ON THE NEWSVENDOR PROBLEM WITH MULTIPLE INPUTS UNDER A CARBON EMISSION CONSTRAINT

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By

Sibel Sözüer

September, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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September, 2012

In this thesis, we consider two problems in the newsvendor setting with multiple inputs, under a carbon emission constraint and non-linear production functions. In the first problem, we assume a strict carbon cap and find the optimal production quantity and input allocation that will maximize the expected profit under this constraint. In the second problem, we consider an emission trading scheme where an advance purchase of carbon emission permits is made at an initial price before the random demand is realized. When the demand is realized and new carbon trade prices are revealed, it is possible to buy additional permits or to sell an excess amount. The aim is to decide on the optimal allocation of the inputs as well as the carbon trading policy so as to maximize the expected profit. In both problems, the production quantity is linked to multiple inputs via the Cobb-Douglas and Leontief production functions. Optimal policy structures are derived and numerical examples are provided.

Keywords: Production, Newsvendor Problem, Inventory Management, Production Function, Multiple Inputs, Carbon Emission, Carbon Cap, Emission Permit Trading, Cobb-Douglas, Leontief.

ÖZET

KARBON EMİSYON KISITI ALTINDA BİRDEN FAZLA GİRDİNİN OLDUĞU GAZETECİ ÇOCUK PROBLEMİ

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Bu çalışmada, karbon kısıtı ve lineer olmayan üretim foksiyonları altında birden fazla girdinin olduğu gazeteci çocuk modeli iki problemde ele alınmıştır. İlk problemde katı bir karbon kotasının olduğu varsayılmış ve bu kısıt altında beklenen karı en iyileyecek üretim miktarı ve girdi dağılımı bulunmuştur. İkinci problemde ise talep belli olmadan önce karbon emisyon permilerinin erkenden alınmasına da olanak tanıyan bir emisyon ticareti mekanizması ele alınmıştır. Talep belli olduktan ve yeni karbon borsası fiyatları açıklandıktan sonra da permi alım-satımı yapmak mümkündür. Amaç beklenen karı en iyileyecek girdi dağılımına ve karbon permi ticareti politikasına karar vermektir. Her iki problemde de üretim miktarı, girdiler ile Cobb-Douglas ve Leontief üretim fonksiyonları aracılığıyla ilişkilendirilmiştir. Politikaların analitik yapısı bulunmuş ve sayısal örnekler verilmiştir.

Anahtar sözcükler: Üretim, Gazeteci Çocuk Problemi, Envanter Yönetimi, Üretim Fonksiyonları, Birden Fazla Girdi, Karbon Emisyonu, Karbon Kotası, Emisyon Permi Borsası, Cobb-Douglas, Leontief .

To my dear sister, father and mother . . .

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Chapter 1

Introduction

Global warming is one of the most critical environmental problems. If not intervined immediately, it may accelerate climate change and eventually lead to a posibble environmental disaster. Ever since the Industrial Revolution, the consumption of fossil fuels is increasing continuously that causes higher emission of greenhouse gases (GHG), which is known to be the main reason for global warming. Of these greenhouse gases, Carbon Dioxide has the most significant affect on global warming. While its concentration in the atmosphere was about 280-290 ppm in the 18^{th} and 19^{th} centuries, it has exceeded 350 ppm in recent years and has been increasing 1 ppm each year [28].

In the last three decades, several initiatives have been started by the authorities in order to reduce global warming. The first of these initiatives is the First World Climate Conference organized by World Meteorological Organization in 1979. This conference later led to the creation of the Intergovernmental Panel on Climate Change (IPCC) in 1988 as well as the establishment World Climate Programme and the World Climate Research Programme. The Second Climate Conference in 1990 was "an important step towards a global climate treaty" and led to the establishment of United Nations Framework Convention on Climate Change (UNFCCC) [53]. The most significant development for climate change intervension is the Kyoto Protocol that is linked to UNFCCC.

Kyoto Protocol is an internationally binding agreement that has tangible targets for reducing GHG emissions. It was adopted in Kyoto, Japan, on 11 December 1997 and entered into force on 16 February 2005. 37 industrialized countries committed themselves to reduce their GHG emissions by an average of 5 % against 1990 levels over the five-year period 2008-2012. Currently, 191 states have ratified the protocol [53]. Even though the primary aim is to meet the emission targets with national measures, Kyoto Protocol offers three market-based mechanisms:

1. Emissions trading, known as 'the carbon market'
2. Clean development mechanism (CDM)
3. Joint implementation (JI).

Kyoto Protocol has established an emission allowance mechanism in order to monitor carbon emission levels. While UNFCCC determines these emission allowances for each state, the states are allowed to spare their emission permits and sell the excess capacity to the countries that exceed their targets. Similarly, in a local setting, the state may allocate its emission allowance between firms, and each firm may trade its permits with other firms. This leads to a national, regional or a global carbon permit market where the players trade their emission allowances.

Clean Development Mechanism allows a country with emission-reduction commitment to implement an emission-reduction project in developing countries. This mechanism enables industrialized countries to earn saleble certified emission reduction (CER) credits while helping sustainable development in other countries.

Joint Implementation offers meeting emission targets through earning emission reduction units (ERUs). In this mechanism, countries may invest or transfer technology to the host country where emission reductions are cheaper compared to reducing emission domestically and meet their targets in a more cost-effective way.

Under Kyoto Protocol, especially carbon trade has been gaining importance. Emission trading schemes have been established in European Union (EU

ETS), New Zealand (NZ ETS) and the USA (Chicago Climate Exchange). Of these schemes, European Union Emission Trading Scheme (EU ETS), which was launched in 2005 under a cap-and-trade scheme by the European Union countries, is the largest multinational emission trading scheme in the world. Under this scheme, EU ETS allocates carbon permits to the firms and the firms with higher emissions than their permits has to buy credit from the market or pay a penalty. In Phase I (2005-2007), the total annual cap was 2.1 billion tonnes CE and it covered more that ten thousand companies in 27 EU member countries. In Phase II (2008-2012), the cap has been cut by 10%. For Phase III (2013-2020), the emission allocation will be reduced by 21% from 2005 [42]. The EU ETS serves as a spot market as well as a futures market. Transaction volume has been increased from 262 million tonnes CE in 2005 to 5 billion tonnes CE in 2009 [8].

Turkey ratified the Kyoto Protocol in 5 February 2009. Turkey's commitment to emission reduction has started in 2012 through Voluntary/Verified Emission Reductions (VER) activities. These activities consist of investment support for renewable energy, forestry, recycling and energy efficiency projects that help reducing carbon emission.

These developments indicate a growing attention to carbon emission and its impacts in the world and in Turkey. Emission trading schemes will create new paradigms in the industry with increasing transaction volumes. In this new setting, the companies will be forced to compete in production, growth and innovation under carbon emission restrictions. Our aim in this thesis is to study two production problems of a single item with multiple inputs under a carbon emission constraint and non-linear production functions in the newsvendor setting.

In Problem 1, we consider the production of a single item with multiple inputs and stochastic demand under a carbon cap regulation and aim to find the production policy that maximizes the expected profit. We construct our problem based on the classical newsvendor problem. In addition, we consider the production costs and emission level which is restricted by the carbon cap. We assume a production model with multiple inputs. The relationship between the inputs and the production quantity is given by a non-linear production function.

The objective in this problem is to maximize the expected profit. The decision variables are the production quantity and the allocation of the inputs.

In our analysis of Problem 1, we study the problem for Cobb-Douglas and Leontief production functions. We consider several subproblems of the main problem and then synthesize our findings. In the Cobb-Douglas production function model, we show the concavity of the objective function under certain conditions for no carbon cap constraint and derive the optimal production policy in general. When we impose a carbon cap restriction, the optimal production policy is found by evaluating the extrema and the limits. For Leontief production function model, we are able to provide a solution set when there is no carbon cap constraint and derive the optimal production policy directly with a binding carbon cap constraint.

In Problem 2, we consider the production of a single item with multiple inputs and stochastic demand under an emission trading scheme and aim to design a carbon permit contract that maximizes the expected profit. We assume an emission trading scheme that allows advance purchase of carbon emission permits at an initial price before the random demand is realized. When the demand is realized and new carbon market prices are revealed, it is possible to buy additional permits or to sell an excess amount. We formulate this problem with backward two-period dynamic programming. The main difference with Problem 1 is that in Problem 2, the production occurs after the demand is realized (and hence known) and carbon emission restriction is not hard in the sense that trading is possible. Moreover, in Problem 2, we assume that the demand is fully satisfied. The objective is to maximize the expected profit. The decision variables are the initial carbon permit amount, traded carbon amount and the allocation of the inputs.

We study Problem 2 for Cobb-Douglas and Leontief production functions. We consider several subproblems of the main problem and then synthesize our findings in order to derive analytical structures for the optimal carbon trading policy. For the Cobb-Douglas production function model, under revealed demand and carbon market prices, we establish that the optimal policy consists of three regions for carbon trading: buy region, sell region and no-trade region. The advance permit purchase contract policy is derived and the objective function is shown to be concave. For the Leontief production function model, the optimal policy consists of two regions for the optimal carbon trading policy: buy region and sell region. We also derive the advance permit purchase contract policy and show that the objective function is concave.

As will be shown in our analysis, the Cobb-Douglas models provide richer problem settings and in some cases subsume the results for the Leontief models. Therefore, our numerical study is based on the Cobb-Douglas production function models for both problems. We present sensitivity analysis for both problems under different parameters in order to provide some managerial insight on the problems. For Problem 1, we additionally provide an illustrative example based on a real (published) study of agricultural production and analyze the optimal production policy performance. In the illustrative agriculture example, we study the greenhouse tomato production where the inputs are fertilizer, chemical, labor, machinery and water for irrigation. Under a carbon cap, we observe that carbon cap tightness and demand variability have significant affects on the optimal production policy. The results indicate that the production quantity, the expected profit and the service level decrease steeply for increasing carbon cap tightness after 30%. Another observation is that higher demand variance yield higher production quantity, but lower expected profit. We also observe that the carbon emission of greenhouse tomato production can be decreased by at least 35% without any significant effect on the expected profit and service level. According to our observations, this result may be obtained by increasing machinery, fertilizer usage, chemical usage and labor by around 5%, 20%, 50% and 100% respectively, and decrease water usage by 70%.

Additionally, we studied in a limited parameter setting the impact of technology improvements. We show that the carbon emission can be reduced by improving the technology level rather than changing the input allocations and decreasing the production quantity. Under a given carbon cap, we infer that, compared to producing with the given technology level, improving technology may even provide side benefits such as higher service levels and possibly higher net expected profits depending on the technology improvement cost. For instance, under a 39% carbon emission reduction by a carbon cap, a 100% improvement in the technology increase the service level from 88.7% to 93.9% and result an additional 12,891 TL expected profit before considering the technology improvement cost.

The sensitivity analysis for Problem 1 indicates that when our production system relies more on the input that emits less carbon, the expected profit is affected less by the carbon cap. In the production systems that rely on high carbon emitting input, the percentage decrease in the expected profit is between 18.7% and 28.2% under a 50% emission reduction by a carbon cap. On the other hand, we only have 3.1% and 1.1% decrease in the expected profit in the production systems that rely on low carbon emitting input. This finding provides evidence on the importance of swiching to less carbon emitting inputs.

The sensitivity study we conduct for Problem 2 shows that relying on high carbon emitting resources leads to become more dependent on the carbon trade market. The results indicate that when the production system mostly depends on the low cost-high carbon emitting input, we tend to purchase twice as much permit with the contract and expect to be twice as active in the emission trading scheme.

The rest of this thesis is organized as follows:

In Chapter 2, we review the literature on the classical newsvendor problem, constrained newsvendor problem, capacity reservations and spot markets and production models involving carbon emission restrictions. In Chapter 3, we present a detailed review on the production function with an emphasis on the Cobb-Douglas and Leontief production functions. In Chapter 4, we introduce Problem 1 in detail, derive the optimal production policy with some structural expressions. In Chapter 5, analytical results of the optimal carbon contract design is derived for Problem 2. In Chapter 6, we work on a numerical example and discuss the sensitivities of the both problems with respect to the problem parameters. Finally, in Chapter 7 concluding remarks are presented.

Chapter 2

Literature Review

In this chapter, we briefly discuss the related literature with emphasis on the works that are directly related with our models either from a methodological perspective or in terms of model settings.

One of the classical problems in the inventory management literature is the newsvendor problem. The analysis of this problem provides key insights for managing inventory in the fashion, airline and hospitality industries. The newsvendor problem is presented by Scarf [44]. It is later developed by other researchers as presented in [25], [41].

In the newsvendor problem, the buyer is interested in determining the optimal inventory policy to satisfy the demand for a single product in a single period under a probabilistic demand framework. The buyer is allowed to replenish his inventory only at the beginning of the the period at a given unit purchasing cost from the supplier. Any remaining inventory at the end of the period is assumed to be disposed of or sold at a discounted price. If the realized demand is greater than the inventory on hand, then the buyer forgoes some profit or even may lose the goodwill of the customer. The expected profit is given by the expected revenue minus the purchasing cost and the expected costs of overestimating and underestimating demand. Under the profit maximization objective, the optimal inventory policy structure is derived.

There are many extensions to the single period newsvendor problem. Researchers studied the newsvendor problem with different objective functions and different supplier pricing policies. They also considered multi-location, multi-period and multi-product extensions of the newsvendor problem. Our scope in this thesis is limited to the newsvendor problem with multiple resources under a constraint on resources. Our models involve production functions that express the production quantity of a product as a function of the resources. However, to our knowledge, there have been no study that considers production functions in the newsvendor setting. Therefore, we consider newsvendor extensions involving multiple resources and multiple products with budget or capacity constraints.

2.1 Newsvendor models with a single resource constraint

Nahmias and Schmidt [36] is among the earlier works that discuss the single period inventory (newsvendor) problem with multiple products under a single constraint on capacity or budget. They provide four heuristics that require fewer computations than the Lagrange multiplier approach in order to find the optimal order quantities for the products and examine the performance of each heuristic.

Gallego and Moon [12] suggest several extensions to the newsvendor model based on [44] one of which is the multi-product case under a linear budget constraint. They form the Lagrangian function and develop a simple solution algorithm. In the algorithm, they first solve the problem without the budget constraint. If the budget constraint is not satisfied, they set an arbitrary value for the Lagrange multiplier. Then they compute the optimal production quantities under that value and evaluate the budget constraint. They increase the value of the Lagrange multiplier and if the cost exceeds the budget; otherwise, they decrease the value of the Lagrange multiplier. They continue with their seach until they find the value of the Lagrange multiplier that satisfies the constraint with equality.

Hadley and Whitin [15] consider the newsvendor problem with multiple products under a capacity constraint. Their solution algoritm is based on a search for the Lagrangian multiplier that satisfies the necessary conditions. First, the problem is solved by ignoring the budget constraint and the optimum order quantity for each product is found. Then, these values are plugged into the budget constraint to see if they satisfy the equation. Otherwise, the budget constaint is assumed to be binding and the Lagrangian approach is used with the same procedure that is proposed in [12]. However, they relax the nonnegativity constraints and neglect to consider the lower bounds of the order quantities in their model.

Abdel-Malek and Montanari [3] address the multi-product problem under stochastic demand framework with a budget restriction and improve the solution methodology for the model in [15]. They divide the solution space by determining two thresholds for the budget ranges: one for relaxing the budget constraint and one for relaxing the nonnegativity constraint. In the first range, the budget is abundant and the optimal solution is given by the unconstrained problem. In the second range, only the budget constaint is binding and the optimal order quantities yielded by the Lagrange approach are all positive. In the last range, the budget is too tight and nonnegativity constraints are binding. They illustrate their procedure in numerical examples.

These papers consider independent demand distribution for each product. There are also studies on the newsvendor model with substitutable products. However, a similar structure can be observed in multi-product and multi-resource problems where some resources are common in multiple products. In addition, these models are closer to the scope of this thesis.

Gerchak and Henig [13] formulate a single period newsvendor model for selecting optimal component stock levels in an assemble-to-order system. They assume a system with multiple products that have common components. They solve this problem in two stages. In the first stage, they determine the allocation of the common components between the products for a given component stock level so as to maximize the revenue. In the second stage, they use his information to select the optimal stock levels.

Jonsson and Silver [21] also deals with assemble-to-order systems with multiple products and multiple components under a budget constraint. Some of the components are assumed to be unique to specific products while others are common to two or more products. The components are ordered before product demands are revealed, but the assembly of the products begin after the demand is known. They also assume normally distributed demand. The aim is to determine component quantities under a budget constraint so as to maximize the expected number of units of end items sold. They consider the assembly of two products with three components: one common component and one unique component for each. They develop a simple heuristic for solving the problem.

Jonsson and Silver [22] later extend their model in [21] to address the problem with multiple components and products. They formulate the problem as a two-stage stochastic programming problem with a recourse which is extremely difficult to solve optimally. They develop three heuristics for solving the problem under some simplifying assumptions. One of the heuristics performed well for the practical case of continuous demand distributions under large budgets.

Harrison and Van Mieghem [17] address the multi-product newsvendor problem with multiple resources which are capital, labor and production technology. They aim to find the optimal investment strategy in resources under uncertain product demand. After the demands are realized, the production quantities for each product are determined given these resources. They also use stochastic programming with recourse to solve the problem and provide structural results of the optimal solution.

The studies presented above share a common approach in the solution methodology. First, they find the optimal production quantities for each product for a given resource level. Then, they use this information in determining the optimal resource level. We also use a similar approach for solving both problems.

These cited works are related to Problem 1 in our study in the sense that they primarily use Lagrangean relaxation methodology. However, in terms of model settings, they are different: These works consider multiple products and a constraint on resources that may be used for all (or some) of these products.

However, the resource usage is linear in all of these models.

2.2 Newsvendor models with resource contract design

In Problem 2, we aim to develop an advance permit purchase policy under random demand and carbon market prices. In this sense, Problem 2 can be treated as a newsvendor problem with a cash constraint or a capacity reservation and spot market alternative. Another similar model is given by options contract. Therefore, it is important to study the literature on options and capacity reservation contracts with spot market alternative.

We begin our discussion on the works with a constraint on cash avaliable for goods replenishment.

Buzacott and Zhang [6] incorporate asset-based financing into production decisions in the multi-period newsvendor setting. They consider a Stackelberg game between the bank and retailer, where the bank is the leader. In this formulation, there is no exogenous budget constraint. Instead, the retailer determines the loan size as well as the order quantity so as to maximize the expected revenue. On the other hand, the bank determines the interest rate and loan limit that maximizes its expected profit. They find three regions for the retailer's optimal loan borrowing policy. In the first region, the retailer does not borrow. In the other regions, the retailer borrows with or without backruptcy risk.

Li et al. [34] examine the simultaneous inventory and financial decisions of a firm under demand uncertainty. They study a discrete time multi-period model. In each period, the firm has to determine how much money to borrow, how much dividend to issue and how much to issue so as to maximize the expected present value of the dividends net of capital subscriptions. They derive a myopic optimal policy.

Xu and Birge [55] consider joint production and financing decisions under demand uncertainty and financial constraints in a single period. The objective is to maximize the expected future value of the firm. They derive structural results for the optimal production and financial policy under market imperfections such as tax and bankruptcy costs. They also examine the debt capacity for the firm.

Lai et al. [30] study the impact of financial constraints on the supply chain contracts in the newsvendor setting. They assume a Stackelberg game between a supplier and a retailer, in which the supplier is the leader and each party aims to maximize its expected profit. They examine the model with three supply chain contracts: preorder, consignment or the combination of both. Under preorder contract, the retailer owns the inventory while in the consignment contract, the supplier takes full inventory risk. The supplier decides on the supply chain contract and determines the wholesale price for the preorder and/or the commission for the consingment order. On the other hand, the retailer determines the optimal order quantity for a given contract and price offer from the supplier. Simultaneously, both parties determine how much loan to borrow from an external financial market in order to finance their production and inventory related decisions. Lai et al. derive the optimal policy structures and show that in the presence of financial constraints, the supplier prefers to share the inventory risk with the retailer.

There are also some works with real options to alleviate either existing or potential resource limitations. Barnes-Schuster et al. [4] studies the role of contracts with options in a two-period production problem with correlated stochastic demand. At the beginning of the first period, the buyer places firm orders for both periods and purchases options to be exercised in the second period. At the end of the first period, the buyer observes the first period demand and updates the demand for period two. Then he decides how much to order through exercising options. On the other hand, the supplier determines the optimal wholesale price as well as option and exercise prices that maximizes the expected profit. They analyze the problem under decentralized system and channel coordination and derive the structure of the optimal policy.

Serel et al. [46] consider sourcing decisions of a firm in the presence of a capacity reservation contract and spot market alternative in the newsvendor setting.

In their models, they allow simultaneous use of both options and assume deterministic contract and spot prices. They investigate the actions of the buyer and the long-term supplier for two types of periodic review policies: the two-number policy and the base stock policy. The supplier determines the contract price. The buyer decides how much capacity to reserve so as to maximize its expected profit under predetermined spot and contract prices. The unused capacity has no value for the supplier or the buyer and it cannot be sold to a third party. They derive an optimal policy structure for the buyer under the base stock policy where they determine a threshold price and order from either the supplier, the spot market or both under given contract and spot market prices.

These papers consider both seller and the buyer. In addition, they both consider deterministic prices for the future capacity purchase. In [4], these are given by the exercise prices of the options whereas in [46], these are given by the spot market prices and the contract prices with the supplier. There are also extensions that assume stochastic price structures for the future purchases.

Serel [45] extends the model in [46] to investigate a multi-period capacity reservation contracts when there is uncertainty on the availability of the resources in the spot market. The supplier guarantees the delivery of the reserved capacity while the buyer may also acquire the input from the spot market if available. In this model, the spot prices are stochastic and dependends on the available quantity. The reservation contract with the long-term supplier involves two terms: capacity price and reservation quantity. The buyer determines the optimal inventory control policy and the reservation quantity while the supplier sets the contract price to maximize its expected profit. He derives the optimal policy structures for the buyer and the supplier. Numerically, he shows that the uncertainty in the input market leads to an increase in the amount of reserved capacity at a lower price.

Wu et al. [54] investigates the capacity contract agreement under a Stackelberg game between a buyer and a seller. The only source of uncertainty is assumed to be the spot market price and the buyer's demand is given as a function of the spot market price. The seller sets the option price and exercise price to maximize its profits. Given these prices, the buyer purchase options from the seller and decides how much options to exercise when the spot market prices are revealed. In this model, the buyer is allowed to sell excess capacity in the spot market. The objective function for the buyer is defined by a utility function which includes the willingness-to-pay function as inverse of the demand function at a given spot price. The solution to the buyer's problem is given in two stages. In the first stage, the spot market price is assumed to be revealed and the buyer determines how many options to exercise and how much capacity to buy from the spot market at a given that price so as to maximize its utility. In the second stage, the buyer finds the optimal amount of options to purchase in order to maximize its expected utility under a spot price distribution. The optimal policy structure seems to be similar to the structure in [46] since it also suggests a threshold price for determining the option exercise and/or spot market purchase policy.

Spinler and Huchzermeier [49] adapt [54] to include more uncertainties in the model. They define a random variable for the state of economy and express all uncertainties in their model as a function of this variable. On top of random spot prices, thet consider random demand for the buyer. They also assume stochastic cost structures of long-term contracts and spot market sales for the seller. In addition, they consider an exogeneous risk of not being able to find a last-minute buyer as a function of spot market price. They show that the structure of the optimal policies in their model share similarities with the policy structures in [54].

2.3 Inventory models with carbon emission considerations

In addition to the constrained newsvendor problem and capacity reservations contracts under stochastic demand, finally we consider the studies on production problems involving carbon emission restrictions. The environmental regulations by the authorities and increasing environmental awareness of the customers force the industry to take some measures to decrase their carbon footprints. The growing interest in sustainable systems in the industry leads many reseaches to focus their attention on this area. In the last three years, there is an increasing number of operations management (OM) researcher that are interested in carbon-emission related issues. They revisit well known problems such as lot-sizing and newsvendor problems, and study them under this new paradigm.

Benjaafar et al. [5] investigate how to involve the carbon emissions concerns into the operational decision-making models. They consider the economic order quantity (EOQ) model in which the objective is to find the optimal ordering quantity so as to minimize the sum of the fixed ordering cost, inventory holding cost and procurement cost under a deterministic demand. They also assume that the carbon emission has a similar structure to the cost function. Through numerical examples, they provide insights that highlight the impact of operational decisions such as procurement, production, and inventory management on the carbon emissions and emphasize the importance of the operational models in assessing the benefits of investments in more carbon-efficient technologies. They suggest other models to be studies further one of which is the newsvendor problem under a carbon emission restriction.

Hua et al [20] consider a lot sizing problem under a cap-and-trade mechanism and aim to discuss how to manage the carbon footprints in the inventory control. They study the EOQ model in this setting, and examine the impacts of carbon trade, carbon price and carbon capacity on the optimal ordering policy, carbon emission and total cost. They derive the optimal ordering policy analytically and provide managerial insights.

Benjaafar et al. [7] apply the EOQ problem with a cap on carbon emissions. They derive the optimal ordering policy under some conditions and analytically support the observations made in [5]. They provide a condition under which it is possible to reduce emissions by modifying order quantities. They also provide conditions under which the relative reduction in emissions is greater than the relative increase in cost and discuss factors that affect the difference in the magnitude of emission reduction and cost increase.

Most studies in the OM literature on carbon emission are based on deterministic demand models. We have found one study that deals with carbon emission restrictions under stochastic demand.

Song and Leng [48] analyze the single-period newsvendor problem under three carbon emission restriction policies: carbon cap, carbon tax and cap-and-trade. They also suggest that the technology investment in the carbon offset is a special case of the cap-and-trade policy. Under each policy, they derive analytical solutions for optimal production quantity and evaluate corresponding expected profit. They provide managerial insights on their analytical results.

2.4 Synthesis of Literature Review

In this thesis, we study the newsvendor problem in a production setting with multiple inputs incorporating production functions in power forms. As such, our model subsumes that in [48], but provides richer and more general results. In addition, we consider two carbon emission restriction policies: carbon cap in Problem 1 and cap-and-trade in Problem 2. We determine the optimal production policy in Problem 1 and optimal carbon permit contract policy in Problem 2 so as to maximize the expected profit.

In Problem 1, we study the production of a single item in the newsvendor setting with multiple inputs under a carbon cap. We extend the model in [48] and consider the inputs that contribute to the production process via production functions. The production cost is included when calculating the expected profit. We aim to find the optimal production policy given by the production quantity and the input allocation. In order to find the production policy, we use a similar approach as in the studies [12], [15], [3]. In these papers, they find the optimal production quantities for each product for a given resource level. Then, they use this information in determining the optimal resource level. In our study, we first determine the optimal allocation of resources for a given production quantity and carbon cap. Then we find the optimal production quantity based on this information.

In Problem 2, we consider an emission trading scheme where an advance permit purchase is made at an initial price. After the demand and carbon market prices are revealed, the producer may buy additional permit or sell some of the permit at hand in order to satify the demand. Therefore, the revelation of the demand and carbon market prices divide the problem into two stages or two periods. In the first period, we determine the optimal carbon trade policy as well as the optimal allocation of resources for a given initial carbon permit under the realized demand and carbon market prices. Then we use the information in order to develop an advance permit purchase policy under random demand and carbon market prices so as to maximize the expected profit.

For Problem 2, we also use the same solution approach as explained for Problem 1 above in order to determine the carbon trade policy after the demand is realized. In this case, we first find the optimal allocation of resources for a given demand and carbon emission level, as in Problem 1. Then we decide on how much permit to buy or sell so as to reach the emission level that maximizes the profit. Note that in Problem 1, the carbon emission level is restricted by the carbon cap and we change the production quantity whereas in Problem 2, we change the carbon emission level and the production quantity is fixed by the demand.

Even though the studies [4] and [46] consider random demand, they assume deterministic prices for the future capacity purchase when deciding the capacity contract. In [4], these are given by the exercise prices of the options whereas in [46], these are given by the spot market prices and the contract prices with the supplier. In Problem 2, we assume stochasticity for the future capacity purchase, carbon market prices in this case, as well and consider that initial carbon permit purchase is made prior to the revelation of these carbon market prices.

The works [45], [54] and [49] are more similar to our formulation of Problem 2 due to stochastic future capacity purchase prices. A common solution method in these studies is to first find the optimal inventory policy under the assumption of deterministic spot market prices and then to decide on the initial capacity purchase under this information. We also follow a similar train of thought in our solution process. As explained above, we consider the optimal production and carbon trade policies when the demand and carbon market prices are revealed. Then we find the amount of carbon permit to purchase prior to this information so as to maximize the expected profit.

Chapter 3

Production Functions

Production can be described as the means of transforming inputs into outputs. It is important to know that a particular output may be produced by alternative combinations of the inputs. A production function is a mean of defining these alternatives. In other words, it gives a set of possible relations between the inputs and outputs at a given technology level.

The production function is at the core of the economic theory of production. Simply, it applies to the production relations within a process, a firm or a plant. In that sense, it is a micro concept. Given a process with a single output and multiple inputs, let Q be the output or production quantity and x_i be the input quantity for resource $i = 1, \ldots, n$. The production function is formally defined as

$$Q = \phi(x_1, \ldots, x_n) \tag{3.1}$$

where $\phi(\ )$ gives the form of the production function.

Before we introduce some forms of the production functions, we present two critical concepts in the theory of production: *Elasticity of Scale* and *Elasticity of Substitution.*

The extent of the affect of changing inputs on the output is given by the *elasticity of scale.* Elasticity of scale is the ratio of the proportionate increase in output to the proportionate increase in inputs. If we assume that all inputs are increased by the same percentage $dx_i/x_i = dx/x$, $\forall i = 1, \ldots, n$, the elasticity of scale, ε is defined by the equation:

$$\varepsilon = (dQ/Q)/(dx/x) \tag{3.2}$$

If $\varepsilon = 1$, then doubling all the inputs will double the output, the case we refer as *constant returns to scale*. If $\varepsilon < 1$, we get less than doubling of the output when we double all the inputs and we have *decreasing returns to scale*. Finally, if $\varepsilon > 1$, this case is called *increasing returns to scale* in which doubling all the inputs will result more than doubling of the output.

Let us assume a production process with two inputs, x_1 and x_2, and let the prices of these inputs be p_1 and p_2 respectively. The cost of production is $\Gamma = p_1x_1 + p_2x_2$. At a constant cost $\Gamma = \overline{\Gamma}$, all combinations of x_1 and x_2 are given by $x_1 = -(p_2/p_1)x_2 + \overline{\Gamma}/p_1$. This line is called isocost line and its slope is $dx_1/dx_2 = -p_2/p_1$. The increase in input 1 price, p_1, may cause two effect: the cost of producing the given output may increase or input 2 may be substituted for input 1. The latter effect is called the *substitution effect*.

The elasticity of substitution is the ratio of the proportionate change in input proportions to the proportionate change in the slope of the isocost. Input proportions are x_1/x_2 and the change in input proportions is $d(x_1/x_2)$ hence the proportionate change in input proportions is $d(x_1/x_2)/(x_1/x_2)$. The slope of the isocost is (dx_1/dx_2) and the change in the slope is $d(dx_1/dx_2)$, hence the proportionate change of the slope is $d\left(\frac{dx_1}{dx_2}\right) / \left(\frac{dx_1}{dx_2}\right)$. To sum up, the elasticity of substitution (σ) is

$$\sigma = \frac{d\left(\frac{x_1}{x_2}\right) / \left(\frac{x_1}{x_2}\right)}{d\left(\frac{dx_1}{dx_2}\right) / \left(\frac{dx_1}{dx_2}\right)} \tag{3.3}$$

Now, we introduce two important production functions that are used in the economic theory of production: *Cobb-Douglas* and *Leontief*.

Cobb-Douglas production function is without a doubt the most widely known production function. Its form was suggested by the mathematician Cobb based on Professor Paul Douglas' observations on the empirical study of capital stock, labor force and GNP for the US manufacturing industries for the period 1899-1922. Even though the suggested form is $Q = Ax_1^{\alpha}x_2^{1-\alpha}$, it may be generalized to multiple inputs

$$Q = A\prod_{i=1}^{n} x_i{}^{\alpha_i} \tag{3.4}$$

where the positive coefficient A is the technology level for the process. Note that it is necessary to have some of each input in the Cobb-Douglas production function since it is not possible to produce unless all the inputs are available.

Elasticity of scale of the Cobb-Douglas function depends solely on α_i (> 0)'s which we will call input elasticities. It is given by $\varepsilon = \sum_{i=1}^{n} \alpha_i$. On the other hand, the elasticity of substitution is $\sigma = 1$ which implies that the inputs are perfectly substitutable.

Cobb-Douglas model is widely applied to a variety of economic systems at different levels from a simple process to a country's economy. Most of the studies that refer to the Cobb-Douglas production function is empirical studies that estimates GNP of a country or an industry with capital and labor as inputs. To illustrate its use in the literature, we present a few of these studies here.

Komiya [27] investigates steam power industry in the US for the period 1938-1956 under the Cobb-Douglas production function. The inputs are capital, labor and fuel while the output is the energy generated.

