

To my Parents, my Grandmother, and my Niece
Hatice Ayas & Nazmi Ayas, Hikmet Demirel, and Seher Hayat Köylüoğlu

TECHNOLOGY IMPROVEMENTS UNDER CARBON EMISSIONS TAXES

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By Onurcan Ayas

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Business Administration.

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ABSTRACT

TECHNOLOGY IMPROVEMENTS UNDER CARBON EMISSIONS TAXES

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Ph.D. in Business Administration

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In this thesis, we consider two technology improvement problems under carbon taxation policies. In the first one, a manufacturer subjected to a carbon tax producing a single product using multiple inputs and employing an initial technology level, which is modeled using the Cobb Douglas production function. We find out that 1) when the output and the technology level are binding, amount of a production input is increasing in carbon tax if the share of the tax due to its carbon footprint in its total variable cost is less than the weighted average of that of all production inputs; 2) when the output is binding, but the technology level is non-binding, while the technology level is always increasing in carbon tax; amount of an input is increasing in carbon tax only if the share of the tax times its carbon footprint in its total variable cost is less than the weighted average of that of all production factors (including technology level); 3) when the output is non-binding, but the technology level is binding, the output, as well as the inputs, is decreasing in carbon tax, at optimality; 4) when the output and the technology level are non-binding, all of the decision variables are decreasing in carbon tax, at optimality. In the second problem, we consider setup reduction activities for a stochastic inventory system, where improvement efforts are modeled via production functions. We derive the optimality quantities of required labor and capital input for improvement, as well as the ordering quantity and reorder point.

Keywords: Carbon Emission Tax, Technology Enhancement, Newsvendor Problem, Production Functions, Multiple Inputs.

ÖZET

KARBON EMİSYON VERGİSİ ALTINDA TEKNOLOJİ İYİLEŞTİRMELERİ

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Bu tezde, karbon emisyon vergisi altında iki teknoloji iyileştirme problemini inceliyoruz. Birinci çalışmada birden fazla girdiyle tek bir ürün üreten üreticinin karbon vergisine tabi olduğu varsayılmış olup bu bağlamda gazeteci çocuk modeli ele alınmıştır ve üretim ilişkileri Cobb Douglas üretim fonksiyonu kullanılarak modellenmiştir. Dahası, üretim miktarının yerine getirilmesi gereken sözleşmeler nedeniyle iki değer arasında olması gerektiği varsayılmıştır. Ayrıca, üreticinin halihazırdaki üretim teknolojisini yükselterek karbon salımını düşürebileceği düşünülmüştür. Sonuç olarak şu bulgular elde edilmiştir: 1) Optimal teknoloji karbon vergisinin artan bir fonksiyonuyken, üretim miktarı sabitse, bir girdinin karbon ayak izinin birim maliyetindeki payı bütün girdilerin aynı paylarının ağırlıklı ortalamasından düşük ise, bu girdinin miktarı karbon vergisinin artan bir fonksiyonudur. 2) Optimal teknoloji ve üretim seviyeleri sabitse, üretim girdileri birinci bulguda sunulan davranışı gösterir. 3) Optimal teknoloji sabitken üretim miktarı karbon vergisi arttıkça düşüyorsa, bütün üretim girdilerinin miktarları karbon vergisinin azalan fonksiyonlarıdır. 4) Optimal teknoloji karbon vergisinin azalan bir fonksiyonu ise, optimal üretim miktarı ve üretim girdileri karbon vergisinin azalan fonksiyonlarıdır. İkinci çalışmada, rassal talepli bir envanter sisteminde sabit ısmarlama maliyetini azaltma/iyileştirme problemini inceliyoruz. Burada iyileştirme çabaları üretim fonksiyonlarıyla tanımlanmaktadır. Analiz sonunda, iyileştirme için gerekli sermaye ve emek miktarları ile optimal sipariş miktarını ve ısmarlama noktasını elde ediyoruz.

Anahtar sözcükler: Karbon Emisyonu Vergisi, Teknoloji Geliştirme, Gazeteci Çocuk Problemi, Üretim Fonksiyonları, Birden Fazla Girdi.

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CHAPTER 1

INTRODUCTION

This thesis studies technology improvements motivated by regulations around carbon emissions. The growing environmental awareness and global warming triggered authorities and regulatory bodies to regulate carbon emissions and several initiatives like First World Climate Conference, Kyoto Protocol and Paris Agreement emerged. These initiatives in establishing the goal of reducing global warming, brought into being several regulatory schemes one of which is the carbon taxation policy. Carbon taxation is a policy under the regulation of governments that determine a price for metric tonne of carbon dioxide emitted by firms.

While technology provides means of living well in general, improper use of technology together with improper allocation of production inputs may increase the use of environmentally harmful materials and decrease products' lifetimes, which in return, may lead to higher carbon emissions and rapid depletion of natural resources, respectively. Since the pace of new product development is getting higher every new day, these issues may be exacerbated in the future. Therefore, the motivation of this study is the objective of inducing improvements in proper technologies and production policies through carbon taxation.

In this study, we consider in detail two settings: In the first one, a price-taker manufacturer uses multiple production inputs to produce a single product in accordance with deterministic contractual obligations in the presence of carbon taxation. The managerial problem faced by the manufacturer is that she has an established technology and she may enhance her technology level to decrease her emissions and, possibly, increase her profit level after the introduction of carbon emissions taxes. The technology improvement can be viewed as acquisition of higher productivity in converting the inputs into the output. The decision in-

volves finding her optimal response to increase in carbon tax, which is composed of optimal production quantity, input mix, emission and technology level. Within this framework, we also attempt to study the decisions that policy makers may be interested in as well: The primary research question in this context is identify taxation levels for specific firm types in order to lead technological advancements and mitigate global warming. The particular contribution of this thesis is that it considers economic production functions to model the transformation process of multiple inputs to generate the output. In the above setting, we focus solely on the Cobb-Douglas (multiplicative) production function. This setting has not been investigated in the literature and our findings provide explicit results on firm's response and policy-maker's taxation rates for technology enhancements.

We investigate the problem of finding the optimal level of technology enhancement through knowledge capital acquisition, which would, in return, enhance "Total Factor Productivity" (TFP). The studies in OR literature dealing with knowledge creation and transfer primarily focus on investments in induced learning regarding process quality improvement which leads to cost reduction due to defect rate enhancement. However, technology enhancement is not considered in this literature as direct productivity improvement. Productivity improvement through knowledge acquisition is studied in endogenous learning and pollution abatement literature, which assumes that GHG emission is a function of the production quantity or service level. Since we considered multiple input production where each input has a carbon footprint, our study can be considered as an extension of this line of research. Moreover, two types of knowledge, tacit and explicit knowledge, are studied in endogenous learning literature. While, explicit knowledge is the type of knowledge that is readily available in the form of insight, tacit knowledge is the type of knowledge that is attached to the level of experience. Thus, while explicit knowledge can be acquired in the short run, tacit knowledge can only be acquired in the long run. Note that, we do not consider a time dependent model. Hence, we assume a convex knowledge acquisition cost function, which is justified, for short run, by Griliches (1998) and Bormotov (2010). Griliches (1998) studies patent statistics and their relationships with economic activities, and suggests that investments on productivity enhancement have diminishing returns in the short-run and Bormotov (2010) states that although total investments in knowledge acquisition do not have diminishing returns because of that total knowledge is unlimited, investment in production and implementation

of a specific knowledge are considered to have diminishing returns. Furthermore, the level of experience is considered as a function of production quantity in the literature (for instance have a look at (Arrow (1962))) and we use a production function that maps knowledge to production quantity. Hence, our model can be adapted to long-term policy making through using convex acquisition cost functions.

In the second setting, we consider a single location inventory system with positive fixed costs of replenishment and normally distributed random demands. The primary research question is determining the optimal reduction in the fixed costs of replenishment (setup reduction). This problem is a reformulation of a long-standing model dating from the early days of lean production. The fundamental novelty in our treatment is that the setup reduction effort is modeled as a multi input R & D activity, which follows specific economic production function forms. Although the basic formulation does not assume carbon taxation, we also present an extension of the model to illustrate how carbon taxation would be incorporated into the model. The contribution of this model to the literature is two-fold: From an inventory theoretic perspective, it is the first work that considers setup reduction efforts with a power investment function in a random demand supply chain under a hard service level constraint. Secondly, it is the first work that incorporates firm level micro-production functions to technology improvement literature. We believe that our theoretical study will initiate empirical work in operations management to better model R / D activities.

The remainder of the thesis is organized as follows: In Chapter 2, we study the problem of finding the optimal level of technology enhancement through knowledge capital acquisition in the presence of carbon tax and constraints on output and technology level; in section 2.1 we provide a literature review regarding the paradigms linked to this study; in section 2.2, we discuss the setting of the problem and formulate the model of the problem; in section 2.3, we approach the problem from the firm's perspective, present the solution procedure, and provide the firm's optimal production policy; in section 2.4 we approach the problem from regulator's perspective, present the variation in the firm's optimal response to increasing carbon tax values in all regions of carbon tax, and provide a discussion regarding our findings. In Chapter 3 we formulate an inventory management model that explicitly integrates process improvement efforts to enhance supply chain sustainability through better use of capital (materials, assets, technology)

and labor (workforce and know-how), in section 3.1 we present a brief literature review regarding the second problem; in section 3.2 we formulate the model in order to solve the problem; in section 3.3, studying the problem for different production functions, we present the optimality and structural results; in section 3.4 we derived insights regarding process improvements for sustainable supply chains. Finally, in Chapter 4 we wrap up our findings and discuss the possible extensions regarding the problems studied in the thesis.



CHAPTER 2

OPTIMAL TECHNOLOGY FOR PRODUCTION FIRMS IN A CARBON ECONOMY

In this chapter we study the problem of finding the optimal level of technology enhancement through knowledge capital acquisition in the presence of carbon tax and constraints on output and technology level. Before providing the solution procedure, optimality results, and analytical findings, we provide a literature review regarding the paradigms linked to this study as follows.

2.1 Literature Review

Our first study regarding finding the optimal level of technology enhancement through knowledge capital acquisition in the presence of carbon tax is related to research in

- carbon reduction mechanisms,
- production functions,
- technology acquisition,
- cost functions,
- pollution abatement,
- endogenous learning and knowledge capital.

Therefore, we discuss the studies regarding these research areas as in the following sub-sections.

