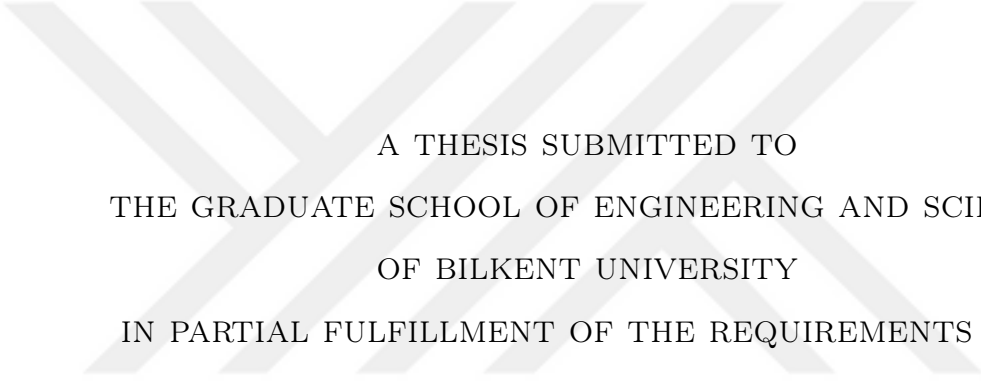


SONIC HAWKING RADIATION FROM A BLACKHOLE LASER



A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
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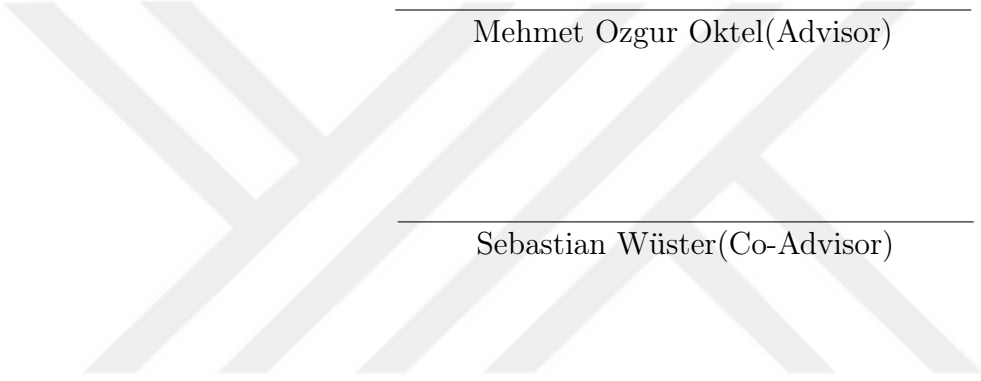
By
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June 2018

Sonic Hawking Radiation from a Blackhole Laser

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June 2018

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.



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ABSTRACT

SONIC HAWKING RADIATION FROM A BLACKHOLE LASER

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M.Sc. in Physics

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The quantum thermal radiation from a black hole (BH) known as “Hawking Radiation” or “Black Hole Evaporation” results from studying quantum fields in the curved space-time of the horizon of a BH. Experimentally, the radiation is difficult if not impossible to be detected from a real black hole with a mass much higher than that of our sun, since the Radiation temperature is substantially below that of microwave background radiation.

However, in 1981 Unruh showed an analogy between the propagation of sound waves in any convergent fluid flow and that of the quantum field in a gravitational field. He showed that if the background fluid is accelerated to higher than the speed of sound then it can develop a horizon (point of no return) for the sound waves. This is the so-called the sonic BH. This horizon will emit thermal radiation in terms of sound wave quanta (phonons) in an analogy to the thermal radiation of black holes (Analogue Hawking Radiation) (AHR). Bose-Einstein Condensates (BECs) can be used as a background fluid developing a sonic horizon for the phonon modes propagating through its background due to the very low temperature of the BEC.

Recently in 2014, Steinhauer has reported the observation of self-Amplifying Hawking radiation from the realization of an accelerated BEC. The experiment reported an exponentially growing signal of modes trapped between a BH and white hole (WH) horizon, where the white hole is the point where sound cannot enter. Experimental signatures of AHR are a growing oscillating perturbation of the condensate mean density and a characteristic pattern in density-density correlation functions. However, the former mentioned oscillations may result from the dynamical instabilities of the classical mean field density.

In this work, we were able to reproduce the experimental results of density modulations in the mean field, and thus without AHR, using only the mean field Gross-Pitaevskii equation (GPE) for the BEC. Furthermore, we include the quantum fluctuation to study the density-density correlation function that is in qualitative agreement with the experiment using the truncated Wigner approximation (TWA). Finally, we then calculate the One Body Density Matrix (OBDM) to distinguish condensed from non-condensed atoms using the Penrose Onsager criterion. We are able to contribute to a discussion in the literature regarding the quantum field or mean field origin of the mean density oscillations in the experiment.

Keywords: Black hole laser, Hawking radiation, Bose Einstein Condensate, Sonic Horizon.

ÖZET

KARADELİK LAZERİNDEN SONİK HAWKİNG IŞINIMI

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Fizik, Yüksek Lisans

Tez Danışmanı: Mehmet Ozgur Oktel

Haziran 2018

"Hawking ışıını" ya da "karadelik buharlaşması" olarak da bilinen karadelikten kuantum termal ışıını, kuantum alanları karadeligin bükülmüş uzay-zaman ufuğunda incelediğimizde ortaya çıkmaktadır. Işınım sıcaklığı mikro-dalga fon (arka plan) ışıınından oldukça düşük olduğundan deneysel olarak bu ışıını gerçek bir karadelikten tespit etmek neredeyse imkansızdır.

Ancak 1981'de Unruh; herhangi bir yakınsak akışkan akışında, ses dalgalarının yayılmasıyla kuantum alanın çekimsel alanda yayılması arasında bir benzetim göstermiştir. Eğer fon akışkanı ses hızından daha yüksek bir hıza hızlandırılırsa, bunun ses dalgaları için bir ufuk (geri dönüşün olmadığı nokta) oluşturduğunu göstermiştir. Bu da sonik karadeliktir. Bu ufuk, karadeliklerin termal ışıınıyla benzer fonon bazında termal ışıını yayır (Analogue Hawking Radiation - AHR). Base-Einstein yoğunlukları (BEC) çok düşük ısıları nedeniyle fonda yayılan fonon modları için sonik bir ufuğa neden olan fon akışkanı olarak kullanılabilir.

2014'te Steinhauer, hızlandırılmış bir BEC'in gerçekleştirilmesinden kendini güçlendiren Hawking ışıınıının gözlemlendiğini bildirmiştir. AHR'nin deneysel işaretleri yoğunluk-yoğunluk korelasyon fonksiyonlarında yoğunluk ortalama yoğunluğunun ve karakteristik bir desenin giderek artan titreşimidir. Bununla birlikte, önceki bahsedilen salınımlar, klasik ortalama alan yoğunluğunun dinamik kararsızlıklarından kaynaklanabilir.

Bu çalışmada, BEC için sadece ortalama alan Gross-Pitaevskii denklemi

(GPE) kullanılarak, AHR olmaksızın, ortalama alandaki yoğunluk modülasyonlarının deneysel sonuçlarını yeniden üretebildik. Ayrıca, Kesikli Wigner yaklaşımı (Truncated Wigner Approximation - TWA) kullanılarak deneyle nitel anlamda olan yoğunluk-yoğunluk korelasyon fonksiyonunu incelemek için kuantum dalgalanmasını da ekledik. Son olarak, daha önce Yoğuşmalı Onsager ölçütünü kullanarak yoğunlaştırılmamış atomlardan yoğunlaşmayı ayırt etmek için tek kütleli yoğunluk matrisini (One Body Density Matrix - OBDM) hesapladık ve literatüre ortalama yoğunluk salınımlarının kuantum alanı veya ortalama alanının çıkış noktası ile ilgili katkıda bulunabiliyoruz.

Anahtar sözcükler: Anahtar Sözcüklerim.

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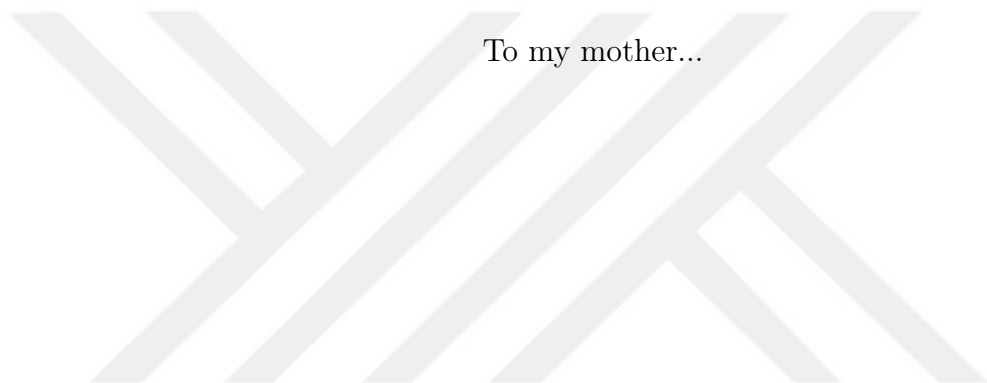
In Bilkent University I have met really lots of nice people. I am so grateful for meeting my wise, hardworking and curious Nigerian friend (Mujaheed) as well as my Tajik friend Murad. I always had great physics talks overnight with both of them. I am also thankful to know, Yahya, Fulya, Mustafa, Büşra nur, Amira Ahmed who always supported me, Shivakant, my nice friend Tarek El-sebai and the lovely couples Nour and Mohamed, Nabeel and Maha with whom I enjoyed a lot during our academic period. I am really grateful to Nermin Karahan and our former dormitory manager (Nimet Kaya) who passed away last year and who was more than a mother for us.

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This work has been done using the Tubitak Truba account super computing facility.



To my mother...

Contents

1	Introduction	1
1.1	Motivation	1
1.2	Thesis Plan	4
2	Theoretical background	5
2.1	Bose-Einstein condensates classical mean field theory	5
2.1.1	Time independent Gross-Pitaevskii equation	5
2.1.2	Ground state for trapped bosons	6
2.1.3	One dimensional Gross-Pitaevskii Equation	8
2.1.4	Hydrodynamics of Bose-Einstein condensates	8
2.2	Microscopic description of Bose-Einstein condensates	10
2.2.1	Classical Fluctuation of Bose-Einstein condensate	10
2.2.2	Quantized excitations	10
2.2.3	Fluctuation vs. Bogoliubov quasi-particles	11

2.3	Truncated Wigner method	12
2.4	Potential field fluctuation and the effective metric	13
3	Hawking radiation in a simplified analogue system	15
3.1	Creating a sonic horizon	15
3.2	The black hole-white hole geometry	16
3.3	Correlation function showing Hawking partners	20
3.3.1	Truncated Wigner approach	20
3.3.2	Simulation of analogue Hawking radiation	21
4	Black hole laser	27
4.1	Brief summary of the Haifa experiment	27
4.1.1	Experiment implementation	27
4.1.2	Main Results of the experiment	28
4.2	Numerical study of the experiment	28
4.2.1	Numerical modeling of the experiment	29
4.2.2	Gross-Pitaevskii simulation of the experiment	35
4.2.3	Density-density correlation function	39
4.2.4	The one-body-density matrix simulation	42
4.3	Conclusion	46

A Numerical methods	47
A.1 Dimensionless formalism	47
A.2 Programming	48



List of Figures

3.1	Density of the condensate in blue is shown at time zero before the horizon creation which shows a constant density with a value of 450 then at time = 18 after the horizon creation. In red, the real part of the wave function of the condensate is shown for time zero and 18 as well	18
3.2	The profile of the Atom-Atom interaction strength in blue and the external potential in red are shown at time zero, which is corresponding to a homogenous system everywhere. We also show it at time equal to 18 long after the horizon creation, which shows the modulation of both potentials.	19
3.3	The profile of the speed of sound in red and the condensate velocity shown in blue in at time zero which shows a subsonic flow everywhere then at time equal to 18 far after the horizon creation, which shows a super sonic region from $x/\eta = 0$ to 80.	19
3.4	Different Bogoliubov modes which are representing the phonon modes on top of the mean field solution are shown in the figure above	21
3.5	Correlation function is shown at different times before the horizon creation where the flow is subsonic everywhere until latest time where the Hawking tongue is present	25

3.6	Comparison between the correlation functions of our results to right and with results of [38]	26
4.1	The upper figure shows the experimental potential profile, the figure in the bottom shows the numerically simulated potential profile. The potential strength is measured in nano Kelvin (nk) . .	32
4.2	Potential profile at different times is shown	33
4.3	The upper figure shows the experimental speed of sound corresponding to the super and subsonic regions, the figure in the bottom shows the numerically simulated speed of sound in red and the velocity of the condensate in blue	34
4.4	Numerical Simulation of the mean field density without quantum fluctuation added to it showing the growth of the standing wave patterns at different times	36
4.5	Experiment vs Numerical simulation of density profile comparison	38
4.6	Simulation of Mean field density with noise added to it at latest time	39
4.7	The upper figure shows the experimental correlation function at different times , the figure in the bottom shows the numerically simulated one at different times of the experiment	41
4.8	The density correlation function is shown at late time developing the fringes and the checkerboard properties are shown as well . . .	42
4.9	The density profiles of the total density in blue the condensed density in red the uncondensed part in green at the beginning of the time evolution which shows the depletion of the condensate mainly	44

4.10	Density profiles at late time showing the total density in blue the condensed part in red and the uncondensed part in green, in fig c it shows that the standing wave pattern is having a great contribution from the condensed part	45
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List of Tables

3.1	System parameters	17
4.1	Essential parameters	29
4.2	Numerical simulation parameters	30
4.3	Potential profile parameters	31
A.1	Normalization units	47
A.2	Dimensionless Quantities	48

Chapter 1

Introduction

1.1 Motivation

Hawking radiation introduced by Stephen Hawking [3] is one of the most fascinating results of theoretical physics in the past few decades. Hawking found that quantum thermal radiation from a black hole (BH) alters the classical picture of the BH being totally black since it emits radiation inversely proportional to its mass. Hawking radiation has implications for our understanding of information and entropy [2]. However the actual observation of Hawking radiation from real black holes seems to be extremely difficult since the radiation temperature for stellar mass BHs is less than of that of cosmic microwave background.