Ozaki [38] examines the developments in Japanese economy from 1955 to 1968. He studies 54 sectors that are categorized in six technology types. For the sectors that use labor intensive, constant earning type technology, he uses the Cobb-Douglas production function with production scale as the output, and labor and capital as the inputs.

Keilbach [23] estimates the value of the marginal product of emission in the German manufacturing industry based on the period 1966-1990. He uses the Cobb-Douglas production function in his model where the output is the value added of the manufacturing industry and the inputs are capital, labor and emission. He considers the model for four different types of emission: SO_2, CO_2, NO_x

and *particular matter*.

Khanna [24] conducts an empirical study on cost of meeting Kyoto Protocol commitments under technology change. In the model, the Cobb-Douglas production function is used to model real GDP as a function of the capital, labor and energy inputs. The data is based on the period 1965-1999 for 23 Annex 1 countries.

Shadbegian and Gray [47] uses the Cobb-Douglas production function in their model to analyze the impact of pollution abatement expenditures on productivity in paper, steel and oil industries for the period 1979-1990. Their output is the value of shipment while the inputs are productive capital stock, pollution abatement capital stock, labor for production, labor for pollution abatement, materials used for production and materials used for pollution abatement.

Hatırlı et al. [18] studies the relationship between energy inputs and crop yield for greenhouse tomato production in Antalya, Turkey. The output is the tomato yield while the inputs are fertilizer, chemicals, machinery, labor force, water for irrigation and seed. They convert the output and the inputs to their energy equivalents. Then they examine the relationship between inputs and the output under the Cobb-Douglas production function.

Wei [52] discusses the impact of energy use efficieny and energy production efficiency on GDP and energy consumption in short and long terms. They assume the Cobb-Douglas model where the output is GDP and the inputs are labor, capital and energy /technology.

Yuan et al. [56] estimates the impact of technologic changes on the energy intensity by assuming a Cobb-Douglas model for the period 1995-2006. The inputs are capital, labor and energy with exogeneous technology progress while the output is the value added of the industry.

Kogan and Tapiero [26] considers a supply chain with N-firms. They assume an aggregate production function where the price is the output, and inputs are the labor force and investment policy. The firms select the optimal level of employment and the level of co-investment in the supply chain infrastucture to maximize their discounted profit. They use the Cobb-Douglas production function in their examples.

In this thesis, we consider Problem 1 and 2 under the Cobb-Douglas production function. In our formulation, we assume that input elasticities are $\alpha_i \leq 1$ for $i = 1, \ldots, n$ and we let *returns to scale* be $r = \sum_{i=1}^{n} \alpha_i$. The production quantity, Q, will be given by Equation 3.4.

Another interesting form of the production functions is the one which focuses on the interdependence of the stages of production. This approach is based on the work of Leontief and is called *putty-clay*, *fixed proportions* or *inter-industry* approach to economic modelling. Leontief focuses on the flows of intermediates between stages and explains the output for these intermediates in terms of the production quantity. In this case, there is a fixed proportion of the inputs and the output decision determines the input quantities. The Leontief production function is given by the equation

$$Q = \min\{Q_i\} \tag{3.5}$$

where Q_i is the intermediate output for stage $i = 1, \ldots, n$ and Q_i is given by some function of the input x_i.

In the Leontief production function, the elasticity of substitution is $\sigma = 0$ which implies that the inputs are perfect complements and the input costs have no affect on input quantities.

While we have perfect substitution in Cobb-Douglas, the Leontief models the production process with perfectly complementing inputs. Even though Leontief is not as popular as the Cobb-Douglas production function, it is significant since it gives an alternative approach to production. To illustrate its use in the literature, we present a few of these studies here.

Komiya [27] examines steam power industry in the US for the period 1938-1956. He uses the Leontief production function in his model. The inputs are capital, labor and fuel while the output is the energy generated. The relationship between input x_i and the output, Y, is given by the following equation, where a_i and b_i are some positive constants: $Y = a_i x_i^{b_i}$.

Haldi and Whitcomb [16] assume the same functional form for the Leontief production function in their study of production process in various industries such as petroleum refining, primary metals and electric power. The output is the capacity while the inputs are capital, labor, energy and raw materials.

Ozaki [38] studies the developments in Japanese economy from 1955 to 1968. He analyzes 54 sectors that are categorized in six technology types. For the sectors that use large quantity processing, large-scale assembly production and capital intensive technology, he uses the Leontief production function with the same functional form in [27] and [16]. He assumes production scale as the output, and labor and capital as the inputs.

Lau and Tamura [32] uses the Leontief production function with a nonhomothetic form which presents a more generalized relation between the inputs by allowing varying returns to scale for the same input. In their empirical study, they present an application of their model to the Japanese petrochemical processing industry for the period 1958-1969. The inputs are capital, labor, energy in the form of fuel and electricity, and raw material. The output is the amount of ethylene produced at the end of the process.

Nakamura [37] introduces a nonhomothetic form of the Leontief cost function. He uses capital, labor and material as the inputs, and applies his model to a pooled data set of Japanese iron and steel industry for the period 1962-1982.

In this thesis, we also consider Problem 1 and 2 under the Leontief production function. In our formulation, the production quantity will be given by Equation 3.5 and we assume a relation between the inputs and the intermediate output which is similar to the input-output relations in [27], [16] and [38]. Thus the intermediate output for any stage is given by the following

$$Q_i = A_i x_i^{\alpha_i} \quad \text{for all } i = 1, \ldots, n \tag{3.6}$$

where Q_i is the intermediate output, A_i (> 0) is the technology level, x_i is the input quantity and α_i (> 0) is the elasticity of scale for stage $i = 1, \ldots, n$.

Chapter 4

Problem 1: Production Policy for a Manufacturer with Multiple Inputs under a Carbon Cap

In this chapter, we study the production decisions of a manufacturer for a single product with multiple inputs and stochastic demand under carbon emission regulation which imposes a strict emission cap.

In the literature, the classical single period inventory control (newsvendor) problem refers to the replenishment/production decision for a single item with random demand. The production quantity is used to satify the demand during the period and the realized demand determines the profit at the end of that period. Each unit of the product is sold at a price per unit, s. If there is remaining inventory at the end of the period, an excess cost per unit, c_e, is incurred for unsold items. If demand exceeds the produced quantity, a shortage cost per unit of demand, c_s, is incurred for the unsatisfied demand. For a nonnegative continuous random demand, D, with p.d.f. $f(.)$ and c.d.f. $F(.)$, and a production quantity, Q, the expected profit, $\tilde{\Pi}(Q)$ is written as

$$\begin{aligned}\tilde{\Pi}(Q) =& sE[min(Q,D)] - c_s E[max(0,(D-Q))] - c_e E[max(0,(Q-D))] \\ =& s\int_0^Q Df(D)dD + s\int_Q^\infty Qf(D)dD - c_s\int_Q^\infty (D-Q)f(D)dD \\ &- c_e\int_0^Q (Q-D)f(D)dD\end{aligned}$$

The classical newsvendor problem refers to the following optimization problem.

$$\begin{aligned}\max_Q \quad & \tilde{\Pi}(Q) \\ s.t. \quad & Q \geq 0\end{aligned}$$

It is a well known fact that the objective function is concave in the production quantity, Q. The optimal solution to this problem is given by

$$F(Q^{**}) = \frac{s+c_s}{s+c_s+c_e} \tag{4.1}$$

where Q^{**} denotes the optimal production quantity [35].

The fractile entity on the righthand side of Equation 4.1 is the so-called desired service level. Note that it is defined through the 'effective' shortage cost per unit $(s+c_s)$ and the excess cost per unit, c_e.

The optimization problem we consider in this chapter relies on this classical newsvendor problem contruct. However, it differs from it in a number of aspects:

(i) production consumes multiple inputs (resources)

(ii) usage of each input is non-linear in the quantity produced

(iii) usage of each input results in carbon emission, which is assumed to be linear in the amount of usage

(iv) there is an externally imposed cap on the total carbon emission during production.

To the best of our knowledge, this is the first work that considers all these aspects. Next, we formally construct our optimization problem.

We assume a production model with $n(\geq 1)$ inputs. The relationship between the inputs and the production quantity is given by $Q = \phi(\vec{x})$ where $\overrightarrow{x} = (x_1, \ldots, x_n)$. Each input, $x_i,\ i = 1, .., n$, has a procurement cost per unit, p_i, and carbon emission per unit β_i. We assume linear production cost and carbon emission in our problem. Hence, the production cost is given as follows

$$\Gamma(\vec{x}) = \sum_{i=1}^{n} p_i x_i \tag{4.2}$$

and the carbon emission is given by

$$\epsilon(\vec{x}) = \sum_{i=1}^{n} \beta_i x_i \tag{4.3}$$

Finally, we assume that a carbon cap κ is imposed by the authorities.

Our objective is to determine the production quantity and input values that will maximize the total expected profit, $\Pi(Q, \vec{x}) = \tilde{\Pi}(Q) - \Gamma(\vec{x})$, under a production function, $Q = \phi(\vec{x})$, and carbon cap restriction, $\epsilon(\vec{x}) \leq \kappa$. Therefore, we consider the below problem which we will address as Problem $P1$:

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \Pi(Q, \vec{x}) = \tilde{\Pi}(Q) - \Gamma(\vec{x}) \\ s.t. \quad & Q = \phi(\vec{x}) \\ & \epsilon(\vec{x}) \leq \kappa \\ & Q \geq 0 \end{aligned} \tag{P1}$$

In our analysis in this chapter, we consider different subproblems of Problem $P1$ in order to gain insight on the problem and develop the solution to the original optimization problem in steps.

First, we analyze the problem without a carbon emission constraint and label this problem as Problem $P1U$.

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \Pi(Q, \vec{x}) = \tilde{\Pi}(Q) - \Gamma(\vec{x}) \\ s.t. \quad & Q = \phi(\vec{x}) \\ & Q \geq 0 \end{aligned} \tag{P1U}$$

In order to solve this problem above, we consider a subproblem for a given $Q \geq 0$, which we label as Problem $SP1U$ and is formally stated as follows.

$$\begin{aligned} \min_{\vec{x}} \quad & \Gamma(\vec{x}) \\ s.t. \quad & Q = \phi(\vec{x}) \end{aligned} \tag{SP1U}$$

The solution to Problem *SP1U* gives us the optimal input allocation for a given $Q \geq 0$, which is $\vec{x}^{**}(Q)$. In this case, the minimum production cost for a given $Q \geq 0$ is $\Gamma^{**}(Q) = \Gamma(\vec{x}^{**}(Q))$. We can substitute $\vec{x}^{**}(Q)$ for $\vec{x}$ in Problem $P1U$ and reduce the problem to an equivalent problem, Problem $P1U^*$, with single varible, Q.

$$\begin{aligned} \max_{Q} \quad & \Pi(Q) = \tilde{\Pi}(Q) - \Gamma^{**}(Q) \\ s.t. \quad & Q \geq 0 \end{aligned} \tag{$P1U^*$}$$

Once we solve this problem, we find the optimal production quantity Q^{**}. Thereby, we can compute the optimal allocation of the inputs $\vec{x}^{**}(Q^{**})$ for the original problem, Problem $P1U$ and the corresponding carbon emission level, $\epsilon_{1U} = \epsilon(\vec{x}^{**}(Q^{**}))$.

So far, we have considered only the unconstrained optimization problem (that is, in the absence of a carbon emission constraint).

Next, we consider a variant problem which provides the minimum emission level for a given $Q \geq 0$. We call this variant Problem $CE1$ and state it formally as follows.

$$\begin{aligned} \min_{\vec{x}} \quad & \epsilon(\overrightarrow{x}) \\ s.t. \quad & Q = \phi(\overrightarrow{x}) \end{aligned} \tag{CE1}$$

The optimal solution to Problem $CE1$ gives the input allocation that will minimize the emission level for a given $Q \geq 0$. The corresponding minimum carbon cap $\kappa_{min}(Q)$ is given by the value of the objective function at the optimal solution, $\epsilon(\vec{x}^{**}(Q))$ resulting in $\kappa_{min}(Q) = \epsilon(\vec{x}^{**}(Q))$. This solution to this problem enables us to develop feasibility conditions which we later use in the construction of the optimal solution stucture in the presence of a carbon emission constraint.

Finally, we study the the original problem Problem $P1$. We first check if $\epsilon_{1U} \leq \kappa$. If this inequality is satisfied, then the optimal solution to Problem $P1$ is given by the optimal solution to Problem $P1U$. Otherwise, we know that the carbon cap constraint is binding and $\epsilon(\vec{x}) = \kappa$. In this case, we introduce a nonnegative scalar Lagrange multiplier, λ, for the carbon cap constraint and write the objective function (the Lagrangean) as $\widehat{\Pi}(Q, \vec{x}) = \tilde{\Pi}(Q) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - \kappa\right)$. The new equivalent problem, Problem $\widehat{P1}$, is

$$\begin{aligned} \max_{Q,\vec{x}} \ & \tilde{\Pi}(Q) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - \kappa\right) \\ s.t. \ & Q = \phi(\vec{x}) \\ & Q \geq 0 \end{aligned} \tag{$\widehat{P1}$}$$

Note that this problem is similar in structure to Problem $P1U$. Therefore we follow the same steps to solve this problem. We first study the allocation of the inputs in Problem $SP1$ for a given λ, $Q \geq 0$.

$$\begin{aligned} \min_{\vec{x}} \ & \widehat{\Gamma(\vec{x})} = \Gamma(\vec{x}) + \lambda\epsilon(\vec{x}) \\ s.t. \ & Q = \phi(\vec{x}) \end{aligned} \tag{SP1}$$

The optimal allocation of the inputs in Problem $SP1$ is $\vec{x}^{**}(\lambda, Q)$ for a given $\lambda, Q \geq 0$. Then we substitute this solution to Problem $\widehat{P1}$ in order to get a new equivalent problem Problem $P1^*$ which reduces to a single variable Q for a given λ. Then we have

$$\begin{aligned} \max_{Q} \ & \widehat{\Pi}(Q) = \tilde{\Pi}(Q) - \Gamma(\vec{x}^{**}(\lambda, Q)) - \lambda\left(\epsilon(\vec{x}^{**}(\lambda, Q)) - \kappa\right) \\ s.t. \ & Q \geq 0 \end{aligned} \tag{$P1^*$}$$

We also enforce $\epsilon(\vec{x}^{**}(\lambda, Q)) = \kappa$ in order to ensure that the carbon cap is binding. We find the optimal production quantity Q^{**} for Problem $P1^*$ and the corresponding λ^{**} value. Then we can obtain the optimal input allocation for the original problem, Problem $P1$.

In the following sections, we analyze Problems $SP1U$, $P1U^*$, $CE1$, $SP1$ and $P1^*$ for two production functions: Cobb-Douglas and Leontief.

4.1 The Cobb-Douglas Production Function Model

In this section, we study Problem 1 under a Cobb-Douglas production function. Recall that in the Cobb-Douglas production function the production quantity, Q, is given by

$$Q = \phi(\vec{x}) = A \prod_{i=1}^{n} {x_i}^{\alpha_i}$$

where x_i is the input quantity and $0 < \alpha_i \leq 1$ for $i = 1, \ldots, n$ and the rate of return is $r = \sum_{i=1}^{n} \alpha_i$.

This relationship enables us to eliminate one of the variables by substitution. We pick arbitrarily a variable, say, x_n to express its value in terms of the production quantity Q and the other inputs x_i for $i = 1, \ldots, n-1$.

Lemma 4.1. *Let $x_n^* = x_n(Q, x_1, \ldots, x_{n-1})$ be a function of production quantity Q and other inputs x_i for $i = 1, \ldots, n-1$. Then the following relation holds*

$$x_n^* = \left(\frac{Q}{A \prod_{j=1}^{n-1} {x_j}^{\alpha_j}} \right)^{\frac{1}{\alpha_n}} \tag{4.4}$$

Proof. Equation 3.4 implies ${x_n}^{\alpha_n} A \prod_{j \neq n} {x_j}^{\alpha_j} = Q$. From which we get, $x_n = \left(\frac{Q}{A \prod_{j \neq n} {x_j}^{\alpha_j}} \right)^{\frac{1}{\alpha_n}}$ □

The relationship given by Equation 4.4 is used to eliminate the production function constraint in the sequel.

Next, we study Problems $SP1U$ and $P1U^*$ (where we ignore the carbon cap constraint) and Problems $CE1$, $SP1$ and $P1^*$ (where we consider the carbon cap) separately .

4.1.1 No Carbon Cap is Imposed

We begin our analysis in the absence of a carbon emission constraint. Our objective is to find the optimal production quantity and input allocation in Problem $P1U$ under the Cobb-Douglas production function. We call this problem $P1Ua$, where 'a' in the label refers to our usage of the Cobb-Douglas function.

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \tilde{\Pi}(Q) - \Gamma(\vec{x}) \\ s.t. \quad & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \\ & Q \geq 0 \end{aligned} \tag{P1Ua}$$

As outlined before, we begin our analysis of production decisions in the presence of the Cobb-Douglas production function by first studying Problems $SP1U$ and $P1U^*$. Their analysis will enable us to solve the optimal production policy for Problem $P1Ua$.

We call Problem $SP1U$ under the Cobb-Douglas Production function, Problem $SP1Ua$, and state it as follows.

$$\begin{aligned} \min_{\vec{x}} \quad & \Gamma(\vec{x}) = \sum_{i=1}^{n} p_i x_i \\ s.t. \quad & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \end{aligned} \tag{SP1Ua}$$

First we establish that the objective function $\Gamma(\vec{x})$ is jointly convex in $\vec{x}$. In order to show this, we provide the following general result.

Lemma 4.2. *Let $G = \{g_{ij}\}$ be an $n \times n$ matrix where*

$$g_{ii} = a_i b_i = a_i(a_i + c_i)$$
$$g_{ij} = a_i a_j \quad \textit{for all } i \neq j$$

Let Δ^n be the determinant of G. Then the following relation holds.

$$\Delta^n = K_n \sum_{i=0}^{n} a_i \prod_{j=1, j\neq i}^{n} c_j$$

where $K_n = \prod_{i=1}^{n} a_i$ and $a_0 = 1$.

Proof. Proof by induction. Details are in Appendix A. □

Noting that Hessian matrix for the objective function of Problem $SP1Ua$ is of the same structure as $G = \{g_{ij}\}$ given above, we directly have the following result.

Lemma 4.3. *The objective function $\Gamma(\vec{x})$ is convex in $\vec{x}$ for a given Q.*

Proof. The Hessian matrix of the objective function of Problem $SP1Ua$ is in the form of G in Lemma 4.2 and has positive definite minors. Details are in Appendix A. □

Having shown the convexity of the objective function, we next, establish the relationship between the optimal allocations of inputs.

Lemma 4.4. *At optimality, for a given production quantity, Q, the input allocations are as follows.*

$$\frac{{x_i}^{**}(Q)}{{x_j}^{**}(Q)} = \frac{\alpha_i}{p_i}\frac{p_j}{\alpha_j} \quad \textit{for all } i,j = 1,\ldots,n \tag{4.5}$$

Proof. Results follow from first order conditions of the objective function in Problem $SP1Ua$. See Appendix A for details. □

Theorem 4.1. *For a given Q,*

(i) the unique optimal solution to Problem $SP1Ua$ is

$${x_i}^{**}(Q) = \frac{\alpha_i}{p_i} A^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \quad \textit{for all } i = 1,..,n \tag{4.6}$$

(ii) the objective function evaluated at the optimal solution ${x_i}^{**}(Q)$ is

$$\Gamma^{**}(Q) \equiv \Gamma(\vec{x}^{**}(Q)) = rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \tag{4.7}$$

(iii) the emission level at the optimal input allocation for a given Q is

$$\epsilon_{U1}^{**}(Q) \equiv \epsilon(\vec{x}^{**}(Q)) = \sum_{i=1}^{n} \frac{\beta_i \alpha_i}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \left(\frac{Q}{A}\right)^{\frac{1}{r}} \tag{4.8}$$

Proof. Equation 4.6 is derived from Equation 3.4 Lemma 4.4. The uniqueness and the optimality follow the convexity of the objective function as shown in 4.3. The objective function value at the optimal solution is evaluated as $\Gamma(\vec{x}^{**}(Q)) = \sum_{j=1}^{n} p_j {x_j}^{**}(Q)$. The emission level at the optimal solution is evaluated as $\epsilon(\vec{x}^{**}(Q)) = \sum_{j=1}^{n} \beta_j {x_j}^{**}(Q)$. □

Remarks.

1. Equation 4.5 suggests a linear relationship between the production inputs which is inversely proportional to their respective unit costs and direclty proportional to their respective elasticities. That is, a resource with a lower unit cost and a higher elasticity is consumed more.

2. Equation 4.6 provides a closed-form expression for the optimal usage of each resource for a given production quantity. Note that ${x_j}^{**}(Q)$ is unique for any Q.

3. Equation 4.6 suggests all inputs are increasing in Q as expected. When $r \leq 1$, the rate of change is also increasing; otherwise, the rate of change is also decreasing.

Corollary 4.1. $\Gamma^{**}(Q)$ *given by Equation 4.7 is a convex function of* Q *for* $r \leq 1$. *For* $r > 1$, $\Gamma^{**}(Q)$ *is concave in* Q.

So far, we have found the optimal input allocation for a given Q and derived the production cost at that allocation. We are ready to find the optimal production quantity, Q^{**}, that maximizes the overall profit, which is given by

$$\Pi(Q) = \tilde{\Pi}(Q) - \Gamma^{**}(Q) = \tilde{\Pi}(Q) - rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \tag{4.9}$$

Now, we move on to Problem $P1U^*a$, which is Problem $P1U^*$ studied under the Cobb-Douglas production function.

$$\max_{Q} \;\; \Pi(Q) = \tilde{\Pi}(Q) - rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \tag{$P1U^*a$}$$
$$s.t. \;\; Q \geq 0$$

We begin our analysis with some preliminary observations on the objective function of $P1U^*a$.

Lemma 4.5. *1.* $\Pi(Q)$ *is a continuous function of* Q.

2. At $Q = 0$, $\Pi(Q = 0) = \tilde{\Pi}(Q = 0) = -c_s E(D)$.

3. As $Q \to \infty$, $\tilde{\Pi}(Q) \to sE(D) - c_e Q$ *and* $\Pi(Q) \to -\infty$.

4. $\Pi(Q)$ *is a concave function of* Q *for* $r \leq 1$*; and, for* $r > 1$*, if*

$$f(Q) \geq \frac{A^{-\frac{1}{r}}}{s + c_s + c_e}\left(1 - \frac{1}{r}\right)\prod_{j=1}^{n}\left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}-2}. \tag{4.10}$$

Proof. The proof of (4) rests on the second order conditions of the objective function with respect to Q. See Appendix A for details. □

We also consider the first derivative of the objective function with respect to Q in Problem $P1U^*a$ in order to gain some insight on the shape of the objective function.

Lemma 4.6. *The first derivative of the objective function with respect to* Q *in Problem* $P1U^*a$ *is given by*

$$\frac{d\Pi(Q)}{dQ} = (s + c_s) - (s + c_s + c_e)F(Q) - A^{-\frac{1}{r}}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}} Q^{\frac{1}{r}-1} \tag{4.11}$$

1. For $r < 1$, $\frac{d\Pi(Q)}{dQ}|_{Q=0} = s + c_s$ *and* $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -\infty$.

2. For $r = 1$, $\frac{d\Pi(Q)}{dQ}|_{Q=0} = s + c_s - \frac{\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{A}$ *and* $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -c_e - \frac{\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{A}$.

3. For $r > 1$, $\frac{d\Pi(Q)}{dQ}|_{Q\to 0} \to -\infty$ *and* $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -c_e$.

Corollary 4.2. *Suppose that the first order condition of the objective function with respect to* Q *in Problem* $P1U^*a$ *exists. Then it is given by*

$$F(Q^*) = \frac{(s + c_s) - A^{-\frac{1}{r}}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}} {Q^*}^{\frac{1}{r}-1}}{s + c_s + c_e} \tag{4.12}$$

Remarks. Equation 4.12 implies the following

1. As technology level, A, increases, Q^* also increases.
2. As purchase price, p_i, for any input i increases, Q^* decreases.

We have already discussed the concavity of the objective function in Problem $P1U^*a$ in Lemma 4.5. Now, we look at the quasi-concavity condition of the objective function in Problem $P1U^*a$ with respect to Q for $r > 1$.

Corollary 4.3. *For $r > 1$, the objective function of Problem $P1U^*a$ is quasi-concave in Q^* under the following condition:*

$$f(Q^*) \geq \left(1 - \frac{1}{r}\right) \frac{1}{Q^*} \left(\frac{s + c_s}{s + c_s + c_e} - F(Q^*)\right) \tag{4.13}$$

Proof. The proof follows from Lemma 4.5 and Corollary 4.2. □

Corollary 4.4. *1. Under uniformly distributed demand over the interval $[l, u]$, Equations 4.12 and 4.10 become*

$$Q^* + \frac{(u-l)A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}}}{s + c_s + c_e} {Q^*}^{\frac{1}{r}-1} - \frac{u(s + c_s) + lc_e}{s + c_s + c_e} = 0$$

$$Q^{2-\frac{1}{r}} \geq (u - l) \frac{A^{-\frac{1}{r}}}{s + c_s + c_e} \left(1 - \frac{1}{r}\right) \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}$$

2. If demand has a newsvendor distribution with $f(Q) = wd'(Q)e^{wd(Q)}$ where $d'(Q)$ is the first derivative of $d(Q)$ with respect to Q, Equations 4.12 and 4.10 become

$$1 - e^{-wd(Q^*)} + \frac{A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}}}{s + c_s + c_e} {Q^*}^{\frac{1}{r}-1} - \frac{s + c_s}{s + c_s + c_e} = 0$$

$$wd'(Q)e^{wd(Q)} \geq \frac{A^{-\frac{1}{r}}}{s + c_s + c_e} \left(1 - \frac{1}{r}\right) \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}-2}$$

Up to this point, we have gained some insight on the shape of the objective function and derived the first order optimality condition for Problem $P1U^*a$. Now, we combine our results to arrive at the following finding.

Lemma 4.7. 1. *For $r < 1$, $\Pi(Q)$ is a concave function with a unique maximum at Q^* that solves Equation 4.12.*

2. *Suppose $r = 1$. If $s + c_s > \frac{\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{A}$, $\Pi(Q)$ is a concave function with a unique global maximum at Q^* that solves Equation 4.12. Otherwise, $\Pi(Q)$ is decreasing in Q.*

3. *For $r > 1$, Equation 4.12 has at most two solutions.*

   (a) *Suppose Equation 4.12 has no solution. Then $\Pi(Q)$ is a decreasing function of Q.*

   (b) *If there exists a unique solution, Q^*, to Equation 4.12, $\Pi(Q)$ is decreasing in Q and has a saddle point at Q^*.*

   (c) *Suppose there exist two solutions to Equation 4.12, Q_1^* and Q_2^*, where $Q_1^* < Q_2^*$. Then Q_1^* is a local minimum and Q_2^* is a local maximum of $\Pi(Q)$ or both are saddle points.*

Proof. 1. For $r < 1$, $\Pi(Q)$ is concave in Q as stated in Lemma 4.5. By Lemma 4.6, there exists a Q^* that solves Equation 4.12 for $r < 1$. RHS of Equation 4.12 decreases in Q^* with the intercept $\frac{(s+c_s)}{s+c_s+c_e}$ which is less than 1 while LHS is an increasing function of Q^* and takes values between 0 and 1. Therefore, they certainly intersect at exactly one point, which yield the unique solution (See Figure 4.1).

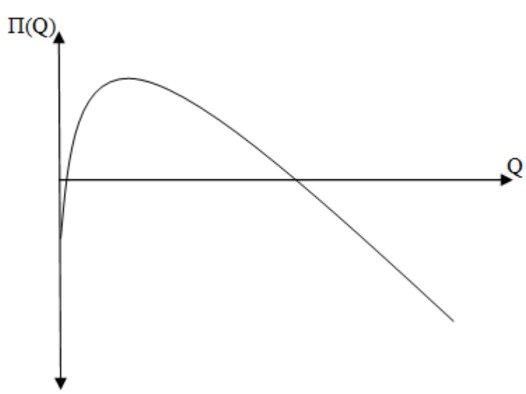

Figure 4.1: Expected Profit as a function of Q, $r < 1$

2. For $r = 1$, $\Pi(Q)$ is concave in Q as stated in Lemma 4.5. By Lemma 4.6, $\frac{d\Pi(Q)}{dQ}$ decreases in Q for $r = 1$. If $s + c_s > \frac{\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{A}$, $\frac{d\Pi(Q)}{dQ}|_{Q=0} > 0$ and there exists a Q^* that solves Equation 4.12 for $r = 1$. RHS of Equation 4.12 is constant and takes the value $\frac{(s+c_s)-A^{-1}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{s+c_s+c_e}$ which is less than 1 while LHS is an increasing function of Q^* and takes values between 0 and 1 (See Figure 4.2). Otherwise, $\Pi(Q)$ is always decreasing in Q (See Figure 4.3).

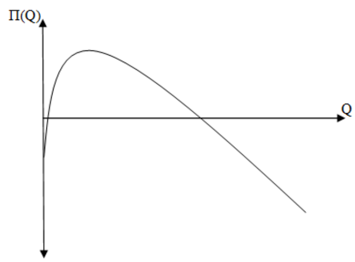

Figure 4.2: Expected Profit as a function of Q, $r = 1$ (a)

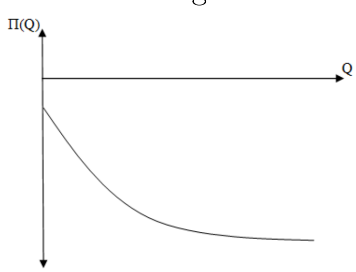

Figure 4.3: Expected Profit as a function of Q, $r = 1$ (b)

3. Note that $\frac{d\Pi(Q)}{dQ} = \frac{s+c_s}{s+c_s+c_e} - \frac{A^{-\frac{1}{r}}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}}}{s+c_s+c_e}Q^{\frac{1}{r}-1} - F(Q)$. For $r > 1$, $\frac{s+c_s}{s+c_s+c_e} - \frac{A^{-\frac{1}{r}}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}}}{s+c_s+c_e}Q^{\frac{1}{r}-1}$ is increasing in Q. It goes to $-\infty$ as $Q \to 0$ and converges to $\frac{(s+c_s)}{s+c_s+c_e}$ as $Q \to \infty$. $F(Q)$ is increasing in Q and takes values between 0 and 1. Therefore, they may have no intersection, exactly one or two intersections depending on the problem parameters.

   (a) Suppose there exists no solution to Equation 4.12. Then $\Pi(Q)$ is always decreasing in Q because $\frac{d\Pi(Q)}{dQ} < 0$ (See Figure 4.4).

   (b) Suppose there exists a unique solution, Q^*, to Equation 4.12. Q^* cannot be a local minimum because $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -c_e$ by Lemma 4.6.

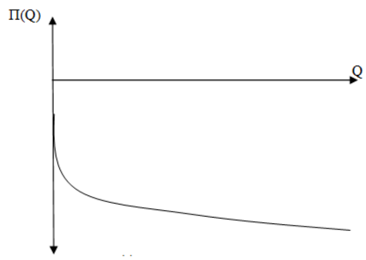

Figure 4.4: Expected Profit as a function of Q, $r > 1$ (a)

Q^* cannot be a local maximum because $\frac{d\Pi(Q)}{dQ}|_{Q\to 0} \to -\infty$ by Lemma 4.6. Then the solution is a saddle point (See Figure 4.5).

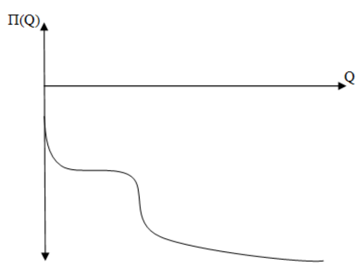

Figure 4.5: Expected Profit as a function of Q, $r > 1$ (b)

(c) Suppose there exist two solutions Q_1^* and Q_2^* where $Q_1^* < Q_2^*$. Q_1^* is either a saddle point or a local minimum because $\frac{d\Pi(Q)}{dQ}|_{Q\to 0} \to -\infty$ by Lemma 4.6. Q_2^* is either a saddle point or a local maximum because $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -c_e$ by Lemma 4.6. Then there are two possible combinations. Either Q_1^* is local minimum and Q_2^* is a local maximum (See Figure 4.6) or both are saddle points (See Figure 4.7).