2.1.1 Research on Carbon Reduction Mechanisms

Xie and Ilic (2010) investigate economic dispatch problem under cap-and-trade, carbon tax, and carbon quota policies. They consider the electricity market in which independent power producers (IPPs) submit supply bids that maximize their profit consisting of convex operational costs over a predefined horizon to an independent system operator (ISO) who collects these bids at each time step and balance the system to maximize social welfare (minimize total generation cost). In the carbon tax based economic dispatch problem, it is assumed that carbon tax increases the short-run operating costs of only fossil fuel based generators, and that of renewable energy producers stays unaffected. The firms, for different price vectors of K periods obtained by perturbing an initial vector up and down, finds the power generation vector that optimizes the total profit consisting of revenue from selling power, cost of generation and tax paid for emissions over the whole planning horizon. In cap-and-trade based model, different from carbon tax based model, as well as the power output levels, the amount of emission allowances traded is also optimized. Required emission balance of the cap-and-trade policy is accounted for through a constraint, which ensures that at the end of the planning horizon the financially traded and physically emitted carbon amount meets the initial carbon allowance imposed. The authors through computationally efficient algorithms solved a numerical example, regarding a modified IEEE 14-bus system that consists of five generators including a wind farm whose operating costs are assumed to be quadratic and whose emission amounts are assumed to be linear in power produced, and test the models and derive analytical results. In the numerical study, the transmission lines are assumed to be lossless and not congested. The data for the 10-minute wind power forecast is obtained from New York ISO website. As a result they find out that the carbon tax based model is temporarily decomposable, and thus, IPPs can focus on just for the next time step in deciding the amount of power generation.

Chen, Kang, Xia and Zhong (2010) propose a comprehensive model, termed as "Low Carbon Power Generation Problem (LCPGE)", which integrates low-carbon factors, including carbon trading mechanisms, low carbon generation technologies, reduction obligations and carbon tax to the power generation expansion model (PGE). The authors discussed that in the literature there does not exist a comprehensive approach that handles the uncertainties in the carbon prices.

They state that the models in the literature are either too macroscopic, for instance models like MARKAL can only be applicable to the whole economy and do not consider the specific properties of power generation systems, or too microscopic, that is, consideration of uncertainties in the carbon emission reduction, carbon prices, and progress of the carbon trading mechanisms, leads to overly complex models from which managerial insights are difficult to derive. Thus, they modeled LCPGE through a compromised modeling approach that reduces computational complexity while keeping important factors and maintaining moderate precision. One of the compromises is that capacity expansion is not considered as unit-based, instead, the power plants with the same type and location are aggregated into a single decision variable. Moreover, instead of power production simulations, a scheme for generation dispatching among different sets of plants is introduced. The details of these compromises are out of the scope of this thesis, thus, they are neglected. The model is tested by employing empirical data and implemented to China's power sector at country-level for the time frame from 2011 to 2030. Among findings of this study, the following results are the more general ones: 1) In the context of generation capacity expansion wind power offering low cost and cleanness is always competitive. 2) The obligations of China who will confront great economical losses due to carbon regulation policies, can be met only after 2029. 3) In meeting obligations clean coal and carbon capture and storage technologies (CCS) is the most promising option for carbon abatement. 4) If carbon reduction is urgent, it is advised that plants are lead to be carbon capture ready (CCR).

Hua, Cheng and Wang (2011) propose an environmental EOQ model for a single-product retailer operating under carbon trading mechanism and, comparing it to the classical EOQ model, reveal the relative effect of carbon price and carbon cap on order quantity decisions, carbon emissions and total cost. They maximize firm's profit through optimizing order quantity, assuming that the emissions stemmed from the warehousing and logistics activities are linear on order quantity and the carbon price is exogenous to the considered firm.

Bonney and Jaber (2011), focusing on descriptions and value based perspectives, study a simple environmental EOQ model through which they, unlike Hua et al. (2011), consider the costs of packaging, wastes due to poor inventory management, and the effect of manufacturing facilities' locations on the costs of transportation and emissions. As a result they find out that greater order quantities

than the classical economical order quantity may turn out to be more advantageous in both economic and environment terms when carbon trading and transportation costs are considered.

Battini, Persona and Sgarbossa (2012) investigate a similar problem to that is studied by Hua et al. (2011). However, they consider also the transportation mode selection and life cycle assessment of material order quantity to account for the environmental impacts arising during the life cycle of a purchasing order, which starts with the handling and transportation phase and finishes with the sale of the last unit of the lot. Furthermore, they model environmental issues as parts of the design objectives rather than considering them as constraints. In the model developed for solving the single-product replenishment problem, it is assumed that product demand is deterministic, price of the product is exogenous to the firms' decisions, and the buyer decides only the order quantity. It is supposed that the process of ordering, warehousing and disposition of waste, as well as transportation and handling of the lot size, lead to emissions. Since, they are considered to be leading to increases in societal costs, the costs associated with these activities are termed as "external costs". They measure the environmental impact of transportation using the techniques discussed by Litman (2009), Ortolani et al. (2009) Ortolani et al. (2011), and Sahin et al. (2009), carbon emissions due to warehousing is computed through employing the technique of Fieldson and Siantonas (2008) and Rai et al. (2011), and for the computation of emissions costs due to waste disposal the authors refer to the software WARM developed by EPA (<http://www.epa.gov>) (all of these studies are cited by Battini et al. (2012)). Through numerical examples, the authors show that if only the road is considered as a transportation mode, the total cost function with externalities and that without them lead to the same optimal solution, and that, the beneficial impact of transportation mode selection on total sustainable cost is greater than that of the decisions on purchasing lot size.

Benjaafar, Li and Daskin (2013) study an EOQ model considering single and multistage lot-sizing, for integrating carbon emission concerns to establish the goal of decision making in operations, production and inventory management and suggest as a further study, modeling the problem in a news-vendor setting. This study differs from the studies of Hua et al. (2011) and Battini et al. (2012) in that while they considered only the order quantity as a decision variable, Benjaafar et al. maximize firm's profit through optimizing order quantity, the indicator

variable regarding the firm's operation choice, the inventory she carries, and the amount of back orders in a given period. More specifically, firstly they model a multi-period production, inventory, and operations planning problem of a firm minimizing the sum of her fixed & variable ordering (production) costs and inventory holding & shortage costs in the presence of a carbon cap constraint on her total emissions which includes fixed emissions per order and variable emissions per each unit ordered (produced) due to transportation (process setup), and handling (production), respectively. Secondly, they extend their model by taking into account the carbon tax, cap-and-trade, and carbon offset mechanisms instead of a strict carbon cap. Thirdly, they further extend their model by considering a cooperation among parties within a supply chain, that is formed in order to reduce carbon footprints. Investigating these models, they derived the following findings: 1) Through operational adjustments alone, carbon emissions can be significantly reduced without increasing the costs too much. 2) Adjusting the operational decisions to meet carbon cap may be more cost effective than investing in energy efficient technologies. 3) Total emissions may become increasing in carbon cap tightness. 4) Carbon offsets enables the imposition of tighter carbon caps. 5) Under cap-and-trade mechanism when carbon price is fixed, total emission is only affected by carbon price. 6) Under cap-and-trade firm's profit may appreciate higher carbon prices. 7) The effect of energy efficient technologies on operational costs depends on the considered carbon regulation policy. 8) Collaborations, while leading to decreases in the costs of emission reduction for some firms in a supply chain, may lead to increased costs and emission levels for other firms left out of the collaboration.

Zhang and Xu (2013) investigate the multi-item production planning problem under carbon trading mechanism. The objective of the model is maximizing the expected profit of a firm, that consists of revenue from selling her products and salvage of leftover products, production costs, and revenues/costs due to selling/buying carbon permits, when she produces n products with a capacity constraint on total production and with a carbon cap on total emissions. Demand for products of the firm is assumed to be stochastic. The effect of decision making on the carbon emissions, trading policy, production decision and total profit is investigated through the proposed solution approach having linear complexity. The effectiveness of cap-and-trade and carbon tax policies is compared and through numerical examples the importance of considering carbon restrictions in

operational decisions like capacity investment is highlighted. Several managerial insights are obtained: 1) Marginal value of the capacity is decreasing in carbon price. 2) there exists a tendency towards carbon efficient product production under the cap-and-trade policy which induces emissions reduction. 3) while optimal profit of the firms with carbon surplus is increasing in carbon price, that of the firms with carbon deficit is decreasing in carbon price. 4) there is an indifference curve of the carbon cap and carbon price pair on which the profit in the absence of carbon market is maintained in the presence of the market. 5) Comparison of the carbon taxation and cap and trade mechanisms reveals that these policies can lead to the same emission or profit levels, but not simultaneously, and that cap and trade mechanism offers a wider range of profitability.

Greenstone (2002) discusses that after the Clean Air Act amended in 1970, the Environmental Protection Agency (EPA) established differential national air quality standards which sets a minimum level of air quality that all counties are required to meet for four criteria pollutants: carbon monoxide (CO), tropospheric ozone (O₃), sulfur dioxide (SO₂), and total suspended particulates (TSPs). Accordingly, the counties of US that can not meet the standards is labeled as non-attainment counties and those that meet the standards are labeled as attainment counties. While non-polluters are free from regulation, emitters of a pollutant in non-attainment counties have tighter regulation than emitters in attainment counties. Studying the effects of these regulations, Clean Air Act Amendments (CAAAAs), on the manufacturers, the author finds out that CAAAs substantially retarded the growth of polluting manufacturers in non-attainment counties, but this negative effect is negligible when entire manufacturing sector is considered. He discusses that the amendments lead to substantial cross sectional and longitudinal variation in regulatory intensity across similar manufacturers at the county level, and thus, can be used to identify the relationship between regulations and industrial activity.

He states that the CAAAs provides three sources of variation in which plants were affected by the non-attainment designations which are 1) cross sectional variation which facilitates differential identification of industry specific shocks and relation effects; 2) longitudinal variation due to the change in the designation of the counties as attainment or non-attainment, which allows inclusion of plant fixed effects, which enables within plant comparisons under these two regimes; 3) intracounty variation that enables the inclusion of unrestricted county by period

effects through which time varying factors common to all plants within a county are not confounded by regulation effects.

The author forms a counterfactual group by restricting it to counties that are CO attainment in 1972 but CO non-attainment in later periods. After comparing this group with unrestricted ones, he finds out that this subset is more similar to non-attainment counties than the set of unrestricted counties was. Therefore, the author concludes that non-attainment status is not orthogonal to observable county-level or plant-level determinants of plant growth in either set of attainment counties. Moreover, through a different analysis the author finds that non-attainment status is not orthogonal to county-level shocks to growth. As a result of these findings the author purged the estimation effects of many likely sources of bias. He states that regulation effects on labor and capital are ambiguous while its theoretically predicted effect on output is negative and test these predictions. They find out that while O₃ and SO₂ non-attainment statuses are basically uncorrelated with the respective growth of emitters, CO and TSP non-attainment designations at the beginning of a period is associated with an 8.4 and 2.4 percent reductions in employment levels in CO-emitting plants five years later. Furthermore, they find out that, five years between observations is a enough period (long run) for establishments to change their capital. Although according to previous literature the regulation effects on new investment in plants and equipment are not significant, the authors find out a negative effect on investments, (since capital stock includes pollution abatement equipment, signaling an upward bias, since, since...). In particular, the estimated regulation effects on investments and shipments from the specification that incorporates plant fixed effects are -0.092 for CO, -0.041 for O₃, -0.063 for SO₂, and -0.043 for TSPs and -0.146 for CO, -0.032 for O₃, -0.010 for SO₂, and -0.032 for TSPs, respectively.

