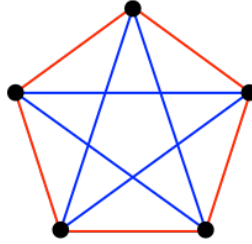


Figure 1.2: A monochromatic triangle-free coloring of K_5

Similar to the finite version, the infinite Ramsey's theorem can be expressed by the following partition relation

$$\aleph_0 \rightarrow (\aleph_0)_m^n \quad \text{for any } n, m \in \mathbb{N}^+. \quad (1.2)$$

We remark that the initial statements of Ramsey's theorems were only given for the case $n = 2$ with the hope that it will invoke more intuition at the reader at the first sight since it corresponds to colorings of edges of graphs. That said, Ramsey's theorems actually hold for all positive integers $n \geq 1$ as stated in equations (1.1) and (1.2).

1.2 Topological and closed partition relations and some basic results

Suppose that the set we are coloring has a specific algebraic or topological structure. In this case, we can seek to find monochromatic subsets having some specific algebraic or topological structures that they inherit from the ambient set. For instance, if the initial set has an order structure on it, then we can investigate monochromatic subsets with specific order types. In particular we can, and will, direct our attention to well-ordered sets. Since well-ordered sets are represented by ordinals, we will be working with colorings of n -subsets of ordinals. This brings us to the following definition introduced by Erdős and Rado, who were the first to introduce the partition calculus on cardinals [5] and ordinals [6].

Definition 2. Let β and α_i be ordinals, μ be a cardinal, $i \in \mu$ and $n \in \mathbb{N}^+$.

$$\beta \rightarrow (\alpha_i)_{i \in \mu}^n$$

means that for every μ -coloring $c : [\beta]^n \rightarrow \mu$, there exist $i \in \mu$ and an i -homogeneous subset $X \subseteq \beta$ such that X is order-isomorphic to α_i . Instead of the last two properties, we will say that X is an i -homogenous copy of α_i .

For the case $n = 1$, the least ordinal β such that $\beta \rightarrow (\alpha_i)_{i \in \mu}^1$ is defined to be the *pigeonhole number* $P(\alpha_i)_{i \in \mu}$. Similarly, for the case $n = 2$, the least ordinal β such that $\beta \rightarrow (\alpha_i)_{i \in \mu}^2$ is defined to be the *ordinal Ramsey number* $R(\alpha_i)_{i \in \mu}$.

In the case μ is finite, say $\mu = k$, we will drop the subscript $i \in \mu$ and simply write $\beta \rightarrow (\alpha_1, \alpha_2, \dots, \alpha_k)^n$ instead of $\beta \rightarrow (\alpha_i)_{i \in \mu}^n$. The same notational simplification will be applied to pigeonhole and Ramsey number notations as well. Note once more that, using this newly introduced notation, the infinite Ramsey's theorem can be stated as $R(\omega, \omega) = \omega$.

In general, the exact calculation of ordinal Ramsey numbers, even for finite ordinals, is a notoriously difficult problem. The values of $R(m, n)$ are not known in general, except for some very limited cases. Indeed, in [27], Joel Spencer tells the following story to stress the difficulty of the problem: "Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens." We refer the reader to [22] for values and bounds of some Ramsey numbers $R(m, n)$ with $m, n \in \mathbb{N}^+$.

Recall that we have shown $6 \rightarrow (3, 3)^2$ and $5 \not\rightarrow (3, 3)^2$. Consequently, we have $R(3, 3) = 6$. In order to make the reader become familiar with some techniques used in proofs, let us now compute an ordinal Ramsey number $R(\alpha, \beta)$ where at least one of α or β is infinite.

Proposition 1. $R(\omega + 1, 3) = \omega \cdot 2 + 1$.

Proof. Let us first prove that $R(\omega + 1, 3) \geq \omega \cdot 2 + 1$. We will do this by providing a coloring $\mathfrak{c} : [\omega \cdot 2]^2 \rightarrow \{0, 1\}$ that induces neither a 0-homogeneous copy of $\omega + 1$ nor a 1-homogeneous copy of 3. Let $W_1 = \{\alpha \mid \alpha < \omega\}$ and $W_2 = \{\alpha \mid \omega < \alpha < \omega \cdot 2\}$. Then $W_1 \cup \{\omega\} \cup W_2$ is a partition of the ordinal $\omega \cdot 2$. Define $\mathfrak{c} : [\omega \cdot 2]^2 \rightarrow \{0, 1\}$ as follows:

$$\mathfrak{c}(\{\alpha, \beta\}) = 0 \quad \text{if and only if} \quad \alpha, \beta \in W_1 \text{ or } \alpha, \beta \in W_2 \cup \{\omega\}$$

There exists no 1-homogeneous copy of 3 because $\mathfrak{c}(\{\alpha, \beta\}) = 1$ if and only if $\alpha \in W_1$ and $\beta \in W_2 \cup \{\omega\}$, or vice versa. We will now show that there exists no 0-homogeneous copy of $\omega + 1$. Assume towards a contradiction that $X \subseteq \omega \cdot 2$ is a 0-homogeneous copy of $\omega + 1$. Let $f : \omega + 1 \rightarrow X$ be the unique order-isomorphism. Observe that, as $X \subseteq \omega \cdot 2$, we have $f(\omega) \leq \omega + n$ for some $n < \omega$. On the one hand, there must be infinitely many elements between $f(0)$ and $f(\omega)$. On the other hand, there are only finitely many elements in $W_2 \cup \{\omega\}$ less than $f(\omega) \leq \omega + n$. Therefore, we must have $f(0) \in W_1$. In this case, we obtain $\mathfrak{c}(\{f(0), f(\omega)\}) = 1$ which contradicts the 0-homogeneity of X .

It remains to prove that $R(\omega + 1, 3) \leq \omega \cdot 2 + 1$. To do so, let $\mathfrak{c} : [\omega \cdot 2 + 1]^2 \rightarrow \{0, 1\}$ be a coloring. Then $W_1 \cup \{\omega\} \cup W_2 \cup \{\omega \cdot 2\}$ is a partition of the ordinal $\omega \cdot 2 + 1$. Assume towards a contradiction that there exists no 1-homogeneous copy of 3 or 0-homogeneous copy of $\omega + 1$ in $\omega \cdot 2 + 1$. By the infinite Ramsey's theorem applied to the restriction of the coloring onto W_1 and W_2 , we can find infinite 0-homogeneous subsets $W'_1 \subseteq W_1$ and $W'_2 \subseteq W_2$.

If the set $\{\alpha \in W'_1 \mid \mathfrak{c}(\{\alpha, \beta\}) = 0\}$ were infinite for some $\omega \cdot 2 \geq \beta \geq \omega$, then there would be a 0-homogeneous copy of $\omega + 1$ in $W'_1 \cup \{\beta\}$ which is a contradiction. So $\{\alpha \in W'_1 \mid \mathfrak{c}(\{\alpha, \beta\}) = 0\}$ is finite for all $\omega \cdot 2 \geq \beta \geq \omega$.

By a similar argument, the set $\{\alpha \in W'_2 \mid \mathfrak{c}(\{\alpha, \omega \cdot 2\}) = 0\}$ is finite. Then we can pick $\beta \in W'_2$ such that $\mathfrak{c}(\{\beta, \omega \cdot 2\}) = 1$. Now the set

$$\{\alpha \in W'_1 \mid \mathfrak{c}(\{\alpha, \beta\}) = 1 \text{ and } \mathfrak{c}(\{\alpha, \omega \cdot 2\}) = 1\}$$

is cofinite as it is the intersection of two cofinite sets in W'_1 . Let $\alpha \in W'_1$ be such that $\mathfrak{c}(\{\alpha, \beta\}) = \mathfrak{c}(\{\alpha, \omega \cdot 2\}) = 1$. Then $\{\alpha, \beta, \omega \cdot 2\}$ is a 1-homogeneous copy of 3 which is a contradiction. $\square$

Ordinal Ramsey numbers have been thoroughly examined. Some pioneering results for ordinal Ramsey numbers came from the works of Sierpiński and Erdős-Milner. In [24], Sierpiński showed that $\alpha \not\rightarrow (\omega + 1, \omega)^2$ for any countable ordinal α . This means that if $\alpha > \omega$ and $R(\alpha, \gamma)$ is countable, then $\gamma < \omega$. On the other hand, Erdős and Milner proved in [4] that, for any countable α and finite k , $R(\alpha, k)$ is indeed countable. Also, for the case $n > 2$, the following theorem of Kruse gives us that $\alpha \not\rightarrow (\omega + 1, n + 1)^n$ for countable α , which drastically limits the diversity of the possible partition relations for countable α and $n > 2$.

Fact 1 ([18]). *Suppose that $n \geq 3$ is an integer. For all ordinals β , if $\alpha < \omega_{\beta+1}$, then $\alpha \not\rightarrow (\omega_\beta + 1, n + 1)^n$.*

For many instances of countable α and finite k , the ordinal Ramsey numbers $R(\alpha, k)$ are either exactly computed or some useful bounds are provided. For example, it was proven by Erdős and Rado [6] that $\omega \cdot m \rightarrow (\omega + 1, m)^2$ and $\omega \cdot m \not\rightarrow (\omega + 1, m + 1)^2$. It was shown by Specker [26] that $\omega^2 \rightarrow (\omega^2, m)^2$ for every finite m and $\omega^n \not\rightarrow (\omega^n, 3)^2$

if $n \geq 3$. Algorithms to compute $R(\alpha, k)$ for some classes of ordinals $\alpha < \omega^\omega$ and all finite k are announced by Haddad and Sabbagh in [7, 8, 9]. It was shown by Chang [3] that $\omega^\omega \rightarrow (\omega^\omega, 3)^2$ and this result was extended by Milner in an unpublished work to $\omega^\omega \rightarrow (\omega^\omega, m)^2$ for all finite m . For more details, see [29, Chapter 7].

Much work has also been done on providing bounds for partition relations on ordinals involving uncountable ordinals. For the case $n = 2$, Erdős and Rado showed in [6] that $\omega_1 \rightarrow (\omega_1, \omega+1)^2$; and in [28] and [10], it has also been shown that $\omega_1 \rightarrow (\omega_1, \alpha)^2$ for all countable α and $\omega_1 \not\rightarrow (\omega_1, \omega+2)^2$ are relatively consistent with ZFC. For the case $n > 2$, some interesting results have also been proven. For example, that $\omega_1 \rightarrow (\omega+1)_k^n$ for all finite n, k and $\omega_1 \not\rightarrow (n+1)_\omega^n$ for $n \geq 2$ was proven by Erdős and Rado [6]. Jones proved in [15] that $\omega_1 \not\rightarrow (\omega+2, \omega)^3$. That $\omega_1 \not\rightarrow (\omega_1, 4)^3$ was proven by Hajnal [11]. Kruse also proved in [18] that $\omega_1 \not\rightarrow (\omega+2, n+1)^n$.

The well-order structure on ordinals automatically induces a topology, namely, the order topology. We may incorporate this topological structure into our investigation. There are two variants of the partition relation in Definition 2 that make use of the topological structure on ordinals. These two versions are called *topological* and *closed* versions.

In what follows, we endow all ordinals with their respective order topologies and subsets of ordinals with the respective subspace topologies. Following Baumgartner [1], we say a subspace X of an ordinal is *order-homeomorphic* to an ordinal α to mean that the order type of X is equal to α and the unique order-isomorphism $f : X \rightarrow \alpha$ is a homeomorphism. Equivalently, this means that X is order-isomorphic to α and closed in its supremum. Before we proceed, let us give an example of an order-isomorphism which is not a homeomorphism. Let $X = \omega \cup \{\omega+1\} \subseteq \omega+2$. Then clearly $\text{ord}(X) = \omega+1$. Consider the unique order-isomorphism $f : X \rightarrow \text{ord}(X)$ given by $f(n) = n$ for all $n < \omega$ and $f(\omega+1) = \omega$. Then f is not a homeomorphism since the sequence $(n)_{n \in \mathbb{N}}$ does not converge to $\omega+1$ in X , but the sequence $(f(n))_{n \in \mathbb{N}}$ converges to $f(\omega+1)$ in $\omega+1$.

Definition 3. Let β and α_i be ordinals, μ be a cardinal, $i \in \mu$ and $n \in \mathbb{N}^+$.

$$\beta \rightarrow_{\text{top}} (\alpha_i)_{i \in \mu}^n$$

means that for every μ -coloring $c : [\beta]^n \rightarrow \mu$, there exists $i \in \mu$ and an i -homogeneous

subspace $X \subseteq \beta$ such that X is homeomorphic to α_i . Instead of the last two properties, we will say that X is an i -homogeneous topological copy of α_i . Likewise,

$$\beta \rightarrow_{cl} (\alpha_i)_{i \in \mu}^n$$

means that for every μ -coloring $c : [\beta]^n \rightarrow \mu$ there exists $i \in \mu$ and an i -homogeneous subspace $X \subseteq \beta$ such that X is order-homeomorphic to α_i . Instead of the the last two properties, we will say that X is an i -homogeneous closed copy of α_i .

For $n = 1$, the least ordinal β such that $\beta \rightarrow_{cl} (\alpha_i)_{i \in \mu}^1$ is defined to be the *closed pigeonhole number* $P^{cl}(\alpha_i)_{i \in \mu}$. For the case $n = 2$, the least ordinal β such that $\beta \rightarrow_{cl} (\alpha_i)_{i \in \mu}^2$ is defined to be the *closed ordinal Ramsey number* $R^{cl}(\alpha_i)_{i \in \mu}$. Similar definitions are made for the *topological pigeonhole numbers* $P^{top}(\alpha_i)_{i \in \mu}$ and *topological ordinal Ramsey numbers* $R^{top}(\alpha_i)_{i \in \mu}$. As before, whenever μ is finite, we will drop the subscript $i \in \mu$ and simply list the ordinals α_i in these notations.

Observe that, by definition, $R(\alpha, \beta) \leq R^{cl}(\alpha, \beta)$ for all ordinals α and β . Thus the result $R(\omega + 1, \omega) \geq \omega_1$ of Si rpiński implies that $R^{cl}(\omega + 1, \omega) \geq \omega_1$. Consequently, if we wish $R^{cl}(\alpha, \beta)$ to be countable and $\omega + 1 \leq \alpha < \omega_1$, then we must have $\beta < \omega$.

For an example of the case $R(\alpha, \beta) < R^{cl}(\alpha, \beta)$, consider $\alpha = \omega + 1$ and $\beta = 3$. We have proven in Proposition 1 that $R(\omega + 1, 3) = \omega \cdot 2 + 1$. On the other hand, as we shall see later, $R^{cl}(\omega + 1, 3) = \omega^2 + 1$.