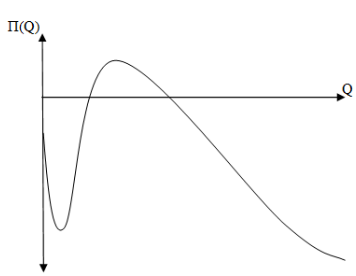

Figure 4.6: Expected Profit as a function of Q, $r > 1$ (c1)

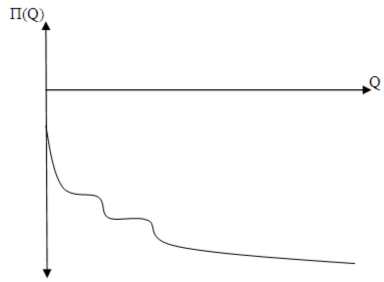

Figure 4.7: Expected Profit as a function of Q, $r > 1$ (c2)

□

Theorem 4.2. *Let Q^{**} be the optimal solution to Problem $P1Ua$.*

1. *For $r < 1$, there exists a unique positive optimal solution, $Q^{**} = Q^*$ where Q^* solves Equation 4.12.*
2. *For $r = 1$, there is a unique optimal solution which is*

$$Q^{**} = \begin{cases} F^{-1}\left(\frac{(s+c_s)-A^{-\frac{1}{r}}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{s+c_s+c_e}\right) & \text{if } s + c_s \geq \frac{\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{A} \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

3. *For $r > 1$,*
   - *(a) If Equation 4.12 has no solution or has a unique solution, then $Q^{**} = 0$.*
   - *(b) Suppose Equation 4.12 has two solutions: Q_1^* and Q_2^*, where $Q_1^* < Q_2^*$. Then $Q^{**} \epsilon \{0, Q_2^*\}$.*

Proof. Follows from Lemma 4.7 □

So far, we have analyzed Problem $P1U$ under the Cobb-Douglas production function. We have found the production cost minimizing input allocation, $x_i^{**}(Q)$ for a given Q. Then we have found the optimal production quantity Q^{**} that maximizes the expected profit. Next, we consider the case where a carbon cap constraint exists.

4.1.2 A Carbon Cap is Imposed

We aim to find the optimal production quantity and input allocation in Problem $P1$ under the Cobb-Douglas production function when a strict carbon cap is imposed. We refer to this problem as Problem $P1a$.

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \tilde{\Pi}(Q) - \Gamma(\vec{x}) \\ s.t. \quad & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \\ & \sum_{i=1}^{n} \beta_i x_i \leq \kappa \\ & Q \geq 0 \end{aligned} \tag{P1a}$$

Theorem 4.3. *Let Q^{**} be the optimal solution to Problem $P1Ua$. Q^{**} is the optimal solution to Problem $P1a$ if the following condition holds:*

$$\sum_{i=1}^{n} \frac{\beta_i \alpha_i}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \left(\frac{Q^{**}}{A}\right)^{\frac{1}{r}} \leq \kappa \tag{4.15}$$

Proof. Follows from Theorem 4.1. □

If the emission level at the optimal solution of the carbon cap unconstrained problem satisfies the carbon cap constraint, then we conclude that it is also optimal for the carbon cap constrained problem. Otherwise, we solve the problem for the case where the carbon cap constraint is binding, $\sum_{i=1}^{n} \beta_i x_i = \kappa$.

Before we begin to work on Problem $P1a$, we look at a subproblem where we minimize the emission level at a given production quanitity, Q. It is given by Problem $CE1a$:

$$\begin{aligned} \min_{\vec{x}} \quad & \epsilon(\vec{x}) = \sum_{i=1}^{n} \beta_i x_i \\ s.t. \quad & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \end{aligned} \tag{CE1a}$$

Note that this problem has the same structure as Problem $SP1Ua$. We have a linear sum of the variables in the objective function and the same Cobb-Douglas production function constraint. Therefore, we can use the results we have derived in the previous section.

Lemma 4.8. *Let $\kappa_{min}(Q)$ be the minimum emission level for a given Q under the Cobb-Douglas production function. Then,*

(i) the optimal solution is given by

$$x_i^{**}(Q) = \frac{\alpha_i}{\beta_i} A^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \quad \text{for all } i = 1, \ldots, n \tag{4.16}$$

(ii) the minimum emission level for a given Q is

$$\kappa_{min}(Q) = r \prod_{j=1}^{n} \left(\frac{\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \left(\frac{Q}{A}\right)^{\frac{1}{r}} \tag{4.17}$$

Proof. The optimal solution of Problem $CE1a$ follows from Theorem 4.1. Then the minimum emission level for a given Q is evaluated as $\kappa_{min}(Q) = \epsilon(\vec{x}^{**}(Q)) = \sum_{i=1}^{n} \beta_i x_i^{**}(Q)$. □

Corollary 4.5. *For given carbon cap κ, there exists an upper bound on the production quantity, $Q \leq Q_{max}(\kappa)$ where*

$$Q_{max}(\kappa) = \frac{A\kappa^r}{r^r} \prod_{j=1}^{n} \left(\frac{\alpha_j}{\beta_j}\right)^{\frac{\alpha_j}{r}} \tag{4.18}$$

Proof. By Lemma 4.8, $\kappa_{min}(Q)$ is the minimum emission level for a given Q. Then, production of any Q units is feasible only if $\kappa_{min}(Q) \leq \kappa$. □

This result establishes the feasibility condition for a given constrained problem. In our numerical study, we use this condition to choose and construct the experimental set.

Now, we move on to Problem $P1a$. We rewrite this problem by introducing a nonnegative scalar Lagrange multiplier, λ, for the carbon cap constraint and name the new equivalent problem for a given λ, Problem $\widehat{P1a}$, which is formally stated as follows

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \tilde{\Pi}(Q) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - \kappa\right) \\ s.t. \quad & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \\ & Q \geq 0 \end{aligned} \tag{$\widehat{P1a}$}$$

Here, we can interpret λ as the penalty cost of exceeding the carbon emission cap.

We proceed in our analysis as in the unconstrained case. That is, we first consider Problems $SP1$ and $P1^*$ for the Cobb-Douglas production function in order to derive the optimal production policy for Problem $\widehat{P1a}$ which is the constrained counterpart of $P1a$.

We construct Problem $SP1$ under the Cobb-Douglas production function called Problem $SP1a$ with the objective function $\Gamma(\vec{x}) + \lambda\epsilon(\vec{x})$ - that is, we consider only the terms containing the decision variable vector $\vec{x}$ and ignore the constant part. Then, Problem $SP1a$ is formally stated as follows.

$$\begin{aligned} \min_{\vec{x}} \quad & \sum_{i=1}^{n} (p_i + \lambda\beta_i)x_i \\ s.t. \quad & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \end{aligned} \tag{SP1a}$$

Noting that Problem $SP1a$ has a structure similar to Problem $SP1Ua$, the result below follows from Theorem 4.1.

Corollary 4.6. *For a given Q and λ,*

(i) the unique optimal solution to Problem $SP1a$ is

$$x_i^{**}(Q,\lambda) = \frac{\alpha_i}{p_i + \lambda\beta_i} A^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \quad \textit{for all } i = 1, \ldots, n \tag{4.19}$$

(ii) the objective function evaluated at the optimal solution $x_i{}^{**}(Q, \lambda)$ is

$$\Gamma^{**}(Q, \lambda) + \lambda \epsilon^{**}(Q, \lambda) = rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \tag{4.20}$$

Next, we consider Problem $P1^*a$, which is Problem $P1^*$ under the Cobb-Douglas production function. After substituting our findings in Corollary 4.6, for a given λ, the objective function is given as follows

$$\widehat{\Pi}(Q) = \tilde{\Pi}(Q) + \lambda \kappa - rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \tag{4.21}$$

$$\begin{aligned} \max_{Q} \quad & \widehat{\Pi}(Q) = \tilde{\Pi}(Q) + \lambda \kappa - rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \\ s.t. \quad & Q \geq 0 \end{aligned} \tag{$P1^*a$}$$

Note that Problem $P1^*a$ differs from Problem $P1^*$ only in the coefficients of in front of the term $Q^{\frac{1}{r}}$ where p_j is replaced by the effective cost $(p_j + \lambda \beta_j)$ and in a constant, λ. The analysis below for the constrained case is along the lines for the unconstrained case.

Corollary 4.7. *For a given λ,*

1. *$\widehat{\Pi}(Q)$ is a continuous function of Q.*

2. *At $Q = 0$, $\widehat{\Pi}(Q = 0) = -c_s E(D) + \lambda \kappa$.*

3. *As $Q \to \infty$, $\tilde{\Pi}(Q) \to sE(D) - c_e Q$ and $\widehat{\Pi}(Q) \to -\infty$.*

4. *$\widehat{\Pi}(Q)$ is a concave function of Q for $r \leq 1$; and, for $r > 1$, if*

$$f(Q) \geq \frac{A^{-\frac{1}{r}}}{s + c_s + c_e} \left(1 - \frac{1}{r} \right) \prod_{j=1}^{n} \left(\frac{p_j + \lambda \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r} - 2}. \tag{4.22}$$

Next, we consider the first derivative of the objective function with respect to Q in Problem $P1^*a$ in order to gain some insight on the shape of the objective function.

Corollary 4.8. *For a given λ, the first derivative of the objective function with respect to Q in Problem $P1^*a$ is given by*

$$\frac{d\widehat{\Pi}(Q)}{dQ} = (s + c_s) - (s + c_s + c_e)F(Q) - A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i + \lambda\beta_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}} Q^{\frac{1}{r}-1} \quad (4.23)$$

1. *For $r < 1$, $\frac{d\widehat{\Pi}(Q)}{dQ}|_{Q=0} = s + c_s$ and $\frac{d\widehat{\Pi}(Q)}{dQ}|_{Q\to\infty} \to -\infty$.*
2. *For $r = 1$, $\frac{d\widehat{\Pi}(Q)}{dQ}|_{Q=0} = s + c_s - \frac{\prod_{i=1}^{n}\left(\frac{p_i+\lambda\beta_i}{\alpha_i}\right)^{\alpha_i}}{A}$ and $\frac{d\widehat{\Pi}(Q)}{dQ}|_{Q\to\infty} \to -c_e - \frac{\prod_{i=1}^{n}\left(\frac{p_i+\lambda\beta_i}{\alpha_i}\right)^{\alpha_i}}{A}$.*
3. *For $r > 1$, $\frac{d\widehat{\Pi}(Q)}{dQ}|_{Q\to 0} \to -\infty$ and $\frac{d\widehat{\Pi}(Q)}{dQ}|_{Q\to\infty} \to -c_e$.*

Corollary 4.9. *Suppose that the first order condition of the objective function with respect to Q in Problem $P1^*a$ exists for a given λ. Then it is given by*

$$F(Q^*) = \frac{(s + c_s) - A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i+\lambda\beta_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}} Q^{*\frac{1}{r}-1}}{s + c_s + c_e} \quad (4.24)$$

Remarks. For a given λ, Equation 4.24 implies the following

1. As technology level, A, increases, Q^* also increases.
2. As purchase price, p_i, for any input i increases, Q^* decreases.

For a given λ, we have already discussed the concavity of the objective function in Problem $P1^*a$ in Corollary 4.7. Now, we look at the quasi-concavity condition of the objective function in Problem $P1^*a$ with respect to Q for $r > 1$ under a given λ.

Corollary 4.10. *Supposer $r > 1$. Then for a given λ, the objective function of Problem $P1^*a$ is quasi-concave in Q^* under the following condition:*

$$f(Q^*) \geq \left(1 - \frac{1}{r}\right) \frac{1}{Q^*} \left(\frac{s + c_s}{s + c_s + c_e} - F(Q^*)\right) \quad (4.25)$$

Corollary 4.11. *1. Under uniformly distributed demand over the interval $[l, u]$, Equations 4.24 and 4.22 become*

$$Q^* + \frac{(u-l)A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i+\lambda\beta_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}}}{s+c_s+c_e} Q^{*\frac{1}{r}-1} - \frac{u(s+c_s)+lc_e}{s+c_s+c_e} = 0$$
$$Q^{2-\frac{1}{r}} \geq (u-l)\frac{A^{-\frac{1}{r}}}{s+c_s+c_e}\left(1-\frac{1}{r}\right)\prod_{j=1}^{n}\left(\frac{p_j+\lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}$$

2. *If demand has a newsvendor distribution with $f(Q) = wd'(Q)e^{wd(Q)}$ where $d'(Q)$ is the first derivative of $d(Q)$ with respect to Q, Equations 4.24 and 4.22 become*

$$1 - e^{-wd(Q^*)} + \frac{A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i+\lambda\beta_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}}}{s+c_s+c_e} Q^{*\frac{1}{r}-1} - \frac{s+c_s}{s+c_s+c_e} = 0$$
$$wd'(Q)e^{wd(Q)} \geq \frac{A^{-\frac{1}{r}}}{s+c_s+c_e}\left(1-\frac{1}{r}\right)\prod_{j=1}^{n}\left(\frac{p_j+\lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}-2}$$

Up to this point, we have gained some insight on the shape of the objective function and derived the first order optimality condition of Problem $P1^*a$ for a given λ. Now, we combine our results to arrive at the following finding.

Corollary 4.12. *For a given λ,*

1. *For $r < 1$, $\widehat{\Pi}(Q)$ is a concave function with a unique maximum at Q^* that solves Equation 4.24.*

2. *Suppose $r = 1$. If $s + c_s > \frac{\prod_{i=1}^{n}\left(\frac{p_i+\lambda\beta_i}{\alpha_i}\right)^{\alpha_i}}{A}$, $\widehat{\Pi}(Q)$ is a concave function with a unique global maximum at Q^* that solves Equation 4.24. Otherwise, $\widehat{\Pi}(Q)$ is decreasing in Q.*

3. *For $r > 1$, Equation 4.24 has at most two solutions.*

   (a) *Suppose Equation 4.24 has no solution. Then $\widehat{\Pi}(Q)$ is a decreasing function of Q.*

   (b) *If there exists a unique solution, Q^*, to Equation 4.24, $\widehat{\Pi}(Q)$ is decreasing in Q and has a saddle point at Q^*.*

(c) *Suppose there exist two solutions to Equation 4.24, Q_1^* and Q_2^*, where $Q_1^* < Q_2^*$. Then Q_1^* is a local minimum and Q_2^* is a local maximum of $\widehat{\Pi}(Q)$ or both are saddle points.*

Note that in this formulation, we assume that the carbon cap constraint is binding. Therefore, we have another equation that we need to satisfy at the optimal solution: $\epsilon(\vec{x}^{**}) = \kappa$. For a given λ and Q, we have derived the optimal allocation of inputs given by Equation 4.19. Therefore, the optimal production quantity Q^{**} found in Problem $P1^*a$ needs to satisfy this equation as well.

Lemma 4.9. *At optimality λ^{**} solves,*

$$\kappa = \left(\frac{Q^{**}(\lambda^{**})}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda^{**}\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n} \frac{\alpha_i \beta_i}{p_i + \lambda^{**}\beta_i} \tag{4.26}$$

Lemma 4.10. *$Q^{**}(\lambda^{**})$ is a decreasing and one-to-one function of λ^{**}.*

Proof. From Equation 4.26, we get

$$\kappa A^{\frac{1}{r}} Q^{**}(\lambda^{**})^{-\frac{1}{r}} = \prod_{j=1}^{n} \left(\frac{p_j + \lambda^{**}\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n} \frac{\alpha_i \beta_i}{p_i + \lambda^{**}\beta_i}$$

Then, we have

$$\begin{aligned}
\frac{dQ^{**}(\lambda^{**})}{d\lambda^{**}} \left(-\frac{\kappa A^{\frac{1}{r}}}{r Q^{**}(\lambda^{**})^{\frac{1}{r}+1}}\right) &= -\prod_{k=1}^{n} \left(\frac{p_k + \lambda^{**}\beta_k}{\alpha_k}\right)^{\frac{\alpha_k}{r}} \left(\sum_{i=1}^{n} \frac{\alpha_i \beta_i^2}{(p_i + \lambda^{**}\beta_i)^2}\right) \\
&\quad + \sum_{i=1}^{n} \frac{\alpha_i \beta_i}{p_i + \lambda^{**}\beta_i} \left(\sum_{j=1}^{n} \frac{\alpha_j}{r} \frac{\beta_j}{\alpha_j} \frac{\alpha_j}{p_j + \lambda^{**}\beta_j} \prod_{k=1}^{n} \left(\frac{p_k + \lambda^{**}\beta_k}{\alpha_k}\right)^{\frac{\alpha_k}{r}}\right) \\
&= r \prod_{k=1}^{n} \left(\frac{p_k + \lambda^{**}\beta_k}{\alpha_k}\right)^{\frac{\alpha_k}{r}} \\
&\quad \times \left(\left(\sum_{j=1}^{n} \frac{\beta_j}{p_j + \lambda^{**}\beta_j} \frac{\alpha_j}{r}\right)^2 - \sum_{i=1}^{n} \left(\frac{\beta_i}{p_i + \lambda^{**}\beta_i}\right)^2 \frac{\alpha_i}{r}\right)
\end{aligned}$$

From which we get,

$$\frac{dQ^{**}(\lambda^{**})}{d\lambda^{**}} = -\frac{r^2 Q^{**}(\lambda^{**})^{\frac{1}{r}+1}}{\kappa A^{\frac{1}{r}}} \prod_{k=1}^{n} \left(\frac{\alpha_k}{p_k + \lambda^{**}\beta_k} \right)^{\frac{\alpha_k}{r}}$$
$$\times \left(\left(\sum_{j=1}^{n} \frac{\beta_j}{p_j + \lambda^{**}\beta_j} \frac{\alpha_j}{r} \right)^2 - \sum_{i=1}^{n} \left(\frac{\beta_i}{p_i + \lambda^{**}\beta_i} \right)^2 \frac{\alpha_i}{r} \right)$$

Let Y be a discrete random variable taking values $\{y_i : i = 1, \ldots, n\}$. Note that $E[Y] = \sum_{i=1}^{n} y_i Pr(Y = y_i)$ and $E[Y^2] = \sum_{i=1}^{n} y_i^2 Pr(Y = y_i)$. Let $y_i = \frac{\beta_i}{p_i + \lambda^{**}\beta_i}$ and $Pr(Y = y_i) = \frac{\alpha_i}{r}$. Then

$$Var(Y) = E[Y^2] - E[Y]^2 = \sum_{i=1}^{n} \left(\frac{\beta_i}{p_i + \lambda^{**}\beta_i} \right)^2 \frac{\alpha_i}{r} - \left(\sum_{j=1}^{n} \frac{\beta_j}{p_j + \lambda^{**}\beta_j} \frac{\alpha_j}{r} \right)^2$$

Since $Var(Y) > 0$, we have $\frac{dQ^{**}(\lambda^{**})}{d\lambda^{**}} < 0$. Hence $Q^{**}(\lambda^{**})$ is a strictly decreasing, consequently, one-to-one function of λ^{**}. □

We have derived the characteristics of the objective function and its roots in Problem $P1^*a$ so far. Now, we are ready to propose a solution for the original problem Problem $P1a$.

Theorem 4.4. *Let Q^{**} and λ^{**} give the optimal solution to Problem $P1a$.*

1. *For $r < 1$, there exists a unique positive optimal solution, $Q^{**} = Q^*$ where Q^* solves Equation 4.24 and λ^{**} solves Equation 4.26.*

2. *For $r = 1$, there is a unique optimal solution. If $s + c_s \geq \frac{\prod_{i=1}^{n} \left(\frac{p_i + \lambda^{**}\beta_i}{\alpha_i} \right)^{\alpha_i}}{A}$,*

$$Q^{**} = F^{-1} \left(\frac{(s + c_s) - A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i + \lambda^{**}\beta_j}{\alpha_i} \right)^{\alpha_i}}{s + c_s + c_e} \right)$$

   and λ^{**} solves Equation 4.26. Otherwise $Q^{**} = 0$ and $\lambda^{**} = 0$.

3. *For $r > 1$, the optimal solution $(Q^{**}, \lambda^{**}) \epsilon \{(0,0),\ (\tilde{Q}^{**}, \tilde{\lambda}^{**})\}$ where $\tilde{\lambda}^{**}$ solves Equation 4.26 and $\tilde{Q}^{**}$ solves Equation 4.24.*

Proof. Follows from Corollary 4.12, Lemmas 4.9 and 4.10. □

Theorem 4.4 gives us the optimal production quantity Q^{**} and corresponding λ^{**} value for Problem $P1a$ where the carbon cap is binding. Then, we can compute the optimal input allocation at (Q^{**}, λ^{**}) from Equation 4.19.

We have derived the optimal production policy for Problem 1 under Cobb-Douglas production function. Now, we move on to Leontief production function in the next section.

4.2 The Leontief Production Function Model

In this section, we study the same optimization problem (Problem 1) under the Leontief production function. Recall that, in our model, the Leontief production function is assumed to have the following form: $Q = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\}$ where $A_i(> 0)$ is the input-specific technology level and $\alpha_i(> 0)$ is the input elasticity of input $i = 1, \ldots, n$.

Next, we study Problems $SP1U$ and $P1U^*$ (where we ignore the carbon cap constraint) and analyze Problems $CE1$, $SP1$ and $P1^*$ (where we consider the carbon cap) separately.

4.2.1 No Carbon Cap is Imposed

In this section, we look at Problem $P1U$ under the Leontief production function, which is labeled as Problem $P1Ub$, and find the optimal production quantity and input allocation.

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \tilde{\Pi}(Q) - \Gamma(\vec{x}) \\ s.t. \quad & Q = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\} \\ & Q \geq 0 \end{aligned} \tag{P1Ub}$$

We first consider Problems $SP1U$ and $P1U^*$ for the Leontief production function to derive the optimal production policy for Problem $P1Ub$.

Problem $SP1U$ under the Leontief Production function, Problem $SP1Ub$, is stated as follows.

$$\min_{\vec{x}} \ \Gamma(\vec{x}) = \sum_{i=1}^{n} p_i x_i \tag{SP1Ub}$$
$$s.t. \ \ Q = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\}$$

For this problem, we have the following result.

Theorem 4.5. *For a given , Q,*

(i) the optimal solution to $SP1Ub$ is given by

$$x_i^{**}(Q) = \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \quad \textit{for all } i = 1, \ldots, n \tag{4.27}$$

(ii) the objective function evaluated at the optimal solution $x_i^{**}(Q)$ is

$$\Gamma^{**}(Q) \equiv \Gamma(\vec{x}^{**}(Q)) = \sum_{i=1}^{n} p_i \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{4.28}$$

(iii) the emission level at the optimal input allocation for a given Q is

$$\epsilon_{U1}^{**}(Q) \equiv \epsilon(\vec{x}^{**}(Q)) = \sum_{i=1}^{n} \beta_i \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{4.29}$$

Proof. See Appendix A for the derivation of $x_i^{**}(Q)$. The objective function value at the optimal solution is evaluated as $\Gamma(\vec{x}^{**}(Q)) = \sum_{j=1}^{n} p_j x_j^{**}(Q)$. The emission level at the optimal solution is evaluated as $\epsilon(\vec{x}^{**}(Q)) = \sum_{j=1}^{n} \beta_j x_j^{**}(Q)$. □

Remarks.

1. Equation 4.27 provides a closed-form expression for the optimal usage of each resource for a given production quantity. Note that $x_j^{**}(Q)$ is unique for any Q.

2. Equation 4.27 suggests the optimal quantity of each input i depends only on the production function parameters, elasticity α_i and the technology A_i of the corresponding input, for any Q.

3. Equation 4.27 also suggests that all inputs are increasing in Q as expected.

The general conditions on convexity of $\Gamma^{**}(Q)$ in Q appear to be quite complex; however, we have been able to obtain a sufficient condition for convexity which we provide below.

Corollary 4.13. *$\Gamma^{**}(Q)$ given by Equation 4.28 is a convex function of Q if $\alpha_i \leq 1, \forall i = 1, \ldots, n$.*

We have derived the optimal solution for a given Q and are ready to find the optimal production quantity, Q^{**}, that maximizes the overall profit. We move on to Problem $P1U^{*}b$, which is Problem $P1U^{*}$ studied under the Leontief production function.

$$\max_{Q} \ \Pi(Q) = \tilde{\Pi}(Q) - \sum_{i=1}^{n} p_i \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \qquad (P1U^{*}b)$$
$$s.t. \ \ Q \geq 0$$

First, we present some preliminary observations on the objective function of $P1U^{*}b$.

Lemma 4.11. *1. $\Pi(Q)$ is a continuous function of Q.*

2. At $Q = 0$, $\Pi(Q = 0) = \tilde{\Pi}(Q = 0) = -c_s E(D)$.

3. As $Q \to \infty$, $\tilde{\Pi}(Q) \to sE(D) - c_e Q$ and $\Pi(Q) \to -\infty$.

4. For $\alpha_i \leq 1, \forall i = 1, \ldots, n$, $\Pi(Q)$ is concave in Q. When any $\alpha_i > 1$ for $i = 1, \ldots, n$, $\Pi(Q)$ is concave in Q if

$$f(Q) \geq \frac{\sum_{i=1}^{n} \frac{p_i(1-\frac{1}{\alpha_i})}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{\frac{1}{\alpha_i}-2}}{s + c_s + c_e} \qquad (4.30)$$

Proof. The details of the derivation in (4) is given in Appendix A. □

We also consider the first derivative of the objective function with respect to Q in Problem $P1U^{*}b$ in order to gain some insight on the shape of the objective function.

Lemma 4.12. *The first derivative of the objective function with respect to Q in Problem $P1U^*b$ is given by*

$$\frac{d\Pi(Q)}{dQ} = s + c_s - (s + c_s + c_e)F(Q) - \sum_{i=1}^{n} \frac{p_i}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{\frac{1}{\alpha_i}-1} \tag{4.31}$$

1. *For $\alpha_i \leq 1, \forall i = 1, \ldots, n$, $\frac{d\Pi(Q)}{dQ}|_{Q=0} = s + c_s$ and $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -\infty$.*
2. *When any $\alpha_i > 1$ for $i = 1, \ldots, n$, $\frac{d\Pi(Q)}{dQ}|_{Q\to 0} \to -\infty$ and $\frac{d\Pi(Q)}{dQ}|_{Q\to\infty} \to -\infty$.*

Now, we look at the first order optimality conditions for Problem $P1U^*b$.

Lemma 4.13. *Suppose that the first order condition of the objective function with respect to Q in Problem $P1U^*b$ exists. Then it is given by*

$$F(Q^*) = \frac{(s + c_s) - \sum_{i=1}^{n} \frac{p_i Q^{*\frac{1}{\alpha_i}-1}}{\alpha_i A_i^{\frac{1}{\alpha_i}}}}{s + c_s + c_e} \tag{4.32}$$

Proof. Follows from Lemma 4.12. □

Remarks. Equation 4.32 implies the following:

1. As technology level, A_i for any input i increases, Q^* also increases.
2. As purchase price, p_i, for any input i increases, Q^* decreases.

Corollary 4.14. 1. *Under uniformly distributed demand over the interval $[l, u]$, Equations 4.32 and 4.30 become*

$$Q^* + \frac{u-l}{s + c_s + c_e} \sum_{i=1}^{n} \frac{p_i Q^{*(1/\alpha_i - 1)}}{\alpha_i A_i^{1/\alpha_i}} - \frac{u(s + c_s) + l c_e}{s + c_s + c_e} = 0$$

$$1 \geq (u - l) \frac{\sum_{i=1}^{n} \frac{p_i(1-\frac{1}{\alpha_i})}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{\frac{1}{\alpha_i}-2}}{s + c_s + c_e}$$

2. *If demand has a newsvendor distribution with $f(Q) = wd'(Q)e^{wd(Q)}$ where $d'(Q)$ is the first derivative of $d(Q)$ with respect to Q,, Equations 4.32 and 4.30 become*

$$1 - e^{-wd(Q^*)}Q^* + \frac{1}{s+c_s+c_e}\sum_{i=1}^{n}\frac{p_i Q^{*(1/\alpha_i - 1)}}{\alpha_i {A_i}^{1/\alpha_i}} - \frac{s+c_s}{s+c_s+c_e} = 0$$

$$wd'(Q)e^{wd(Q)} \geq \frac{\sum_{i=1}^{n}\frac{p_i(1-\frac{1}{\alpha_i})}{\alpha_i {A_i}^{\frac{1}{\alpha_i}}}Q^{\frac{1}{\alpha_i}-2}}{s+c_s+c_e}$$

Theorem 4.6. *Let Q^{**} be the optimal solution to Problem $P1Ub$.*

1. *For $\alpha_i \leq 1, \forall i = 1,\ldots,n$, there exists a unique positive optimal solution, $Q^{**} = Q^*$ where Q^* solves Equation 4.32.*

2. *Suppose $\exists\alpha_i > 1, i = 1,\ldots,n$. Let $\vec{Q}^*$ be the set of solutions that satisfy Equations 4.32 and 4.30. Then the unique optimal solution $Q^{**}\epsilon\{0, \vec{Q}^*\}$.*

Proof. Follows from Lemmas 4.11, 4.12 and 4.13. □

Remarks. For any $\alpha_i > 1, i = 1,\ldots,n$ case, we cannot guarantee that the largest Q^* that solves Equation 4.32 is the optimal. Hence we need to check each and every extremum for optimality.

We have analyzed Problem $P1U$ under Leontief production function. We have found the input allocation that minimizes the production cost for a given Q, $x_i^{**}(Q)$. Then we have found the optimal production quantity Q^{**} that maximizes the expected profit. Next, we consider the problem with carbon cap constraint.

4.2.2 A Carbon Cap is Imposed

Our objective is to find the optimal production quantity and input allocation in Problem $P1$ under Leontief production function. We refer to this problem as

Problem $P1b$.

$$\begin{aligned}
\max_{Q,\vec{x}} \quad & \tilde{\Pi}(Q) - \Gamma(\vec{x}) \\
s.t. \quad & Q = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\} \\
& \sum_{i=1}^{n} \beta_i x_i \leq \kappa \\
& Q \geq 0
\end{aligned} \tag{P1b}$$

Theorem 4.7. *Let Q^{**} be the optimal solution to Problem $P1Ub$. Q^{**} is the optimal solution Problem $P1b$ if the following condition holds:*

$$\sum_{i=1}^{n} \beta_i \left(\frac{Q^{**}}{A_i} \right)^{\frac{1}{\alpha_i}} \leq \kappa \tag{4.33}$$

Proof. Follows from Theorem 4.5. □

If the emission level at the optimal solution of the carbon cap unconstrained problem satisfies the carbon cap constraint, then the optimal for the carbon cap constrained problem is the same as carbon cap unconstrained problem. Otherwise, we need to solve the problem with binding carbon cap constraint, $\sum_{i=1}^{n} \beta_i x_i = \kappa$.

Before moving on with Problem $P1b$, we consider a subproblem where we minimize the emission level at a given production quanitity, Q. It is given by Problem $CE1b$:

$$\begin{aligned}
\min_{\vec{x}} \quad & \epsilon(\vec{x}) = \sum_{i=1}^{n} \beta_i x_i \\
s.t. \quad & Q = \min_i\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\}
\end{aligned} \tag{CE1b}$$

We can observe that this problem has the same structure as Problem $SP1Ub$ with a linear sum of the variables in the objective function and the same Leontief production function constraint. Therefore, we can use the results we have derived in the previous section.

Lemma 4.14. *For Problem $CE1b$,*

(i) the optimal solution for a given Q is

$$x_i^{**}(Q) = \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \quad \text{for all } i = 1, .., n \tag{4.34}$$

(ii) the minimum emission level for a given Q is

$$\kappa_{min}(Q) \equiv \epsilon(\vec{x}^{**}(Q)) = \sum_{i=1}^{n} \beta_i \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{4.35}$$

Proof. Follows from Theorem 4.5. □

Now, we continue working on Problem $P1b$. We can rewrite this problem by introducing a nonnegative scalar Lagrange multiplier, λ, for the carbon cap constraint and name the new equivalent problem for a given λ, Problem $\widehat{P1b}$, is

$$\begin{aligned} \max_{Q,\vec{x}} \quad & \tilde{\Pi}(Q) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - \kappa\right) \\ s.t. \quad & Q = \min_i\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\} \\ & Q \geq 0 \end{aligned} \tag{$\widehat{P1b}$}$$

Here, λ can be interpreted as a penalty cost of exceeding carbon emission. We first study Problems $SP1$ and $P1^*$ for Leontief production function in order to derive the optimal production policy for Problem $\widehat{P1b}$ which is equivalent to $P1b$.

Problem $SP1$ under Leontief Production function, Problem $SP1b$, is

$$\begin{aligned} \min_{\vec{x}} \quad & \Gamma(\vec{x}) + \lambda\epsilon(\vec{x}) = \sum_{i=1}^{n}(p_i + \lambda\beta_i)x_i \\ s.t. \quad & Q = \min_i\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\} \end{aligned} \tag{SP1a}$$

Lemma 4.15. *For a given Q and λ, the optimal solution to Problem $SP1b$ is*

$$x_i^{**}(Q) = x_i^{**}(Q, \lambda) = \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \quad \text{for all } i = 1, .., n \tag{4.36}$$

Proof. Problem $SP1b$ has similar structure to Problem $SP1Ub$. The results follow from Theorem 4.5. □

Remarks. Equation 4.36 that the optimal input allocation for Problem $SP1b$ only depends on Q. It implies that optimal allocation of the inputs is not affected by the value that we give to the carbon constraint.

