CHARACTERISTIC BISETS AND LOCAL FUSION SUBSYSTEMS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Fusion systems are categories that contain the p-local structure of a finite group. Bisets are sets endowed with two coherent group actions. We investigate the relation between fusion systems and bisets in this thesis.

Fusion systems that mimic the inclusion of a Sylow p-subgroup of a finite group are called saturated. Similarly, if S is a Sylow p-subgroup of G, then G regarded as an (S, S)-biset has special properties, which make it a characteristic biset for the p-fusion of G. These two concepts are linked in that a fusion system is saturated if and only if it has a characteristic biset. We give a proof for this result by following the work in [1] and [2].

Fusion systems have a notion of normalizer and centralizer subsystems, mimicking the notion for finite group theory. This thesis reviews a proof by Gelvin and Reeh [3] of a result of Puig [2] asserting that normalizer and centralizer fusion subsystems of a saturated fusion system are saturated. This result comes from the connection between saturation of fusion systems and the existence of characteristic bisets.

Keywords: Fusion, bisets.

ÖZET

KARAKTERİSTİK İKİLİ KÜMELER VE LOKAL FÜZYON ALT SİSTEMLERİ

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Füzyon sistemleri sonlu bir grubun p-lokal yapılarını içeren kategorilerdir. İkili kümeler iki grubun uyumlu etkisine sahip kümelerdir. Biz bu tezde, füzyon sistemleri ve ikili kümeler arasındaki ilişkiyi araştıracağız.

Bir sonlu grup ile Sylow-p altgrubu arasındaki ilişkiyi taklit eden füzyon sistemlerine doygun füzyon sistemi denir. Eğer S sonlu bir grup olan G'nin Sylow p-altgrubu ise, (S, S)-ikili kümesi olarak düşünülebilen G grubu kendine has özelliklere sahiptir ki bu özellikler, onu G'nin p-füzyonunda karakteristik ikili bir küme yapar. Bu iki kavram şu şekilde ilişkilidir: Bir füzyon sistemi ancak ve ancak karakteristik bir ikili kümeye sahipse doygundur. Bu sonucu, [1] ve [2]'deki çalışmaları takip ederek ispatlıyoruz.

Füzyon sistemleri sonlu grup teorisindeki normalleyici ve sabitleyici kavramlarını taklit ederek kendine has normalleyici ve sabitleyici alt sistemlere sahiptir. Bu tez, Puig tarafından [2]'de kanıtlanan doygun füzyon sistemlerinin normalleyici ve sabitleyici füzyon alt sistemlerinin de doygun olması sonucunu, [3]'deki bir ispatı gözden geçirirek tekrar inceliyor. Bu sonuç karakteristik ikili kümelerin varlığından ve füzyon sistemlerini arasındaki ilişkiden gelmektedir.

Anahtar sözcükler: Füzyon, ikili küme.

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Chapter 0

Introduction

Let G be a finite group, p a prime number that divides the order of G, and S a Sylow p-subgroup of G.

A *fusion system* $\mathcal{F}$ on S is a category whose objects are the subgroups of S and whose morphisms are contained in the set of injective group homomorphisms between subgroups. Conjugating a subgroup P of S to a subgroup Q of S by an element of G is a *G-fusion*, and we say that P and Q are *fused* in G.

Fusion systems have shown up in finite group theory during the last thirty years, but the ideas of fusion theory and the term fusion itself can be traced back to Frobenius and Burnside as they talked about fusion of p-elements. Frobenius also investigated the relation between the fusion in a finite group and its normal p-complement subgroups. His normal p-complement theorem states that finding a subgroup of G that is normal p-complement to S is equivalent to all morphisms induced by G can be induced by S.

Alperin's Fusion Theorem was an important step toward think of fusion systems as objects in their own right. It says that any G-fusion between p-subgroups of a finite group can be generated by using the normalizers of certain subgroups.

A major contribution to fusion theory was made by Puig. He abstracted the notion of a G-fusion on S, and wrote the axioms of saturated fusion systems [4] with the motivation of discovering some relations between fusion systems and modular representation theory. Then, several mathematicians contributed to development of fusion theory. For instance, Broto, Levi, and Oliver used the relation between the theory of fusion systems and homotopy theory managed and proved some results related to p-completed classifying spaces of finite groups, the Martino-Priddy Conjecture being among the most famous of these results.

The classification of the finite simple groups is one of the biggest projects in finite group theory. This big project is a compilation of hundereds of journal paper written by many mathematician over the 50 years. Gorenstein, Lyons and Solomon revised and simplified these results and collected them in [5]. The project came to an end after the announcment of Aschbacher and Smith in their 1221 page long work that finishes the classification of quasithin groups [6][7]. Aschbacher also has a considerable contribitution to the local theory of fusion systems. As an important collection of results in the local theory of fusion systems can be found in [8]. After the developments in local theory of fusion systems, today fusion theory is considered as a promising tool in order to simplify the classification of finite simple groups.

0.1 Flow Of The Thesis

The first chapter contains definitions, examples and many useful tools related to fusion systems. It mostly draws upon the work in [1], but presents some alternative definitions to some key concepts, and presents proofs to some important theorems with a different approach.

In the second chapter, we start exploring the relations between bisets and fusion systems. We define the stabilizer fusion systems that come from bisets and try to understand when these stabilizer fusion systems are actually saturated by following the work given in [2].

In the third chapter, we introduce characteristic bisets. We will prove that for a given saturated fusion system, we can always find a related characteristic biset. On the other hand, a characteristic biset of a fusion system encodes the fusion data of this fusion system, therefore characteristic bisets maybe considered as an alternative definition for saturated fusion systems. Most of our work in this section is taken from [1].

For the fourth chapter, we review of some results related to local subsystems given by Puig in [2] by using the work of Gelvin and Reeh in [3]. After introducing normalizer and centralizer fusion systems, we give sufficient conditions that make those fusion systems on local subgroups saturated.

0.2 Notation

Throughout this thesis, p is always a prime number.

Let G be a finite group. By $|G|$ we refer the order of G. A subgroup H of G is denoted by $H \leq G$. If H is proper or normal subgroup of G, we write $H < G$ or $H \trianglelefteq G$, respectively. The symbols $N_G(H)$ and $C_G(H)$ will denote the G-normalizer and G-centralizer of H respectively.

We denote the set of the Sylow p-subgroups of a finite group by $\mathrm{Syl}_p(G)$. Given

$g, h \in G$, the left conjugate of h by g is ghg^{-1} and is denoted by ${}^g h$, and the right conjugate of h by g is $g^{-1}hg$ and is denoted h^g. For $P \leq G$, ${}^g P$ is the subgroup $\{gag^{-1} : a \in P\}$, and again swap g and g^{-1} to obtain P^g.

Let $P, Q \leq G$. The set of group injections from P to Q is denoted by $\mathrm{Inj}(P, Q)$. We write $\mathrm{Hom}_G(P, Q)$ for the set of group homomorphisms from P to Q induced by left conjugation in G. We denote the automorphism group of S in G by $\mathrm{Aut}_G(S)$. The symbols $\mathrm{Out}_G(S)$ and $\mathrm{Inn}(S)$ denote the outer automorphism group of S in G and the inner automorphism group of S, respectively. Recall that $\mathrm{Inn}(S) = \mathrm{Aut}_S(S)$ and $\mathrm{Out}_G(S) = \mathrm{Aut}_G(S)/\mathrm{Inn}(S)$.

Chapter 1

Fusion Systems

In this chapter, we introduce basic concepts of fusion systems. We first define fusion systems for a finite group G and its Sylow p-subgroup S. Then we remove G and define fusion systems on S without talking about any over group of S. We also review Alperin's Fusion Theorem and give a proof of it. At the end of this chapter, we look at the direct products of fusion systems, as we will use them to prove the core theorem in Chapter 3.

1.1 The Transporter of A Finite Group

Definition 1.1.1. Let G be a finite group and P, Q subgroups of G. The *transporter* of P to Q in G is the set $\mathcal{T}_G(P,Q) = \{g \in G : {}^gP \leq Q\}$. If $n \in \mathcal{T}_G(P,Q)$, the n-conjugation map is defined by $c_n : P \to Q$ such that $c_n(a) = {}^na$ for all $a \in P$. Thus, each element of $\mathcal{T}_G(P,Q)$ gives rise to a conjugation map from P to Q.

A quick comment on transporters is that if n and m are in $\mathcal{T}_G(P,Q)$, and

$m^{-1}n \in C_G(P)$, then c_n and c_m turn out to be the same map. If we consider quotienting $\mathcal{T}_G(P,Q)$ by $C_G(P)$, we get the maps from P to Q induced by conjugations in G. Therefore, $\mathrm{Hom}_G(P,Q) \cong \mathcal{T}_G(P,Q)/C_G(P)$, and that result, in fact, gives a sense of fusion systems that we are about to define.

1.2 The p-Local Structure Of A Finite Group

Let G be a finite group whose order can be divided by p. By the p-local structure of G, we consider a Sylow p-subgroup of G and G-conjugacy morphisms between the subgroups of S. Since all the Sylow p-subgroups of G are isomorphic, they hold the same p-local structure of G, so which Sylow-p subgroup of G we choose does not matter.

Definition 1.2.1. Let G be a finite group and S a Sylow p-subgroup of G. We call $R \leq S$ a *p-local* subgroup of S whenever there exists a non-trivial subgroup P of G such that $R = N_G(P)$.

One can consider the next definition as an encoding of p-local structure of a finite group into a category.

1.3 The Fusion System Of A Finite Group

Definition 1.3.1. Let S be a Sylow p-subgroup of a finite group G. The *fusion system* on S induced by G is the category $\mathcal{F}_S(G)$ whose objects are all subgroups of S, and whose morphisms $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q)$ are

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \{\varphi \in \mathrm{Hom}_G(P,Q) \,:\, \varphi = c_g|_P \text{ for some } g \in G\}.$$

Remark 1.3.2. We could also define fusion systems on a non-Sylow p-subgroup of the ambient group, and the definition above would still be valid on any subgroup of G. In fact, the fusion systems we handle in Chapter 2 will usually be on subgroups that are not Sylow in the groups that induce their morphisms.

To visualize fusions systems, we provide an easy example of three different fusion systems on a finite p-group.

Example 1.3.3. Recall that D_8 is the group of symmetries of the square. It is generated by r, the 90 degrees clockwise rotation of the square, and f, a flipping of the square by a fixed axis perpendicular to its one of the edges, so that $D_8 = \langle r, f \,:\, r^4 = f^2 = 1, fr = r^{-1}f \rangle$.

Initially, we illustrate the fusion system on D_8 induced by D_8. After that, we compare $\mathcal{F}_{D_8}(D_8)$ with the fusion systems $\mathcal{F}_{D_8}(\Sigma_4)$ and $\mathcal{F}_{D_8}(A_6)$, where Σ_4 is the symmetric group of order 24 and A_6 is the alternating group of order 360. Note that D_8 is Sylow-2 subgroup of all these groups.

Consider the following lattice diagram of D_8 with three different types of arrows. In the diagram, all the arrows except dotted and dashed ones are morphisms of $\mathcal{F}_{D_8}(D_8)$. Specifically, the one directional arrows are injective non-isomorphisms, and the double directional ones are isomorphisms. We omit using arrows for the identity morphisms and the automorphisms of subgroups for the clarity of the diagram.

$$
\begin{tikzcd}
 & & D_8 & & \\
 & \langle r^2, f\rangle \arrow[ur] & \langle r \rangle \arrow[u] & \langle r^2, rf\rangle \arrow[ul] & \\
\langle f \rangle \arrow[ur] \arrow[r, leftrightarrow, "c_r"] & \langle r^2 f\rangle \arrow[u] \arrow[r, leftrightarrow, dotted, red] & \langle r^2 \rangle \arrow[ul] \arrow[u] \arrow[ur] \arrow[r, leftrightarrow, dashed, blue] & \langle rf \rangle \arrow[u] \arrow[r, leftrightarrow, "c_{r^3}"] & \langle r^3 f\rangle \arrow[ul] \\
 & & 1 \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

It is important to keep in mind that every morphism in the diagram can be factored as an isomorphism followed by an inclusion. One can immediately observe that there are no isomorphisms between the center $\langle r^2\rangle$ and any other subgroups of size 2 of D_8.

On the other hand, if we consider the fusion system on D_8 whose morphisms induced by Σ_4, we have $\langle r^2 f\rangle$ and $\langle r^2\rangle$ are fused, represented by the dashed arrow. Setting $r = (1234)$ and $f = (13)$ generates an isomorphic copy of D_8 in Σ_4, and the map $c_{(143)}$ fuses $\langle rf\rangle$ to $\langle r^2\rangle$. Yet we do not have that $\langle rf\rangle$ and $\langle r^2\rangle$ are fused .

Now we examine $\mathcal{F}_{D_8}(A_6)$ by setting $r = (1234)(56)$ and $f = (13)(56)$. Define $\beta_1 := \langle r^2 f\rangle \to \langle r^2\rangle$ to be $\beta_1 = c_{(125)(346)}$. Then define $\beta_2 := \langle r^2\rangle \to \langle rf\rangle$ to be $\beta_2 = c_{(1324)(56)}$, represented by the dotted arrow. In this case, all the subgroups of size 2 of D_8 are fused.