Highlighting that the percentage of production shifted abroad would be a good measure of the national costs of the regulations, he states it is impossible since the target of shift (attainment counties or foreign countries) can not be determined. Therefore regulation effects on national loss of activity and losses in non-attainment counties are probably overstated. Finally, he concludes that since, in a freely functioning market economy, jobs and capital are not lost or made obsolete, the losses due to regulation are tractable adjustment costs associated with the shifting of resources to new sectors, which may lead to wage losses due to displacement of workers and capital stock adjustment costs of firms.

2.1.2 Studies Employing Production Functions

In this sub-section, we present the details of the production function we use for the formulation of the model regarding optimization of input allocation, technology, and output of a firm subjected to a carbon tax and provide a taxonomy of production functions used in the literature.

In the presence of carbon regulation mechanisms, there may exist a competition among production inputs depending on their carbon footprints and unit costs and the Cobb-Douglas production function allows substitution among production inputs. Hence, in order to investigate the trade-offs between production inputs we use Cobb-Douglas production function which has an elasticity of substitution that is greater than zero.

Note that standard Cobb Douglass production function is of the following form.

$$Q = AK^{\delta}L^{\varphi},$$

where Q , A , K , and L are output amount, technology level, capital, and labor, respectively. However, in order to leave room for further expansion and derive universal results, we study the following version of this function

$$Q = A \prod_{i=1}^N x_i^{\alpha_i}, \quad (2.1)$$

where x_i represents the amount of production input i and α_i is the output elasticity of this input. Accordingly, elasticity of scale of this function is:

$$r = \sum_{i=1}^N \alpha_i,$$

and marginal rate (technical rate when production quantity is assumed to be fixed) of substitution of x_i for x_j is:

$$MRS_{x_i x_j} = \frac{\alpha_i x_j}{\alpha_j x_i}.$$

The following table presents a taxonomy of production functions used in the literature.

Table 2.1: The Production Functions Used in the Literature.

Type	Form	Description	Paper
Cobb Douglas	$Q = A \prod_{i=1}^n x_i^{\alpha_i}$	Q is the prod. amount, x_i 's are inputs (or factors) of prod., A is the total factor productivity or technology factor and α_i 's are the input elasticities.	Komiya (1962), Courville (1974), Baron and Taggart Jr (1977), Kopp and Smith (1980), Lans and Goulder Lawrence (1996), Welsch (1998), Ebert and Welsch (2007), Belegri-Roboli and Michaelides (2010).
CES	$Y_t = \left(aK_t^{\rho\alpha} L_t^{\rho(1-\alpha)} + bE_t^{\rho\beta} N_t^{\rho(1-\beta)} \right)^{1/\rho}$	Y_t is the prod. amount at time t , K_t and L_t are the capital and labor inputs at time t respectively and N_t and E_t are the non-electric energy and electricity demand at time t respectively. $\rho = (1 - 1/\sigma)$, a and b are calibration parameters. α is the share of capital in capital-labor aggregate and β is the share of electricity in the energy aggregate.	Kopp and Smith (1980), Hamilton et al. (1992), Manne and Wene (1992), Lans and Goulder Lawrence (1996), Messner and Schratzenholzer (2000), Timilsina and Shrestha (2002), Edmonds et al. (2004), McFarland et al. (2004), Sands (2004), Fawcett and Sands (2005), Timilsina and Shrestha (2006), Wing (2006), Zhang et al. (2013), Proença and St Aubyn (2013).
General CES	$Q = A(\sum_{i=1}^n \alpha_i X_i^{\beta_i})^{1/\gamma}$	is a generalization of the CES production function. This function need not be homogeneous of degree one. Q is the production amount, X_i is the i^{th} input of the prod. process, β_i is its output elasticity, and α_i is a parameter.	Dhrymes and Kurz (1962).
Leontief	$Q = \min_{i \in 1, \dots, n} [A_i x_i^{\alpha_i}]$	Q is the prod. amount, x_i 's are inputs (or factors) of prod., A_i is the factor productivity of input i and α_i 's are input elasticities.	Bergman (1991), Böhringer (1998), Brenkert et al. (2004), Edmonds et al. (2004), Sands (2004), Fawcett and Sands (2005), Schumacher and Sands (2006), Wing (2006), Fabrizio et al. (2007), Pettersson et al. (2012).
Hybrid	$Q = \min \left[g(L), \left(\alpha_f F^{\beta_f} + \alpha_K K^{\beta_K} \right)^{(1/\gamma)} \right]$	is a hybrid of Leontief and neoclassical production functions. Q is the production amount, $g(L)$ is the labor requirement, F and K represent fuel and capital inputs respectively and α 's and β 's are the parameters. There are also other types of production functions which are composite of Leontief, Cobb Douglas, and CES prod. func.'s, where contribution of labor and capital is modeled by Cobb Douglas prod. func. and called <i>KL</i> composite, the contribution of energy inputs are aggregated with <i>KL</i> composite through a CES prod. func. to form <i>KLE</i> composite, which together with non-energy inputs produces electricity through a Leontief prod. func.	Dhrymes and Kurz (1962), Proença and St Aubyn (2013).
VES	$E = \beta_0 F + \beta_1 K - \beta_2 K^2 E^{-1} + \beta_3 K^3 E^{-2}$	where E is the annual quantity of energy in joules required to produce a domestic hot water (DHW) service (temperature and volume), F and K represents annual quantities of fuel (in KWHs) and solar process capital (in square meters of solar collector surface.) inputs respectively and β 's are the parameters of the production function.	Sav (1984)
Translog	$\log Q = \log \alpha_0 + \beta_1 \log K + \beta_2 \log L + \beta_3 \log F + \beta_4 (\log K)^2 + \beta_5 (\log L)^2 + \beta_6 (\log F)^2 + \beta_7 \log K \log F + \beta_8 \log K \log L + \beta_9 \log F \log L$	where Q stands for the electricity production quantity, K, L, F denotes capital, labor and fuel inputs of electricity production, and β 's are the output elasticities of the corresponding factors and α_0 is the technology factor of the electricity generation	Spann (1974), Kopp and Smith (1980), Gollop and Roberts (1983), Atkinson and Halvorsen (1984), Karlson (1986), Burney and Al-Matrouk (1996), Shanmugam and Kulshreshtha (2002), Saleem (2007), See and Coelli (2012).
CRES	$Q = (\alpha_K K^{-p_K} + \alpha_L L^{-p_L} + \alpha_M M^{-p_M} + \alpha_F F^{-p_F})^{-1/p}$	more general than CES production function but assumes partial elasticities of substitution to be constant in ratio form. K, L, M , and F stands for capital, labor, material and fuel inputs respectively, p 's are the elasticities of the corresponding factors and α 's are their coefficients.	Boyes (1976).
Other	$y_i = h_i(\alpha) b_i x_i$	where $\bar{Y}$ is the design maximum capacity of the generation unit, $0 \leq c_i \leq 1$ is a factor that determines the actual maximum capacity of the generating unit when it uses coal type i , that is, the term $c_i \bar{Y}$ defines the actual output capacity of the unit, $y_i \leq c_i \bar{Y}$ is the production amount of the generating unit with coal quality i , $h_i(\alpha) \equiv \alpha$ represents the energy efficiency of the unit, x_i stands for the quantity of coal of type i used annually by the generating unit with technology i , and the term $(b_i x_i)$ is the heat energy input used in the electricity generation process.	Khanna and Zilberman (1999), Khanna and Zilberman (2001).

2.1.3 Research on Technology Acquisition

In this section we present a review of the contributions regarding technology selection in the context of production planning.

Weitzman (1967) addresses the problem of selecting the optimal choice among producing on-site and outsourcing options. He uses the term "Production System" which, he suggests, is an input-output model modified to incorporate alternative production processes, exogenous commodities such as labor services or raw materials which can be bought by a firm in any amount, and capital items in production relations. He formulates this production system with a model in which a manufacturing firm has n departments, associated with n commodities (each department deals with one commodity) that are involved in the production system. (By "commodity" he is referring to the production factors like intermediate goods and raw materials.) He defines the k_i^{th} activity of the department i as

$$\{B_i^{k_i}, \mu_i^{k_i}, c_i^{k_i}\},$$

where $B_i^{k_i}$ is an n component non-negative column vector of annual stock requirements j^{th} component of which is the capital stock j required for producing a unit of product i ; $\mu_i^{k_i}$ is that of amortization, j^{th} component of which is the amortization of capital stock j when it is involved in the producing a unit of product i ; $c_i^{k_i}$ is a scalar that stands for the cost of operating the department i at the unit level, by activity k_i . He then formulates the following composite requirement vector each component of which defines the amount of commodities which are amortized (used-up) in the production of a unit of commodity i by technique k_i annually:

$$P_i^{k_i} = B_i^{k_i} \times \mu_i^{k_i}.$$

Finally, he introduces a minimum cost linear programming model in which the firm decides whether to buy or produce related commodities and determines the level of production activities carried out by each department and attaches the solution procedure of the problem. However, he does not test the model and the solution procedure by means of a numerical example.