Next, we consider Problem $P1^*b$, which is Problem $P1^*$ under Leontief production function. We substitute our findings in Lemma 4.15 and for a given λ, we have

$$\begin{aligned} \max_{Q} \quad & \widehat{\Pi}(Q) = \tilde{\Pi}(Q) + \lambda\kappa - \sum_{i=1}^{n} (p_i + \lambda\beta_i) \left(\frac{Q}{A_i}\right)^{\frac{1}{\alpha_i}} \\ s.t. \quad & Q \geq 0 \end{aligned} \tag{$P1^*b$}$$

In this formulation, we assume that the carbon cap constraint is binding. Therefore, we have another equation that we need to satisfy at the optimal solution: $\epsilon(\vec{x}^{**}) = \kappa$. For a given λ and Q, we have derived the optimal allocation of inputs in Equation 4.36. We also observed that the optimal allocation of inputs depends only on Q. Therefore, we can find the optimal production quantity Q^{**} for Problem $P1^*b$ from the equality given by the carbon cap constraint. Note this solution is also the optimal solution to Problem $P1b$.

Theorem 4.8. *Let Q^{**} be the optimal production quantity for Problem $P1b$. Q^{**} solves*

$$\kappa = \sum_{i=1}^{n} \beta_i \left(\frac{Q^{**}}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{4.37}$$

Proof. The righthand side of the equation is always increasing in Q. Therefore, there exists a unique solution Q^{**} for any carbon cap κ. $\square$

Theorem 4.8 gives us the optimal production quantity Q^{**} for Problem $P1a$ where the carbon cap is binding. Then we can compute the optimal input allocation at Q^{**} from Equation 4.36.

Finally, we consider the value of λ^{**} at the optimal solution.

Corollary 4.15. *At optimality, λ^{**} solves*

$$\lambda^{**} = \frac{(s+c_s) - (s+c_s+c_e)F(Q^{**}) - \sum_{i=1}^{n} \frac{p_i}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{**\frac{1}{\alpha_i}-1}}{\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{**\frac{1}{\alpha_i}-1}}$$

Proof. See Appendix A. □

Chapter 5

Problem 2: Carbon Permit Contract Design for a Manufacturer with Multiple Inputs

In this chapter, we study the production of a single item with multiple inputs and stochastic demand under an emission permit trading scheme.

In our problem, we analyze a series of decisions on the timeline (Fig. 5.1). The first decision is advance carbon permit purchase, κ, at price c per unit carbon emitted. At this decision epoch, which we call as Period 1, the demand, D, is not revealed but has a known distribution with p.d.f. $f(.)$ and c.d.f. $F(.)$. In addition, the future carbon market prices, p_b (carbon permit purchase price) and p_s (carbon permit selling price), remain unknown, but have a joint distribution with p.d.f $g(.,.)$ and c.d.f. $G(.,.)$, which may be dependent on demand.

After the initial carbon permit purchase, we move on to the second decision epoch which we call Period 2. At this epoch, the demand and the carbon market prices are revealed. We assume that the demand needs to be fully satisfied. We also assume that the carbon permit is bought at price p_b and sold at p_s, and that $p_b \geq p_s$ in order to prevent arbitrage. Under the revealed information, we decide on the allocation of the inputs, $\vec{x}$ where $\vec{x} = (x_1, \ldots, x_n)$, and carbon trade policy, (q_b, q_s) where q_b is the carbon permit bought at price p_b and q_s is the carbon permit sold at price p_s, in order to produce the given demand. Note that, by construct $q_b q_s = 0$.

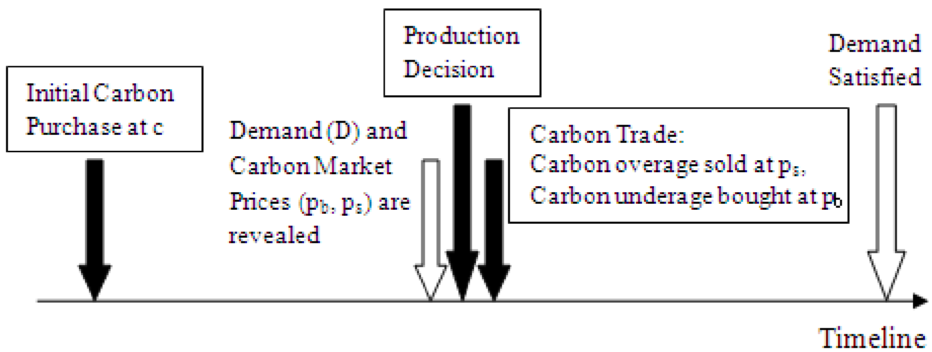

Figure 5.1: Decision timeline for Problem 2

The output of the production process, D, is given by a production function in the form $D = \phi(\vec{x})$. Each input, x_i, $i = 1, .., n$, has a procurement cost per unit, p_i, and carbon emission per unit β_i. We assume linear production cost and carbon emission in our problem. Hence, the production cost is

$$\Gamma(\vec{x}) = \sum_{i=1}^{n} p_i x_i \tag{5.1}$$

and the carbon emission is

$$\epsilon(\vec{x}) = \sum_{i=1}^{n} \beta_i x_i \tag{5.2}$$

Our objective is to find the initial carbon permit, carbon permit trade policy and allocation of the inputs that will maximize the total expected profit. We solve this problem with a backward two-period dynamic programming formulation. That is, we begin with Period 2 and obtain the optimal carbon trade policy given an initial carbon permit amount κ. Then we consider Period 1 in which we optimize over κ. Next, we discuss each of these steps in detail.

In Period 2, we decide on the allocation of the inputs $\vec{x}$ and carbon trade policy (q_b, q_s) that will maximize the profit in Period 2 under a production function, $D = \phi(\vec{x})$, and carbon emission constraint, $\epsilon(\vec{x}) = \kappa + q_b - q_s$. For a given initial carbon permit κ, we let

$$\Pi(q_b, q_s) = sD + p_s q_s - p_b q_b \tag{5.3}$$

Then the Period 2 problem, Problem $P2.2$, becomes

$$\begin{aligned} \max_{(q_b,q_s),\vec{x}} \quad & \Pi_2((q_b, q_s), \vec{x}|\kappa) = \Pi(q_b, q_s) - \Gamma(\vec{x}) \\ s.t. \quad & D = \phi(\vec{x}) \\ & \epsilon(\vec{x}) = \kappa + q_b - q_s \\ & q_b \geq 0 \\ & 0 \leq q_s \leq \kappa \end{aligned} \tag{P2.2}$$

We introduce a nonnegative scalar Lagrange multiplier, λ, for the carbon emission constraint and write the objective function as such $\widehat{\Pi}((q_b, q_s), \vec{x}|\kappa) = \Pi(q_b, q_s) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - (\kappa + q_b - q_s)\right)$ in order to remove the carbon emission constraint from the problem. Then the new equivalent problem, Problem $\widehat{P}2.2$, is

$$\begin{aligned} \max_{(q_b,q_s),\vec{x}} \quad & \widehat{\Pi}((q_b, q_s), \vec{x}|\kappa) = \Pi(q_b, q_s) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - (\kappa + q_b - q_s)\right) \\ s.t. \quad & D = \phi(\vec{x}) \\ & q_b \geq 0 \\ & 0 \leq q_s \leq \kappa \end{aligned} \tag{$\widehat{P}2.2$}$$

First, we study the optimal allocation of the inputs for a given λ and $q_b, q_s \geq 0$. We call this Problem $SP2$.

$$\begin{aligned} \min_{\vec{x}} \quad & \widehat{\Gamma(\vec{x})} = \Gamma(\vec{x}) + \lambda\epsilon(\vec{x}) \\ s.t. \quad & D = \phi(\vec{x}) \end{aligned} \tag{SP2}$$

We denote the optimal allocation of the inputs in Problem $SP2$ for a given λ and $q_b, q_s \geq 0$ by $\vec{x}^{**}(\lambda, q_b, q_s)$. Then, we substitute this solution to Problem $\widehat{P}2.2$ in order to get a new equivalent problem Problem $P2.2^*$ which reduces to two variables q_b and q_s for a given λ and implicitly, D, p_b and p_s. Then we have

$$\begin{aligned}
\max_{(q_b,q_s)} \quad & \widehat{\Pi}((q_b,q_s)|\kappa) = \Pi(q_b,q_s) - \Gamma(\vec{x}^{**}(\lambda,q_b,q_s)) \\
& \qquad - \lambda\left(\epsilon(\vec{x}^{**}(\lambda,q_b,q_s)) - (\kappa + q_b - q_s)\right) \\
s.t. \quad & q_b \geq 0 \\
& 0 \leq q_s \leq \kappa
\end{aligned} \tag{$P2.2^*$}$$

We also enforce $\epsilon(\vec{x}^{**}(\lambda,q_b,q_s)) - (\kappa + q_b - q_s) = 0$ in order to ensure that the carbon emission constraint is binding. We find the optimal carbon trade policy (q_b^{**}, q_s^{**}) for Problem $P1^*$ and the corresponding λ^{**} value. Then we can derive the production policy and carbon permit trade policy for the original problem, Problem $P2.2$, and evaluate the value of the objective function for a given κ, at the optimal solution $\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)$. This concludes the decision making process at the beginning of Period 2.

Next, we move on to Period 1. Here, our objective is to determine the initial carbon permit, κ, that will maximize the total expected profit computed for the entire problem horizon, which is given as follows

$$\Pi_1(\kappa) = -c\kappa + E_{D,p_b,p_s}\left(\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)\right) \tag{5.4}$$

. Therefore we consider the problem below which we will address as Problem $P2.1$.

$$\begin{aligned}
\max_{\kappa} \quad & \Pi_1(\kappa) \\
s.t. \quad & \kappa \geq 0
\end{aligned} \tag{P2.1}$$

Note that $\Pi((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)$ gives us the maximum profit for a given κ under one instance of realized demand, D, and carbon market prices, p_b and p_s. Therefore, we take the expectation with respect to these random variables in order to capture the expected profit in Period 2 for a given κ.

In the following sections, we analyze Problems $SP2$, $P2.2^*$ and $P2.1$ for two production functions: Cobb-Douglas and Leontief.

5.1 The Cobb-Douglas Production Function

In this section, we study the Problem 2 under the Cobb-Douglas production function. Recall that in Cobb-Douglas Production Function the production quantity, D, is given by

$$D = \phi(\vec{x}) = A \prod_{i=1}^{n} {x_i}^{\alpha_i}$$

where x_i is the input quantity and $\alpha_i \leq 1$ for $i = 1, \ldots, n$ and the rate of return is $r = \sum_{i=1}^{n} \alpha_i$.

Now, we proceed with Problems $SP2$, $P2.2^*$ and $P2.1$. We will analyze Problems $SP2$ and $P2.2^*$ under the heading of Period 2 and analyze Problem $P2.1$ under the heading of Period 1.

5.1.1 Period 2

Our objective is to find the optimal carbon trade policy and input allocation in Problem $P2.2$ under the Cobb-Douglas production function when demand and carbon market prices are revealed. We refer to this problem as $P2.2a$.

$$\begin{aligned}
\max_{(q_b, q_s), \vec{x}} \quad & \Pi_2((q_b, q_s), \vec{x} | \kappa) = \Pi(q_b, q_s) - \Gamma(\vec{x}) \qquad \text{(P2.2a)} \\
s.t. \quad & D = A \prod_{i=1}^{n} {x_i}^{\alpha_i} \\
& \epsilon(\vec{x}) = \kappa + q_b - q_s \\
& q_b \geq 0 \\
& 0 \leq q_s \leq \kappa
\end{aligned}$$

We rewrite this problem by introducing a nonnegative scalar Lagrange multiplier, λ, for the carbon emission constraint and name the new equivalent problem for a given λ, Problem $\widehat{P}2.2a$, is

$$\max_{(q_b,q_s),\vec{x}} \ \widehat{\Pi}((q_b,q_s),\vec{x}|\kappa) = \Pi(q_b,q_s) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - (\kappa + q_b - q_s)\right) \tag{$\widehat{P}2.2a$}$$

$$\begin{aligned} s.t.\ \ & D = A\prod_{i=1}^{n} {x_i}^{\alpha_i} \\ & q_b \geq 0 \\ & 0 \leq q_s \leq \kappa \end{aligned}$$

We can interpret λ as a penalty cost of exceeding carbon emission. In our analysis, we first consider Problems $SP2$ and $P2.2^*$ for the Cobb-Douglas production function in order to derive the optimal production policy for Problem $\widehat{P}2.2a$ which is equivalent to $P2.2a$.

Problem $SP2$ under the Cobb-Douglas Production function, Problem $SP2a$, is

$$\min_{\vec{x}} \ \widehat{\Gamma(\vec{x})} = \Gamma(\vec{x}) + \lambda\epsilon(\vec{x}) \tag{SP2a}$$

$$s.t.\ \ D = A\prod_{i=1}^{n} {x_i}^{\alpha_i}$$

Lemma 5.1. *For a given λ and $q_b, q_s \geq 0$ under demand D, the optimal solution to Problem $SP2a$ is*

$$x_i{}^{**}(\lambda, q_b, q_s, D) = \frac{\alpha_i}{p_i + \lambda\beta_i}\prod_{j=1}^{n}\left(\frac{p_j + \lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}\left(\frac{D}{A}\right)^{\frac{1}{r}} \quad \textit{for all } i = 1,\ldots,n \tag{5.5}$$

Proof. Problem $SP2a$ has similar structure to Problem $SP1Ua$. The results follow from Theorem 4.1. □

Now, we move on to Problem $P2.2^*a$, which is Problem $P2.2^*$ under Cobb-Douglas production function. After substituting our findings in Lemma 5.1, for a given λ, we have

$$\max_{(q_b,q_s)} \quad \widehat{\Pi}((q_b,q_s)|\kappa) = \Pi(q_b,q_s) - \lambda(\kappa + q_b - q_s) - rA^{-\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}} \tag{$P2.2^*a$}$$

$$\begin{aligned} s.t. \quad & q_b \geq 0 \\ & 0 \leq q_s \leq \kappa \end{aligned}$$

Note that in this formulation, we assume that the carbon emission constraint is binding. Therefore, we have another equation that we need to satisfy at the optimal solution: $\epsilon(\vec{x}^{**}(\lambda, q_b, q_s)) = \kappa + q_b - q_s$. For a given λ and $q_b, q_s \geq 0$, we have derived the optimal allocation of inputs given by Equation 5.5. Therefore, the carbon trade policy (q_b^{**}, q_s^{**}) found in Problem $P2.2^*a$ needs to satisfy this equation as well.

Corollary 5.1. *For a given λ under demand D, (q_b^{**}, q_s^{**}) solves*

$$\kappa + q_b^{**} - q_s^{**} = \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n} \frac{\alpha_i\beta_i}{p_i + \lambda\beta_i} \tag{5.6}$$

Now, we make some preliminary observations that we will be useful for deriving the optimal policy later on.

Lemma 5.2. *Suppose the relation between Λ and ϵ is given as follows*

$$\epsilon = \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \Lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n} \frac{\alpha_i\beta_i}{p_i + \Lambda\beta_i} \tag{5.7}$$

for $\Lambda, D, A \geq 0$, $\alpha_i, p_i, \beta_i > 0, \forall i = 1, \ldots, n$ and $\sum_i \alpha_i = r$. Then, ϵ is a decreasing and one-to-one function of Λ.

Proof.

$$\frac{d\epsilon}{d\Lambda} = \left(\frac{D}{A}\right)^{\frac{1}{r}} \{\sum_{i=1}^{n} \frac{\alpha_i\beta_i}{p_i+\Lambda\beta_i} \left(\sum_{j=1}^{n} \frac{\alpha_j}{r}\frac{\beta_j}{\alpha_j}\frac{\alpha_j}{p_j+\Lambda\beta_j} \prod_{k=1}^{n}\left(\frac{p_k+\Lambda\beta_k}{\alpha_k}\right)^{\frac{\alpha_k}{r}}\right)$$
$$-\prod_{k=1}^{n}\left(\frac{p_k+\Lambda\beta_k}{\alpha_k}\right)^{\frac{\alpha_k}{r}} \left(\sum_{i=1}^{n}\frac{\alpha_i\beta_i^2}{(p_i+\Lambda\beta_i)^2}\right)\}$$
$$= r\left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{k=1}^{n}\left(\frac{p_k+\Lambda\beta_k}{\alpha_k}\right)^{\frac{\alpha_k}{r}}$$
$$\times\left(\left(\sum_{j=1}^{n}\frac{\beta_j}{p_j+\Lambda\beta_j}\frac{\alpha_j}{r}\right)^2 - \sum_{i=1}^{n}\left(\frac{\beta_i}{p_i+\Lambda\beta_i}\right)^2\frac{\alpha_i}{r}\right)$$

Let Y be a discrete random variable taking values $\{y_i : i = 1, \ldots, n\}$. Note that $E[Y] = \sum_{i=1}^{n} y_i Pr(Y = y_i)$ and $E[Y^2] = \sum_{i=1}^{n} y_i^2 Pr(Y = y_i)$. Let $y_i = \frac{\beta_i}{p_i+\Lambda(\epsilon)\beta_i}$ and $Pr(Y = y_i) = \frac{\alpha_i}{r}$. Then

$$Var(Y) = E[Y^2] - E[Y]^2 = \sum_{i=1}^{n}\left(\frac{\beta_i}{p_i+\Lambda\beta_i}\right)^2\frac{\alpha_i}{r} - \left(\sum_{j=1}^{n}\frac{\beta_j}{p_j+\Lambda\beta_j}\frac{\alpha_j}{r}\right)^2$$

Since $Var(Y) > 0$, we have $\frac{d\epsilon}{d\Lambda} < 0$. Hence ϵ is a strictly decreasing, and consequently, one-to-one function of Λ. □

Definition 5.1. Given p_b, p_s and D, let $\kappa_{up}(D, p_b, p_s)$ and $\kappa_{down}(D, p_b, p_s)$ be given by

$$\kappa_{up}(D, p_b, p_s) = \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n}\left(\frac{p_b\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n}\frac{\alpha_i\beta_i}{p_i+p_b\beta_i} \tag{5.8}$$

$$\kappa_{down}(D, p_b, p_s) = \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n}\left(\frac{p_s\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n}\frac{\alpha_i\beta_i}{p_i+p_s\beta_i} \tag{5.9}$$

For brevity, we suppress the arguments of the above defined carbon emission thresholds and use κ_{up} and κ_{down} to refer to them.

Corollary 5.2. $\kappa_{up} < \kappa_{down}$

Proof. Note that, $p_b > p_s$ by assumption. Then, by Lemma 5.2, the result follows as we replace Λ by p_b and p_s. □

Corollary 5.3. *Let κ_{min} be minimum carbon emission required to produce D. Then, $\kappa_{min} < \kappa_{up}$ for any given p_b, p_s and D.*

Proof. $\kappa_{min} \equiv \kappa_{min}(D) = r\prod_{j=1}^{n}\left(\frac{\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}\left(\frac{D}{A}\right)^{\frac{1}{r}}$ by Lemma 4.8. In Equation 5.7, as $\epsilon \to \kappa_{min}$, $\Lambda(\epsilon) \to \infty$. Since $p_b < \infty$, the result follows. □

Now, we are ready to derive the optimal solution to Problem $P2.2^*a$. First, we consider the first derivatives of the objective function with respect to the decision variables q_b and q_s.

Lemma 5.3. *For a given λ under carbon market prices p_b and p_s,*

$$\frac{d\widehat{\Pi}((q_b, q_s)|\kappa)}{dq_b} = \lambda - p_b \tag{5.10}$$

$$\frac{d\widehat{\Pi}((q_b, q_s)|\kappa)}{dq_s} = p_s - \lambda \tag{5.11}$$

Next, we observe the optimal policies for Problem $P2.2^*a$ in three scenarios: (i) $\kappa \leq \kappa_{up}$, (ii) $\kappa_{up} \leq \kappa \leq \kappa_{down}$ and (iii) $\kappa \geq \kappa_{down}$. As noted above, q_b and q_s do not take positive values at the same time since we assume $p_b > p_s$. Therefore, we look for the optimal solution in the following regions: $R_1 : \{q_b = 0\}$ and $R_2 : \{q_s = 0\}$. In R_1, we are not allowed to buy anymore carbon permit and we maximize the profit with respect to the single variable q_s. In R_2, we maximize the profit with respect to the single variable q_b under the condition that we do not sell any permit. After we find the optimal solutions in both regions, we compare them to find the the optimal solution for that scenario.

Lemma 5.4. *Given demand D and carbon market prices p_b and p_s, $\widehat{\Pi}((q_b, q_s)|\kappa)$ attains its optimum as follows*

1. *For $\kappa < \kappa_{up}$ and $q_b = 0$, $q_s^* = 0$.*

2. *For $\kappa < \kappa_{up}$ and $q_s = 0$, $q_b^* = \kappa_{up} - \kappa$.*

3. *For $\kappa_{up} \leq \kappa \leq \kappa_{down}$ and $q_b = 0$, $q_s^* = 0$.*

4. *For $\kappa_{up} \leq \kappa \leq \kappa_{down}$ and $q_s = 0$, $q_b^* = 0$.*

5. *For* $\kappa > \kappa_{down}$ *and* $q_b = 0$, $q_s^* = \kappa - \kappa_{down}$.

6. *For* $\kappa > \kappa_{down}$ *and* $q_s = 0$, $q_b^* = 0$.

Proof. See Appendix B. □

Next, we derive the optimal policies for each scenario for Problem $P2.2^*a$.

Corollary 5.4. *The optimal policy for Problem* $P2.2^*a$ *is given by*

1. *For* $\kappa < \kappa_{up}$, $(q_b^{**}, q_s^{**}) = (\kappa_{up} - \kappa, 0)$.

2. *For* $\kappa_{up} \leq \kappa \leq \kappa_{down}$, $(q_b^{**}, q_s^{**}) = (0, 0)$.

3. *For* $\kappa_{down} < \kappa$, $(q_b^{**}, q_s^{**}) = (0, \kappa - \kappa_{down})$.

Proof.

1. By Lemma 5.4, $(q_b^{**}, q_s^{**}) \epsilon \{(0,0), (\kappa_{up} - \kappa, 0)\}$ and $q_b = \kappa_{up} - \kappa$ yields higher profit when $q_s = 0$. Thus, $\widehat{\Pi}((0,0)|\kappa) \geq \widehat{\Pi}((\kappa_{up} - \kappa, 0)|\kappa)$.

2. Result immediately follows from Lemma 5.4.

3. By Lemma 5.4, $(q_b^{**}, q_s^{**}) \epsilon \{(0,0), (0, \kappa - \kappa_{down})\}$ and $q_s = \kappa - \kappa_{down}$ yields higher profit when $q_b = 0$. Thus, $\widehat{\Pi}((0,0)|\kappa) \geq \widehat{\Pi}((0, \kappa - \kappa_{down})|\kappa)$.

□

Regarding the λ values for the carbon emission constraint, we have the following result.

Corollary 5.5. *The optimal* λ^{**} *values for the carbon emission constraints are given by*

1. *For* $\kappa < \kappa_{up}$, $\lambda^{**} = p_b$.

2. *For* $\kappa_{up} \leq \kappa \leq \kappa_{down}$, $\lambda^{**} = \overline{\lambda} \equiv \overline{\lambda}(D, \kappa)$ *where* $\overline{\lambda}$ *is defined as*

$$\kappa = \left(\frac{D}{A}\right)^{1/r} \prod_{j=1}^{n} \left(\frac{\overline{\lambda}\beta_j + p_j}{\alpha_j}\right)^{\alpha_j/r} \sum_{i=1}^{n} \frac{\alpha_i}{\overline{\lambda} + \frac{p_i}{\beta_i}} \tag{5.12}$$

3. For $\kappa_{down} < \kappa$, $\lambda^{**} = p_s$.

Proof. Suppose $\kappa < \kappa_{up}$. By Corollary 5.4, $(q_b^{**}, q_s^{**}) = (\kappa_{up} - \kappa, 0)$. Given this solution, λ^{**} that solves the equation in Corollary 5.1 is equal to p_b by Definition 5.1.

The other results similarly follow from Corollaries 5.4 and 5.1, and Definition 5.1. □

So far, we have considered Problem $\widehat{P}2.2a$ in two subproblems: Problem $SP2a$ and Problem $P2.2^*a$. In Problem $SP2a$, we have found the optimal input allocation for a given carbon trade policy. Then, we have substituted our findings in the objective function of $\widehat{P}2.2a$ and obtained Problem $P2.2^*a$. Thus, the optimal carbon trade policy that we have derived in Problem $P2.2^*a$ gives us the optimal solution to Problem $\widehat{P}2.2a$. As we have stated above, Problem $\widehat{P}2.2a$ is equivalent to Problem $P2.2a$.

Now, we are ready to propose the optimal solution to Problem $P2.2a$.

Theorem 5.1. *Under demand D and carbon market prices p_b and p_s, the optimal solution to Problem $P2.2a$ is*

$$(q_b{}^{**}, q_s{}^{**}) = \begin{cases} (\kappa_{up} - \kappa, 0) & , if \kappa < \kappa_{up} \\ (0,0) & , if \kappa_{up} \leq \kappa \leq \kappa_{down} \\ (0, \kappa - \kappa_{down}) & , if \kappa > \kappa_{down} \end{cases} \tag{5.13}$$

$$x_i{}^{**} = \frac{\alpha_i}{\lambda^{**}\beta_i + p_i} \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{\lambda^{**}\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \tag{5.14}$$

where

$$\lambda^{**} = \begin{cases} p_b & , if \kappa < \kappa_{up} \\ \overline{\lambda} & , if \kappa_{up} \leq \kappa \leq \kappa_{down} \\ p_s & , if \kappa > \kappa_{down} \end{cases} \tag{5.15}$$

and $\overline{\lambda}$ is defined as in Corollary 5.5

Proof. The optimal carbon trade policy for Problem $P2.2^*a$ is stated in Corollary 5.4. The optimal allocation of the inputs for Problem $SP2a$ follows from Lemma 5.1. The Lagrange multiplier value of the carbon emission constraint for Problem $P2.2^*a$ is given in Corollary 5.5. As stated above, these problems are the sub-problems of Problem $\widehat{P}2.2a$, Problem $P2.2^*a$ and Problem $SP2a$, and Problem $\widehat{P}2.2a$ is equivalent to Problem $P2.2a$. □

Finally, we look at the emission levels and profit at the optimal solution.

Corollary 5.6. *Let* $\epsilon^{**} \equiv \kappa + q_b^{**} - q_s^{**}$ *be the emission level at the optimal solution for Problem* $P2.2a$*. Then*

$$\epsilon^{**} = \begin{cases} \kappa_{up} & : if \kappa < \kappa_{up} \\ \kappa & : if \kappa_{up} \leq \kappa \leq \kappa_{down} \\ \kappa_{down} & : if \kappa > \kappa_{down} \end{cases} \tag{5.16}$$

Corollary 5.7. *The value of the objective function in Problem* $P2.2a$ *at the optimal solution is*

$$\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa) = sD + \lambda^{**}\kappa - r\left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{\lambda^{**}\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \tag{5.17}$$

where λ^{**} *solves Equation 5.15.*

So far, we have derived the optimal solution in Period 2 under the Cobb-Douglas production function and evaluated the corresponding profit for a revelation of demand and market prices. Now, we move on to Period 1 where we decide on the initial carbon permit purchase.

5.1.2 Period 1

Our aim in the Period 1 is to determine the initial carbon permit purchase, κ, that will maximize the total expected profit. Therefore, we look at Problem $P2.1$

under the Cobb-Douglas production function, which we call Problem $P2.1a$.

$$\begin{aligned} \max_{\kappa} \ \ & \Pi_1(\kappa) = -c\kappa + E_{D,p_b,p_s}\left(\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)\right) \\ s.t. \ \ & \kappa \geq 0 \end{aligned} \tag{P2.1a}$$

Even if the Cobb-Douglas production function is not shown explicitly in the formulation, it is considered in the Period 2 Problem and is in $\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)$ implicitly. First, we will derive the first order conditions with respect to κ. Then we will look at concavity of the objective function in κ.

Before we move on with the problem, we define some functions that will help us in deriving the solution.

Corollary 5.8. *Define $e(\gamma, \kappa)$ be a function of γ and κ such that*

$$e(\gamma, \kappa) = A\kappa^r \frac{\prod_{j=1}^n \left(\frac{\alpha_j}{\gamma\beta_j + p_j}\right)^{\alpha_j}}{\left(\sum_{i=1}^n \frac{\alpha_i\beta_i}{p_i + \gamma\beta_i}\right)^r} \tag{5.18}$$

for $\gamma, \kappa, A > 0$, $\alpha_i, p_i, \beta_i > 0, \forall i = 1, \ldots, n$ and $\sum_i \alpha_i = r$. Given a demand D and carbon market prices p_b and p_s, the following relations hold for a given κ

1. *$D > e(p_b, \kappa)$ for $\kappa < \kappa_{up}$.*

2. *$D < e(p_s, \kappa)$ for $\kappa > \kappa_{down}$.*

Proof. Let $\kappa < \kappa_{up}$. Recall that κ_{up} is a function of D (as well as p_b and p_s); hence, for clarity, we now explicitly show the argument, $\kappa_{up}(D)$. Then, by Definition 5.1,

$$\kappa < \kappa_{up}(D) = \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^n \left(\frac{p_j + p_b\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^n \frac{\alpha_i\beta_i}{p_i + p_b\beta_i}$$

From which we get,

$$D > A\kappa^r \frac{\prod_{j=1}^n \left(\frac{\alpha_j}{p_j + p_b\beta_j}\right)^{\alpha_j}}{\left(\sum_{i=1}^n \frac{\alpha_i\beta_i}{p_i + p_b\beta_i}\right)^r} = e(p_b, \kappa)$$

The other result follows similarly. □

$e(p_b,\kappa)$ and $e(p_s,\kappa)$ denote the upper and lower limits respectively on the demand level in Period 2 within which the entire carbon permit (κ) is used for production (without any purchase or sale of permit). Hence, $\overline{\lambda}(D,\kappa)$, which has been defined above in Corollary 5.5, corresponds to the effective shadow price of carbon permit for a given demand level D which lies between $e(p_b,\kappa)$ and $e(p_s,\kappa)$.

Lemma 5.5. *For a given κ, $\overline{\lambda}(D,\kappa)=p_b$ when $D=e(p_b,\kappa)$ and $\overline{\lambda}(D,\kappa)=p_s$ when $D=e(p_s,\kappa)$.*

Proof. Let $D=e(p_b,\kappa)$. Then, by Corollary 5.8, $\kappa=\kappa_{up}$. By Corollary 5.5,

$$\kappa_{up}=\kappa=\left(\frac{e(p_b,\kappa)}{A}\right)^{\frac{1}{r}}\prod_{j=1}^{n}\left(\frac{p_j+\overline{\lambda}(e(p_b,\kappa),\kappa)\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}\sum_{i=1}^{n}\frac{\alpha_i}{p_i+\overline{\lambda}(e(p_b,\kappa),\kappa)\beta_i}$$

Then, $\overline{\lambda}(e(p_b,\kappa),\kappa)=p_b$ by Definition 5.1. □

In the above Lemma, we have established that the shadow price of carbon permit for a given κ, the upper and lower demand limits ($e(p_b,\kappa)$ and $e(p_s,\kappa)$) is corresponding purchase or selling price of carbon permits in the market. Now, we can derive the first order optimality conditions and comment on the shape of the objective function in Problem $P2.1a$.

Lemma 5.6. *The first order condition of the objective function in Problem $P2.1a$ with respect to κ gives*

$$\int_{p_b,p_s}\left(p_sF(e(p_s,\kappa^*))+p_b\overline{F}(e(p_b,\kappa^*))+\int_{e(p_s,\kappa^*)}^{e(p_b,\kappa^*)}\overline{\lambda}(D,\kappa^*)f(D)dD\right)g(p_b,p_s)dp_bps=c \tag{5.19}$$

Proof. See Appendix B. □

Lemma 5.7. *The objective function of Problem $P2.1a$ is concave in κ.*

Proof. See Appendix B. □

Theorem 5.2. *The unique optimal solution to Problem $P2.1a$, $\kappa^{**}=\kappa^*$ where κ^* solves Equation 5.6*

Proof. Follows from Lemmas 5.6 and 5.7. $\square$

We have found the optimal production and carbon trade policy for Problem 2 under the Cobb-Douglas production function. In the next section, we will study the problem under Leontief production function.

5.2 The Leontief Production Function Model

In this section, we study Problem 2 under the Leontief production function. In our model, recall that we assume that Leontief production function has the following form: $D = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\}$ where A_i is the technology level and α_i is the input elasticity of input $i = 1, \ldots, n$.

Now, we study Problems $SP2$, $P2.2^*$ and $P2.1$. We will analyze Problems $SP2$ and $P2.2^*$ under the heading of Period 2 and analyze Problem $P2.1$ under the heading of Period 1.