While the topological and closed versions coincide on some cases, they do not coincide in general. For example, $\omega + 1 \rightarrow_{top} (\omega + n)_1^1$ for all $n < \omega$ because $\omega + 1$ is homeomorphic to $\omega + n$ for all $n < \omega$; however, $\omega + 1 \not\rightarrow_{cl} (\omega + 2)_1^1$ simply because no subset of $\omega + 1$ is order-isomorphic to $\omega + 2$. On the other hand, the next proposition is a good example of some cases where topological and closed pigeonhole numbers coincide.

Proposition 2 ([2, Proposition 3.4]). $P^{top}(\omega + 1)_k = P^{cl}(\omega + 1)_k = \omega^k + 1$.

More generally, the topological and closed pigeonhole numbers coincide for *order-reinforcing ordinals*, i.e., the ordinals α such that for every set of ordinals X that is homeomorphic to α , there is a subspace $Y \subseteq X$ such that Y is order-homeomorphic

to α . We also have $R^{cl}(\alpha, k) = R^{top}(\alpha, k)$ for all order-reinforcing ordinals α and all $k < \omega$. Order-reinforcing ordinals were classified by Hilton in [12, Corollary 2.17].

In [2], Caicedo and Hilton proved a topological version of the Erdős-Milner theorem and showed that both $R^{top}(\alpha, k)$ and $R^{cl}(\alpha, k)$ are countable for all countable α and finite k .

We would like to end this subsection by mentioning some results on pigeonhole numbers, a subject that was postponed until now. Milner and Rado calculated classical pigeonhole numbers in [21], Hilton calculated the topological pigeonhole numbers in [12] and Caicedo and Hilton calculated the closed pigeonhole numbers in [2]. On many occasions, pigeonhole numbers turn out to be essential in the calculation of Ramsey numbers. Even though we do not explicitly use the calculation of pigeonhole numbers in our core work, we will need to use them to establish an upper bound later.

1.3 The closed ordinal Ramsey numbers $R^{cl}(\alpha, k)$ for $\alpha < \omega^2$

Caicedo and Hilton provided bounds for topological and closed Ramsey numbers for various pairs of countable ordinals in [2]. Among other results, they provided detailed bounds for topological and closed Ramsey numbers $R^{top}(\alpha, k)$ and $R^{cl}(\alpha, k)$ where α is a successor ordinal with $\omega < \alpha < \omega^2$ and k is a positive integer. Note that such α are of the form $\alpha = \omega \cdot m + \omega \cdot n + 1$ for some natural number n and positive integer m . For example, we have the following theorem.

Fact 2 ([2, Proposition 5.5]). *For all positive integers m, n, k with $k \geq 2$,*

$$R^{cl}(\omega \cdot m + n + 1, k + 1) \leq R^{cl}(\omega \cdot m + 1, k + 1) + P^{cl}((R^{cl}(\omega \cdot m + n + 1, k))_{2m}, R(n, k + 1))$$

where the notation $((x)_{2m}, y)$ is used for $(x, x, \dots, x, y)$ with $2m$ -many x 's.

Indeed, for the case $n = 0$ and the case $n = 0$ and $m = 1$, they provide the following more detailed values in [2, Theorem 4.1, Theorem 6.1].

$$R^{top}(\omega \cdot m + 1, k + 1) = R^{cl}(\omega \cdot m + 1, k + 1) < \omega^\omega$$

$$R^{top}(\omega + 1, k + 1) = R^{cl}(\omega + 1, k + 1) = \omega^k + 1$$

For the case $\alpha = \omega^2$, they proved in [2, Theorem 7.1] that

$$R^{top}(\omega^2, k) = R^{cl}(\omega^2, k) \leq \omega^\omega$$

Later on, in [20], the exact value of $R^{cl}(\omega^2, 3)$ is computed to be ω^6 by Mermelstein. Mermelstein also proved in [19] that $R^{cl}(\omega \cdot 2, 3) = \omega^3 \cdot 2$. Both of these results are based on canonical colorings, which is a concept introduced in [19]. We will also adopt and make extensive use of the notion of a canonical coloring in our main results.

Besides these upper bounds, Caicedo and Hilton also provided the exact values of the closed ordinal Ramsey numbers $R^{cl}(\omega + n, 3)$ for the cases $n = 1$ and $n = 2$.

Fact 3 ([2, Theorem 4.1, Lemma 5.2, Lemma 5.3]).

$$R^{cl}(\omega + 1, 3) = \omega^2 + 1 \tag{1.3}$$

$$R^{cl}(\omega + 2, 3) = \omega^2 \cdot 2 + \omega + 2 \tag{1.4}$$

For the closed ordinal Ramsey numbers $R^{cl}(\omega + n, 3)$ for $n \geq 3$, the following upper bound can also be obtained as a corollary of Fact 2. However, since this upper bound was not explicitly stated by Caicedo and Hilton, we will derive it here for completeness using the results of [2].

Fact 4. *For any positive integer $n \geq 3$,*

$$R^{cl}(\omega + n, 3) \leq \omega^2 \cdot (R(n - 1, 3) + 1) + \omega \cdot (n - 1) + n$$

Proof. Let $n \geq 3$. Substituting $m = 1$ and $k = 2$ and replacing n by $n - 1$ in Fact 2, we get,

$$R^{cl}(\omega + n, 3) \leq R^{cl}(\omega + 1, 3) + P^{cl}((R^{cl}(\omega + n, 2))_2, R(n - 1, 3)) \tag{1.5}$$

We know from Fact 3 that $R^{cl}(\omega + 1, 3) = \omega^2 + 1$ and it is clear that $R^{cl}(\alpha, 2) = \alpha$ for any ordinal α . When we substitute these results in (1.5), we get

$$R^{cl}(\omega + n, 3) \leq \omega^2 + 1 + P^{cl}((\omega + n)_2, R(n - 1, 3)) \tag{1.6}$$

We claim the following.

Claim 1. *For $M \in \mathbb{N}^+$, $P^{cl}(\omega + n, \omega + n, M) = \omega^2 \cdot M + \omega \cdot (n - 1) + n$.*

Proof of claim. We proceed by induction on M . For the base case $M = 1$, we have $P^{cl}(\omega + n, \omega + n, 1) = P^{cl}(\omega + n, \omega + n) = \omega^2 + \omega \cdot (n - 1) + n$ which can be obtained by applying [2, Fact 3.3 (7)]. For the inductive step, assume that the claim holds for M . It follows from the computations of [2, Theorem 3.2 (5)] that

$$P^{cl}(\omega + n, \omega + n, M + 1) = P^{cl}(\omega + 1, \omega + 1) + \max\{P^{cl}(n - 1, \omega + n, M + 1), P^{cl}(\omega + n, \omega + n, M)\} \quad (1.7)$$

By the iterative application of [2, Theorem 3.2 (5)] on the closed pigeonhole numbers with three terms $P^{cl}(\alpha_1, \alpha_2, k)$ and $P^{cl}(\alpha, k_1, k_2)$ where $\omega < \alpha, \alpha_1, \alpha_2 < \omega \cdot 2$ and $k, k_1, k_2 < \omega$, we observe that the first pigeonhole number has the leading term $\omega^2 \cdot n_1$ and the second pigeonhole number has the leading term $\omega \cdot n_2$ for some positive integers n_1 and n_2 . This observation lets us conclude that

$$P^{cl}(\omega + n, \omega + n, M) \geq P^{cl}(n - 1, \omega + n, M + 1)$$

On the other hand, $P^{cl}(\omega + 1, \omega + 1) = \omega^2 + 1$ by Proposition 2. Substituting these results on 1.7 and using the inductive assumption now gives,

$$\begin{aligned} P^{cl}(\omega + n, \omega + n, M + 1) &= \omega^2 + 1 + P^{cl}(\omega + n, \omega + n, M) \\ &= \omega^2 + 1 + \omega^2 \cdot M + \omega \cdot (n - 1) + n = \omega^2 \cdot (M + 1) + \omega \cdot (n - 1) + n \end{aligned}$$

□

Now that we have proven the claim, (1.6) immediately gives us

$$\begin{aligned} R^{cl}(\omega + n, 3) &\leq \omega^2 + 1 + \omega^2 \cdot R(n - 1, 3) + \omega \cdot (n - 1) + n \\ &\leq \omega^2 \cdot (R(n - 1, 3) + 1) + \omega \cdot (n - 1) + n \end{aligned}$$

□

1.4 Contributions of this thesis

The main contribution of this thesis is to improve the previously known lower and upper bounds for the closed ordinal Ramsey numbers $R^{cl}(\omega + n, 3)$ with $n \geq 3$. Our first main is the following general lower bound.

Theorem 1. *For every positive integer $n \geq 3$, we have*

$$R^{cl}(\omega + n, 3) \geq \omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + n$$

It is stated by Caicedo and Hilton in [2, Chapter 5] that the best general lower bound for $R^{cl}(\omega + n, 3)$ which can be derived from their general results is $P^{cl}(\omega + n)_2 = \omega^2 + \omega \cdot (n - 1) + n$. However, as they note, this general lower bound is even behind the calculated value $R^{cl}(\omega + 2, 3) = \omega^2 \cdot 2 + \omega + 2$. Thus, no non-trivial lower bounds for $R^{cl}(\omega + n, 3)$ with $n \geq 3$ were known prior to our result.

Our second main result is a strengthening of Fact 4.

Theorem 2. *For every positive integer $n \geq 3$, we have*

$$R^{cl}(\omega + n, 3) \leq \omega^2 \cdot n + \omega \cdot (R(2n - 3, 3) + 1) + 1$$

Theorem 1 and Theorem 2 together show that the coefficient of ω^2 in $R^{cl}(\omega + n, 3)$ is n . While we were able to determine the exact coefficient of ω^2 , we believe that we have not reached natural limits within the scope of our tools in the arguments providing the coefficients of ω in the lower and upper bounds for $R^{cl}(\omega + n, 3)$. Thus we expect that these bounds for the coefficient of ω can be improved with more refined approaches.

The classical Ramsey numbers $R(n, 3)$ are known to have an asymptotic order of magnitude $n^2 / \ln(n)$, see [17]. On the other hand, the exact computation of $R(n, 3)$ is a notorious combinatorial problem. For this reason, we also prove the following (asymptotically weaker) upper bound to get rid of the Ramsey numbers. This upper bound also gives better results for small n values (at least, for $3 \leq n \leq 7$), even though it is asymptotically worse than the upper bound of Theorem 2. Our third and last main result is the following.

Theorem 3. *For every positive integer $n \geq 3$, we have*

$$R^{cl}(\omega + n, 3) \leq \omega^2 \cdot n + \omega \cdot (n^2 - 4) + 1$$

1.5 Outline of this thesis

The structure of the thesis is as follows. In Chapter 2, we will give some preliminary notions and definitions that we will need in the remainder of this thesis. More specifically, in Section 2.1, we will introduce the Cantor-Bendixson rank of an ordinal and related definitions; and a special partial orderings on ordinals which will allow us to see some ordinals as forests. In Section 2.2, we will introduce some special types of colorings on ordinals which will be vital to our arguments. In Chapter 3, we will prove Theorem 1. Our proof will be based on the construction of a special triangle-free graph on a partition of $\omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + (n - 1)$. This approach is inspired by the proof of $R^{cl}(\omega + 2, 3) \geq \omega^2 \cdot 2 + \omega + 2$ in [2, Lemma 5.3], which the authors credit to Mermelstein. In Chapter 4, we will prove Theorem 2; and in Chapter 5, we will prove the asymptotically worse upper bound given in Theorem 3. Both of these proofs use the fact that, instead of arbitrary colorings, one can work with canonical colorings which are in some sense determined by finitary information. Finally, in Chapter 6, we will offer our ideas regarding potential future research directions.

1.6 Some notational and terminological remarks

Throughout this thesis, the small-case greek letters α, β, γ will be used to denote ordinals and Latin letters will generally be used to denote natural numbers.

Let γ be an ordinal. A function $\mathfrak{c} : [\gamma]^2 \rightarrow n$ is called a *pair coloring* of γ with n colors and a function $\mathfrak{r} : \gamma \rightarrow n$ is called a *singleton coloring* of γ with n colors. For simplicity, given a pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$, we will say that X is a *red homogeneous closed copy of α* if X is order-homeomorphic to α and is homogeneous of color 0. Similarly, we will say that X is a *blue homogeneous closed copy of α* if X is order-homeomorphic to α and is homogeneous of color 1.



CHAPTER 2

PRELIMINARIES

2.1 Cantor-Bendixson Rank

It is well-known that every non-zero ordinal number γ can be written in the following form,

$$\gamma = \omega^{\sigma_1} \cdot n_1 + \cdots + \omega^{\sigma_m} \cdot n_m \quad (2.1)$$

where $n_i \in \mathbb{N}^+$ for all $1 \leq i \leq m$ and σ_i, σ_j are ordinals with $\sigma_i < \sigma_j$ for $i > j$. This is usually what is referred to as the Cantor normal form of γ . By eliminating n_i and composing a representation in which each ω^{σ_i} component is explicitly shown, we can obtain another unique representation for γ as follows,

$$\gamma = \omega^{\beta_1} + \cdots + \omega^{\beta_{k_\gamma}} \quad (2.2)$$

where β_i are ordinals with $\beta_i \leq \beta_j$ for all i, j such that $1 \leq j < i \leq k_\gamma$, $k_\gamma \in \mathbb{N}^+$. In this thesis, we shall call this second representation of γ its *Cantor normal form*.

Given ordinals θ and γ with $\theta < \gamma$ where γ has the Cantor normal form in (2.2), $CNF_\gamma(\theta)$ represents the smallest $1 \leq i \leq k_\gamma$ such that $\theta \leq \omega^{\beta_1} + \cdots + \omega^{\beta_i}$. We also set

$$CC_\gamma(i) = \{\theta < \gamma \mid CNF_\gamma(\theta) = i\}. \quad (2.3)$$

Let γ be as in (2.1) and (2.2). The *Cantor-Bendixson rank* of γ is defined to be σ_m in (2.1) and shown by $CB(\gamma) = \sigma_m$, which is also equal to β_{k_γ} in (2.2). We shall also denote the number n_m by $L(\gamma)$. To avoid trivialities we define $CB(\gamma) = 0$ and $L(\gamma) = 1$ for $\gamma = 0$.

The well-known α -th *Cantor-Bendixson derivative* of a topological space X is defined by transfinite recursion on ordinals α as follows: We set $X^{(0)} = X$. For every

ordinal α , we set $X^{(\alpha+1)} = (X^{(\alpha)})'$ where S' denotes the set of limit points of S . For a limit ordinal α , we set $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$.

The relationship between the Cantor-Bendixson rank of an ordinal and the Cantor-Bendixson derivative is given in the following proposition.

Proposition 3 ([13, Prop 2.3.3]). *Let γ be an ordinal and $\alpha \in \gamma$. The ordinal $CB(\alpha)$ is the greatest ordinal β such that $\alpha \in [0, \gamma)^{(\beta)}$.*

In other words, the Cantor-Bendixson rank of an ordinal x is the number of iterations of deleting the isolated points of the class of ordinals for which x can stand without *disappearing*.