Li and Tirupati (1994) address the issue of locating the trade-off between dedicated and flexible technologies. They investigate under which conditions one technology dominates the other, through studying a multi-period capacity expansion scheduling problem when a manufacturer produces multiple product families and want to decide between flexible and dedicated technologies, in the presence of dynamically growing and known demand. It is assumed that while the cost of flexible technology acquisition is more than that of dedicated technology. However, while dedicated technologies permit the production of only a single product family, flexible technologies enables the production of a wide range of product families with small changeover costs. It is highlighted in the study that while studies in the literature consider linear technology acquisition costs, they are assumed to be concave in this study. The structure of the functional form for these costs are very similar to that of Erlenkotter (1977a). The solution approach is a two stage decomposition based heuristic the first phase of which is associated with solving a sequence of sub-problems with short planning horizons which restricts the firm to invest in each technology at most once, in order to get an initial solution. (In the original problem the firm can invest multiple times for each technology.) These sub-problems are also solved through heuristics and the quality of their solution, as stated by the authors, depends on the length of planning horizon T . In the second phase of the heuristic, initial solution is improved through several methods. The authors also provide an integer programming formulation in order to present a proxy for adjusting the parameters of the heuristic for small scaled problems. They carried out several numerical studies in which different parameters for power (Cobb Douglas) and fixed investment cost functions are considered and find out that 1) adopting a flexible technology expansion policy leads firms to adjust their product mix in conjunction with the demands of the market which in return enhance their bottom-lines, 2) investments in flexible technology is increasing in scale economies, which contradicts with the common observation that scale economies lead to larger investments in dedicated technologies,

Debo, Toktay and Van Wassenhove (2005) investigate the effect of technology choice and market segmentation on the decision of determining optimal remanufacturability level of the product a manufacturer produces. They model technology cost, $k(q) = kq$, which incurs before the production starts, as a convex increasing function of remanufacturability level q . Considering them as functions

of remanufacturability level, they modeled manufacturing cost of new products and remanufacturing cost of used products as

$$c_n(q) = c_{n0} - c_{n1} \ln(1 - q)$$

and

$$c_r(q) = c_{r0} + c_{r1} q,$$

which are convex increasing and non-increasing in q , respectively. Furthermore, they model consumer willingness to purchase a new and re-manufactured product by a random variable. They study a multi-period, infinite-horizon, discrete-time, discounted profit optimization model in which the firm decides remanufacturability level of her new product before production starts. They assume that each period is assumed to be equal to the life duration of the new product and that if the new product is sold in the n^{th} period, it will be bought from the consumer, remanufactured only once and sold in the $(n+1)^{st}$ period. In their extended model, they consider a game theoretical setting in which the firm is assumed to hold monopoly in the new product market, but there are several remanufacturers competing on the market for remanufactured product. They suppose that the strategy of players in period n solely depend on the used-products at the beginning of that period. Hence, the history of the game is not important for the strategy to be taken in any period. As a result of their analysis, they find out that 1) high production cost of single-use products, low remanufacturing costs, and low cost of conversion of single-use products to remanufacturable ones are the main drivers of technology enhancement, 2) if the volume of low-end consumers is higher, the potential for remanufacturing is lower, 3) Higher fixed costs lead to lower optimal remanufacturability levels at which the concentration of consumers with high willingness to pay is higher in the market.

Elhedhli and Gzara (2008) consider a tri-echelon (warehouse, plant, customer zone), multi-commodity flow, capacitated facility location problem in which a technology selection decision is involved. Every plant located at point i and warehouse located at point j is assumed to be equipped with technologies p and q , has fixed costs g_{ip} and f_{jq} , and has capacities M_{ip} and M_{jq} , respectively. Furthermore, every commodity has a capacity absorption rate of r_{ip} and r_{jq} at corresponding plant and warehouse, respectively. While the location and technologies of both

plants and warehouses, affect the average unit costs of producing and shipping the commodities from plants to warehouses, the locations and technologies of only warehouses designate the shipping and handling costs due to the flow of the commodities from warehouses to customer zones. The authors propose a mixed integer programming multi-commodity-flow formulation for the problem and using a primal heuristic and an interior-point Lagrange Multipliers method, compute upper and lower bounds, respectively.

Aghion, Dechezleprêtre, Hemous, Martin and Van Reenen (2016) analyze directed technical change in the auto sector. They consider fuel prices as a proxy for carbon tax and number of clean and dirty patents as proxies for clean and dirty innovation level. Furthermore, they disaggregated dirty patents into gray (related to efficiency of the combustion engine) and purely dirty patents (related to part of the combustion engine). They modeled number of clean patents applied by firm i in year t as

$$PAT_{C,it} = \exp(\beta_{C,p} \ln FP_{it-1} + A_{C,it-1}) + u_{C,it},$$

which depends on her clean knowledge stock, $A_{C,it-1}$, expressed as

$$A_{C,it} = \beta_{C,1} \ln SPILL_{C,it} + \beta_{C,2} \ln SPILL_{D,it} + \beta_{C,3} \ln K_{C,it} + \beta_{C,4} \ln K_{D,it},$$

and the fuel price in year $t - 1$, FP_{it-1} , expressed as

$$\ln FP_{it} = \sum_c w_{ic0}^{FP} \ln FP_{ct},$$

where w_{ic0} is the share of firm i 's patents in country c (these weights are time invariant and uses information only prior to the regression sample period).

Firm i 's clean knowledge stock is assumed to be a function of firm's own knowledge stock (clean and dirty) and spillovers of knowledge (stocks of economy-wide patents) from other firms in both clean and dirty technologies. In weighting country spillover stocks they used number of inventors in the specific country. They expect that the positive effect of dirty spillovers and dirty knowledge stocks on dirty innovation would be larger than the effects of clean spillovers and clean knowledge stocks.

They consider Triadic patents as their outcome measure, which have been taken

out in all three of the world's major patent offices in the United States, Europe, and Japan USPTO, EPO, and JPO. This has a number of advantages: 1) triadic patents provide a common measure of innovation worldwide, which is not affected from administrative idiosyncrasies of patent offices (If the same invention is covered by one and two patents in the US and Japan, respectively, which are part of the same triadic patent family, it is considered as a single invention). 2) triadic patents cover only the most valuable inventions that capture high-quality patents, 3) they protect inventions with a potential worldwide application, which makes them relatively independent of the countries in which they are filed.

They construct a fuel price variable for each firm as a weighted average of fuel prices across countries based on a proxy of where the firm expects its future market to be, which is the fraction of a firm patents taken out in a specific country, since they thought that a firm will protect its intellectual property in a country if they expect it to be their future market. Furthermore, they discussed that evolution of fuel prices can be modeled as a random walk process and they observe a high variability in the fuel price over time across countries, which is primarily driven by cross-country differences in tax policies. Assuming that the country distribution of a firm's patent portfolio is a good indicator of the firm's expectation of where its markets will be in the future, we can use this distribution to construct a firm-specific fuel price whose value is computed as the weighted mean of the fuel price in the relevant markets, with weights equal to the shares of the corresponding countries in the firm's patent portfolio. They further weight by each country's average GDP to account for the greater importance of larger countries.

They calculated firms' patent stocks using the perpetual inventory method and modeled

$$K_{z,it} = PAT_{z,it} + (1 - \delta)K_{z,it-1},$$

where $PAT_{z,it}$ are the patent flows and δ is the depreciation of R&D capital (20 percent, as is often assumed in the literature). Spillovers are modeled as

$$SPILL_{z,it} = \sum_c w_{ic0}^S SPILL_{z,ct},$$

where the spillover weight is the share of all firm i 's inventors that worked in

country c , between 1965 and 1985. Furthermore, $SPILL_{z,ct}$ is

$$SPILL_{z,ct} = \sum_{j \neq i} w_{jc0}^S SPILL_{z,jt},$$

which is the spillover pool of a country that is the sum of all other firms' patent stocks with a weight that depends on how many inventors the other firm has in that country.

They assume that a type of clean car is more substitutable (has higher elasticity of substitution) to another type of clean car than with any dirty car.

Firstly, they predict and verify that an increase in fuel prices stimulates clean innovations and decreases dirty innovations. Secondly, they find that firms that are innovative in clean (dirty) development or that are surrounded by these type of firms (spillover effect), tend to innovate in clean (dirty) technologies. Moreover, a firm's clean knowledge stock has a positive effect on future clean innovations. Interestingly, it is found out that her dirty knowledge stock has also positive effect on clean innovations. Every finding regarding clean innovations holds also for dirty innovations symmetrically, except that firm's clean knowledge stock has a negative effect on future dirty innovations. Because of this path dependence the authors conclude that regulatory bodies should take an action to shift incentives to clean technologies as soon as possible. Furthermore they state that mild carbon price increases and *R&D* subsidies have almost no role in shifting attention on clean development, and thus, substantial amount of carbon taxes should be set in order to lead to such a shift. They further find that, the electricity price have a negative effect on clean innovation and a positive effect on dirty innovation in automobile industry, while the coefficients are less precisely determined than those for the fuel price.

Through simulation experiments they study conditions under which the clean knowledge stock for the aggregate economy exceeds the dirty knowledge stock. They find out that 40 percent increase in fuel prices achieves a parity between clean and dirty innovations. However when income growth rate is decreased by 0.25 percent, since income growth has a stronger positive effect on clean innovation than dirty innovation, crossover year is postponed by 2 years, from 2020 to 2022, which is considered by the authors as not so significant compared to its benefits in clean development.

2.1.4 Cost Functions Used in the Literature

In this section we present a taxonomy for the structures of capacity expansion cost functions.

While Anderson (1972), Fong and Rao (1975), Giglio (1973), and Rao (1976) consider concave cost functions for capacity expansion, Leondes and Nandi (1975) consider a convex one, Freidenfelds (1980) considers a fixed expansion cost, and concave, convex, and fixed cost functions are employed in Erlenkotter (1973), Erlenkotter (1974), Erlenkotter and Trippi (1976), and Erlenkotter (1977b).

Bogle and O'Sullivan (1980) consider the following functional form:

$$c_b(i, j) = \frac{q_k(i)}{(1 + \rho)^{t(i)}},$$

where $t(i)$ is the critical time associated with network state i , $q_k(i)$ is the cost of adding a standard capacity element on arc a_k when its capacity is specified by the k^{th} component i_k of i and ρ is the cost of capital.

Barber (1975), Fong and Srinivasan (1981a), Fong and Srinivasan (1981b), Freidenfelds and McLaughlin (1979), Giglio (1970), Sinden (1960), and Smith (1979) consider cost functions like the following one:

$$c_b(x) = \begin{cases} 0, & \text{if } x = 0 \\ A_1 + B_1(x - 0), & \text{if } 0 < x \leq a_1 \\ A_n + B_n(x - a_{n-1}), & \text{if } a_{n-1} < x \leq a_n \end{cases},$$

where A_n and B_n are the fixed and variable costs of capacity acquisition when there is an upper-bound a_n for each technology level n .

Erlenkotter (1972), Erlenkotter (1977b); Giglio (1970), Giglio (1973), Kalotay (1973), Kalotay (1974), and Snow (1975) employ power cost functions for capacity expansion, one of which is presented below:

$$c_b(xD) = k(xD)^\alpha.$$

$c_b(xD)$ is the cost of investing in capacity of $x.D$ where x is the number of periods and D is the demand for each period and k and α are constants.

Erlenkotter and Trippi (1976) employ the following quadratic form:

$$C_i(q_i) = c_i + b_{i1}q_i + b_{i2}q_i^2,$$

where $C_i(q_i)$ is the cost of acquiring and operating the capacity expansion project i of size q_i , c_i is the fixed cost and b_{i1} and b_{i2} are coefficients of the operating cost of the project i .

Hinomoto (1965) uses the following cost function for capacity expansion:

$$c_b(t) = (K(z) + q(t)X[y(t), z])e^{ht_\alpha},$$

where $K(z)$ is the purchasing cost of the facility of size z , $q(t)$ is the operating cost per unit time, X is the current input for a facility available at time 0 where $X[y(t), z]$ is the input function that depends on the production amount at time t , $y(t)$ and the size of the facility z . $q(t)$ stands for the operating cost per unit input and h is the decrease in the input requirement of a future facility.