5.2.1 Period 2

We aim to find the optimal carbon trade policy and input allocation in Problem $P2.2$ under the Leontief production function when demand and carbon market prices are revealed. We refer to this problem as $P2.2b$.

$$\begin{aligned}
\max_{(q_b,q_s),\vec{x}} \quad & \Pi_2((q_b, q_s), \vec{x}|\kappa) = \Pi(q_b, q_s) - \Gamma(\vec{x}) \\
s.t. \quad & D = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\} \\
& \epsilon(\vec{x}) = \kappa + q_b - q_s \\
& q_b \geq 0 \\
& 0 \leq q_s \leq \kappa
\end{aligned} \tag{P2.2b}$$

We rewrite this problem by introducing a nonnegative scalar Lagrange multiplier, λ, for the carbon emission constraint and construct the new equivalent problem for a given λ, called Problem $\widehat{P}2.2b$, as follows.

$$\max_{(q_b,q_s),\vec{x}} \ \widehat{\Pi}((q_b,q_s),\vec{x}|\kappa) = \Pi(q_b,q_s) - \Gamma(\vec{x}) - \lambda\left(\epsilon(\vec{x}) - (\kappa + q_b - q_s)\right) \tag{$\widehat{P}2.2b$}$$

$$\begin{aligned} s.t.\ \ & D = \min_i\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\} \\ & q_b \geq 0 \\ & 0 \leq q_s \leq \kappa \end{aligned}$$

Here again, we can interpret λ as a penalty cost of exceeding carbon emission. Next, we consider Problems $SP2$ and $P2.2^*$ for the Leontief production function in order to derive the optimal production policy for Problem $\widehat{P}2.2b$ which is equivalent to $P2.2b$.

Problem $SP2$ under the Leontief Production function referred to as Problem $SP2b$ is stated as follows

$$\min_{\vec{x}} \ \widehat{\Gamma(\vec{x})} = \Gamma(\vec{x}) + \lambda\epsilon(\vec{x}) \tag{SP2b}$$
$$s.t.\ \ D = \min\{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\}$$

Lemma 5.8. *For a given λ, q_b, q_s under demand D, the optimal solution to Problem $SP1b$ is*

$${x_i}^{**}(D) = \left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} \quad \textit{for all } i = 1, .., n \tag{5.20}$$

Proof. Problem $SP2b$ has similar structure to Problem $SP1Ub$. The result follows from Theorem 4.5. □

Note that we assume that the carbon emission constraint is binding in the formulation of Problem $\widehat{P}2.2b$. Therefore, we have another equation that we need to satisfy at the optimal solution: $\epsilon(\vec{x}^{**}(D)) = \kappa + q_b - q_s$. For a given λ and $q_b, q_s \geq 0$, we have derived the optimal allocation of inputs given by Equation 5.20. Therefore, the carbon trade policy (q_b^{**}, q_s^{**}) found in Problem $P2.2^*b$ needs to satisfy this equation as well.

Lemma 5.9. *For a given λ under demand D, (q_b^{**}, q_s^{**}) solves*

$$\kappa + q_b^{**} - q_s^{**} = \sum_{i=1}^{n} \beta_i \left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{5.21}$$

Lemma 5.9 suggests that optimal carbon trade policy, (q_b^{**}, q_s^{**}) in Problem $\widehat{P}2.2b$, only depends on the realized demand and is independent of carbon market prices. Also note that q_b and q_s do not take positive values at the same time since $p_b > p_s$. Therefore, we can directly incur the optimal policy for Problem $P2.2b$ from this relation.

Theorem 5.3. *For a given κ under demand D and carbon market prices p_b and p_s, the optimal solution to Problem $P2.2b$ is*

$$({q_b}^{**}, {q_s}^{**}) = \left(\left(\sum_{i=1}^{n} \beta_i \left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} - \kappa\right)^{+}, \left(\kappa - \sum_{i=1}^{n} \beta_i \left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}}\right)^{+}\right) \tag{5.22}$$

$${x_i}^{**} = \left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} \quad \textit{for all } i = 1, .., n \tag{5.23}$$

$$\lambda^{**} \,\epsilon\, \Re \tag{5.24}$$

Proof. The optimal solution to Problem $SP2b$ is given in Lemma 5.8. Since it is a subproblem of Problem $\widehat{P}2.2b$ for a given λ, q_b and q_s, it gives the optimal allocation of inputs for Problem $\widehat{P}2.2b$ as well. Since Problem $\widehat{P}2.2b$ is equivalent to Problem $P2.2b$, the result holds for Problem $P2.2b$ as well. The optimal carbon trade policy follows from Lemma 5.9. □

Finally, we look at the emission levels and profit at the optimal solution.

Corollary 5.9. *Let $\epsilon^{**} \equiv \epsilon^{**}(D) = \kappa + q_b^{**} - q_s^{**}$ be the emission level under demand D at the optimal solution for Problem $P2.2b$. Then ,*

$$\epsilon^{**} = \sum_{i=1}^{n} \beta_i \left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{5.25}$$

Corollary 5.10. *The value of the objective function in Problem $P2.2b$ at the optimal solution is*

$$\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa) = sD + \begin{cases} p_s\kappa - \sum_{i=1}^{n}(p_i + p_s\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} & : if \epsilon^{**}(D) < \kappa \\ p_b\kappa - \sum_{i=1}^{n}(p_i + p_b\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} & : if \epsilon^{**}(D) \geq \kappa \end{cases} \tag{5.26}$$

Up to this point, we have found the optimal solution in Period 2 under the Leontief production function and evaluated the corresponding profit for a given revelation of demand and carbon market prices. Next, we study Period 1 when we decide on the initial carbon permit purchase.

5.2.2 Period 1

Our objective in Period 1 is to find the initial carbon permit purchase, κ, that will maximize the total expected profit. In this model, we assume a joint distribution for demand and carbon market prices with a known p.d.f. $f(.,.,.)$ and c.d.f. $F(.,.,.)$. Therefore, we look at Problem $P2.1$ under the Leontief production function, which we call Problem $P2.1b$.

$$\begin{aligned} \max_{\kappa} \quad & \Pi_1(\kappa) = -c\kappa + E_{D,p_b,p_s}\left(\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)\right) \\ s.t. \quad & \kappa \geq 0 \end{aligned} \tag{P2.1b}$$

Even if the Leontief production function is not shown explicitly in the formulation, it is considered in Period 2 and is in $\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)$ implicitly. First, we will derive the first order conditions with respect to κ. Then we will look at concavity of the objective function in κ.

First, we give some preliminary definitions that will help us in deriving the solution.

Corollary 5.11. *Let ω be a function of κ such that*

$$\kappa = \sum_{i=1}^{n} \beta_i \left(\frac{\omega(\kappa)}{A_i}\right)^{\frac{1}{\alpha_i}} \tag{5.27}$$

for $\alpha_i, \beta_i, A_i > 0, \forall i = 1, \ldots, n$ and $\kappa > 0$. Then, $\omega(\kappa)$ is increasing in κ.

Lemma 5.10. *The first order condition of the objective function in Problem $P2.1b$ with respect to κ gives*

$$\int_{p_b, p_s} \left(p_s \int_0^{\omega(\kappa^*)} f(p_b, p_s, D) dD + p_b \int_{\omega(\kappa^*)}^{\infty} f(p_b, p_s, D) dD \right) dp_b p_s = c \quad (5.28)$$

Proof. See Appendix B. □

Lemma 5.11. *The objective function of Problem $P2.1b$ is concave in κ.*

Proof. See Appendix B. □

Theorem 5.4. *The unique optimal solution to Problem $P2.1b$, $\kappa^{**} = \kappa^*$ where κ^* solves Equation 5.28.*

Proof. Follows from Lemmas 5.10 and 5.11. □

We have found the optimal production and carbon trade policy for Problem 2 under the Leontief production function.

Chapter 6

Numerical Study

In this chapter, we present a set of computational experiments in which we evaluate the optimal policies for *Problem 1* and *Problem 2* and discuss the sensitivity of the optimal policy parameters in each problem with respect to various system parameters. We construct our numerical analysis based on the Cobb-Douglas Production Function. This chapter consists of three sections.

In Section 6.1, we focus on the important features of the optimal production policy under a carbon cap in Problem 1 for the Cobb-Douglas production function. In Section 6.1.1, we construct an example based on agricultural production. This example is motivated by a real life problem which is presented in the study of Hatırlı et al. [18]. We study this example under different carbon cap tightnesses, demand variabilities and cost structures. We examine their affects on the production policy parameters given by the production quantity Q^{**}, input quantities x_i^{**}, input allocation ratios, λ^{**}, production cost, realized service level and expected profit. We also consider the affect of changing technology level on the production policy parameters under a carbon cap. In Section 6.1.2, we evaluate the sensitivity of the production policy parameters with respect to changes in input elasticities, input costs and carbon coefficients under a carbon cap.

In Section 6.2, we aim to study the optimal carbon trade policy in Problem

2 under the Cobb-Douglas production function. We construct a numerical analysis to examine the sensitivity of the optimal policy parameters such as initial carbon permit purchase κ^{**}, expected carbon permit trade in the market and improvement over the no-contract policy ($\kappa = 0$) with respect to change in input elasticities, input costs, demand distibution.

6.1 Production System and Sensitivity Analysis under Carbon Cap Restrictions

In this section, we study on the essential features of the optimal production policy under a carbon cap in Problem 1 for Cobb-Douglas production function. We analyze the production policy parameters given by the production quantity Q^{**}, input quantities x_i^{**}, input allocation ratios, λ^{**}, production cost, realized service level and expected profit. We first construct a numerical example based on a real life problem and then we conduct a sensitivity analysis under a carbon cap for two inputs.

6.1.1 Agricultural Production System Analysis under Carbon Cap Restrictions

In this study, we use the work by Hatırlı et al. [18], who study the relationship between energy inputs and crop yield for greenhouse tomato production in Antalya, Turkey. They collect the data by conducting face to face surveys in 2001 with forty randomly selected greenhouse operators in Central and Serik districts of Antalya where the greenhouse farming is one of the most common economic activities. In the study, the output is given by the tomato yield while the inputs are fertilizer, chemicals, machinery, labor force, water for irrigation and seed. They convert the output and the inputs to their energy equivalents. Then they examine the relationship between the inputs and the output under the Cobb-Douglas production function.

We construct an illustrative example based on the work of Hatırlı et al. [18]. In this example, we consider greenhouse tomato production, Q. The inputs are fertilizer (x_1), chemicals (x_2), labor (x_3), machinery (x_4) and water for irrigation (x_5). We omit the input seed since the elasticity is found to be negative for this input in [18]. However, while calculating the technology level, the contribution of the seed (the input we have excluded in this example) will be considered. For the sake of consistancy, the measured unit for each input have been taken as their energy equivalents in mega joules (MJ).

The input elasticities (α_i) are directly taken from the data provided in [18]. We rely on different sources for the derivation of the production costs (p_i) with the measured unit of TL/MJ and carbon coefficients (β_i) with the measured unit of kg Carbon Emitted/MJ. The details of our constructions are provided in Appendix C. Parameter values for each input are given in Table 6.1.

We estimate the technology level $A = 1.34$ from the data in [18] as explained in Appendix C. The selling price for tomato is calculated as $s = 2.875\ TL/MJ$ as shown in Appendix C. We assume uniformly distributed demand with mean $D = 130000$ MJ.

In the analysis, we study this example under various cost ratios, coefficients of variation and carbon cap tightnessess.

In the classical newsvendor problem, the service level, the probability of satisfying all of the demand, is given by $\frac{s+c_s}{s+c_s+c_e}$. In this study, we work with a base service level 95%. We assume that the stockout cost is given by a ratio of the selling price and calculate the excess cost based on these values. We conduct our analysis with $c_s/s \epsilon \{0.1,\ 0.2,\ 0.4\}$. The corresponding values of stockout and excess costs as a result of these calculations are shown in Table 6.2.

We also analyze the example under different coefficients of variance, which are given by $COV \epsilon \{0.58,\ 0.5,\ 0.33,\ 0.2\}$ (Table 6.3).

In our analysis, we also study the affect of the carbon cap tightness. We first solve Problem 1 without carbon cap constraint (Problem $P1Ua$)for the given parameters and calculate the carbon emission, ϵ_{1U}. Then we tighten the carbon cap by a percentage δ. In this case, the carbon cap is given by $\kappa = (1-\delta)\epsilon_{1U}$. In this example, we consider the percentage carbon cap tightnesses, $\delta \, \epsilon \, \{0\%,\ 2\%,\ 5\%,\ 10\%,\ 15\%,\ 25-50\%\}$.

In this study, we examine a total of $3 \times 4 \times 31 = 372$ experiments. Below, we provide the pseudocode that we use in MATLAB for the analysis. This process takes less than three minutes of CPU time.

Algorithm 1: Optimal solution under carbon cap restriction

```
BEGIN
INITIALIZE parameters s, D, A and α_i, p_i, β_i, ∀i = 1,...,n
SET c_s/s ϵ {0.1, 0.2, 0.4} and calculate c_s and c_e
SET COV ϵ {0.58, 0.5, 0.33, 0.2} and calculate minimum (D_l) and maximum demand (D_h)
SET FoundRoot1 := 0 and FoundRoot2 := 0
    FOR Q = 0 → 2D_h with ΔQ = 0.1
        IF FoundRoot1 == 0 and dΠ(Q)/dQ > 0 (Eqn. 4.11)
            Set Q*_1 := Q and FoundRoot1 := 1
        IF FoundRoot1 :== 1 and dΠ(Q)/dQ < 0 (Eqn. 4.11)
            Set Q*_2 := Q, FoundRoot2 := 1 and EXIT FOR LOOP
    IF FoundRoot2 = 1
        Set Q**_unc := 0 and END
    ELSE
        Compute Π(Q*_2) and Π(0) (Eqn. 4.9)
        IF Π(Q*_2) > Π(0)
            Set Q**_unc := Q*_2 and compute F(Q**_unc), x**_i(Q**_unc) (Eqn. 4.6), Γ**(Q**_unc)
            (Eqn. 4.7) and ϵ**_U1(Q**_unc) (Eqn. 4.8)
        ELSE
            Set Q**_unc := 0 and END
    SET δ ϵ {0%, 2%, 5%, 10%, 15%, 25 − 50%}
        Compute κ = (1 − δ)ϵ**_U1(Q**_unc)
        FOR λ = 0 → 700 with Δλ = 0.0001
            Compute Q (Eqn. 4.26)
            IF dΠ̂(Q)/dQ < 0 (Eqn 4.23)
```

```
            Set Q* := Q, λ* := λ and EXIT FOR LOOP
    Compute Π̂(Q*) (Eqn. 4.21)
    IF Π̂(Q*) > Π(0)
        Set Q** := Q* and λ** := λ*
        Compute F(Q**), x_i**(Q**) (Eqn. 4.19) and Γ(x⃗**(Q**)) (Eqn. 4.2)
    ELSE
        Set Q** := 0 and λ** := 0
END
```

In this algorithm, we first initialize parameters and set the values for COV and c_s/s ratio that we use in the experiment. Then, we conduct a linear search for Q with the increment $\Delta Q = 0.1$ on the interval $[0, 2D_h]$ where $D_h = D + COV\sqrt{3}$. Next, we inspect the behavior of the objective function $\Pi(Q)$ with respect to Q and find the extremum point(s) that change the sign of the first derivative of the objective function. Note that Q^{**}_{unc} may take a positive value if and only if Q^*_1 is a local minimum and Q^*_2 is a local maximum by Theorem 4.2 since $r > 1$ in the example we consider. Therefore, we construct this algorithm so that the first root we find Q^*_1 gives the local minimum and the second root Q^*_2 gives the local maximum, if they exist.

After the seach for the extrema points ends, we evaluate whether we have found two roots. If not, we conclude that $Q^{**}_{unc} = 0$. It implies that we should not produce at all when there is no carbon constraint and further investigation with the carbon emission constraint is unnecessary. Thus, we end the algorithm. Otherwise, by Theorem 4.2, we conclude that $Q^{**}_{unc} \epsilon \{0, Q^*_2\}$. Then, we compute the expected profits and set Q^{**}_{unc} as the solution that yields the maximum profit. If $Q^{**}_{unc} = 0$, we do not investigate the case with carbon emission contraint as explained above. If $Q^{**}_{unc} = Q^*_2$, then we compute $\mathrm{F}(Q^{**}_{unc})$ and $x^{**}_i(Q^{**}_{unc})$, $\Gamma^{**}(Q^{**}_{unc})$ and $\epsilon^{**}_{U1}(Q^{**}_{unc})$ as given in Theorem 4.1.

Next, we set a percentage carbon cap tightness on the emission level under no carbon cap. Then, for each tightness, we compute the corresponding carbon cap,

κ. In our preliminary numerical study based on a search for λ on the interval [0,10000], we have observed that there exists a unique λ^{**} and corresponding $Q^{**}(\lambda^{**})$ (given by Equation 4.26) that solve Equation 4.24. Therefore, in this algorithm, we conduct a linear search on λ with the increment $\Delta\lambda = 0.0001$ on the interval $[0, 700]$. For each λ value, we compute the corresponding Q by Lemma 4.9. Note that by Lemma 4.10, there exists a unique Q for each λ. Then, we find the local maximum point (λ^*, Q^*) where the sign of $\frac{d\widehat{\Pi}(Q)}{dQ}$ changes from positive to negative. The algorithm ends with the comparison of the objective function values at (λ^*, Q^*) and $(0, 0)$ in order to find the optimal solution under a carbon cap, κ.

In order to double check, we consider Corollary 4.9 for a given λ. We find two positive roots, Q_1^* and Q_2^*, that solve Equation 4.12. For instance, under $COV = 0.5$ and $c_s/s = 0.1$, we find $Q_1^* = 0.4$ and $Q_2^* = 225340.2$ for $\lambda = 1.5995$, and $Q_1^* = 0.6$ and $Q_2^* = 224737.4$ for $\lambda = 2.2166$. Our algorithm gives (λ^{**}, Q^{**})=$(1.5995, 225340.2)$ under $\delta = 75\%$ and (λ^{**}, Q^{**})=$(2.2166, 224737.4)$ under $\delta = 72\%$. Thus, we conclude that the local maximum point Q_2^* that satisfy Equation 4.12 for λ^{**} is the optimal solution we find in the algorithm.

Now, we discuss the impacts of the change in c_s/s ratio, COV and carbon cap tightness on the optimal policy parameters: the expected profit, λ^{**}, Q^{**}, x_i^{**} $\forall i = 1, \ldots, n$, input allocation ratios, the production cost and the realized service level given by $F(Q^{**})$.

First, we aim to observe the affects of the change in c_s/s ratios on the optimal production policy parameters. Figure 6.1 shows the expected profit as a function of the percentage carbon cap tightness for every instance of $c_s/s \,\epsilon\, \{0.1,\ 0.2,\ 0.4\}$ and $COV \,\epsilon\, \{0.58,\ 0.2\}$. We observe that as c_s/s increases, the expected profit decreases and this affect is more profound under higher demand variances. This result is expected since an increase in the stockout cost and excess cost leads to a decrease in the expected profit, especially under the presence of higher uncertainty in demand. However, we should note that the impact of c_s/s on the expected profit does not seem to be significant. When we move from $c_s/s = 0.1$ to $c_s/s = 0.4$ under $COV = 0.2$, we observe that the decrease in profit is under 3% for all percentage carbon cap tighnesses.

We can interpret λ^{**} as the shadow price of increasing the carbon cap by one unit, thus it implies the significance of the carbon cap constraint. In Figure 6.2, we see λ^{**} as a function of the percentage carbon cap tightness for every instance of $c_s/s \ \epsilon \ \{0.1,\ 0.2,\ 0.4\}$ and $COV \ \epsilon \ \{0.58,\ 0.2\}$. We observe that λ^{**} increases in c_s/s for all COV values. This increase is more profound under a tight carbon cap. For instance, when we change $c_s/s = 0.1$ to $c_s/s = 0.4$, the change in λ^{**} is approximately 28% for all COV values at carbon cap tightness of $\delta = 50\%$ while we observe no change at all upto $\delta = 15\%$. This implies meeting the carbon cap requirement becomes more difficult for a tight carbon cap under higher stockout and excess costs as expected.

Even though the change in c_s/s ratio have some affect on the expected profit and λ^{**} values, it does not seem to have any affect on the production quantity. Figure 6.3 shows the production quantity as a function of the percentage carbon cap tightness for every instance of $c_s/s \ \epsilon \ \{0.1,\ 0.2,\ 0.4\}$ and $COV \ \epsilon \ \{0.58,\ 0.5,\ 0.33\ 0.2\}$. We observe that the optimal production quantity does not show a notable change (less than 1%) in c_s/s for a given COV and δ. In Table 6.4, we see the input quantities for every instance of the parameters $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$, $c_s/s = 0.1\ 0.2,\ 0.4$ and $\delta = 0\%,\ 5\%,\ 10\%,\ 15\%,\ 30\%,\ 40\%,\ 50\%$. We observe that the input quantities do not change significantly for alternative c_s/s ratios under a given COV and δ. It is observed from the Table 6.4, that an increase in c_s/s from 0.1 to 0.4 leads to less than 1% increase in x_1^{**}, less than 3% increase in x_2^{**}, less than 6% increase in x_3^{**}, less than 0.5% change in x_4^{**} and less than 3% change in x_5^{**} for all instances with the parameters $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$ and $\delta = 0\%,\ 5\%,\ 10\%,\ 15\%,\ 30\%,\ 40\%,\ 50\%$.

We conclude that c_s/s does not have a significant affect on the production quantity Q^{**} and input quantities $x_i^{**},\ \forall i = 1, \ldots, n$. Thus the change in c_s/s does not affect the production cost. For instance, when we move from $c_s/s = 0.1$ to $c_s/s = 0.4$, the increase in the production cost for all COV and δ is observed to be at most 3%. When we consider the expected profit-production cost ratio, the values change do not chage significantly; for instance, at $COV = 0.2$ and $\delta = 45\%$, the ratio gives 16.95 and 16.38 for $c_s/s = 0.1$ and $c_s/s = 0.4$, respectively.

Next, we observe the affect of coefficient of variation on the optimal production policy parameters. Since we have already covered the affect of c_s/s, we conduct our further analysis based on one instance of $c_s/s = 0.1$.

In Figure 6.1, we observe that the expected profit decreases in demand variance. When we increase the coefficient of variation from $COV = 0.2$ to $COV = 0.58$, we observe an approximately 5% decrease for all δ values under $c_s/s = 0.1$. This result is expected since highly varying demand leads to a decrease in the expected profit.

In Figure 6.2, we observe that when the coefficient of variation is increased from $COV = 0.2$ to $COV = 0.58$, λ^{**} decreases. This decrease is more profound in tighter carbon cap restrictions. For instance, the change in coefficient of variation from $COV = 0.2$ to $COV = 0.58$ under $c_s/s = 0.1$ leads 45% decrease in λ^{**} at $\delta = 50\%$ while we observe less than 1% decrease in λ^{**} for $\delta < 15\%$. We can infer from this result that when the demand is highly variable, the carbon cap restriction becomes less significant as expected.

Figure 6.3 shows that the production quantity Q^{**} increases by around 42% for all δ values under $c_s/s = 0.1$ when we increase the coefficient of variation from $COV = 0.2$ to $COV = 0.58$. This result implies that demand uncertainty is high, we tend to produce more as expected.

In Table 6.4, we observe that the input quantities used in production are greater for higher demand variances. This is expected since the we produce more under higher demand variances as concluded previously. When we compare the input quantities for $COV = 0.58$ to those for $COV = 0.2$, we observe that we use 14-38% more inputs in the production process depending on the δ values.

Figure 6.4 shows the production cost as a function of the percentage carbon cap tightness for every instance of $c_s/s = 0.1$ and $COV \in \{0.58,\ 0.5,\ 0.33,\ 0.2\}$. We observe that the expected production cost is increasing in demand variance.

This result is also expected as increasing input quantities lead to a higher production cost. When we compare the production costs for $COV = 0.58$ and $COV = 0.2$, we observe that at higher coefficient of variation the production cost increase between 24-30% depending on the δ values. When we consider the expected profit-production cost ratio, the values do not chage significantly; for instance, at $c_s/s = 0.1$ and $\delta = 45\%$, the ratio decreases in COV and takes values between 13.09 and 16.95.

In Figure 6.5, we see the realized service level as a function of the percentage carbon cap tightness for every instance of $c_s/s = 0.1$ and $COV \epsilon \{0.58,\ 0.5,\ 0.33,\ 0.2\}$. When we consider the realized service level under the optimal production policy, we observe that higher demand variance yield higher service levels as expected due to the increase in the production quantity. While the difference between the realized service levels for $COV = 0.58$ and $COV = 0.2$ is less than 1% for $\delta < 35\%$, at $\delta = 50\%$, this difference reaches 21% under $c_s/s = 0.1$.

Finally, we consider the optimal production policy parameters with respect to the percentage carbon cap tightness. In Figure 6.1, we observe that the expected profit decreases in the percentage carbon cap tightness as expected. This decrease becomes steep after 30% carbon cap tightness.

We observe in Figure 6.2 that λ^{**} values increase in the percentage carbon cap tightness. The increase in λ^{**} is observed to be steeper for $\delta > 30\%$. This result is expected since the tighter the carbon cap, more difficult meeting the carbon cap restriction.

For further investigation, we consider Table 6.5, where λ^{**} values are compared to $\frac{\Delta Exp.Profit}{\Delta Emission}$ for all δ, $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$ and $c_s/s = 0.1$. We observe that $\frac{\Delta Exp.Profit}{\Delta Emission}$ takes close, but slightly smaller values to λ^{**}. The difference is caused by the large intervals. If we consider smaller intervals than 1% for the percentage carbon cap tightness, we expect $\frac{\Delta Exp.Profit}{\Delta Emission}$ to take almost identical values as λ^{**}. It implies that the shadow price of the carbon cap constraint is given by the incremental change in the expected profit with respect to the incremental change in the carbon emission at the given carbon cap.

Figure 6.3 shows that as the percentage carbon cap tightness increases, production quantity Q^{**} decreases. We observe a steeper decrease in Q^{**} for $\delta > 30\%$. This result implies that for tighter carbon cap restrictions, we tend to produce less to meet the carbon cap as expected.

In order to analyze the impact of the percentage carbon cap tightness on the expected profit and production quantity Q^{**} in depth, we plot the percentage decreases in the expected profit and Q^{**} with respect to the expected profit and Q^{**}_{unc} values found in the carbon cap unconstrained problem as a function of the percentage carbon cap tightness. The percentage decreases in the expected profit and Q^{**} are found to be almost identical for all instances of $c_s/s \in \{0.58,\ 0.5,\ 0.33,\ 0.2\}$ and $COV \in \{0.1,\ 0.2,\ 0.4\}$. Therefore, we only present our findings for the case where $COV = 0.58$ and $c_s/s = 0.1$ in Figure 6.6. Figure 6.6 show that the percentage decrease in Q^{**} is always greater than the percentage decrease in the expected profit under the increasing carbon cap tightness. We also observe that the difference between the percentage changes of the expected profit and Q^{**} grows in δ. For instance, at the percentage decreases in the expected profit and Q^{**} is 0.5% and 1.4%, respectively for $\delta = 30\%$ while they become 7.8% and 22.7% for $\delta = 50\%$.

The percentage decrease in the realized service level with respect to the carbon cap tightness is almost identical to the change in Q^{**}. Therefore, the comments for the change in Q^{**} with respect to the carbon cap tightness also hold for the realized service level.

Next, we consider the change in the input quantities and the input allocation with respect to the percentage carbon cap tightness. In Table 6.4, we observe the impact of the carbon cap tightness on the input quantities. We observe that the input quantities behave in a similar way in changing carbon cap tightness for all COV and c_s/s parameter values. For better illustration, we examine Figure 6.7 where the input quantities are given as a function of the percentage carbon cap tightness at $COV = 0.58$ and $c_s/s = 0.1$. We observe that as carbon cap tightness increases, we use the irrigations systems (x_5^{**}) less while relying on labor (x_3^{**}), fertilizer (x_1^{**}) and chemicals (x_2^{**}). On the other hand, there is no significant change in machinery (x_4^{**}) use. When the carbon cap tightness exceeds 35%, the use of fertilizer and chemicals begin to decrease due to smaller production quantities.

The input allocation ratios are given in Table 6.6 for $\delta = 0\%,\ 5\%,\ 10\%,\ 15\%,$ $30\%,\ 40\%,\ 50\%$ $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$ and $c_s/s = 0.1$. In this table, we observe the change in input allocation with respect to the normalizing input x_1^{**}. We observe that the allocations of chemicals (x_2^{**}) and labor (x_3^{**}) are always increasing while the allocations of machinery (x_4^{**}) and water for irrigation (x_5^{**}) are decreasing in the percentage carbon cap tightness for all instances under $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$ and $c_s/s = 0.1$.

Figure 6.4 shows that the production cost increases in the percentage carbon cap tightness upto 45% and then begins to decrease. This decrease results from the significant decrease in the production quantity in the percentage carbon cap as discussed previously. For $COV = 0.58$ and $c_s/s = 0.1$, the production cost increases upto $\delta = 45\%$ and reaches an increase by 20% compared to $\delta = 0\%$ level. When we consider the production cost at $\delta = 50\%$, it is only 16% greater than the production cost at $\delta = 0\%$. We observe that the expected profit-production cost ratio do not chage significantly in the carbon tightness; for instance, at $COV = 0.58$ and $c_s/s = 0.1$, the ratio decreases in δ and takes values between 12.99 and 16.36.

We also consider the change in the realized service level with respect to the change in the percentage carbon cap tightness. In Figure 6.5, we observe that the realized service level steeply decreases in the percentage carbon cap tightness for $\delta > 35\%$. For instance, at $COV = 0.2$ and $c_s/s = 0.1$, the realized service level under no carbon cap is 93% while it drops to 92% at $\delta = 30\%$ and 51% at $\delta = 50\%$.

Based on our observations so far, we conclude that the carbon emission under no cap restrictions can be decreased by at least 35% without any significant effect on the expected profit and the realized service level for all COV and c_s/s values considered in this analysis. At carbon cap tightness of 35%, the expected profit only decreases by approximately 1% while the realized service level only decreases from 93% to 90%. This change can be created by decresing the production quantity by approximately 3% and altering the input allocation as suggested in Table 6.7.

Another alternative in dealing with carbon cap restrictions is improvement in production technology. In our analysis, we study the affect of technology improvement on production system and expected profit for the base case with parameters $COV = 0.58$ and $c_s/s = 0.1$. We first find the optimal production policy parameters under technology improvements of 25%, 50%, 75% and 100%. We also calculate the emission levels and percentage emission reductions compared to the base case. Then we empose these reduction levels by a carbon cap under no technology improvement in order to compare the effect of technology improvement under a carbon cap. The results are presented in Table 6.8.

With technology improvements, not only we decrease our emission level, but also we slightly improve on the realized service levels and increase the expected profit compared to the base case. For instance, at 50% technology improvement, we reduce the emission by 25% while increasing the expected profit by 1.6% and the realized service level from 93.2% to 93.6%. We also note that the input quantities decrease with the increasing technology level while keeping the ratio of the input quantities constant. For 50% technology improvement, we observe this decrease to be 25% for each input.

We also observe that the carbon cap restriction can be met by improving the technology level rather than changing the input allocations and decreasing the production quantity. Under a given carbon cap, we may produce with the given technology by adjusting our production system or we may improve the technology to meet the carbon cap. For instance, for 33% reduction in the emission level, the expected profit with technology improvement is 9013 TL greater than the expected profit with no improvement. It implies that we should improve the technology level by 75% if this improvement costs less than 9013 TL. Moreover, with 75%technology improvement under 33% carbon cap tightness, the realized service level is increased from 91.4% to 93.8%.

In Table 6.9, we present the expected benefit of technology improvement over no improvement case under a given emission reduction by a carbon cap. These results are inferred from the data in Table 6.8. For a strict carbon cap, we observe that the expected benefit of technology improvement is an increasing and convex function of the technology level under the given carbon cap. It implies that under technology improvement cost is linear, the net expected benefit increases as the carbon cap decreases. It suggests that technology improvement is more important under more strict carbon cap restrictions.

6.1.2 Sensitivity Analysis of the Production Policy under a Carbon Cap

In this section, we focus on the sensitivity of the optimal production policy parameters with respect to change in input elasticities, input costs, carbon coefficients under a carbon cap.

We consider a production system with two inputs and constant rate of return, $r = 1$. We assume technology level $A = 1$. For input 1, we assume $p_1 = 1$ and $\beta_1 = 1$. We set the selling price, $s = 50$, stockout cost $c_s = 5$ and we calculate the excess cost based on the assumption of 95% the service level in the classical newsvendor problem, i.e. $\frac{s+c_s}{s+c_s+c_e} = 95\%$. The demand is assumed to be uniformly distributed over $[0,\ 1000]$. We analyze this system under the parameter set in Table 6.10 that gives a total of $2 \times 2 \times 2 \times 2 = 16$ experiments.

For computations, we use MATLAB with a similar procedure explained in the previous section. In this example, we have assumed $r = 1$. Note that our choice of parameters satisfies the condition: $s + c_s \geq \frac{\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}}{A}$. By Theorem 4.2, a positive Q^{**}_{unc} is directly given by Equation 4.14. The search for λ^{**} proceeds as in the given algorithm.

The results are presented in Table 6.11. We consider the case $\alpha_2 = 0.9$. It implies that the production process relies on the input 2.

First, we observe the change in the optimal production policy parameters for

$(p_2,\ \beta_2) = (0.25,\ 0.25)$. This case implies that the production process relies on the low cost-low carbon emitting input. We observe that the input quantity is significantly higher for input 2 under no carbon cap and the input allocation given by x_2^{**}/x_1^{**} does not change under a carbon cap tightness of 50%. We also observe that the production quantity decreases by 50% while the decrease in the expected profit is 27.7%.