In the quest for testing the prediction of Hawking, Unruh [4] has introduced an analogy between the propagation of quantum fields in a curved space time and the propagation of sound waves in a background flow turning supersonic. He showed that the equation of motion governing the propagation of a velocity potential field in an irrotational, viscosity free fluid is similar to the propagation of a massless scalar field propagating in a geometry with an effective metric having the same form as Schwarzschild metric near the horizon. Through the same math as the one of Hawking, he thus predicted the creation of analogue Hawking radiation

(AHR), from a sonic black hole. The analogy introduced by Unruh assumes a smooth background flow that varies only on length scales that are large compared to the wavelengths of the created sound wave radiation.

Since the proposal made by Unruh, lots of studies have introduced media and systems in which AHR could be investigated [20]. Systems such as low group velocity light in a moving nonlinear media [13], superfluid ^3He [8], surface waves in moving water [42], Fermi degenerate liquid [15], and Bose-Einstein Condensates(BEC) [35, 25, 12, 27].

In this thesis we focus our attention on the study of the BEC as a system in which analogue Hawking radiation can be investigated through looking at the propagation of quantized sound waves (phonons) when the BEC flow develops a sonic horizon by turning its speed of sound from subsonic to supersonic. A first numerical investigation of the behavior of phonons near a sonic horizon in a BEC has been introduced in [38]. The result of that work has introduced important features developed from calculations resulting from microscopic theory of the BEC without gravitational analogy. Those authors found clear signatures of AHR in the numerical simulation when inspecting condensate density-density correlation functions. These were predicted in [36] to arise due to correlation between the emitted Hawking phonon and its “absorbed” partner particle.

In recent years an experiment that has attracted much attention among the community [44] has reported the observation of analogue Hawking Radiation in a harmonically trapped BEC accelerated through a water fall- like potential. The experiment reports the observation of a phenomena called the black hole laser as an evidence of observing the AHR. The black hole laser was introduced in the context of general relativity in [41], and arises when considering quantized fields in a geometry that contains both, a black hole (particles are trapped inside) and a white hole (WH, particles cannot enter). In that case certain field modes reflect back and forth between the BH and WH, being amplified by the Hawking effect on each bounce. A standing wave pattern was observed in the experiment [44] and was interpreted as evidence of the black hole laser effect resulting from the modulation of the density by the amplified phonon modes of the condensate.

However, numerical studies of the experiment [48, 47] have indicated that the standing wave pattern in the supersonic region could be fully recovered using only a classical mean field picture of the condensate without the need of introducing quantum fluctuations which represents the phonon modes.

A more detailed study of the quantum fluctuations in the experiment thus need to be performed specifically to study the density-density correlation function that was investigated theoretically in [41]. In addition to that, there is no clear differentiation in the literature between the classical and the quantum fluctuation of the experimental BEC flow.

In this thesis we study the density-density correlation function using the Truncated Wigner approximation (TWA) and compare it with the experimental results of the experiment [44]. The TWA offers a way to study quantum fluctuations around a coherent BEC background by the controlled addition of numerical noise into a mean-field type simulation. We -to the best of our knowledge- propose the first idea to differentiate the contribution coming from the classical and quantum fluctuations using Penrose-Onsager criteria proposed in [1]. The idea is implemented by means of the one body density matrix (OBDM) to figure out the condensed and uncondensed parts of the total density of the BEC and its contribution to the standing wave pattern reported in the experiment [44].

1.2 Thesis Plan

In chapter 2, we introduce the basic theory of the BEC and the equations governing its dynamics in the classical mean field picture which also include the hydrodynamic picture of the BEC as a fluid. We then introduce methods for studying of the quantum fluctuation dynamics using the Truncated Wigner method and describe how to apply it to extract density-density correlation functions. Also the use of density-density correlation functions of BEC is introduced as a signature of Hawking radiation.

In chapter 3, we revisit the same simplified system proposed for numerically simulating the analogue Hawking radiation in a BEC [38]. In order to benchmark our numerical apparatus we obtain the expected density-density correlation pattern for the Hawking partners.

In chapter 4, we introduce a full numerical studies of the experiment in [44]. Starting with a mean field simulation of the experiment, we show the evolution of the condensate density. We then proceed to the quantum fluctuations using the TWA to get the total density. Both densities (condensed and uncondensed) are then compared with the experimentally measured density.

Chapter 2

Theoretical background

2.1 Bose-Einstein condensates classical mean field theory

Bose-Einstein condensation (BEC) is a phenomenon that happens when a large number of Bose particles are cooled down below some critical temperature. At this critical temperature the atomic de-broglie wavelength of individual atoms starts to overlap so the particles begin to occupy the same quantum state. In the following, we describe the main equations governing the condensate behavior and dynamics basically. We will build on this description in the following chapters for investigating the analogue systems that are based on the BEC as a simulator.

2.1.1 Time independent Gross-Pitaevskii equation

A description of a weakly interacting Bose gas forming a BEC is simplified by realising that the interaction between the particles takes the form of the S-wave scattering only. That regime can be described with an effective contact interaction

$$U(x) = U_0 \delta(r - r') \tag{2.1}$$

where $U_0 = \frac{4\pi\hbar^2 a_s}{m}$ which depends on the S-wave scattering length a_s . Since the condensation happens when a large number of atoms are in the same quantum state then, a mean field description is introduced to describe the macroscopically occupied state. The equation governing this behaviour is the Gross-Pitaevskii (GP) equation which is obtained by replacing the field operator by its expectation values [39]. The time independent GP equation is

$$\frac{-\hbar^2}{2m}\nabla^2\phi(r) + V(r)\phi(r) + U_0|\phi(r)|^2\phi(r) = \mu\phi(r) \quad (2.2)$$

here μ is the chemical potential that assures a constant total number of particles. $V(r)$ is the external applied potential applied to the condensate. The normalization of the wave function is equal to the total number of particles $N = \int dr|\phi(r)|^2$. The time evolution from a stationary state ϕ_0 of condensate wave function is obtained by the equation

$$\phi(r, t) = e^{\frac{-i\mu t}{\hbar}}\phi_0(r) \quad (2.3)$$

If we consider the case where we have a homogeneous Bose gas and no potential, then the GP equation reduces to

$$\mu = U_0|\phi(r)|^2 = U_0\rho_0. \quad (2.4)$$

where ρ_0 here is the uniform density of the condensate.

We now introduce the time dependent Gross-Pitaevskii equation which is governing the mean field behaviour of the BEC

$$i\hbar\frac{\partial\phi(r, t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V(r, t) + U_0(r, t)|\phi(r, t)|^2\right)\phi(r, t) \quad (2.5)$$

2.1.2 Ground state for trapped bosons

Finding the ground state of the BEC is of a specific importance since it is providing the stationary state for the dynamics of the condensate that will be studied throughout different systems. Of particular importance we look at a BEC in a trap. The Thomas-Fermi approximation provides a simple estimate of the cloud

size and shape of a BEC in a harmonic trap. The approximation neglects the kinetic energy term then considers only the weak interaction and the external potential terms in the BEC Hamiltonian. Following Thomas-Fermi approximation the time independent GP equation becomes

$$V(r)\phi(r) + U_0|\phi(r)|^2\phi(r) = \mu\phi(r) \quad (2.6)$$

which gives the solution

$$|\phi(r)|^2 = \frac{\mu - V(r)}{U_0}, \mu > v(r) \quad (2.7)$$

The Thomas Fermi approximation gives a good estimate of the the size of the cloud by equating the extension of the harmonic trap to the chemical potential of the BEC. The cloud size is given by [39]

$$R_{TF} = \sqrt{\frac{2\mu}{m\omega}} \quad (2.8)$$

where ω is the radial frequency of the trap.

A method well established numerically to find more exact ground state of a BEC is the imaginary time evolution. Introducing the functional energy out of the BEC hamiltonian as defined in [34]

$$E(\phi(r)) \equiv \langle \hat{H} \rangle = \int d^3r \left(\phi^*(r) \left[-\frac{\hbar^2 \nabla^2}{2m} + V(r) \right] \phi(r) + U_0 |\phi(r)|^4 \right) \quad (2.9)$$

then by evolving the wave function $\phi(r)$ toward the "steepest descent in energy" or simply finding the minimum of energy functional by going in the opposite direction of the gradient, we can look for a local energy minimum [17]. We can quantitatively write the formulation of the method by substituting $t = -i\tau$ in equation 2.1.1 to get the following equation

$$\hbar \frac{\partial \phi(r, t)}{\partial t} = \left(-\frac{\hbar^2 \nabla^2 \phi}{2m} + V(r, t) + U_0(r, t) |\phi(r, t)|^2 \right) \phi(r, t) \quad (2.10)$$

By re-normalizing $\phi(r)$ to N after each time-step, it will then converge towards the ground state.

2.1.3 One dimensional Gross-Pitaevskii Equation

We now introduce the one dimensional version of equation 2.1.1 because it will be the basis of many calculations throughout the course of this thesis. We consider the case of a BEC in a three dimensional harmonic trap of the form $V(x, y, z) = \frac{1}{2}m(\Omega_x^2 x^2 + \Omega_y^2 y^2 + \Omega_z^2 z^2)$ where we have $\Omega_z \gg \Omega_x, \Omega_y \gg \Omega_x$ we may write $V(R) = \frac{1}{2}m(\Omega_R(x^2 + y^2))$ to be the trap in the Radial (transverse direction). The dynamics of the condensate in the radial direction would not be affected compared to the one in the x (elongated) direction as a result of the higher trapping energy in the radial direction which has the scale $(\hbar\Omega_R)$. The dynamics of the condensate is then constrained to one dimension while the radial direction dynamics stays in the ground state of the harmonic oscillator trap. In this regime, the wave function may be written as the product of the following

$$\phi(r, t) = \phi(x, y, z, t) = \phi(x, t)\phi(R) \quad (2.11)$$

$\phi(R)$ is the gaussian wave function corresponding to the ground state of the harmonic oscillator [18, 15, 23, 22]. Inserting equation 2.1.3 into equation 2.1.1 then multiplying it by the conjugate $\phi^*(R)$ and integrating over the radial direction we reduce equation 2.1.1 to the following:

$$i\hbar \frac{\partial \phi(x, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) + U_1(x, t)|\phi(x, t)|^2 \right) \phi(x, t) \quad (2.12)$$

Here; $V(x, t)$ is the external potential in the axial (x) dimension and U_1 is the one dimensional version of the atom-atom interaction strength where $U_1 = \frac{U_0}{2\pi a_R^2}$ and $a_R = \sqrt{\frac{\hbar}{m\Omega_R}}$ is the standard harmonic oscillator length.

2.1.4 Hydrodynamics of Bose-Einstein condensates

The wave function of the condensate may be defined in terms of a constant number density and a phase in the form $\phi = \sqrt{\rho}e^{i\varrho}$ while ρ is real and positive, ϱ is real. We define the the velocity to be $v = \frac{\hbar \nabla \varrho}{m}$ then inserting those into Eq. 2.1.1 [39]

we find the hydrodynamic equations of the BEC to be

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v) \quad (2.13)$$

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla)v - \nabla \left(\frac{V + U_0 \rho}{m} \right) + \frac{\hbar^2}{2m^2} \nabla \left(\frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right) \quad (2.14)$$

equation 2.13 is equivalent to the continuity equation while equation 2.14 is equivalent to the zero viscosity Euler equation except for an additional term named the quantum pressure. The definition of the quantum pressure is

$$P_q = -\frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m \sqrt{\rho}} \quad (2.15)$$

Thus, if we neglect the quantum pressure term the BEC can be viewed as a viscosity free flow. The BEC also is an irrotational fluid. To be able to neglect the effect of the quantum pressure, we have to consider condensates that vary only on length scales larger than the healing length of the BEC

$$\eta = \frac{\hbar}{\sqrt{2mc}} \quad (2.16)$$

where the local speed of sound is:

$$c = \sqrt{\frac{U_0 \rho}{m}} \quad (2.17)$$

For purposes of our later numerical calculations, we define the flux to be

$$\vec{v} = \frac{\hbar}{2mi} \frac{\phi^* \nabla \phi - \phi \nabla \phi^*}{|\phi(r)|^2} \quad (2.18)$$

which is equivalent to our earlier definition as well.