Kalotay (1974), Sinden (1960), and Giglio (1973) consider non-linear cost functions one of which is as follows.

$$y(t_1, t_2) = e^{-rt_1} [g(t_2) - g(t_1)],$$

where $y(\cdot)$ is the total cost of installing at time t_1 a quantity $g(t_2) - g(t_1)$ of capacity, $g(t) = 1 - e^{-dt}$ is the demand at time t , r stands for the discount rate and d is a constant.

2.1.5 Research on Endogenous Learning and Knowledge Capital

2.1.5.1 Economics Literature

Arrow (1962) points out that "learning" is a process of knowledge acquisition and defines technical change as a prolonged process of learning about our environment. He maintains that learning from repetition of the same process will lead to diminishing returns. His hypothesis is that technical change can generally be attributed to experience from production.

Zimmerman (1982) advocates that firms not only learn by doing, they but also transfer knowledge from other institutions operating in their environment. He develops a Cobb Douglas type model that incorporates technology transfer from industry to the specific firm as well as considering learning-by-doing. He assumes that accumulation of experience reduces construction costs and increase the accuracy of forecasts about these costs. He further assumes that, learning takes place at the end of the construction, since most of the learning comes in the last stages of construction.

Romer (1990) considers technology as a non-rival and partially excludable good. He suggests that since it is a non-rival good, it is a non-convex function of human capital devoted to research and development. We can consider Romer as the pioneer of the paradigm that considers technology as an endogenous variable that is a function of research and development. He tests his model by a theoretical data. He proposes a variant of the Cobb Douglas production function that incorporates technological change as an endogenous variable as below.

$$Y(H_Y, L, \vec{x}) = H_Y^\alpha L^\beta \sum_{i=1}^{\infty} x_i^{1-\alpha-\beta},$$

where $\vec{x}$ is the vector of production inputs, Y is the final output, L stands for physical labor, and H_Y represents human capital. It is assumed that there exists a number A representing the state of the technology, such that $x_i = 0$ for all $i \geq A$ (For more information about similar production functions used in literature see Buonanno, Carraro and Galeotti (2003), Gillingham, Newell and Pizer (2008), Grossmann, Steger and Trimborn (2013), and Ornaghi (2006)). Romer's work contains a comprehensive review of the literature regarding technological growth.

Hanel (2000) studies the relationship between productivity and R&D investments of Canadian manufacturers in the presence of international and inter-industry spillovers. He points out that process oriented R&D contributes more to the enhancement of productivity than product oriented R&D and confirm the validity of (Griliches (1994)). He modifies the Cobb Douglas production function in order to incorporate endogenous technological progress and spillovers as below:

$$Q_t = (\phi K_t^\gamma e^{\lambda t}) \left(\prod_{i=1}^m X_{it}^{\alpha_i} \right),$$

where first parenthesis represents the technology term of the the production function and the second one stands for the contribution of each production input. X represents production inputs, m represents number of production inputs, K stands for the stock of productive knowledge, γ stands for the output elasticity of the knowledge capital, ϕ is a constant and λ is the trend of technological change (For more information regarding similar modifications on Cobb Douglas production function, see Griliches (1986), Goto and Suzuki (1989), Lichtenberg and Siegel (1991), Wakelin (2001) and Wang and Tsai (1998)). He empirically tests the model and finds out that the effect of intranational interindustry spillovers on the growth of total factor productivity in Canada is much larger than that of international spillovers.

Kim (1998) studies the effect of induced learning, that is established within the firm through R&D efforts, and autonomous learning, acquired through external technology acquisition, on operations improvement. Referring to (Dosi (1982)), he considers technology as a "a set of pieces of knowledge, both directly practical and theoretical, know-how, methods, procedures, experience of successes and failures, and physical devices and equipment" (Kim (1998): 40).

In order to find the optimal technological growth level, Kim develops an optimal control model, in which the control variable is represented by $u(\tau, \bar{w})$ where $1 \leq \tau \leq T$ is the time of the external technology acquisition, T is the length of the planning horizon, $1 \leq \bar{w} \leq W$ is the age of the technology and W is the required age of the technology to be fully applicable in the production system. In the model proposed, firstly, the following cost function is considered.

$$P(\bar{\omega}) = K\bar{\omega}^m,$$

where $P(\bar{\omega})$ is the cost associated with induced learning, K is the base nurturing price of the adopted least-aged technology, $\bar{\omega}$ is the conceptual age of the technology, $m = \ln \phi_m / \ln 2$ and ϕ_m represents internal learning rate.

Secondly, the author considered the following cost function.

$$P(\tau, \bar{\omega}) = K_p \bar{\omega}^\alpha \tau^n,$$

where $P(\tau, \bar{\omega})$ is the cost of external technology acquisition, K_p is the base purchasing price of the least aged technology, $\bar{\omega}$ is the conceptual age of the external

technology, τ is the period of the acquisition, $0 \leq \alpha \leq 1$, $n = \ln \phi_n / \ln 2$ and ϕ represents external learning rate.

Since the study is a theoretical one, no empirical data is used to test the proposed model.

Fine (2000) reviews the literature regarding endogenous growth theory. Being a comprehensive literature review, this paper can be used as a reference for those interested in further information related to the evolution of the related Economics literature.

2.1.5.2 OR/MS Literatures

In OR/MS literatures, research on productive knowledge acquisition in the way we define it can be considered as scarce. There are several studies that deal with knowledge creation and transfer. However, these studies pioneered by Porteus (1986), Tapiero (1987), Fine (1988), and Fine and Porteus (1989) primarily focus on investments in induced learning that lead to a process quality improvement directly linked to production cost reduction stemming from improved product defect rates. Unlike these studies, we directly considered process improvement as productivity improvement through knowledge acquisition that amplifies the output of the production process, which constitutes one of our key contributions to the literature.

Huber (1991), in his detailed qualitative account regarding the process of organizational learning, argues that firms not only learn from their experiences, they but also transfer knowledge from other institutions operating in their environment and that if more of the components of the organization acquires the knowledge and if this knowledge is recognized as useful, then organizational learning occurs.

Dorroh, Gullledge and Womer (1994) study a cost minimization problem regarding a producer of specialized products to contractual order that is to decide how much she should devote her resources to learning-before-doing in the form of training to enhance production process. They distinguish between learning and knowledge and refer to learning as an endogenous by-product of production process while referring to knowledge acquisition as a direct decision that the management should consciously make. They use following functional forms in order

to formulate the problem: Total output and total stock of knowledge at time t are modeled as

$$Q(t) = \int_0^t q(\tau) d\tau,$$

$$L(t) = \int_0^t l(\tau) d\tau,$$

where $q(\cdot)$ and $l(\cdot)$ are production rate and knowledge production rate respectively.

The rate of production and knowledge generation are modeled by the following functional forms:

$$q(t) = L(t)f\left[\frac{x_q(t)}{L(t)}\right],$$

$$l(t) = L(t)g\left[\frac{x_l(t)}{L(t)}\right],$$

where x_q and x_l stands for resources that is devoted for production and knowledge generation respectively. Thus total resources are represented by x where $x = x_q + x_l$.

Note that, $f(\cdot)$ and $g(\cdot)$ are production functions. If we use CES production functions and get rid of time dependency then above formulation becomes:

$$q = La_1 \left[\alpha \left(\frac{x_q}{L} \right)^{-\gamma} + (1 - \alpha) \right]^{-1/\gamma},$$

$$l = La_2 \left[\beta \left(\frac{x_l}{L} \right)^{-\delta} + (1 - \beta) \right]^{-1/\delta}.$$

The authors, by using a theoretical data, test their model and derive managerial insights at the end of the paper.

Darr, Argote and Epple (1995) study knowledge acquisition (through learning-by-doing), depreciation and transfer within organizations. The model proposed is a learning curve model.

Carrillo and Gaimon (2000), like Dorroh, Gulledge and Womer (1994), study a profit maximizing firm's decision to devote what portion of her resources to process change established through learning-before-doing, attributed to investment

in training, and learning-by-doing which is related to learning from experience or replacement of software and hardware or modification of manufacturing procedures. In their model, cumulative level of knowledge at time t , $S(t)$, and the effective level of capacity at time t , $K(t)$, are considered as the state variables. Furthermore, $s(t)$ that represents the rate of preparation and training (learning-before-doing) at time t and $p(t)$ that stands for the rate of process change implemented at time t are taken to be as control variables. It is supposed that effective capacity decreases following the initiation of the process change, then increases with a decreasing rate (diminishing returns) and ultimately exceeds its level before the initiation of the process change because of its benefits. Thus the increase of the effective capacity does not take place instantly. The objective of the model is maximizing the firm's profits, which is considered as a function of effective capacity and cumulative knowledge and assumed to increase with a decreasing rate as they increase. It is also assumed that salvage value of knowledge capital and effective capacity affects the firm's profits. The cost associated with the process change is considered as the investment in preparation and training, which is assumed to increase with an increasing rate with respect to $s(t)$.

Biskup and Simons (2004) incorporate autonomous learning that is associated with learning-by-doing, and induced learning which is a consequence of efforts on process improvement, in their due date scheduling model. Mentioning the contributions of (Lapr , Mukherjee and Van Wassenhove (2000), Lapr , Mukherjee and Van Wassenhove (2000), Lapr  and Van Wassenhove (2001), Hatch and Mowery (1998) and Mukherjee, Lapr  and Van Wassenhove (1998)) in the modeling of induced learning, they offer a functional form for learning curve that cures some previous shortcomings and incorporates induced learning. The functional form is as below:

$$p_{ir} = p_i r^{\log_2 u},$$

$$u = (1 - x)s \quad \text{s.t. } 0 \leq x \leq x_{max} < 1,$$

where p_i is the processing time for each job i out of n jobs that are indexed according to the Longest Processing Time Rule. Processing of these jobs will be carried out by a single machine.

V r s (2006) studies a single item producer where a monopolistic competition takes place. Demand of the product is a function of its performance, quality and

price. Productivity knowledge, decreasing the unit production cost, and quality knowledge can be enhanced through induced and autonomous learning.

Gaimon, Özkan and Napoleon (2011) formulate models that incorporate learning-before-doing and learning-by-doing. He states that without explicit investment from the manager in general training, the positive effect of the existing workforce knowledge to net income declines over time, which is referred as workforce knowledge depreciation. Hence, level of workforce knowledge is assumed to decrease by a depreciation rate and increase by the rate of general training.

