$f(R)$ THEORIES OF DARK GRAVITY

by

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This is to certify that I have examined this copy of a master's thesis by

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and that any and all revisions required by the final

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ABSTRACT

A review of f(R) theories of Dark Gravity is given. The variational field equations are derived for space-time geometries both with and without torsion. The method of Lagrange multipliers is used. Possible modifications to cosmological solutions are discussed in general. A specific solution to modified Friedmann equations with an exponential expansion is obtained. Finally further generalizations of theories of dark gravity are pointed out.

ÖZETÇE

Karanlık kütleçekiminin $f(R)$ teorileri gözden geçirilmiştir. Varyasyonel alan denklemleri burkulmalı ve burkulmasız uzay-zaman geometrilerinde Lagrange çarpanları yöntemiyle çıkarılmıştır. Karanlık kütleçekiminin kozmolojik çözümlere getirdiği katkılar genel olarak tartışıldıktan sonra genişletilmiş Friedmann denklemlerinin üstel olarak genleşen özel bir çözümü bulunmuştur. En sonunda karanlık kütleçekimi teorilerinin genelleştirilmesi üzerinde durulmuştur.

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Chapter 1

INTRODUCTION

General Relativity(GR) is one of the most succesful theories of the last century. When Einstein introduced his theory of general relativity in 1916 in its final form[1], it replaced Newton's theory of gravity -another succesful theory of gravitation- which at the time had survived for more than 200 years and was still continuing its impact. Although today GR explains many physical phenomena with success and covers shortcomings of Newtonian gravity, it has its own shortcomings in the very large and very small scales. Therefore taking our hint from the history of physics, we think that GR might also be modified by a more general theory of gravitation that reduces to GR in a certain limit. In this thesis we will concentrate on certain modifications of GR at cosmological length scales.

If we look at the sky, by observing the stars we realize that matter is not uniformly distributed in the universe. On the average 10^{11} stars constitute a galaxy. Moreover, when we increase our observational range we realize the galaxies separated from each other by at least ˜3×10^6 light years from clusters. For sufficiently large distances we see that such galaxy clusters are evenly distributed through the universe with a distance of 10^8 light years between them on avarage. Therefore beyond this scale we see a homogeneous distribution of matter in the universe.

Cosmological principle states that the structure of universe is homogeneous and isotropic at such large scales. The universe model obeying this principle is called the Friedmann-Robertson-Walker(FRW) model of the universe. This model is characterized by the metric

$$g = -dt \otimes dt + a^2(t) dx \otimes dx \tag{1.1}$$

called flat Robertson-Walker metric [2]. Here $a(t)$ is a scalar function known as the scale factor. From this metric and the Einstein equations with a perfect fluid source the Friedmann equations below follow:

$$\left(\frac{\dot{a}}{a}\right)^2 = \kappa^2 \frac{\rho}{3}$$

and

$$3\frac{\ddot{a}}{a} = -\kappa^2(\rho + 3p) \ .$$

where κ is related to the gravitational constant as $\kappa^2 = \frac{8\pi G}{c^4}$. The viability of such a characterization and the derivation of the Friedmann equations are discussed in Chapter 3. Let us now just focus here just on the physical results that follow from these equations. Here the matter and radiation content of the universe is described by a cosmological fluid with energy density ρ and pressure p. Since recent observations show the universe is expanding at an accelerated rate, that is $\ddot{a} > 0$, we must have

$$p < -\frac{\rho}{3} \ .$$

This condition implies that the universe must be filled with an unusual kind of matter/energy with negative pressure called dark energy to account for its accelerated expansion. In fact cosmological data estimates that %72 of the universe consists of dark energy [3].Evidently a new physics is needed [4], [5], [6], [7]. The simplest candidate for dark energy is a cosmological constant which has the equation of state $p = -\rho$. Since we assume energy density is positive, the corresponding pressure turns out to be negative [8], [9]. An alternative approach to dark energy problem consists of modifying the LHS of the Einstein field equations instead of the energy-momentum tensor. This approach is called the Dark Gravity [10], [11]. The common way to modify GR is to add higher/lower orders of the Ricci scalar to the Einstein-Hilbert action. One of the choices is to add a $\frac{1}{R}$ term to the action [4], [12]. This kind of modification becomes dominant in low curvature regions. Another choice is to add higher order terms to the action , which will be effective in high curvature regions[13].

In this thesis we focus on $f(R)$ theories of dark gravity. In the Chapter 2 we give the derivation of f(R) field equations with or without torsion [14]. While doing this we use tensor valued differential forms instead of tensors [15]. Therefore we give a brief introduction to the tools of exterior differential forms over space-time in appendix. In Chapter 3 we introduce cosmological spacetimes and derive the generic cosmological metric implied by the cosmological principle. We derive the modified Friedmann equations and apply it to a specific f(R) model with a negative power of curvature to explain dark gravity. In chapter

4 we discuss some further generalizations of theories of dark gravity. Finally we give a brief conclusion in Chapter 5.

Chapter 2

$f(R)$ GRAVITY FIELD EQUATIONS

The action for an $f(R)$ theory of gravity in $N+1$ dimensions is given by

$$S[e,\omega,\Phi]=\int_M \mathcal{L}(e,\omega)+S_m \ , \tag{2.1}$$

where M is the $N+1$ dimensional space-time manifold without boundary and S_m is the matter action describing the gravitational couplings of matter fields involved in the usual action. We take

$$\mathcal{L}(e,\omega)=f(R)*1 \tag{2.2}$$

in terms of an arbitrary function $f(R)$ of the Ricci scalar R. Let us first describe the forms involved in the above Lagrangian. The definitions and properties for these mathematical tools are explained in the Appendix. $*1$ is the volume form defined as $*1=e^0\wedge e^1\wedge...\wedge e^N$ where $*$ is the Hodge star map and $\{e^a\}$ are the co-frame 1-forms from which the space-time metric can be obtained as $g=\eta_{ab}e^a\otimes e^b$. η_{ab} label the orthonormal components of the flat metric expressed as $diag(-1,1,1,1)$. Equivalently we can write $*1$ as

$$*1=\frac{1}{(N+1)!}\epsilon_{a_0....a_N}e^{a_0}\wedge e^{a_1}...e^{a_N}$$

where $\epsilon_{a_0....a_N}$ is the totally anti-symmetric Levi-Civita symbol .
The Ricci scalar R is obtained by contracting the curvature 2-forms $R^a{}_b$ twice:

$$R_a=i_bR^b{}_a=\mathcal{R}_{ab}e^b \tag{2.3}$$

are the Ricci 1-forms and

$$R=i_aR^a \ . \tag{2.4}$$

We can relate the curvature 2-forms $R^a{}_b$ with the usual Riemann tensor of type (1,3) as follows:

$$R^a{}_b=\frac{1}{2}\mathcal{R}^a{}_{bcd}e^c\wedge e^d \ . \tag{2.5}$$

The second rank covariant, symmetric Ricci tensor $\mathcal{R}_{ab}$ is defined by a contraction as

$$\mathcal{R}_{ab} = \mathcal{R}^{c}{}_{acb} \; . \tag{2.6}$$

Then we further note that in terms of the Riemann tensor the Ricci scalar can be expressed as

$$R = \mathcal{R}^{ab}{}_{ab}. \tag{2.7}$$

The curvature 2-forms themselves are obtained from Cartan's second structure equations:

$$R^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} \tag{2.8}$$

where $\{\omega^{a}{}_{b}\}$ are the connection 1-forms which appear in Cartan's first structure equations

$$De^{a} = de^{a} + \omega^{a}{}_{b} \wedge e^{b} = T^{a} \; . \tag{2.9}$$

Here D denotes the exterior covariant derivative and d denotes the exterior derivative. $\{T^{a}\}$'s are the torsion 2-forms of space-time. If we set $T^{a} = 0$, the connection is called torsion-free. There is another condition that we impose on the connection 1-forms, called metric compatibility. It amounts to the requirement that the exterior covariant derivative of the metric tensor vanishes, that is

$$Dg = 0 \; . \tag{2.10}$$

This condition is equivalent to taking $\omega_{ab} = -\omega_{ba}$. The unique torsion-free, metric compatible connection, called the Levi-Civita connection, is determined up to local Lorentz transformations by the metric tensor. In this case, and only in this case, the space-time geometry will be completely determined by the metric tensor.

If $\{X_{a}\}$ is an orthonormal frame such that $e^{a}(X_{b}) = \delta^{a}{}_{b}$, then the connection 1-forms can be given in terms of the affine connection coefficients as

$$\Gamma^{a}{}_{bc} \quad = \quad \omega^{a}{}_{b}(X_{c}). \tag{2.11}$$

For the torsion-free, metric compatible Levi-Civita connection, these coefficients are also known as the Christoffel symbols of the second kind.

In the following sections we are going to obtain $f(R)$ field equations in cases with or without torsion. However, before accomplishing these tasks let us first say a few words on Einstein field equations for future reference.