For further discussion, let $\psi := \beta_2 \circ \beta_1 : \langle r^2 f\rangle \to \langle rf\rangle$ be an isomorphism. Note that $c_{(125)(346)} \in \mathrm{Aut}(\langle r^2, f\rangle)$ and $c_{(1324)(56)} \in \mathrm{Aut}(\langle r^2, rf\rangle)$. We call these two automorphisms α_1 and α_2, respectively. Since $\alpha_1|_{\langle r^2 f\rangle} = \beta_1$ and $\alpha_2|_{\langle r^2\rangle} = \beta_2$, the composition $\alpha_2|_{\langle r^2\rangle} \circ \alpha_1|_{\langle r^2 f\rangle}$ gives ψ, which is generated as restrictions of automorphisms of subgroups of D_8 whose size is larger than 2. This is an example of the "local-to-global" principle, and we will explain how this principle works for fusion systems with Alperin's Fusion Theorem.

Notation 1.3.4. Let $\mathcal{F}$ be a fusion system of a finite p-group S. We write $\mathcal{F}(P,Q)$ for $P,Q \leq S$ for the morphisms from P to Q in $\mathcal{F}$ and $\mathcal{F}(P,Q)_{\text{iso}}$ will mean the set of isomorphisms from P to Q in $\mathcal{F}$.

Definition 1.3.5. Let $\mathcal{F}$ be a fusion system on a finite p-group S. If there exits an isomorphism between P and Q in $\mathcal{F}$, then P is said to be *$\mathcal{F}$-conjugate* or *$\mathcal{F}$-isomorphic* to Q, and this isomorphism is called an *$\mathcal{F}$-isomorphism.*

For a fusion system $\mathcal{F}_S(G)$, if we consider the $\mathcal{F}_S(G)$-isomorphism class of $P \leq S$, not all subgroups in this class enjoy the same properties relative to S and G. As an example, we could find subgroups of S which are $\mathcal{F}$-conjugate P, and their S-normalizers are not Sylow in their G-normalizers, while for others this does hold. This situation motivates us to use following terminology.

Definition 1.3.6. For a fusion system $\mathcal{F}$ on a finite p-group S and $P \leq S$ a subgroup,

i. P is *fully $\mathcal{F}$-normalized* if $|N_S(P)| \geq |N_S(Q)|$ whenever Q is $\mathcal{F}$-conjugate to P.
ii. P is *fully $\mathcal{F}$-centralized* if $|C_S(P)| \geq |C_S(Q)|$ whenever Q is $\mathcal{F}$-conjugate to P.

Here is another characterization of fully normalized and centralized subgroups.

Lemma 1.3.7. *Let G be a finite group, S a Sylow-p subgroup of G, P a subgroup of S, and $\mathcal{F} = \mathcal{F}_S(G)$. Then P is fully $\mathcal{F}$-normalized if and only if $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$. Furthermore, P is fully $\mathcal{F}$-centralized if and only if $C_S(P)$ is a Sylow p-subgroup of $C_G(P)$.*

Proof. Suppose that P is fully $\mathcal{F}$-normalized. Let T be a Sylow p-subgroup of $N_G(P)$ that contains $N_S(P)$. Since S is a Sylow p-subgroup of G, there exist $g \in G$

such that ${}^gT \leq S$, and we have also ${}^gT \leq {}^gN_G(P) = N_G({}^gP)$. It follows that ${}^gT \leq S \cap {}^gN_G(P)$. Note that $c_g|_P$ defines an $\mathcal{F}$-isomorphism from P to ${}^gP$, so P and ${}^gP$ are $\mathcal{F}$-isomorphic. We can now form the following inequality:

$$|T| = |{}^gT| \leq |S \cap {}^gN_G(P)| = |S \cap N_G({}^gP))| = |N_S({}^gP)| \leq |N_S(P)| \leq |T|$$

Thus, $N_S(P)$ and T are of the same order, hence $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$.

Conversely, suppose $N_S(P)$ is a Sylow p-subgroups of $N_G(P)$. $N_G(P)$ and $N_G({}^gP)$ are of the same order, so the order of their Sylow p-subgroups is $|N_S(P)|$. Since $N_S({}^gP)$ is contained in a Sylow p-subgroup of $N_G(P)$, we must have $|N_S(P)| \geq |N_S({}^gP)|$ where P and ${}^gP$ are $\mathcal{F}$-conjugate. Hence, $N_S(P)$ is fully $\mathcal{F}$-normalized.

The proof of the latter condition follows similar arguments. $\square$

To sum up briefly, morphisms in a fusion system are induced by finite groups, thus all morphisms are injective. Recall that in Example 1.3.3, each morphism was an isomorphism followed by an inclusion. In the next section, we will define fusion systems without a reference to an ambient group by considering our observations in this section.

1.4 Saturated Fusion Systems

When talking about fusion in a finite p-group S, we always gave a reference to an ambient group inducing the fusion between the subgroups of S. In [4, 2.3], Puig defined the divisible P-categories that encode the data of a fusion system without the information of a finite group which induce the morphisms of the fusion system. Elsewhere in the literature, divisible P-categories are known as (abstract) fusion systems, and we follow this presentation of the topic.

Definition 1.4.1. ([1, Definition 1.1]) An *abstract fusion system* on a finite p-group S is a category $\mathcal{F}$ that has all the subgroups of S as its objects and the morphisms of $\mathcal{F}$ are group homomorphisms that satisfy

i. $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$
ii. If $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$, then $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, \varphi(P))$ and $\varphi^{-1} \in \mathrm{Mor}_{\mathcal{F}}(\varphi(P), P)$.

Composition in $\mathcal{F}$ is composition of group homomorphisms. The second axiom implies that every morphism in an abstract fusion system is an isomorphism followed by an inclusion.

Remark 1.4.2. From now on, any fusion system will mean an abstract fusion system on a finite p-group.

Example 1.4.3. $\mathcal{F}_S(G)$ is clearly an abstract fusion system.

Definition 1.4.4. If $\varphi \in \mathcal{F}(P, Q)_{\text{iso}}$, the *extender subgroup* of φ is

$$N_\varphi := \{n \in N_S(P) \,:\, \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(Q)\}.$$

Remark 1.4.5. Let $\varphi \in \mathcal{F}(P, Q)_{\text{iso}}$, $R \leq N_S(P)$, and suppose that φ extends to $\tilde{\varphi} \in \mathcal{F}(R, S)$. Given $a \in N_S(P)$, $Q = \tilde{\varphi}(aPa^{-1}) = \tilde{\varphi}(a)\tilde{\varphi}(P)\tilde{\varphi}(a^{-1}) = \tilde{\varphi}(a)\varphi(P)\tilde{\varphi}(a^{-1})$, meaning $c_{\tilde{\varphi}(a)}|_Q \in \mathrm{Aut}_S(Q)$, hence $\varphi \circ c_a \circ \varphi^{-1} \in \mathrm{Aut}_S(Q)$, so $R \leq N_\varphi$. Therefore, N_φ is the largest subgroup of $N_S(P)$ to which φ could possibly extend.

Before defining saturated fusion systems, it would be good to remark that in [2], Puig calls them Frobenius p-categories. On the other hand, there are several equivalent definitions of saturated fusion systems, but we stick with the definition provided by Broto, Levi, and Oliver as it is the most useful for the remaining part of our work. However, we will provide an alternative definition for saturated fusion systems with a minor change at the end of this chapter.

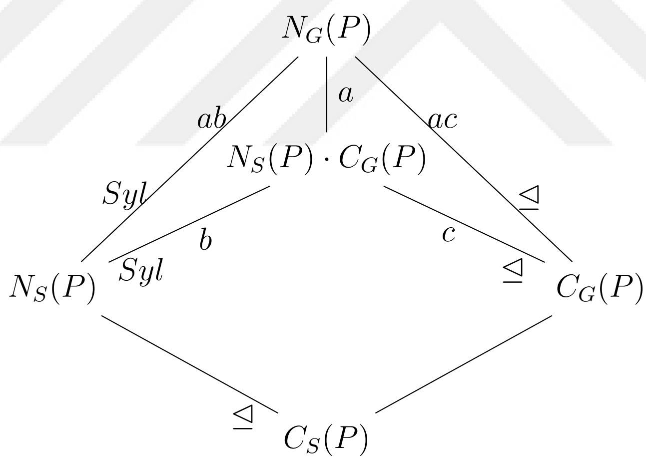
Definition 1.4.6. ([1, Definition 1.2]) Let $\mathcal{F}$ be a fusion system on a finite p-group S. We say that $\mathcal{F}$ is a *saturated fusion system* whenever the following axioms are satisfied:

(i) (Sylow axiom) If $P \leq S$ is fully $\mathcal{F}$-normalized, then $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ and P is fully $\mathcal{F}$-centralized.

(ii) (Extension axiom) If φ is an isomorphism of $\mathcal{F}(P,Q)$ and Q is fully $\mathcal{F}$-centralized, φ extends to a morphism $\tilde{\varphi} \in \mathcal{F}(N_\varphi, S)$ (i.e., $\tilde{\varphi}|_P = \varphi$).

Proposition 1.4.7. [1, Proposition 1.3] *If S is a Sylow p-subgroup of a finite group G, then $\mathcal{F}_S(G)$ is saturated.*

Proof. Set $\mathcal{F} := \mathcal{F}_S(G)$. Suppose that $P \leq S$ is fully $\mathcal{F}$-normalized. By Lemma 1.3.7, $N_S(P) \in \mathrm{Syl}_p(N_G(P))$. Define $a := [N_G(P) : N_S(P) \cdot C_G(P)]$, $b := [N_S(P) \cdot C_G(P) : N_S(P)]$, and $c := [N_S(P) \cdot C_G(P) : C_G(P)]$, which all can be seen in the following diagram:



Since the index of $N_S(P)$ in $N_G(P)$ is prime to p, the index of $N_S(P)$ in $N_S(P) \cdot N_G(P)$ is also prime to p, so $N_S(P) \in \mathrm{Syl}_p(N_S(P) \cdot C_G(P))$. By the Isomorphism Theorems for finite groups, $N_S(P) \cdot C_G(P)/C_G(P) \cong N_S(P)/C_S(P)$, and the index of $C_S(P)$ in $N_S(P)$ is also c. By counting the indices over the left and the right lines in the lattice diagram, we calculate that $[C_G(P) : C_S(P)] = b$, meaning $C_S(P)$

is a Sylow p-subgroup of $C_G(P)$ since b is prime to p. By Lemma 1.3.7, P is fully $\mathcal{F}$-centralized.

Observe that $|\mathrm{Aut}_G(P)| = |N_G(P)/C_G(P)| = |N_G(P)|/|C_G(P)| = ac$ and $|\mathrm{Aut}_S(P)| = |N_S(P)/C_S(P)| = |N_S(P)|/|C_S(P)| = c$. It follows that $[\mathrm{Aut}_G(P) : \mathrm{Aut}_S(P)] = a$, so $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_G(P))$ as a is prime to p, and Lemma 1.3.7 implies that P is fully $\mathcal{F}$-normalized.

For the second saturation axiom, fix an isomorphism $\varphi \in \mathcal{F}(P, Q)$ for $P, Q \leq S$ with Q is fully $\mathcal{F}$-centralized. Fix $g \in G$ such that $\varphi = c_g|_P$. Let $n \in N_\varphi$, so $c_{gng^{-1}} \in \mathrm{Aut}_S(Q)$. Then there exists $m \in N_S(Q)$ such that $c_{gng^{-1}}(q) = c_m(q)$ for all $q \in Q$. Therefore, $m^{-1}gng^{-1}$ must centralize Q. Thus, there is a $z \in C_G(Q)$ such that $m^{-1}gng^{-1} = z$, and $gng^{-1} = mz$, which implies ${}^gN_\varphi \leq N_S(Q) \cdot C_G(Q)$.

By Lemma 1.3.7, $C_S(Q) \in \mathrm{Syl}_p(C_G(P))$, thus one can draw the lattice diagram of $N_G(Q)$ that is similar to the lattice diagram of $N_G(P)$ after replacing P with Q. Then it is immediate that $N_S(Q) \in \mathrm{Syl}_p(N_S(Q) \cdot C_G(Q))$. Then we can conjugate ${}^gN_\varphi$ into $N_S(P)$ by an element in $C_G(Q)$. Fix an $h \in C_G(Q)$ such that ${}^{hg}N_\varphi \leq N_S(P)$ and define $\tilde{\varphi} : N_\varphi \to S$ by $n \mapsto {}^{hg}n$. Since $\tilde{\varphi}|_P = \varphi$, we see that φ extends to $\tilde{\varphi}$. $\square$

After Proposition 1.4.7, we need to distinguish the saturated fusion systems whose morphisms can be induced by a finite group from those whose morphisms cannot be induced by a finite group, so we give the following definition.

Definition 1.4.8. Let $\mathcal{F}$ be a fusion system on a finite p-group S. If there exists a finite group G such that $\mathcal{F} = \mathcal{F}_S(G)$, then $\mathcal{F}$ is *weakly realizable*. Moreover, if $S \in \mathrm{Syl}_p(G)$, we call $\mathcal{F}$ *realizable*. In case $\mathcal{F}$ is not realizable, then $\mathcal{F}$ is said to be *exotic*.

Today we know that every saturated fusion system is weakly realizable due to Park [10] and in Chapter 3, we will talk about this result.

For $p = 2$, in [11], Solomon's work shows that there are three saturated fusion systems on the Sylow 2-subgroups of the group $\mathrm{Spin}_7(3)$, and one of the is exotic. We call this exotic fusion system Solomon fusion system, and it is the smallest example of exotic fusion systems on prime 2.

For $p \neq 2$, Ruiz and Viruel have classified the fusion systems of all extra-special groups of order p^3 in [12]. We know that some fusion systems of the extra-special group of order 7^3, i.e., 7_+^{1+2}, are exotic.

The following notation is for the next proposition.