2.1.6 Research on Pollution Abatement

Löscher (2002) and Gillingham, Newell and Pizer (2008) review the economic models that formulate technological change (TC) and its effect on environmental policy. Löscher (2002) states that although the majority of the environmental policy models consider technological change as an exogenous variable, there is tremendous evidence that it is an endogenous one. The authors of both studies point out that the researchers considering technological change as an endogenous variable formulate it as a function of following main drivers: 1) energy prices, 2) investment in R&D, 3) spillovers from R&D, 4) technology learning. They discussed some of the macroeconomic studies in the Economics literature in order to find the roots of the endogenous and induced technological change paradigms emerged. They state that while most of the studies in the environmental policy literature considers endogenous technological change based on natural technological change and aggregate R&D expenditures, there are only a few studies that considers endogenous technological change as a direct consequence of investment in innovation or knowledge and review these studies.

Goulder and Schneider (1999), in their climate policy model of multi-sector general equilibrium that incorporates induced technological change through knowledge stock growth which is led by investments in R&D, provide a variant of CES production function that incorporates the details of the CO₂ abatement policies. The importance of this paper for our research is that they consider induced technological change at sector level. Thus we will investigate the functional forms used in this study as below.

The production function used in the study is expressed as

$$\begin{aligned}
 Y &= \gamma(\bar{H})(\alpha_H H^\rho + \alpha_G G^\rho)^{1/\rho}, \\
 G &= (\alpha_K K^{\rho_G} + \alpha_L L^{\rho_G} + \alpha_E E^{\rho_G} + \alpha_M M^{\rho_G})^{1/\rho_G}, \\
 E &= (\alpha_{EC} EC^{\rho_E} + \alpha_{EA} EA^{\rho_E})^{1/\rho_E}, \\
 M &= (\alpha_{MI} MI^{\rho_M} + \alpha_{MN} MN^{\rho_M})^{1/\rho_M},
 \end{aligned}$$

where Y is production quantity, G represents aggregation of all production inputs other than H and $\bar{H}$ that stand for excludable and non-excludable knowledge stock respectively. (Excludable good refer to a good that can be appropriated and whose consumption is limited to paying customers.) EC and EA stand for conventional and alternative energy fuels, MI and MN stand for energy-intensive and other materials respectively. α and ρ are parameters where $\rho = (\sigma - 1)/\sigma$.

Nordhaus (2002) develops a model of world economy, regarding R&D induced technological change, known as R&DICE for profit-oriented enterprises that seek for improvements in energy and environmental production technology. The decision variables of the model are consumption, the rate of investment in tangible capital, and investment in emission reduction. It is assumed that each region of the world produces a single commodity. Changes in welfare are incorporated in the consumption of this single utility for that region. It is assumed that there is no international trade between regions except for carbon emission permits and that each region has an initial level of labor and capital. R&DICE model is a modification of the DICE model which is proposed in one of his previous studies (Nordhaus (1999)). While DICE model deals with environmental impact of technological change by considering it as an exogenous variable, R&DICE model considers technological change as a function of investments in R&D increasing knowledge stock. Furthermore, while in DICE model it is assumed that some level of substitution between carbon-energy, which can be considered as the energy services derived from fossil-fuel consumption, and other production factors is present, in R&DICE model production inputs and carbon-energy are assumed to be perfect complements.

Nordhaus and Boyer (2000) present updated versions of DICE and R&DICE models which are DICE-99 and RICE-99. We investigate the details of the RICE-99 model. The authors suppose that knowledge can be decomposed into two types, idiosyncratic and generic knowledge, which are further decomposed into three

sub-types that are basic, applied and engineering knowledge. They define the term "innovation" as an improvement in knowledge level, which is established by combining sector's idiosyncratic R&D efforts to the pool of idiosyncratic and generic knowledge. Generic technological change is modeled as Hicks-neutral exogenous variable and idiosyncratic technological change is considered as an endogenous one derived from the efforts for reducing the ratio of CO_2 emissions to carbon-energy inputs.

Technological change and associated production function (Cobb Douglas production function) is formulated as follows:

$$\begin{aligned}\frac{\dot{\sigma}_i(t)}{\sigma_i(t)} &= \phi_1(R_i(t), H_t) = \kappa_i R_i(t)^{\beta_i} - \sigma_i, \\ Q_i^*(t) &= A_i(t) f_i(t)(\cdot) = \left\{ A_i(t) K_i(t)^{\gamma_i} L_i(t)^{1-\beta_i-\gamma_i} ES_i(t)^{\beta_i} - c_i^E(t) ES_i(t) \right\}, \\ Q_i(t) &= \Omega_i(t) Q_i^*(t),\end{aligned}$$

where the first equation is called as deterministic "innovation-possibility frontier". This deterministic structure is considered by Nordhaus and Boyer (2000) as a disconcerting one, in that, distribution of returns to inventive activity is known to be highly skewed. $\dot{\sigma}_i(t)/\sigma_i(t)$ is the rate of technological change (here this corresponds to rate of change of carbon intensity or carbon-output ratio) which can be discrete or continuous, $\sigma_i(t)$ stands for the level of technological knowledge in sector i in year t , $\dot{\sigma}_i(t) = \delta\sigma_i(t)/\delta t$ represents technological change, $R_i(t)$ is the research carried out in industry i in year t and H_t denotes generic knowledge which is assumed to be an exogenous variable that impacts the rate of technological change. κ_i , β_i and ϕ_i are parameters that would be determined according to the industry i assuming optimized R&D in the past periods. $Q_i(t)$ is the production amount in sector i in year t , $f_i(t)(\cdot)$ is a function of inputs of production such as capital and labor which is assumed to be equal to gross world output $Q_i^*(t)$. $Q_i^*(t)$ is calculated through employing the DICE-99 model to derive the optimal values of the production factors $K_i(t)$ and $L_i(t)$ and the interest rates for the previous periods. Thus, $Q_i^*(t)$ has a fixed value for R&DICE model. $ES_i(t) = \varsigma_i(t)E_i(t)$ stands for energy services. $E_i(t)$ and $\varsigma_i(t)$ represent carbon energy and level of carbon-augmenting-technology respectively, $[c_i^E(t)ES_i(t)]$ is the cost of producing carbon-energy, and finally, $\Omega_i(t)$ represents an output scaling factor due to damages from climate change.

The authors suppose that production incorporates several elemental goods and associated elemental processes i 's and every process has a constant returns to scale. They further assume that at any period t , (a period in this model corresponds to the effective life of a new invention.) in the planning horizon there exists a dominant technology for each process that sets the upper-limit of the prices for the next period and formulate this as below:

$$P_i^T(t) = C_i^T(t) + \alpha [C_{i,t-1}^T - C_i^T(t)],$$

where $P_i^T(t)$ and $C_i^T(t)$ are the price and cost of the dominant technology in industry i for period t , $\alpha [C_{i,t-1}^T - C_i^T(t)]$ represents the margin of the innovator due to her contribution of increasing the technology level or carbon intensity level to $\sigma_i(t)$ and decreasing the costs from $C_i^T(t)$ to $C_{i,t-1}^T$, where appropriability ratio, α , defines this margin.

Total consumption which is incorporated in the objective function of the model is formulated as below:

$$C_i(t) = Q_i(t) - I_i(t) - 4R_i(t) - R(t)E_i(t),$$

where $C_i(t)$ represents the consumption, $I_i(t)$ is the total capital formation, $R_i(t)$ is the R&D inputs into the carbon-energy sector, $R(t) = \prod_{v=0}^t [1 + \rho(v)]^{-10}$ represents the pure rate of time preference which implies that current generations are favored over future ones, and $E_i(t)$ stands for the carbon-energy inputs which are equal to industrial CO_2 emissions.

The objective function of the model is the "social welfare function" which is a function of per capita consumption $c_i(t) = C_i(t)/L_i(t)$, where $L_i(t)$ corresponds to total population at time t which is also equal to labor inputs, and it is as follows:

$$\max_{\{c_i(t)\}} = \sum_t U[c_i(t), L_i(t)] \rho(t) = \sum_t L_i(t) \{\log[c_i(t)]\} r_s(t).$$

As can be seen from the equation "Social welfare function" is increasing in the consumption of the generation at time t with decreasing marginal utility of consumption.

They also present the outcomes of the numerical studies and associated sensitivity analysis carried out to test R&DICE model.

Goulder and Mathai (2000) study a partial equilibrium model of knowledge accumulation where the decision to take is the selection of the optimal pollution abatement and R&D investment time paths. Abatement cost is assumed to be a function of knowledge stock, H , and the level of abatement, A .

Following is the knowledge accumulation function they use in the numerical study:

$$\psi(I_t, H_t) = M_\psi I_t^\gamma H_t^\phi,$$

where I_t and H_t stand for the investment in R&D and knowledge stock at time t and M_ψ , γ and ϕ are the parameters to be fit according to the industry.

Van der Zwaan, Gerlagh, Schrattenholzer et al. (2002) study the impact of the endogenous technological change on the optimal CO_2 abatement and carbon tax levels. They use the following production function.

$$Q(t) = \left[A(t) \left(K(t)^\alpha L(t)^{1-\alpha} \right)^\gamma + B(t) \left((F(t)^\chi N(t)^\chi)^{1/\chi} \right)^\gamma \right]^{1/\gamma},$$

where $Q(t)$ is the cumulative production quantity, $A(t)$ is technological progress that is neutral in the use of labor and capital, $K(t)$ and $L(t)$ are capital stock and labor input respectively, $B(t)$ is energy technology stock, $F(t)$ and $N(t)$ are fossil and non-fossil energy inputs respectively as functions of time t . Other non-mentioned parameters are the elasticity of substitution of the related inputs. They also incorporate learning by doing effects in their later models.

Buonanno, Carraro, Castelnovo and Galeotti (2000) develop models of environmental technological change through modifying the RICE model by incorporating the effect of "endogenous technological change" (Base Model) and "induced technological change" (Second model) due to the increase in the stock of knowledge established by investments in R&D on factor productivity and emission output ratio respectively, and accounting for international technological spillovers (ETC-RICE model) by assuming a non-cooperative Nash game played by countries in a dynamic setting.

Every country generates its output through a Cobb Douglas production function

as below.

$$Q(n, t) = A(n, t)K_R(n, t)^{\beta_n}WK_R(n, t)^{\epsilon_n} \left[L(n, t)^\gamma K_F(n, t)^{1-\gamma} \right],$$

$$WK_R(j, t) = \sum_{j \neq i} K_R(i, t),$$

where $Q(n, t)$ is the portion of the total output that is affected by environmental technological change (overall climate change effects), $A(n, t)$ is the exogenous portion of the technology and $K_R(j, t)$, $L(n, t)$ and $K_F(n, t)$ are knowledge capital, labor and physical capital in country n at time t respectively. $WK_R(n, t)$ is the stock of world knowledge which is used to represent the effect of spillovers on the county n . β_n , ϵ_n , γ and $(1 - \gamma)$ are the output elasticities of knowledge capital, world knowledge, labor and physical capital of country n respectively. The authors denote that as long as β_n is positive the output production process is characterized by increasing returns to scale.