2.2 Microscopic description of Bose-Einstein condensates

2.2.1 Classical Fluctuation of Bose-Einstein condensate

We now introduce a perturbation to the BEC by inserting

$$\phi_0(x) = e^{\frac{-i\mu t}{\hbar}} \left[\phi_0(x) + U(x)e^{-i\Omega t} - \Upsilon^*(x)e^{i\Omega t} \right] \quad (2.19)$$

into Eq. 2.1.1 and then using Eq. 2.1.1 to get to the Bogoliubov equations that describe the elementary excitations of the BEC

$$\left[\frac{-\hbar^2}{2m} \nabla^2 + V(x) + 2U_0|\phi_0|^2 - \mu - \hbar\Omega_l \right] U(x) - U_0\phi(x)^2\Upsilon(x) = 0 \quad (2.20)$$

$$\left[\frac{-\hbar^2}{2m} \nabla^2 + V(x) + 2U_0|\phi_0|^2 - \mu + \hbar\Omega_l \right] \Upsilon(x) - U_0\phi^*(x)^2U(x) = 0 \quad (2.21)$$

where Ω_l is the frequency of the mode. The fluctuation term of the wave function is defined as

$$\delta\phi(x, t) = \sum_l \left(U_l(x)e^{-i\Omega_l t} - \Upsilon_l(x)e^{i\Omega_l t} \right) \quad (2.22)$$

2.2.2 Quantized excitations

We have treated the condensate in a classical mean field picture while considering all the atoms are condensed. We now consider the situation where some atoms are not condensed by introducing the field operator [26]

$$\hat{\phi}(x) = \phi_0(x) + \sum_l \left(U_l(x)\hat{\beta}_l - \Upsilon_l^*(x)\hat{\beta}_l^\dagger \right) \quad (2.23)$$

into the Hamiltonian

$$\hat{H} = \int dx \left[\hat{\phi}^\dagger(x) \left(\frac{-\hbar^2}{2m} \nabla^2 + V(x) \right) \hat{\phi}(x) + \frac{1}{2} U_0 \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}(x) \right] \quad (2.24)$$

then we diagonalize the Hamiltonian in 2.2.2 which is accomplished when U_l and Υ_l satisfy both Eq.2.21 and Eq.2.21. We then obtain the quasi-particles hamiltonian

$$\hat{H} = E(\phi(x)) + \sum_l (\mu\hbar\Omega_l) \hat{\beta}_l^\dagger \hat{\beta}_l \quad (2.25)$$

We define also the fluctuation operator as

$$\hat{\chi} = e^{\frac{-i\mu t}{\hbar}} \sum_l U_l(x) \hat{\beta}_l - \Upsilon_l^*(x) \hat{\beta}_l^\dagger \quad (2.26)$$

with the operator equation [26] defined as

$$i\hbar \frac{\partial}{\partial t} \hat{\chi}(x) = \hat{H}_0(x) \hat{\chi}(x) + U_0 \left(2|\phi(x)|^2 \hat{\chi}(x) + \phi(x)^2 \hat{\chi}^\dagger(x) \right) \quad (2.27)$$

In order for $\hat{\beta}_l$ and $\hat{\beta}_l^\dagger$ to satisfy the commutation relations for creation and annihilation operators they should satisfy the following condition [39]

$$\int d^3x \left[|U_l(x)|^2 - |\Upsilon_l(x)|^2 \right] = 1. \quad (2.28)$$

Here we define the density of the uncondensed atoms to be $\rho_{unc} = \langle \hat{\chi}^\dagger(x) \hat{\chi}(x) \rangle$. The quantum state of the BEC in this case is supposed to be the Bogoliubov vacuum $|0\rangle$ that is defined as

$$\hat{\beta}_l |0\rangle = 0 \quad \forall l \quad (2.29)$$

2.2.3 Fluctuation vs. Bogoliubov quasi-particles

The classical fluctuation of the BEC wave function as well as the quantum fluctuation satisfy the same Eq. 2.21. The difference between a classical fluctuation of the condensate or the quantum fluctuation represented by the field operator $\hat{\chi}$ is based on whether there is a coherent phase relation with the condensate or not [9]. An incoherent excitation would have an expectation value of $\langle \hat{\beta}_l \rangle = 0$ while a coherent one would have $\langle \hat{\beta}_l \rangle = z$ where $z \in \mathbb{C}$.

2.3 Truncated Wigner method

To obtain the dynamics of the BEC a time evolution of the quantum field has to be computed. However, a system with m number of atoms and q number of modes occupies a Hilbert space of dimension q^m . Solving such a system is of great difficulty specially when q and m becomes large. There are techniques well established in the field of quantum optics to get the time evolution of the quantum field [37, 28]. The system density operator is expressed in terms of the coherent state basis using the phase-space distribution. The master equation governing the time evolution of the density matrix is converted to a partial differential equation (PDE) describing the evolution of the distribution. The PDE equations then can be transferred to classical stochastic differential equations (SDE) under certain conditions [37, 28]. Those SDE can give an equivalence of the quantum expectation values by evaluating the averages of the different moments of the phase-space distribution. We will consider the case of a one dimensional BEC because of its importance to our work.

A detailed description of the Truncated Wigner method for a quasi one dimensional BEC is discussed in detail in [10]. Here, we shall draw the basic picture and equations necessary for our purpose. The master equation of the BEC system can be mapped to a Fokker-Blank equation for the Wigner distribution. The wigner distribution is defined in [10] then time evolution of the Wigner distribution then can be written as follows

$$\frac{\partial W(\phi, \phi^*)}{\partial \tau} = \int_{-\infty}^{\infty} dx i \left\{ \frac{\delta}{\delta \phi} [\zeta_0 \phi + \Gamma(|\phi|^2 - 1)\phi] - \frac{1}{4} \frac{\delta^3}{\delta^2 \phi \delta \phi^*} \phi \right\} W(\phi, \phi^*) + c.c. \quad (2.30)$$

where the terms $\zeta_0 = -\frac{1}{2} \frac{d}{dx^2} + \frac{1}{2} x^2 - \mu$ and $\Gamma = \frac{2a}{\lambda a_0}$ In order to map the previous Eq.2.3 into a SDE, a truncation of 2.3 at second order derivative is done. The truncation leads to a stochastic differential equation of a classical field represented by

$$i\partial_{\tau} \phi_W(x, \tau) = \zeta_0 \phi_W(x, \tau) + \Gamma |\phi_W(x, \tau)|^2 \phi_W(x, \tau). \quad (2.31)$$

where there is noise included in the initial state represented by the distribution function for $\phi_W(x, 0)$

A suitable choice of the initial state should resemble closely the ground state of the BEC. The noise can be added to all the modes so that the initial state is in the form

$$\phi_W(x, 0) = \phi_{GP}(x) + \sum_{m=0}^N \frac{1}{2} \eta_m \phi_m(x) \quad (2.32)$$

where each trajectory realization is starting with a different form of Eq.2.3. N is taken to be as large a possible so that the result does not depend on N (number of modes). Since the uncondensed atoms from the BEC is of a later interest the representation of the noise in terms of the Bogoliubov basis is a reasonable choice. The noise is added according to the following equation [30]

$$\phi(x, 0) = \phi_0(x) + \sum_l [\sqrt{\gamma_l} \eta_l U_l(x) - \sqrt{\gamma_l} \eta_l^* \Upsilon_l^*(x)] \quad (2.33)$$

where $\sqrt{\gamma_l} = \left[2 \tanh\left(\frac{\varepsilon_l - \mu}{2K_B T}\right) \right]^{\frac{-1}{2}}$ while $\varepsilon_l = \hbar \Omega_l$ which corresponds to the bogoliubov mode energy that we get from Eq. 2.21

If we define $\eta(x) = \eta_l U_l(x) - \eta_l^* \Upsilon_l^*(x)$ then it is resembling a Gaussian distributed complex random function satisfying the following stochastic average $\overline{\eta(x)\eta(x')} = 0$ and $\overline{\eta^*(x)\eta(x')} = \delta(x - x')$.

Now we may define how to get the quantum correlations from the stochastic averaging over the variables $\phi_W(x)$ as defined in [28]

$$\langle \hat{\psi}^\dagger(x) \hat{\psi}(x') \rangle = \overline{\phi_W(x) \phi_W(x')} - \frac{1}{2} \delta(x - x') \quad (2.34)$$

2.4 Potential field fluctuation and the effective metric

The definition of the quantum fluctuation in 2.2.2 together with the hydrodynamic formulation in 2.1.4 can be used to develop an important analogy. Writing $\phi(x) = \sqrt{\rho(x)} e^{i\varphi(x)}$ so that the velocity is defined as $v = \frac{\hbar \nabla \varphi}{m}$ We can then write

the Hermitian fields

$$\hat{\rho}_1(x) = \sqrt{\rho(x)} \left(\hat{\chi}(x) e^{-i\varrho(x)} + \hat{\chi}^\dagger(x) e^{i\varrho(x)} \right) \quad (2.35)$$

$$\hat{\varrho}(x) = \frac{i}{2\sqrt{\rho(x)}} \left(\hat{\chi}(x) e^{-i\varrho(x)} - \hat{\chi}^\dagger(x) e^{i\varrho(x)} \right) \quad (2.36)$$

Then the fluctuation field operator can be represented as

$$\hat{\chi}(x) = e^{i\varrho(x)} \left\{ \frac{\hat{\rho}_1(x)}{2\sqrt{\rho(x)}} - i\sqrt{\rho(x)} \hat{\varrho}(x) \right\} \quad (2.37)$$

Now by substituting Eq. 2.4 into Eq. 2.2.2 we get the following equations [24]

$$\frac{\partial \hat{\rho}_1(x)}{\partial t} = -\nabla \cdot \left\{ \nabla \cdot v(x) \hat{\rho}_1(x) - \frac{\hbar \rho(x)}{m} \nabla \hat{\varrho}(x) \right\}. \quad (2.38)$$

$$\frac{\partial \hat{\varrho}(x)}{\partial t} = \frac{U_0}{\hbar} \hat{\rho}_1(x) - v(x) \cdot \nabla \hat{\varrho}(x) - \frac{\hbar^2}{2m} D_2 \hat{\rho}_1(x). \quad (2.39)$$

Where

$$D_2 \hat{\rho}_1(x) = -\frac{1}{2} \frac{\nabla^2 \sqrt{\rho(x)}}{\sqrt{\rho^3(x)}} \hat{\rho}_1(x) + \frac{1}{2} \frac{1}{\sqrt{\rho(x)}} \nabla^2 \left(\frac{\hat{\rho}_1(x)}{\sqrt{\rho(x)}} \right) \quad (2.40)$$

while in the hydrodynamic regime the first term in Eq. 2.4 is neglected. Combining Eq. 2.39 and Eq. 2.39 we arrive at a second order differential equation for $\hat{\varrho}(x)$:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \hat{\varrho}(x, t) = 0. \quad (2.41)$$

where $g^{\mu\nu}$ is the effective inverse metric, $g = \det(g) = \frac{\rho^2(x)}{c(x)}$ Where Eq. 2.4 is the Klein-Gordon equation which characterizes the minimally coupled massless scalar quantum field propagation in a curved space time having a metric $g_{\mu\nu}$ [5, 4] and the effective inverse metric is

$$g^{\mu\nu} = \frac{1}{\rho(x)c(x)} \begin{bmatrix} -1 & \vdots & -\nu_n \\ \dots & \cdot & \dots \\ -\nu_m & \vdots & (c(x)^2 \delta_{mn} - \nu_m \nu_n) \end{bmatrix}. \quad (2.42)$$

These results are only applied to potential fluctuation field which has wave number shorter than $\frac{1}{\eta}$.

Chapter 3

Hawking radiation in a simplified analogue system

In this chapter we try to introduce a simple system in which we can use a one dimensional BEC to probe a sonic horizon by turning the BEC flow supersonic we then look at the signature of the Hawking radiation using the technique of density-density correlation functions.

3.1 Creating a sonic horizon

In order to establish the analogy of an effective metric in BEC flow, Eq. 2.4 needs to have the background fluid velocity turning supersonic. An idea to make the BEC flow turn supersonic and exceed the speed of sound while having uniform density was discussed by [14, 16] where the flow velocity stays constant while the speed of sound changes spatially by means of changing the interaction strength of Eq. 2.1.1 and according to Eq.2.1.4. Using a Galilean transformation, the speed of sound would be dependent on both time and position. We consider a spatial dimension x in which at some point $x > x_0$ the the (flow velocity) $v > c$ (speed of sound) and at $x < x_0$, $v < c$. Then the sound waves traveling to left at $x >$

x_0 will not be able to escape from the horizon.

3.2 The black hole-white hole geometry

Following the study of [38] we introduce a simple one dimensional BEC of an originally constant density flowing at a constant velocity v_0 , then by varying the atom-atom interaction strength we can vary the speed of sound across the spatial dimension. This could be realized with Feshbach resonance by spatially varying the magnetic field applied to the system [46].

We start by introducing a one dimensional BEC described by the time dependent GP equation given in 2.1.3. The atom-atom interaction strength $U_{1d} = U_{1d,1}$ and the external potential $V(x) = V_1$ is constant initially so the solution to the GP equation is given by the plane wave solution

$$\phi(x, t) = \sqrt{\rho_0} e^{i(k_0 x - \Omega_0 t)} \quad (3.1)$$

where ρ_0 is the constant density, $k_0 = \frac{mv_0}{\hbar}$ is the momentum of the moving density.

After time $t > t_0$ we tune the atom-atom interaction strength and the external potential to be $U_{1d}(x > 0) = U_{(1d,2)}$, $U_{1d}(x < 0) = U_{(1d,1)}$ and $V(x > 0) = V_2$, $V(x < 0) = V_1$ while we keep the condition of $\rho_0 U_{(1d,1)} + V_1 = \rho_0 U_{(1d,2)} + V_2$. Through this construction the uniform plane wave solution of the condensate is valid in both spatial regions $x < x_0$ and $x > x_0$, with $\Omega_0 = \frac{\hbar k_0^2}{2m} + \frac{V_{1,2}}{\hbar} + \frac{\mu_{1,2}}{\hbar}$ where μ is the chemical potential of the condensate defined as $\mu = \rho_0 U_{1d}$ so that $\mu_1 = \rho_0 U_{(1d,1)}$, $\mu_2 = \rho_0 U_{(1d,2)}$. Following the modulation of the interaction strength and the external potential after $t > t_0$, the result is a modulation of the speed of sound realizing the following hierarchy $c(x > 0) < v_0 < c(x < 0)$ This gives rise to a sonic horizon at $x = 0$.

In the following we show the system parameters that were used in the numerical

simulation of the system. All the parameters are in the dimensionless formalism described in Sec. A.1. It should be noted that we could not regenerate the exact parameters of [38] due to lack of information. Nonetheless, we regenerated the main features reported in [38] represented in the density-density correlation function. The system parameters are shown in the following table 3.1

Table 3.1: System parameters

Normalized Parameters	Values
Density $\tilde{\rho}_0$	450
Momentum of the moving density $\tilde{k}_0$	1.6
Flow velocity $\tilde{v}_0$	1.6
$\tilde{U}_{(1d,1)}$	81×10^{-4}
$\tilde{U}_{(1d,2)}$	36×10^{-4}
$\tilde{V}_1$	0
$\tilde{V}_2$	2.2025
$\tilde{t}_0$	.5
Potentials modulation width $\tilde{\sigma}_x$	0.5
time evolution $\tilde{t}$	30
Supersonic region width	80

A numerical simulation of the condensate density as well as the wave function is shown in Fig. 3.1. It is shown that at later time after the change of the potentials, the density stays almost the same and is not much affected. This validates our approximation earlier to expect a solution with uniform density on both sides despite the potential modulation. The wave function corresponds to a plane wave solution with a momentum equal to $\tilde{k}_0$. Periodic boundary conditions are applied and the periodicity of the wave function has been checked numerically.