Notation 1.4.9. Let G be a finite group, $P \leq S$, and φ be an isomorphism from P to a subgroup of G. We define

$$^{\varphi}\mathrm{Aut}_G(P) := \{\varphi \circ \alpha \circ \varphi^{-1} \,:\, \text{for all } \alpha \in \mathrm{Aut}_G(P)\}.$$

The next proposition presents a case where the extender subgroup has maximal order. We also use the next proposition several times during our work.

Proposition 1.4.10. *Let $\varphi \in \mathcal{F}(P,Q)$ be an isomorphism. If Q is fully $\mathcal{F}$-normalized, then there exists a $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\chi\varphi$ extends to some $\vartheta \in \mathcal{F}(N_S(P), N_S(Q))$ (i.e., $\vartheta|_P = \chi\varphi$).*

Proof. By the Sylow axiom, $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. Consider $^{\varphi}\mathrm{Aut}_S(P)$, which is clearly a subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, so $^{\varphi}\mathrm{Aut}_S(P) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$. It follows from the Sylow theorem that there exist some $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $^{\chi\varphi}\mathrm{Aut}_S(P) \leq \mathrm{Aut}_S(Q)$. Then given $n \in N_S(P)$, we have $n \in N_{\chi\varphi}$ since $c_n \in \mathrm{Aut}_S(P)$ and $^{\chi\varphi}\mathrm{Aut}_S(P) \leq \mathrm{Aut}_S(Q)$. Therefore, $N_{\chi\varphi} = N_S(P)$. By the extension axiom, φ extends to a morphism $\vartheta \in \mathcal{F}(N_S(P), S)$. It remains to show that $\vartheta(N_S(P)) \leq N_S(Q)$. Let $n \in N_S(P)$, then $\vartheta(nPn^{-1}) = Q$. Therefore, $\vartheta(n) \cdot Q \cdot \vartheta(n)^{-1} = Q$. This means that $\vartheta(n)$ normalizes Q. $\square$

1.5 Alperin's Fusion Theorem

In [9], Alperin shows that fusion of p-elements (elements of order a power of p) is determined by p-local subgroups. In other words, he introduces one of the key aspects of the "Local-to-Global principle" for finite groups, which enable us to write the same fusion between subgroups by using restrictions of automorphisms of larger subgroups.

In [13], Goldschimdt moved Alperin's work one step further by identifying a subset of local subgroups that generate the fusion. We will call these special local subgroups $\mathcal{F}$-essential. This result today is known as the Alperin-Goldschimdt Theorem.

Puig has adapted Alperin's Theorem to (abstract) fusion systems in [4]. In part of our work, we present Alperin's Fusion Theorem for fusion systems, but first we need to define $\mathcal{F}$-essential subgroups.

Definition 1.5.1. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. For $P \lneq S$ and $\mathcal{C} := \{R \leq S \mid |P| \lneq |R|\}$ we define the subgroup $\mathrm{Aut}^{+}_{\mathcal{F}}(P)$ that consists of the elements of $\mathrm{Aut}_{\mathcal{F}}(P)$ that can be written as the composite of restrictions of automorphisms of the elements of $\mathcal{C}$.

There are some alternative definitions to the next one, the reason we prefer Puig's version is that it eases the work to prove Alperin's Fusion Theorem.

Definition 1.5.2. ([2, 5.7]) Let $\mathcal{F}$ be a fusion system on a finite p-group S and P be a subgroup of S. P is *$\mathcal{F}$-essential* if $\mathrm{Aut}^{+}_{\mathcal{F}}(P) \neq \mathrm{Aut}_{\mathcal{F}}(P)$.

In Definition 1.5.1, if we let $P \leq S$, S would be an $\mathcal{F}$-essential. However, most sources on fusion systems avoid doing that, so we do not consider S as an F-essential subgroup.

Theorem 1.5.3 (Alperin's Fusion Theorem)**.** [4, Corollary 5.10] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $\mathcal{F}^{nes}$ be the set of fully $\mathcal{F}$-normalized, $\mathcal{F}$-essential subgroups of S together with S itself. Then every isomorphism of $\mathcal{F}$ can be written as a composite of restrictions of automorphisms of subgroups in $\mathcal{F}^{nes}$.*

Proof. If $\theta \in \mathcal{F}(S, S)$, the theorem trivially holds, so we work on proper subgroups of S. Suppose the theorem does not hold for all $\mathcal{F}$-isomorphisms and let $\mathcal{A}$ be the set of isomorphisms of $\mathcal{F}$ that can be written as a composite of restrictions of automorphisms of subgroups in $\mathcal{F}^{nes}$. We proceed by a downward induction on the size of the source of the isomorphisms not in $\mathcal{A}$. Thus, pick an $\mathcal{F}$-isomorphism $\varphi : P \to Q$ such that $\varphi \notin \mathcal{A}$ and $|P|$ is maximal among the orders of the sources of the other isomorphisms not in $\mathcal{A}$.

As an initial step, suppose that Q is fully $\mathcal{F}$-normalized. By Proposition 1.4.10, there is some $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\chi\varphi$ extends to $\vartheta \in \mathcal{F}(N_S(P), N_S(Q))$. Since $P \lneq S$ and S is a p-group, we have $P \lneq N_S(P)$. The maximality of $|P|$ implies that $\vartheta \in \mathcal{A}$. If $\chi \in \mathrm{Aut}^+_{\mathcal{F}}(Q)$, then $\chi \in \mathcal{A}$, meaning $\varphi \in \mathcal{A}$ as $\varphi = \chi\vartheta|_P$. If $\chi \notin \mathrm{Aut}^+_{\mathcal{F}}(Q)$, we must have that $\mathrm{Aut}_{\mathcal{F}}(Q) \neq \mathrm{Aut}^+_{\mathcal{F}}(Q)$. In this case, Q would be $\mathcal{F}$-essential and $\chi \in \mathcal{A}$.

If Q is not fully $\mathcal{F}$-normalized, to have the diagram below we can pick an R in the $\mathcal{F}$-isomorphism class of P such that R is fully $\mathcal{F}$-normalized and $\psi : Q \to R$ is an $\mathcal{F}$-isomorphism. Now we have

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\varphi"'] & Q \arrow[r, "\psi"] & R \end{tikzcd}$$

Note that $\psi\varphi$ in the figure above is an $\mathcal{F}$-isomorphism. So by repeating the preceding argument, we get $\psi, \psi\varphi \in \mathcal{A}$. Writing $\psi^{-1} \circ \psi\varphi = \varphi$ implies that $\varphi \in \mathcal{A}$. That contradicts our initial assumption. $\square$

By following similar steps in the proof of the previous theorem, we show that if an $\mathcal{F}$-conjugacy class contains an $\mathcal{F}$-essential subgroup, then all subgroups in this class are $\mathcal{F}$-essential.

Proposition 1.5.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and P an $\mathcal{F}$-essential subgroup of S. Then every subgroup that is $\mathcal{F}$-isomorphic to P is also $\mathcal{F}$-essential.*

Proof. Let $Q \leq S$ be $\mathcal{F}$-isomorphic to P. Suppose that Q is not $\mathcal{F}$-essential. Since P is $\mathcal{F}$-essential, we have $\mathrm{Aut}_{\mathcal{F}}(P) \neq \mathrm{Aut}^{+}_{\mathcal{F}}(P)$. Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \setminus \mathrm{Aut}^{+}_{\mathcal{F}}(P)$, and we aim to get a contradiction.

First suppose that Q is fully $\mathcal{F}$-normalized. By Proposition 1.4.10 there is an $\mathcal{F}$-isomorphism $\varphi \in \mathcal{F}(P, Q)$ that extends to $\tilde{\varphi} \in \mathcal{F}(N_S(P), N_S(Q))$. Let $\beta = \varphi \circ \alpha \circ \varphi^{-1}$, clearly $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Since Q is not $\mathcal{F}$-essential, $\beta \in \mathrm{Aut}^{+}_{\mathcal{F}}(Q)$, so β can be generated by the restrictions of automorphisms of subgroups larger than Q. Note that $\tilde{\varphi}$ is an $\mathcal{F}$-isomorphism from $N_S(Q)$ to $\tilde{\varphi}(N_S(Q))$. By Alperin's Fusion Theorem, we can write $\tilde{\varphi}$ and $\tilde{\varphi}^{-1}$ as composite of restrictions of automorphisms of $\mathcal{F}$-essential, fully $\mathcal{F}$-normalized subgroups of S. The size of each of these subgroups is larger than $|Q|$ because $Q < N_S(Q)$. Since $\alpha = \tilde{\varphi}|_P \circ \beta \circ \tilde{\varphi}^{-1}|_Q$, we get $\alpha \in \mathrm{Aut}^{+}_{\mathcal{F}}(Q)$, and this leads to a contradiction.

To handle the case that Q is not fully $\mathcal{F}$-normalized, let $\vartheta \in \mathcal{F}(Q, R)_{\mathrm{iso}}$ and $\theta \in \mathcal{F}(P, R)_{\mathrm{iso}}$ where R is fully $\mathcal{F}$-normalized subgroup of S, then examine the figure below.

$$\begin{tikzcd} P \arrow[r, "\vartheta^{-1} \circ \theta"] \arrow[rr, bend right, "\theta"'] & Q \arrow[r, "\vartheta"] & R \end{tikzcd}$$

By Proposition 1.4.10, we can extend θ and ϑ to $\tilde{\theta}$ and $\tilde{\vartheta}$ to S-normalizers P and Q respectively. We also have that $\vartheta^{-1} \circ \theta$ is an $\mathcal{F}$-isomorphism from P to Q. Since R is fully $\mathcal{F}$-normalized, it follows from the argument above that θ and ϑ can be generated by the automorphisms of larger subgroups, so can $\vartheta^{-1} \circ \theta$, and this completes the proof. □

1.6 Direct Products Of Fusion Systems

In Chapter 3, we will prove a result that is shared by all saturated fusion systems, and at some point when proving it, we will need the work that is done in this section.

In this section, we will see that direct products of fusion systems generate a fusion system and examine the case where those fusion systems are saturated.

Definition 1.6.1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems on given finite p-groups S_1 and S_2. The category $\mathcal{F}_1 \times \mathcal{F}_2$ has subgroups of $S_1 \times S_2$ as objects and the set of morphisms between $P, Q \leq S_1 \times S_2$ is

$$\mathcal{F}_1 \times \mathcal{F}_2(P, Q) = \{(\varphi_1, \varphi_2)|_P \,:\, P \leq P_1 \times P_2, Q \leq Q_1 \times Q_2, \varphi_i \in \mathcal{F}_i(P_i, Q_i)\}$$

where P_i and Q_i are the projections onto the ith coordinate of $S_1 \times S_2$ for $i = 1, 2$.

$\mathcal{F}_1 \times \mathcal{F}_2$ is a fusion system on $S_1 \times S_2$. We will show that $\mathcal{F}_1 \times \mathcal{F}_2$ is saturated if $\mathcal{F}_1$ and $\mathcal{F}_2$ are individually saturated, but first we need to prove a lemma that provides an alternative definition for saturated fusion systems.

Lemma 1.6.2. ([1, Lemma 1.4]) *Let $\mathcal{F}$ a fusion system on a p-group S. The Sylow Axiom of Definition 1.4.6 can be changed to the following:*

- *Any subgroup P of S is $\mathcal{F}$-conjugate to a subgroup Q of S which is fully $\mathcal{F}$-centralized and* $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

Proof. The Sylow Axiom in Definition 1.4.6 directly implies our new condition. We claim that our condition also implies the Sylow Axiom, so suppose it holds. Let P be a fully $\mathcal{F}$-normalized subgroup of S. We want to show that P is fully $\mathcal{F}$-centralized.

There is a subgroup $Q \leq S$ which is $\mathcal{F}$-conjugate to P such that $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ and Q is fully $\mathcal{F}$-centralized by the condition. Since $P \cong_{\mathcal{F}} Q$, we also have $\mathrm{Aut}_{\mathcal{F}}(P) \cong_{\mathcal{F}} \mathrm{Aut}_{\mathcal{F}}(Q)$. By the definition of fully $\mathcal{F}$-normalized subgroups, $|N_S(P)| \geq |N_S(Q)|$. Thus, we can write the last inequality as

$$|\mathrm{Aut}_S(P)||C_S(P)| \geq |\mathrm{Aut}_S(Q)||C_S(Q)|. \tag{1.1}$$

Since $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$, we must have $|\mathrm{Aut}_S(P)| \leq |\mathrm{Aut}_S(Q)|$. This forces $|C_S(P) \geq |C_S(Q)|$ by (1.1), but since Q is fully $\mathcal{F}$-centralized, we must have that $|C_S(P)| = |C_S(Q|$ and $|\mathrm{Aut}_S(P)| = |\mathrm{Aut}_S(Q)|$, meaning P is fully $\mathcal{F}$-centralized, and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. □

Proposition 1.6.3. ([1, Lemma 1.5]) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems on given p-groups S_1 and S_2. Then $\mathcal{F}_1 \times \mathcal{F}_2$ is a saturated fusion system on $S_1 \times S_2$.*

Proof. Set $S := S_1 \times S_2$ and $\mathcal{F} := \mathcal{F}_1 \times \mathcal{F}_2$. Additionally, for any $P \leq S$ and $\varphi \in \mathcal{F}(P, Q)$, let P_i and Q_i be projections of P and Q onto the ith coordinates respectively. We first prove that $\mathcal{F}$ satisfies the Sylow axiom by using Lemma 1.6.2.