2.1 Einstein's Field Equations

The Einstein-Hilbert action is given by

$$S_{EH} = \int_M \mathcal{L}_{EH} = -\frac{1}{2\kappa^2}\int_M R * 1 \ . \tag{2.12}$$

where $\kappa^2 = \frac{8\pi G}{c^4}$ is Newton's universal gravitational constant. We choose units such that $\kappa = c = 1$ throughout this thesis. Due to the identity

$$R * 1 = R^a{}_b \wedge *(e_a \wedge e^b) \ , \tag{2.13}$$

we can express the variation of $\mathcal{L}_{EH}$ as follows:

$$\delta\mathcal{L}_{EH} = -\frac{1}{2}\delta R^a{}_b \wedge *(e_a \wedge e^b) - \frac{1}{2}R^a{}_b \wedge \delta * (e_a \wedge e^b)) \ . \tag{2.14}$$

Let us investigate each term above separately. We have

$$\begin{aligned}
\delta R^a{}_b \wedge *(e_a \wedge e^b) &= (d\delta\omega^a{}_b + \delta\omega^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge \delta\omega^c{}_b) \wedge *(e_a \wedge e^b) \\
&= d(\delta\omega^a{}_b \wedge *(e_a \wedge e^b)) + \delta\omega^a{}_b \wedge D * (e_a \wedge e^b) \ , \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
\delta * (e^a \wedge e^b) &= \delta\left(\frac{1}{2}\epsilon^{ab}{}_{cd} e^c \wedge e^d\right) \\
&= \delta e^c \epsilon^{ab}{}_{cd} e^d \\
&= \delta e^c \wedge *(e^a \wedge e^b \wedge e_c) \ . \qquad (2.16)
\end{aligned}$$

As a result

$$\begin{aligned}
\delta S_{EH} &= -\int_M \frac{1}{2} d(\delta\omega^a{}_b \wedge *(e_a \wedge e^b)) \\
&\quad - \int_M \left(\frac{1}{2}\delta\omega^a{}_b \wedge D * (e_a \wedge e^b) + \frac{1}{2}R^a{}_b \wedge \delta e^c \wedge *(e_a \wedge e^b \wedge e_c)\right) \ .
\end{aligned} \tag{2.17}$$

By Stokes' theorem we have

$$\int_M \frac{1}{2} d(\delta\omega^a{}_b \wedge *(e_a \wedge e^b)) = \int_{\partial M} \delta\omega^a{}_b \wedge *(e_a \wedge e^b) \tag{2.18}$$

and since we assumed M has no boundary, the above integral vanishes. Next, we consider the identity

$$\begin{aligned} D*(e^a \wedge e^b) &= D\left(\frac{1}{2}\epsilon^{ab}{}_{cd} \wedge e^c \wedge e^d\right) \\ &= \epsilon^{ab}{}_{cd} De^c \wedge e^d \\ &= T^c \wedge *(e^a \wedge e^b \wedge e_c) \ . \end{aligned} \tag{2.19}$$

At this point there are two ways that we may proceed. We can either vary the action with respect to co-frames and connection forms independently and then solve for the torsion. Alternatively, we can set the torsion equal to zero at the beginning and consider the zero-torsion constrained variations of the action. For the Einstein-Hilbert action without any coupling to matter, both of these cases imply the same set of field equations. Namely, the source-free, vacuum Einstein field equations. To prove this statement, let us first give the zero-torsion case. In this case we write the source-free Einstein field equations as

$$-\frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b \wedge e_c) = 0 \tag{2.20}$$

For the non-zero torsion case, the independent variations of the co-frames and connection 1-forms give

$$-\frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b \wedge e_c) = 0 \ , \tag{2.21}$$

$$T^c \wedge *(e^a \wedge e^b \wedge e_c)) = 0 \ , \tag{2.22}$$

respectively. The unique solution of (2.22) for the torsion 2-forms is $T^a = 0$. Therefore (2.21) is exactly the same as (2.20) before.

Now let us introduce the Einstein tensor G_{ab} that is a second rank covariant, symmetric tensor defined by

$$\mathcal{R}_{ab} - \frac{1}{2}g_{ab}R = G_{ab} \ . \tag{2.23}$$

The Einstein tensor is covariantly constant by construction. That is

$$DG_{ab} = 0 \tag{2.24}$$

Now we re-express the Einstein field equations in terms of the Einstein tensor:

$$\begin{aligned}
\frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b \wedge e_c) &= \frac{1}{4}\mathcal{R}^a{}_{bdf} e^d \wedge e^f \wedge *(e_a \wedge e^b \wedge e_c) \\
&= \frac{1}{4}\mathcal{R}^a{}_{bdf} e^d \wedge (\eta^f_c * (e_a \wedge e^b) + \eta^f_a * (e^b \wedge e_c) - \eta^f b * (e_c \wedge e^a)) \\
&= \frac{1}{4}\mathcal{R}^a{}_{bdf} (\eta^f_c (\eta^d_a * e^b - \eta^{db} * e_a) + \eta^f_a (\eta^{db} * e_c - \eta^d_c * e^b) \\
&- \eta^{fb} (\eta^d_c * e_a - \eta^d_a * e_c) \\
&= -(\mathcal{R}_{bc} * e^b - \frac{1}{2}R * e_c) \\
&= -(\mathcal{R}_{bc} - \frac{1}{2}R\eta_{bc}) * e^b = -G_{bc} * e^b \ . \qquad (2.25)
\end{aligned}$$

Therefore we rewrite (2.20) as

$$G_{ca} * e^a = -\frac{1}{2}R^a{}_b \wedge *(e_a \wedge e^b \wedge e_c) = 0 \ . \qquad (2.26)$$

Up to this point we considered Einstein field equations without matter fields. To couple matter fields to gravity we include in the total action, the Lagrangian density for matter $\mathcal{L}_m$ that is given in terms of matter fields Φ and their first derivatives $d\Phi$. Here a matter field Φ is considered to be a p-form. We should take $p = 0$ for scalar fields and $p = 1$ for vector fields. In this thesis these will be the only cases we consider. Furthermore we always assume that the matter action $S_m = \int_M L_m$ depends on the metric tensor g and the matter field Φ with no explicit dependence on the connection 1-forms $\omega^a{}_b$. We vary the total action

$$S_T = S_{EH} + S_m \qquad (2.27)$$

with respect to the metric, connection and matter fields to obtain the corresponding system of Einstein coupled with matter field equations. The variation of the Lagrangian density $\mathcal{L}_m$ gives us

$$\delta L_m = \delta e^a \wedge \frac{\delta \mathcal{L}_m}{\delta e^a} + \delta\Phi \wedge \frac{\mathcal{L}_m}{\delta\Phi} + \delta d\Phi \wedge \frac{\delta\mathcal{L}_m}{\delta d\Phi} \qquad (2.28)$$

Here we make use of the fact that the exterior derivative d and the variation operator δ commute. Then

$$\delta\mathcal{L}_m = \delta e^a \wedge \frac{\delta\mathcal{L}_m}{\delta e^a} + \delta\Phi \wedge \left(\frac{\delta\mathcal{L}_m}{\delta\Phi} + (-1)^p d\frac{\delta\mathcal{L}_m}{\delta d\Phi}\right) + d\left(\delta\Phi \wedge \frac{\delta\mathcal{L}_m}{\delta d\Phi}\right) \qquad (2.29)$$

We thus find the Euler- Lagrange equation satisfied by a matter field Φ as

$$\frac{\delta\mathcal{L}_m}{\delta\Phi} + (-1)^p d\frac{\delta\mathcal{L}_m}{\delta d\Phi} = 0. \qquad (2.30)$$

We further identify the energy-momentum tensor of matter from the co-frame variation of $\mathcal{L}_m$:

$$\frac{\delta \mathcal{L}_m}{\delta e^a} \equiv \mathcal{T}_a[\Phi] = T_{ab}[\Phi] * e^b. \tag{2.31}$$

Given the definitions above, we introduce the covariant second rank, symmetric tensor fields

$$G = G_{ab} e^a \otimes e^b$$

and

$$T(\text{matter}) = T_{ab}[\Phi] e^a \otimes e^b$$

and write the Einstein field equations with a source as

$$G = \kappa^2 T(\text{matter}) + \Lambda g. \tag{2.32}$$

Here Λ is a cosmological constant that may carry either sign.

2.2 Derivation of f(R) Gravity Field Equations

In this section we are going to obtain $f(R)$ field equations in both cases with or without torsion.

2.2.1 Zero-Torsion Constrained Variations

We will vary the action (1) that is modified by a Lagrangian constraint to impose the condition that our connection is torsion-free. In order to do this we introduce Lagrange multiplier $(N-1)$-forms λ_a and write the Lagrangian density as:

$$\mathcal{L}(e,\omega) = f(R) * 1 + (de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a\ . \tag{2.33}$$

The variation of this Lagrangian gives

$$\delta\mathcal{L}(e,\omega) = \delta f(R) * 1 + f(R)\delta * 1 + \delta(de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a \\ + (de^a + \omega^a{}_b \wedge e^b) \wedge \delta\lambda_a\ . \tag{2.34}$$

Let us compute each term separately.