The aggregation of knowledge capital which is induced by investment in research and development is formulated as

$$K_R(n, t + 1) = R\&D(n, t) + (1 - \delta_R)K_R(n, t),$$

where $R\&D$ stands for the investment in R&D and δ_R is the rate of knowledge depreciation.

Environmental technological change is assumed to be endogenous and induced by the investment in research and development. (Note that both environmental and non-environmental technological change are assumed to be induced by the same R&D effort.) Then the corresponding emission output relationship is formulated as

$$E(n, t) = \left[\sigma_n + \chi_n e^{(-\sigma_n K_R(n, t) - \theta_n WK_R(n, t))} \right] [1 - \mu(n, t)] Q(n, t),$$

where E stands for the emission amount, α_n represents the elasticity of reduction in the emission-output ratio induced by increase in knowledge capital, χ_n is a scaling factor, μ is the rate of abatement effort and σ_n is the asymptotic value the emission-output ratio will tend as the knowledge stock goes to infinity.

The portion of the output that is affected by non-environmental technological

change but unresponsive to environmental technological change, Y , is formulated as

$$Y(n, t) = c_b(n, t) + I(n, t) + R\&D(n, t) + p(t)NIP(n, t),$$

where C , I , NIP and p stands for consumption, total fixed capital formation (total transfer of savings to the business in the form of investment), net demand for emission permits and the price of a unit of emission permit and, finally, following constraint is used in order to account for carbon trading mechanism in the model.

$$E(n, t) = Kyoto(n) + NIP(n, t),$$

where $Kyoto$ is the emission targets set in the Kyoto Protocol for the signatory countries and BAU levels for non-signatory ones.

Shadbegian and Gray (2005) empirically study the effect of investment in pollution abatement on productivity. Modifying the Cobb Douglas production function by dis-aggregating the input vector X into productive, X_P , and pollution abatement, X_A , inputs, they could distinctively incorporate the investments in productivity and pollution abatement. The production function is expressed as follows.

$$\begin{aligned} Q &= F(X_P, X_A, O), \\ \ln Q &= \alpha + \beta_{KP} \ln K_P + \beta_{KA} \ln K_A + \beta_{LP} \ln L_P + \beta_{LA} \ln L_A \\ &\quad + \beta_{MP} \ln M_P + \beta_{MA} \ln M_A + YEARS + U, \end{aligned}$$

where K_P and K_A are real productivity and abatement capital stocks respectively, L_P and L_A are number of worker hours for power generation and pollution abatement respectively, M_P and M_A are the real materials used for power generation and pollution abatement respectively, $YEARS$ is the set of year dummies, and Q represents overall power generation, and O stands for the macroeconomic factors other than production inputs that may affect the power output. They find that investments in abatement does not contribute to production and that their negative effect on productivity inputs is negligible.

Zhang and Schoengold (2010) in the context of pollution abatement study China's power generation industry, by formulating an analytical model which incorpo-

rates joint production functions to model the production and emission processes and investment opportunities for emission abatement. Joint production function they employ is as follows.

$$Y = A^a K_1^\alpha X^{\alpha'},$$

$$E = A^b K_1^\beta X^{\beta'} e^{-K_2 \gamma},$$

where Y and E stand for the generation and emission amounts respectively, K_1 and K_2 are capital investments in production productivity and emission efficiency respectively, X is the input, fossil fuel, A is the technology, α and β represents marginal productivity of the capital K_1 and γ stands for the abatement efficiency rate of removing emission flows after power generation.

The objective of the model is utility maximization in which production output increases but emission level decreases utility, which is presented as fallows.

$$\max_{X, I_1, I_2} \int U(V, S) e^{-\rho_0 t},$$

where ρ_0 is the social discount rate. The authors carry out a numerical study the data of which is derived according to IPCC (2005) report and inspect the relationship between the emission tax policy and the level of abatement technology. They find that improvements in abatement technology may lead to considerable decreases in the required emissions taxes.

Telli, Voyvoda and Yeldan (2008), through a general equilibrium model, investigate the economic impacts of possible environmental policies of compliance with the Kyoto Protocol. The model they develop considers ten production sectors: 1) agricultural production (AG), 2) coal mining (CO), 3) petroleum and gas (PG), 4) refined petroleum (RP), 5) electricity production (EL), 6) cement production (CE), 7) paper production (PA), 8) iron and steel Production (IS), 9) transportation (TR), and 10) a composite of remaining primary sectors of the economy (OE). It is assumed that government is operating in an open macroeconomic environment over the period 2006 – 2020 and the factors of production are 1) labor for sector i , (L_i), 2) capital for sector i , (K_i), 3) primary energy composite, that is composed of electricity (EL), coal (CO), and petroleum and gas (PG), (ENG_i) for sector i , 4) intermediary inputs excluding primary energy inputs (ID). Accord-

ingly, gross product for sector i , XS_i , is modeled as

$$XS_i = AX_i \left[K_i^{\lambda_{K,i}} L_i^{\lambda_{L,i}} \left(\prod_j ID^{\lambda_{ID,j,i}} \right) ENG_i^{\lambda_{ENG,i}} \right],$$

where AX_i is the technology level parameter, and $\lambda_{K,i}$, $\lambda_{L,i}$, and $\lambda_{ENG,i}$ represent the share of capital, labor, and energy in the value of gross output in sector i , respectively, which are functions of the primary energy composite in sector i , ENG_i , that can be expressed as

$$ENG_i = AE_i \left[\kappa_{CO,i} ID_{CO,i}^{-\rho_{x_i}} + \kappa_{PG,i} ID_{PG,i}^{-\rho_{x_i}} + \kappa_{EL,i} ID_{EL,i}^{-\rho_{x_i}} \right].$$

2.1.7 Further Review and Justification

In our setting, the manufacturer produces a single item by employing N inputs. We suppose that some level of substitutability is present among production inputs. Hence "Cobb Douglas" or "CES" production functions would be appropriate. In this case, a unit productivity enhancement in one factor will contribute to the whole process. Furthermore, we assume that technology (or knowledge) acquisition cost is a convex increasing function of technology level.

A strong justification of our assumption that cost of technology acquisition is convex in knowledge comes from the findings of Bormotov (2010). His detailed study on the aspects of the modern knowledge based economy has conclusions regarding the structure of knowledge acquisition cost. He states that although it is true that total investments in knowledge acquisition do not have diminishing return because of the fact that total knowledge is unlimited, investment in production and implementation of a specific knowledge (technological concept, technology, machinery, etc.) have diminishing return. Moreover, Griliches (1998), studying the relationship between patent statistics and economic activities, suggests that R&D investments in productivity enhancement have diminishing returns in general. (For more information regarding studies suggesting diminishing returns from R&D see Posner and Zuckerman (1992), Fine and Porteus (1989), Gaimon (1985), Gaimon (1997), Gaimon and Ho (1994), Gaimon (1989), Gaimon, Özkan and Napoleon (2011), and Carrillo and Gaimon (2000) Carrillo and Gaimon (2002)).

2.2 The Setting of the Problem

In this section, we investigate the multi-input production optimization and technology enhancement problem under carbon tax policy when production quantity is a function of production inputs and technology level. The decision maker considered is a single item manufacturer and the decision to be made is selecting the optimal value of technology level, production inputs, and production quantity. Manufacturer produces Q units, by using $N \geq 2$ inputs. The amount of production input i ($i \in \{1, 2, \dots, N\}$) to be used is denoted by x_i . Each input has a cost of p_i per unit and a carbon footprint of β_i metric tonne per unit. The product has a selling price of s per unit. We suppose that the manufacturer has contracts to fulfill that requires a minimum of Q_l units of production and her capacity or contracted upper bound is equal to Q_h units. Thus, she has to produce an amount that is in between $Q_l > 0$ and $Q_h > 0$; $Q \in [Q_l, Q_h]$.

The setting we envision can be found in a number of industries - particularly, in utilities industry, where the utility price is set by some regulator and the producer would be obliged to satisfy the demands within a certain range of values. As for the inputs, they may be, for example, labor, capital (equipment/machinery) and energy. Different types of energy use may be directly substitutable within each other.

We assume that the production quantity, Q , depends on the technology level K and requires N inputs, where each input is denoted by x_i 's. For the integration of these quantities with the model, we consider a production function of the form

$$F(\vec{x}, A) = Q,$$

the form of which will be discussed later. The revenue from selling the item is

$$\Pi(Q) = Qs,$$

The cost of selecting technology level K , details of which is discussed further in the thesis, is denoted by $c_T(K)$.

Remember that, the input i has a procurement cost of p_i per unit and the carbon tax policy requires firms to pay a unit price, ρ , for metric tonne of carbon they emitted, which means that $\rho\beta_i$ is variable cost of input i due to its carbon

footprint. Hence, total variable cost of input i , v_i , can be formulated as

$$v_i = p_i + \rho\beta_i.$$

We assume that the total cost and emission of production inputs are linear in the input amounts. Then, the procurement cost of input i and the emission due its carbon footprint can be represented as $p_i x_i$ and $\beta_i x_i$, respectively. Then, the total procurement cost, $c_p(\vec{x})$, is

$$c_p(\vec{x}) = \sum_{i=1}^N p_i x_i,$$

the total emission of the firm, $em(\vec{x})$, is

$$em(\vec{x}) = \sum_{i=1}^N \beta_i x_i,$$

and the total carbon tax paid by the manufacturer, $c_e(\vec{x})$, is

$$c_e(\vec{x}) = \rho em(\vec{x}) = \rho \sum_{i=1}^N \beta_i x_i.$$

Hence, the total cost of production inputs, $c_T(\vec{x}) = c_p(\vec{x}) + c_e(\vec{x})$, can be expressed as

$$c(\vec{x}) = \sum_{i=1}^N (p_i + \rho\beta_i) x_i.$$

The model is formally stated as

$$(P) \quad \max_{\vec{x}, K, Q} \quad \Pi(\vec{x}, K, Q) = Qs - c_p(\vec{x}) - c_e(\vec{x}) + c_T(K), \quad (2.2)$$

$$s.t. \quad Q = F(\vec{x}, A), \quad (2.3)$$

$$Q \in [Q_l, Q_h], \quad (2.4)$$

$$x_i \geq 0 \quad for \quad i = \{1, 2, \dots, N\}, \quad (2.5)$$

where, Expression (2.2) represents the firm's profit, Equality (2.3) ensures that Cobb-Douglas production function constraint is satisfied, Constraint (2.4) en-

sures that production quantity is higher than the contracted amount and lower than the maximum output, and Constraint (2.5) ensures the non-negativity of the amount of inputs.

In the following two sub-sections we will present the details of the production function and the technology cost function we use in this chapter.

2.2.1 Production Function

We consider the Cobb-Douglas production function which allows the inputs of production to be substitutable. That is, elasticity of substitution is greater than zero.