The profiles of the external potential as well as the interaction strength are shown

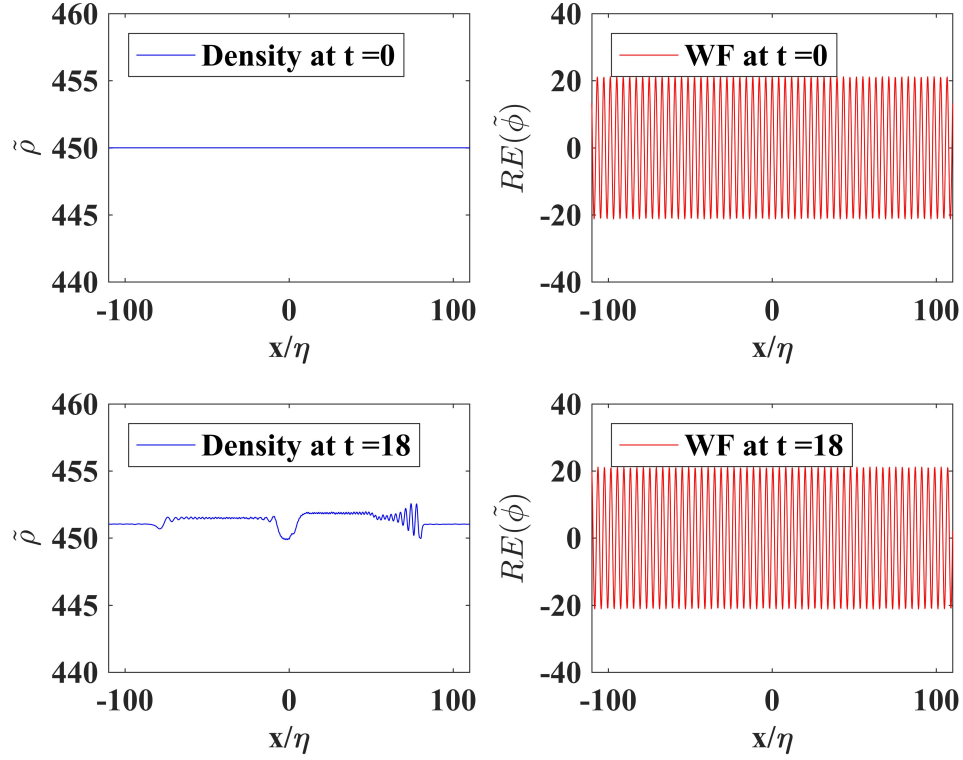


Figure 3.1: Density of the condensate in blue is shown at time zero before the horizon creation which shows a constant density with a value of 450 then at time = 18 after the horizon creation. In red, the real part of the wave function of the condensate is shown for time zero and 18 as well

in Fig.3.2 when there were constant potentials as well as at later time showing the modulation. The width of the supersonic region (modulation region) is equal to 80 but an extended region is shown to show the periodicity of the system mentioned earlier. For numerical convenience the step like change in the profiles is being implemented using tanh function with step width $\tilde{\sigma}_x = 0.5$ to avoid sharp transition in the potential. The local speed of sound of the condensate is shown in Fig.3.3 thus indicating the location of the sonic horizon at $x = 0$. The profile of the speed of sound is calculated according to equation 2.1.4.

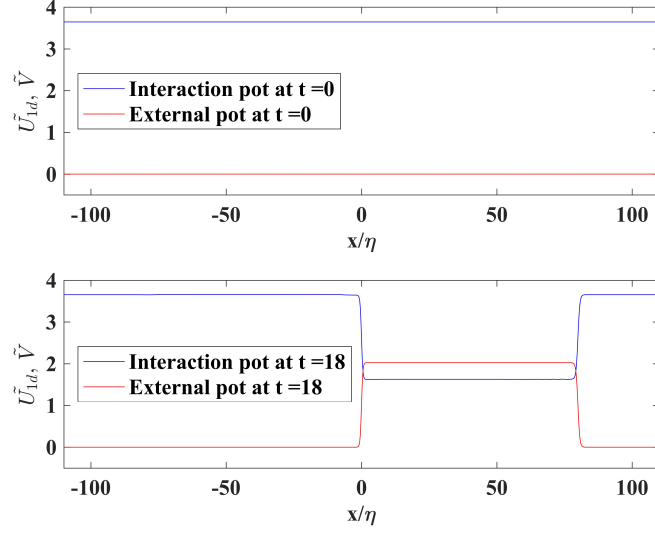


Figure 3.2: The profile of the Atom-Atom interaction strength in blue and the external potential in red are shown at time zero, which is corresponding to a homogenous system everywhere. We also show it at time equal to 18 long after the horizon creation, which shows the modulation of both potentials.

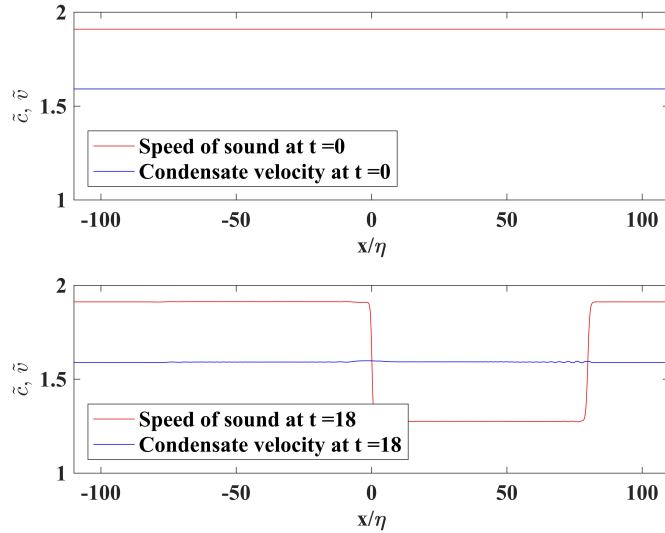


Figure 3.3: The profile of the speed of sound in red and the condensate velocity shown in blue in at time zero which shows a subsonic flow everywhere then at time equal to 18 far after the horizon creation, which shows a super sonic region from $x/\eta = 0$ to 80.

3.3 Correlation function showing Hawking partners

3.3.1 Truncated Wigner approach

To capture the behavior of quantum fluctuation around the mean field wave function of the condensate we use the Truncated Wigner method TWM explained earlier in Sec. 2.3 to get the complete time evolution. The TWM uses the same time dependent differential equation as mean-field (GP) theory, but with a random initial state that is then being averaged over many trajectories of evolution. Quantum fluctuation are represented as the phonon modes known as the Bogoliubov modes. The phonon modes here represent the sound wave perturbation on top of the background flow. The dynamical behavior of these in the presence of a sonic horizon is the main focus of this thesis.

To implement the TWM we need to evolve an initially randomized state of the BEC and then get an average over many trajectories. The initial random state corresponds to the ground state of the BEC mean field with random noise. We choose the random noise to be represented in terms of Bogoliubov modes. For a uniform system of a constant interaction strength and external potential at time $t < t_0$ the Bogoliubov modes are plane waves. The initial stochastic wave function for the TWM then can be written as:

$$\phi_0(x) = e^{i(k_0 x - \Omega_0 t)} \left\{ \sqrt{\rho_0} + \sum_{k \neq 0} \left[\beta_k U_k e^{ikx} + \beta_k^* \Upsilon_k e^{-ikx} \right] \right\} \quad (3.2)$$

where the terms Υ_k and U_k are the Bogoliubov coefficients satisfying Eq. 2.2.2 and are defined as

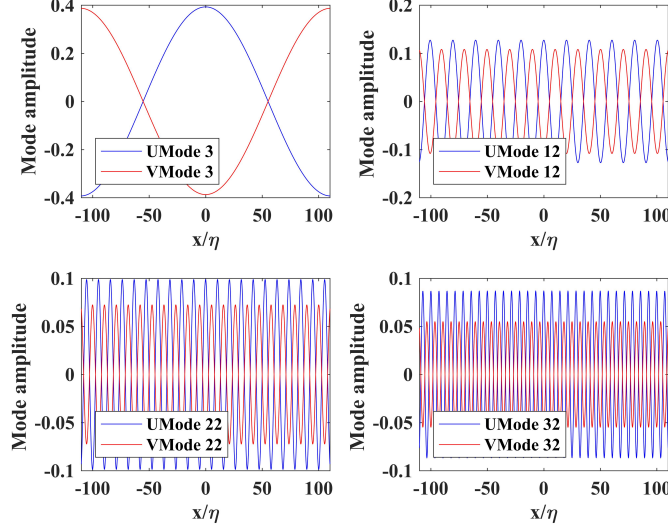


Figure 3.4: Different Bogoliubov modes which are representing the phonon modes on top of the mean field solution are shown in the figure above

$$\Upsilon_k \pm U_k = \pm \sqrt{\frac{\epsilon_k}{\epsilon}} \quad (3.3)$$

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} \quad (3.4)$$

$$\epsilon_k = \sqrt{\epsilon_k(\epsilon_k + 2\rho_0 U_{(1d,1)})} \quad (3.5)$$

We use a total number of 64 modes for implementing the Wigner method in terms of the Bogoliubov basis, where the momentum mode inserted in the plane wave solution is $K_{mode} = \frac{n\pi}{L}$ where L is the maximum spatial grid size, n is the mode number. Different modes are shown.

3.3.2 Simulation of analogue Hawking radiation

As discussed in [36] the phonons corresponding to the Hawking particle and its partner are going to result in a long range density correlations with specific features. The features were derived in [36] by graviational analogy. Here, using the microscopic theory of BEC we can verify these predictions. It is shown in

[36] that Hawking particles are going to result in “a valley shaped feature” on the lines of $\frac{x'}{c_l - v_0} = \frac{x}{c_r - v_0}$ while here $c_r = c(x > 0)$ and $c_l = c(x < 0)$.

A definition of the density-density correlation function is given in [38] as:

$$G^{(2)}(x, x') = \frac{\langle : \rho(x) \rho(x') : \rangle}{\langle \rho(x) \rangle \langle \rho(x') \rangle} \quad (3.6)$$

where $\hat{\rho}(x) = \hat{\phi}^\dagger(x) \hat{\phi}(x)$. Using the commutation relation for bosons

$$[\hat{\phi}(x), \hat{\phi}^\dagger(x')] = \delta(x - x') \quad (3.7)$$

then inserting the definition of $\hat{\phi}(x)$ from equation.3.2 we arrive at the following definition of a delta function in terms of bogoliubov modes.

$$\delta(x - x') = e^{ik_0(x - x')} \left[\sum_{k \neq 0} \left(U_k(x) U_k^*(x') \right) - \left(\Upsilon_k^*(x) \Upsilon_k(x') \right) + \sum_{k \neq 0} \left(U_k^*(x) U_k(x') \right) - \left(\Upsilon_k(x) \Upsilon_k^*(x') \right) \right] \quad (3.8)$$

where $U_k(x) = U_k e^{ikx}$ and $U_k^*(x) = U_k e^{-ikx}$ and the same for $\Upsilon_k(x)$ and $\Upsilon_k^*(x)$. For infinite number of modes the right hand side would also be delta function, but since numerics has finite number only, we use it with finite number of modes, where $\delta(x - x')$ is non local.

Inserting the commutation relation of bosons 3.3.2 into equation.3.3.2 we arrive at a definition of the correlation function for implementation:

$$\begin{aligned} G^{(2)}(x, x') &= \langle \rho(x) \rho(x') \rangle - \frac{1}{2} \langle \rho(x) \rangle \delta(x') - \frac{1}{2} \langle \rho(x') \rangle \delta(x) \\ &\quad - \frac{1}{2} \overline{\phi_0(x) \phi_0(x')} \delta(x' - x) - \frac{1}{2} \overline{\phi_0(x') \phi_0(x)} \delta(x - x') + \\ &\quad \frac{1}{4} \delta(x - x') \delta(x' - x) + \frac{1}{4} \delta(x) \delta(x') \end{aligned} \quad (3.9)$$

A numerical simulation of the dynamics of the condensate fluctuations is done using the TWM mentioned earlier. Different observables in the correlation function are obtained by averaging over many trajectories which were 504000 trajectories for the current result. The numerical simulation of the density-density correlation

function at different times is shown in Fig.3.6. In the following we will have a brief discussion on the important features appearing in the correlation function.

1. The correlation function of a coherent state is always equal to 1.
2. At time equal to zero in Fig.3.5a before the horizon creation there is a strip along the diagonal line $(x - x')$ where $G^{(2)}$ is approximately 0.9, <1 . This strip corresponds to the anti-bunching coming from the repulsive interaction between atoms [11].
3. There is a strip along the line $\frac{x'}{c_l - v_0} = \frac{x}{c_r - v_0}$ mentioned earlier (perpendicular to the diagonal), which lengthens with time after the creation of the horizon as in Fig.3.5b. It stays till the end of the time evolution as indicated in Fig.3.5a. That is the Hawking radiation.
4. Shown in Fig.3.5b, diagonal strips parallel to the line $(x - x')$ are being emitted after the horizon creation. It is attributed in ref [38] to be due to the modulation in the atom-atom interaction strength. It results in the analogue of cosmological particle creation during inflation [31, 6, 29, 33, 21].

To illustrate more on feature 2 we consider the theory developed in [40] for a BEC exposed to a step modulation shape in the external potential and the atom-atom interaction strength. It has been shown that the scattering solution of the Bogoliubov theory after the quantization of the Bogoliubov modes is going to result in a spectrum of emission of phonons of a negative norm ingoing mode in the region $x > x_0$ to a positive norm outgoing mode in the region $x < x_0$ while ingoing means directed toward the horizon. The reason behind the contribution from the negative norm ingoing mode is that it comes from the vacuum fluctuation since the spectrum of the emission shows no occupation of the +ve norm modes at zero temperature. The formula of correlation function presented in [36] shows that the strongest contribution is due to the -ve norm ingoing mode which results in two +ve norm outgoing mode that escape from the horizon along lines $\frac{x}{v_0 - c(x < 0)}$ and $\frac{x'}{v_0 - c(x > 0)}$.