(i) We have

$$\delta f(R) * 1 = f'(R)\delta R * 1 \tag{2.35}$$

where prime denotes differentiation with respect to R. Using $\delta(R*1) = (\delta R)*1 + R\delta*1$ we can express (2.35) as

$$\delta f(R)*1 = f'(R)\delta(R*1) - f'(R)R\delta*1 \ . \tag{2.36}$$

Let us note the following identities

$$\begin{aligned} \delta*1 &= \delta\frac{1}{(N+1)!}\epsilon_{a_0....a_N}e^{a_0}\wedge e^{a_1}...e^{a_N}) &(2.37)\\ &= \delta e^{a_0}\wedge\frac{1}{(N)!}\epsilon_{a_0a_1....a_N}e^{a_1}\wedge ...e^{a_N}) &(2.38)\\ &= \delta e^a\wedge *e_a \ , &(2.39)\end{aligned}$$

$$\begin{aligned} f'(R)\delta(R*1) &= f'(R)\delta(R^a{}_b)\wedge*(e_a\wedge e^b) + f'(R)R^a{}_b\wedge\delta*(e_a\wedge e^b)\\ &= f'(R)\delta(d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b)\wedge*(e_a\wedge e^b)\\ &\quad +f'(R)R^a{}_b\wedge\delta e^c\wedge*(e_a\wedge e^b\wedge e_c) \ , \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} f'(R)\delta(d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b)\wedge*(e_a\wedge e^b) &= f'(R)\delta(d\omega^a{}_b)\wedge*(e_a\wedge e^b)\\ &\quad +f'(R)\delta(\omega^a{}_c)\wedge\omega^c{}_b)\wedge*(e_a\wedge e^b)\\ &\quad +f'(R)\omega^a{}_c\wedge\delta(\omega^c{}_b)\wedge*(e_a\wedge e^b) \ . \end{aligned} \tag{2.41}$$

At this point we make use of the Leibniz rule

$$\begin{aligned} d(f'(R)\delta\omega^a{}_b\wedge*(e_a\wedge e^b)) &= df'(R)\wedge\delta\omega^a{}_b\wedge*(e_a\wedge e^b)\\ &\quad +f'(R)d(\delta\omega^a{}_b)\wedge*(e_a\wedge e^b))\\ &\quad -f'(R)\delta\omega^a{}_b\wedge d(*(e_a\wedge e^b)) \end{aligned} \tag{2.42}$$

and write

$$\begin{aligned} f'(R)\delta(d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b)\wedge*(e_a\wedge e^b) &= d(f'(R)\delta\omega^a{}_b\wedge*(e_a\wedge e^b))\\ &\quad -df'(R)\wedge\delta\omega^a{}_b\wedge*(e_a\wedge e^b)\\ &\quad +f'(R)\delta\omega^a{}_b\wedge d(*(e_a\wedge e^b))\\ &\quad +f'(R)\delta(\omega^a{}_b)\wedge\omega^b{}_c)\wedge*(e_a\wedge e^c)\\ &\quad +f'(R)\omega^c{}_a\wedge\delta(\omega^a{}_b)\wedge*(e_c\wedge e^b) \ , \end{aligned} \tag{2.43}$$

where we re-named dummy indices.

In fact (2.43) can be written in a more compact way by introducing the exterior covariant derivative D of $*(e_a \wedge e^b)$:

$$\begin{aligned}\delta f'(R)(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge *(e_a \wedge e^b) &= d(f'(R)\delta\omega^a{}_b \wedge *(e_a \wedge e^b)) \\ &\quad -df'(R) \wedge \delta\omega^a{}_b \wedge *(e_a \wedge e^b) \\ &\quad +\delta\omega^a{}_b \wedge f'(R) D * (e_a \wedge e^b) \ . \qquad (2.44)\end{aligned}$$

Finally (2.40) becomes

$$\begin{aligned}f'(R)\delta R * 1 &= d(f'(R)\delta\omega^a{}_b \wedge *(e_a \wedge e^b)) - df'(R) \wedge \delta\omega^a{}_b \wedge *(e_a \wedge e^b) \\ &\quad +\delta\omega^a{}_b \wedge f'(R) D * (e_a \wedge e^b) + f'(R) R^a{}_b \wedge \delta e^c \wedge *(e_a \wedge e^b \wedge e_c) \\ &\quad -f'(R) R \delta e^a \wedge *e^a \ . \qquad (2.45)\end{aligned}$$

(ii) Now, we consider the third term in equation (2.34). Since $d(\delta e^a \wedge \lambda_a) = d(\delta e^a) \wedge \lambda_a - \delta e^a \wedge d\lambda_a$, we have

$$\begin{aligned}\delta(de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a &= d(\delta e^a \wedge \lambda_a) + \delta e^a \wedge d\lambda_a + \delta\omega^a{}_c \wedge e^b \wedge \lambda_a + \omega^b{}_c \wedge \delta e^a \wedge \lambda_b \\ &= d(\delta e^a \wedge \lambda_a) + \delta\omega^a{}_b \wedge e^c \wedge \lambda_a + \delta e^a \wedge D\lambda_a \ , \qquad (2.46)\end{aligned}$$

where we renamed the dummy indices and obtained a term that involves the covariant derivative of λ_a. Also we note that

$$\delta\omega^a{}_b \wedge e^b \wedge \lambda_a = \delta\omega_{ab} \wedge \frac{1}{2}\left(e^b \wedge \lambda^a - e^a \wedge \lambda^b\right) \qquad (2.47)$$

that follows from the metric-compability condition $\omega_{ab} = -\omega_{ba}$.

If we now write all the terms in the variation together,

$$\begin{aligned}&\delta\mathcal{L} = d(f'(R)\delta\omega^a{}_b \wedge *(e_a \wedge e^b)) + d(\delta e^a \wedge \lambda_a) + (de^a + \omega^a{}_b \wedge e^b) \wedge \delta\lambda_a \\ &+\delta e^a \wedge \Big(f'(R) R^c{}_b \wedge *(e_c \wedge e^b \wedge e_a) - f'(R) R * e_a + f(R) * e_a + d\lambda_a\Big) + \\ &\delta\omega_{ab} \wedge \left(df'(R) \wedge *(e^a \wedge e^b) + f'(R) D * (e^a \wedge e^b) + \frac{1}{2}\left(e^b \wedge \lambda^a - e^a \wedge \lambda^b\right)\right) \ .(2.48)\end{aligned}$$

By Stokes' theorem

$$\int_M d\left[f'(R)\delta\omega^a{}_b \wedge *(e_a \wedge e^b) + d(\delta e^a \wedge \lambda_a)\right] = \int_{\partial M} f'(R)\delta\omega^a{}_b \wedge *(e_a \wedge e^b) + \delta e^a \wedge \lambda_a$$

Since at the beginning we assumed that M is a manifold without boundary, the above integral vanishes. Therefore, the variational principle states that

$$\delta S = \int_M (\delta\mathcal{L} + \delta L_M) = 0 \ .$$

and consequently we continue with the remaining terms in the variation of the action and obtain three independent sets of equations:

$$f'(R)R^c{}_b \wedge *(e_c \wedge e^b \wedge e_a) - f'(R)Re_a + f(R)e_a + d\lambda_a = -G_{ab} * e^b \ , \tag{2.49}$$

$$-df'(R) \wedge *(e^a \wedge e^b) + f'(R)D * (e^a \wedge e^b) + \frac{1}{2}\left(e^b \wedge \lambda^a - e^a \wedge \lambda^b\right) = 0 \ , \tag{2.50}$$

$$de^a + \omega^a{}_b \wedge e^b = 0 \ . \tag{2.51}$$

Equation (2.51) is the constraint that we imposed at the beginning for a torsion-free connection. The Lagrange multipliers $\{\lambda_a\}$ are to be solved from (2.50). Since

$$D * (e^a \wedge e^b) = 0 \tag{2.52}$$

by the zero-torsion constraint equation, (2.50) reduces to

$$-df'(R) \wedge *(e^a \wedge e^b) = \frac{1}{2}\left(e^b \wedge \lambda^a - e^a \wedge \lambda^b\right) \ . \tag{2.53}$$