Note that there exists another definition of Cantor-Bendixson rank, referring to a property of a topological space X . The Cantor-Bendixson rank of a topological space X is the least ordinal β such that $X^{(\beta)} = X^{(\beta+1)}$. But in this thesis, while talking about the Cantor-Bendixson rank of an ordinal α , we will be referring to $CB(\alpha)$ as given in Proposition 3.

The following strict partial ordering $<^*$ on the class of ordinals was first brought up by Hilton and Caicedo in [2].

Definition 4. *Let α and β be ordinals with $\beta > 0$ with $\beta = \eta + \omega^\gamma$, where η is a multiple of ω^γ . We define $<^*$ as follows.*

$$\alpha <^* \beta \text{ if and only if } \alpha = \eta + \zeta \text{ for some } 0 < \zeta < \omega^\gamma$$

We also write $\alpha \triangleleft^ \beta$ if $\alpha <^* \beta$ and there is no ordinal δ with $\alpha <^* \delta <^* \beta$.*

Equivalently, $\alpha <^* \beta$ if and only if $\beta = \alpha + \omega^\gamma$ for some $\gamma > CB(\alpha)$; and, $\alpha \triangleleft^* \beta$ if and only if $\beta = \alpha + \omega^{CB(\alpha)+1}$. In other words, $\alpha \triangleleft^* \beta$ if and only if β is the immediate successor of α with respect to the ordering $<^*$. The following properties of $<^*$ can easily be checked:

- $\alpha <^* \beta$ implies $\alpha < \beta$.
- $\alpha <^* \beta$ implies $CB(\alpha) < CB(\beta)$.

- $\alpha \triangleleft^* \beta$ implies $CB(\alpha) + 1 = CB(\beta)$.
- For each α , there exists a unique ordinal β such that $\alpha \triangleleft^* \beta$.

For a graphical representation of the relation $<^*$ on the ordinal $\omega^2 \cdot n + \omega \cdot K + 1$ as a forest, see Figure 2.1. We will now present some new notations related to $<^*$.

$$T(\alpha) = \{\beta \mid \beta <^* \alpha\} \cup \{\alpha\}$$

$$T^{=k}(\alpha) = \{\beta \in T(\alpha) \mid CB(\beta) = k\}$$

When we consider an ordinal as a forest as in Figure 2.1, the set $T(\alpha)$ represents the tree below α in this graph and $T^{=k}(\alpha)$ represents the elements of $T(\alpha)$ with the Cantor-Bendixson rank of k .

For an ordinal $\theta < \omega^\omega$, we define $[\theta; i, j]$ as follows.

$$[\theta; i, j] = \{\alpha \in \theta : CNF_\theta(\alpha) = i, CB(\alpha) = j\}$$

2.2 Special Types of Colorings

The notion of a skeleton of an ordinal first appeared in Mermelstein [19]. We will be using this notion to introduce some special colorings on ordinals.

Definition 5. A skeleton of an ordinal $\gamma < \omega^\omega$ is a subset $I \subseteq \gamma$ such that,

- I is order-homeomorphic to γ ,
- For all $x, y \in I$, $x <^* y$ if and only if $\rho(x) <^* \rho(y)$, where $\rho : I \rightarrow \gamma$ is the (unique) order-homeomorphism.

For two sets of ordinals $I \subseteq J$, we say that I is a skeleton of J if $\rho[I]$ is a skeleton of $ord(J)$, where $ord(J)$ denotes the order-type of J and $\rho : J \rightarrow ord(J)$ is the unique order-preserving bijection. The notation $I \subseteq^{sk} J$ means that I is a skeleton of J . Given a skeleton $I \subseteq^{sk} \gamma$, a pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$ and a singleton coloring $\mathfrak{r} : \gamma \rightarrow n$, we define the induced colorings of $\mathfrak{c}$ and $\mathfrak{r}$ with respect to I as the colorings $\mathfrak{c}_I : [\gamma]^2 \rightarrow n$ and $\mathfrak{r}_I : \gamma \rightarrow n$ respectively as follows,

$$\mathfrak{c}_I(\{\alpha, \beta\}) = \mathfrak{c}(\{\rho^{-1}(\alpha), \rho^{-1}(\beta)\}), \quad \mathfrak{r}_I(\alpha) = \mathfrak{r}(\rho^{-1}(\alpha))$$

where $\rho : I \rightarrow \gamma$ is the unique order-homeomorphism. It is straightforward to check that the image of any homogeneous closed copy of θ in γ with respect to $\mathfrak{c}_I$ under the map ρ^{-1} is a homogeneous closed copy of θ in $I \subseteq \gamma$ with respect to $\mathfrak{c}$. We shall later use this observation to assume without loss of generality that our colorings have special properties.

Definition 6. For each ordinal $\alpha < \omega^\omega$, let $\hat{\alpha}$ be the unique ordinal such that $\alpha \triangleleft^* \hat{\alpha}$. For each $r \in \mathbb{N}$ and $I \subseteq^{sk} J$, we say that I is an r -**skeleton** of J if

$$\{\alpha \in J \mid \{\beta \mid \beta \triangleleft^* \hat{\alpha}\} \cap I \neq \emptyset \text{ and } L(\rho_J(\alpha)) \leq r\} \subseteq I$$

and denote it by $I \subseteq_r^{sk} J$.

Informally, if I is an r -skeleton of J , for each set $\mathfrak{L}_\alpha := \{\beta \mid \beta \triangleleft^* \hat{\alpha}\}$ for each $\alpha \in J$, while passing from J to I , we expect $\mathfrak{L}_\alpha$ to either be completely dropped in I , or for its first r elements to be kept. Next, we will define certain "large" subsets of ordinals less than ω^ω .

Definition 7. Let $k, r \in \mathbb{N}$. For each $0 \leq m \leq k$, the sets $F(\omega^k)_m^r$ are defined recursively as follows.

$$F(\omega^k)_m^r = \begin{cases} \{\omega^k\} & \text{if } m = k, \\ \bigcup_{\alpha \in F(\omega^k)_{m+1}^r} \{\beta \in T^{=m}(\omega^k) : \beta \triangleleft^* \alpha, L(\beta) > r\} & \text{if } m < k. \end{cases}$$

This definition is extended to all ordinals less than ω^ω as follows. For every $\theta < \omega^\omega$ with $CB(\theta) = 0$, we set $F(\theta)_0^r = \{\theta\}$ and for every $\theta < \omega^\omega$ with $CB(\theta) \neq 0$, we define

$$F(\theta)_m^r = \rho^{-1} \left[F(\omega^{CB(\theta)})_m^r \right]$$

where $\rho : \{\alpha : \alpha \triangleleft^* \theta\} \cup \{\theta\} \rightarrow \omega^{CB(\theta)} + 1$ is the unique order preserving map. For a pictorial representation of these "large" sets in the partial ordering $\triangleleft^*$, see Figure 2.1.

Lastly we define some "container" sets for these large sets. For some $\theta < \omega^\omega$ and $m \leq CB(\theta)$, we set

$$\begin{aligned} \mathfrak{F}(\theta)_m^r &= \{A \subseteq T^{=m}(\theta) \mid F(\theta)_m^r \subseteq A\} \\ \mathfrak{F}(\theta)_m &= \bigcup_{r \in \mathbb{N}} \mathfrak{F}(\theta)_m^r \end{aligned}$$

2.2.1 Good Colorings

The main theorem of the following subsections is Proposition 5, which will allow us to take a "canonical" skeleton $I \subseteq^{sk} \gamma$ for each pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$ and work on I instead of γ . This result first appeared in [19, Proposition 3.11]. However, we made slight modifications on some definitions and lemmas related to [19, Proposition 3.11] to meet our needs and hence, we will hereby include proofs of these modified versions.

For the rest of this chapter, we will take γ to be a successor ordinal for our purposes. That is, $\gamma = \omega^{\beta_1} + \dots + \omega^{\beta_{k_\gamma}} + 1$ for ordinals β_i, β_j with $\beta_i \leq \beta_j$ for $i > j$.

First we will introduce r -good skeletons and show that each singleton coloring $\mathfrak{r} : \gamma \rightarrow n$ has an r -good skeleton.

Definition 8. Let $\mathfrak{r} : \gamma \rightarrow n$ be a singleton coloring and $r \in \mathbb{N}$. We say that $\mathfrak{r}$ is **r -good** if for every $\theta < \gamma$ and $l \leq CB(\theta)$, there is some color $c < n$ such that,

$$\{\alpha \in T^{=l}(\theta) \mid \mathfrak{r}(\alpha) = c\} \in \mathfrak{F}(\theta)_l^r$$

We say that $I \subseteq^{sk} \gamma$ is **r -good with respect to $\mathfrak{r}$** if $\mathfrak{r}_I$ is r -good.

In other words, a coloring $\mathfrak{r}$ is r -good if $\mathfrak{r}$ is constant on a "large" set of each level on the tree below each ordinal $\theta < \gamma$. We will now show that every singleton coloring $\mathfrak{r} : \gamma \rightarrow n$ has an r -good skeleton.

Lemma 1 ([19, Lemma 3.2]). Let $\mathfrak{r} : \gamma \rightarrow n$ be a singleton coloring. Then, for any $r \in \mathbb{N}$, there exists some $I \subseteq_r^{sk} \gamma$ which is r -good with respect to $\mathfrak{r}$.

Proof. Let $r \in \mathbb{N}$. By [19, Lemma 2.7], we know that if we can find a skeleton $J_i \subseteq_r^{sk} CC_\gamma(i)$ for every $1 \leq i \leq k_\gamma$, then $I \subseteq_r^{sk} \gamma$ where $I = \bigcup_{1 \leq i \leq k_\gamma} J_i$. Clearly, for a coloring $\mathfrak{r} : \gamma \rightarrow n$, if all $\mathfrak{r} \upharpoonright_{J_i}$ considered as colorings on $ord(J_i)$ are r -good, then $\mathfrak{r}_I$ is also r -good. Thus, in order to prove that there exists an r -good r -skeleton for γ , it is enough to prove the existence of an r -good r -skeleton for each $CC_\gamma(i)$. Observe that each $CC_\gamma(i)$ is order-homeomorphic to $\omega^k + 1$ for some $k \in \mathbb{N}^+$ or to $1 = \{0\}$. For the case $CC_\gamma(i)$ is order-homeomorphic to $1 = \{0\}$, the result is clear. So we will prove the claim for $\gamma = \omega^k + 1$ for all $k \in \mathbb{N}^+$.

We will prove the claim by induction on k . For the base case, consider $\gamma = \omega + 1$. By the infinite Ramsey's theorem, there exists a color $c < n$ such that

$$S = \{\alpha < \omega \mid \mathfrak{r}(\alpha) = c\}$$

is an infinite set. Let $S^+ = S \cup \{0, \dots, r\}$. Then, $I = S^+ \cup \{\omega\}$ is an r -good r -skeleton of γ .

Let $k \in \mathbb{N}^+$ and suppose that the claim holds for k . Then, for each $i \in \mathbb{N}^+$, the subtree of the form $T_i := T(\omega^k \cdot i)$, whose order type is $\omega^k + 1$, has an r -good r -skeleton with respect to $\mathfrak{r} \upharpoonright_{T_i}$ considered as a coloring on $\text{ord}(T_i)$. For each i , let C_i be the set of colors $\{c_i^0, \dots, c_i^k\}$ where c_i^j is the unique color for which

$$\{\alpha \in T^{=j}(\omega^k \cdot i) \mid \mathfrak{r} \upharpoonright_{T_i}(\alpha) = c_i^j\} \in \mathfrak{F}(\omega^k \cdot i)_j^r$$

for each $0 \leq j \leq k$. By pigeonhole principle, there is an infinite set $S \subseteq \mathbb{N}$ where $C_i = C_j$ for all $i, j \in S$. Let $S^+ = S \cup \{1, \dots, r\}$. Then one can check that $I = \{\omega^{k+1}\} \cup \{T_i \mid i \in S^+\}$ is an r -good r -skeleton of $\gamma = \omega^{k+1} + 1$. $\square$

Next, we define r -goodness and goodness of ordinals $\alpha < \gamma$ with respect to pair colorings $\mathfrak{c} : [\gamma]^2 \rightarrow n$ and show that every pair coloring $\mathfrak{c}$ has a skeleton for which every α is good.

In what follows, for a pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$ and $\alpha < \gamma$, we write $\mathfrak{c}_\alpha$ for any singleton coloring $\mathfrak{c}_\alpha : \gamma \rightarrow n$ assigning $\mathfrak{c}_\alpha(\beta) = \mathfrak{c}(\{\alpha, \beta\})$ for every $\beta \in \gamma - \{\alpha\}$.

Definition 9 ([19, Definition 3.5]). *Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be some pair coloring. For $\alpha < \gamma$ and $r \in \mathbb{N}$, if $\mathfrak{c}_\alpha$ is r -good, say that α is **r -good for $\mathfrak{c}$** . We say that α is **good for $\mathfrak{c}$** whenever α is r -good for $\mathfrak{c}$ for some $r \in \mathbb{N}$.*

Lemma 2 ([19, Lemma 3.7]). *Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be some pair coloring. Then, there exists some $I \subseteq^{sk} \gamma$ such that every $\alpha \in I$ is good with respect to I for $\mathfrak{c}$.*

Proof. The proof of this lemma is exactly the same as the proof of [19, Lemma 3.7]. $\square$

Now we will define the goodness function t_α for each ordinal $\alpha < \gamma$ which is good with respect to a pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$.

Definition 10. Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be a pair coloring. Let α be good for $\mathfrak{c}$. We set $i_\alpha = \text{CNF}_\gamma(\alpha)$ and $\epsilon_m = \sup CC_\gamma(m)$ for every $1 \leq i_\alpha \neq m \leq k_\gamma$. For each $1 \leq m \neq i_\alpha \leq k_\gamma$ and for each $l \leq CB(\epsilon_m)$, let $t_\alpha(m, l)$ denote the unique color for which

$$\{\beta \in T^{\leq l}(\epsilon_m) \mid \mathfrak{c}(\{\alpha, \beta\}) = t_\alpha(m, l)\} \in \mathfrak{F}(\epsilon_m)_l.$$

Observe that such $t_\alpha(m, l)$ in Definition 10 exists since $\mathfrak{c}_\alpha$ is r -good for some $r \in \mathbb{N}$. The function $t_\alpha(\cdot, \cdot)$ is called the goodness function of α . The goodness function of an element α which is good with respect to $\mathfrak{c}$, takes (m, l) , where m is the index of a different non-empty connected component (in the tree representation) than α is in (i.e. $i_\alpha \neq m \leq k_\gamma$) and l is a Cantor-Bendixson rank less than or equal to $CB(\sup CC_\gamma(m))$. Then it maps (m, l) to the color $\mathfrak{c}_\alpha$ takes on a large set over the elements $\beta < \gamma$ with $\text{CNF}_\gamma(\beta) = m$ and $CB(\beta) = l$.