Finally, a comparison between our results and the results of [38] is shown in Fig.3.6. The comparison shows that we are able to capture all the features that have been found qualitatively, with quantitative differences due to slightly different parameters.



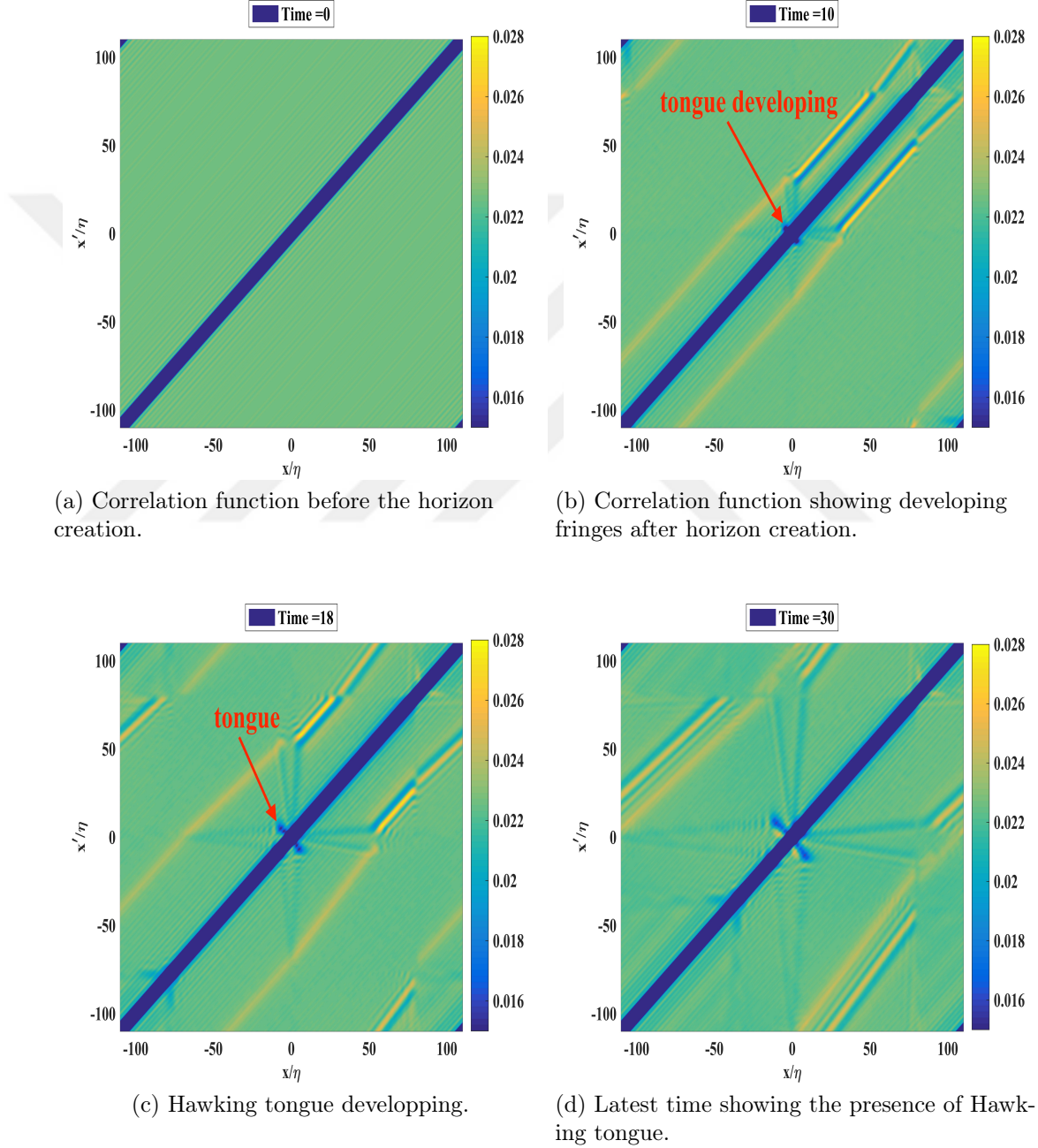
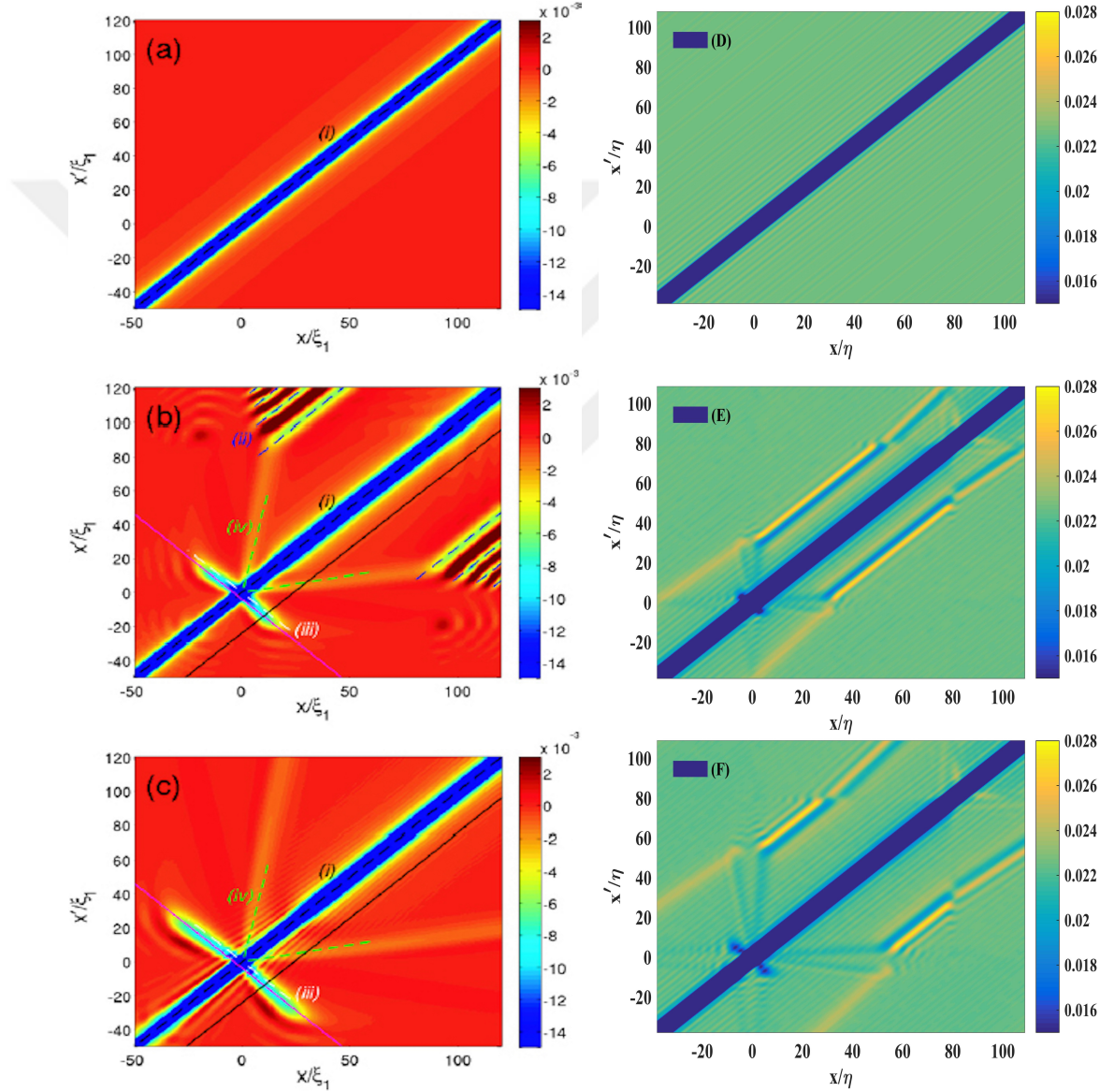


Figure 3.5: Correlation function is shown at different times before the horizon creation where the flow is subsonic everywhere until latest time where the Hawking tongue is present



(a) Original paper Correlation function at different times.

(b) Correlation function of the current work.

Figure 3.6: Comparison between the correlation functions of our results to right and with results of [38]

Chapter 4

Black hole laser

4.1 Brief summary of the Haifa experiment

In this section the experiment in Technion [44] that has reported an observation of a self amplifying Hawking radiation is being introduced. The experimental implementation of the experiment as well as the main results are presented briefly.

4.1.1 Experiment implementation

The experiment [44] starts with ^{87}Rb atoms in the zeeman state $F = 2, m_f = 2$ in a magnetic trap with a radial harmonic trap of a frequency 123 Hz. The condensate is assumed to have a cigar shaped configuration reducing the dimensionality of the dynamics to one dimension.

The black hole and white hole geometry is created by an additional negative step which is swept across the shallow potential by means of a Gaussian laser beam half blocked. The step width is $2\mu\text{m}$. The step is swept at a speed of $0.21 \frac{\text{mm}}{\text{s}}$ from left to right which accelerates the condensate to the left to higher

than the speed of sound and thus creates the black hole geometry. Because of the shallow trap potential the condensate turns subsonic further to the left ($x < 0$) again which will create a black hole white-hole geometry.

4.1.2 Main Results of the experiment

The experiment [44] is reporting observation of a standing wave pattern present in the condensate density between the black hole and white hole geometry. The standing wave pattern is referred to be the self-amplifying Hawking radiation known as the "Black hole laser" effect. The black hole laser effect is due to certain modes of sound waves which are trapped between the black hole and the white hole. Those modes are interfering then as a result they get amplified to give rise to a standing wave pattern which is exponentially growing [41]. The condensate density is measured using phase contrast imaging with an ensemble of 80 images averaged over for each time of measurement.

A study of the the density-density correlation function is given in the paper [44] which shows definite features inside the lasing region. The features are the checkerboard that are claimed to be due to the fact that the quantum fluctuations are having definite wavelength in agreement with the prediction in [41]. Another feature of the density-density correlation function is the horizontal and vertical bands extending outside the black and white holes, the one outside the black hole is referred as the escaping flux and is claimed to be the Hawking radiation.

4.2 Numerical study of the experiment

In this section we numerically study the experiment starting from precisely modeling the setup. The aim is to reproduce the experimental results for the condensate density and the standing wave pattern that we will show it can be reproduced by mean field simulations of the condensate only. Then, we try to study the density-density correlation function and compare it with the experimental results. At the

end of the section we try to investigate the standing wave pattern to differentiate the parts which are coming from the condensed density and the one coming from the uncondensed density then we compare this to the total density of the condensate.

4.2.1 Numerical modeling of the experiment

Since the experiment corresponds to a very good approximation to a quasi-1D BEC, we solve the time dependent one dimensional Gross-Pitaevskii (GP) equation 2.1.3. The numerical solution of the GP equation is obtained by setting up the ground state of the condensate in a harmonic trap by means of the imaginary time evolution method. We subsequently start the real time evolution at $-50ms$. At which time the potential step is initially far away from the condensate. The total duration of the experiment is 120 ms and given the fact that we start the evolution earlier by 50 ms then the total simulated time of our simulation is 170 ms. A list of the values of the basic constants is shown in table 4.1 as well as the numerical simulation parameters which are shown in table 4.2 and table 4.3. The numerical simulation parameters follow the dimensionless formalism stated in Sec. A.1.

Table 4.1: Essential parameters

Essential Parameters	Values
S-wave scattering length a_s	5.5×10^{-9} m
Healing length η	8.35×10^{-7} m
Atomic Mass of ^{87}Rb	1.44×10^{-25} kg
Radial potential trap frequency	153.6 Hz
Chemical potential μ	8 nk $\times K_B$
Number of atoms N	4670

Table 4.2: Numerical simulation parameters

Normalized numerical Parameters	Values
Interaction potential $\tilde{g}_{1D}$	121×10^{-4}
Time evolution	179.05
Potential step velocity $\tilde{v}$	24×10^{-2}
Potential step width $\tilde{x}_{wid}$	2.39
Spatial dimension L	130

where we choose the notation of the interaction strength here to be $\tilde{g}_{1D}$ for purpose of notation convenience.

The black hole-white hole configurations are being created by means of manipulating the external longitudinal trap potential as mentioned earlier in 4.1.1. In order to model the experimental implementation of the external potential and the negative potential step sweeping across it we follow the same model given in [47]. A description of the potential model as well numerical results compared to the experiment is given in the next subsection 4.2.1.1

4.2.1.1 Potential Profile

The potential profile used is in the form

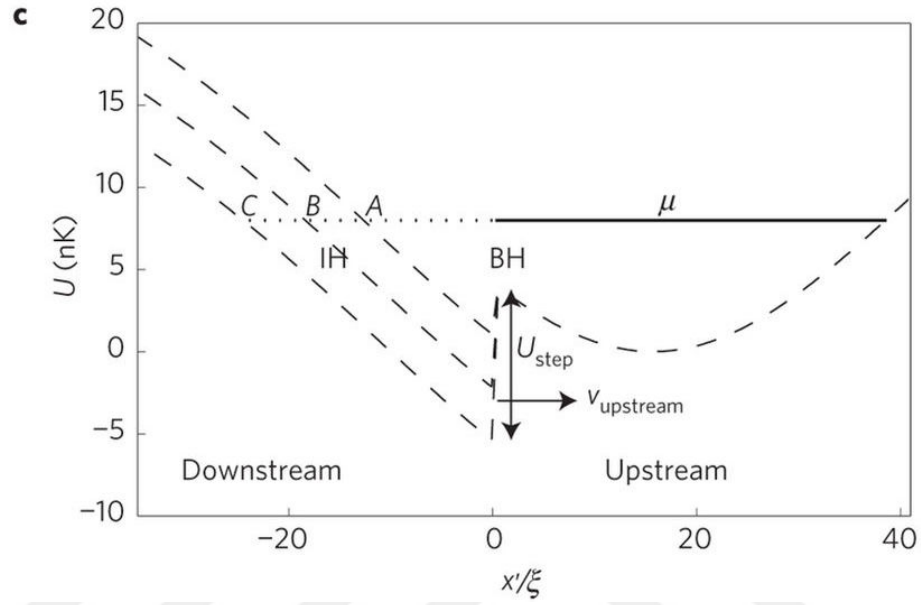
$$\tilde{U}(\tilde{x}, \tilde{t}) = \tilde{U}_0 \sqrt{(\tilde{x} - \tilde{x}_0 + \tilde{v}\tilde{t})^2 + b^2} - \frac{\tilde{U}_{st}}{2} [1 - \tanh(\frac{\tilde{x}}{\tilde{x}_{wid}})] + \tilde{U}_{Wf}$$

with the normalized parameters shown in the following table 4.3 The step potential is of width $\tilde{x}_{wid}$ to resemble the smooth step transition in the experiment and $\tilde{U}_{Wf}$ is chosen to fit the waterfall potential in the experiment. In this implementation we move the shallow trap in the $-ve$ x direction (to the left) while the step potential is kept at rest at $\tilde{x} = 0$ which is equal to the experimental procedure of sweeping the step to the right direction.