We take the interior product i_a of both sides to get

$$\begin{aligned} -i_a(df') \wedge *(e^a \wedge e^b) + df' \wedge i_a * (e^a \wedge e^b) \\ = \frac{1}{2}\left(i_a e^b \wedge \lambda^a - e^b \wedge i_a\lambda^a - i_a e^a \wedge \lambda^b + e^a \wedge i_a\lambda^b\right) \end{aligned} \tag{2.54}$$

and consider each term above separately. Obviously we have

$$i_a * (e^a \wedge e^b) = 0 \ . \tag{2.55}$$

Furthermore

$$i_a e^b = \delta^a{}_b \ , \tag{2.56}$$

$$i_a e^a = N + 1 \ , \tag{2.57}$$

$$e^a \wedge i_a\lambda^b = (N-1)\lambda^b \ , \tag{2.58}$$

since λ_b is a $N-1$ form. Thus (2.55) becomes

$$i_a(df') \wedge *(e^a \wedge e^b) = \frac{1}{2}\left(\lambda^b + e^b \wedge i_a\lambda^a\right) \ . \tag{2.59}$$

We take again the interior product i_b of the above equation and similarly obtain

$$i_b i_a(df') \wedge *(e^a \wedge e^b) = \frac{1}{2}\left(4i_a \wedge \lambda^a\right) \ . \tag{2.60}$$

Since $i_a df'$ is a zero-form, the LHS of the above equation is zero. Therefore, $i_a\lambda^a$ is also zero. We apply this condition to (2.59) and find

$$i_a(df') \wedge *(e^a \wedge e^b) = \frac{1}{2}\lambda^b \tag{2.61}$$

$$*(i_a df' \wedge e^a \wedge e^b) = \frac{1}{2}\lambda^b \tag{2.62}$$

$$\lambda^b = 2i^b * (df') \tag{2.63}$$

The remaining field equations (2.49)

$$f'(R)R^c{}_b \wedge *(e_c \wedge e^b \wedge e_a) + (f(R) - f'(R)R) * e_a + 2D(i_a * df') = 0 \tag{2.64}$$

are the source-free $f(R)$ gravity field equations. In covariant index-free notation we write the $f(R)$ gravity field equations coupled with matter as

$$-f'(R)G + \frac{1}{2}(f - Rf'(R))g + \Sigma_a \otimes e^a + e^a \otimes \Sigma_a = T \tag{2.65}$$

where we defined the 1-forms

$$\Sigma_a = *(Di_a * df').$$

This is the main result of this section.

Before we go on to discuss the cosmological solutions to $f(R)$ theories of gravity, we wish to take the trace of the $f(R)$ gravity field equations that will be of use in the coming sections. Trace is obtained by taking the wedge product of e^a with (2.64):

$$\begin{aligned} e^a \wedge f'(R)R^c{}_b \wedge *(e_c \wedge e^b \wedge e_a) &= f'(R)R^c{}_b e^a \wedge i_a * (e_c \wedge e^b) \\ &= (N-1)f'(R)R^c{}_b * (e_c \wedge e^b) \\ &= (N-1)f'(R)R * 1 \ . \end{aligned} \tag{2.66}$$

Since

$$e^a \wedge *e_a = (N+1) * 1 \ , \tag{2.67}$$

we have

$$\begin{aligned} e^a \wedge D(i_a * df') &= De^a \wedge i_a * df'(R) - D(e^a \wedge i_a * df'(R)) \\ &= -D(e^a \wedge i_a * df'(R)) \\ &= -ND * df'(R) \\ &= -Nd * df'(R) \end{aligned} \tag{2.68}$$

where we used the fact that torsion 1-forms vanish. Finally we find the traced equation as

$$(N-1)f'(R)R + (N+1)(f(R) - f'(R)R) + 2N\triangle f'(R) = 0 \ . \tag{2.69}$$

2.2.2 Field Equations with Torsion

In this section we will derive the field equations in cases when the space-time connection has torsion. The variation of the Einstein-Hilbert action with no couplings to matter gives the same set of gravitational field equations in both cases with or without torsion. However, for $f(R)$ gravity action this need not be the case. Here the variations with torsion give a different set of field equations than the variations with zero-torsion as already given in the above subsection.

In the case with non-zero torsion, the independent variations of co-frames and connection forms give

$$df'(R) \wedge *(e_a \wedge e^b) + f'(R)T^c \wedge *(e_a \wedge e^b \wedge e_c) = 0 \tag{2.70}$$

and

$$f'(R)R^c{}_b \wedge *(e_c \wedge e^b \wedge e^a) + (f - f'(R)R) * e_a = 0 \ . \tag{2.71}$$

We can solve (2.70) for the torsion 2-forms as follows:

$$T^a = \frac{df'(R)}{2f'(R)} \wedge e^a \ . \tag{2.72}$$

Although this looks like an explicit expression for the torsion 2-forms, we should remember that the Ricci scalar R appearing in the argument of the function $f(R)$ on the RHS of these equations are obtained by contracting the curvature 2-forms of a connection with torsion. Hence, we have above a highly non-linear system of differential equations for the components of the torsion tensor. Eqn. (2.72) cannot be solved consistently for the torsion 2-forms in general. It should also be remarked that setting the torsion equal to zero in the above field equations (2.71) leads to field equations that are not in general equivalent to field equations (2.64) obtained under the constraint of vanishing torsion given in the previous subsection.

Chapter 3

COSMOLOGICAL SPACETIMES

3.1 FRW Universe

The cosmological principle simply states that the universe is homogeneous and isotropic. Here we use the word universe in the sense of a dynamically evolving 3-dimensional space [1]. Obviously the cosmological principle is valid for large length scales greater than $\sim 10^8$ light-years. This lower limit is determined by the average distance between galaxy clusters. Although the meaning of homogeneity and isotropy is intuitive, let us first give some mathematical definitions required in order to be more rigorous.

Let (M, g) be a Riemannian manifold with a metric g. Throughout this section we will call this manifold as space. Let x^i be a local coordinate system on M. A coordinate transformation $x \to x'$ that leaves the metric form invariant is called an isometry. Since we are interested in infinitesimal isometries, we define Killing vectors first.

A vector field $V \in TM$ satisfying $L_V g = 0$ is called a Killing vector field. V generates an infinitesimal isometry that takes a coordinate point $x \to x + \epsilon v$ such that $\epsilon^2 \sim 0$. It is known that an N-dimensional (pseudo-)Riemannian space (M, g) can admit at most $\frac{N(N+1)}{2}$ Killing vector fields [2]. We call a space maximally symmetric if it admits the maximal number $\frac{N(N+1)}{2}$ of Killing vectors.

In particular, a space is said to be isotropic about a point $p \in M$ if there exists $\frac{N(N-1)}{2}$ Killing vectors that fixes p. Moreover, a space isotropic if it is isotropic about all the points of M.

A space is said to be homogeneous if for every pair of points $p, q \in M$, there exists an isometry carrying p to q.

A space for which the curvature 2-forms are given by $R^a{}_b = Ke^a \wedge e_b$ for some constant K is called a space of constant curvature. A space of constant positive curvature with $K > 0$ is closed. A space of constant negative curvature with $K < 0$ is open. It can be easily shown that the Weyl conformal curvature 2-forms

$$C^a{}_b := R^a{}_b + \frac{R}{(N-2)(N-1)} e^a \wedge e_b + \frac{1}{N-2}(e^a \wedge R_b - e_b \wedge R^a) \tag{3.1}$$

vanish for constant curvature spaces. This implies that spaces of constant curvature are conformally flat [15].

Given an N-dimensional (pseudo-)Riemannian space (M, g), the following statements are equivalent:

(1) M is maximally symmetric,

(2) M has constant curvature,

(3) M is isotropic about a point and homogeneous,

(4) M is isotropic.

The proof can be found in [2].

We will now construct a generic metric for cosmological space-times by making use of constant curvature spaces. Since spaces of constant curvature are conformally flat, there exists an orthogonal basis such that $e^i = \frac{1}{\varphi} dx^i$ where φ is some function of x^i's. In this basis, the connection 1-forms $\omega^i{}_j$ can be solved from the Cartan's first structure equations as

$$\omega^i{}_j = \partial^i \varphi e_j - \partial_j \varphi e^i \ .$$

Then the use of Cartan's second structure equations give

$$R^i{}_j = \varphi(\partial_k \partial^i \varphi e^k \wedge e_j - \partial_k \partial_j \varphi e^k \wedge e_i) - \partial_k \varphi \partial^k \varphi e^i \wedge e_j \ .$$

By the definition of constant curvature spaces we have $R^i{}_j = k e_i \wedge e^j$ and therefore the second order partial derivatives $\partial_i \partial_j \varphi$ vanish for all $i \neq j$. Thus ϕ is of the form $\sum_{i=1}^m f_i(x_i)$ and $f_j''(x_j) + f_i''(x_i) = constant = \varphi^{-1}(k + f_i' f_j' \delta^{ij})$. Consequently the metric for this 3-space is given by

$$g = \frac{\delta_{ij} dx^i \otimes dx^j}{(1 + \frac{k}{4} r^2)^2} \tag{3.2}$$

where $r^2 = x^i x^j \delta_{ij}$.