If the P_i are not fully $\mathcal{F}_i$-normalized, we pick another subgroup from the

$\mathcal{F}$-conjugacy class, so that its projections on the first and second coordinates are fully $\mathcal{F}_i$-normalized. We now move our attention to the subgroup $P_1 \times P_2$. It clearly includes P, and since $C_S(P) = C_S(P_1 \times P_2) = C_S(P_1) \times C_S(P_2)$, we have P and $P_1 \times P_2$ are fully $\mathcal{F}$-centralized. Definition 1.6.1 implies that $\mathrm{Aut}_{\mathcal{F}}(P_1 \times P_2) = \mathrm{Aut}_{\mathcal{F}_1}(P_1) \times \mathrm{Aut}_{\mathcal{F}_2}(P_2)$, repeating the same arguments for S-automorphisms yields $\mathrm{Aut}_S(P_1 \times P_2) = \mathrm{Aut}_{S_1}(P_1) \times \mathrm{Aut}_{S_2}(P_2)$. Since $\mathrm{Aut}_{S_i}(P_i)$ are Sylow in $\mathrm{Aut}_{S_i}(P_i)$ (by the Sylow axiom) for $i = 1, 2$; then $\mathrm{Aut}_S(P_1 \times P_2)$ is Sylow in $\mathrm{Aut}_{\mathcal{F}}(P_1 \times P_2)$.

By Definition 1.6.1, we have $\mathrm{Aut}_{\mathcal{F}}(P) \leq \mathrm{Aut}_{\mathcal{F}}(P_1 \times P_2)$. For a given $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P_1 \times P_2)$, consider the set $\alpha \circ \mathrm{Aut}_{\mathcal{F}}(P) \circ \alpha^{-1}$, and let $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$. We see that $\alpha \circ \beta \circ \alpha^{-1}|_{\alpha(P)} = \alpha \circ \beta|_P = \alpha|_P \in \mathrm{Aut}_{\mathcal{F}}(\alpha(P))$, so $\alpha \circ \mathrm{Aut}_{\mathcal{F}}(P) \circ \alpha^{-1} = \mathrm{Aut}_{\mathcal{F}}(\alpha(P))$. Then by the Sylow Theorem, there is an $\theta \in \mathrm{Aut}_{\mathcal{F}}(P_1 \times P_2)$ such that $\mathrm{Aut}_S(P_1 \times P_2)$ contains the Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\alpha(P))$. This shows that $\mathrm{Aut}_S(\alpha(P))$ is Sylow in $\mathrm{Aut}_{\mathcal{F}}(\alpha(P))$ since $\mathrm{Aut}_S(\alpha(P)) = \mathrm{Aut}_S(P_1 \times P_2) \cap \mathrm{Aut}_{\mathcal{F}}(P_1 \times P_2)$. Then $\mathrm{Aut}_{\mathcal{F}}(\alpha(P))$ is fully $\mathcal{F}$-centralized, so it follow by Lemma 1.6.2 that Sylow axiom is satisfied.

For the extension axiom, take $\varphi \in \mathcal{F}(P, S)$ and suppose that $\varphi(P)$ is fully $\mathcal{F}$-centralized. Since $C_S(P) = C_{S_1}(P_1) \times C_{S_2}(P_2)$, P_i are fully $\mathcal{F}_i$-centralized for $i = 1, 2$. As φ_1 and φ_2 extend to N_{φ_1} and N_{φ_2}, φ extends to $N_{\varphi_1} \times N_{\varphi_2}$. This proves the proposition.

□

Chapter 2

Bisets

In this chapter, we review the structure of bisets and try to express group homomorphisms induced by the elements of bisets. Then we will show that bisets encodes morphisms of a specific fusions system, which we call a stabilizer fusion system.

The sections of this chapter are based on the ideas of Puig, which appear in Chapter 21 of [2].

2.1 Bisets

Definition 2.1.1. Let G and H be finite groups. A (G, H)-*biset* is a set Ω with a left G-action and a right H-action such that whenever $g \in G$, $h \in H$, $\omega \in \Omega$, we have

$$g \cdot (\omega \cdot h) = (g \cdot \omega) \cdot h.$$

Let $\varphi : K \to G$ be an injection, and Ω a (G, H)-biset. By ${}^{\varphi}_{K}\Omega_H$, we mean the

(K, H)-biset Ω on which K acts via φ, meaning if $\odot$ is the left K action on Ω, $k \odot \omega = \varphi(k) \cdot \omega$ for all $k \in K$, where $\cdot$ is the left G-action.

We call Ω *left free* if the left G-action is free, *right free* if the right H-action is free, and *bifree* if Ω is both left and right free.

The (G, H)-biset Ω can also be viewed as a left $(G \times H)$-set: For $(g, h) \in G \times H$, set

$$(g, h) \odot \omega = g \cdot \omega \cdot h^{-1},$$

where $\cdot$ is the biset action. We can also make a (G, H)-biset from a $(G \times H)$-set by writing

$$g \cdot \omega \cdot h = (g, h^{-1}) \odot \omega.$$

Hence, when working with bisets, we can view a biset as a left $(G \times H)$-set via this correspondence.

Definition 2.1.2. If Ω and Ψ are (G, H)-bisets, *a biset morphism* $\varphi : \Omega \to \Psi$ is a map such that $\varphi(g \cdot \omega \cdot h) = g \cdot \varphi(\omega) \cdot h$ for any $g \in G$, $h \in H$ and $\omega \in \Omega$.

2.2 Stabilizer Fusion Systems

Definition 2.2.1. Let Ω be a (G, H)-biset. The opposite biset Ω^{op} is the (H, G)-biset such that $\Omega = \Omega^{\mathrm{op}}$ as sets, and for all $g \in G$, $h \in H$, and $\omega \in \Omega$ the (H, G)-action is given by

$$h \odot \omega \odot g = g^{-1} \cdot \omega \cdot h^{-1}.$$

where $\cdot$ is the (G, H)-biset action. If $G = H$ and $\Omega \cong \Omega^{\mathrm{op}}$ as (G, G)-bisets, we say Ω is *symmetric*.

Let S be a finite p-group and Ω be a bifree and symmetric (S, S)-biset. Let G_Ω be $\mathrm{Aut}({}_1\Omega_S)$, the group permutations of Ω that respect the right S-action, meaning for all $\sigma \in G_\Omega$, $\omega \in \Omega$, and $a \in S$ we have $\sigma(\omega \cdot a) = \sigma(\omega) \cdot a$.

Since the elements of Ω are permuted by the left S-action, and the left and right S actions commute, each element of S induces an element of G_Ω: For any $a \in S$, let $\sigma_a \in G_\Omega$ be given by $\omega \mapsto a \cdot \omega$. The map $\varepsilon : S \to G_\Omega$: $a \mapsto \sigma_a$ embeds S in G_Ω. If $P \leq S$, we denote $\varepsilon(P)$ by P_Ω.

Next, for $P, Q \in S$, we can map $\mathcal{T}_{G_\Omega}(P_\Omega, Q_\Omega)$ (the G_Ω transporter of P_Ω to Q_Ω) to $\mathrm{Hom}(P, Q)$. Let Ψ be the map $\mathcal{T}_{G_\Omega}(P_\Omega, Q_\Omega) \to \mathrm{Hom}(P, Q)$: $\sigma \mapsto \varphi$ such that $\sigma \circ \sigma_a \circ \sigma = \sigma_{\varphi(a)}$ for each $a \in P$.

Definition 2.2.2. Let Ω be a symmetric (S, S)-biset. We call the category $\mathcal{F}_\Omega := \mathcal{F}_{S_\Omega}(G_\Omega)$ the *stabilizer* fusion system of Ω.

The first observation about $\mathcal{F}_\Omega$ is that since $S_\Omega \cong S$, we can view $\mathcal{F}_\Omega$ as a fusion system on S. On the other hand, S_Ω does not need to be Sylow in G_Ω, so $\mathcal{F}_\Omega(G_\Omega)$ is not a priori saturated.

Remark 2.2.3. Letting Ω be symmetric says that constructing the stabilizer fusion system $\mathcal{F}_S(\mathrm{Aut}({}_S\Omega_1))$ gives that $\mathcal{F}_S(\mathrm{Aut}({}_S\Omega_1))$ and $\mathcal{F}_S(\mathrm{Aut}({}_1\Omega_S))$ are isomorphic.

A better description for the morphisms of $\mathcal{F}_\Omega$ is required, and the following lemma enables us to do that.

Lemma 2.2.4. *Let Ω be a bifree-symmetric (S, S) biset. Given $P, Q \leq S$ and $\varphi \in \mathrm{Hom}(P, Q)$, $\varphi \in \mathcal{F}_\Omega$ if and only if ${}_P\Omega_S$ and ${}^{\varphi}_{P}\Omega_S$ are isomorphic as (P, S)-bisets.*

Proof. We first prove the necessity part of the lemma. For $\varphi \in \mathcal{F}_\Omega(P, Q)$, there is some $\sigma \in G_\Omega$ such that $\sigma \circ \sigma_a \circ \sigma^{-1} = \sigma_{\varphi(a)}$ for all $a \in P$. Then for all $\omega \in \Omega$, we have $\sigma(\sigma_a(\omega)) = \sigma_{\varphi(a)}(\sigma(\omega))$, and it follows that $\sigma(a \cdot \omega) = \varphi(a) \cdot \sigma(\omega)$, therefore σ is an isomorphism of (P, S)-bisets ${}_P\Omega_S \cong {}^{\varphi}_{P}\Omega_S$.

The sufficiency part is easy, since if we are given the isomorphism $\psi : {}_P\Omega_S \mapsto {}^{\varphi}_{P}\Omega_S$, for all $a \in S, \omega \in \Omega$, we have $\psi(a \cdot \omega) = a \odot \psi(\omega) = \psi(a) \cdot \psi(\omega)$ where $\odot$ is the left P action on the (P, S)-biset ${}^{\varphi}_{P}\Omega_S$. $\square$

2.3 Fixed-point Morphisms

In this section, we investigate the relation between fixed-point morphisms and stabilizer fusion systems. As a result of our work, in the next chapter, we review Park's result in [10]. We will see that stabilizer fusion systems are saturated if their morphisms match up with the corresponding fixed-point morphisms.

Definition 2.3.1. Let Ω be a (G, H)-biset. Given $\omega \in \Omega$, the *point-stabilizer* of ω is the set

$$\mathrm{Stab}_{S\times S}(\omega) = \{(g, h) \in G \times H \,:\, g \cdot \omega \cdot h^{-1} = \omega\}.$$

Clearly $(1, 1) \in \mathrm{Stab}_{S\times S}(\omega)$. Given (g, h) and (k, m) in $\mathrm{Stab}_{S\times S}(\omega)$, we have $(gk, hm) \in \mathrm{Stab}_{S\times S}(\omega) \leq G \times H$. Therefore, $\mathrm{Stab}_{S\times S}(\omega)$ is a subgroup of $G \times H$.

Definition 2.3.2. Let S be a finite p-group, $P \leq S$ a subgroup, and $\varphi : P \hookrightarrow S$ an injective group homomorphism. The twisted diagonal subgroup of φ and P is

$$(\varphi, P) := \{(\varphi(a), a) \,:\, a \in P \leq S\}.$$

Definition 2.3.3. Let Ω be an (S, S)-biset, $P \leq S$ a subgroup, $\varphi : P \hookrightarrow S$ a group injection, and $\omega \in \Omega$. For each $a \in P$, if $\varphi(a) \cdot \omega \cdot a^{-1} = \omega$, then we say (φ, P) fixes ω. The set of all points of Ω fixed by (φ, P) is denoted by $\Omega^{(\varphi,P)}$.

With the following lemma, we will get closer to relating point-stabilizers with the corresponding morphisms of a fusion system.

Lemma 2.3.4. *Let Ω be a bifree (S, S)-biset. Then all point-stabilizers in Ω are of twisted-diagonal form.*

Proof. Let $\omega \in \Omega$. Since $\mathrm{Stab}_{S\times S}(\omega) \leq S \times S$, let Q and P be the projections of $\mathrm{Stab}_{S\times S}(\omega)$ onto the first and second coordinates respectively. For each $a \in P$, there exists a $b \in Q$ such that $(a, b) \in \mathrm{Stab}_{S\times S}(\omega)$, so that $b \cdot \omega \cdot a^{-1} = \omega$. Note that b is here uniquely determined as the left S-action on Ω is free. Then we can define the map $\Phi_\omega : P \to Q$ such that $(\Phi_\omega(a), a) \in \mathrm{Stab}_{S\times S}(\omega)$ for each $a \in P$. Furthermore, Φ_ω is injective due to the free right S-action on Ω. Since

$$(\Phi_\omega(a))^{-1} \cdot \omega \cdot a = \Phi_\omega(b) \cdot \omega \cdot b^{-1},$$

we conclude that $\Phi_\omega(a)\Phi_\omega(b) \cdot \omega \cdot (ab)^{-1}$, meaning $\Phi_\omega(ab) = \Phi_\omega(a)\Phi_\omega(b)$. Hence, $\Phi_\omega \in \mathrm{Hom}(P, Q)$. □

Indeed, we are interested in twisted diagonal forms whose group homomorphisms come from a fusion system $\mathcal{F}$.

Definition 2.3.5. Let $\mathcal{F}$ be a fusion system on a finite p-group S. The bifree (S, S)-biset Ω is *$\mathcal{F}$-generated* if for each $\omega \in \Omega$ we have

$$\mathrm{Stab}_{S\times S}(\omega) = (\varphi, P), \text{where } P \leq S \text{ and } \varphi \in \mathcal{F}(P, S).$$

It will be useful to represent all fixed-point morphisms in a set.