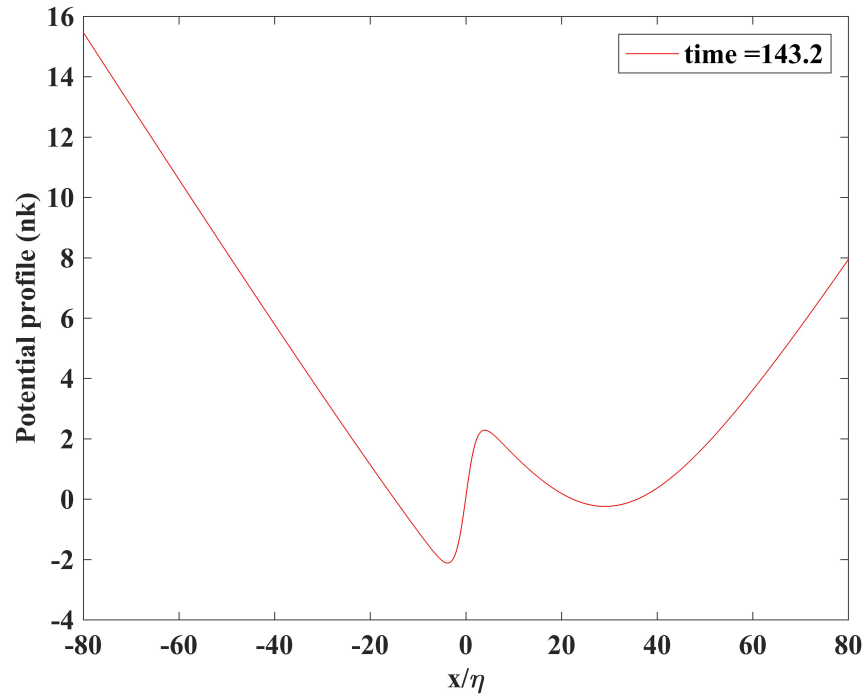
Table 4.3: Potential profile parameters

Normalized Parameters	Values
$\tilde{U}_0$	313×10^{-4}
$\tilde{x}_0$	63.48
b	23.27
Potential step height $\tilde{U}_{st}$	0.8
Water fall potential $\tilde{U}_{Wf}$	-0.75

A comparison between the experimental potential and the trap potential is shown in Fig. 4.1, the numerical model shown in Fig. 4.1b. We see good agreement with the shape of the potential trap of the experiment. The potential profile at different times of the simulation is shown in Fig. 4.2. The figure shows a shallow trap potential in the beginning then the introduction of the negative step potential at later times.



(a) Experiment Potential profile.



(b) Numerically created potential profile.

Figure 4.1: The upper figure shows the experimental potential profile, the figure in the bottom shows the numerically simulated potential profile. The potential strength is measured in nano Kelvin (nk)

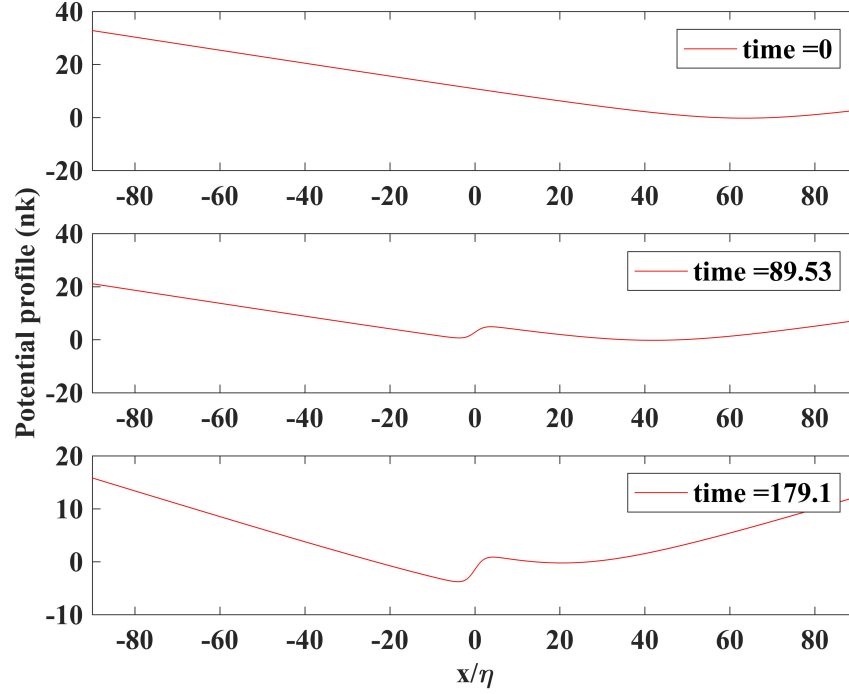
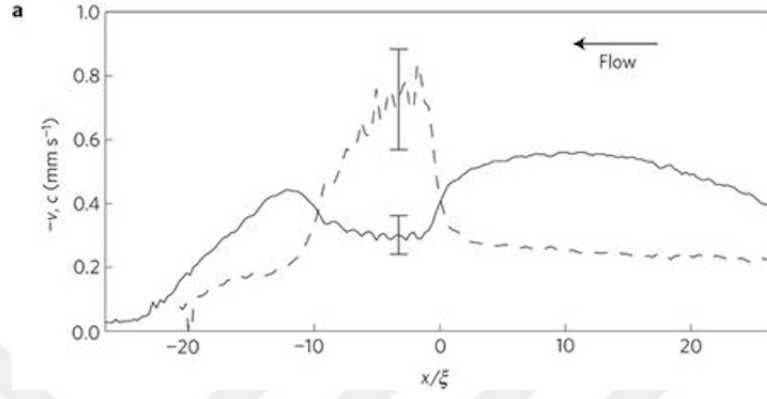


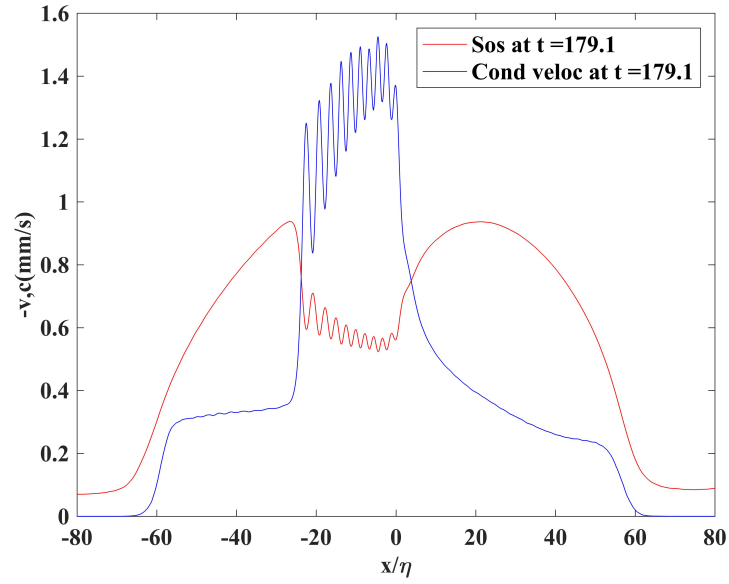
Figure 4.2: Potential profile at different times is shown

4.2.1.2 Black hole and White hole Creation

After introducing the negative step potential the flow starts to turn supersonic and then subsonic creating a black hole-white hole geometry in a region across x direction. A demonstration of that configuration is done by showing the speed of sound of the condensate compared to the velocity of it. The speed of sound is calculated according to Eq. 2.1.4. The velocity of the condensate is calculated using the definition given by Eq. 2.1.4. The speed of sound as well as the condensate velocity is shown in Fig. 4.3 where a comparison between the experimental and the numerical results is indicated. It should be mentioned that the condensate velocity is multiplied by -1 as the condensate is moving to the left.



(a) Experiment velocity profiles.



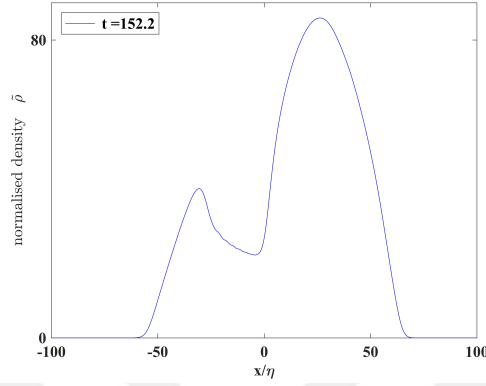
(b) Numerically created velocity profiles.

Figure 4.3: The upper figure shows the experimental speed of sound corresponding to the super and subsonic regions, the figure in the bottom shows the numerically simulated speed of sound in red and the velocity of the condensate in blue

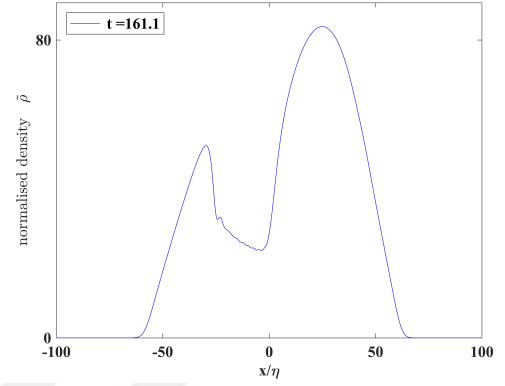
4.2.2 Gross-Pitaevskii simulation of the experiment

A solution of the GP equation gives us the mean field density of the condensate. The density profile is shown in Fig. 4.4 for different times of the dynamics starting from the normalized time = 152.2 which is equivalent to the time = 80 ms in the experiment. The frame corresponding to the dynamics of the experimental density at time = 80 ms is shown in the next subsection in Fig. 4.5a at the 4th frame. It is important to mention that the standing wave pattern inside the supersonic region is fully recovered already by the mean-field density simulation. The growing standing wave pattern as mentioned earlier is referred in the [44] as evidence for the self amplifying Hawking radiation. It is claimed in the paper that the pattern is due to the interference between two negative norm phonon modes. One of the phonon modes is the Hawking partner generated from the emission of the *+ve* norm Hawking radiation that travels away from the black hole. Moreover, the growing standing wave pattern which is exhibiting an exponential behavior over time is claimed to be due to the round trips of the hawking partner between the two holes which acts as a cavity for the modes. However, we show here that the lasing effect as well as the standing wave pattern is generated using a mean field simulation of the condensate and without any quantum fluctuation in terms of phonons.

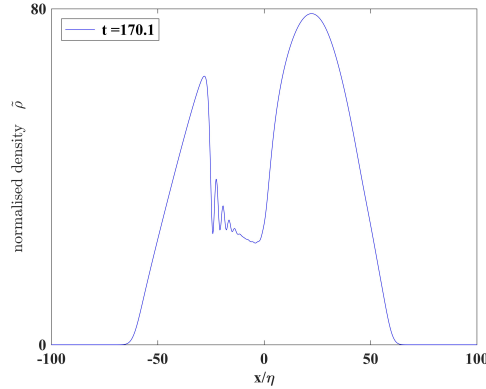
A further detailed discussion about the origin of the growing standing wave pattern has been introduced in [48] where it is said that the wave pattern is due to classical modulation of the mean field density of the condensate. Furthermore, it is shown that the standing wave results from the Bogoliubov-Cerenkov radiation effect where the saturated amplitude of this standing wave is proportional to the strength of the density of the background flow. As shown in Fig. 4.4a, 4.4b the pattern starts to appear at the white-hole position then it travels toward the black hole position which is another property of Bogoliubov-Cerenkov radiation that happens when a condensate traveling at supersonic speed is facing a weak obstacle [32, 45].



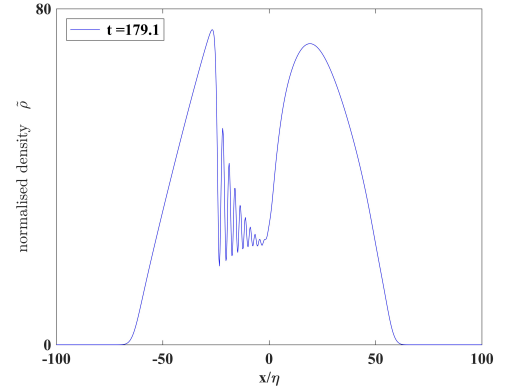
(a) Wave pattern appearing.



(b) Wave pattern traveling from WH.



(c) Growth of the wave pattern.



(d) Standing wave pattern at latest time.