As the last definition of this subsection, we will define *uniform goodness* of a pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$. We will use uniform goodness while proving the existence of a canonical skeleton for every pair coloring $\mathfrak{c}$, which will be defined later.

Definition 11 ([19, Definition 3.9]). Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be a pair coloring. We say that $\mathfrak{c}$ is **uniformly good** if the followings hold:

- Every $\alpha < \gamma$ is good with respect to $\mathfrak{c}$.
- For any $\alpha < \gamma$, the goodness function t_α of α depends only on $\text{CNF}_\gamma(\alpha)$ and $CB(\alpha)$. Namely, there is a function $\tilde{\mathfrak{c}}$ independent of α such that

$$t_\alpha = \tilde{\mathfrak{c}}(\text{CNF}_\gamma(\alpha), CB(\alpha))$$

We say that $I \subseteq^{sk} \gamma$ is **uniformly good with respect to $\mathfrak{c}$** if $\mathfrak{c}_I$ is uniformly good.

Notation: For the coloring $\tilde{\mathfrak{c}}$ given in Definition 11, we will abuse the notation and write $\tilde{\mathfrak{c}}(i_1, j_1; i_2, j_2)$ to denote the color $t_\alpha(i_2, j_2)$ whenever $\alpha < \gamma$ is such that $\text{CNF}_\gamma(\alpha) = i_1$ and $CB(\alpha) = j_1$.

Proposition 4 ([19, Proposition 3.11]). Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be a pair coloring. Then, there exists some $I \subseteq^{sk} \gamma$ which is uniformly good with respect to $\mathfrak{c}$.

Proof. By Lemma 2, the existence of some $J \subseteq^{sk} \gamma$ such that every $\alpha \in J$ is good with respect to J for $\mathfrak{c}$ is guaranteed. Thus, by replacing γ with J , we may assume without loss of generality that every $\alpha \in \gamma$ is good for $\mathfrak{c}$. We will show that there exists $I \subseteq^{sk} \gamma$ such that the goodness function t_α depends only on $CB(\alpha)$ and $CC_\gamma(\alpha)$ for each $\alpha \in I$.

Observe that there are only finitely many goodness functions t_α for $\alpha \in \gamma$. Consider the singleton coloring on γ given by $\mathfrak{r}(\alpha) = t_\alpha$ for all $\alpha \in \gamma$. It now follows from Lemma 1 that there exists a skeleton $I \subseteq_0^{sk} \gamma$ which is 0-good with respect to $\mathfrak{r}$. This means that, for every $1 \leq m \leq k_\gamma$ and $0 \leq l \leq CB(\epsilon_m)$, the goodness functions t_α are the same for every $\alpha \in T(\epsilon_m)$ with $CB(\alpha) = l$. $\square$

2.2.2 Normal Colorings

In this subsection, we will give the definition of a normal coloring.

Definition 12 ([19, Definition 3.3]). *Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be some pair coloring. Say $\mathfrak{c}$ is **normal** if whenever $\beta_1, \beta_2 < \gamma$ with $\beta_1 <^* \beta_2$, the value $\mathfrak{c}(\{\beta_1, \beta_2\})$ depends only on $CB(\beta_1), CB(\beta_2)$ and $CNF_\gamma(\beta_2)$ ($= CNF_\gamma(\beta_1)$). Namely, there is a function $\hat{\mathfrak{c}}$, independent of β_1, β_2 such that*

$$\mathfrak{c}(\{\beta_1, \beta_2\}) = \hat{\mathfrak{c}}(CNF_\gamma(\beta_2), CB(\beta_2), CB(\beta_1))$$

*Say that $I \subseteq^{sk} \gamma$ is **normal with respect to $\mathfrak{c}$** if $\mathfrak{c}_I$ is normal.*

Lemma 3 ([19, Lemma 3.4]). *Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be some pair coloring. Then, there exists a $I \subseteq^{sk} \gamma$ normal with respect to $\mathfrak{c}$.*

2.2.3 Canonical Colorings

In this subsection we will introduce canonical colorings and prove our main proposition Proposition 5 stating the existence of a canonical skeleton for every pair coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$.

Definition 13. *Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be a pair coloring. We say that $I \subseteq^{sk} \gamma$ is **canonical with respect to $\mathfrak{c}$** if I is normal and uniformly good with respect to $\mathfrak{c}$. If γ itself is canonical with respect to $\mathfrak{c}$ we call $\mathfrak{c}$ a **canonical coloring**.*

Proposition 5 ([19, Proposition 3.11]). *Let $\mathfrak{c} : [\gamma]^2 \rightarrow n$ be a pair coloring. Then, there exists some $I \subseteq^{sk} \gamma$ canonical with respect to $\mathfrak{c}$.*

Proof. This immediately follows from an application of Proposition 4 and a subsequent application of Lemma 3. □

To summarize what we have obtained in the past subsections, given a canonical coloring $\mathfrak{c} : [\gamma]^2 \rightarrow n$, we can find two special functions $\hat{\mathfrak{c}}$ and $\tilde{\mathfrak{c}}$ such that:

For $\alpha <^* \beta < \gamma$, in which case $CNF_\gamma(\beta) = CNF_\gamma(\alpha)$, we have

$$\hat{\mathfrak{c}}(CNF_\gamma(\beta), CB(\beta), CB(\alpha)) = \mathfrak{c}(\{\alpha, \beta\})$$

For $\alpha, \beta < \gamma$ with $CNF_\gamma(\beta) \neq CNF_\gamma(\alpha)$, we have

$$\tilde{\mathfrak{c}}(CNF_\gamma(\alpha), CB(\alpha); CNF_\gamma(\beta), CB(\beta)) = t_\alpha(CNF_\gamma(\beta), CB(\beta))$$

which is the color α takes on a large subset of $[\gamma; CNF_\gamma(\beta), CB(\beta)]$. Unpacking the above definition, one can check that, for a canonical coloring $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$, the following are equivalent

- $\tilde{\mathfrak{c}}(i, j; k, \ell) = c$
- For all α with $CNF_\gamma(\alpha) = i$ and $CB(\alpha) = j$, there exists $r \in \mathbb{N}$ such that $\mathfrak{c}(\{\alpha, \beta\}) = c$ for every $\beta \in F(\epsilon_k)_\ell^r$.

In our arguments, we shall use this equivalence whenever we have $\tilde{\mathfrak{c}}(i, j; k, \ell) = c$. Also note that, $\mathfrak{c}$ being normal carries information about the colors $\mathfrak{c}(\{\alpha, \beta\})$ where $CNF_\gamma(\alpha) = CNF_\gamma(\beta)$; whereas, $\mathfrak{c}$ being uniformly good carries information about the colors $\mathfrak{c}(\{\alpha, \beta\})$ where $CNF_\gamma(\alpha) \neq CNF_\gamma(\beta)$.

2.2.4 ω -homogeneous Colorings

Next we will introduce a new kind of special coloring, ω -homogeneous coloring, from which we will take advantage of while finding upper bounds.

A coloring $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$ is said to be ω -homogeneous if for all $\alpha < \gamma$ there exists $c_\alpha \in \{0, 1\}$ such that $\mathfrak{c}(\{\beta, \theta\}) = c_\alpha$ for all $\beta, \theta \triangleleft^* \alpha$. In other words, an ω -homogeneous coloring is a coloring for which the children of each node in the tree representation of γ with respect to $<^*$ form a homogeneous copy of ω .

Lemma 4. *Let $\gamma < \omega^\omega$ be an ordinal. For every coloring $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$ there exists a skeleton $I \subseteq \gamma$ such that $\mathfrak{c}_I : [I]^2 \rightarrow \{0, 1\}$ is ω -homogeneous.*

Proof. We will first prove the result for ordinals of the form $\omega^k + 1$ with $k \in \mathbb{N}^+$. Let $k \in \mathbb{N}^+$ and let $\mathfrak{c} : [\omega^k + 1]^2 \rightarrow \{0, 1\}$ be a coloring. For every $\delta \leq \omega^k$, we define $H(\delta) \subseteq \delta + 1$ inductively on the Cantor-Bendixson rank of δ as follows.

- If $CB(\delta) = 0$, then we set $H(\delta) = \{\delta\}$.
- If $CB(\delta) > 0$, then choose some infinite homogeneous $J_\delta \subseteq \{\alpha : \alpha \triangleleft^* \delta\}$. Observe that such a set J_δ must exist by the infinite Ramsey's theorem. Now set $H(\delta) = \{\delta\} \cup \bigcup_{\alpha \in J_\delta} H(\alpha)$.

A straightforward induction on $1 \leq i \leq k$ implies that, for all $\lambda \leq \omega^k$ such that $CB(\lambda) = i$, the set $H(\lambda)$ is a skeleton of

$$\{\theta : \theta <^* \lambda\} \cup \{\lambda\}$$

Consequently, $I = H(\omega^k)$ is a skeleton of $\omega^k + 1$. That $\mathfrak{c}_I$ is ω -homogeneous trivially follows from the choice of J_δ 's.

To finish the proof, let $\gamma < \omega^\omega$ be an ordinal. The claim clearly holds for $\gamma = 0$. So suppose that $\gamma = \omega^{m_1} + \dots + \omega^{m_n}$ where $m_1 \geq \dots \geq m_n$ are natural numbers. Let $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$ be a coloring. For each $1 \leq i \leq n$, consider the set

$$\{\omega^{m_1} + \dots + \omega^{m_{i-1}} + \alpha : \alpha \leq \omega^{m_i}\}$$

which is a copy of $\omega^{m_i} + 1$ if $m_i \neq 0$, or, a copy of $1 = \{0\}$ if $m_i = 0$. Let I_i be a skeleton obtained by applying the argument above with the restriction of $\mathfrak{c}$ to the copy of $\omega^{m_i} + 1$ if $m_i \neq 0$, and, let I_i be the copy of $1 = \{0\}$ itself if $m_i = 0$. Then it is easily verified that

$$I = \left(\bigcup_{i=1}^n I_i \right) - \{\gamma\}$$

is a skeleton for which $\mathfrak{c}_I$ is ω -homogeneous. □

We should mention that, our use of ω -homogeneous colorings in the proofs is non-essential and is due to our not wanting to apply the infinite Ramsey's theorem repeatedly.

Observation. It follows from Lemma 4 and Proposition 5 that, in order to prove an inequality of the form $R(\alpha, \beta) \leq \gamma$, it suffices to prove that any ω -homogeneous canonical coloring of γ has a red homogeneous copy of α or a blue homogeneous copy of β . The reason is that, given a coloring $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$, we can first find a skeleton $I \subseteq \gamma$ for which $\mathfrak{c}_I : [\gamma]^2 \rightarrow \{0, 1\}$ is ω -homogeneous and then, find a skeleton $J \subseteq \gamma$ for which $(\mathfrak{c}_I)_J : [\gamma]^2 \rightarrow \{0, 1\}$ is both canonical and ω -homogeneous. (For the latter claim, observe that the induced coloring of an ω -homogeneous coloring with respect to a skeleton is ω -homogeneous.) But then, any homogeneous subset of γ with respect to $(\mathfrak{c}_I)_J$ can be pulled back to a homogeneous subset of γ with respect to $\mathfrak{c}$ of the same order type.

Lemma 5. *Let $n \geq 2$ be an integer and let $\mathfrak{c} : [\omega^2]^2 \rightarrow \{0, 1\}$ be a normal ω -homogeneous coloring with no red homogeneous closed copy of $\omega + n$ and no blue homogeneous closed copy of ω . Then*

(a) $\hat{\mathfrak{c}}(1, 1, 0) = 1$ or

(b) For every $i < \omega$, we have that

$$\{\beta \in [\omega^2; 1, 1] : \{\alpha \in W_i : \mathfrak{c}(\{\alpha, \beta\}) = 1\} \text{ is cofinal in } W_i\}$$

is cofinal in $[\omega^2; 1, 1]$, where $W_i = \{\omega \cdot i + m : 0 < m < \omega\}$.

Proof. Assume towards a contradiction that $\hat{\mathfrak{c}}(1, 1, 0) = 0$ and that, for some $i < \omega$,

$$\{\beta \in [\omega^2; 1, 1] : \{\alpha \in W_i : \mathfrak{c}(\{\alpha, \beta\}) = 1\} \text{ is cofinal in } W_i\}$$

is not cofinal in $[\omega^2; 1, 1] \cong \omega$ and hence, is finite. Since $W_i \cong \omega$, the complement of this set

$$\{\beta \in [\omega^2; 1, 1] : \{\alpha \in W_i : \mathfrak{c}(\{\alpha, \beta\}) = 1\} \text{ is finite}\}$$

is cofinite in $[\omega^2; 1, 1]$. Thus there exist ordinals $\omega \cdot (i + 1) = \beta_0 < \beta_1 < \dots < \beta_{n-1}$ in $[\omega^2; 1, 1]$ such that $\{\alpha \in W_i : \mathfrak{c}(\{\alpha, \beta_j\}) = 1\}$ is finite for each $1 \leq j \leq n - 1$.

Hence the set

$$H_i = \{\alpha \in W_i : \mathfrak{c}(\{\alpha, \beta_j\}) = 0 \text{ for all } 1 \leq j \leq n-1\}$$

is cofinite in W_i . Since $\mathfrak{c}$ is ω -homogeneous and there exists no blue homogeneous copy of ω , the sets W_i and $\{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ are both red homogeneous. Also, $\hat{\mathfrak{c}}(1, 1, 0) = 0$ implies that $c(\{\alpha, \omega \cdot (i+1)\}) = 0$ for all $\alpha \in W_i$. It follows that the set

$$H_i \cup \{\omega \cdot (i+1), \beta_1, \beta_2, \dots, \beta_{n-1}\}$$

is a red homogeneous closed copy of $\omega + n$, which is a contradiction. □





CHAPTER 3

PROOF OF THEOREM 1

In this chapter, we will prove Theorem 1. To do that, we shall first construct a special triangle-free graph whose vertices are subsets of ordinals and whose edges shall induce a coloring that witnesses $\omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + (n - 1) \not\rightarrow_{cl} (\omega + n, 3)^2$. Throughout this chapter, we let

$$\gamma = \omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + (n - 1)$$

Let $n \geq 3$ be an integer and set $K = R(n, 3) - n$. For each integer $1 \leq i \leq n$, set

$$\begin{aligned} A_1 &= [\gamma; \mathbf{1}, \mathbf{0}] = \{\alpha + 1 : \alpha < \omega^2\} \cup \{0\} \\ A_i &= [\gamma; \mathbf{i}, \mathbf{0}] = \{\omega^2 \cdot (i - 1) + \alpha + 1 : \alpha < \omega^2\} && \text{if } i \neq 1 \\ B_i &= [\gamma; \mathbf{i}, \mathbf{1}] = \{\omega^2 \cdot (i - 1) + \omega \cdot (k + 1) : k < \omega\} \\ L_i &= [\gamma; \mathbf{i}, \mathbf{2}] = \{\omega^2 \cdot i\} \end{aligned}$$

and for each integer $n < i \leq n + K$, set

$$\begin{aligned} C_i &= [\gamma; \mathbf{i}, \mathbf{0}] = \{\omega^2 \cdot n + \omega \cdot (i - n - 1) + \alpha + 1 : \alpha < \omega\} \\ L_i &= [\gamma; \mathbf{i}, \mathbf{1}] = \{\omega^2 \cdot n + \omega \cdot (i - n)\} \end{aligned}$$

In addition, set

$$R = \{\omega^2 \cdot n + \omega \cdot K + m : 1 \leq m \leq n - 2\}$$

It is easily seen that

$$V = \{A_i, B_i, L_i : 1 \leq i \leq n\} \cup \{C_i, L_i : n < i \leq n + K\} \cup \{R\}$$

is a partition of the ordinal $\gamma = \omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + (n - 1)$. Observe that, by the definition of Ramsey numbers, there exists a coloring

$$\mathbf{c} : [\{L_i : 1 \leq i < R(n, 3)\}]^2 \rightarrow \{0, 1\}$$

with no red homogeneous sets of size n and no blue homogeneous set of size 3. We want to find a red homogeneous set of size $n - 1$ for this coloring. The following proposition guarantees the existence of such a homogeneous set.