Now, we consider the case $(p_2,\ \beta_2) = (0.25,\ 4.00)$. In this case, the production process relies on the low cost-high carbon emitting input. We observe that even though the input quantity is significantly higher for input 2 under no carbon cap and the input allocation changes in favor of the less carbon emitting input with higher cost, input 1, under a carbon cap tightness of 50%. Another observation is that the producition quantity decreases by 40.3% while the expected profit decreases by 18.7%. The decrease in the expected profit is less compared to the previous case since this case includes a trade-off of using low cost-high carbon emitting input or high cost-low carbon emitting input in the production. On the other hand, in the previous case, the production system is already at its top performance in terms of the input allocation by relying on the low cost-low carbon emitting input, and only high reductions in the production quantity help meeting the carbon cap, which reduces the expected profit due to lost sales and stockout costs.

Next, we examine the case $(p_2,\ \beta_2) = (4.00,\ 0.25)$. This case considers a production process that relies on the high cost-low carbon emitting input. We observe that the input quantities are more balanced under no carbon cap compared to the low cost cases we have studied above. However, the input allocation is still in favor of input 2 and input 2 becomes more dominant in the production under a carbon cap tightness of 50%. The production quantity decreases only by 9.2% whereas the decrease in the expected profit is 3.1%. Since the input allocation under no carbon cap is more balanced compared to the previous case, there exists a room for improvement by increasing the allocation of the less carbon emitting input, input 2 in this case.

Lastly, we study the case $(p_2,\ \beta_2) = (4.00,\ 4.00)$. In this case, the production process relies on the high cost-high carbon emitting input. In our analysis, we observe that the input quantities are more balanced and in favor of the input 2 under no carbon cap. We also observe that the allocation of the inputs do not change under a carbon cap tightness of 50% even though the production quantity is decreased by 50%. A change in the allocation does not occur since there does not exist a trade-off for changing the input allocation under the given parameter values at the presense of a carbon cap. Another observation is that the expected profit decreases by 28.2% for this case.

We do not examine the other cases since they are equivalent to the scenarios that we discussed above and yield similar results. For instance, $(\alpha_2, p_2,\ \beta_2) = (0.9, 4.00,\ 0.25)$ is equivalent to $(\alpha_2, p_2,\ \beta_2) = (0.1, 0.25,\ 4.00)$ in the sense that the input elasticity of the high cost-low carbon emitting input is higher, i.e. the production process relies more on this input.

So far, we have observed that the optimal input quantity is significantly high when the elasticity of the the low cost-low carbon emitting input is higher. These results are expected since we always prefer to use cheaper and less carbon emitting resource with greater contribution on the production process. We have also observed that under a carbon cap, the input allocation changes in favor of the input with carbon coefficients.

Our final observation is that the least percentage decrease in the expected profit under a carbon cap is obtained in two cases: $(\alpha_2, p_2,\ \beta_2) = (0.9, 4.00,\ 0.25)$ *and* $(0.1, 0.25,\ 4.00)$. While we only have 3.1% and 1.1% decrease in the expected profit under a carbon cap, respectively, the percentage decrease is between 18.7% and 28.2% for other cases. This result indicates that when our production system relies more on the less carbon emitting input, the expected profit is affected less by the carbon cap. This finding provides evidence on the importance of swiching to less carbon emitting inputs.

6.2 Sensitivity Analysis of the Carbon Permit Contract Policy under Emission Trading Scheme

In this section, we present an illustrative example in order to study the carbon permit contract policy parameters under different sets of input elasticities, input costs and demand distributions for the Cobb-Douglas production function.

We analyze a production system with two resources and constant rate of return, $r = 1$. We set a fixed technology level $A = 50$. The carbon coefficients are assumed to be $(\beta_1,\ \beta_2) = (15,\ 45)$. This implies that the input 1 is the low carbon emitting input while the input 2 is the high carbon emitting input. The selling price is taken as $s = 3$. We assume the carbon contract $c = 0.5$ and the following independent distributions for carbon market prices:

$$p_b = \begin{cases} 0.5 & w/prob 0.2 \\ 0.55 & w/prob 0.4 \\ 0.6 & w/prob 0.4 \end{cases} \qquad p_s = \begin{cases} 0.4 & w/prob 0.4 \\ 0.45 & w/prob 0.5 \\ 0.5 & w/prob 0.1 \end{cases}$$

For our analysis, we choose three demand distributions with mean 1000 and standard deviation 288. These are UNI: *Uniform(500, 1500)*, TRIA 1: *Triangular(44.3, 543.3, 1813.3)* and TRIA 2: *Triangular(186.7, 1356.7, 1456.7)*. We choose UNI since each demand in the interval is equally probable. The reason we consider TRIA1 and TRIA2 is that the mass of the distribution is concentrated at either sides of the mean for each distribution. TRIA1 is a right-skewed distribution with relatively few high demand values whereas TRIA2 is a left-skewed distribution with relatively few low demand values. In addition, we study this example under the experiment instances given in Table 6.12.

In this study, we examine a total of $5 \times 3 = 15$ instances. We use MATLAB for computations and the pseudocode we use is provided below. This process takes less than 5 minutes of CPU time.

Algorithm 2: Optimal solution under emission trading scheme

```
BEGIN
INITIALIZE parameters s, A, c, β_i ∀i = 1,...,n and distributions for p_b, p_s
SET the demand distribution ϵ {UNI, TRIA1, TRIA2}
SET α_i, p_i, ∀i = 1,...,n for the experiment instance ϵ {1, 2, 3, 4, 5}
  FOR κ = 0 → 5000 with Δκ = 0.01
    FOR p_b ϵ {0.5, 0.55, 0.6}
      Compute e(p_b, κ) (Eqn. 5.18)
      FOR p_s ϵ {0.4, 0.45, 0.5}
        Compute e(p_s, κ) (Eqn. 5.18)
        FOR D = e(p_s, κ) → e(p_b, κ) with ΔD = 0.1
          FOR λ = p_s → p_b with Δλ = 0.0001
            Compute κ̃(λ) (Eqn. 5.12)
            IF κ̃(λ) <= κ
              Set λ**_{p_b,p_s,D} := λ and EXIT FOR LOOP
    Compute dΠ_1/dκ
    IF dΠ_1/dκ < 0
      Set κ** := κ, compute Π_1 (Eqn. 5.4) and EXIT FOR LOOP
END
```

In this algorithm, we first initialize the parameters and set the experiment. Then, we begin on a search for initial carbon contract purchase κ on the interval [0,5000] with the increment $\Delta\kappa = 0.01$. Next, we consider the distributions of the carbon market prices. For each possible value that p_b and p_s may take, we determine the demand limits $e(p_b, \kappa)$ and $e(p_s, \kappa)$ from Equation 5.18. Note that between these limits, the optimal carbon trade policy in Period 2 suggests to produce with carbon emission κ by Theorem 5.1 and Corollary 5.8. Then, for each demand point between $e(p_b, \kappa)$ and $e(p_s, \kappa)$ under a given p_b and p_s, we search for λ on the interval [p_s,p_b] with the increment $\Delta\lambda = 0.01$ that satisfies Equation 5.12. Next, we evaluate the first derivative of the objective function

Π_1 with respect to κ. By Lemma 5.7, the objective function is concave in κ. Therefore, we consider the sign change in the first derivative, we conclude that $\kappa^{**} := \kappa$ when the sign changes from positive to negative and stop our search for κ.

Now, we discuss the initial permit purchase with the carbon permit contract, the expected carbon trade at the contract purchase decision epoch and the comparison with the no-contract policy.

First, we study the advance carbon permit purchase with the contract under the given parameters. The results are presented in Table 6.13 for each experiment instance under the distributions UNI, TRIA1 and TRIA2. When we compare the experiment instances 1, 2, 4 and 5, we observe that as the elasticity of the low cost-high carbon emitting input increases with respect to the high cost-low carbon emitting input, we tend to purchase more with the carbon permit contract. For instance, by observing the experiment instances 1 and 5, we conclude that switching from high-cost-low carbon emitting input to low cost-high carbon emitting input may increase the amount of carbon permit purchased by more than twice as much.

We can also observe that the amount of the carbon purchase is the greatest at TRIA2 and the smallest at TRIA1 compared to the other distributions. It implies that when the demand is concentrated at lower values, the carbon permit contract becomes less significant.

Another observation we can infer from this table is that when the input prices are more dispersed, the amount of carbon purchased with the contract increases as seen in the comparison of the experiment instances 2 and 3. It suggests that we tend to rely on the carbon permit contract more when the difference between the costs of inputs increases.

Next, we evaluate how much carbon permit we expect to trade in the future when we decide on the initial carbon purchase with the contract. In Table 6.14, these results are presented under all experiment instances and demand distributions: UNI, TRIA1, TRIA2. In order to compute these values, we assume that we have purchased κ^{**} given in Table 6.13 with the carbon permit contract. Then, we calculate the expected permit to buy and expected permit to sell under the given carbon market price distributions for each demand distribution: UNI, TRIA1, TRIA2.

When we compare the results we have found for the experimental instances 1, 2, 4 and 5 under all demand distributions in Table 6.14, we observe that both the expected permit to buy and sell values increase as the elasticity of the low cost-high carbon emitting input increases. These increases are observed to be more than twice as much when we consider the experimental instances 1 and 5. It suggests that when the production process relies on the high carbon emitting input, we expect to become more active in the carbon trade market.

Another observation from Table 6.14 is that the dispersion of the input costs results in higher expected permit to buy and sell values as seen in comparison of the experimental instances 2 and 3. This result is expected since we tend to use more of high carbon emitting input when its cost is relatively lower than the low carbon emitting input and it increases the activity in the carbon trade market.

It is important to note that in Table 6.14, we observe that the expected permit to buy is the greatest for TRIA1 and the smallest for TRIA2 under all the experiment instances. Inversely, the expected permit to sell is the greatest for TRIA2 and the smallest for TRIA1. It implies when the demand is concentrated on the lower values, we expect to buy more permit in the future. On the other hand, when the demand is concentrated on the higher values, we expect to sell more permit in the future. These results are intuitive. For instance, we purchase more permit with the contract under TRIA2 demand distribution as seen in Table 6.13; therefore, we expect to buy less and sell more permit compared to the other demand distributions UNI and TRIA1.

When we examine the net difference between the expected permit to buy in the future and the expected permit to sell for each experiment instances, we observe that the expected permit to buy in the future exceeds the expected permit to sell for demand distributions UNI and TRIA1, and this difference is more significant for TRIA1 demand distribution. It implies that the optimal carbon contract policy suggests further carbon purchase in the future when the demand is equally distributed over the interval or concentrated on lower values. For TRIA2 distribution, we observe that the expected permit to sell is greater than the expected permit to buy. This result is expected since it implies that when the demand is concentrated at higher values, the optimal carbon contract policy suggests to buy more permit with the contract than we may expect to use. For instance, we expect to buy 1.9% more permit for UNI and 6.9% more for TRIA1 distribution under the experiment instance 5. For TRIA2 distribution under the experiment instance 5, we expected to sell 3.1% of the permit we have purchased with the contract.

Finally, we observe the impact of introducing a carbon permit contract. We evaluate the expected profit and the carbon emission under the optimal carbon contract policy, κ^{**}, and no contract policy, $\kappa = 0$. Then, we calculate the percentage improvement over the no contract policy as well as the increase in emission. The results are presented in Table 6.15.

We observe in Table 6.15 that the expected profit decreases and the expected emission increases as the elasticity of the low cost-high carbon emitting input increases. For instance, changing the input elasticity of the low cost-high carbon emitting input from 0.1 (experimental instance 1) to 0.8 (experimental instance 5) decreases the expected profit by 11% for the optimal carbon contract policy. When we consider the same experiment instances, we observe that the expected emission increases more than twice as much.

Another observation is that the expected profit decreases and the expected emission increases when the input costs become dispersed. When we compare the experiment instances 2 and 3 for the optimal carbon contract policy, we observe that the dispersion of the input costs decreases the expected profit by approximately 12% while increasing the expected emission by 11%.

It is also important to note that the expected profit and the expected emission do not depend on the type of demand distribution. From Table 6.15, we can observe that there is no significant change in the values for different demand distributions.

When we compare the expected profit and carbon emission of the optimal contract policy and the no-contract policy in Table 6.15, we observe that we obtain higher profits with the contract while emitting more carbon. The greatest expected profit improvement over the no-contract policy (3.64%) is obtained for the experimental instance 4. The corresponding increase in the expected emission is only 0.88%.

The results of this sensitivity analysis indicate that carbon permit contract provides an opportunity for increasing the expected profit, but it also leads to higher carbon emission. We also conclude that when the elasticity of the low cost-high carbon emitting input is significantly greater than the elasticity of the high cost-low carbon emitting input, in other words, when our production system mostly depends on the low cost-high carbon emitting input, the carbon permit contract and the carbon trade gain importance. We tend to purchase more permit with the contract and expect to be more active in the carbon trade market while obtaining lower expected profit. This result is expected since relying on high carbon emitting resources leads to become more dependent on the carbon trade market.

Figures of the Numerical Study

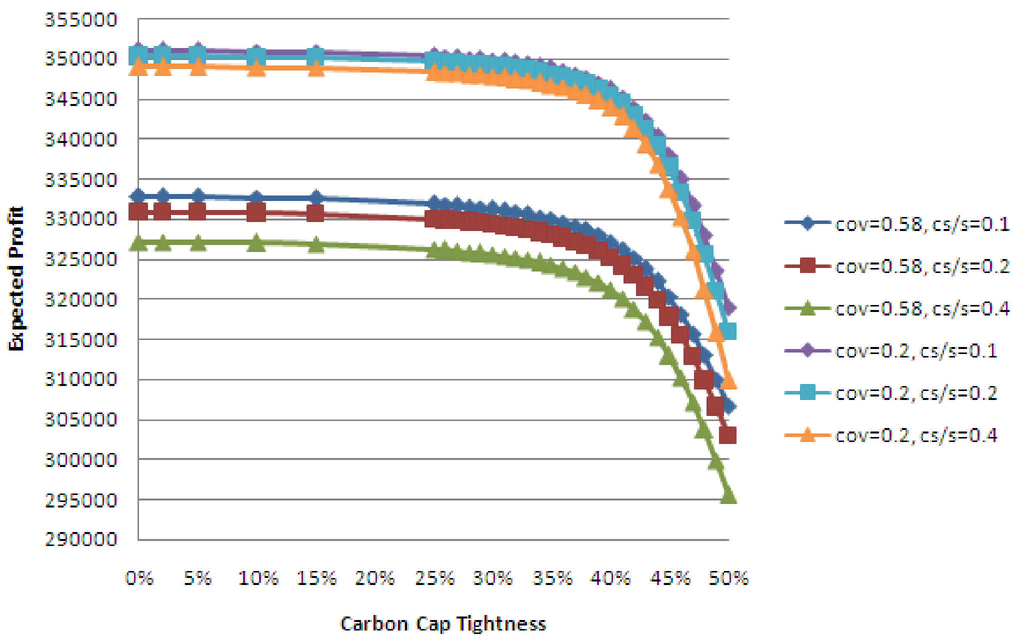

Figure 6.1: Expected profit as a function of carbon cap tightness at $COV = 0.58,\ 0.2,\ c_s/s = 0.1,\ 0.2,\ 0.4$

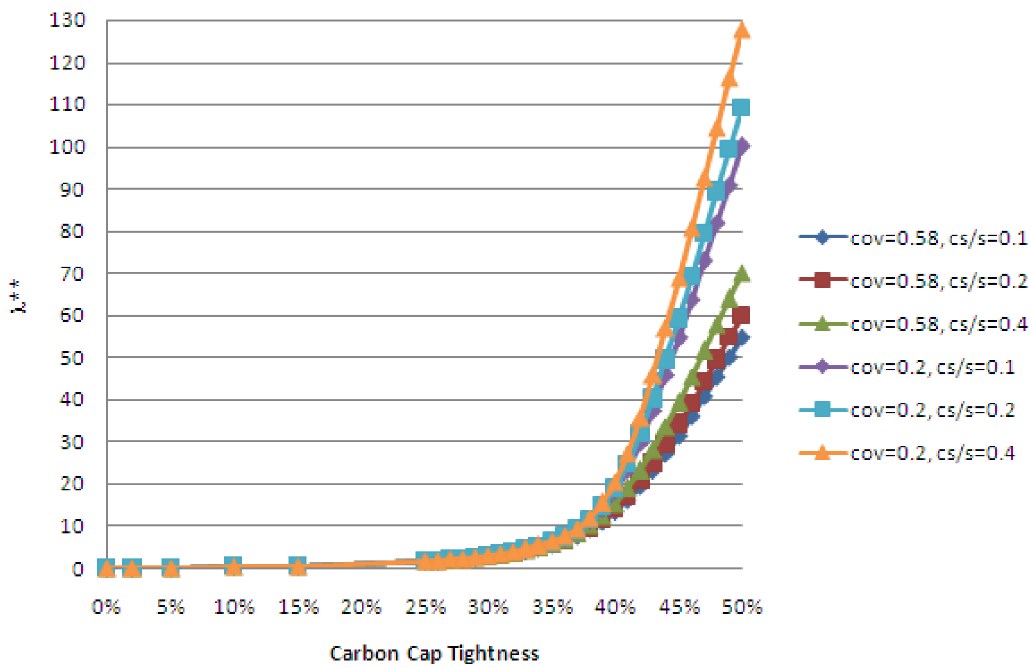

Figure 6.2: Optimal λ^{**} as a function of carbon cap tightness at $COV = 0.58,\ 0.2$, $c_s/s = 0.1,\ 0.2,\ 0.4$

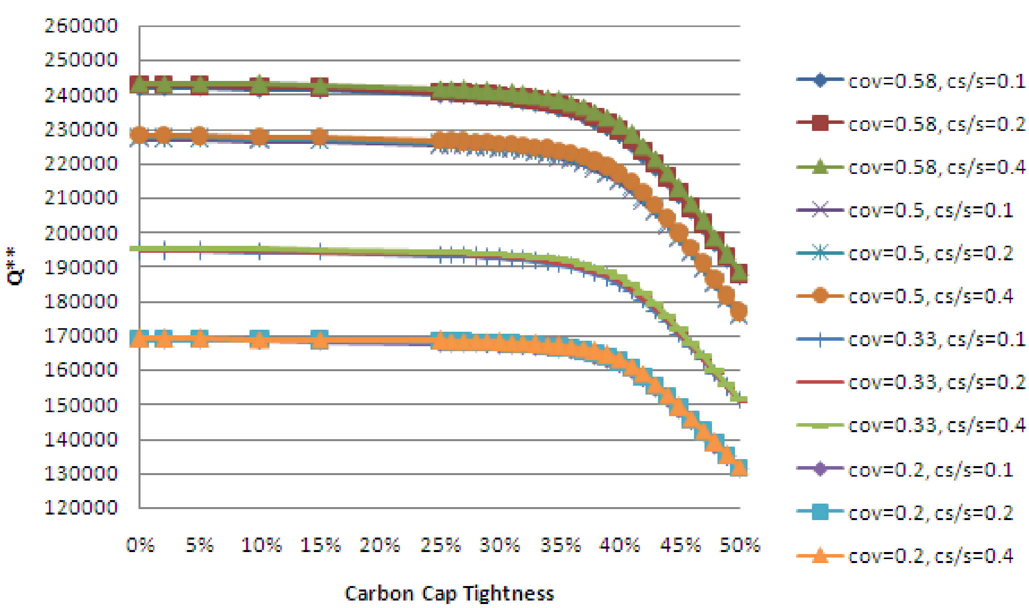

Figure 6.3: Optimal production quantity as a function of carbon cap tightness at $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$, $c_s/s = 0.1,\ 0.2,\ 0.4$

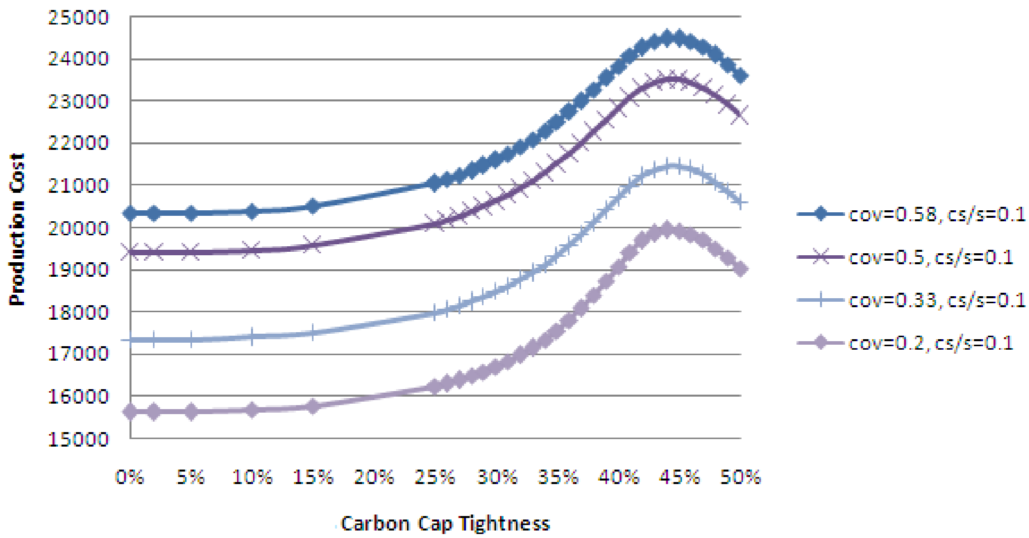

Figure 6.4: Expected production cost as a function of carbon cap tightness at $COV = 0.58,\ 0.5,\ 0.33,\ 0.2,\ c_s/s = 0.1$

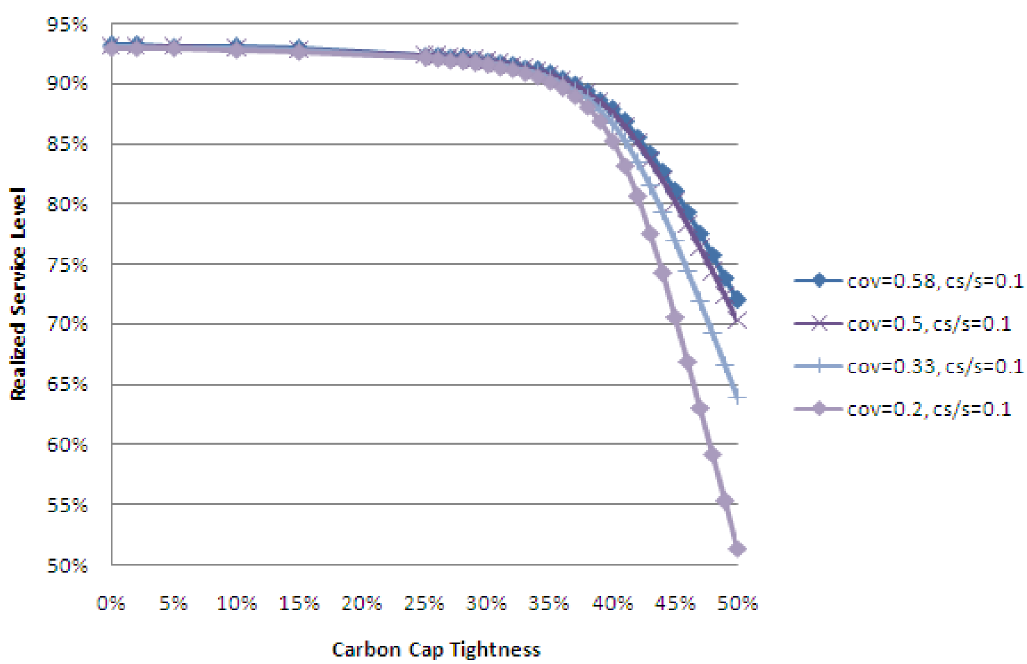

Figure 6.5: Service Level as a function of carbon cap tightness at $COV = 0.58,\ 0.5,\ 0.33,\ 0.2,\ c_s/s = 0.1$

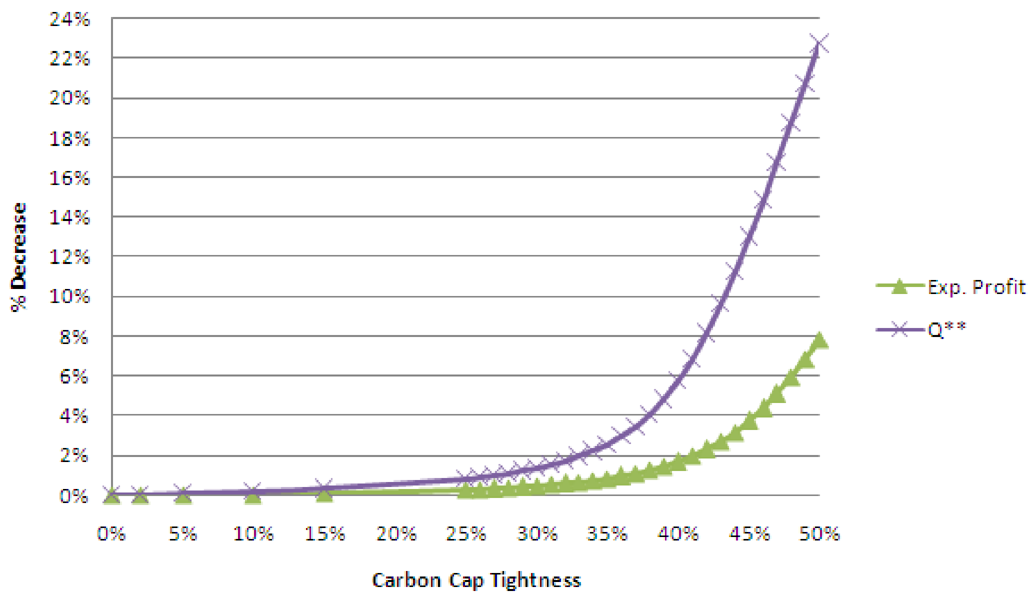

Figure 6.6: The percentage decrease in the expected profit and the production quantity with respect to the values found in the carbon cap unconstrained problem as a function of carbon cap tightness at $COV = 0.58$, $c_s/s = 0.1$

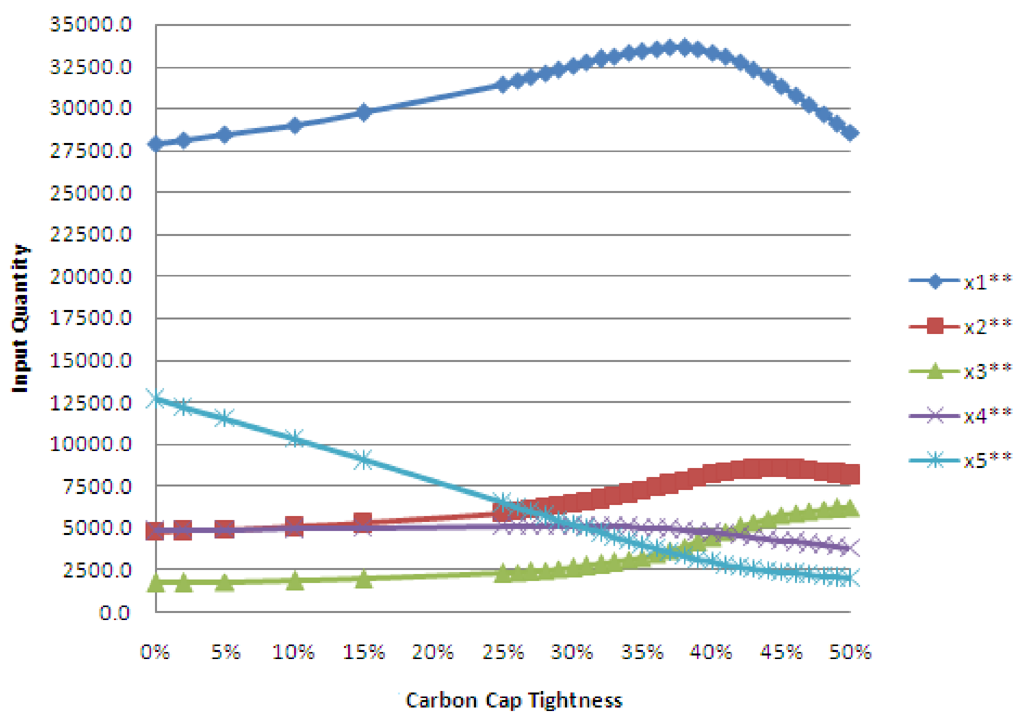

Figure 6.7: Input allocation as a function of carbon cap tightness at $COV = 0.58$, $c_s/s = 0.1$

Tables of the Numerical Study

Parameter	Elasticity	Input Cost (TL/MJ)	Carbon Coef. (kg CE /MJ)
fertilizer	0.24	0.1278	0.0197
chemical	0.19	0.5929	0.0504
labor	0.15	1.3043	0.0398
machinery	0.56	1.7217	0.3542
water for irrigation	0.23	0.2689	0.2951

Table 6.1: Parameter values for the inputs

c_s/s	c_s (TL/MJ)	c_e (TL/MJ)
0.1	0.144	0.0834
0.2	0.288	0.0909
0.4	0.576	0.1061

Table 6.2: Parameter set for the stockout and excess costs base on a base service level of 95%

coefficient of variation	minimum demand	maximum demand
0.20	84967	175033
0.33	54951	205048
0.50	17417	242583
0.58	0	260000

Table 6.3: Parameter set for demand values

COV	δ	$c_s/s = 0.1$					$c_s/s = 0.2$					$c_s/s = 0.4$				
		x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}
0.58	0%	27873.3	4756.4	1707.0	4827.7	12695.4	27907.0	4762.2	1709.0	4833.5	12710.7	27959.8	4771.2	1712.3	4842.7	12734.8
	5%	28411.4	4893.6	1769.3	4887.1	11486.7	28447.0	4899.8	1771.5	4893.2	11500.2	28504.8	4909.9	1775.2	4903.0	11520.8
	15%	29719.1	5264.3	1947.6	5010.4	9024.7	29763.2	5272.4	1950.7	5017.6	9033.2	29833.3	5285.4	1955.7	5029.1	9046.4
	30%	32518.0	6414.7	2624.8	5117.0	5213.1	32598.9	6435.9	2635.7	5127.3	5208.4	32726.6	6469.5	2653.1	5143.4	5200.7
	40%	33345.9	8160.8	4452.1	4711.5	2960.5	33465.4	8242.7	4547.0	4715.6	2936.4	33656.1	8381.3	4714.5	4720.8	2895.5
	50%	28523.7	8126.1	6173.7	3802.7	2007.7	28549.4	8183.2	6324.9	3798.2	1994.2	28581.9	8273.7	6581.0	3789.8	1972.3
0.5	0%	26593.5	4538.0	1628.6	4606.0	12112.4	26623.6	4543.2	1630.4	4611.2	12126.2	26671.0	4551.3	1633.3	4619.4	12147.8
	5%	27108.0	4669.2	1688.1	4662.8	10958.9	27140.1	4674.7	1690.1	4668.3	10970.9	27192.2	4683.8	1693.4	4677.2	10989.4
	15%	28359.9	5023.7	1858.7	4781.1	8608.7	28399.8	5031.1	1861.5	4787.6	8616.3	28462.8	5042.8	1866.0	4797.9	8628.0
	30%	31052.9	6129.2	2509.5	4884.8	4966.3	31125.5	6148.3	2519.3	4894.0	4962.0	31240.4	6178.6	2535.0	4908.5	4955.1
	40%	31867.8	7834.0	4307.0	4494.1	2806.1	31975.9	7910.3	4397.3	4497.5	2783.6	32148.3	8039.4	4556.7	4501.5	2745.5
	50%	27208.4	7784.7	5986.0	3622.0	1904.8	27229.2	7836.2	6127.0	3617.6	1892.5	27254.8	7917.8	6365.2	3609.5	1872.6
0.33	0%	23759.5	4054.4	1455.0	4115.2	10821.6	23781.3	4058.1	1456.4	4118.9	10831.6	23815.6	4064.0	1458.5	4124.9	10847.2
	5%	24222.6	4172.3	1508.5	4166.5	9790.0	24246.0	4176.3	1510.0	4170.5	9798.7	24283.2	4182.8	1512.3	4176.8	9812.2
	15%	25353.5	4491.8	1662.1	4273.9	7686.7	25382.3	4497.1	1664.1	4278.6	7692.2	25427.9	4505.6	1667.3	4286.0	7700.7
	30%	27818.2	5500.3	2256.0	4371.5	4417.3	27871.5	5514.4	2263.4	4378.3	4414.0	27955.6	5536.7	2275.0	4388.8	4408.7
	40%	28615.8	7138.2	4027.6	4011.0	2454.5	28696.3	7201.1	4107.6	4012.4	2436.0	28824.2	7307.4	4248.5	4013.6	2404.6
	50%	24284.7	7040.1	5624.0	3218.5	1672.8	24294.2	7077.9	5739.5	3214.4	1663.4	24304.0	7137.6	5932.5	3207.1	1648.4
0.2	0%	21405.9	3652.8	1310.9	3707.5	9749.7	21420.1	3655.2	1311.8	3710.0	9756.1	21442.2	3659.0	1313.1	3713.8	9766.2
	5%	21827.2	3759.8	1359.4	3754.4	8819.1	21842.7	3762.5	1360.3	3757.0	8824.6	21866.3	3766.5	1361.8	3761.0	8833.4
	15%	22858.8	4050.5	1499.0	3852.9	6920.5	22877.7	4054.0	1500.3	3856.0	6924.0	22906.9	4059.4	1502.4	3860.8	6929.6
	30%	25146.1	4982.8	2048.4	3946.6	3957.6	25181.1	4992.1	2053.3	3951.0	3955.4	25236.4	5006.9	2061.1	3957.9	3951.7
	40%	25945.4	6612.5	3883.6	3605.1	2145.4	25999.2	6662.2	3954.2	3604.7	2130.7	26084.0	6746.2	4078.4	3603.3	2105.9
	50%	21832.2	6436.6	5416.1	2877.3	1474.0	21832.9	6461.3	5503.7	2873.8	1467.5	21830.9	6499.9	5648.0	2867.9	1457.2