Figure 4.4: Numerical Simulation of the mean field density without quantum fluctuation added to it showing the growth of the standing wave patterns at different times

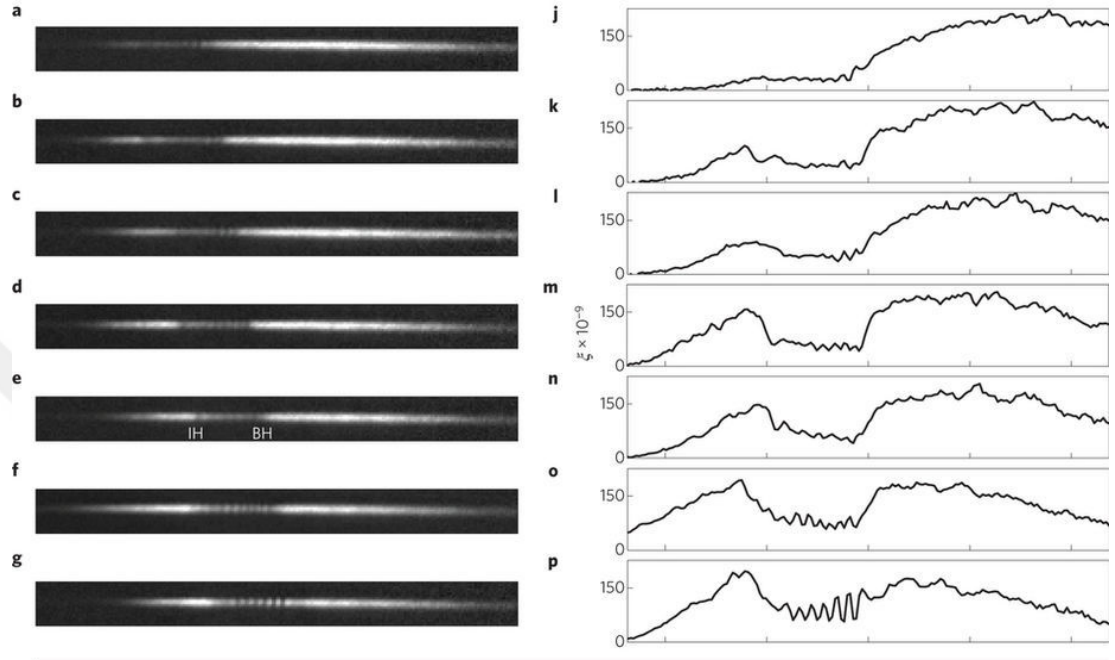
4.2.2.1 Quantum fluctuation Simulation

We now add the quantum fluctuations to the mean field GP solution of the condensate and obtain their time evolution. In this regime we are dealing with the fluctuation as a small perturbation on top of the mean field. Here we again follow the same methodology of the TWA mentioned and illustrated earlier in 3.3.2 and 2.3. The initial noise is added in terms of Bogoliubov modes which we include in terms of plane wave solution written as the following equation:

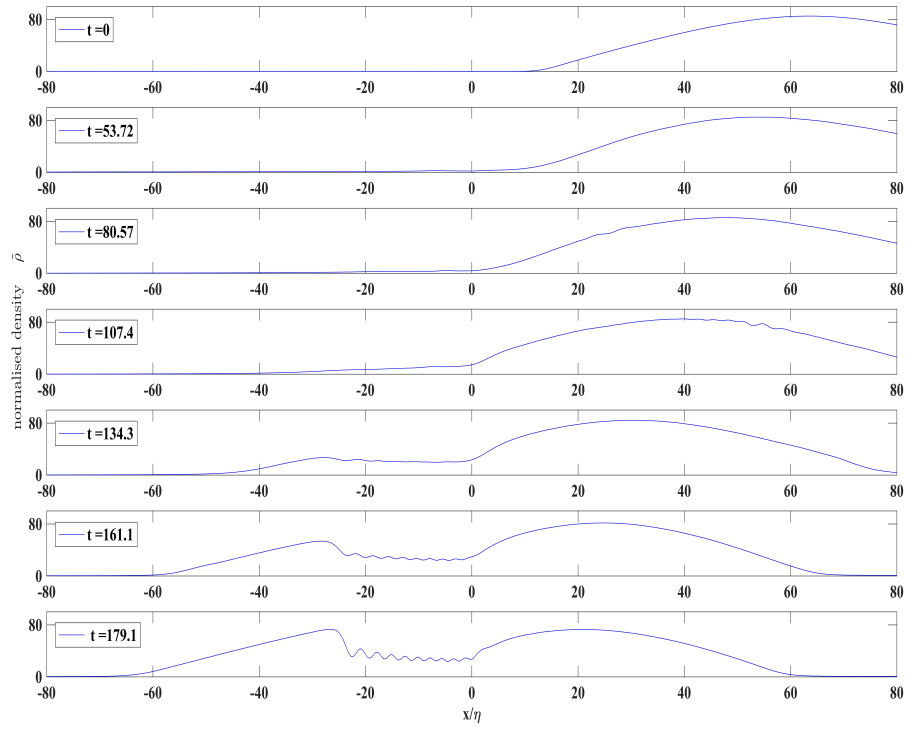
$$\psi_0(x) = \psi_{MF} + \sum_{k \neq 0} \left[\beta_k U_k e^{ikx} + \beta_k^* \Upsilon_k e^{-ikx} \right] \quad (4.1)$$

Where β_k, U_k, Υ_k are defined according to the same definition in 2.21. ψ_{MF} is the stationary initial mean field solution of the condensate wave function. We define the total density of the condensate here according to Eq. 2.3 We have sampled the quantum fluctuations numerically in terms of 64 modes. We evolve the wavefunction of the condensate with an initial random noise then it is averaged over different many realisations. The different realisations are termed trajectories and we have used 58800 trajectories. Result of the time evolution of the condensate with the quantum fluctuation is shown in Fig. 4.5 where a comparison between the experimentally reported density profile (4.5a) and the numerically simulated one (4.5b) at different times is shown. The numerical simulation is able to reproduce the growing standing wave pattern within the supersonic region as in the experiment.

It is worth mentioning that the standing wave pattern in Fig.4.5b is not as clear as in the case of the meanfield simulation shown earlier. There are also weak oscillations at time = 161.1 shown in Fig.4.5b in addition to the previous ones appearing in the mean field simulation in Fig.4.4b. The density profile of the latest time is showed in Fig.4.6 that has a lower amplitude of the standing wave pattern than the one in 4.4d. A discussion behind these observations is left to Subsec. 4.2.3



(a) Experiment condensate density at different time.



(b) Simulation result of the density at different time.

Figure 4.5: Experiment vs Numerical simulation of density profile comparison

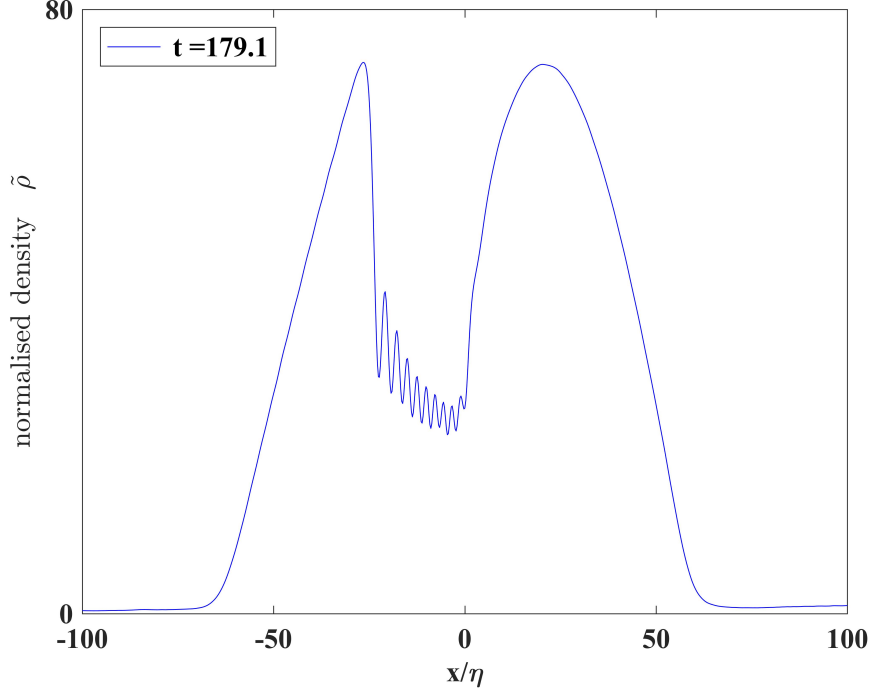


Figure 4.6: Simulation of Mean field density with noise added to it at latest time

4.2.3 Density-density correlation function

In this subsection we introduce the study of the density-density correlation function. The correlation function was reported in [44] to have the checker board feature that was introduced by [41] and the the extending bands which is attributed to be the Hawking radiation.

4.2.3.1 Density-Density Correlation Function Simulation

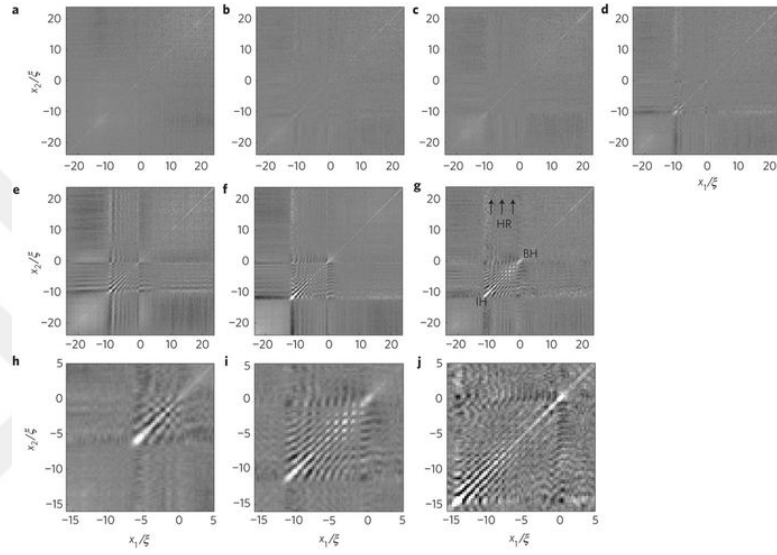
We implement the density-density correlation function using the TWA method discussed earlier. In the experiment the density-density correlation was realized by averaging over 80 ensembles. We follow the same definition given in [44] for the correlation function

$$G^{(2)}(x, x') = \rho_L^{-2} [\langle \rho(x) \rho(x') \rangle - \langle \rho(x) \rangle \langle \rho(x') \rangle - \langle \rho(x) \rangle \delta(x - x')] \quad (4.2)$$

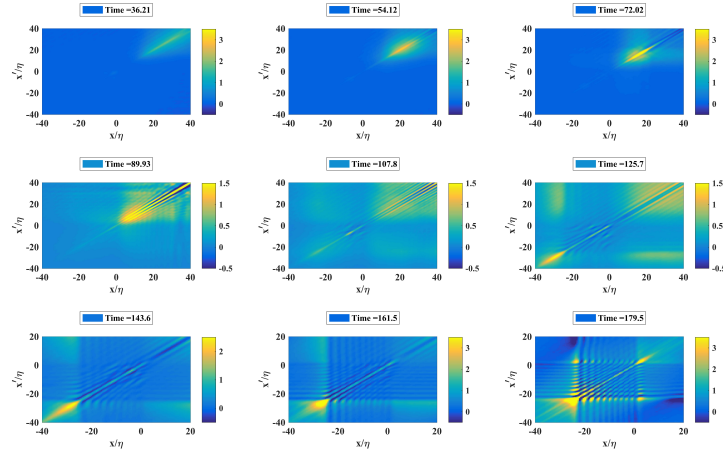
We choose the value of ρ_L to be the density in the middle of the supersonic region, while it was defined in [44] as the average density of the supersonic region. Our value is $\rho_L = 26.32$.

A comparison at all times between the experimental density-density correlation function and the numerically generated one is showed in Fig.4.7. As showed in Fig.4.8 the numerical simulation indicate the checker board feature that were present in the experiment as well as the extending bands outside the horizon region. In Fig. 4.8c, The number of maxima is 10 which is the same reported in the experiment for the largest step potential profile. Near the white-hole location the correlation function is showing fringes parallel to the diagonal which was attributed in the experiment to be due to the fact that the white-hole location is determined by the hydrodynamics.

The correlation function was smoothened by a Gaussian smoothing with an average length of 1.5 to get a better resolution. The correlation function is rescaled so that the details of the figure can be clearer.

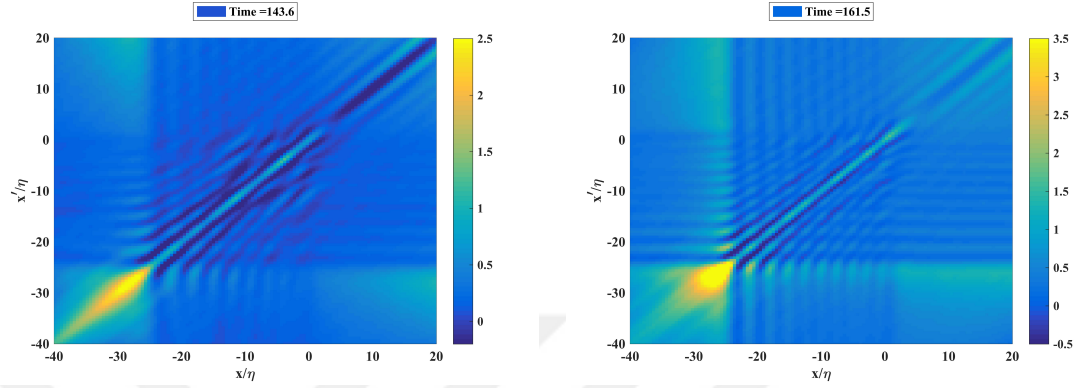


(a) Experiment correlation function at different times.