Proposition 6. $R(n - 1, 3) \leq R(n, 3) - 1$.

Proof. Set $N = R(n, 3)$ and let $f : [X]^2 \rightarrow \{0, 1\}$ be a coloring of a set X with $N - 1$ elements, say, $X = \{v_1, v_2, \dots, v_{N-1}\}$. We wish to show that there exists a red homogeneous subset of X of size $n - 1$ or a blue homogeneous subset of X of size 3, which would prove that $R(n - 1, 3) \leq N - 1$.

Let v_N be not in X and extend f to a coloring $\hat{f} : [X \cup \{v_N\}]^2 \rightarrow \{0, 1\}$ by defining

$$\hat{f}(\{v_i, v_j\}) = \begin{cases} f(\{v_i, v_j\}) & \text{if } 1 \leq i, j \leq N - 1 \\ 0 & \text{if } 1 \leq i \leq N - 1 \text{ and } j = N \end{cases}$$

By the definition of $R(n, 3)$, there exists a red homogeneous subset of $X \cup \{v_N\}$ of size n or a blue homogeneous subset of $X \cup \{v_N\}$ of size 3. Observe that no blue homogeneous subset of $X \cup \{v_N\}$ can contain the element v_N . It then follows that there exists a red homogeneous subset of X of size at least $n - 1$ or a blue homogeneous subset of X of size 3. $\square$

It follows from Proposition 6 that there exists a red homogeneous set of size $n - 1$ with respect to $\mathbf{c}$. By relabeling if necessary, we may assume without loss of generality that the set $\{L_1, L_2, \dots, L_{n-1}\}$ is red homogeneous. Fix such a coloring $\mathbf{c}$.

We shall next construct a graph $\mathbf{G}_n$ on the vertex set V using $\mathbf{c}$. Define the edge sets E_1, E_2, E_3 and E_4 as follows.

$$E_1 = \{\{L_i, A_j\}, \{A_i, B_j\} : 1 \leq i < j \leq n\} \cup \{\{A_i, B_i\}, \{B_i, L_i\} : 1 \leq i \leq n\}$$

$$E_2 = \{\{C_i, L_i\} : n < i < n + K\}$$

$$E_3 = \{\{L_i, L_j\} : \mathbf{c}(\{L_i, L_j\}) = 1, 1 \leq i \neq j < R(n, 3)\}$$

$$E_4 = \{\{X, A_i\} : X \in W, 1 \leq i \leq n\}$$

where $W = \{C_{n+1}, C_{n+2}, \dots, C_{n+K}, L_{n+K}, R\}$. Consider the graph

$$\mathbf{G}_n = (V, E_1 \cup E_2 \cup E_3 \cup E_4)$$

We shall not attempt to perform the currently impossible task of drawing a diagram representation of G_n , simply because there is no known way to find such a map c for an arbitrary n . However, in Figure 3.1, a diagram representation of G_3 is given for some appropriate choice of c . In Figure 3.2, we do provide a diagram representation of the subgraph $(V, E_1 \cup E_2)$ for arbitrary n so that the reader may follow the arguments on this diagram if necessary.

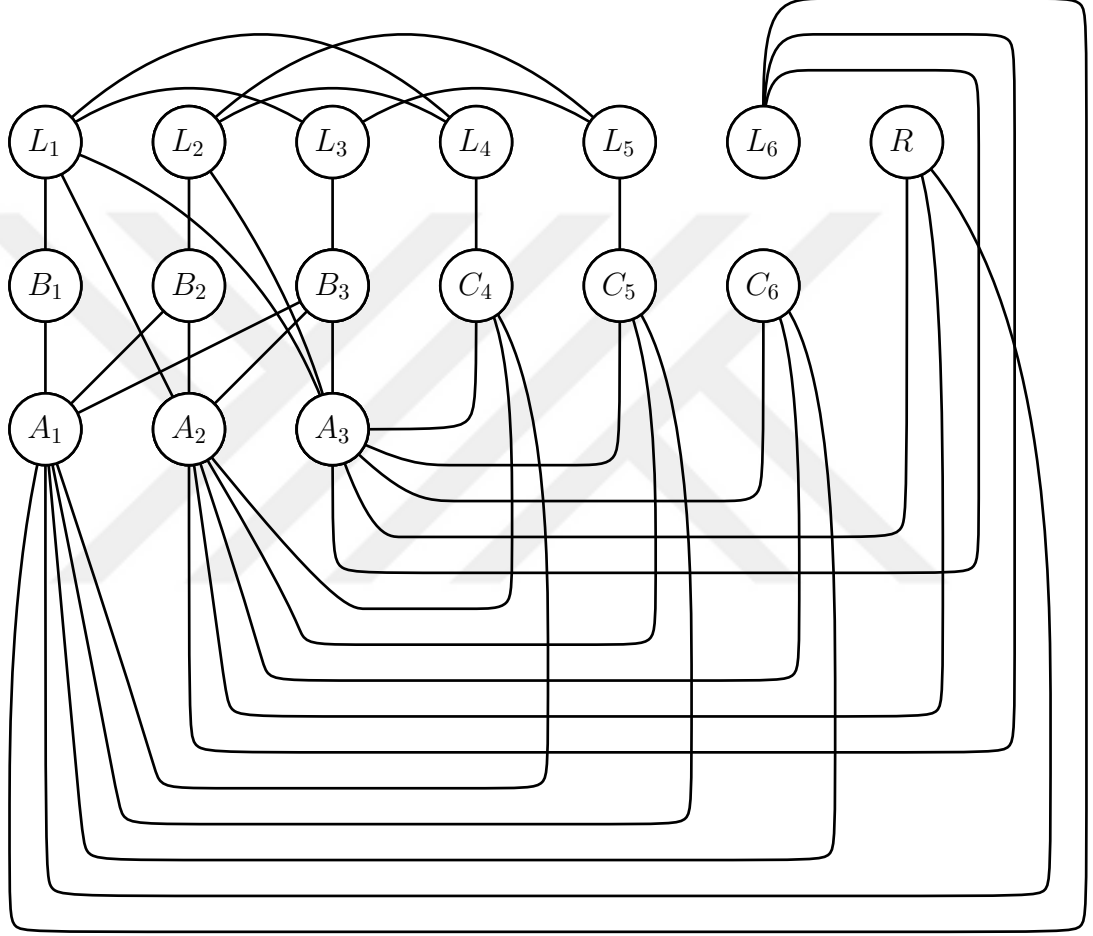


Figure 3.1: A diagram representation of G_3 for a choice of c .

Lemma 6. *The graph G_n is triangle-free.*

Proof. We shall first prove that (V, E_1) is triangle-free. Assume towards a contradiction that there exists a triangle $T \subseteq V$ in the graph (V, E_1) . Since no two $2B_i$'s are adjacent and no two L_i 's are adjacent in (V, E_1) , we must have that there exists

$1 \leq j \leq n$ with $A_j \in T$. On the other hand, the set of neighbors of A_j in (V, E_1) is

$$\{B_k : j \leq k \leq n\} \cup \{L_i : 1 \leq i < j\}$$

Moreover, for $1 \leq i, k \leq n$, we have that L_i is adjacent to B_k if and only if $i = k$. It follows that no two neighbors of A_j are adjacent, which is a contradiction.

Having proven that (V, E_1) is triangle-free, it is easily verified that $(V, E_1 \cup E_2)$ is triangle-free. This follows from the fact that the edges in E_2 are

- not incident with vertices that are incident to the edges in E_1 , and
- not incident with each other.

We shall next prove that $(V, E_1 \cup E_2 \cup E_3)$ is triangle-free. Suppose that there exists a triangle $T = \{X, Y, Z\} \subseteq V$ in the graph $(V, E_1 \cup E_2 \cup E_3)$. Since $(V, E_1 \cup E_2)$ is triangle-free, we must have that some edge in E_3 are incident to vertices in T , say, $X = L_i$ and $Y = L_j$ for some $1 \leq i < j < R(n, 3)$. Since $\{L_1, \dots, L_{n-1}\}$ was arranged to be red homogeneous with respect to $\mathfrak{c}$, we must have $n \leq j$.

Then, by construction, we have that $Z = B_n$, $Z = C_j$ or $Z = L_k$ for some $1 \leq k \neq j < R(n, 3)$. The first case leads to a contradiction as the only neighbor of B_n among L_m 's in this graph is L_n . The second case leads to a contradiction as C_j 's only neighbor in this graph is L_j . The third case leads to a contradiction because there are no edges between L_m 's in the graph $(V, E_1 \cup E_2)$ and consequently, the third case happening would imply that all the edges of this triangle are from E_3 , in which case we would have $\mathfrak{c}(\{L_i, L_j\}) = \mathfrak{c}(\{L_j, L_k\}) = \mathfrak{c}(\{L_i, L_k\}) = 1$, creating a blue homogeneous set of size 3 with respect to $\mathfrak{c}$. Thus $(V, E_1 \cup E_2 \cup E_3)$ is triangle-free.

Finally, we shall prove that $\mathbf{G}_n = (V, E_1 \cup E_2 \cup E_3 \cup E_4)$ is triangle-free. Suppose that there exists a triangle $T = \{X, Y, Z\} \subseteq V$ in $\mathbf{G}_n$. Since $(V, E_1 \cup E_2 \cup E_3)$ is triangle-free, we must have that some edge in E_4 are incident to vertices in T , say, $X \in W$ and $Y = A_i$ for some $1 \leq i \leq n$. The set of neighbors of X is a subset of $\{L_j : n < j < R(n, 3)\}$ and the set of neighbors of Y is a subset of $\{L_j : 1 \leq j < i \leq n\} \cup \{B_k : i \leq k \leq n\} \cup W$. However, these sets do not intersect and hence, we have a contradiction. Therefore, $\mathbf{G}_n$ is triangle-free. $\square$

We are now ready to prove the first main result.

Proof of Theorem 1. Let $n \geq 3$ be a positive integer and set

$$\gamma = \omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + (n - 1)$$

Consider the coloring $c : [\gamma]^2 \rightarrow \{0, 1\}$ given by $c(\{\alpha, \beta\}) = 1$ if and only if the vertices containing α and β in $\mathbf{G}_n$ are adjacent.

Since $\mathbf{G}_n$ is triangle-free by Lemma 6, there does not exist a blue homogeneous copy of $3 = \{0, 1, 2\}$. We shall next show that there exists no red homogeneous closed copy X of $\omega + n$. Assume to the contrary that there exists such a set $X \subseteq \gamma$. Observe that, by definition, the vertices to which the elements of X belong are not adjacent in $\mathbf{G}_n$. Let $h : \omega + n \rightarrow X$ be the order-homeomorphism and let us denote $h(\alpha)$ by $\dot{\alpha}$ for all $\alpha \in \omega + n$. So we can write X as

$$\dot{0} < \dot{1} < \dots < \dot{\omega} < \omega \dot{+} 1 < \dots < \omega + \dot{n} - 1$$

Since $\dot{\omega}$ is a limit ordinal, we have that $\dot{\omega} \in B_i$ for some $1 \leq i \leq n$, or, $\dot{\omega} \in L_i$ for some $1 \leq i \leq R(n, 3)$. We now analyze these cases.

Suppose that $\dot{\omega} \in B_i$ for some $1 \leq i \leq n$. Then, for cofinitely many $k \in \omega$, we have $\dot{k} \in A_i$. On the other hand, A_i and B_i are adjacent in $\mathbf{G}_n$, which leads to a contradiction.

Suppose that $\dot{\omega} \in L_i$ for some $1 \leq i \leq n$. Then, for cofinitely many $k \in \omega$, we have $\dot{k} \in A_i$ or $\dot{k} \in B_i$. But, as B_i and L_i are adjacent, we obtain that $\dot{k} \in A_i$ for cofinitely many $k \in \omega$. Recall that

- A_i is adjacent to each vertex in $\{B_j : i \leq j \leq n\} \cup W$, and
- L_i is adjacent to each vertex in $\{A_j : i < j \leq n\}$.

So the vertices in $\{A_j : i < j \leq n\} \cup \{B_j : i \leq j \leq n\} \cup W$ cannot contain the elements of X greater than $\dot{\omega}$. It follows that

$$\{\dot{\omega}, \omega \dot{+} 1, \omega \dot{+} 2, \dots, \omega + \dot{n} - 1\} \subseteq \bigcup_{j=i}^{R(n,3)-1} L_j$$

Since each L_j is a singleton and the set on the left-hand side is red homogeneous, we obtain that $\{L_j : i \leq j < R(n, 3)\}$ has a subset of size n , no two vertices of which are adjacent. Recall that the edges between L_j 's in $\mathbf{G}_n$ come from E_3 . Consequently, there exists a red homogeneous set of size n with respect to the coloring $\mathfrak{c}$, which is a contradiction.

Suppose that $\dot{\omega} \in L_i$ for some $n < i < R(n, 3)$. Then, for cofinitely many $k \in \omega$, we have $\dot{k} \in C_i$. On the other hand, C_i and L_i are adjacent in $\mathbf{G}_n$, which leads to a contradiction.