Now we are ready to give the form of a generic metric for cosmological space-times. In other words we write a metric for an $(N+1)$-dimensional space-time with a maximally symmetric N-dimensional submanifold which describes the space. This metric is called the Robertson-Walker metric and written as

$$g = -dt^2 + a^2(t)\left(\frac{dx^i dx_i}{(1+\frac{k}{4}r^2)^2}\right) \tag{3.3}$$

in terms of a cosmic time variable t. The space indices $i, j = 1, .., N$ in isotropic coordinates. The curvature index can take three values $k = 0, \pm 1$ and for each we have a different cosmological model. We call the function $a(t)$ as the expansion function of the universe that will be determined by the field equations. We introduce the following set of orthonormal co-frames:

$$e^0 = dt \quad , \quad e^i = a(t)\frac{dx^i}{(1+\frac{k}{4}r^2)}.$$

We will determine the corresponding curvature 2-forms in order to solve the Einstein field equations for this metric. The connection 1-forms are found from (2.9) as

$$\omega^0{}_i = \omega^i{}_0 = \frac{\dot{a}}{a}e^i \quad , \quad \omega^i{}_j = -\omega^j{}_i = \frac{k}{2a}(x^i e^j - x^j e^i) \ .$$

Then the curvature 2-forms are as follows :

$$R^0{}_i = -R^i{}_0 = \frac{\ddot{a}}{a}e^0 \wedge e_i \quad , \quad R^i{}_j = R^j{}_i = \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] e^i \wedge e_j \ . \tag{3.4}$$

The corresponding Ricci scalar is

$$R = 2N\frac{\ddot{a}}{a} + N(N-1)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \ . \tag{3.5}$$

In particular, for $N = 3$ we have

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{3.6}$$

There are two parameters used for comparison with observational data. The first one

$$H(t) \equiv \frac{\dot{a}}{a}$$

is called the Hubble parameter which is the measure of the expansion rate of the universe. The current value of the Hubble parameter is $H_0 = 72 \pm 8$ (km/s)/Megaparsec [3]. The other parameter

$$q(t) \equiv \frac{a\ddot{a}}{\dot{a}^2} \quad .$$

measures the rate of acceleration of the expansion of the universe.

3.1.1 Energy-Momentum Tensor of Matter

In General Relativity, we assume cosmological space-times to be filled with a uniform distribution of matter that can be modelled by a perfect fluid. Therefore we describe the RHS of the Einstein field equations with matter in terms of the energy-momentum tensor of a perfect fluid given as

$$T = (\rho + p)\tilde{U} \otimes \tilde{U} + pg = T_{ab} e^a \otimes e^b \tag{3.7}$$

where U is a unit time-like velocity vector field of an inertial observer co-moving with the fluid. Note that $g(U, U) = -1$.
Then in component form

$$T_{ab} \; = (\rho + p)\, U_a U_b + p\eta_{ab} \tag{3.8}$$

where $U_a = \tilde{U}(X_a)$ and $\eta_{ab} = g(X_a, X_b)$. It is sometimes convenient to write the fluid energy-momentum tensor in matrix form as

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} .$$

Here ρ denotes the energy density and p denotes the pressure of the fluid . ρ is always positive and p may have either sign. The perfect fluid will further be specified by an equation of state $p = p(\rho) = w\rho$ where w is a real constant. In particular if $w = 0$ we will have a dust filled universe. Then it is said that the universe is matter dominated. On the other hand if $w = \frac{1}{3}$ then the trace of the energy-momentum tensor vanishes. Then it is said the universe is radiation dominated.

3.1.2 Friedmann Equations

Now we are ready to obtain Friedmann equations by inserting the Robertson-Walker metric into the Einstein field equations (2.20). We work in 4-dimensions so that $N = 3$ from now on. Let us substitute the curvature 2-forms (3.4) and the energy-momentum tensor (3.8) in the Einstein field equations (2.20). The resulting ordinary differential equations

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = \kappa^2 \rho \tag{3.9}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -\kappa^2 p \tag{3.10}$$

are called the Friedmann equations. These are to be supplemented by an equation of state

$$p = p(\rho). \tag{3.11}$$

There are two simple results that follow from the Friedmann equations:

$$\kappa^2(\rho + 3p) = -6\frac{\ddot{a}}{a} \tag{3.12}$$

and

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{3.13}$$

Therefore in a dust filled universe ($p = 0$) we will have

$$\kappa^2 \rho = -6\frac{\ddot{a}}{a}$$

so that the expansion of the universe should be decelerating: $q(t) < 0$. Next we may integrate

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0$$

to get

$$a^3 \rho = M : constant$$

that is (up to factors that fix the units) nothing but the total mass of the universe.

In order to be consistent with the observations that the universe is expanding at an accelerated rate, we have to justify the condition $\ddot{a} > 0$. In the light of the acceleration equation (3.12) we can achieve this by taking

$$\rho + 3p < 0\ . \tag{3.14}$$

Since we always assume energy density is positive , the corresponding pressure turns out to be negative. This means that we must be considering an unusual fluid with negative pressure to describe the extra energy that is needed to explain the observed accelerated expansion of the universe. The simplest possibility is to introduce a dark energy provided by a cosmological constant which corresponds to constant ρ and constant p with an equation of state parameter $w = -1$. We take here an alternative approach that consists of modifying the LHS of the Einstein field equations instead of the energy-momentum tensor. This approach is called the Dark Gravity in the literature [10].

3.2 Modified Friedmann Equations

We have seen above the Friedmann equations obtained from the Einstein field equations of General Relativity. Now we are ready to modify these using $f(R)$ gravity field equations. The basic motivation for such a modification is the dark energy problem mentioned in the introduction. By such a modification we expect to give an alternative explanation of the unknown physics of the so called dark energy.

Let us first start with the most general case. We substitute the Robertson-Walker metric (3.3) into the $f(R)$ gravity field equations (2.65) and by using (3.6) we obtain

$$-6F\left((\frac{\dot{a}}{a})^2+\frac{k}{a^2}\right)-6\frac{\dot{a}}{a}\dot{F}-(f-f'(R)R)=\rho \tag{3.15}$$

and

$$2F\left(2\frac{\ddot{a}}{a}+(\frac{\dot{a}}{a})^2+\frac{k}{a^2}\right)+\left(2\ddot{F}+4\frac{\dot{a}}{a}\dot{F}\right)+(f-f'(R)R)=p \tag{3.16}$$

where we set $f'(R)\equiv F(t)$. We will call these the modified Friedmann equations. It can be easily checked by taking $f(R)=-\frac{1}{2\kappa^2}R$ that they reduce to the Friedmann equations (3.9) and (3.10), respectively, as it should be. We supplement the equations (3.15) (3.16) by an equation of state

$$p=p(\rho). \tag{3.17}$$

In order to proceed further at this point, we consider a specific function $f(R)$ given below and ask whether it provides a viable solution to the dark energy problem:

$$f(R)=-\frac{1}{2\kappa^2}R+\frac{\alpha}{\kappa^6 R} \tag{3.18}$$

where α is a dimensionless, real coupling constant. We now solve the field equations (3.15) and (3.16) in a source-free space-time with $\rho=0$ and $p=0$. We further simplify the problem by considering the case $k=0$. Then the coupled equations to be solved become

$$6F(\frac{\dot{a}}{a})^2+6\frac{\dot{a}}{a}\dot{F}+\frac{2\alpha}{\kappa^6 R}=0 \tag{3.19}$$

and

$$4F\frac{\ddot{a}}{a}+2F(\frac{\dot{a}}{a})^2+2\ddot{F}+4\frac{\dot{a}}{a}\dot{F}+\frac{2\alpha}{\kappa^6 R}=0. \tag{3.20}$$

We will look for inflationary solutions of the form

$$a(t)=a_0e^{\gamma t}$$

where the exponent γ is to be determined by the modified Friedmann equations above. We obtain

$$\frac{\ddot{a}}{a} = (\frac{\dot{a}}{a})^2 = \gamma^2$$

and furthermore the curvature scalar is a constant:

$$R = 12\gamma^2.$$

Therefore both (3.19) and (3.20) are satisfied provided

$$6\gamma^2 \left(\frac{1}{2\kappa^2} + \frac{\alpha}{144\kappa^6 R} \right) = \frac{\alpha}{6\kappa^6 R}. \tag{3.21}$$

We should take γ as a positive real root of the quartic equation

$$6\gamma^4 + \frac{\alpha}{144\kappa^4}\gamma^2 = 4\frac{\alpha}{144\kappa^4}$$

to get a solution that describes an exponentially expanding universe. This simple solution backs up the Dark Gravity approach by providing a model for an exponentially expanding flat universe without introducing a cosmological constant.

Chapter 4

SOME POSSIBLE GENERALIZATIONS OF $f(R)$ THEORIES OF GRAVITY

f(R) theory is not the only way to modify general relativity. There are also other attempts to find a more general theory of gravity. Although they may serve for different purposes, the common property of these theories is that they are designed to overcome some shortcomings of GR.