Definition 2.3.6. For each $P \leq S$ we define $\mathrm{Hom}^{\Omega}(P, S)$ to be those group injections $\varphi : P \hookrightarrow S$ such that $\Omega^{(\varphi,P)}$ is non-empty.

We do not know yet that how stabilizer fusion systems and fixed-point morphisms are related. We will first show that the existence of a possible composition between them. Then the containment $\mathcal{F}_\Omega(P,S) \subseteq \mathrm{Hom}^\Omega(P,S)$ will be proved in some cases.

Proposition 2.3.7. [2, 21.3.2] *Let Ω be a bifree (S,S)-biset. If $\Omega^{(\mathrm{id}_S,S)}$ is non-empty, then $\mathcal{F}_\Omega(P,S) \subseteq \mathrm{Hom}^\Omega(P,S)$ for each $P \leq S$.*

Proof. Let $P,Q,R \leq S$, $\varphi \in \mathrm{Hom}^\Omega(P,Q)$ and $\psi \in \mathcal{F}_\Omega(Q,R)$. We claim that $\psi \circ \varphi \in \mathrm{Hom}^\Omega(P,R)$.

Since $\varphi \in \mathrm{Hom}^\Omega(P,Q)$, there exists an $\omega \in \Omega$ such that $\varphi(a) \cdot \omega = \omega \cdot a$. On the other hand, by Lemma 2.2.4, there exists a (Q,S)-biset isomorphism $\sigma : {}_Q\Omega_S \to {}^{\psi}_{Q}\Omega_S$ such that $\sigma(x \cdot \omega) = \psi(x)\sigma(\omega)$ for all $x \in Q$. Observe that

$$\begin{aligned}\sigma(\omega) &= \sigma(\varphi(a) \cdot \omega \cdot a^{-1}) \\ &= \psi(\varphi(a)) \cdot \sigma(\omega) \cdot a^{-1}.\end{aligned}$$

Hence, $\psi \circ \varphi \in \mathrm{Hom}^\Omega(P,R)$. Note that for a given any subgroup $P \leq S$, $\mathrm{Hom}^\Omega(\mathrm{id}_P,P)$ is non-empty as $\mathrm{Hom}^\Omega(\mathrm{id}_S,S)$ is non-empty. This implies that we have $\psi \circ \mathrm{id}_P = \psi \in \mathrm{Hom}^\Omega(P,S)$ by our claim whenever $\psi \in \mathcal{F}_\Omega(P,S)$. $\square$

Chapter 3

Characteristic Bisets Of Fusion Systems

In this chapter, we will explicitly describe the properties of certain bisets that exist for every saturated fusion system. As we pointed out earlier, exotic fusion systems cannot be realized by a finite group unless we remove the Sylow condition for realizability. Therefore, this important difference between exotic and realizable fusion systems leads us to work with different structures that can hold all the information of a saturated fusion system. At this point, characteristic bisets will be our new way of seeing saturated fusion systems.

3.1 Characteristic Bisets

Characteristic bisets and fusions systems together were first investigated by Linckelmann and Webb in their unpublished notes as a result of their work on the $\mathbb{F}_p$-cohomology of fusion systems. Since the homotopy theory of fusion systems is not a subject of this thesis, we avoid saying more about this material, but the details about how their biset idea emerged can be found in [14].

Before introducing characteristic bisets, we need the following definitions.

Definition 3.1.1. Let $\mathcal{F}$ be a fusion system on a finite p-group S. An (S,S)-biset Ω is

i. *Left $\mathcal{F}$-stable* if $|\Omega^{(\varphi,P)}| = |\Omega^{(\iota_P^S,P)}|$ (ι_P^S stands for the inclusion of P in S) for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.

ii. *Right $\mathcal{F}$-stable* if $|\Omega^{(\iota_P^S,P)}| = |\Omega^{(\varphi^{-1},\varphi(P))}|$ for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.

iii. *$\mathcal{F}$-stable* if Ω is both left and right $\mathcal{F}$-stable.

Remark 3.1.2. If Ω is an bifree (S,S)-biset and $\mathcal{F}$ is its stabilizer fusion system, Ω is always $\mathcal{F}$-stable.

Fact 3.1.3. For a bifree (S,S)-biset Ω and a fusion system $\mathcal{F}$ on S, being $\mathcal{F}$-generated and $\mathcal{F}$-stable is equivalent to ${}^{\varphi}_{P}\Omega_S \cong {}_{P}\Omega_S$ for all $P \leq S$ and $\varphi : P \hookrightarrow S$ in $\mathcal{F}$.

Definition 3.1.4. Let $\mathcal{F}$ be a fusion system on the finite p-group S, and Ω an (S,S)-biset. We call Ω an *$\mathcal{F}$-characteristic biset* if the following conditions hold:

i. Ω is $\mathcal{F}$-generated (see Definition 2.3.5).

ii. Ω is $\mathcal{F}$-stable.

iii. $|\Omega|/|S|$ is prime to p.

Remark 3.1.5. The first condition of characteristic bisets implies that each twisted diagonal subgroup has trivial intersection with the left $(1 \times S)$ and $(S \times 1)$ actions, hence we get free left and right S actions on Ω, in other words Ω is bifree.

We continue with the key example of characteristic bisets.

Example 3.1.6. Suppose that S is a Sylow p-subgroup of G and $\mathcal{F} := \mathcal{F}_S(G)$ is the induced fusion system on S. Recall that G is an (S,S) biset as S acts on G from the left and right via the group operation. Given $x \in G$, let $(a, b) \in \mathrm{Stab}_{S\times S}(x)$, then $axb^{-1} = x$ or $b = x^{-1}ax$. This implies that $(c_x, S \cap S^x) = \mathrm{Stab}_{S\times S}(x)$, and G is $\mathcal{F}$-generated as $c_x \in \mathcal{F}$.

For a subgroup $P \leq S$ and a morphism $c_g \in \mathcal{F}(P, S)$, define the maps:

$$\theta_1 : G^{(c_g,P)} \to G^{(c_g^{-1},c_g(P))},\ x \mapsto x^{-1},$$
$$\theta_2 : G^{(c_g,P)} \to G^{(\iota_P^S,P)},\ x \mapsto g^{-1}x.$$

Note that for $x \in G^{(c_g,P)}$ and each $a \in P$, since $c_g(a)(x)a^{-1} = x$, we have that

$$x^{-1} = (c_g(a)(x)a^{-1})^{-1} = a(x^{-1})(c_g(a))^{-1} = c_g^{-1}(c_g(a))(x^{-1})(c_g(a))^{-1}$$
$$g^{-1}x = g^{-1}c_g(a)(x)a^{-1} = a(g^{-1}x)a^{-1},$$

hence both maps are well defined. On the other hand, the inverse maps are given by

$$\theta_1^{-1} : G^{(c_g^{-1},c_g(P))} \to G^{(c_g,P)},\ x \mapsto x^{-1},$$
$$\theta_2^{-1} : G^{(\iota_P^S,P)} \to G^{(c_g,P)},\ x \mapsto gx.$$

Therefore, G is $\mathcal{F}$-stable as θ_1 and θ_2 are bijection.

The third axiom follows immediately as S is Sylow in G. Therefore G is an (S, S)-characteristic biset of the saturated fusion system $\mathcal{F}_S(G)$.

3.2 Saturated Fusion Systems From A Different Perspective

If $\mathcal{F}$ is a realizable fusion system, we know that $\mathcal{F}$ is always saturated by Proposition 1.4.7. Example 3.1.6 then motivates us to ask a natural question, is a fusion system saturated if and only if it holds a characteristic biset? We will need the following lemma before we respond to the first part of this question. In this lemma, for a finite p-group S, we prefer to work only with left S-sets rather than using both the left and right group actions to handle things more easily. It is also an example to see how one can manipulate and construct new S-sets by adding orbits.

Lemma 3.2.1. [1, Lemma 5.4] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $\mathcal{H}$ be an arbitrary collection of subgroups of S such that $\mathcal{H}$ is closed under taking subgroups of S and $\mathcal{F}$-subconjugacy. Let X_0 be an S-set such that $|X_0^P| = |X_0^Q|$ whenever $P \cong_{\mathcal{F}} Q$ and $P \notin \mathcal{H}$. Then there exists an S-set X that contains X_0 and has the following properties for all $P \leq S$:*

i. $|X^P| = |X^{\varphi(P)}|$ *whenever* $\varphi \in \mathcal{F}(P, S)$.
ii. *If* $P \notin \mathcal{H}$, *then* $X^P = X_0^P$.

Proof. If $\mathcal{H} = \emptyset$, simply let X be X_0. Otherwise, we construct X to satisfy the lemma by adding related orbits to X_0.

We apply induction on the number of subgroups in $\mathcal{H}$. Let P be maximal in $\mathcal{H}$, and without lose of generality, we can assume that P is fully $\mathcal{F}$-normalized in its $\mathcal{F}$-conjugacy class. After that we define $\mathcal{H}_0$ such that it contains all the subgroups in $\mathcal{H}$ but P and the subgroups $\mathcal{F}$-conjugate to P. By the inductive hypothesis, the lemma is satisfied for $\mathcal{H}_0$.

We aim to have that $|X_0^P| = |X_0^Q|$ for all $Q \leq S$ that are $\mathcal{F}$-conjugate to P. If this equality fails, we will add orbits of type $[S/P]$ or $[S/Q]$ (using brackets to indicate orbit type). However, when adding $[S/Q]$-orbits we need Proposition 1.4.10. For this reason, by adding $[S/P]$-orbits we first guarantee the inequality $|X_0^P| \geq |X_0^Q|$.

Now fix $Q \leq S$ that is $\mathcal{F}$-conjugate to P. By Proposition 1.4.10, $\varphi \in \mathcal{F}(N_S(Q), N_S(P))$ restricts to an $\mathcal{F}$-isomorphism from Q to P. Then consider the $N_S(Q)/Q$ action on X_0^Q and X_0^P (via φ). A quick observation about the orbits of this action is that if the size of the orbit of an element $x_0 \in X_0^Q$ is $|N_S(Q)/Q|$, then x_0 must have trivial stabilizer, so the action of $N_S(Q)/Q$ on this type of orbit is free. However, if the orbit of x_0 has size less than $|N_S(Q)/Q|$, then $N_S(Q)/Q$ acts non-freely on these orbits, meaning the points live in the orbit ω_0 must be fixed by some subgroups of S properly containing Q. Before making use of that fact, we define two subsets ${}^fX_0^Q$ and ${}^{nf}X_0^Q$, namely free and non-free parts of X^Q. Hence, $X_0^Q = {}^fX_0^P \coprod {}^{nf}X_0^P$, similarly $X_0^{\varphi(Q)} = {}^fX_0^{\varphi(Q)} \coprod {}^{nf}X_0^{\varphi(Q)}$. We claim that the non-free parts can be written as below:

$$ {}^{nf}X_0^Q = \bigcup_{Q \lneq R \leq N_S(Q)} X_0^R \quad \text{and} \quad {}^{nf}X_0^{\varphi(Q)} = \bigcup_{\varphi(Q) \lneq R \leq N_S(\varphi(Q))} X_0^{\varphi(R)}. $$

The containment $\subseteq$ is obvious. Conversely, given $x_0 \in {}^{nf}X_0^Q$, x_0 should be stabilized by some subgroup of S that contains Q as it lives in an orbit whose size is smaller than $|N_S(Q)/Q|$ by Orbit-Stabilizer Theorem, and similar arguments follows for second equality just above. This also implies that $|X_0^R| = |X_0^{\varphi(R)}|$ by the induction hypothesis.

Hence, we have the same number of elements living inside the non-free orbits of X_0^Q and X_0^P for the action of $N_S(Q)/Q$ on X_0^Q and (via φ) $X_0^{\varphi(Q)}$, and the difference $|X_0^Q| - |X_0^{\varphi(Q)}|$ will give the number of elements living inside all orbits where the $N_S(Q)/Q$ action is free. We can then divide $|X_0^Q| - |X_0^{\varphi(Q)}|$ by $|N_S(Q)/Q|$ to obtain the number of these orbits.

Let c_Q be $\frac{|X_0^P| - |X_0^Q|}{|N_S(Q)/Q|}$, and define a new S-set by

$$ X_1 = X_0 \coprod \Bigg(\coprod_{Q \in [P]_{\mathcal{F}} \setminus P} c_Q \cdot [S/Q] \Bigg), $$

where $c_Q \cdot [S/Q]$ can be seen as the c_Q different disjoint union of $[S/Q]$ orbits. Since we did not change the balance of the points fixed by $T \notin \mathcal{H}$, we still have $X_1^T = X_0^T$.

The proof follows inductively for the sets X_1 and $\mathcal{H}_1$. $\square$

Just as in the above lemma, we would like to be able to work with transitive subsets of the bisets we are acting on. With the help of the next definition, we will have transitive sub-bisets, so that we can write any given bifree (S, S)-biset as a union of that kind of subbisets.

Definition 3.2.2. Let $\varphi : P \to S$ be an injective group homomorphism. Set $S \times_{(\varphi,P)} S = S \times S / \sim$, where $\sim$ is given by $(s_1\varphi(a)), s_2) \sim (s_1, as_2)$ for $s_1, s_2 \in S$, $a \in P$. Note that $S \times_{\varphi,P} S$ is an (S, S)-biset.

We can also relate these sub-bisets with the stabilizers as we need this relation in the next theorem.