Finally, suppose that $\dot{\omega} \in L_{R(n,3)}$. In this case, we must have

$$\{\dot{\omega} + 1, \dot{\omega} + 2, \dots, \dot{\omega} + \dot{n} - 1\} \subseteq R$$

This is a contradiction as the left-hand side has $n - 1$ elements, whereas, the right-hand side has $n - 2$ elements. We obtained contradictions in all cases. Therefore, there exists no such set X and so

$$\omega^2 \cdot n + \omega \cdot (R(n, 3) - n) + (n - 1) \not\rightarrow_{cl} (\omega + n, 3)^2$$

This completes the proof. □

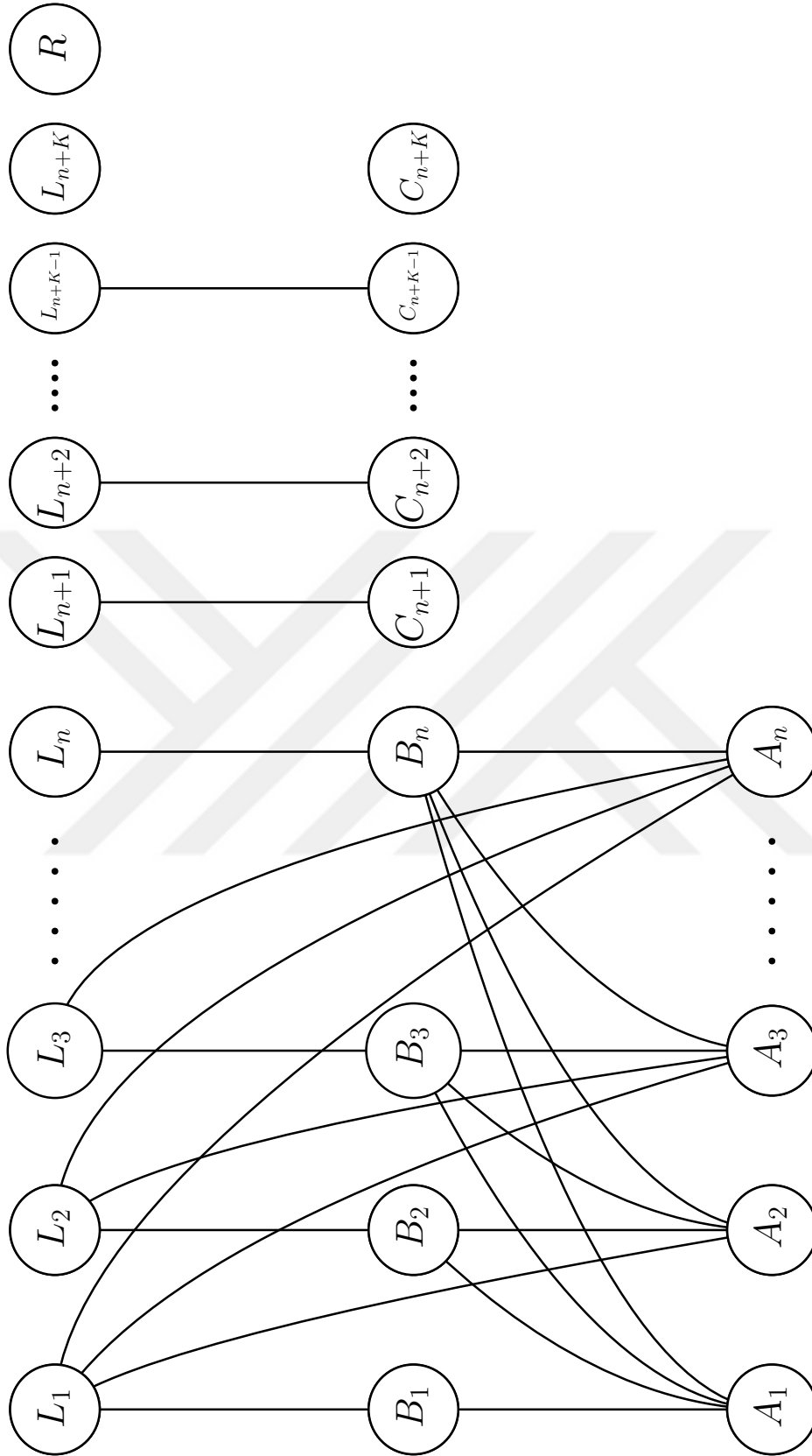


Figure 3.2: A diagram representation of $(V, E_1 \cup E_2)$.



CHAPTER 4

PROOF OF THEOREM 2

In this chapter, we will prove Theorem 2. In order to do this, we will need several technical lemmas. For the following lemmas, fix integers $n, K \geq 2$, the ordinal

$$\gamma = \omega^2 \cdot n + \omega \cdot K + 1$$

and a canonical ω -homogeneous coloring $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$ with no red homogeneous closed copy of $\omega + n$ and no blue homogeneous copy of $3 = \{0, 1, 2\}$. Recall that, since $\mathfrak{c}$ is canonical, there exist functions $\hat{\mathfrak{c}}$ and $\tilde{\mathfrak{c}}$ as in Chapter 2.

The proof of Theorem 2 will be a convoluted case-by-case proof that uses these lemmas which essentially show that certain values of $\tilde{\mathfrak{c}}$ and $\hat{\mathfrak{c}}$ are automatically determined by the non-existence of a red homogeneous $\omega + n$ and a blue homogeneous 3. We shall see that, intuitively speaking, some of the patterns in the graph G_n were unavoidable and had to appear if we are to avoid certain homogeneous sets. To keep track of what is going on, the reader may want to "visualize" the statements and arguments of these lemmas. In order to do this, the reader may pretend that we are constructing a directed graph on the partition

$$\gamma = \bigsqcup_{1 \leq i \leq n, 0 \leq j \leq 2} [\gamma; i, j] \sqcup \bigsqcup_{n+1 \leq i \leq n+K, 0 \leq j \leq 1} [\gamma; i, j]$$

of γ by putting an edge

- from $[\gamma; i, j]$ to $[\gamma; k, \ell]$ if $\tilde{\mathfrak{c}}(i, j; k, \ell) = 1$.
- from $[\gamma; i, 2]$ to $[\gamma; i, \ell]$ if $\hat{\mathfrak{c}}(i, 2, \ell) = 1$.
- from $[\gamma; i, 1]$ to $[\gamma; i, 0]$ for $1 \leq i \leq n$.

We would like to note that $\tilde{c}(i, j; k, \ell) = 1$ does not imply that every pair of ordinals coming from the corresponding vertices have the color 1 under c . It only implies that "most" pairs have the color 1. We would also like to remark that the edges from $[\gamma; i, 1]$ to $[\gamma; i, 0]$ are automatically added regardless of the value of $\hat{c}$, due to Lemma 5, which says that either $\hat{c}(i, 1, 0) = 1$ or there are "many" pairs coming from the corresponding vertices for which c has value 1. Finally, we wish to emphasize that whether we are using a directed graph or an undirected graph has absolutely no role in the proofs. Indeed, our arguments do not refer to any graphs at all. We are simply suggesting this "supplementary" approach if the reader wishes to do more than line-by-line proof checking. With this graph interpretation in mind, the following lemma prevents the existence of certain triangles in this graph. The first four items of this lemma essentially appeared in [19, Lemma 4.3]. Nevertheless, we include the proofs for self-containment.

Lemma 7. *Let $1 \leq k \leq n$ and $0 \leq \ell \leq 2$.*

(a) *For every $1 \leq i \neq k \leq n$ and $0 \leq j \leq 2$,*

$$\tilde{c}(i, j; k, 0) = 0 \text{ or } \tilde{c}(i, j; k, 1) = 0.$$

(a') *For every $n+1 \leq i \leq n+K$ and $0 \leq j \leq 1$,*

$$\tilde{c}(i, j; k, 0) = 0 \text{ or } \tilde{c}(i, j; k, 1) = 0.$$

(b) *For every $1 \leq i \neq k \leq n$,*

$$\tilde{c}(i, 0; k, \ell) = 0 \text{ or } \tilde{c}(i, 1; k, \ell) = 0.$$

(c) *For every $1 \leq i \leq n$, $\hat{c}(i, 2, 0) = 0$ or $\hat{c}(i, 2, 1) = 0$.*

(d) *For every $n+1 \leq i \neq m \leq n+K$, if $\tilde{c}(m, 0; k, \ell) = \tilde{c}(i, 0; k, \ell) = 1$, then $\tilde{c}(m, 0; i, 0) = 0$.*

Proof. To prove (a) and (a'), assume to the contrary that $\tilde{c}(i, j; k, 0) = 1$ and $\tilde{c}(i, j; k, 1) = 1$ for some such i and j . By definition of $\tilde{c}$, we know that for all $\theta \in [\gamma; i, j]$ there exist $r_1, r_2 \in \mathbb{N}^+$ such that for all $\beta_1 \in F(\omega^2 \cdot k)_0^{r_1}$ and $\beta_2 \in F(\omega^2 \cdot k)_1^{r_2}$ we have

$\mathfrak{c}(\{\theta, \beta_1\}) = \mathfrak{c}(\{\theta, \beta_2\}) = 1$. Let $r = \max\{r_1, r_2\}$ and fix $\theta \in [\gamma; \mathbf{i}, \mathbf{j}]$. Applying Lemma 5 to the closed copy $\{\beta \in \gamma : \beta <^* \omega^2 \cdot k\}$ of ω^2 , we may split into two cases.

Suppose that $\hat{\mathfrak{c}}(k, 1, 0) = 1$. Choose some $\beta \in F(\omega^2 \cdot k)_1^r$ and $\beta' \in F(\omega^2 \cdot k)_0^r$ with $\beta' <^* \beta$. Then $\mathfrak{c}(\{\beta, \beta'\}) = 1$ and hence, the set $\{\theta, \beta, \beta'\}$ is a blue homogeneous copy of 3, which is a contradiction.

Suppose that, for every $i < \omega$,

$$\{\beta \triangleleft^* \omega^2 \cdot k : \{\alpha \in W_i : \mathfrak{c}(\{\alpha, \beta\}) = 1\} \text{ is cofinal in } W_i\}$$

is cofinal in $\{\beta \in \gamma : \beta \triangleleft^* \omega^2 \cdot k\}$, where $W_i = \{\omega^2 \cdot (k-1) + \omega \cdot i + m : 0 < m < \omega\}$. In particular, this claim holds for $i = r$. We can then find some $\beta \in F(\omega^2 \cdot k)_1^r$ and $\beta' \in F(\omega^2 \cdot k)_0^r$ with $\mathfrak{c}(\{\beta, \beta'\}) = 1$. In this case, the set $\{\theta, \beta, \beta'\}$ is a blue homogeneous copy of 3, which is a contradiction. This completes the proof of (a) and (a').

To prove (b), assume to the contrary that $\tilde{\mathfrak{c}}(i, 0; k, \ell) = 1$ and $\tilde{\mathfrak{c}}(i, 1; k, \ell) = 1$ for some $1 \leq i \neq k \leq n$ and $0 \leq \ell \leq 2$. Applying Lemma 5 to the closed copy $\{\beta \in \gamma : \beta <^* \omega^2 \cdot i\}$ of ω^2 , we see that, in either case, there exist some $\beta \in [\gamma; \mathbf{i}, 1]$ and $\beta' \in [\gamma; \mathbf{i}, 0]$ with $\mathfrak{c}(\{\beta, \beta'\}) = 1$.

Then, by the definition of $\tilde{\mathfrak{c}}$, there exists $r_1 \in \mathbb{N}^+$ such that $\mathfrak{c}(\{\beta, \theta\}) = 1$ whenever $\theta \in F(\omega^2 \cdot k)_\ell^{r_1}$; and there exists $r_2 \in \mathbb{N}^+$ such that $\mathfrak{c}(\{\beta', \theta\}) = 1$ whenever $\theta \in F(\omega^2 \cdot k)_\ell^{r_2}$. Choose $\theta \in F(\omega^2 \cdot k)_\ell^r$ where $r = \max\{r_1, r_2\}$. Then the set $\{\theta, \beta, \beta'\}$ is a blue homogeneous copy of 3, which is a contradiction. This completes the proof of (b).

To prove (c), assume to the contrary that $\hat{\mathfrak{c}}(i, 2, 0) = 1$ and $\hat{\mathfrak{c}}(i, 2, 1) = 1$ for some $1 \leq i \leq n$. As before, applying Lemma 5 to the closed copy $\{\beta \in \gamma : \beta <^* \omega^2 \cdot i\}$ of ω^2 gives us $\beta \in [\gamma; \mathbf{i}, 1]$ and $\beta' \in [\gamma; \mathbf{i}, 0]$ such that $\mathfrak{c}(\{\beta, \beta'\}) = 1$. Then the set $\{\omega^2 \cdot i, \beta, \beta'\}$ is a blue homogeneous copy of 3, which is a contradiction. This completes the proof of (c).

To prove (d), assume to the contrary that

$$\tilde{\mathfrak{c}}(m, 0; i, 0) = \tilde{\mathfrak{c}}(m, 0; k, \ell) = \tilde{\mathfrak{c}}(i, 0; k, \ell) = 1$$

for some such i and m . Let $\alpha \in [\gamma; \mathbf{m}, 0]$. Since $\tilde{\mathfrak{c}}(m, 0; i, 0) = 1$, as before,

we can find $\alpha \in [\gamma; \mathbf{m}, \mathbf{0}]$ and $\beta \in [\gamma; \mathbf{i}, \mathbf{0}]$ such that $\mathbf{c}(\{\alpha, \beta\}) = 1$. The other equations now imply that there exist $r_1, r_2 \in \mathbb{N}^+$ such that for all $\beta_1 \in F(\omega^2 \cdot k)_\ell^{r_1}$ and $\beta_2 \in F(\omega^2 \cdot k)_\ell^{r_2}$ we have $\mathbf{c}(\{\alpha, \beta_1\}) = \mathbf{c}(\{\beta, \beta_2\}) = 1$. Choose $\theta \in F(\omega^2 \cdot k)_\ell^r$ where $r = \max\{r_1, r_2\}$. Then the set $\{\alpha, \beta, \theta\}$ is a blue homogeneous copy of 3, which is a contradiction. This finishes the proof. $\square$

Lemma 8. *Let $1 \leq k \leq n$ and $0 \leq \ell \leq 1$.*

(a) *For all $k < i \leq n$ and $0 \leq j \leq 1$,*

if $\hat{\mathbf{c}}(k, 2, \ell) = 0$, then $\tilde{\mathbf{c}}(i, j; k, \ell) = 1$ or $\tilde{\mathbf{c}}(i, j; k, 2) = 1$.

(b) *For all $n + 1 \leq i \leq n + K$ and $j = 0$,*

if $\hat{\mathbf{c}}(k, 2, \ell) = 0$, then $\tilde{\mathbf{c}}(i, j; k, \ell) = 1$ or $\tilde{\mathbf{c}}(i, j; k, 2) = 1$.