Table 6.4: Optimal input quantities at $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$, $c_s/s = 0.1\ 0.2,\ 0.4$ and δ=0%, 5%, 10%, 15%, 30%, 40%, 50%

δ	$COV = 0.58$		$COV = 0.50$		$COV = 0.33$		$COV = 0.20$	
	λ^{**}	$\frac{\Delta Exp.Profit}{\Delta Emission}$	λ^{**}	$\frac{\Delta Exp.Profit}{\Delta Emission}$	λ^{**}	$\frac{\Delta Exp.Profit}{\Delta Emission}$	λ^{**}	$\frac{\Delta Exp.Profit}{\Delta Emission}$
0%	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2%	0.0505	0.0248	0.0505	0.0248	0.0506	0.0248	0.0507	0.0249
5%	0.1370	0.0925	0.1371	0.0925	0.1374	0.0927	0.1378	0.0930
10%	0.3201	0.2238	0.3205	0.2240	0.3214	0.2246	0.3225	0.2253
15%	0.5771	0.4407	0.5778	0.4412	0.5801	0.4427	0.5828	0.4446
25%	1.5949	1.0023	1.5995	1.0044	1.6133	1.0105	1.6304	1.0181
26%	1.7725	1.6818	1.7782	1.6869	1.7953	1.7023	1.8164	1.7213
27%	1.9750	1.8714	1.9821	1.8778	2.0034	1.8969	2.0300	1.9206
28%	2.2077	2.0885	2.2166	2.0965	2.2436	2.1205	2.2773	2.1505
29%	2.4771	2.3389	2.4885	2.3490	2.5230	2.3796	2.5663	2.4179
30%	2.7917	2.6302	2.8065	2.6432	2.8512	2.6825	2.9078	2.7321
31%	3.1626	2.9719	3.1819	2.9888	3.2408	3.0402	3.3160	3.1056
32%	3.6040	3.3767	3.6297	3.3990	3.7085	3.4673	3.8104	3.5550
33%	4.1345	3.8608	4.1693	3.8908	4.2769	3.9831	4.4178	4.1033
34%	4.7787	4.4459	4.8266	4.4868	4.9763	4.6141	5.1761	4.7824
35%	5.5684	5.1598	5.6356	5.2167	5.8485	5.3959	6.1392	5.6378
36%	6.5450	6.0391	6.6409	6.1197	6.9497	6.3771	7.3847	6.7343
37%	7.7599	7.1301	7.8990	7.2462	8.3557	7.6236	9.0248	8.1660
38%	9.2750	8.4896	9.4787	8.6589	10.1638	9.2216	11.2186	10.0675
39%	11.1572	10.1827	11.4563	10.4311	12.4901	11.2786	14.1787	12.6246
40%	13.4689	12.2750	13.9042	12.6385	15.4530	13.9140	18.1517	16.0707
41%	16.2509	14.8198	16.8712	15.3434	19.1398	17.2341	23.3431	20.6406
42%	19.5071	17.8404	20.3630	18.5745	23.5654	21.2927	29.7983	26.4693
43%	23.1979	21.3189	24.3342	22.3117	28.6557	26.0602	37.3476	33.4934
44%	27.2500	25.1974	28.7000	26.4883	34.2716	31.4264	45.6881	41.4653
45%	31.5749	29.3934	33.3591	31.0092	40.2553	37.2390	54.5155	50.0723
46%	36.0863	33.8183	38.2147	35.7741	46.4659	43.3467	63.5938	59.0414
47%	40.7105	38.3916	43.1854	40.6933	52.7933	49.6236	72.7617	68.1753
48%	45.3884	43.0469	48.2075	45.6943	59.1568	55.9747	81.9137	77.3423
49%	50.0752	47.7325	53.2332	50.7217	65.4995	62.3317	90.9816	86.4570
50%	54.7371	52.4094	58.2272	55.7339	71.7810	68.6467	99.9201	95.4630

Table 6.5: Comparison of λ^{**} and $\frac{\Delta Exp.Profit}{\Delta Emission}$ for $COV = 0.58,\ 0.5,\ 0.2,\ 0.1$ and $c_s/s = 0.1$

COV	δ	$c_s/s = 0.1$					$c_s/s = 0.2$					$c_s/s = 0.4$				
		x_1^{**}	$\frac{x_2^{**}}{x_1^{**}}$	$\frac{x_3^{**}}{x_1^{**}}$	$\frac{x_4^{**}}{x_1^{**}}$	$\frac{x_5^{**}}{x_1^{**}}$	x_1^{**}	$\frac{x_2^{**}}{x_1^{**}}$	$\frac{x_3^{**}}{x_1^{**}}$	$\frac{x_4^{**}}{x_1^{**}}$	$\frac{x_5^{**}}{x_1^{**}}$	x_1^{**}	$\frac{x_2^{**}}{x_1^{**}}$	$\frac{x_3^{**}}{x_1^{**}}$	$\frac{x_4^{**}}{x_1^{**}}$	$\frac{x_5^{**}}{x_1^{**}}$
0.58	0%	27873.3	0.1706	0.0612	0.1732	0.4555	27907.0	0.1706	0.0612	0.1732	0.4555	27959.8	0.1706	0.0612	0.1732	0.4555
	5%	28411.4	0.1722	0.0623	0.1720	0.4043	28447.0	0.1722	0.0623	0.1720	0.4043	28504.8	0.1722	0.0623	0.1720	0.4042
	15%	29719.1	0.1771	0.0655	0.1686	0.3037	29763.2	0.1771	0.0655	0.1686	0.3035	29833.3	0.1772	0.0656	0.1686	0.3032
	30%	32518.0	0.1973	0.0807	0.1574	0.1603	32598.9	0.1974	0.0809	0.1573	0.1598	32726.6	0.1977	0.0811	0.1572	0.1589
	40%	33345.9	0.2447	0.1335	0.1413	0.0888	33465.4	0.2463	0.1359	0.1409	0.0877	33656.1	0.2490	0.1401	0.1403	0.0860
	50%	28523.7	0.2849	0.2164	0.1333	0.0704	28549.4	0.2866	0.2215	0.1330	0.0698	28581.9	0.2895	0.2303	0.1326	0.0690
0.5	0%	26593.5	0.1706	0.0612	0.1732	0.4555	26623.6	0.1706	0.0612	0.1732	0.4555	26671.0	0.1706	0.0612	0.1732	0.4555
	5%	27108.0	0.1722	0.0623	0.1720	0.4043	27140.1	0.1722	0.0623	0.1720	0.4042	27192.2	0.1722	0.0623	0.1720	0.4041
	15%	28359.9	0.1771	0.0655	0.1686	0.3036	28399.8	0.1772	0.0655	0.1686	0.3034	28462.8	0.1772	0.0656	0.1686	0.3031
	30%	31052.9	0.1974	0.0808	0.1573	0.1599	31125.5	0.1975	0.0809	0.1572	0.1594	31240.4	0.1978	0.0811	0.1571	0.1586
	40%	31867.8	0.2458	0.1352	0.1410	0.0881	31975.9	0.2474	0.1375	0.1407	0.0871	32148.3	0.2501	0.1417	0.1400	0.0854
	50%	27208.4	0.2861	0.2200	0.1331	0.0700	27229.2	0.2878	0.2250	0.1329	0.0695	27254.8	0.2905	0.2335	0.1324	0.0687
0.33	0%	23759.5	0.1706	0.0612	0.1732	0.4555	23781.3	0.1706	0.0612	0.1732	0.4555	23815.6	0.1706	0.0612	0.1732	0.4555
	5%	24222.6	0.1722	0.0623	0.1720	0.4042	24246.0	0.1722	0.0623	0.1720	0.4041	24283.2	0.1723	0.0623	0.1720	0.4041
	15%	25353.5	0.1772	0.0656	0.1686	0.3032	25382.3	0.1772	0.0656	0.1686	0.3031	25427.9	0.1772	0.0656	0.1686	0.3028
	30%	27818.2	0.1977	0.0811	0.1571	0.1588	27871.5	0.1978	0.0812	0.1571	0.1584	27955.6	0.1981	0.0814	0.1570	0.1577
	40%	28615.8	0.2494	0.1407	0.1402	0.0858	28696.3	0.2509	0.1431	0.1398	0.0849	28824.2	0.2535	0.1474	0.1392	0.0834
	50%	24284.7	0.2899	0.2316	0.1325	0.0689	24294.2	0.2913	0.2363	0.1323	0.0685	24304.0	0.2937	0.2441	0.1320	0.0678
0.2	0%	21405.9	0.1706	0.0612	0.1732	0.4555	21420.1	0.1706	0.0612	0.1732	0.4555	21442.2	0.1706	0.0612	0.1732	0.4555
	5%	21827.2	0.1723	0.0623	0.1720	0.4040	21842.7	0.1723	0.0623	0.1720	0.4040	21866.3	0.1723	0.0623	0.1720	0.4040
	15%	22858.8	0.1772	0.0656	0.1686	0.3028	22877.7	0.1772	0.0656	0.1685	0.3027	22906.9	0.1772	0.0656	0.1685	0.3025
	30%	25146.1	0.1982	0.0815	0.1569	0.1574	25181.1	0.1982	0.0815	0.1569	0.1571	25236.4	0.1984	0.0817	0.1568	0.1566
	40%	25945.4	0.2549	0.1497	0.1389	0.0827	25999.2	0.2562	0.1521	0.1386	0.0820	26084.0	0.2586	0.1564	0.1381	0.0807
	50%	21832.2	0.2948	0.2481	0.1318	0.0675	21832.9	0.2959	0.2521	0.1316	0.0672	21830.9	0.2977	0.2587	0.1314	0.0667

Table 6.6: Optimal input allocation ratios at $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$, $c_s/s = 0.1$ and δ=0%, 5%, 10%, 15%, 30%, 40%, 50%

input	35% cap tightness minimum change	maximum change
x_1^{**}	+ 20%	+ 22%
x_2^{**}	+ 51%	+ 56%
x_3^{**}	+ 91%	+ 102%
x_4^{**}	+ 4%	+ 5%
x_5^{**}	- 69%	- 70%

Table 6.7: Percentage change in input quantities under 35% carbon emission reduction at $COV = 0.58,\ 0.5,\ 0.33,\ 0.2$, $c_s/s = 0.1,\ 0.2,\ 0.4$

BASE CASE	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	Q^{**}	Service Level	Expected Prod. Cost	Expected Profit	Emission	
no carbon cap, no technology improvement	27873.3371	4756.4258	1706.9561	4827.6872	12695.3773	242214.0001	93.2%	20334.29619	332712.815	6313.1381	
TECHNOLOGY IMPROVEMENT, NO CARBON CAP											
% improvement on technology (A)	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	Q^{**}	Service Level	Expected Prod. Cost	Expected Profit	Emission	Emission Reduction
25%	23735.40	4050.31	1453.55	4110.99	10810.68	242937	93.4%	17315.6	335772.5	5375.9	15%
50%	20809.53	3551.03	1274.37	3604.23	9478.05	243445	93.6%	15181.1	337931.8	4713.2	25%
75%	18616.17	3176.74	1140.05	3224.34	8479.05	243825	93.8%	13581.0	339548.3	4216.4	33%
100%	16902.27	2884.28	1035.09	2927.49	7698.42	244121	93.9%	12330.6	340810.1	3828.3	39%
CARBON CAP, NO TECHNOLOGY IMPROVEMENT											
Emission reduction by a carbon cap (δ)	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	Q^{**}	Service Level	Expected Prod. Cost	Expected Profit	λ^{**}	
15%	29719.14	5264.29	1947.60	5010.42	9024.65	241430	92.9%	20512.7	332482.4	0.5771	
25%	31468.29	5891.34	2289.47	5112.74	6492.55	240213	92.4%	21049.2	331849.6	1.5949	
33%	33127.04	6848.67	2949.50	5076.44	4461.41	237502	91.4%	22081.0	330535.2	4.1345	
39%	33519.91	7984.64	4165.13	4791.80	3135.28	230554	88.7%	23543.6	327918.6	11.1572	

Table 6.8: Comparison of technology improvement and no technology improvement cases under carbon cap restrictions for the base case with $COV = 0.58$ and $c_s/s = 0.1$

% emission reduction by a carbon cap	required % of technology improvement	expected benefit of technology improvement
15%	25%	3290.1
25%	50%	6082.1
33%	75%	9013.0
39%	100%	12891.4

Table 6.9: Expected benefit of technology improvement over no improvement case under a carbon cap at $COV = 0.58$, $c_s/s = 0.1$

Parameter	Values
Elasticity of input 2 (α_2)	(0.9, 0.1)
Purchase cost of input 2 (p_2)	(0.25, 4)
Carbon coef. of input 2 (β_2)	(0.25, 4)
Carbon cap tightness (δ)	(0%, 50%)

Table 6.10: Parameter set

α_2	p_2	β_2	δ	x_1^{**}	x_2^{**}	$\frac{x_2^{**}}{x_1^{**}}$	λ^{**}	Q^{**}	$\frac{Q^{**}}{Q_{unc}^{**}}$	Exp. Profit	$\frac{Exp.\ Profit}{Exp.\ Profit_{unc}}$
0.9	0.25	0.25	0%	37.49	1349.60	36.00	0.0000	929.187	100.0%	23248.75	100.0%
			50%	18.74	674.80	36.00	68.6839	464.594	50.0%	16811.56	72.3%
		4.00	0%	37.49	1349.60	36.00	0.0000	943.134	100.0%	23248.75	100.0%
			50%	229.37	622.14	2.71	4.4993	563.059	59.7%	18906.00	81.3%
	4.00	0.25	0%	417.76	939.96	2.25	0.0000	866.748	100.0%	19246.75	100.0%
			50%	68.19	1032.73	15.14	9.8920	786.978	90.8%	18656.57	96.9%
		4.00	0%	417.76	939.96	2.25	0.0000	866.748	100.0%	19246.75	100.0%
			50%	208.88	469.98	2.25	5.2056	433.374	50.0%	13810.06	71.8%
0.1	0.25	0.25	0%	1007.68	447.86	0.44	0.0000	929.187	100.0%	22492.82	100.0%
			50%	503.84	223.93	0.44	22.3222	873.535	50.0%	16244.61	72.2%
		4.00	0%	1007.68	447.86	0.44	0.0000	922.537	100.0%	22492.82	100.0%
			50%	1202.04	49.38	0.04	1.8954	544.082	94.0%	22241.36	98.9%
	4.00	0.25	0%	1320.12	36.67	0.03	0.0000	943.134	100.0%	22136.37	100.0%
			50%	626.33	153.25	0.24	17.3730	471.567	59.0%	17616.03	79.6%
		4.00	0%	1320.12	36.67	0.03	0.0000	922.537	100.0%	22136.37	100.0%
			50%	660.06	18.34	0.03	16.7960	461.269	50.0%	15977.27	72.2%

Table 6.11: Sensitivity analysis under parameter set in Table 6.10

Experiment instance	Input elasticities $(\alpha_1,\ \alpha_2)$	Input costs $(p_1,\ p_2)$
1	(0.9, 0.1)	(30, 15)
2	(0.8, 0.2)	(30, 15)
3	(0.8, 0.2)	(40, 10)
4	(0.5, 0.5)	(30, 15)
5	(0.2, 0.8)	(30, 15)

Table 6.12: Experiment instances

Experiment instance	Demand Distribution		
	UNI	TRIA1	TRIA2
1	488.3	465.6	514.2
2	679.0	647.5	715.0
3	754.0	719.4	794.2
4	1176.2	1121.6	1238.6
5	1260.6	1202.0	1327.5

Table 6.13: Initial carbon permit purchase with the contract, κ^{**}, under the experiment instances and demand distributions: UNI, TRIA1, TRIA2

Expected Permit to Buy in Period 2			
Experiment instance	Demand Distribution		
	UNI	TRIA1	TRIA2
1	65.2	73.8	49.7
2	89.1	101.0	67.5
3	94.7	107.8	70.7
4	153.6	174.2	115.9
5	169.2	191.4	129.1

Expected Permit to Sell in Period 2			
Experiment instance	Demand Distribution		
	UNI	TRIA1	TRIA2
1	56.0	41.7	65.8
2	76.8	57.0	90.6
3	82.3	60.4	97.5
4	132.5	98.3	156.7
5	145.3	108.5	170.8

Table 6.14: Expected carbon permit trade in Period 2 when making the initial permit purchase under the experiment instances and demand distributions: UNI, TRIA1, TRIA2

Expected profit						
Experiment instance	No-contract policy			Optimal contract policy		
	UNI	TRIA 1	TRIA 2	UNI	TRIA 1	TRIA 2
1	1931.17	1931.49	1932.78	1952.72	1953.42	1954.51
2	1720.83	1721.12	1722.26	1750.40	1751.19	1752.00
3	1501.32	1501.57	1502.57	1533.09	1533.79	1534.20
4	1427.70	1427.94	1429.04	1478.93	1479.94	1480.38
5	1685.15	1685.43	1686.63	1741.17	1742.47	1743.08

Expected carbon emission						
Experiment instance	No-contract policy			Optimal contract policy		
	UNI	TRIA 1	TRIA 2	UNI	TRIA 1	TRIA 2
1	494.8	494.9	495.2	497.5	497.7	498.2
2	685.0	685.0	686.0	691.2	691.5	691.9
3	754.5	754.6	755.1	766.4	766.8	767.4
4	1186.0	1187.0	1187.0	1197.3	1197.5	1197.8
5	1279.0	1279.0	1280.0	1284.6	1284.9	1285.7

Table 6.15: Comparison of no-carbon contract policy and the optimal contract policy, $\kappa = 0$, under the experiment instances and demand distributions: UNI, TRIA1, TRIA2

Chapter 7

Conclusion

To the best of our knowledge, this is the first work that considers production functions in the newsvendor setting. Additionally, there has been no study on the production with multiple inputs under a carbon emission constraint. Our work is significant in the sense that it incorporates all of these aspects.

In this thesis, we analyze two problems in the newsvendor setting for the production of a single item with multiple inputs, under a carbon emission constraint and non-linear production functions. In both problems, the production quantity is linked to multiple inputs via the Cobb-Douglas and Leontief production functions which are widely used in the economic theory of production.

In the first problem, we assume a strict carbon cap and find the optimal production quantity and input allocation that will maximize the expected profit under this constraint. We construct our problem based on the classical newsvendor problem where production costs are generated via production functions. We derive the optimal production policy that gives the production quantity and input allocation.

In the second problem, we consider an emission trading scheme that allows for advance purchase of carbon emission permits made at an initial price before the realization of random demand is revealed. When the demand and market carbon trade prices are revealed, it is possible to buy additional permits or to sell an excess amount. We provide structural results on the optimal allocation of the inputs under demand commitment as well as the carbon trading policy that maximizes the expected profit.

In our numerical study, we analyze both problems under the Cobb-Douglas production function. In the illustrative example based on a real-life agricultural production application, we observe that carbon emission can be decreased by at least 35% without any significant impact on the expected profit and service level. We also observe that the carbon emission can be reduced by improving the technology level rather than changing the input allocations and decreasing the production quantity. Under a given carbon cap, we infer that compared to producing with the given technology level, improving technology may even provide side benefits such as higher service levels and possibly higher net expected profit depending on the technology improvement cost.

The sensitivity analysis on Problem 1 indicates that when our production system relies more on the input which emits less carbon, the expected profit is affected less by the carbon cap. This finding provides evidence on the importance of swiching to less carbon emitting inputs.

The sensitivity study we conduct for Problem 2 shows that relying on high carbon emitting resources leads to become more dependent on the carbon trade market. The results present that when the production system mostly depends on the low cost-high carbon emitting input, we tend to purchase more permit with the contract and expect to be more active in the emission trading scheme while obtaining lower expected profit.

For further study, we propose several extensions to our problems. For Problem 1, it would be interesting to consider the multi-echelon extension to our problem. In the multi-echelon setting, the emission policy makers that set the carbon cap for the producer may be integrated in the system and the optimal carbon cap policy that maximizes the overall social utility may be derived. It can also be extended to include multiple players where the authority may allocate the national or regional carbon cap to several producers under an objective. This problem may also be analyzed under multiple period production systems with technology investments.

For future study, Problem 2 may be considered without demand commitment. It may also be analyzed under production prior to demand realization. In this case, the producer needs to consider the optimal production policy as well as the optimal carbon contract and trading policies. This problem may also be considered under options contract design where the producer buys the ability to purchase emission permits rather than the permits itself. Analyzing this problem in multiple period horizon with technology improvement may be also interesting. In our problem, we assume that carbon permits are always available in the market. It would be interesting to consider the case where there is a fixed amount of emission available in the market. Finally, we suggest that an interesting extension may be analyzing this problem with multiple players where the players are given an initial carbon permit by the authorities and are allowed to trade this permit with each other.

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Appendix A

Problem 1

Proof of Lemma 4.2

We will prove this by induction. First, we look at the base case where $n = 2$.

$$G^2 = \begin{pmatrix} a_1 b_1 & a_1 a_2 \\ a_2 a_1 & a_2 b_2 \end{pmatrix}$$

$$\begin{aligned}
\Delta^2 &= a_1 b_1 a_2 b_2 - (a_1 a_2)^2 \\
&= a_1 a_2((a_1 + c_1)(a_2 + c_2) - a_1 a_2) \\
&= a_1 a_2(a_1 a_2 + a_1 c_2 + a_2 c_1 + c_1 c_2 - a_1 a_2) \\
&= a_1 a_2(c_1 c_2 + a_1 c_2 + a_2 c_1) \\
&= K_2 \sum_{i=0}^{2} a_i \prod_{j=1, j\neq i}^{2} c_j
\end{aligned}$$

By induction hypothesis, we claim that $\Delta^n = K_n \sum_{i=0}^{n} a_i \prod_{j=1,j\neq i}^{n} c_j$. Then we prove this equation for $n+1$. We partition G^{n+1} as follows

$$G^{n+1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$A_{11} = G^n$$

$$A_{22} = a_{n+1}b_{n+1}$$

$$A_{12} = A_{21}^T = \begin{pmatrix} a_{n+1}a_1 & a_{n+1}a_2 & ... & a_{n+1}a_n \end{pmatrix}$$

Then, we compute $\Delta^{n+1} = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}|$. Note that $A_{22}^{-1} = \frac{1}{a_{n+1}b_{n+1}}$.

$$\begin{aligned}
A_{12}A_{22}^{-1}A_{21} &= \frac{1}{a_{n+1}b_{n+1}} \begin{pmatrix} a_1a_{n+1} \\ a_2a_{n+1} \\ . \\ . \\ a_na_{n+1} \end{pmatrix} \begin{pmatrix} a_{n+1}a_1 & a_{n+1}a_2 & ... & a_{n+1}a_n \end{pmatrix} \\
&= \frac{a_{n+1}}{b_{n+1}} \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \\
&= \frac{a_{n+1}}{b_{n+1}} \begin{pmatrix} {a_1}^2 & a_1a_2 & ... & a_1a_n \\ a_2a_1 & {a_2}^2 & ... & a_2a_n \\ . & & & \\ . & & & \\ a_na_1 & a_na_2 & ... & {a_n}^2 \end{pmatrix}
\end{aligned}$$

Then we consider the following matrix

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{pmatrix} a_1b_1 & a_1a_2 & ... & a_1a_n \\ a_2a_1 & a_2b_2 & ... & a_2a_n \\ . & & & \\ . & & & \\ a_na_1 & a_na_2 & ... & a_nb_n \end{pmatrix} - \frac{a_{n+1}}{b_{n+1}} \begin{pmatrix} {a_1}^2 & a_1a_2 & ... & a_1a_n \\ a_2a_1 & {a_2}^2 & ... & a_2a_n \\ . & & & \\ . & & & \\ a_na_1 & a_na_2 & ... & {a_n}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1 - \frac{a_{n+1}}{b_{n+1}}{a_1}^2 & a_1a_2\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & ... & a_1a_n\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \\ a_2a_1\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & a_2b_2 - \frac{a_{n+1}}{b_{n+1}}{a_2}^2 & ... & a_2a_n\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \\ . & & & \\ . & & & \\ a_na_1\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & a_na_2\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & ... & a_nb_n - \frac{a_{n+1}}{b_{n+1}}{a_n}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1\left(b_1 - \frac{a_{n+1}}{b_{n+1}}a_1\right) & a_1a_2\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & ... & a_1a_n\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \\ a_2a_1\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & a_2\left(b_2 - \frac{a_{n+1}}{b_{n+1}}a_2\right) & ... & a_2a_n\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \\ . & & & \\ . & & & \\ a_na_1\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & a_na_2\left(1 - \frac{a_{n+1}}{b_{n+1}}\right) & ... & a_n\left(b_n - \frac{a_{n+1}}{b_{n+1}}a_n\right) \end{pmatrix}$$

In the next step, we manipulate the elements in the matrix. We let

$$\begin{aligned} \overline{b_i} &= \frac{b_i - a_i\frac{a_{n+1}}{b_{n+1}}}{1 - \frac{a_{n+1}}{b_{n+1}}} \\ &= \frac{b_ib_{n+1} - a_ia_{n+1}}{b_{n+1} - a_{n+1}} \\ &= \frac{(a_i + c_i)(a_{n+1} + c_{n+1}) - a_ia_{n+1}}{a_{n+1} + c_{n+1} - a_{n+1}} \\ &= \frac{a_ia_{n+1} + c_ia_{n+1} + a_ic_{n+1} + c_ic_{n+1} - a_ia_{n+1}}{c_{n+1}} \\ &= a_i + c_i\left(1 + \frac{a_{n+1}}{b_{n+1}}\right) \end{aligned}$$

We let $\gamma = 1 - \frac{a_{n+1}}{b_{n+1}} = \frac{c_{n+1}}{a_{n+1}+c_{n+1}}$. We also let $\overline{c_i} = c_i\left(1 + \frac{a_{n+1}}{c_{n+1}}\right) = c_i\left(\frac{a_{n+1}+c_{n+1}}{c_{n+1}}\right) =$

$\frac{c_i}{\gamma}$, then $\overline{b_i} = a_i + \overline{c_i}$. Next, we rewrite the above matrix as follows:

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = \left(1 - \frac{a_{n+1}}{b_{n+1}}\right) \begin{pmatrix} a_1\overline{b_1} & a_1a_2 & ... & a_1a_n \\ a_2a_1 & a_2\overline{b_2} & ... & a_2a_n \\ . & & & \\ . & & & \\ a_na_1 & a_na_2 & ... & a_n\overline{b_n} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} a_1(a_1 + \overline{c_1}) & a_1a_2 & ... & a_1a_n \\ a_2a_1 & a_2(a_2 + \overline{c_2}) & ... & a_2a_n \\ . & & & \\ . & & & \\ a_na_1 & a_na_2 & ... & a_n(a_n + \overline{c_n}) \end{pmatrix}$$

This matrix is an $n \times n$ matrix in the form of the matrix G; therefore, we can use the induction hypothesis to determine the determinant of this matrix.

$$\begin{aligned} |A_{11} - A_{12}A_{22}^{-1}A_{21}| &= \gamma^n K_n \sum_{i=0}^{n} a_i \prod_{j=1, j\neq i}^{n} \overline{c_j} \\ &= \gamma^n K_n \sum_{i=0}^{n} a_i \prod_{j=1, j\neq i}^{n} \frac{c_j}{\gamma} \\ &= \gamma^n K_n \left(\frac{\prod_{k=1}^{n} c_k}{\gamma^n} + \frac{1}{\gamma^{n-1}} \sum_{i=1}^{n} a_i \prod_{j=1, j\neq i}^{n} c_j \right) \\ &= K_n \left(\prod_{k=1}^{n} c_k + \gamma \sum_{i=1}^{n} a_i \prod_{j=1, j\neq i}^{n} c_j \right) \end{aligned}$$

Finally, we evaluate the determinant of the $(n+1)\times(n+1)$ matrix.

$$\begin{aligned}
\Delta^{n+1} &= |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}| \\
&= a_{n+1}(a_{n+1}+c_{n+1})K_n\left(\prod_{k=1}^{n} c_k + \gamma\sum_{i=1}^{n} a_i \prod_{j=1,j\neq i}^{n} c_j\right) \\
&= K_{n+1}\left(\prod_{k=1}^{n+1} c_k + a_{n+1}\prod_{k=1}^{n} c_k + \frac{(a_{n+1}+c_{n+1})c_{n+1}}{a_{n+1}+c_{n+1}}\sum_{i=1}^{n} a_i \prod_{j=1,j\neq i}^{n} c_j\right) \\
&= K_{n+1}\left(\prod_{k=1}^{n+1} c_k + a_{n+1}\prod_{k=1}^{n} c_k + c_{n+1}\sum_{i=1}^{n} a_i \prod_{j=1,j\neq i}^{n} c_j\right) \\
&= K_{n+1}\sum_{i=1}^{n+1} a_i \prod_{j=1,j\neq i}^{n+1} c_j
\end{aligned}$$

Hence we showed that $\Delta^n = K_n \sum_{i=0}^{n} a_i \prod_{j\neq i} c_j$.

Proof of Lemma 4.3

In order to show the convexity of the objective function, we first aim to derive the second order conditions for Problem $SP1Ua$ given by

$$\begin{aligned}
\min_{\vec{x}}\ \ & \Gamma(\vec{x}) = \sum_{i=1}^{n} p_i x_i \\
s.t.\ \ & Q = A\prod_{i=1}^{n} {x_i}^{\alpha_i}
\end{aligned}$$

We eliminate the production function constraint by substituting x_n^* for x_n where x_n^* is a function of $Q, x_1, \ldots, x_{n-1}$ by Lemma 4.1. Then we have

$$\min_{x_1,\ldots,x_{n-1}}\ \ \Gamma(x_1,\ldots,x_{n-1}) = \sum_{i=1}^{n-1} p_i x_i + p_n x_n^*$$

Next, we derive the first order conditions of the objective function.

$$\begin{aligned}
\frac{\partial\Gamma(x_1,\ldots,x_{n-1})}{\partial x_i} &= p_i + p_n\left(\frac{Q}{A\prod_{j=1}^{n-1} {x_j}^{\alpha_j}}\right)^{\frac{1}{\alpha_n}}\left(-\frac{\alpha_i}{\alpha_n}\right){x_i}^{-1} \\
&= p_i - p_n\frac{\alpha_i}{\alpha_n}\frac{x_n^*}{x_i}\ \ \text{for all } i = 1,\ldots,n-1
\end{aligned}$$

The the second order derivatives are given as follows

$$\frac{\partial^2 \Gamma(x_1, \ldots, x_{n-1})}{\partial {x_i}^2} = p_n x_n^* \left(\frac{\alpha_i}{\alpha_n}\right)\left(1 + \frac{\alpha_i}{\alpha_n}\right) {x_i}^{-2} \quad \forall i = 1, \ldots, n-1$$

$$\frac{\partial^2 \Gamma(x_1, \ldots, x_{n-1})}{\partial x_i \partial x_j} = p_n x_n^* \left(\frac{\alpha_i}{\alpha_n}\right)\left(\frac{\alpha_j}{\alpha_n}\right) {x_i}^{-1} {x_j}^{-1} \quad \forall i \neq j,\ i, j = 1, \ldots, n-1$$

We let

$$a_i = \frac{\alpha_i}{\alpha_n} {x_i}^{-1}$$

$$c_i = {x_i}^{-1}$$

$$b_i = a_i + c_i$$

Then

$$\frac{\partial^2 \Gamma(x_1, \ldots, x_{n-1})}{\partial {x_i}^2} = p_n x_n^* a_i (a_i + c_i) \quad \text{for all } i = 1, \ldots, n-1$$

$$\frac{\partial^2 \Gamma(x_1, \ldots, x_{n-1})}{\partial x_i \partial x_j} = p_n x_n^* a_i a_j \quad \text{for all } i \neq j,\ i, j = 1, \ldots, n-1$$

Then the Hessian matrix is $H = \{h_{ij}\}$ where $h_{ij} = p_n x_n g_{ij}$ for all $i, j = 1, .., n-1$ where

$$g_{ii} = a_i b_i = a_i (a_i + c_i) \quad \text{for all } i = 1, \ldots, n-1$$

$$g_{ij} = a_i a_j \qquad \text{for all } i \neq j,\ i, j = 1, \ldots, n-1$$

and $G = \{g_{ij}\}$. Since all elements in H contain $p_n x_n^*$, we can write $det(H) = p_n x_n^* \times det(G)$. By Lemma 4.2,

$$det(H) = p_n x_n^* \left(\prod_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} {x_i}^{-1}\right) \sum_{i=0}^{n-1} \frac{\alpha_i}{\alpha_n} {x_i}^{-1} \prod_{j=1, j \neq i}^{n-1} {x_i}^{-1}$$

Since $\alpha_i > 0$ and $x_i > 0$ for all $i = 1, .., n$, $det(H)$ is positive definite and has positive definite minors.