4.1 Scalar-tensor $f(\phi, R)$ theories of gravity

A scalar-tensor theory of gravity was introduced by Brans and Dicke in 1961 [16]. It is described by the action

$$S_{BD} = \int \left(\frac{1}{2}\phi R^a{}_b \wedge *(e_a \wedge e^b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi \right) \tag{4.1}$$

where ϕ is a real scalar field and ω is a real parameter. The origin of the idea of coupling a scalar field to the Einstein-Hilbert action comes from the dimensionful gravitational constant. In Brans-Dicke theory the gravitational coupling constant is treated as a local variable. That is, $\frac{1}{\kappa^2}$ is replaced by a scalar field ϕ [17].

The above action is said to be written in Jordan frame because the scalar field ϕ is non-minimally coupled to curvature scalar. On the other hand it is possible by using conformal scalings of the metric to write the above action in the Einstein frame in which the scalar field couples minimally. Here we consider a scalar through a scalar-tensor modification of the Brans-Dicke action by changing the kinetic term $\frac{1}{2}\phi R$ by $f(\phi, R)$ that is assumed to be given by

$$f(\phi, R) = \sum_{m,n \in Z} \alpha_{mn} \phi^m R^n \ . \tag{4.2}$$

Therefore we write a new Lagrangian density appearing in the action as

$$L_\phi = f(\phi, R) * 1 - \frac{\omega}{2\phi} d\phi \wedge *d\phi \ . \tag{4.3}$$

This form of the Lagrangian is said to be written in the Jordan frame in which the scalar field is non-minimally coupled. We assume again that the connection is metric compatible and torsion free. In order to motivate the choice of a function $f(\phi, R)$, we will consider the conformal scale transformations of the metric

$$g \rightarrow \tilde{g} = e^{2\sigma} g \ . \tag{4.4}$$

So the coframes transform as

$$e^a \rightarrow \tilde{e}^a = e^{\sigma} e^a \ . \tag{4.5}$$

Since we want the transformed connection to remain torsion-free we make use of the structure equation

$$d\tilde{e}^a + \tilde{\omega}^a{}_b \wedge \tilde{e}^a = 0 \ , \tag{4.6}$$

and assume the transformation law

$$\tilde{\omega}^a{}_b = \omega^a{}_b + \Omega^a{}_b \ . \tag{4.7}$$

Then

$$\begin{aligned} de^{\sigma} e^a + (\omega^a{}_b + \Omega^a{}_b) \wedge e^{\sigma} e^b &= 0 \\ e^{\sigma} d\sigma \wedge e^a e^{\sigma} d\sigma \wedge e^a + e^{\sigma}(de^a + \omega^a{}_b \wedge e^b) + \Omega^a{}_b \wedge e^{\sigma} e^b &= 0 \\ d\sigma \wedge e^a + \Omega^a{}_b \wedge e^b &= 0 \end{aligned} \tag{4.8}$$

Now we can solve the above equation to obtain the $\Omega^a{}_b$'s. We take the interior product of both sides twice and obtain

$$\Omega^a{}_b = (i_b d\sigma) e^a - (i_a d\sigma) e_b \tag{4.9}$$

Therefore

$$\omega^a{}_b \rightarrow \tilde{\omega}^a{}_b = \omega^a{}_b + (i_b d\sigma) e^a - (i_a d\sigma) e_b \tag{4.10}$$

Moreover by plugging the transformed connection 1-forms into the second Cartan structure equations we find that the curvature 2-forms transform according to

$$R^a{}_b \rightarrow \tilde{R}^a{}_b = R^a{}_b + D\Omega^a{}_b + \Omega^a{}_c \wedge \Omega^c{}_b \tag{4.11}$$

Finally we transform the Hodge star operator. Let α be a p-form which can be written as $\alpha_{a_1 a_2 \ldots a_p} e^{a_1} \wedge e^{a_2} \ldots \wedge e^{a_p}$. Then

$$\begin{aligned}
*\alpha &= \alpha_{a_1 \ldots a_p} \frac{1}{(N+1-p)!} \epsilon^{a_1 .. a_p}{}_{a_{p+1} \ldots a_{N+1}} * (e^{a_{p+1}} \ldots e^{a_{N+1}}) && (4.12)\\
&= e^{-(N+1-p)\sigma} \alpha_{a_1 \ldots a_p} \frac{1}{(N+1-p)!} \epsilon^{a_1 .. a_p}{}_{a_{p+1} \ldots a_{N+1}} \tilde{*}(e^{a_{p+1}} \ldots \tilde{e}^{a_{N+1}}) && (4.13)\\
&= e^{-(N+1-p)\sigma} \alpha_{a_1 \ldots a_p} \frac{1}{(N+1-p)!} \epsilon^{a_1 .. a_p}_{a_{p+1} \ldots a_{N+1}} * (\tilde{e}^{a_{p+1}} \ldots \tilde{e}^{a_{N+1}}) && (4.14)\\
& && (4.15)
\end{aligned}$$

Hence if $\alpha \rightarrow \tilde{\alpha} = e^{l\sigma} \alpha$ then

$$*\alpha \rightarrow \tilde{*}\tilde{\alpha} = e^{(-(N+1)+2p-l)\sigma} \alpha \qquad (4.16)$$

Now we have enough tools to transform the whole action. We start with the most simple term of the function $f(\phi, R)$ which is also the common term with Brans-Dicke action.

$$\phi R * 1 = \phi R^a_b \wedge *(e_a \wedge e^b) = \phi(\tilde{R}^a{}_b - D\Omega^a{}_b - \Omega^a{}_c \wedge \Omega^c{}_b) \wedge e^{(1-N)\sigma} \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b) \qquad (4.17)$$

We can get rid of the ϕ in the term $\phi R * 1$ by choosing $\phi = e^{(N-1)\sigma}$. In addition to that if we take σ as a constant the terms $\Omega^a{}_b$ vanish. Therefore we may choose the following globally scale invariant combination

$$\begin{aligned}
f(\phi, R) &= \sum_{m+n=2} \alpha_{mn} \phi^m R^n \\
&= \ldots + b_{-2} \frac{\phi^4}{R^2} + b_{-1} \frac{\phi^3}{R} + \Lambda \phi^2 + \phi R + b_2 R^2 + \ldots . && (4.18)
\end{aligned}$$

4.1.1 Field equations:

In order to find source and torsion free field equations we impose again the condition that torsion is zero to the action. The variation of the Lagrangian density gives

$$\delta(L_\Phi + T^a \wedge \lambda_a) = \delta\phi \frac{\partial f}{\partial \phi} * 1 + \frac{\partial f}{\partial R} \delta R * 1 + f \delta * 1 - \frac{\omega}{2} \delta \left(d\phi \wedge * \frac{d\phi}{\phi} \right) + \delta T^a \wedge \lambda_a + T^a \wedge \delta \lambda_a \ .$$

We have calculated most of the terms in Chapter 2. We calculate the remaining terms here:

$$\begin{aligned}
\delta \left(d\phi \wedge * \frac{d\phi}{\phi} \right) &= \delta d\phi \wedge * \frac{d\phi}{\phi} + d\phi \wedge * \frac{\delta d\phi}{\phi} - \frac{\delta \phi}{\phi^2} d\phi \wedge * d\phi \\
&\quad - \frac{1}{\phi} \delta e^a \wedge (i_a d\phi \wedge * d\phi + d\phi \wedge i_a * d\phi) \ . && (4.19)
\end{aligned}$$

Using the property $d\delta\phi \wedge *\frac{d\phi}{\phi} = d\phi \wedge *\frac{d\delta\phi}{\phi}$ and the Leibniz rule

$$d\left(\delta\phi * \frac{d\phi}{\phi}\right) = d\delta\phi \wedge *\frac{d\phi}{\phi} + \delta\phi d * \left(\frac{d\phi}{\phi}\right)$$

we get

$$\delta\left(d\phi \wedge *\frac{d\phi}{\phi}\right) = 2\left(d(\delta\phi * \frac{d\phi}{\phi}) - \delta\phi d(*\frac{d\phi}{\phi})\right) - \frac{\delta\phi}{\phi^2} d\phi \wedge *d\phi$$
$$-\frac{1}{\phi}\delta e^a \wedge (i_a d\phi \wedge *d\phi + d\phi \wedge i_a * d\phi) \ . \tag{4.20}$$

Therefore the variational field equations are

$$\frac{\partial f}{\partial \phi} * 1 + \omega d(*\frac{d\phi}{\phi}) - \frac{\omega}{\phi^2} d\phi \wedge *d\phi = 0 \tag{4.21}$$
$$\frac{\partial f}{\partial R} R^c{}_b \wedge *(e_c \wedge e^b \wedge e_a) - \frac{\partial f}{\partial R} R * e_a + f(R) * e_a + D\lambda_a = 0 \tag{4.22}$$
$$\frac{\partial f}{\partial R} * (e^a \wedge e^b) + f'(R) D * (e^a \wedge e^b) + \frac{1}{2}\left(e^b \wedge \lambda^a - e^a \wedge \lambda^b\right). \tag{4.23}$$