Proof. We shall prove both parts at once. Assume to the contrary that

$$\hat{\mathbf{c}}(k, 2, \ell) = \tilde{\mathbf{c}}(i, j; k, \ell) = \tilde{\mathbf{c}}(i, j; k, 2) = 0$$

for some such i and j . Then, by the definition of $\tilde{\mathbf{c}}$, for every $\alpha \in [\gamma; \mathbf{i}, \mathbf{j}]$ there exists $r_\alpha \in \mathbb{N}$ such that

$$\mathbf{c}(\{\alpha, \beta\}) = \mathbf{c}(\{\alpha, \omega^2 \cdot k\}) = 0$$

for all $\beta \in F(\omega^2 \cdot k)_\ell^{r_\alpha}$. Since $\mathbf{c}$ is ω -homogeneous and there is no homogeneous copy of 3, we can find a red homogeneous set $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\} \subseteq [\gamma; \mathbf{i}, \mathbf{j}]$ with $\alpha_1 < \dots < \alpha_{n-1}$. Set

$$r = \max\{r_{\alpha_1}, r_{\alpha_2}, \dots, r_{\alpha_{n-1}}\}$$

It is now straightforward to check using the infinite Ramsey's theorem that the set

$$F(\omega^2 \cdot k)_\ell^r \cup \{\omega^2 \cdot k, \alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$$

contains a red homogeneous closed copy of $\omega + n$, which leads to a contradiction. $\square$

Lemma 9. *Let $1 \leq k < n$.*

(a) *For any $k < i \leq n$ and $0 \leq j \leq 1$, we have that*

$$\tilde{\mathbf{c}}(i, j; k, 0) = 1 \text{ or } \tilde{\mathbf{c}}(i, j; k, 1) = 1 \text{ or } \tilde{\mathbf{c}}(i, j; k, 2) = 1.$$

(b) For any $n + 1 \leq i \leq n + K$ and $j = 0$, we have that

$$\tilde{c}(i, j; k, 0) = 1 \text{ or } \tilde{c}(i, j; k, 1) = 1 \text{ or } \tilde{c}(i, j; k, 2) = 1.$$

Proof. As before, we shall prove both parts at once. Assume towards a contradiction that, for some, for some such i and j ,

$$\tilde{c}(i, j; k, 0) = \tilde{c}(i, j; k, 1) = \tilde{c}(i, j; k, 2) = 0$$

By Lemma 7.c, we have $\hat{c}(k, 2, 0) = 0$ or $\hat{c}(k, 2, 1) = 0$. In both cases, we get a contradiction by Lemma 8. $\square$

Lemma 10. For any $1 \leq i < n$, either $\hat{c}(i, 2, 0) = 1$ or $\hat{c}(i, 2, 1) = 1$.

Proof. It is clear by Lemma 7.c that we do not have $\hat{c}(i, 2, 0) = 1$ and $\hat{c}(i, 2, 1) = 1$. So assume towards a contradiction that $\hat{c}(i, 2, 0) = 0$ and $\hat{c}(i, 2, 1) = 0$ for some $1 \leq i \leq n$.

It follows from Lemma 7.b, $\tilde{c}(i + 1, j; i, 2) = 0$ for some $0 \leq j \leq 1$. Now, by Lemma 7.a, $\tilde{c}(i + 1, j; i, \ell) = 0$ for some $0 \leq \ell \leq 1$. But then, we have that $\hat{c}(i, 2, \ell) = \tilde{c}(i + 1, j; i, \ell) = \tilde{c}(i + 1, j; i, 2) = 0$, which contradicts Lemma 8.a. $\square$

Before we state the next lemma, we will introduce some notation in light of Lemma 10. For each $1 \leq i < n$, let us denote

- the (unique) pair (i, j) for which $\hat{c}(i, 2, j) = 0$ by A_i and
- the (unique) pair (i, j) for which $\hat{c}(i, 2, j) = 1$ by B_i

We shall also denote

- the pair $(i, 2)$ by L_i for each $1 \leq i \leq n$, and
- the pair $(i, 1)$ by L_i for each $n + 1 \leq i \leq n + K$.

Using the ideas in the proof of Lemma 8 and this newly introduced notation, we next prove a technical lemma that will be used multiple times in the main proof.

Lemma 11. *Let $1 \leq k \leq n$ and $0 \leq \ell \leq 1$ and let $k < i_1 < i_2 < \dots < i_{n-1} \leq n+K$.*

If we have

$$\tilde{\mathbf{c}}(L_{i_{t'}}; L_{i_t}) = \tilde{\mathbf{c}}(L_{i_t}; k, \ell) = \tilde{\mathbf{c}}(L_{i_t}; L_k) = 0$$

for all $1 \leq t < t' \leq n-1$, then $\hat{\mathbf{c}}(k, 2, \ell) = 1$.

Proof. Assume to the contrary that

- (i) $\tilde{\mathbf{c}}(L_{i_{t'}}; L_{i_t}) = 0$ for all $1 \leq t < t' \leq n-1$,
- (ii) $\tilde{\mathbf{c}}(L_{i_t}; k, \ell) = 0$ for all $1 \leq t \leq n-1$,
- (iii) $\tilde{\mathbf{c}}(L_{i_t}; L_k) = 0$ for all $1 \leq t \leq n-1$, and
- (iv) $\hat{\mathbf{c}}(k, 2, \ell) = 0$.

Let α_t be the unique element of $[\gamma; \mathbf{L}_{i_t}]$ for each $1 \leq t \leq n-1$. Using (i) and (iii), one can easily show that

$$\{\omega^2 \cdot k, \alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$$

is red homogeneous. It follows from (ii) and the definition of $\tilde{\mathbf{c}}$ that, for every $1 \leq t \leq n-1$, there exists $r_t \in \mathbb{N}$ such that $\mathbf{c}(\{\alpha_t, \beta\}) = 0$ for all $\beta \in F(\omega^2 \cdot k)_\ell^{r_t}$. Set $r = \max\{r_1, r_2, \dots, r_{n-1}\}$. It is now straightforward to check using the infinite Ramsey's theorem, the previous observations and (iv) that the set

$$F(\omega^2 \cdot k)_\ell^r \cup \{\omega^2 \cdot k, \alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$$

contains a red homogeneous closed copy of $\omega + n$, which leads to a contradiction. $\square$

Lemma 12. *Let $1 \leq k < n$. Then*

- a. *For any $k < i \leq n$, we have $\tilde{\mathbf{c}}(i, 0; B_k) = \tilde{\mathbf{c}}(i, 1; B_k) = 0$.*
- b. *For any $n+1 \leq i \leq n+K$, we have $\tilde{\mathbf{c}}(i, 0; B_k) = 0$.*

Proof. We will prove both parts at once. Assume towards a contradiction that $\tilde{\mathbf{c}}(i, j; B_k) = 1$ for some such i and j . Then it follows from Lemma 7.a that $\tilde{\mathbf{c}}(i, j; A_k) = 0$.

By the assumption that $\tilde{c}(i, j; B_k) = 1$, for every $\alpha \in [\gamma; \mathbf{i}, \mathbf{j}]$, there exists $r \in \mathbb{N}^+$ such that $c(\{\alpha, \beta\}) = 1$ for every $\beta \in F(\omega^2 \cdot k)_\ell^r$ where $B_k = (k, \ell)$. Choose such α and β . Since $\hat{c}(k, 2, \ell) = 1$, if it were the case that $\tilde{c}(i, j; L_k) = 1$, then the set $\{\alpha, \beta, \omega^2 \cdot k\}$ would be a blue homogeneous copy of 3. Thus $\tilde{c}(i, j; L_k) = 0$. But this contradicts Lemma 8, as we have $\tilde{c}(i, j; L_k) = \tilde{c}(i, j; A_k) = 0$. $\square$

Corollary 1. *For every $1 \leq k < n < i \leq n + K$, we have that $\tilde{c}(i, 0; A_k) = 1$ or $\tilde{c}(i, 0; L_k) = 1$.*

Proof. This easily follows from Lemma 9.b and Lemma 12.b. $\square$

Corollary 2. *For any $1 \leq k < i \leq n$, exactly one of the following holds.*

- i. $\tilde{c}(i, 1; L_k) = \tilde{c}(i, 0; A_k) = 1$ and $\tilde{c}(i, 1; A_k) = \tilde{c}(i, 0; L_k) = 0$.
- ii. $\tilde{c}(i, 1; L_k) = \tilde{c}(i, 0; A_k) = 0$ and $\tilde{c}(i, 1; A_k) = \tilde{c}(i, 0; L_k) = 1$.

Proof. Let $1 \leq k < i \leq n$. We split into two cases.

Suppose that $\tilde{c}(i, 1; L_k) = 1$. Then, by Lemma 7.b, we get $\tilde{c}(i, 0; L_k) = 0$. Now, applying Lemma 9.a and Lemma 12.a, we see that $\tilde{c}(i, 0; A_k) = 1$. Since $\tilde{c}(i, 0; A_k) = 1$, it follows from Lemma 7.b that $\tilde{c}(i, 1; A_k) = 0$. Therefore, we are in Case (i).

Suppose that $\tilde{c}(i, 1; L_k) = 0$. Applying Lemma 9.a and Lemma 12.a, we obtain that $\tilde{c}(i, 1; A_k) = 1$. Then $\tilde{c}(i, 0; A_k) = 0$ by Lemma 7.b. Since $\tilde{c}(i, 0; A_k) = 0$, another application of Lemma 9.a, and Lemma 12.a, gives us that $\tilde{c}(i, 0; L_k) = 1$. Therefore, we are in Case (ii). That these cases are mutually exclusive is clear. $\square$

With our graph interpretation in mind, this last corollary basically explains why, in the proof of Theorem 1, the edges in E_1 of $\mathbf{G}_n$ were chosen as they are. Up to certain choices, the edge structure of $\mathbf{G}_n$ was already mostly determined by the non-existence of a red homogeneous $\omega + n$ and a blue homogeneous 3. We shall need two more lemmas before we prove the second main theorem and conclude this section.

Lemma 13. *Let $1 \leq i < n$ and $0 \leq j \leq 2$. If we have that $\tilde{c}(m, 0; i, j) = 1$ for all $n + 1 \leq m \leq n + K$, then $\tilde{c}(L_m; i, j) = 0$ for all $n + 1 \leq m < n + K$.*

Proof. Assume that $\tilde{c}(m, 0; i, j) = 1$ whenever $n + 1 \leq m \leq n + K$. It follows from Lemma 7.d that $\tilde{c}(n + K, 0; m, 0) = 0$ for all $n + 1 \leq m \leq n + K$.

We claim that $\hat{c}(m, 1, 0) = 1$ or $\tilde{c}(n + K, 0; L_m) = 1$ for all $n + 1 \leq m \leq n + K$. Suppose towards a contradiction that $\hat{c}(m, 1, 0) = \tilde{c}(n + K, 0; L_m) = 0$ for some $n + 1 \leq m \leq n + K$. Then, using the ideas in the proof of Lemma 8, one can construct a red homogeneous closed copy of $\omega + n$ inside the set

$$[\gamma; \mathbf{m}, 0] \cup \{\omega^2 \cdot n + \omega \cdot m\} \cup [\gamma; \mathbf{n} + \mathbf{K}, 0]$$

which leads to a contradiction.

Let $n + 1 \leq m < n + K$. If $\hat{c}(m, 1, 0) = 1$, then we must have $\tilde{c}(L_m; i, j) = 0$ since, otherwise, having $\tilde{c}(m, 0; i, j) = \tilde{c}(L_m; i, j) = \hat{c}(m, 1, 0) = 1$ would create a blue homogeneous copy of 3. If $\tilde{c}(n + K, 0; L_m) = 1$, then we must have $\tilde{c}(L_m; i, j) = 0$ since, otherwise, having $\tilde{c}(n + K, 0; L_m) = \tilde{c}(n + K, 0; i, j) = \tilde{c}(L_m; i, j) = 1$ would create a blue homogeneous copy of 3. (We do not explicitly write the arguments for these claims as they can be done imitating the proof of Lemma 7.) Therefore, in either case, we have that $\tilde{c}(L_m; i, j) = 0$. $\square$

In what follows, for each $1 \leq i < n$, we set $\overline{A}_i = L_i$ and $\overline{L}_i = A_i$.

Lemma 14. *Let $1 \leq k < i \leq n$ and $X \in \{A_k, L_k\}$. If $\tilde{c}(L_i; X) = 1$, then $\tilde{c}(L_m; \overline{X}) = 0$ for all $n + 1 \leq m < n + K$.*

Proof. Assume that $\tilde{c}(L_i; X) = 1$. We split into two cases depending on whether or not we have $\hat{c}(i, 2, j) = 1$ for some $0 \leq j \leq 1$.

Suppose that $\hat{c}(i, 2, 0) = \hat{c}(i, 2, 1) = 0$, in which case we must have $i = n$ by Lemma 10. Now, Lemma 8.b implies that, $\tilde{c}(m, 0; L_n) = 1$ or $\tilde{c}(m, 0; n, 0) = 1$, and $\tilde{c}(m, 0; L_n) = 1$ or $\tilde{c}(m, 0; n, 1) = 1$. But from Lemma 7.a', we have $\tilde{c}(m, 0; n, 0) = 0$ or $\tilde{c}(m, 0; n, 1) = 0$. So $\tilde{c}(m, 0; L_n) = 1$ for all $n + 1 \leq m \leq n + K$. Consequently, in order to avoid a blue homogeneous copy of 3, we must have $\tilde{c}(m, 0; X) = 0$ for all $n + 1 \leq m \leq n + K$. But then, by Lemma 8.b, we must have $\tilde{c}(m, 0; \overline{X}) = 1$ for all $n + 1 \leq m \leq n + K$. Consequently, Lemma 13 implies that $\tilde{c}(L_m; \overline{X}) = 0$ for all $n + 1 \leq m < n + K$.

Now suppose that $\hat{c}(i, 2, 0) = 1$ or $\hat{c}(i, 2, 1) = 1$. We would like to remark that we may or may not have $i = n$. Even if $i = n$, let us name the pairs $(i, 0)$ and $(i, 1)$ by A_i and B_i in such a way that $\tilde{c}(L_i; A_i) = 0$ and $\tilde{c}(L_i, B_i) = 1$. By Lemma 8.b, we must have that $\tilde{c}(m, 0; A_i) = 1$ or $\tilde{c}(m, 0; L_i) = 1$ for all $n + 1 \leq m \leq n + K$.

On the other hand, as $\tilde{c}(L_i; X) = 1$, we cannot have $\tilde{c}(B_i; X) = 1$ in order for there not to be a blue homogeneous 3. So, by Corollary 2, we have $\tilde{c}(A_i; X) = 1$. By Lemma 8.b, we have $\tilde{c}(m, 0; L_i) = 1$ or $\tilde{c}(m, 0; A_i) = 1$ for all $n + 1 \leq m \leq n + K$. We also have $\tilde{c}(A_i; X) = \tilde{c}(L_i; X) = 1$ and hence, the non-existence of a blue homogeneous copy of 3 now gives us that $\tilde{c}(m, 0; X) = 0$ for all $n + 1 \leq m \leq n + K$. Now applying Lemma 8.b, we must have $\tilde{c}(m, 0; \overline{X}) = 1$ for all $n + 1 \leq m \leq n + K$. Subsequently, by Lemma 13, we have $\tilde{c}(L_m; \overline{X}) = 0$ for all $n + 1 \leq m < n + K$. $\square$

We are now ready to prove our second main theorem.