Proof of Lemma 4.4

In the previous proof, we already derived the first order condition:

$$\frac{\partial \Gamma(x_1, \ldots, x_{n-1})}{\partial x_i} = p_i - p_n \frac{\alpha_i}{\alpha_n} \frac{x_n^*}{x_i} \quad \text{for all } i = 1, \ldots, n-1$$

At the extrema, we have $\frac{\partial \Gamma(x_1,\ldots,x_{n-1})}{\partial x_i} = 0$ for all $i = 1, \ldots, n-1$; therefore, we obtain the following equations:

$$\frac{p_i x_i^*}{\alpha_i} = \frac{p_n x_n^*}{\alpha_n} \quad \text{for all } i = 1, \ldots, n-1$$

from which we get

$$\frac{{x_i}^*(Q)}{{x_j}^*(Q)} = \frac{\alpha_i}{p_i}\frac{p_j}{\alpha_j} \quad \text{for all } i, j = 1, \ldots, n$$

By Lemma 4.3, this solution holds at optimality.

Proof of Lemma 4.5

We need to check the second derivative of $\Pi(Q)$ with respect to Q in order to evaluate the convavity. The first derivative of $\Pi(Q)$ with respect to Q is

$$\frac{d\Pi(Q)}{dQ} = (s + c_s) - s + c_s + c_e F(Q) - A^{-\frac{1}{r}} \prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{\frac{\alpha_i}{r}} Q^{\frac{1}{r}-1}$$

For $r > 1$, the second derivative of$\Pi(Q)$ with respect to Q is

$$\frac{d^2\Pi(Q)}{dQ^2} = -(s + c_s + c_e) f(Q) - \sum_{i=1}^{n} p_i \frac{A^{-\frac{1}{r}}}{r} \left(\frac{1}{r} - 1\right) \frac{\alpha_i}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}-2}$$

For concavity, we need $\frac{d^2\Pi(Q)}{dQ^2} \leq 0$. Then,

$$-(s + c_s + c_e) f(Q) - A^{-\frac{1}{r}} \left(\frac{1}{r} - 1\right) \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}-2} \leq 0$$

$$f(Q) \geq \frac{A^{-\frac{1}{r}}}{s + c_s + c_e} \left(1 - \frac{1}{r}\right) \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} Q^{\frac{1}{r}-2}$$

If this condition is satisfied, then the objective function turns out to be concave in Q.

Proof of Theorem 4.5

Problem $SP1Ub$ is

$$\begin{aligned}
&\min_{\vec{x}} \;\; \Gamma(\vec{x}) = \sum_{i=1}^{n} p_i x_i \\
&\; s.t. \;\; Q = \min_i \{Q_i : Q_i = A_i {x_i^*}^{\alpha_i}, \forall i\}
\end{aligned}$$

We pick an arbitrary input $n = argmin\{Q_i\}$. Then for a given Q, $Q_n = Q$ and $Q_i = Q + \epsilon_i$, $\epsilon_i \geq 0$ for all resource $i = 1, \ldots, n-1$. Then we have

$$\begin{aligned}
x_n &= \left(\frac{Q}{A_n}\right)^{\frac{1}{\alpha_n}} \\
x_i &= \left(\frac{Q_i}{A_i}\right)^{\frac{1}{\alpha_i}} \\
&= \left(\frac{Q + \epsilon_i}{A_i}\right)^{\frac{1}{\alpha_i}} \quad \text{for all } i = 1, \ldots, n-1
\end{aligned}$$

Then, Problem $SP1Ub$ turns out to be

$$\begin{aligned}
&\min_{\epsilon_1, \ldots, \epsilon_{n-1}} \;\; \Gamma(\epsilon_1, \ldots, \epsilon_{n-1}) = p_n \left(\frac{Q}{A_n}\right)^{\frac{1}{\alpha_n}} + \sum_{i=1}^{n-1} p_i \left(\frac{Q + \epsilon_i}{A_i}\right)^{\frac{1}{\alpha_i}} \\
&\qquad s.t. \;\; \epsilon_i \geq 0 \;\; \text{for all } i = 1, \ldots, n-1
\end{aligned}$$

Now, we can look at the first order derivative of the objective function with respect to ϵ_i, for all $i = 1, \ldots, n-1$.

$$\frac{\partial \Gamma(\epsilon_1, \ldots, \epsilon_{n-1})}{\partial \epsilon_i} = \frac{p_i}{A_i \alpha_i} \left(\frac{Q + \epsilon_i}{A}\right)^{\frac{1}{\alpha_i} - 1} \quad \text{for all } i = 1, \ldots, n-1$$

The expression above is always positive for $\epsilon_i \geq 0$, for all $i = 1, \ldots, n-1$, which implies the objective function is strictly increasing in ϵ_i. Therefore, in order to minimize the objective function, we have $\epsilon_i^* = 0$, for all $i = 1, \ldots, n-1$. Then we get

$${x_i}^{**}(Q) = \left(\frac{Q}{A}\right)^{\frac{1}{\alpha_i}}$$

Proof of Lemma 4.11

First we look at the second derivative of the objective function $\Pi(Q)$ with respect to Q.

$$\frac{d\Pi(Q)}{dQ} = s + c_s - (s + c_s + c_e)F(Q) - \sum_{i=1}^{n} \frac{p_i}{\alpha_i} A_i^{-\frac{1}{\alpha_i}} Q^{\frac{1}{\alpha_i}-1}$$

Then we have

$$\frac{d^2\Pi(Q)}{dQ^2} = -(s + c_s + c_e)f(Q) - \sum_{i=1}^{n} \frac{p_i(\frac{1}{\alpha_i} - 1)}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{\frac{1}{\alpha_i}-2}$$

$\Pi(Q)$ is concave in Q only if $\frac{d^2\Pi(Q)}{dQ^2} \leq 0$. Then we have

$$-(s + c_s + c_e)f(Q) - \sum_{i=1}^{n} \frac{p_i(\frac{1}{\alpha_i} - 1)}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{\frac{1}{\alpha_i}-2} \leq 0$$

Then, the objective function is concave if the following condition holds:

$$f(Q) \geq \frac{\sum_{i=1}^{n} \frac{p_i(1-\frac{1}{\alpha_i})}{\alpha_i A_i^{\frac{1}{\alpha_i}}} Q^{\frac{1}{\alpha_i}-2}}{s + c_s + c_e}$$

Proof of Corollary 4.15

The first derivative of the objective function $\widehat{\Pi}(Q)$ with respect to Q is

$$\frac{d\widehat{\Pi}(Q)}{dQ} = s + c_s - (s + c_s + c_e)F(Q) - \sum_{i=1}^{n} \frac{p_i + \lambda\beta_i}{\alpha_i} A_i^{-\frac{1}{\alpha_i}} Q^{\frac{1}{\alpha_i}-1}$$

At the extrema, we have $\frac{d\Pi(Q)}{dQ} = 0$. Then, we obtain the following equation at the optimal solution Q^{**}:

$$\lambda^{**} = \frac{(s + c_s) - (s + c_s + c_e)F(Q^{**}) - \sum_{i=1}^{n} \frac{p_i Q^{*\frac{1}{\alpha_i}-1}}{\alpha_i A_i^{\frac{1}{\alpha_i}}}}{\sum_{i=1}^{n} \frac{\beta_i Q^{*\frac{1}{\alpha_i}-1}}{\alpha_i A_i^{\frac{1}{\alpha_i}}}}$$

Appendix B

Problem 2

Proof of Lemma 5.4

In this proof, we will be using Lemmas 5.3 and 5.2, Corollary 5.1 and Definition 5.1. Set $\epsilon \equiv \kappa + q_b - q_s$. Then by Corollary 5.1, we have

$$\epsilon \equiv \kappa + q_b - q_s = \left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_j + \lambda\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} \sum_{i=1}^{n} \frac{\alpha_i\beta_i}{p_i + \lambda\beta_i}$$

1. Suppose $\kappa < \kappa_{up}$ and $q_b = 0$. Then the only decision variable is q_s and $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_b}$ do not exist while $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda$ for a given λ by Lemma 5.3. Set $\Delta\kappa = \kappa_{up} - \kappa$. Then we have $\epsilon = \kappa_{up} - \Delta\kappa - q_s < \kappa_{up}$. It implies $\lambda > p_b$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda < p_s - p_b < 0$. It suggests that $\hat{\Pi}((0, q_s)|\kappa)$ is strictly decreasing in q_s. Since $0 \leq q_s \leq \kappa$, $q_s^* = 0$.

2. Suppose $\kappa < \kappa_{up}$ and $q_s = 0$. Then the only decision variable is q_b and $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_s}$ do not exist while $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_b} = \lambda - p_b$ for a given λ by Lemma 5.3. Set $\Delta\kappa = \kappa_{up} - \kappa$.

   First, suppose $q_b \leq \Delta\kappa$. Then we have $\epsilon = \kappa_{up} - \Delta\kappa + q_b \leq \kappa_{up}$. It implies $\lambda \geq p_b$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_b} = \lambda - p_b \geq 0$. It suggests that $\hat{\Pi}((q_b, 0)|\kappa)$ is nondecreasing in q_b. Since $0 \leq q_b \leq \Delta\kappa$, $q_b^* = \Delta\kappa$.

   Now, suppose $q_b \geq \Delta\kappa$. Then we have $\epsilon = \kappa_{up} - \Delta\kappa + q_b \geq \kappa_{up}$. It implies

$\lambda \leq p_b$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_b} = \lambda - p_b \leq 0$. It suggests that $\hat{\Pi}((q_b,0)|\kappa)$ is nonincreasing in q_b. Since $q_b \geq \Delta\kappa$, $q_b^* = \Delta\kappa$.

In conclusion, $q_b^* = \Delta\kappa = \kappa_{up} - \kappa$.

3. Suppose $\kappa_{up} \leq \kappa \leq \kappa_{down}$ and $q_b = 0$. Then the only decision variable is q_s and $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_b}$ do not exist while $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda$ for a given λ by Lemma 5.3. Set $\Delta\kappa = \kappa_{down} - \kappa$. Then we have $\epsilon = \kappa_{down} - \Delta\kappa - q_s \leq \kappa_{down}$. It implies $\lambda \geq p_s$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda \leq 0$. It suggests that $\hat{\Pi}((0,q_s)|\kappa)$ is nonincreasing in q_s. Since $0 \leq q_s \leq \kappa$, $q_s^* = 0$.

4. Suppose $\kappa_{up} \leq \kappa \leq \kappa_{down}$ and $q_s = 0$. Then the only decision variable is q_b and $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_s}$ do not exist while $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_b} = \lambda - p_b$ for a given λ by Lemma 5.3. Set $\Delta\kappa = \kappa - \kappa_{up}$. Then we have $\epsilon = \kappa_{up} + \Delta\kappa + q_b \geq \kappa_{up}$. It implies $\lambda \leq p_b$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_b} = \lambda - p_b \leq 0$. It suggests that $\hat{\Pi}((q_b,0)|\kappa)$ is nonincreasing in q_b. Since $q_b \geq 0$, $q_b^* = 0$.

5. Suppose $\kappa > \kappa_{down}$ and $q_b = 0$. Then the only decision variable is q_s and $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_b}$ do not exist while $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda$ for a given λ by Lemma 5.3. Set $\Delta\kappa = \kappa - \kappa_{down}$.

   First suppose $q_s \leq \Delta\kappa$. Then we have $\epsilon = \kappa_{down} + \Delta\kappa - q_s \geq \kappa_{down}$. It implies $\lambda \leq p_s$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda \geq 0$. It suggests that $\hat{\Pi}((0,q_s)|\kappa)$ is nondecreasing in q_s. Since $0 \leq q_s \leq \Delta\kappa$, $q_s^* = \Delta\kappa$.

   Now, suppose $q_s \geq \Delta\kappa$. Then we have $\epsilon = \kappa_{down} + \Delta\kappa - q_s \leq \kappa_{down}$. It implies $\lambda \geq p_s$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((0,q_s)|\kappa)}{dq_s} = p_s - \lambda \leq 0$. It suggests that $\hat{\Pi}((0,q_s)|\kappa)$ decreases in q_s. Since $\Delta\kappa \leq q_s \leq \kappa$, $q_s^* = \Delta\kappa$.

   In conclusion, $q_s^* = \Delta\kappa = \kappa - \kappa_{down}$.

6. Suppose $\kappa > \kappa_{down}$ and $q_s = 0$. Set $\Delta\kappa = \kappa - \kappa_{down}$. Then we have $\epsilon = \kappa_{down} + \Delta\kappa + q_b > \kappa_{down}$. It implies $\lambda < p_s$ by Lemma 5.2. Then $\frac{d\hat{\Pi}((q_b,0)|\kappa)}{dq_b} = \lambda - p_b < p_s - p_b < 0$. It suggests that $\hat{\Pi}((q_b,0)|\kappa)$ strictly decreases in q_b. Since $q_b \geq 0$, $q_b^* = 0$.

Proof of Lemma 5.6

By Corollary 5.7, we have

$$\Pi_2{}^{**} = sD + \lambda^{**}\kappa - r\left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{\lambda^{**}\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}$$

In Theorem 5.1, λ^{**} values are given as follows:

$$\lambda^{**} = \begin{cases} p_b & : if \kappa > \kappa_{up} \\ \overline{\lambda} & : if \kappa_{up} \leq \kappa \leq \kappa_{down} \\ p_s & : if \kappa > \kappa_{down} \end{cases}$$

where $\overline{\lambda} \equiv \overline{\lambda}(D, \kappa)$ solves Equation 5.12. We can rewrite the conditions for λ^{**} using Corollary 5.8. Then,

$$\lambda^{**} = \begin{cases} p_b & : if D > e(p_b, \kappa) \\ \overline{\lambda}(D, \kappa) & : if e(p_s, \kappa) \leq D \leq e(p_b, \kappa) \\ p_s & : if D < e(p_s, \kappa) \end{cases}$$

where $\overline{\lambda}(D, \kappa)$ solves Equation 5.12. First, we derive $E_{D,p_b,p_s}[\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)]$. We suppress this notation for simplicity as follows: $E[\Pi_2^{**}]$.

$$\begin{aligned} E[\Pi_2^{**}] =& sE[D] + \int_{p_b,p_s} \{ \int_0^{e(p_s,\kappa)} p_s \kappa f(D) dD \\ & - \int_0^{e(p_s,\kappa)} r\left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_s\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} f(D) dD \\ & + \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \overline{\lambda}(D,\kappa)\kappa f(D) dD \\ & - \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} r\left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{\overline{\lambda}(D,\kappa)\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} f(D) dD \\ & + \int_{e(p_b,\kappa)}^{\infty} p_b \kappa f(D) dD \\ & - \int_{e(p_b,\kappa)}^{\infty} r\left(\frac{D}{A}\right)^{\frac{1}{r}} \prod_{j=1}^{n} \left(\frac{p_b\beta_j + p_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}} f(D) dD \} g(p_b, p_s) dp_b ps \end{aligned}$$

Now we evaluate $\frac{dE[\Pi_2^{**}]}{d\kappa}$ using Leibniz Rule.

$$
\begin{aligned}
\frac{dE[\Pi_2^{**}]}{d\kappa} =& \int_{p_b,p_s} \{ \frac{de(p_s,\kappa)}{d\kappa} p_s \kappa f(e(p_s,\kappa)) + \int_0^{e(p_s,\kappa)} p_s f(D) dD \\
& - \frac{de(p_s,\kappa)}{d\kappa} r \left(\frac{e(p_s,\kappa)}{A} \right)^{\frac{1}{r}} \prod_{j=1}^n \left(\frac{p_s \beta_j + p_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} f(e(p_s,\kappa)) \\
& + \frac{de(p_b,\kappa)}{d\kappa} \overline{\lambda}(e(p_b,\kappa),\kappa) \kappa f(e(p_b,\kappa)) + \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \frac{d\overline{\lambda}(D,\kappa)}{d\kappa} \kappa f(D) dD \\
& + \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \overline{\lambda}(D,\kappa) f(D) dD - \frac{de(p_s,\kappa)}{d\kappa} \overline{\lambda}(e(p_s,\kappa),\kappa) \kappa f(e(p_s,\kappa)) \\
& - \frac{de(p_b,\kappa)}{d\kappa} r \left(\frac{e(p_b,\kappa)}{A} \right)^{\frac{1}{r}} \prod_{j=1}^n \left(\frac{\overline{\lambda}(e(p_b,\kappa),\kappa) \beta_j + p_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} f(e(p_b,\kappa)) \\
& - \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \frac{d\overline{\lambda}(D,\kappa)}{d\kappa} \left(\frac{D}{A} \right)^{1/r} \prod_{j=1}^n \left(\frac{\overline{\lambda}(D,\kappa) \beta_j + p_j}{\alpha_j} \right)^{\alpha_j/r} \\
& \qquad \times \sum_{i=1}^n \frac{\alpha_i \beta_i}{p_i + \overline{\lambda}(D,\kappa) \beta_i} f(D) dD \\
& + \frac{de(p_s,\kappa)}{d\kappa} r \left(\frac{e(p_s,\kappa)}{A} \right)^{\frac{1}{r}} \prod_{j=1}^n \left(\frac{\overline{\lambda}(e(p_s,\kappa),\kappa) \beta_j + p_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} f(e(p_s,\kappa)) \\
& - \frac{de(p_b,\kappa)}{d\kappa} p_b \kappa f(e(p_b,\kappa)) + \int_{e(p_b,\kappa)}^{\infty} p_b f(D) dD \\
& + \frac{de(p_b,\kappa)}{d\kappa} r \left(\frac{e(p_b,\kappa)}{A} \right)^{\frac{1}{r}} \prod_{j=1}^n \left(\frac{p_b \beta_j + p_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} f(e(p_b,\kappa)) \} g(p_b,p_s) dp_b ps
\end{aligned}
$$

By Lemma 5.5, $\overline{\lambda}(e(p_b,\kappa),\kappa) = p_b$ and $\overline{\lambda}(e(p_s,\kappa),\kappa) = p_s$. Then, we obtain the following result after simplifications:

$$
\begin{aligned}
\frac{dE[\Pi_2^{**}]}{d\kappa} =& \int_{p_b,p_s} \{ \int_0^{e(p_s,\kappa)} p_s f(D) dD + \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \overline{\lambda}(D,\kappa) f(D) dD \\
& + \int_{e(p_b,\kappa)}^{\infty} p_b f(D) dD \} g(p_b,p_s) dp_b ps \\
=& \int_{p_b,p_s} \{ p_s F(e(p_s,\kappa^*)) + \int_{e(p_s,\kappa^*)}^{e(p_b,\kappa^*)} \overline{\lambda}(D,\kappa^*) f(D) dD \\
& + p_b \overline{F}(e(p_b,\kappa^*)) \} g(p_b,p_s) dp_b ps
\end{aligned}
$$

Then the first order derivative of the objective function in Problem $P2.1a$ with respect to κ is given by

$$\begin{aligned}\frac{d\Pi_1(\kappa)}{d\kappa} =& -c + \int_{p_b,p_s} \{p_s F(e(p_s,\kappa)) + \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \overline{\lambda}(D,\kappa) f(D) dD \\ &+ p_b \overline{F}(e(p_b,\kappa))\} g(p_b,p_s) dp_b ps\end{aligned}$$

Finally we look at the first order condition $\frac{d\Pi_1(\kappa)}{d\kappa} = 0$ as shown below:

$$\int_{p_b,p_s} \left(p_s F(e(p_s,\kappa^*)) + p_b \overline{F}(e(p_b,\kappa^*)) + \int_{e(p_s,\kappa^*)}^{e(p_b,\kappa^*)} \overline{\lambda}(D,\kappa^*) f(D) dD \right) g(p_b,p_s) dp_b ps = c$$

Proof of Lemma 5.7

From above we have

$$\begin{aligned}\frac{d\Pi_1(\kappa)}{d\kappa} =& -c + \int_{p_b,p_s} \{p_s F(e(p_s,\kappa)) + \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \overline{\lambda}(D,\kappa) f(D) dD \\ &+ p_b \overline{F}(e(p_b,\kappa))\} g(p_b,p_s) dp_b ps\end{aligned}$$

Now we observe the second derivative with respect to κ.

$$\begin{aligned}\frac{d^2\Pi_1(\kappa)}{d\kappa^2} =& \int_{p_b,p_s} \{e'(p_s,\kappa) p_s f(e(p_s,\kappa)) + e'(p_b,\kappa)\overline{\lambda}(e(p_b,\kappa),\kappa) f(e(p_b,\kappa)) \\ &+ \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \frac{d\overline{\lambda}(D,\kappa)}{d\kappa} f(D) dD - e'(p_s,\kappa)\overline{\lambda}(e(p_s,\kappa),\kappa) f(e(p_s,\kappa)) \\ &- e'(p_b,\kappa) p_b f(e(p_b,\kappa))\} g(p_b,p_s) dp_b p_s\end{aligned}$$

By Lemma 5.5, $\overline{\lambda}(e(p_b,\kappa),\kappa) = p_b$ and $\overline{\lambda}(e(p_s,\kappa),\kappa) = p_s$. Then

$$\frac{d^2\Pi_1(\kappa)}{d\kappa^2} = \int_{p_b,p_s} \int_{e(p_s,\kappa)}^{e(p_b,\kappa)} \frac{d\overline{\lambda}(D,\kappa)}{d\kappa} f(D) g(p_b,p_s) dD dp_b p_s$$

Since $\frac{d\overline{\lambda}(D,\kappa)}{d\kappa} < 0$ by Lemma 5.2, $\frac{d^2\Pi_1(\kappa)}{d\kappa^2} < 0$. The objective function in Problem $P2.2a$ is concave in κ.

Proof of Lemma 5.10

By Corollary 5.10. We have

$$\Pi_2{}^{**} = sD + \begin{cases} p_s\kappa - \sum_{i=1}^{n}(p_i + p_s\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} & : if \epsilon^{**}(D) < \kappa \\ p_b\kappa - \sum_{i=1}^{n}(p_i + p_b\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} & : if \epsilon^{**}(D) \geq \kappa \end{cases}$$

By using the relation given in Corollary 5.11, we rewrite the equation above as follows:

$$\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa) = sD + \begin{cases} p_s\kappa - \sum_{i=1}^{n}(p_i + p_s\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} & : if D < \omega(\kappa) \\ p_b\kappa - \sum_{i=1}^{n}(p_i + p_b\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}} & : if D \geq \omega(\kappa) \end{cases}$$

Now, we derive $E_{D,p_b,p_s}[\Pi_2((q_b^{**}, q_s^{**}), \vec{x}^{**}|\kappa)]$. We suppress the notation for simplicity as follows: $E[\Pi_2^{**}]$

$$\begin{aligned} E[\Pi_2^{**}] =& sE[D] + \int_{p_b,p_s}\{\int_0^{\omega(\kappa)}\left(p_s\kappa - \sum_{i=1}^{n}(p_i + p_s\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}}\right) f(D)dD \\ &+ \int_{\omega(\kappa)}^{\infty}\left(p_b\kappa - \sum_{i=1}^{n}(p_i + p_b\beta_i)\left(\frac{D}{A_i}\right)^{\frac{1}{\alpha_i}}\right) f(D)dD\} g(p_b, p_s) dp_b p_s \end{aligned}$$

Next we evaluate $\frac{dE[\Pi_2^{**}]}{d\kappa}$ using Leibniz Rule.

$$\begin{aligned}
\frac{dE[\Pi_2^{**}]}{d\kappa} &= \int_{p_b,p_s} \{ \frac{d\omega(\kappa)}{d\kappa} \left(p_s\kappa - \sum_{i=1}^{n} (p_i + p_s\beta_i) \left(\frac{\omega(\kappa)}{A_i} \right)^{\frac{1}{\alpha_i}} \right) f(\omega(\kappa)) \\
&\quad + \int_0^{\omega(\kappa)} p_s f(D)dD + \int_{\omega(\kappa)}^{\infty} p_b f(D)dD \\
&\quad - \frac{d\omega(\kappa)}{d\kappa} \left(p_b\kappa - \sum_{i=1}^{n} (p_i + p_b\beta_i) \left(\frac{\omega(\kappa)}{A_i} \right)^{\frac{1}{\alpha_i}} \right) f(\omega(\kappa)) \} g(p_b,p_s) dp_b p_s \\
&= \int_{p_b,p_s} \{ \frac{d\omega(\kappa)}{d\kappa} p_s \left(\kappa - \sum_{i=1}^{n} \beta_i \left(\frac{\omega(\kappa)}{A_i} \right)^{\frac{1}{\alpha_i}} \right) f(\omega(\kappa)) \\
&\quad - \frac{d\omega(\kappa)}{d\kappa} \sum_{i=1}^{n} p_i \left(\frac{\omega(\kappa)}{A_i} \right)^{\frac{1}{\alpha_i}} f(\omega(\kappa)) \\
&\quad + \int_0^{\omega(\kappa)} p_s f(D)dD + \int_{\omega(\kappa)}^{\infty} p_b f(D)dD \\
&\quad - \frac{d\omega(\kappa)}{d\kappa} p_b \left(\kappa - \sum_{i=1}^{n} \beta_i \left(\frac{\omega(\kappa)}{A_i} \right)^{\frac{1}{\alpha_i}} \right) f(\omega(\kappa)) \\
&\quad + \frac{d\omega(\kappa)}{d\kappa} \sum_{i=1}^{n} p_i \left(\frac{\omega(\kappa)}{A_i} \right)^{\frac{1}{\alpha_i}} f(\omega(\kappa)) \} g(p_b,p_s) dp_b p_s \\
&= \int_{p_b,p_s} \{ \int_0^{\omega(\kappa)} p_s f(D)dD + \int_{\omega(\kappa)}^{\infty} p_b f(D)dD \} g(p_b,p_s) dp_b p_s
\end{aligned}$$

Then the first order derivative of the objective function in Problem $P2.1b$ with respect to κ is given by

$$\frac{d\Pi_1(\kappa)}{d\kappa} = \int_{p_b,p_s} \{ p_s \left(\int_0^{\omega(\kappa)} f(D)dD \right) + p_b \left(\int_{\omega(\kappa)}^{\infty} f(D)dD \right) \} g(p_b,p_s) dp_b p_s - c$$

We look at the first order condition $\frac{d\Pi_1(\kappa)}{d\kappa} = 0$ as shown below:

$$\int_{p_b,p_s} \{ p_s \int_0^{\omega(\kappa)} f(D)dD + p_b \int_{\omega(\kappa)}^{\infty} f(D)dD \} g(p_b,p_s) dp_b p_s = c$$

Proof of Lemma 5.11

The first order derivative of the objective function in Problem $P2.1b$ with respect to κ is deriven above.

$$\frac{d\Pi_1(\kappa)}{d\kappa} = \int_{p_b,p_s} \{p_s \int_0^{\omega(\kappa)} f(D)dD + p_b \int_{\omega(\kappa)}^{\infty} f(D)dD\} g(p_b,p_s)dp_b p_s - c$$

Now we observe the second derivative with respect to κ.

$$\begin{aligned}\frac{d^2\Pi_1(\kappa)}{d\kappa^2} &= \int_{p_b,p_s} \{\frac{d\omega(\kappa)}{d\kappa} p_s f(\omega(\kappa)) - \frac{d\omega(\kappa)}{d\kappa} p_b f(\omega(\kappa))\} g(p_b,p_s)dp_b p_s \\ &= -\int_{p_b,p_s} \{\frac{d\omega(\kappa)}{d\kappa}(p_b - p_s) f(\omega(\kappa))\} g(p_b,p_s)dp_b p_s\end{aligned}$$

Since $\frac{d\omega(\kappa)}{d\kappa} > 0$ by Corollary 5.11 and $p_b > p_s$ by assumption, $\frac{d^2\Pi_1(\kappa)}{d\kappa^2} < 0$. The objective function in Problem $P2.1b$ is concave in κ.

Appendix C

Numerical Study

Derivation of Parameters p_1 and b_1

In the study conducted by Hatırlı et. al [18], the quantity of total fertilizer used in greenhouse tomato production is found to be 940.9 kg/ha and its energy equivalent is 29443.9 MJ/ha. Thus, the energy equivalent of fertilizer is calculated as $29443.9/940.9 = 31.3\ MJ/kg$.

In [51], the fertilizer price is given as 4 TL/kg. Then,

$$p_1 = \frac{4\ TL/kg}{31.3\ MJ/kg} = 0.1278\ TL/MJ$$

We base our calculations on the nitrogen fertilizer for the estimation of the carbon coefficient of the fertilizer. The mean carbon emission of the nitrogen fertilizer is estimated as 1.3 $kg\ CE/kg$ [31]. The energy equivalent of nitrogen is 66.14 MJ/kg [18]. Then,

$$\beta_1 = \frac{1.3\ kg\ CE/kg}{66.14\ MJ/kg} = 0.0197\ kg\ CE/MJ$$

Derivation of Parameters p_2 and b_2

We base our calculations on the insecticides for the estimation of the price and carbon coefficient of the chemicals. The energy equivalent of chemicals is 101.2 MJ/kg [18]. The price of the insecticide is 60 TL/lt [9]. Assuming that 1 lt is equal to 1 kg, the price is 60 TL/kg. The average carbon emission of the insecticide is estimated as 5.1 $kg\ CE/kg$ [31]. Then,

$$p_2 = \frac{60\ TL/kg}{101.2\ MJ/kg} = 0.5929\ TL/MJ$$

$$\beta_2 = \frac{5.1\ kg\ CE/kg}{101.2\ MJ/kg} = 0.0504\ kg\ CE/MJ$$

Derivation of Parameter p_3 and b_3

The energy equivalent of labor is 2.3 MJ/h [18]. The daily salary of workers is 24 $TL/day = 3\ TL/h$ [1].

$$p_3 = \frac{3\ TL/h}{2.3\ MJ/h} = 1.3043\ TL/MJ$$

We base our estimation of carbon coefficient of the labor on the plant/sow/drill operations. Lal [31] estimates the average carbon emission of theses operations to be 3.2 $kg\ CE/ha$. Manually operated seed drill operation takes 35 h/ha [11].

$$\beta_3 = \frac{3.2\ kg\ CE/ha}{35\ h/ha \times 2.3\ MJ/h} = 0.0398\ kg\ CE/MJ$$

Derivation of Parameter p_4 and b_4

The energy equivalent of machinery is 64.8 MJ/h [18]. The cost of diesel fuel is 3.83 TL/lt [40]. A diesel tractor rated at 100 maximum PTO-hp operating at full load is 7.69 gal/h [10].

$$p_4 = \frac{3.83\ TL/lt \times 7.69\ gal/h}{0.264 gal/lt \times 64.8\ MJ/h} = 1.7217\ TL/MJ$$

For diesel fuel, $1gal = 7lb$ [50]. Lal [31] estimates the mean carbon emission of diesel fuel to be 0.94 $kg\ CE/kg$. Then,

$$\beta_4 = \frac{0.94\ kg\ CE/kg \times 0.4536\ kg/lb \times 7\ lb/gal \times 7.69\ gal/h}{64.8\ MJ/h} = 0.3542\, kg\, CE/MJ$$

Derivation of Parameter p_5 and b_5

The energy equivalent of water for irrigation is 0.63 MJ/m^3 [18]. We base our estimation of price and carbon coefficient of water for irrigation on the furrow irrigation systems. The operating cost of water for irrigation is 36 $TL/da/year$ [2]. Robinson et. al. [43] estimate the water consumption in furrow irrigation to be $1850 - 2400\ m^3/ha/year$. We use the mean value in our calculations which is 2125 $m^3/ha/year$. The carbon emission of furrow irrigation is 395 $kg\ CE/ha/year$ [31]. Then,

$$p_5 = \frac{36\ TL/da/year \times 10\ da/ha}{2125\ m^3/ha/year \times 0.63\ MJ/m^3} = 0.2689\ TL/MJ$$

$$\beta_5 = \frac{395\ kg\ CE/ha/year}{2125\ m^3/ha/year \times 0.63\ MJ/m^3} = 0.2951\ TL/MJ$$

Derivation of Parameter A

Parameter	Elasticity	Energy equivalent (MJ)
fertilizer	0.24	29443.9
chemical	0.19	10872.5
labor	0.15	9222.6
machinery	0.56	3041.1
water for irrigation	0.23	389.7
tomato	-	127748.6

Table C.1: Energy equivalents of the inputs and yield in greenhouse tomato production given in [18]

We derive the value of the technology level A from Equation 3.4 based on the values provided in [18] (Table C). Then, A is found to be 1.34.

Derivation of Parameter s

The tomato selling price is assumed to be 2.3 TL/kg. The energy equivalent of tomato is 0.8 MJ/kg [18]. Then,

$$s = \frac{2.3\ TL/kg}{0.8\ MJ/kg} = 2.875\ TL/MJ$$