Lemma 3.2.3. *If $P \leq S$ and $\varphi \in \mathrm{Hom}(P, S)$, then $S \times_{(P,\varphi)} S$ is isomorphic to $(S \times S)/(\varphi, P)$ as (S, S)-bisets.*

Proof. Define $f : S \times_{(P,\varphi)} S \to (S \times S)/(\varphi, P)$ by $(g, h) \mapsto (g, h^{-1})$. We only need to show that this map is well defined. Note that for $g \in G$, $h \in H$ and $a \in P$, the elements equivalent to (g, h) are of the form $(g\varphi(a), a^{-1}h)$, and they will be mapped to $(g\varphi(a), ah^{-1})$, which lives with (g, h^{-1}) inside the same coset. □

In the next theorem, we will construct a characteristic biset for a given saturated fusion system. In this construction, we will use similar arguments given in the previous lemma.

Theorem 3.2.4. [1, Proposition 5.5] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then $\mathcal{F}$ has a characteristic (S, S)-biset.*

Proof. As we mentioned earlier, every $(G \times H)$-biset corresponds to a left (G, H)-set. We will construct a left $(S \times S)$-set, and use Lemma 3.2.1 to finish the proof.

Start by defining the set

$$\Omega_0 = \coprod_{[\varphi]\in \mathrm{Out}_{\mathcal{F}}(S)} (S \times_{(\varphi,S)} S).$$

Note that given φ_1 and $\varphi_2 \in \mathcal{F}$, we have that (φ_1, S) and (φ_2, S) are conjugate in $S \times S$ whenever φ_1 and φ_2 are in the same equivalence class in $\mathrm{Aut}_F(P)/\mathrm{Inn}(S)$. Hence, Ω_0 is well defined as an $(S \times S)$-set.

Since S is fully $\mathcal{F}$-normalized we have $\mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$ by the saturation axioms, so $|\mathrm{Out}_{\mathcal{F}}(S)|$ is prime to p. As each subgroup (φ, S) has order p, Then by Lemma 3.2.3, $|S \times_{(\varphi,S)} S| = |(S \times S)/(\varphi, S)| = |S|$, thus $|\Omega_0|/|S| = |\mathrm{Out}_{\mathcal{F}}(S)|$ which is prime to p.

Let $\mathcal{H}$ be the set of subgroups of $S \times S$ of the form (φ, P) for all proper subgroups P of S and $\varphi \in \mathcal{F}(P, S)$. Since $\mathcal{F}$ is saturated, $\mathcal{F} \times \mathcal{F}$ is a saturated fusion system on $S \times S$ by Propostion 1.6.3. Observe that if $P \leq S \times S$ and $P \notin \mathcal{H}$, we have $\Omega_0^P = \emptyset$ because Ω_0 is a bifree (S, S)-biset and φ factors as an $\mathcal{F}$-isomorphism followed by an inclusion in $\mathcal{F}$, so all stabilizers are of twisted diagonal form by a morphism in $\mathcal{F}$. In addition, the case $P = (\psi, R)$ for $\psi \in \mathcal{F}(R, S)$ is not our concern as we use $\mathcal{F}$-morphisms in the construction of Ω_0. Therefore, $\mathcal{H}$, as a left $(S \times S)$-set, satisfies the conditions given in Lemma 3.2.1.

Lemma 3.2.1 implies that there exits $(S \times S)$-set $\Omega \supseteq \Omega_0$ such that $|\Omega^P| = |\Omega^{P'}|$ whenever P and P' are $\mathcal{F} \times \mathcal{F}$-conjugate, and $\Omega^Q = \Omega_0^Q$ if $Q \notin \mathcal{H}$.

Lastly, since the orbits of Ω/Ω_0 are of the form $(S \times S)/(\varphi, S)$ for $P \lneq S$, they are multiple of $p|S|$, we get $\Omega \equiv \Omega_0 \bmod (p|S|)$. This implies that $|\Omega/S|$ is prime to p. $\square$

As we promised at the end of the previous chapter, here is Park's result.

Theorem 3.2.5. [10, Theorem 1] *Every saturated fusion system is weakly realizable.*

Proof. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. By Theorem 3.2.4, the fusion system $\mathcal{F}$ has an $\mathcal{F}$-characteristic (S,S)-biset Ω. Let $\mathcal{F}_\Omega = \mathcal{F}_{S_\Omega}(G_\Omega)$ be the stabilizer fusion system of Ω where $G_\Omega = \mathrm{Aut}({}_1\Omega_S)$ and $S_\Omega \cong S$. We claim that $\mathcal{F} = \mathcal{F}_\Omega$.

Let $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$. By Fact 3.1.3, we have ${}^{\varphi}_{P}\Omega_S \cong {}_P\Omega_S$, so that $\varphi \in \mathcal{F}_\Omega$, meaning $\mathcal{F} \subseteq \mathcal{F}_\Omega$.

For the containment $\mathcal{F} \supseteq \mathcal{F}_\Omega$, let $P \leq S$ and $\psi \in \mathcal{F}_\Omega$. Consider the right S-action on $S\backslash\Omega$. All the non-trivial orbits of this S-action have length divisible by p, so the set of fixed points $(S\backslash\Omega)^S$ is not empty as p does not divide $|S\backslash\Omega|$. For $P \leq S$, an orbit containing a point with point stabilizer (φ, P) has order $|S|^2/|P|$. If all points have such a point stabilizer with $P \lneq S$, this would force $|S\backslash\Omega| \equiv_p 0$. Therefore, since Ω is $\mathcal{F}$-generated, there is an $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\Omega^{(\alpha,S)}$ is non-empty. Since Ω is $\mathcal{F}$-stable, we have $\mathrm{Hom}^{\Omega}(\mathrm{id}_S, S) \neq \emptyset$. Then Proposition 2.3.7 implies that $\psi \in \mathcal{F}_\Omega$. Hence, $\mathcal{F} = \mathcal{F}_\Omega$. □

To conclude our work in this chapter with characteristic bisets, we need to show that for a fusion system $\mathcal{F}$, the existence of $\mathcal{F}$-characteristic bisets implies saturation. To give a proof for this result, the technical lemma will be needed. Ragnarsson and Stancu expanded the ideas given in [15, Proposition 1.16] and had some results that we collect in the following lemma.

Lemma 3.2.6. [14, Lemma 6.1 and Lemma 6.2] *Let $\mathcal{F}$ be a fusion system on a finite p-group S, Ω an $\mathcal{F}$-characteristic (S,S)-biset, and $P \leq S$. Let $\pi : \Omega \to S\backslash\Omega$ be the projection and Ω_0 be the pre-image $\pi^{-1}((S\backslash\Omega)^P)$. Then we have*

(i) $|\Omega_0| = \sum_{\varphi \in \mathcal{F}(P,S)} |\Omega^{(\varphi,P)}|$ *and* $|\Omega_0|/|S| \not\equiv_p 0$,

(ii) Given $\varphi \in \mathcal{F}(P,S)$, *then* $\varphi(P)$ *is fully* $\mathcal{F}$*-centralized if and only if* $\frac{|\Omega^{(\iota_P^S,P)}|}{|C_S(\varphi(P))|} \not\equiv_p 0$.

Proof. (i) We first review the structure of Ω_0. It is naturally an (S, P) biset, and if $\omega \in \Omega_0$, then for all $a \in P$, there is some $b \in S$ such that $b \cdot \omega \cdot a^{-1} = \omega$. Here b is uniquely determined since Ω is bifree by Remark 3.1.5. Therefore, for each $\omega \in \Omega_0$, we have injective group homomorphisms $\Phi_\omega : P \to S$ such that

$$\mathrm{Stab}_{S\times P}(\omega) = \{(\Phi_\omega(a), a)\}_{a\in P}.$$

Since Ω is $\mathcal{F}$-generated, $\Phi_\omega \in \mathcal{F}(P, S)$. Then let us define

$$\Phi : \Omega_0 \to \mathcal{F}(P, S)$$

by $\omega \mapsto \Phi_\omega$. If $\varphi \in \mathcal{F}(P, S)$, the pre-image $\Phi^{-1}(\varphi)$ is given by $\Omega_0^{(\varphi,P)} = \Omega^{(\varphi,P)}$. Then

$$\Omega_0 = \coprod_{\varphi\in\mathcal{F}(P,S)} \Omega^{(\varphi,P)},$$

and the order of each side is given by

$$|\Omega_0| = \sum_{\varphi\in\mathcal{F}(P,S)} |\Omega^{(\varphi,P)}|. \tag{3.1}$$

Consider the right P-action on $S\backslash\Omega$. All non-trivial orbits have length divisible by p. Since $|S\backslash\Omega|$ is prime to p, then by the Orbit-Stabilizer Theorem, the set of P-fixed points has size prime to p. Therefore, we have

$$|S\backslash\Omega| \equiv_p |(S\backslash\Omega)^P| = |S\backslash\Omega_0| \not\equiv_p 0.$$

(ii) Given $\omega \in \Omega_0$, let φ be Φ_ω. For all $a \in P, b \in S$ we have the following equality:

$$b \cdot \omega = b \cdot (\varphi(a) \cdot \omega \cdot a^{-1}) = b\varphi(a)b^{-1} \cdot (b \cdot \omega) \cdot a^{-1}. \tag{3.2}$$

We conclude that $\Phi_{b\cdot\omega} = c_b \circ \Phi_\omega$. Therefore, Φ induces a map $\bar{\Phi} : S\backslash\Omega_0 \to \mathrm{Rep}_{\mathcal{F}}(P, S)$, where $\mathrm{Rep}_{\mathcal{F}}(P, S) := \mathrm{Inn}(S)\backslash\mathcal{F}(P, S)$. We also define the projection $\hat{\pi} : \mathcal{F}(P, S) \to \mathrm{Rep}_{\mathcal{F}}(P, S)$, and let $\theta = \bar{\pi} \circ \Phi$. All these maps give the commutative diagram below.

$$\begin{tikzcd} \Omega_0 \arrow[r, "\Phi"] \arrow[d, "\pi"] \arrow[dr, "\theta"] & \mathcal{F}(P,S) \arrow[d, "\bar{\pi}"] \\ S\backslash\Omega_0 \arrow[r, "\bar{\Phi}"] & \mathrm{Rep}_{\mathcal{F}}(P,S) \end{tikzcd}$$

Note that if $\varphi \in \mathcal{F}(P,S)$, the size of its $\mathrm{Inn}(S)$-coset in $\mathrm{Rep}_{\mathcal{F}}(P,S)$ is $|S/C_S(\varphi(P))|$ by (3.2). With this in mind, for each $[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,S)$, the size of each pre-image $\theta^{-1}([\varphi])$ can be written as

$$|\theta^{-1}([\varphi])| = \frac{|S||\Omega_0^{(\varphi,P)}|}{|C_S(\varphi(P))|}.$$

On the other hand, each $S \times P$-orbit with a point-stabilizer (φ, P) has order $|S|$ as the left S-action is free on Ω_0 , hence

$$\frac{|\theta^{-1}([\varphi])|}{|S|} = \frac{|\Omega_0^{(\varphi,P)}|}{|C_S(\varphi(P))|} = \frac{|\Omega_0^{(\iota_P^S,P)}|}{|C_S(\varphi(P))|}.$$

The latter equality holds since Ω is $\mathcal{F}$-stable. Then we can partition the sum (3.1) for each $[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,S)$, and with the information of from (i), we get

$$0 \not\equiv_p |\Omega_0|/|S| = \sum_{[\varphi]\in\mathrm{Rep}_{\mathcal{F}}(P,S)} \frac{|\Omega_0^{(\iota_P^S,P)}|}{|C_S(\varphi(P))|}.$$

Since $\theta^{-1}([\varphi])$ is not empty, $|C_S(\varphi(P))|$ divides $|\Omega_0^{(\iota_P^S,P)}|$ for each $\varphi \in \mathcal{F}(P,S)$. Let $\varphi, \psi \in \mathcal{F}(P,S)$ such that $\varphi(P)$ is fully $\mathcal{F}$-centralized and $\psi(P)$ is not, then $|C_S(\varphi(P)| = p^k|C_S(\psi(P))|$ for some positive integer k. Since $|\Omega_0^{(\iota_P^S,P)}|$ is constant in the summand, only the order of fully $\mathcal{F}$-centralized subgroups cancel out the p-factor of $|\Omega_0^{(\iota_P^S,P)}|$, and this finishes the proof of this lemma.

□

The following Theorem is due to Puig.

Theorem 3.2.7. [2, Proposition 21.p] *Let $\mathcal{F}$ be a fusion system on a finite p-group S and Ω an $\mathcal{F}$-characteristic (S,S)-biset. Then $\mathcal{F}$ is saturated.*

Proof. For a given fully $\mathcal{F}$-normalized subgroup P of S, we need to show that P is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. We keep the same setup given in Lemma 3.2.6.