Proof of Theorem 2. Let $n \geq 3$ be an integer. Set $K = R(2n - 3, 3) + 1$ and $\gamma = \omega^2 \cdot n + \omega \cdot K + 1$. Let $\mathfrak{c} : [\gamma]^2 \rightarrow \{0, 1\}$ be a coloring. We wish to show that there exist a red homogeneous closed copy of $\omega + n$ or a blue homogeneous (necessarily closed) copy of $3 = \{0, 1, 2\}$. By the remarks at the end of Section 2, we may assume without loss of generality that $\mathfrak{c}$ is canonical and ω -homogeneous. Assume to the contrary that such homogeneous sets do not exist. Then Lemma 7-14 and Corollary 1-2 all hold.

It is clear that $[\gamma; \mathbf{A}_1] \cup \{\omega^2\}$ contains a red homogeneous closed copy of $\omega + 1$. By assumption, the set $[\gamma; \mathbf{A}_1] \cup \{\omega^2 \cdot 1, \omega^2 \cdot 2, \dots, \omega^2 \cdot n\}$ does not contain a red homogeneous closed copy of $\omega + n$. Consequently, there exists $2 \leq i \leq n$ such that $\tilde{c}(L_i; X) = 1$ for some $X \in \{A_1, L_1, L_2, \dots, L_{i-1}\}$, since, otherwise, we would obtain a contradiction by Lemma 11.

By Corollary 2, we have $\tilde{c}(Y, X) = 1$ and $\tilde{c}(Y, \overline{X}) = 0$ where $Y = (i, j)$ for some $0 \leq j \leq 1$. Since there is no blue homogeneous copy of 3 in γ , by the definition of $R(2n - 3, 3)$, there exist $2n - 3$ indices $n + 1 \leq m_1 < m_2 < \dots < m_{2n-3} < n + K$ such that

$$\bigcup_{t=1}^{2n-3} [\gamma; \mathbf{L}_{m_t}]$$

is red homogeneous. Now, by the pigeonhole principle, we can find $n - 1$ indices $n + 1 \leq m'_1 < m'_2 < \dots < m'_{n-1} < n + K$ such that

- $\tilde{c}(L_{m'_t}, X) = 0$ for all $1 \leq t \leq n - 1$ or
- $\tilde{c}(L_{m'_t}, X) = 1$ for all $1 \leq t \leq n - 1$.

Applying Lemma 14, we see that $\tilde{c}(L_m; \overline{X}) = 0$ for all $n + 1 \leq m < n + K$. So the first case directly contradicts Lemma 11. Thus the second case holds. But then, since $\tilde{c}(Y, X) = 1$, we must have $\tilde{c}(L_{m'_t}, Y) = 0$ for all $1 \leq t \leq n - 1$. Similarly, since $\tilde{c}(L_i, X) = 1$, we must have $\tilde{c}(L_{m'_t}, L_i) = 0$ for all $1 \leq t \leq n - 1$. (Otherwise, one can create a blue homogeneous copy of 3.) On the other hand, since $\tilde{c}(L_i; X) = \tilde{c}(Y; X) = 1$, in order for there not to be a blue homogeneous copy of 3, we must have $\hat{c}(i, 2, j) = 0$ where $Y = (i, j)$. Together with the previous observation that $\tilde{c}(L_{m'_t}, Y) = \tilde{c}(L_{m'_t}, L_i) = 0$ for all $1 \leq t \leq n - 1$, this contradicts Lemma 11, which completes the proof. $\square$

CHAPTER 5

PROOF OF THEOREM 3

In this chapter, we will prove Theorem 3. Recall that Theorem 3 provides an asymptotically weaker upper bound eliminates the use of Ramsey numbers and provides better upper bound for small n values, at least for the n values satisfying $3 \leq n \leq 7$.

In this section, we shall prove Theorem 3 using the ideas that are employed in Chapter 4. For this reason, we retain all the notation introduced in Chapter 4.

Analyzing the proof of Theorem 2, one sees that the whole proof is based on the following phenomenon: If $[\gamma; \mathbf{A}_1] \cup \{\omega^2 \cdot 1, \omega^2 \cdot 2, \dots, \omega^2 \cdot n\}$ fails to contain a red homogeneous copy of $\omega + n$, then having sufficiently many L_i 's with $n < i$ whose corresponding elements form a red homogeneous set automatically creates a red homogeneous closed copy of $\omega + n$. In that proof, such L_i 's were extracted using Ramsey numbers. In the proof of Theorem 3, we shall take another approach to create such L_i 's.

Proof of Theorem 3. Let $n \geq 3$ be an integer and set $\gamma = \omega^2 \cdot n + \omega \cdot (n^2 - 4) + 1$. Let $c : [\gamma]^2 \rightarrow \{0, 1\}$ be a canonical ω -homogeneous coloring. Assume to the contrary no red homogeneous closed copy of $\omega + n$ and no blue homogeneous copy of 3 exist. Then Lemma 7-14 and Corollary 1-2 all hold.

As before, there must exist $2 \leq i \leq n$ and $X \in \{A_1, L_1, L_2, \dots, L_{i-1}\}$ such that $\tilde{c}(L_i; X) = 1$. Take the least such i and such X . We shall now split into two cases depending on whether $i = n$ or $i < n$.

Suppose that $i < n$. By the non-existence of a blue homogeneous 3, we must have $\tilde{c}(B_i; X) = 0$ since $\tilde{c}(L_i; X) = 1$. As $\tilde{c}(B_i; X) = 0$, we obtain $\tilde{c}(A_i; X) = 1$ by

Corollary 2. On the other hand, we also have by Corollary 2 that

$$\tilde{\mathfrak{c}}(i+1, j; L_i) = \tilde{\mathfrak{c}}(i+1, 1-j; A_i) = 1$$

for some $0 \leq j \leq 1$. Thus

$$\tilde{\mathfrak{c}}(i+1, j; L_i) = \tilde{\mathfrak{c}}(L_i; X) = \tilde{\mathfrak{c}}(i+1, 1-j; A_i) = \tilde{\mathfrak{c}}(A_i; X) = 1$$

But then, it follows that

$$\tilde{\mathfrak{c}}(i+1, j; X) = \tilde{\mathfrak{c}}(i+1, 1-j, X) = 0$$

because, otherwise, one can construct a blue homogeneous copy of 3. However, the last equality contradicts Corollary 2.

Now suppose that $i = n$. Set $W_X = \{L_j : \tilde{\mathfrak{c}}(L_j; X) = 1, n+1 \leq j \leq n + (n^2 - 4)\}$ and consider the sets

$$W = \{L_j : \tilde{\mathfrak{c}}(L_j; X) = 0, n+1 \leq j \leq n + (n^2 - 4)\}$$

$$W_{n-1} = \{L_j \in W : \tilde{\mathfrak{c}}(L_j; A_{n-1}) = 1 \text{ or } \tilde{\mathfrak{c}}(L_j; L_{n-1}) = 1\}$$

$$W_{n-2} = \{L_j \in W : L_j \notin W_{n-1}, \tilde{\mathfrak{c}}(L_j; A_{n-2}) = 1 \text{ or } \tilde{\mathfrak{c}}(L_j; L_{n-2}) = 1\}$$

...

$$W_k = \left\{ L_j \in W : L_j \notin \left(\bigcup_{m=k+1}^{n-1} W_m \right), \tilde{\mathfrak{c}}(L_j; A_k) = 1 \text{ or } \tilde{\mathfrak{c}}(L_j; L_k) = 1 \right\}$$

...

$$W_1 = \left\{ L_j \in W : L_j \notin \left(\bigcup_{m=2}^{n-1} W_m \right), \tilde{\mathfrak{c}}(L_j; A_1) = 1 \text{ or } \tilde{\mathfrak{c}}(L_j; L_1) = 1 \right\}$$

Recall that i was chosen to be the least integer with its property. It follows that the set $[\gamma; \mathbf{A}_1] \cup \{\omega^2 \cdot 1, \omega^2 \cdot 2, \dots, \omega^2 \cdot (n-1)\}$ contains a red homogeneous closed copy of $\omega + (n-1)$. Consequently, in order for there not be a red homogeneous closed copy of $\omega + n$, we must have that, for every $n+1 \leq j \leq n + (n^2 - 4)$, there exists $Y \in \{A_1, L_1, L_2, \dots, L_{n-1}\}$ such that $\tilde{\mathfrak{c}}(L_j; Y) = 1$. It follows that

$$W_X \cup \left(\bigcup_{m=1}^{n-1} W_m \right) = \{L_j : n+1 \leq j \leq n + (n^2 - 4)\}$$

Next will be made some important observations.

- Let $1 \leq k \leq n - 2$. Suppose that W_k has at least $(2k + 1)$ elements. Then, by the pigeonhole principle, we must have that $\tilde{c}(L_j; A_k) = 1$ for at least $k + 1$ of these elements or $\tilde{c}(L_j; L_k) = 1$ for at least $k + 1$ of these elements. In either case, since there is no blue homogeneous copy of 3, there exists $k + 1$ elements $L_{m_1}, L_{m_2}, \dots, L_{m_{k+1}}$ in W_k such that $\tilde{c}(L_{m_t}; L_{m_{t'}}) = 0$ whenever $1 \leq t \neq t' \leq k + 1$. Recall that $\tilde{c}(L_j; L_{j'}) = 0$ for all $1 \leq j \neq j' < n$. Moreover, because $L_{m_t} \in W_k$, we have that $\tilde{c}(L_{m_t}; A_{k+1}) = 0$ and that $\tilde{c}(L_{m_t}; L_j) = 0$ for all $1 \leq t \leq k + 1$ and all $k + 1 \leq j \leq n - 1$. This contradicts Lemma 11.
- Suppose that W_{n-1} has at least $(2n - 3)$ elements. Then, as above, there are $L_{m_1}, L_{m_2}, \dots, L_{m_{n-1}}$ in W_{n-1} such that $\tilde{c}(L_{m_t}; L_{m_{t'}}) = 0$ whenever $1 \leq t \neq t' \leq n - 1$. Since $W_{n-1} \subseteq W$, we have that $\tilde{c}(L_{m_t}; X) = 0$ for all $1 \leq t \leq n - 1$. Moreover, by Lemma 14, we have $\tilde{c}(L_{m_t}; \overline{X}) = 0$ for all $1 \leq t \leq n - 1$. This contradicts Lemma 11.
- Suppose that W_X has at least $n - 1$ elements, say, $L_{m_1}, L_{m_2}, \dots, L_{m_{n-1}}$. Then, since $\tilde{c}(L_{m_t}; X) = \tilde{c}(L_n; X) = 1$ for all $1 \leq t \leq n - 1$, we must have that $\tilde{c}(L_{m_t}; L_{m_{t'}}) = \tilde{c}(L_{m_t}; L_n) = 0$ whenever $1 \leq t \neq t' \leq n - 1$ because there is no blue homogeneous 3. By Corollary 2, there exists $0 \leq \ell \leq 1$ such that $\tilde{c}(n, \ell; X) = 1$. Again, by the non-existence of a blue homogeneous 3, we obtain that $\tilde{c}(L_{m_t}; n, \ell) = \hat{c}(n, 2, \ell) = 0$ for all $1 \leq t \leq n - 1$. Together with previous observations, this contradicts Lemma 11.

Therefore, W_k has at most $2k$ elements for all $1 \leq k \leq n - 2$, W_{n-1} has at most $2n - 4$ elements and W_X has at most $n - 2$ elements. This means that $W_X \cup (\cup_{m=1}^{n-1} W_m)$ can have at most

$$\left(\sum_{k=1}^{n-2} 2k \right) + (2n - 4) + (n - 2) = (n - 2)(n - 1) + (2n - 4) + (n - 2) = n^2 - 4$$

elements. We will now argue that it can indeed have at most $n^2 - 5$ elements. Recall that $X \in \{A_1, L_1, L_2, \dots, L_{n-1}\}$. Consequently, the corresponding W_k is either empty or equal to $\{L_{n^2-4}\}$, because, by Lemma 14, we have $\tilde{c}(L_m; \overline{X}) = 0$ for all $n + 1 \leq m < n + (n^2 - 4)$. Thus, the corresponding W_k can have at most one element instead of at most $2k$ elements. So the union $W_X \cup (\cup_{m=1}^{n-1} W_m)$ can indeed have at most $n^2 - 4 - (2k - 1)$ elements.

Letting $k = 1$, we obtain that $W_X \cup (\cup_{m=1}^{n-1} W_m)$ can actually have at most $n^2 - 5$ elements. However, we had $W_X \cup \cup_{m=1}^{n-1} W_m = \{L_j : n + 1 \leq j \leq n + (n^2 - 4)\}$ which now gives a contradiction since the right-hand side has $n^2 - 4$ elements. This completes the proof. $\square$



CHAPTER 6

CONCLUSION AND FUTURE RESEARCH DIRECTIONS

In this thesis, we provided new lower and upper bounds for the closed ordinal Ramsey numbers $R^{cl}(\omega + n, 3)$ for $n \geq 3$. We believe that, for small values of n , the bounds we have found can further be improved to get exact values of $R^{cl}(\omega + n, 3)$ by following the same construction as we did, while applying more refined combinatorial arguments.

Furthermore, we suspect that an efficient general algorithm may be generated for the calculation of closed Ramsey numbers $R^{cl}(\omega \cdot m + n, 3)$ for $n, m \in \mathbb{N}^+$. We think this can be achieved by algorithmically reducing the problem to a finite one through canonical colorings and the corresponding finitary functions $\hat{c}$ and $\tilde{c}$. We strongly believe that canonical colorings can be adopted to introduce some new upper bounds for the closed ordinal Ramsey numbers $R^{cl}(\omega + n, k)$ with $k > 3$, which possibly involve the classical Ramsey numbers $R(n, k)$.

Besides these generic open problems, it does not seem to be easy to compute the exact values of $R^{cl}(\omega + n, 3)$ even for small concrete cases. For example, for $n = 3$, by using Theorem 1 and Theorem 3, we get

$$\omega^2 \cdot 3 + \omega \cdot 3 + 3 \leq R^{cl}(\omega + 3, 3) \leq \omega^2 \cdot 3 + \omega \cdot 5 + 1$$

However, our current approach does not seem to be sufficient to provide an insight for what the coefficient of ω in $R^{cl}(\omega + 3, 3)$ is. As it is the simplest open case, we would like to pose the question of determining $R^{cl}(\omega + 3, 3)$ exactly.



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