We solve (4.23) for multipliers λ_a and obtain

$$\lambda_a = 2 D i_a * d\left(\frac{\partial f}{\partial R}\right). \tag{4.24}$$

The final field equations are

$$\frac{\partial f}{\partial \phi} * 1 + \omega d(*\frac{d\phi}{\phi}) - \frac{\omega}{\phi^2} d\phi \wedge *d\phi = 0 \ , \tag{4.25}$$
$$\frac{\partial f}{\partial R} R^c{}_b \wedge *(e_c \wedge e^b \wedge e_a) - \frac{\partial f}{\partial R} R * e_a + f(R) * e_a + 2D(i_a * d\left(\frac{\partial f}{\partial R}\right)) = 0 \ . \tag{4.26}$$

4.2 f(G) Theories of Gravity

In Chapter 2 we added to the usual Einstein-Hilbert action higher order curvature scalars and obtained a different theory. There are also other invariants that can be used to modify the action called the Euler-Poincaré densities

$$R_{ab} \wedge *e^{ab} \ , \tag{4.27}$$

$$R_{ab} \wedge R_{cd} \wedge *e^{abcd} \ , \tag{4.28}$$

$$R_{a_1a_2} \wedge R_{a_3a_4} ... R_{a_{n-1}a_n} \wedge *e^{a_1...a_n} \tag{4.29}$$

where we denoted $e^{a_1} \wedge ... \wedge e^{a_k}$ by $e^{a_1a_2...a_k}$ for any integer k. The above forms are closed in 2-dimensions, 4-dimensions and n-dimensions respectively. Actually the first Euler-Poincaé

density is also called the Gauss-Bonnet density. The most interesting property of the Euler-Poincaré densities is that variational equations that follow from the zero torsion constrained variations of all them give second order differential equations [19], [20].
We now define now the Gauss-Bonnet scalar as

$$G * 1 = R_{ab} \wedge R_{cd} * e^{abcd} \tag{4.30}$$

and take an arbitrary function of G in the action as our the Langrangian density [18].

$$L_G = f(G) * 1 \tag{4.31}$$

The variation of L_G gives

$$\begin{aligned} \delta L_G &= f'(G)\delta(G) * 1 + f(G)\delta * 1 \\ &= f'(G)\delta(G * 1) - f'(G)G\delta * 1 + f(G)\delta * 1 \ . \end{aligned} \tag{4.32}$$

Since

$$\begin{aligned} f'(G)\delta(G * 1) &= 2f'(G)\delta R_{ab} \wedge R_{cd} \wedge *e^{abcd} + f'(G)\delta e^a \wedge R^{bc} \wedge R^{df} \wedge *e_{abcdf} \ , \\ &= 2\delta\omega^{ab} \wedge df'(G) \wedge R^{cd} \wedge *e_{abcd} + f'(G)\delta e^a \wedge R^{bc} \wedge R^{df} \wedge *e_{abcdf} \end{aligned} \tag{4.33}$$

with the usage of the Bianchi identity $DR_{ab} = 0$, the variations with respect to coframes and connections results in

$$\begin{aligned} \delta L_G \quad &= \delta e^a \wedge \{(f(G) - f'(G)G) * e_a + f'(G)R^{bc} \wedge R^{df} \wedge *e_{abcdf}\} \\ &+\delta\omega^{ab}\{2df'(G) \wedge R^{cd} \wedge *e_{abcd}\} \ . \end{aligned} \tag{4.34}$$

Since L_G is a modification of a theory of gravity we suppose we are given L_{main} as the main Lagrangian density. Again we do the calculations in a torsion free space-time therefore we write the total Lagrangian density as

$$L_{total} = L_{main} + L_G + T^a \wedge \lambda_a \ . \tag{4.35}$$

As a result we obtain three equations:

$$(f(G) - f'(G)G) * e_a + f'(G)R^{bc} \wedge R^{df} \wedge *e_{abcdf} + \frac{\delta L_{main}}{\delta e^a} + D\lambda_a = 0 \tag{4.36}$$

$$2df'(G) \wedge R^{cd} \wedge *e_{abcd} + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b) + \frac{\delta L_{main}}{\delta \omega^{ab}} = 0 \tag{4.37}$$

$$T^a = 0 \tag{4.38}$$

We can solve multipliers λ_a using the equation (4.37) as

$$\lambda_b = 4i^a(df')R^{cd} * e_{abcd} + -4df' i^a(R^{cd}) * e_{abcd} + 2i^a \left(\frac{\delta L_{main}}{\delta \omega^{ab}}\right) - \Gamma \wedge e_b \tag{4.39}$$

where Γ is given by

$$\Gamma = df' i^b i^a(R^{cd}) \wedge *e_{abcd} + \frac{1}{2} i^b i^a \left(\frac{\delta L_{main}}{\delta \omega^{ab}}\right) \tag{4.40}$$

In order to find field equations we have to give the total Lagrangian modified by L_G . Since we do not want to restrict ourselves to a specific action we don't go further at this point.

Chapter 5

CONCLUSION

In this thesis, we derive the variational field equations for $f(R)$ theories of gravity. We briefly discussed the FRW universes and obtain a solution for the expansion function of the universe for a specific function $f(R)$. We found an exponential solution which gives the Hubble parameter as a real constant. In Chapter 3 we presented some other generalized dark gravity actions. Especially for the $f(\phi, R)$ case we gave the necessary form of the action in order to obtain a globally scale invariant theory.

Naturally we were not able to cover all the details of the theories of dark gravity. Since recently the interest on the modified theories of gravity has increased , there is a wide literature written on this subject. However, it is not as complete and reliable as GR yet. For example, some forms of the function $f(R)$ suffers from the instabilities as discussed in [21]. The further investigation on the viability of f(R) would be worthwhile to study. Also for the Gauss-Bonnet gravity the Kaluza-Klein reduction of the theory would be considered for further study.

Appendix A

EXTERIOR ALGEBRA

In this section we will review some part of the Exterior algebra over the space-time. For further details see [2] and [15].

Let M be a n-dimensional differentiable manifold. For a point $p \in M$ $T_p(M)$ denotes the tangent space and $T_p^*(M)$ denotes the cotangent space at p. If $x^1, ..., x^n$ is a local coordinate system in a neighborhood of p then the coordinate differentials $\{dx^1, ..., dx^n\}$ form a basis for $T_p^*(M)$ Any element ω of $T_p^*(M)$ is called a 1-form and can be written as

$$\omega = \sum_i f_i dx^i$$

in a unique way . Here $\{f_i\}$ are scalar functions defined in the neighborhood of p. The corresponding basis for the dual vector space $T_p(M)$ is given by $\{\partial_1, ...\partial_n\}$ where we write $\partial_i = \frac{\partial}{\partial x^i}$, i=1,2...n.

We know that a basis for a vector space is not unique so we can choose different bases. $\{X_a = E^b{}_a \partial_b\}$ is said to be the non coordinate basis for $T_p(M)$ and the elements are called frame vectors. $\{e^a = e^a{}_b dx^b\}$ is said to be non coordinate basis for $T_p^*(M)$ and the elements are called coframe 1-forms. Moreover they satisfy the property $e^a(X_b) = \delta^a{}_b$. A tensor of type (r, s) defined at point $p \in M$ is a multilinear map

$$\underbrace{T_p^*(M) \times T_p^*(M) \times ... \times T_p^*(M)}_{s\ times} \times \underbrace{T_p(M) \times ... \times T_p(M)}_{s\ times} \to \mathbb{R} \ .$$

Such a tensor is said to be a contravariant of degree r and covariant of degree s. If we assign a multilinear map at each point of M then we have a tensor field.

A p-form ω on a manifold M is a completely antisymmetric tensor field of type $(0, p)$ and can be written uniquely in terms of the basis 1- forms of $T^*(M)$ as

$$\omega = \frac{1}{p!} \omega_{i_1 i_2 ... i_p} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_p} \tag{A.1}$$

where

$$dx^{i_1} \wedge dx^{i_2} ... \wedge dxi_p = \sum_{P \in S_p} sgn(P) dx^{i_{P(1)}} \otimes dx^{i_{P(2)}} ... \otimes dx^{i_{P(p)}}$$

and the exterior(wedge) product of a p-form α and q-form β is defined as

$$(\alpha \wedge \beta)(X_1, ..., X_{p+q}) = \frac{1}{p!q!} \sum_{P \in S_{p+q}} sgn(P) \alpha(X_P(1) ... X_P(p)) \beta(X_{P(p+1)}, ..., X_{P(p+q)}) \quad \text{(A.2)}$$

where $X_i \in T(M)$. We note that if $(p+q) > n$ for a n-dimensional manifold M then the resulting form is zero.