Let π be the projection $\Omega \to S\backslash\Omega$ and set $\Omega_0 := \pi^{-1}((S\backslash\Omega)^P)$, so Ω_0 is the disjoint union of the points fixed by (φ, P) for each $\varphi \in \mathcal{F}(P,S)$. Therefore,

$$\Omega_0 = \coprod_{\varphi\in\mathcal{F}(P,S)} \Omega^{(\varphi,P)}.$$

By Lemma 3.2.6, the order of $S\backslash\Omega_0$ is given by the sum

$$0 \not\equiv_p |S \setminus \Omega_0| = \frac{1}{|S|} \sum_{\varphi\in\mathcal{F}(P,S)} |\Omega^{(\varphi,P)}|.$$

Since Ω is $\mathcal{F}$-stable we can change $|\Omega^{(\varphi,P)}|$ to $|\Omega^{(\iota_P^S,P)}|$ in the summand and take advantage of Lemma 3.2.6. However, this sum says nothing about $C_S(P)$. At this point, to have $|C_S(P)|$ in the summand, we will manipulate this sum by changing its partition. In the first place, note that the number of $\mathcal{F}$-morphisms from P to S is the sum of the number of $\mathcal{F}$-morphisms from P to each subgroup that is $\mathcal{F}$-conjugate to P. Hence,

$$0 \not\equiv_p \frac{1}{|S|} \sum_{Q\cong_{\mathcal{F}}P} |\mathcal{F}(P,Q)| \cdot |\Omega^{(\iota_P^S,P)}|.$$

Whenever Q is $\mathcal{F}$-conjugate to P, then P is automatically $\mathcal{F}$-conjugate to the subgroups S-conjugate to Q by the definition of fusion systems. We want to partition the sum for the representatives of each S-conjugacy class, so multiply the summand by $|S/N_S(Q)|$ which is the number of subgroups S-conjugate Q. Let's denote the S-conjugacy class of Q by $[Q]_S$ and write the sum as:

$$0 \not\equiv_p \frac{1}{|S|} \sum_{[Q]_S\cong_{\mathcal{F}}P} \frac{|S|}{|N_S(Q)|} \cdot |\mathcal{F}(P,Q)| \cdot |\Omega^{(\iota_P^S,P)}|.$$

Since the whole sum is prime to p, and the summand can be divided by p whenever Q is not fully $\mathcal{F}$-normalized, we can omit the non-fully $\mathcal{F}$-normalized subgroups from the sum. After canceling out each $|S|$, since $N_S(Q)/C_S(Q) \cong \mathrm{Aut}_S(Q)$, we can replace $|N_S(Q)|$ by $|\mathrm{Aut}_S(Q)||C_S(Q)|$. In addition, since $P \cong_{\mathcal{F}} Q$ we also replace $|\mathcal{F}(P,Q)|$ by $|\mathrm{Aut}_{\mathcal{F}}(Q)|$.

$$0 \not\equiv_p \sum_{\substack{[Q]_S \cong_{\mathcal{F}} P \\ Q \in \mathcal{F}^{\mathrm{fn}}}} \frac{|\mathrm{Aut}_{\mathcal{F}}(Q)|}{|\mathrm{Aut}_S(Q)| \cdot |C_S(Q)|} \cdot |\Omega^{(\iota_P^S, P)}| = \sum_{\substack{[Q]_S \cong_{\mathcal{F}} P \\ Q \in \mathcal{F}^{\mathrm{fn}}}} \frac{|\mathrm{Aut}_{\mathcal{F}}(Q)|}{|\mathrm{Aut}_S(Q)|} \cdot \frac{|\Omega^{(\iota_P^S, P)}|}{|C_S(Q)|},$$

where $\mathcal{F}^{\mathrm{fn}}$ is the set of subgroups fully $\mathcal{F}$-normalized in $\mathcal{F}$. Then,

$$0 \not\equiv_p \sum_{\substack{[Q]_S \cong_{\mathcal{F}} P \\ Q \in \mathcal{F}^{\mathrm{fn}}}} [\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)] \cdot \frac{|\Omega^{(\iota_P^S, P)}|}{|C_S(Q)|},$$

Since the sum above is prime to p, both $[\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)]$ and $|\Omega^{(\iota_P^S, P)}|/|C_S(Q)|$ must be prime to p too in the summand. Hence, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ and by Lemma 3.2.6, $C_S(P)$ is fully $\mathcal{F}$-centralized.

To show that the extender axiom is satisfied, let $\varphi \in \mathcal{F}(P,S)$ and suppose that $\varphi(P)$ is fully $\mathcal{F}$-centralized.

Let N be the $S \times S$ normalizer of the twisted diagonal subgroup (φ, P). Given $(m,n) \in N$ and $(\varphi(a), a) \in (\varphi, P)$, there is some $(\varphi(b), b) \in (\varphi, P)$ such that $c_{(m,n)}(\varphi(a), a) = (\varphi(b), b)$. Then,

$$c_m(\varphi(a)) = \varphi(b), \tag{3.3}$$

$$c_n(a) = b, \tag{3.4}$$

and (3.3) and (3.4) together yield $c_m(\varphi(a)) = \varphi(c_n(a))$ for all $a \in P$, so $c_m|_{\varphi(P)} = \varphi \circ c_n|_P \circ \varphi^{-1}$. Note that since $n \in N_S(P)$, we have c_m and $\varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))$. With this in mind, recall that N_φ is the extender of φ given by the set

$$N_\varphi = \{n \in N_S(P) | \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}.$$

This shows that if we define the projection π from N onto its second coordinate, we get $\pi(N) = N_\varphi$. We see that the kernel of this projection is $C_S(\varphi(P)) \times 1$ by (3.3), so $C_S(\varphi(P)) \times 1 \trianglelefteq N$.

We now consider the action of N on $\Omega^{(\varphi,P)}$. To evaluate this action, let $(m, n) \in N$ and $\omega \in \Omega^{(\varphi,P)}$. $(m, n) \cdot \omega = m \cdot \omega \cdot n^{-1}$, and

$$\begin{aligned}\varphi(b)m \cdot \omega \cdot n^{-1} &= m\varphi(a) \cdot \omega_0 \cdot n^{-1} \text{ by (3.3)} \\ &= m \cdot \omega \cdot an^{-1} \\ &= m \cdot \omega \cdot n^{-1}b \text{ by (3.4)},\end{aligned}$$

so $(m, n) \cdot \omega_0 \in \Omega^{(\varphi,P)}$. It follows that N acts on $C_S(\varphi(P)) \setminus \Omega^{(\varphi,P)}$. Since Ω is $\mathcal{F}$-stable, we have $|\Omega^{(\varphi,P)}| = |\Omega^{(\iota_P^S,P)}|$. As $\varphi(P)$ is fully $\mathcal{F}$-centralized, by Lemma 3.2.6, $\frac{|\Omega^{(\iota,P)}|}{|C_S(\varphi(P))|}$ is prime to p. Therefore there must be trivial orbits of this action, so for all $(m, n) \in N$, there is some $z \in C_S(\varphi(P))$ such that $m \cdot \omega \cdot n^{-1} = z \cdot \omega$, $z^{-1}m \cdot \omega = \omega \cdot n$, and we end up with $(z^{-1}m, n) \in \mathrm{Stab}_{S\times S}(\omega)$. Let $\mathrm{Stab}_{S\times S}(\omega) = (\psi, Q)$ where $Q \leq S$ and $\psi \in \mathcal{F}(Q, S)$, thus $P \leq Q$ and $\psi|_P = \varphi$. Since for all $(m, n) \in N$, there is some such $z \in C_S(\varphi(P))$, we have $N_\varphi \leq Q$, and φ extends to $\psi|_{N_\varphi}$. □

As a result of Theorem 3.2.7, we have the following corollary.

Corollary 3.2.7.1. [15, Proposition 1.16] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and P be any subgroup of S. The order of S-conjugacy classes of subgroups that are fully $\mathcal{F}$-normalized and $\mathcal{F}$-conjugate to P is prime to p.*

Proof. Since $\mathcal{F}$ is saturated, there exists an $\mathcal{F}$-characteristic (S, S)-biset Ω by Theorem 3.2.4. By Theorem 3.2.7 we have the following sum that is prime to p.

$$0 \not\equiv_p \sum_{\substack{[Q]_S \cong_{\mathcal{F}} P \\ Q \in \mathcal{F}^{\mathrm{fn}}}} [\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)] \cdot \frac{|\Omega^{(\iota_P^S,P)}|}{|C_S(Q)|}.$$

The result is immediate. □

Chapter 4

Local Fusion Subsystems

Let $\mathcal{F}$ be a fusion system. In [4], Puig introduced the $\mathcal{F}$-normalizer subsystem and showed some essential results about this topic. In this chapter, we first introduce $\mathcal{F}$-normalizer and $\mathcal{F}$-centralizer fusion systems, which are also known as $\mathcal{F}$-*local subsystems*. Of course as we are interested in saturated fusion systems, saturated $\mathcal{F}$-normalizer and $\mathcal{F}$-centralizer systems will be our main interest. At the end of this chapter, we will focus on the saturation of $\mathcal{F}$-local subsystems.

4.1 Normalizer And Centralizer Fusion Systems

Definition 4.1.1. Let $\mathcal{F}$ be a fusion system on a finite p-group S. We define the $\mathcal{F}$-*normalizer* $N_{\mathcal{F}}(P)$ of P as a subcategory of $\mathcal{F}$, its objects are all subgroups of $N_S(P)$, and for $U, V \leq N_S(P)$, $N_{\mathcal{F}}(P)$ has morphisms given by

$$\begin{aligned} N_{\mathcal{F}}(P)(U,V) := \{\varphi \in & \mathcal{F}(U,V) : \\ & \exists \tilde{\varphi} \in \mathcal{F}(PU,PV) \text{ s.t. } \tilde{\varphi}|_U = \varphi,\ \tilde{\varphi}|_P \in \mathrm{Aut}_{\mathcal{F}}(P)\}. \end{aligned}$$

Similarly we define the $\mathcal{F}$*-centralizer* $C_{\mathcal{F}}(P)$ of P as a subcategory of $\mathcal{F}$ with all subgroups of $C_S(P)$ as objects, and morphisms for $U, V \leq C_S(P)$ are given by

$$C_{\mathcal{F}}(P)(U,V) := \{\varphi \in \mathcal{F}(U,V) \,:\, \exists \tilde{\varphi} \in \mathcal{F}(PU,PV) \text{ s.t. } \tilde{\varphi}|_U = \varphi,\ \tilde{\varphi}|_P = \mathrm{id}_P\}.$$

Lemma 4.1.2. *Let $\mathcal{F}$ be fusion system on finite p-group S, and $P \leq S$. Then, $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ are fusion systems.*

Proof. All the morphisms of $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ are in $\mathcal{F}$ by definition, so they are all injections. Let $a \in N_S(P)$ and $U, V \leq N_S(P)$. If $c_a(U) = V$, then $c_a(PU) = PV$, so c_a extends to PU. Thus, we have

$$\mathrm{Hom}_{N_S(P)}(U,V) \subseteq N_{\mathcal{F}}(P)(U,V) \subseteq \mathrm{Inj}(U,V).$$

This result is similar for $C_{\mathcal{F}}(P)$. For the second axiom, given an isomorphism $\varphi \in N_{\mathcal{F}}(P)(U,V)$, the extension $\tilde{\varphi} \in \mathcal{F}(PU,PV)$ of φ is an isomorphism too by Proposition 1.4.10. Therefore, $\tilde{\varphi}^{-1}|_V \in N_{\mathcal{F}}(P)(V,U)$, so it is the inverse of φ. $\square$

We again use bisets to investigate the saturation of normalizer and centralizer fusion systems.

Definition 4.1.3. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, Ω an $\mathcal{F}$-characteristic (S,S)-biset, and $P \leq S$. We define the subset $N_\Omega(P) \subseteq \Omega$ to be

$$N_\Omega(P) = \coprod_{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)} \Omega^{(\alpha,P)}.$$

Lemma 4.1.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, Ω an $\mathcal{F}$-characteristic (S,S)-biset, and $P \leq S$. Then $N_\Omega(P)$ is an $(N_S(P), N_S(P))$-biset.*

Proof. Let $\omega \in N_\Omega(P)$, then there is an $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\varphi(a) \cdot \omega = \omega \cdot a$ for all $a \in P$. Let $m, n \in N_S(P)$. Since $c_m \circ \varphi \circ c_n \in \mathrm{Aut}_{\mathcal{F}}(P)$ we claim that

$(c_m \circ \varphi_\omega \circ c_n, P) \leq \mathrm{Stab}_{N_S(P)\times N_S(P)}(m\cdot\omega\cdot n)$, then this will imply that $m\cdot\omega\cdot n \in N_\Omega(P)$. Given $a \in P$, there is $b \in P$ such that $a = nbn^{-1}$. It follows that

$$\begin{aligned} c_m \circ \varphi \circ c_n(b) m \cdot \omega \cdot n &= c_m \circ \varphi \circ c_n(n^{-1}an) m \cdot \omega \cdot n \\ &= c_m(\varphi(a)m) \cdot \omega \cdot n \\ &= m\varphi(a) \cdot \omega \cdot n \\ &= m \cdot \omega \cdot an \\ &= m \cdot \omega \cdot nb. \end{aligned}$$

Thus, $N_\Omega(P)$ is an $(N_S(P), N_S(P))$-biset. □

For the next theorem, we adopt some arguments given by Gelvin and Reeh in [3]. They prove a similar result for semi-characteristic bisets (characteristic bisets without the p-prime condition).

Theorem 4.1.5. [2, Proposition 21.11] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If $P \leq S$ is fully $\mathcal{F}$-normalized, then $N_\mathcal{F}(P)$ is saturated.*

Proof. [3, Proposition 9.10] Fix a fully $\mathcal{F}$-normalized subgroup P of S. Since $\mathcal{F}$ is saturated there exists an $\mathcal{F}$-characteristic (S, S)-biset Ω by Theorem 3.2.7. Given $\omega \in N_\Omega(P) \subseteq \Omega$, there is some $T \leq S$ and $\psi \in \mathcal{F}(T, S)$ such that $\psi|_P \in \mathrm{Aut}_\mathcal{F}(P)$, and $\mathrm{Stab}_{S\times S}(\omega) = (\psi, T)$ as Ω is $\mathcal{F}$-generated.