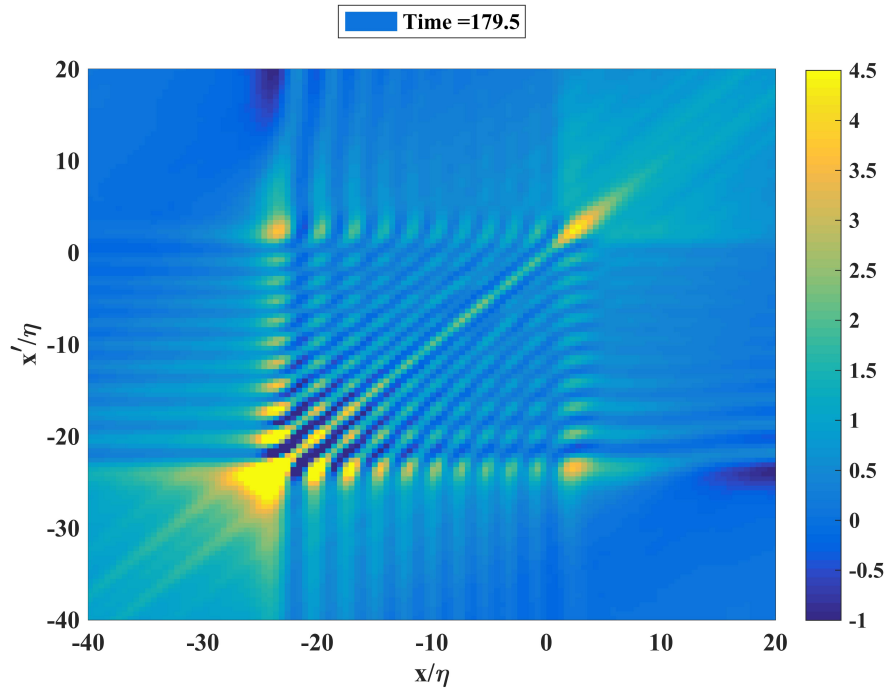


(b) Numerically simulated correlation function.

Figure 4.7: The upper figure shows the experimental correlation function at different times, the figure in the bottom shows the numerically simulated one at different times of the experiment



(a) Correlation function showing the fringes. (b) Correlation function showing the fringes.



(c) Correlation function showing the checkerboard feature.

Figure 4.8: The density correlation function is shown at late time developing the fringes and the checkerboard properties are shown as well

4.2.4 The one-body-density matrix simulation

4.2.4.1 Condensed Vs Uncondensed Densities

A criteria to differentiate between the condensed and uncondensed parts of the total density of the condensate is set by Pensore-Onsager [1]. We can expand the one body density matrix $\hat{\rho}(x - x')$ in terms of eigen values λ_k which represent the occupation number and eigen vectors ϑ_k that represent the condensate wave function.

$$\rho(r, r') = \sum_k \lambda_k \vartheta_k(r') \vartheta_k(r) \quad (4.3)$$

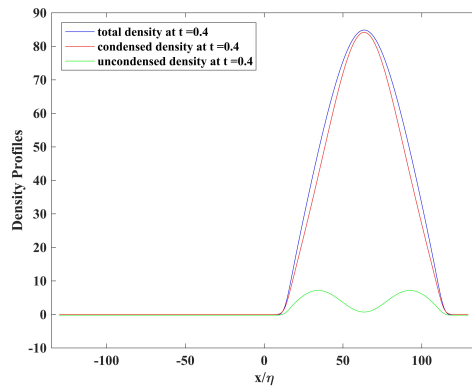
Then we define the condensed density as

$$\rho_{cond} = \lambda_k |\vartheta_k(r)|^2 \quad (4.4)$$

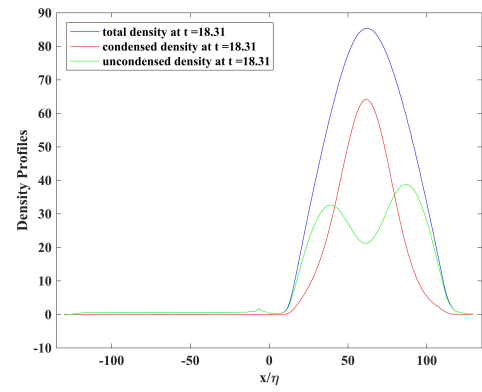
consequently the uncondensed density is defined as $\rho_{uncond} = \rho_{tot} - \rho_{cond}$

We used the previous equations to solve numerically for the different components of the density. The results are supplied in Fig.4.9 and Fig.4.10. We shall comment on the results as follows:

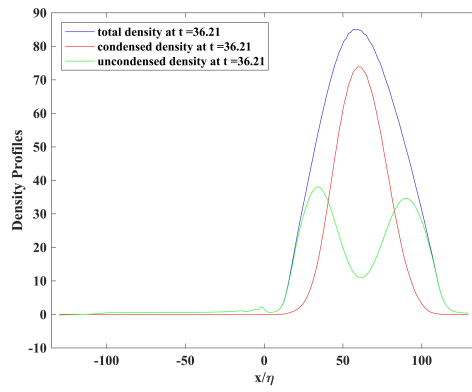
1. The condensate starts heating by a noticable amount after the time evolution just starts.
2. In Fig.4.10 Our results showed that the main contribution to the standing wave pattern is coming from the classical condensed density not from the uncondensed part of the density that resembles the quantum fluctuation.
3. Moreover, The uncondensed part of the density is maintaining a constant phase relation over different trajectories. The constant phase relation shows that there is an interaction between the condensed and the uncondensed parts.
4. We should mention here that the classical fluctuation of the BEC and the quantum excitations are obeying the same eq.2.21.



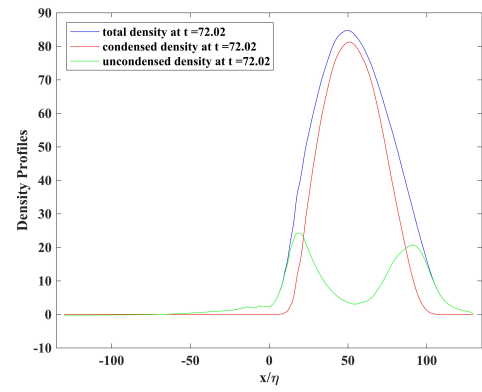
(a) .



(b) .

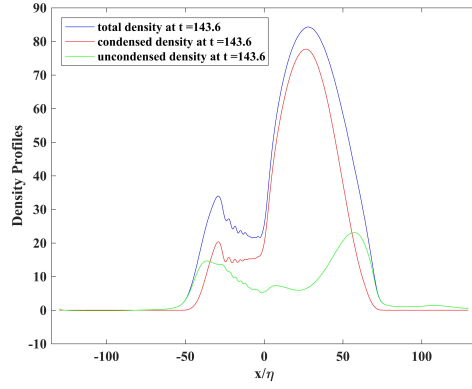


(c) .

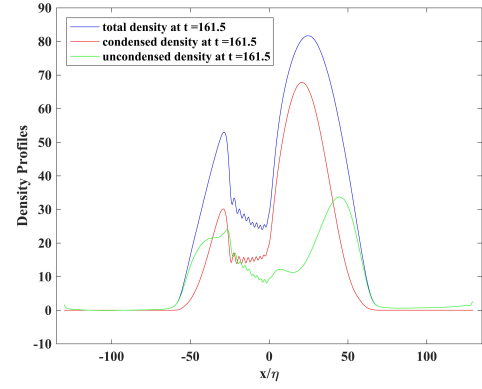


(d) .

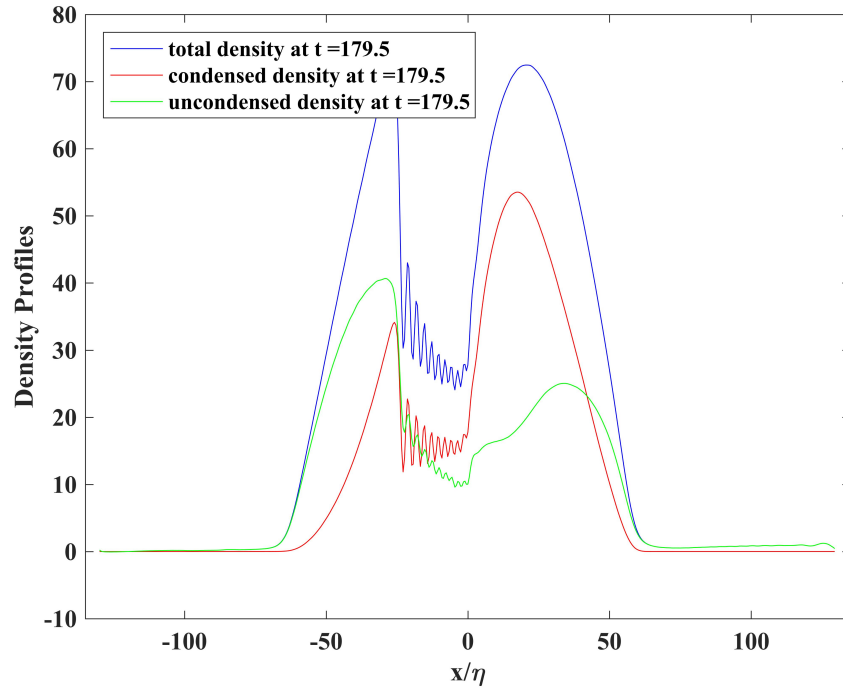
Figure 4.9: The density profiles of the total density in blue the condensed density in red the uncondensed part in green at the beginning of the time evolution which shows the depletion of the condensate mainly



(a)



(b)



(c)

Figure 4.10: Density profiles at late time showing the total density in blue the condensed part in red and the uncondensed part in green, in fig c it shows that the standing wave pattern is having a great contribution from the condensed part

4.3 Conclusion

In this thesis we have numerically studied systems to simulate the analogue Hawking radiation (AHR) in Bose-Einstein condensate. The first question was to propose a simple system in which we can simulate the Analogue Hawking radiation. We have showed the possibility of simulating (AHR) in simple system turning supersonic through looking at the long range density-density correlation function.

The second question was toward the recent black hole lasing experiment [44]. The question mainly was whether the lasing phenomenon observed in the experiment is a signature of Hawking pohnons. We have indicated that the lasing phenomena is not purely a quantum fluctuation or a contribution coming from the vaccum fluctuation of the condensate but it is more of a classical density perturbation. We also found that the density of the condensate is having a significance modulation due to the white hole position uncertainty. The current experimental realisation therefore may not say that the lasing effect is purely due to the zero point vaccum fluctuation corresponding to Hawking radiation.

A further detailed study to seperate the contribution coming from the quantum and the classical fluctuations need to be carried out. Furthermore, a critical attention in the experimental realization should be paid to the the fluctuation in the position of the White-hole. More studies are needed to provide better system parameters and regimes to enhance the experimental realization.

Appendix A

Numerical methods

A.1 Dimensionless formalism

In order to guarantee generic numerical calculations that does not depend on specific units, a dimensionless formalism was used during the numerical calculations. It is convenient to normalize the calculated quantities with parameters related to Bose-Einstein condensates. The following table indicate the natural units used for normalization.

Table A.1: Normalization units

Quantity	Natural unit
Length	$\eta = \frac{\hbar}{\sqrt{m\mu}}$ healing length
Energy	μ chemical potential
Time	$\frac{\hbar}{\mu}$

Using the natural units in Tab. A.1, we convert the quantities used in the numerical calculation of the one-dimensional Gross-Pitaevskii Eq. 2.1.3 to the following dimensionless units.

Table A.2: Dimensionless Quantities

Parameter	Dimensionless Form
x	$\tilde{x} = \frac{x}{\eta}$
t	$\tilde{t} = \frac{t\mu}{\hbar}$
$V(x, t)$	$\tilde{V}(\tilde{x}, \tilde{t}) = \frac{V(x, t)}{\mu}$
U_1	$\tilde{U}_1 = \frac{U_1}{\eta\mu}$
$\phi(x, t)$	$\tilde{\phi}(\tilde{x}, \tilde{t}) = \frac{\phi(x, t)}{\sqrt{\eta}}$
ρ	$\tilde{\rho} = \rho \times \eta$

Inserting the parameters of Tab.A.2 into Eq.2.1.3 we obtain the dimensionless one-dimensional GP equation to be

$$\frac{\partial \tilde{\phi}}{\partial \tilde{t}} = -i \frac{1}{2} \frac{\partial^2 \tilde{\phi}(\tilde{x}, \tilde{t})}{\partial \tilde{x}^2} - i \left(\tilde{U}_1(\tilde{x}, \tilde{t}) |\tilde{\phi}(\tilde{x}, \tilde{t})|^2 + \tilde{V}(\tilde{x}, \tilde{t}) \right) \tilde{\phi}(\tilde{x}, \tilde{t}) \quad (\text{A.1})$$

A.2 Programming

Numerical simulations have been done using XMDS2 high-level language [43, 7, 19]. XMDS2 is a tool for solving initial value ordinary and partial differential equations (PDE). Basically, the problem of interest is written in an XML script which describes the parameters and the variables involved. The XML then generates C++ codes which solve the problems using different algorithms stated earlier in the XML script. The result of the simulation is written in a data file that could be processed using different post-processing programs. Throughout this thesis Matlab has been used for post-processing and visualisation.

Obtaining the dynamics of the GP equation requires a start from a ground state which is obtained using the imaginary time evolution method discussed in Subsec.

2.1.2 and described by Eq. 2.1.2. The imaginary time evolution equation is solved using the Runge-Kutta (RK4) algorithm which is a fourth order in the step size.

The dynamics of the condensate is obtained using the Adaptive Runge-Kutta algorithm ARK89. The implementation of truncated-Wigner approach described theoretically in Sec. 2.3 beside the results of 4.2.2.1 and 3.3.1 was obtained by running multiple trajectories simulation. The multiple trajectories simulation is done by specifying the use of multiple processors from XMDS2 scripts. The parallelization job is to run the simulation multiple times then averaging over all the runs. It should be mentioned that in the context of truncated-Wigner simulation, averaging initial noise in one trajectory over the grid is the same as averaging noise value on one grid point over many trajectories.

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