Properties of the wedge product:

For $\alpha \in \Lambda^p(M)$, $\beta \in \Lambda^q(M)$ and $\gamma \in \Lambda^s(M)$ wedge product is
(i) associative

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \ ,$$

(ii) graded commutative

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \ ,$$

(iii) bilinear

$$(a\alpha + b\beta) \wedge \gamma = a(\alpha \wedge \gamma) + b(\beta \wedge \gamma) \ for \ a, b \in \mathbb{R} \ .$$

If we denote the vector space of p-forms by $\Lambda^p(T^*M)$ and consider the wedge product of the forms above, then the exterior algebra on M is the direct sum

$$\Lambda(T^*(M)) = \Lambda^0(T^*(M)) \oplus \Lambda^1(T^*M) \oplus \Lambda^2(T^*M) \oplus \cdots \oplus \Lambda^n(T^*M) \ .$$

It is not difficult to show that the dimension of $\Lambda(T^*M)$ is 2^n.

Exterior Derivative

Exterior derivative of a differentiable p-form on M is a map $d : \Lambda^p(T^*M) \to \Lambda^{p+1}(T^*M)$ satisfying

(i) Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ where α is a p-form and β is a q-form.
(ii) Poincaré lemma: $d^2\alpha = 0$

For a p-form ω given in (A.1)

$$d\omega = \frac{1}{p!}\frac{\partial}{\partial x^a}\omega_{i_1 i_2 \ldots i_p} dx^a \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \tag{A.3}$$

A p-form α is said to a closed form if $d\alpha = 0$ and it is said to be exact if there exists a (p-1)-form such that $\alpha = d\beta$. It follows from the property (i) that every exact form is closed.

Interior Product(Contractions)

For a given vector field $X \in TM$, the interior product with respect to X is a linear map $i_X : \Lambda^p(T^*M) \rightarrow \Lambda^{p-1}(T^*M)$ such that for a p-form ω given in (A.1)

$$i_X\omega = \frac{1}{p!}\sum_{k=1}^{p} X^{i_k}\omega_{i_1 \ldots i_k \ldots i_p} dx^{i_1} \ldots \wedge dx^{i_{k-1}} \wedge dx^{i_{k+1}} \wedge \ldots \wedge dx^{i_p}$$

If X_a is a frame vector, we sometimes write i_a instead of i_{X_a}. The interior product satisfies
(i) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X\beta$ for a p-form α and a q-form β .
(ii) $(i_X)^2\alpha = 0$
Both the interior product and exterior derivative satisfies the properties of a graded derivation. In fact they are called antiderivations.

Lie Derivative

Lie derivative of a form ω with respect to a vector field X on M is a type-preserving map which can be simply given by the Cartan formula

$$L_X\omega = di_X\omega + i_X d\omega$$

Hodge star map

Given a metric $g = \eta_{\alpha\beta}e^\alpha \otimes e^\beta$ as a second rank, covariant, symmetric, non-degenerate tensor field on a n-dimensional manifold M, the Hodge star map is a linear bijection $* : \Lambda^p(T^*M) \rightarrow \Lambda^{n-p}(T^*M)$ such that for ω given in (A.1)

$$*\omega = \frac{1}{p!(n-p)!}\omega_{i_1 \ldots i_p}\varepsilon^{i_1 \ldots i_p}{}_{i_{p+1} \ldots i_n} e^{p+1} \wedge \ldots \wedge e^n$$

where ε is called the Levi-Civita symbol and defined as

$$\epsilon_{i_1...i_n} = \begin{cases} +1 & \text{if } (i_1...i_n) \text{ is an even permutation} \\ -1 & \text{if } (i_1...i_n) \text{ is an odd permutation} \\ 0 & \text{if any two indices are the same} \end{cases} \quad . \tag{A.4}$$

Let M be a manifold with a metric g on it and ω be a p-form then

(i) $**\omega = \begin{cases} (-1)^{p(n-p)}\omega \ , & \text{if (M,g) is Riemannian} \\ (-1)^{1+p(n-p)}\omega \ , & \text{if (M,g) is Lorentzian ,} \end{cases}$

(ii)$*1 = \frac{1}{n!}\varepsilon_{abc...}e^a \wedge ...e^n$,

(iii) For any p-forms α and β , $\alpha \wedge *\beta = \beta \wedge *\alpha$.

The Covariant Laplace-Beltrami operator

The Laplace- Beltrami operator $\triangle$ is a type preserving linear second order differential operator such that for a p-form ω

$$\triangle\omega = *d * d\omega + d * d * \omega \ .$$

Integration

Let M be an n-dimensional orientable manifold. Suppose there exists a n-form ω on M called the volume form which vanishes nowhere. We can write $\omega = \mu dx^1 \wedge ... \wedge dx^n$ in a neighbourhood U of p for the coordinate chart (U, φ) with $x = \varphi(p)$. Then for a smooth function $f \in C^\infty(M)$;

$$\int_U f\omega = \int_{\varphi(U_i)} f(\varphi^{-1}(x))\mu(\varphi^{-1}(x))dx^1 \wedge ... \wedge dx^n \tag{A.5}$$

Let $\{U_i\}$ be a covering of M and $\{\epsilon_i\}$ be a partition of unity subordinate to U_i. Then we can define the integral of f on the manifold as

$$\int_M f\omega = \sum_i \int_{U_i} f_i\omega \tag{A.6}$$

where $f_i(p) = \epsilon_i(p)f(p)$ for a point $p \in U_i$.

Stokes' Theorem: Let ω be an (n-1) form on M with a compact support. Then

$$\int_M d\omega = \oint_{\partial M} \omega$$

where ∂M denotes the boundary of M.

BIBLIOGRAPHY

[1] Rindler, Wolfgang, Relativity , *Oxford University Press*, New York (2006).

[2] Thirring, Walter, Classical Mathematical Physics, *Springer*, New York(1997).

[3] Freedman W. L. *et al.*(2001), Final Results from the Hubble Space Telescope Key Project to Measure the Hubble Constant, *Astrophys. J.* 553: 47.

[4] Carroll, Sean M. *et al.*(2004),Is cosmic speed-up due to new gravitational physics?, *Phys.Rev. D.* 70: 043528.

[5] Capozziello, S. and Francaviglia, M.(2007), Extended theories of gravity and their cosmological and astrophysical applications, *Gen. Relativ. Gravit.* 40: 357-420.

[6] Alam, U., Sahni, V. and Starobinsky A. A.(2004), The case for dark energy revisited, *JCAP* 06: 008.

[7] Uzan, Jean-Philippe (2007), The acceleration of the universe and the physics behind it, *Gen. Relativ. Gravit.* 39: 307-342.

[8] Sahni V., Starobinsky(2000), The case for a positive cosmological *Lambda*-term *Int. J. Mod. Phys. D* 9: 373443.

[9] Starobinsky, A. A.(2007), Dissappering Cosmological Constant in f(R) Gravity, *JETP Letters.* 86: 157-163.

[10] Durrer, Ruth and Maartens, Roy (2008),Dark energy and dark gravity: theory overview, *Gen. Relativ. Gravit.* 40: 301-328.

[11] Faraoni, V. and Sotiriou, T.P.(2008), $f(R)$ theories of gravity, arXiv:0805.1726v3.

[12] Woodard, R.P.(2007), Avoiding dark energy with 1/R modifications of gravity, *Lect.Notes Phys.* 720: 403-433.

[13] Nojiri, S. and Odintsov, S.(2003), Modified gravity with positive and negative powers of curvature: Unification of inflation and cosmic acceleration, *Phys. Rev.D.* 68: 123512.

[14] Sotiriou, Thomas P.(2009), $f(R)$ gravity, torsion and non-metricity, *Class. Quantum Grav.* 26: 152001.

[15] Nakahara, Mikio , Geometry, Topology and Physics, *IoP*, London(2003).

[16] Brans, Carl(1967), The roots of scalar-tensor theory: an approximate history, arxiv: gr-gc/050663.

[17] Dereli, T. and Tucker, R.W.(1982), Weyl scalings and spinor matter interactions in scalar-tensor theories of gravitation, *Phys. Lett. B* 110: 206.

[18] Nojiri, S., Odintsov, S. D.and Tretyakov P.V.(2007), Dark energy from modified $f(R)$-scalar-Gauss-Bonnet gravity *Phys.Lett.B*, 651: 224-231.

[19] Arık, M. and Dereli, T.(1987), Euler-Poincare Lagrangians and Kaluza-Klein theory, *Phys. Lett. B* 189: 96.

[20] Arık, M. and Dereli, T. (1989), Euler-Form Actions and the Vanishing of the Cosmological Constant, *Phys. Rev. Lett.* 62: 5.

[21] Dolgov, A.D. and Kawasaki, M. (2003), Can modified gravity explain accelerated cosmic expansion ? , *Phys.Lett. B* 573: 1.