We claim that $\mathrm{Stab}_{N_S(P)\times N_S(P)}(\omega) = (\psi|_{T\cap N_S(P)}, T \cap N_S(P))$. Let $n \in T \cap N_S(P)$. For $a, b \in P$, we have $nan^{-1} = b$, and

$$\begin{aligned} \psi|_{T\cap N_S(P)}(nan^{-1}) &= \psi|_{T\cap N_S(P)}(n)\psi|_{T\cap N_S(P)}(a)(\psi|_{T\cap N_S(P)}(n))^{-1} \\ &= \psi|_{T\cap N_S(P)}(b). \end{aligned}$$

This implies that $\psi|_{T\cap N_S(P)}(n) \in N_S(P)$. Then our claim holds, so $N_\Omega(P)$ is $N_\mathcal{F}(P)$-generated.

For the stability condition, let $R, Q \leq N_S(P)$ and $\varphi \in N_{\mathcal{F}}(P)(R,Q)_{\text{iso}}$. We first deal with the easy case $P \leq R$. Since each point fixed by (φ, R) is fixed by (φ, P), then $\Omega^{(\varphi,R)} \subseteq \Omega^{(\varphi,P)}$, and $\Omega^{(\varphi,R)} \subseteq N_\Omega(P)$, so $(N_\Omega(P))^{(\varphi,R)} = \Omega^{(\varphi,R)}$. By using the $\mathcal{F}$-stability of Ω we get that

$$|(N_\Omega(P))^{(\varphi,R)}| = |(N_\Omega(P))^{(\iota,R)}| = |(N_\Omega(P))^{(\varphi^{-1},\varphi R)}|. \tag{4.1}$$

For the case $P \not\leq S$, we focus on the subgroup PR to make use of the argument above. Let $\{\tilde{\varphi}_i\}_{i=1}^n$ be the set of maps such that $\tilde{\varphi}_i \in N_{\mathcal{F}}(P)(PR, N_S(P))$ and each $\tilde{\varphi}_i|_R = \varphi$. Note that for each φ_i, we have $\varphi_i(PR) = \varphi_i(P)\varphi_i(R) = PQ$. It is clear that $(N_\Omega(P))^{(\tilde{\varphi}_i,PR)} \subseteq (N_\Omega(P))^{(\varphi,R)}$. Let $\omega_0 \in (N_\Omega(P))^{(\varphi,R)}$. Since Ω is $\mathcal{F}$-generated, there is a subgroup $Q \leq S$ and $\theta \in \mathcal{F}(Q,S)$ with $\text{Stab}_{S\times S}(\omega_0) = (\theta, Q)$. Notice that $PR \leq Q$. Since $\omega_0 \in N_\Omega(P)$, then $\theta|_P \in \text{Aut}_{\mathcal{F}}(P)$ and $\theta|_R = \varphi$. Then φ extends to $\theta|_{PR}$, so $\theta|_{PR}$ has to be one of these $\tilde{\varphi}_i$. This shows that we can write

$$(N_\Omega(P))^{(\varphi,R)} = \coprod_{i=1}^{n} (N_\Omega(P))^{(\tilde{\varphi}_i,PR)}.$$

Since $P \leq PR$, we have $(N_\Omega(P))^{(\tilde{\varphi}_i,PR)} = \Omega^{(\tilde{\varphi}_i,PR)}$, and this gives

$$|(N_\Omega(P))^{(\varphi,R)}| = \sum_{i=1}^{n} |\Omega^{(\tilde{\varphi}_i,PR)}|.$$

We are left to verify that $|(N_\Omega(P))^{(\varphi,R)}| = |(N_\Omega(P))^{(\iota_R^{N_S(P)},R)}|$. Let $\{\alpha\}_{i=1}^m$ be the set of extensions in $N_{\mathcal{F}}(P)$ of id_R with source PR. Note that since $\alpha(P) = P$ and $\alpha(R) = R$, then $\alpha(PR) = PR$. Therefore, each $\alpha_i \in \text{Aut}_{N_{\mathcal{F}}(P)}(PR)$, meaning $\{\alpha\}_{i=1}^m$ is a group. Given $\tilde{\varphi}_k, \tilde{\varphi}_l \in \{\tilde{\varphi}_i\}_{i=1}^n$, then the composition $(\tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_l) \in \text{Aut}_{N_{\mathcal{F}}(P)}(PR)$ as $(\tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_l)|_R = \text{id}_R$. Then $(\tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_l)$ must be in $\{\alpha_i\}_{i=1}^m$. Therefore, the right $\{\alpha\}_{i=1}^m$ action on the set $\{\tilde{\varphi}_i\}_{i=1}^n$ given by composition is free and transitive, so $m = n$.

Following similar arguments for $(N_\Omega(P))^{(\varphi,R)}$ gives

$$|(N_\Omega(P))^{(\mathrm{id}_R,R)}| = \sum_{i=1}^{n} |\Omega^{(\alpha_i,PR)}|.$$

By the $\mathcal{F}$-stability of Ω, we conclude that $|(N_\Omega(P))^{(\varphi,R)}| = |(N_\Omega(P))^{(\iota_R^{N_S(P)},R)}|$. One can also show that $|(N_\Omega(P))^{(R,\varphi^{-1})}| = |(N_\Omega(P))^{(\iota_R^{N_S(P)},R)}|$, hence $N_\Omega(P)$ is $N_\mathcal{F}(P)$-stable.

Lastly, we need to show that $|N_\Omega(P)|/|N_S(P)|$ is prime to p. Let $n = |N_\Omega(P)|/|N_S(P)|$. Recall that

$$N_\Omega(P) = \coprod_{\alpha\in \mathrm{Aut}_\mathcal{F}(P)} \Omega^{(\alpha,P)}.$$

Then the size of the left or right $N_S(P)$-cosets of $N_\Omega(P)$ is given by

$$n = \frac{1}{|N_S(P)|} \sum_{\alpha\in \mathrm{Aut}_\mathcal{F}(P)} |\Omega^{(\alpha,P)}|$$

The rest of this proof is quite similar to the proof given for 3.2.7. Remember that Ω is $\mathcal{F}$-stable, hence $|\Omega^{(\alpha,P)}| = |\Omega^{(\iota_P^S,P)}|$ for each $\alpha \in \mathrm{Aut}_\mathcal{F}(P)$, so we can write that

$$n = \frac{1}{|N_S(P)|} \cdot |\mathrm{Aut}_\mathcal{F}(P)| \cdot |\Omega^{(\iota_P^S,P)}|$$

Since $N_S(P)/C_S(P) \cong \mathrm{Aut}_S(P)$, we can replace $|N_S(P)|$ by $|\mathrm{Aut}_S(P)||C_S(P)|$, so

$$n = \frac{|\mathrm{Aut}_\mathcal{F}(P)|}{|N_S(P)/C_S(P)|} \cdot \frac{|\Omega^{(\iota_P^S,P)}|}{|C_S(P)|} = [\mathrm{Aut}_\mathcal{F}(P) : \mathrm{Aut}_S(P)] \cdot \frac{|\Omega^{(\iota_P^S,P)}|}{|C_S(P)|}.$$

By the Sylow axiom of saturated fusion systems, P is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}(P))$. Since P is fully $\mathcal{F}$-centralized, Lemma 3.2.6 implies that $\frac{|\Omega^{(\iota_P^S,P)}|}{|C_S(P)|}$ is prime to p. Furthermore, the index $[\mathrm{Aut}_\mathcal{F}(P) : \mathrm{Aut}_S(P)]$ is also prime to p as $\mathrm{Aut}_S(P)$ is Sylow p-subgroup of $\mathrm{Aut}_\mathcal{F}(P)$, thus n is prime to p, and we are done.

$\square$

By using arguments which are similar to the proof of Theorem 4.1.5, we show the same result for the centralizer fusion systems.

Theorem 4.1.6. [2, Proposition 21.11] *Let $\mathcal{F}$ be saturated fusion on a finite p-group S. If $P \leq S$ is fully $\mathcal{F}$-centralized, then $C_\Omega(P)$ is a saturated.*

Proof. By Therem 3.2.7, let Ω be an $\mathcal{F}$-characteristic (S, S)-biset. Set up a subset $C_\Omega(P) \subseteq \Omega$ such that

$$C_\Omega(P) = \Omega^{(\iota_P^S, P)}.$$

We first observe that $C_\Omega(P)$ is an $(C_S(P), C_S(P))$-biset. Given $\omega \in C_\Omega(P)$, and for each $a \in P$, we have $a \cdot \omega \cdot a^{-1} = \omega$. Let $c, d \in C_S(P)$, the following equality shows that C_Ω is an $(C_S(P), C_S(P))$-biset:

$$ac \cdot \omega \cdot da^{-1} = ca \cdot \omega \cdot a^{-1}d = c \cdot \omega \cdot d.$$

As P is fully $\mathcal{F}$-centralized, Lemma 3.2.6 implies that $|\Omega^{(\iota_P^S, P)}|/|C_S(P)|$ is prime to p, in other words, $|C_\Omega(P)|/|C_S(P)|$ is prime to p.

Since Ω is $\mathcal{F}$-generated, for $K \leq S$, we can suppose that $\mathrm{Stab}_{S\times S}(\omega) = (\psi, K)$ such that $\psi \in \mathcal{F}(K, S)$ and $\psi|P = \mathrm{id}_P$. For $c \in C_S(P)$ and $k \in K \cap C_S(P)$, the following equality

$$\psi|_{K\cap C_S(P)}(k) = \psi|_{K\cap C_S(P)}(cac^{-1}) = \psi|_{K\cap C_S(P)}(c)\psi|_{K\cap C_S(P)}(a)\psi|_{K\cap C_S(P)}(c^{-1})$$

yields that $\psi(k) \in C_S(P)$, hence $\mathrm{Stab}_{C_S(P)\times C_S(P)} = (\psi|_{K\cap C_S(P)}, K \cap C_S(P))$. As this is true for each $\omega \in C_\Omega(P)$, we have that $C_\Omega(P)$ is $C_{\mathcal{F}}(P)$-generated.

We are left to show that $C_\Omega(P)$ is $C_{\mathcal{F}}(P)$-stable. Let $L, Q \leq C_S(P)$ and $\varphi \in C_{\mathcal{F}}(P)(L, Q)_{\mathrm{iso}}$. If $P \leq L$, then $\varphi = \mathrm{id}_L$, meaning $C_{\mathcal{F}}(P)$ is $C_S(P)$-stable. If this is not the case, let $\{\tilde{\varphi}_i\}_{i=1}^n$ be the set of $C_{\mathcal{F}}(P)$-morphisms from PL to PQ

such that $\tilde{\varphi}_i|_P = \mathrm{id}_P$ and $\tilde{\varphi}_i|_L = \varphi$. It is easy to see that

$$(C_\Omega(P))^{(\varphi,L)} \supseteq \coprod_{i=1}^{n} (C_\Omega(P))^{(\tilde{\varphi}_i,PL)}.$$

To see the above containment is an equality, let $\omega_0 \in C_{\mathcal{F}}(P)$. For $R \leq S$ and $\theta \in \mathcal{F}(R,S)$, suppose that $\mathrm{Stab}_{S\times S} = (\varphi, R)$. Since $PL \leq R$ we have that $\theta|_R = \varphi$, and $\theta|_P = \mathrm{id}_P$. This result implies that $\theta \in \{\tilde{\varphi}_i\}_{i=1}^n$, and we now have

$$(C_\Omega(P))^{(\varphi,L)} = \coprod_{i=1}^{n} (C_\Omega(P))^{(\tilde{\varphi}_i,PL)}.$$

Since $(C_\Omega(P))^{(\varphi_i,PL)} = \Omega^{(\varphi_i,PL)}$, we get

$$|(C_\Omega(P))^{(\varphi,L)}| = \sum_{i=1}^{n} |\Omega^{(\tilde{\varphi}_i,PL)}|. \tag{4.2}$$

On the other hand, we need to show that $|(C_\Omega(P))^{(\varphi_i,PL)}| = |(C_\Omega(P))^{(\varphi_i,PL)}|$. Let $\{\alpha_i\}_{i=1}^m$ be the set of extensions of id_L in $C_{\mathcal{F}}(P)$ with source PL. Since $\alpha_i(P) = P$ and $\alpha_i(L) = L$, we get $\alpha_i \in \mathrm{Aut}_{C_{\mathcal{F}}(P)}(PL)$. Observe that given $\tilde{\varphi}_k, \tilde{\varphi}_l \in \{\tilde{\varphi}_i\}_{i=1}^n$, then $\tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_l \in \mathrm{Aut}_{C_{\mathcal{F}}(P)}(PL)$. Since $(\tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_l)|_L = \mathrm{id}_L$, the set $\{\alpha_i\}_{i=1}^m$ must contain $(\tilde{\varphi}_k^{-1} \circ \tilde{\varphi}_l)$. This shows that $\{\alpha_i\}_{i=1}^m$ has a free and transitive action on $\{\tilde{\varphi}_i\}_{i=1}^n$. Hence, $m = n$.

Then using similar arguments for finding $(C_\Omega(P))^{(\varphi,L)}$ gives

$$|(C_\Omega(P))^{(\mathrm{id}_L,L)}| = \sum_{i=1}^{m} |\Omega^{(\alpha_i,PL)}|.$$

As Ω is $\mathcal{F}$-stable, we have $|(C_\Omega(P))^{(\varphi,L)}| = |(C_\Omega(P))^{(\iota_L^{C_{\mathcal{F}}(P)},L)}|$. Similar arguments for $(C_\Omega(P))^{(\varphi^{-1},\varphi(L))}$ yields $|(C_\Omega(P))^{(\varphi^{-1},\varphi(L))}| = |(C_\Omega(P))^{(\iota_L^{C_{\mathcal{F}}(P)},L)}|$, thus $C_{\mathcal{F}}(P)$ is $C_S(P)$-stable.

$\square$

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