Note that, by Douglas (1976), standard Cobb-Douglas production function is formulated as follows.

$$Q = AK^\delta L^\varphi,$$

where Q , A , K , and L are production quantity, technology level, capital, and labor, respectively. However, in this thesis, in order to leave room for further expansion and derive universal results, we study the generalized version of this function which is of the form

$$Q = A \prod_{i=1}^N x_i^{\alpha_i}, \tag{2.6}$$

where x_i represents the amount of production input i and α_i is the output elasticity of this input.

Accordingly, elasticity of scale of the Cobb-Douglas production function is defined as

$$r = \sum_{i=1}^N \alpha_i.$$

2.2.2 Technology Acquisition

We consider the term "technology" as Total Factor Productivity (TFP). A portion of TFP can be considered as explicit knowledge regarding productivity, which is associated with learning-before-doing and acquired through investments in R&D or training programs. Consequently, we formulate technology level, A , as

$$A = aK^{\alpha_K},$$

where K is the knowledge level, α_K is the elasticity of K , which is assumed to be less than or equal to one, and a is the uncontrollable part of the Total Factor Productivity (For instance it may represent cumulative technical change, as in the knowledge stock function used by Hanel (2000)).

In order to account for the technology enhancement cost, we use a power function that is a modified version of the cost function used by Porteus (1985). The function in Porteus (1985) represents the cost of **decreasing** setup cost in an EOQ setting. However, what we have to derive is the cost of **increasing** technology level from its initial level K_0 to K . Thus technology enhancement cost may be expressed of the form

$$c_T(K) = bK^g + d \quad \text{for } 0 < K_0 \leq K. \quad (2.7)$$

Following the construction of Porteus (1985), we represent the cost of technology acquisition as in the following proposition.

Proposition 2.1. *Technology acquisition cost function can be represented as*

$$c_T(K) = \frac{c_K \gamma'}{\gamma'' + 1} \left(K^{(\gamma''+1)} - K_0^{(\gamma''+1)} \right), \quad (2.8)$$

where,

$$\gamma' = \frac{\ln(1.1 + \gamma)}{K_0^{\frac{\ln(1.1+\gamma)}{\ln(1.1)}} \ln(1.1) \left(1.1^{\frac{\ln(1.1+\gamma)}{\ln(1.1)}} - 1 \right)}, \quad (2.9)$$

$$\gamma'' = \frac{\ln(1.1 + \gamma)}{\ln(1.1)} - 1, \quad (2.10)$$

c_K is the cost of increasing technology level 10% from its initial level, K_0 , and γ represents diminishing returns from investing in technology enhancement.

Proof. Substituting K_0 for K in Expression (2.7) and setting it equal to zero, we get

$$d = -bK_0^e.$$

Considering a ten percent improvement in technology level, we have

$$c_T(1.1K_0) = b(1.1K_0)^e - bK_0^e, \quad (2.11)$$

and, then,

$$b = \frac{c_K}{K_0^e(1.1^e - 1)}, \quad (2.12)$$

where c_K stands for the cost of increasing technology level by 10% from K_0 to $1.1K_0$. Accordingly, the cost of increasing technology level by 10% from $1.1K_0$ to $1.21K_0$ can be expressed as

$$c_T(1.1K_0)(2.1 + \gamma) = c_T(1.21K_0),$$

where γ ensures that technology enhancement has diminishing returns. Using (2.12) and substituting (2.11) for $c_T(1.1K_0)$ in the above equation, we get

$$\frac{c_K}{1.1^e - 1}(1.1^e - 1)(2.1 + \gamma) = \frac{c_K}{1.1^e - 1}(1.21^e - 1),$$

which gives

$$e = \frac{\ln(1.1 + \gamma)}{\ln(1.1)}.$$

Substituting the above entities,

$$c_T(K) = \frac{c_K}{K_0^{\frac{\ln(1.1+\gamma)}{\ln(1.1)}} \left(1.1^{\frac{\ln(1.1+\gamma)}{\ln(1.1)}} - 1 \right)} \left(K^{\frac{\ln(1.1+\gamma)}{\ln(1.1)}} - K_0^{\frac{\ln(1.1+\gamma)}{\ln(1.1)}} \right).$$

Finally, letting

$$\gamma' = \frac{\ln(1.1 + \gamma)}{K_0^{\frac{\ln(1.1 + \gamma)}{\ln(1.1)}} \ln(1.1) \left(1.1^{\frac{\ln(1.1 + \gamma)}{\ln(1.1)}} - 1 \right)}$$

and

$$\gamma'' = \frac{\ln(1.1 + \gamma)}{\ln(1.1)} - 1,$$

we get

$$c_T(K) = \frac{c_K \gamma'}{\gamma'' + 1} \left(K^{(\gamma'' + 1)} - K_0^{(\gamma'' + 1)} \right),$$

which completes the proof. $\square$

Since power cost function we employ requires that technology level, K , is greater than or equal to its initial level, K_0 , we should impose the constraint $K_0 \leq K$ to the model, which is presented in the next section.

2.3 The Model

Considering the constraint on technology level, the problem (P) can be rewritten as follows.

$$(P) \max_{\vec{x}, K, Q} \Pi(\vec{x}, K, Q) = Qs - c_p(\vec{x}) - c_e(\vec{x}) - c_T(K), \quad (2.13)$$

$$s.t. \quad Q = aK^{\alpha_K} \prod_{j=1}^N x_j^{\alpha_j}, \quad (2.14)$$

$$Q \in [Q_l, Q_h], \quad (2.15)$$

$$K \geq K_0, \quad (2.16)$$

$$x_i \geq 0 \quad for \quad i = \{1, 2, \dots, N\}, \quad (2.17)$$

where, Expression (2.13) represents the profit of the firm, Constraint (2.14) ensures that relationships among the inputs, the technology level, and the output

are integrated into the model by using the Cobb-Douglas production function, Constraint (2.15) ensures that production quantity is higher than the contracted amount and lower than the production capacity, Constraint (2.16) ensures that the selected technology level can not be less than its initial level, and Constraint (2.17) ensures the non-negativity of the amount of each input.

Note that we will approach the problem by considering firstly the firm's perspective, and then, the social planner's perspective. The former related to finding the optimal production policy for a given carbon tax value, and the latter is related to finding the change in the optimal response of the firm to increases in the carbon tax levels, which will help uncover the trade-offs between the technology improvements, the output level, the input allocation, and the total emission.

In the following section, we provide the structural properties of the firm's problem, give the details of the solution procedure, and present the optimal policy of the firm for a given carbon tax value.

2.4 Firm's Problem

Firm's problem is maximizing the profit over the amounts of production inputs, $\vec{x}$, the technology level, K , and the production quantity, Q , for a given carbon tax, ρ . Let the total cost of production inputs and technology enhancement be denoted by $C(\vec{x}, K) = c_p(\vec{x}) + c_e(\vec{x}) + c_T(K)$. Then, the problem of maximizing (2.13) for given Q , is equivalently solved by minimizing $C(\vec{x}, K)$ over $\vec{x}$ and K subjected to Constraints (2.16) and (2.17). Furthermore, the problem of minimizing $C(\vec{x}, K)$ for given K , is equivalently solved by minimizing $C(\vec{x}|K) = c_p(\vec{x}) + c_e(\vec{x})$ over $\vec{x}$ subjected to Constraint (2.17). Therefore, we will use a backward substitution approach in solving Problem P .

We know from Representation Theorem that any point in a convex set can be represented by a convex combination of the extreme points of this set.

Remember that the Cobb Douglas production function we use is of the form

$$Q = aK^{\alpha_K} \prod_{j=1}^N x_j^{\alpha_j}. \quad (2.18)$$

Then, since its natural logarithm is

$$\ln\left(\frac{Q}{aK^{\alpha_K}}\right) = \sum_{j=1}^N \alpha_j \ln(x_j), \quad (2.19)$$

letting $y_i = \ln(x_i)$ and Y^{+N} be the set of vectors satisfying $\sum_{j=1}^N \alpha_j y_j = \ln\left(\frac{Q}{aK^{\alpha_K}}\right)$ s.t. $y_j \in \mathbb{R}^+$, any vector $\vec{y} \in Y^{+N}$ can be represented by the convex combinations of the extreme points of Y^{+N} , for given K and Q . Note that we can express the i^{th} extreme point of Y^{+N} as $e_i = \left(o_1, o_2, \dots, \frac{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)}{\alpha_i}, \dots, o_N\right)$ for $i \in \{1, 2, \dots, N\}$, where $o_j = 0$ for $j \neq i$. Hence, vector $\vec{y}$ can be represented as follows.

$$y_i = \ln(x_i) = \frac{\lambda_i}{\alpha_i} \ln\left(\frac{Q}{aK^{\alpha_K}}\right) \quad \text{for } i \in \{1, 2, \dots, N\} \quad \text{s.t.} \quad \sum_{j=1}^N \lambda_j = 1. \quad (2.20)$$

Then, x_i ($i \in \{1, 2, \dots, N\}$) for given K and Q can be represented by λ_i as

$$x_i = \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_i}{\alpha_i}} \quad \text{for } i \in \{1, 2, \dots, N\} \quad \text{s.t.} \quad \sum_{j=1}^N \lambda_j = 1. \quad (2.21)$$

Accordingly, we will optimize λ_i , instead of x_i , for $i \in \{1, 2, \dots, N\}$, by transforming the model into the following form.

$$(\check{P}_0) \quad \max_{\vec{\lambda}, K, Q} \quad \check{\Pi}_0(\vec{\lambda}, K, Q) = Qs - \sum_{i=1}^N (p_i + \rho\beta_i) \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_i}{\alpha_i}} - \frac{c_K \gamma'}{\gamma'' + 1} \left(K^{(\gamma''+1)} - K_0^{(\gamma''+1)}\right), \quad (2.22)$$

$$\text{s.t.} \quad \sum_{j=1}^N \lambda_j = 1, \quad (2.23)$$

$$K \geq K_0, \quad (2.24)$$

$$Q \in [Q_l, Q_h], \quad (2.25)$$

$$\lambda_i \in (0, 1) \quad \text{for } i \in \{1, 2, \dots, N\}, \quad (2.26)$$

where, $\check{\Pi}_0(\vec{\lambda}, K, Q)$ is the firm's profit, Equation (2.23) is the constraint that maps λ_i to x_i for $i \in \{1, 2, \dots, N\}$, Constraint (2.24) ensures that selected technology level does not fall below its initial level, Constraint (2.25) ensures that production quantity is in between the maximum and minimum contract levels, Expression

(2.26) ensures the non-negativity of the amount of each input.

In order to solve this problem we relax Constraint (2.23) via the Lagrange multiplier w . Thus, problem $(\check{P}_0)$ transforms into

$$(\check{P}) \min_w \max_{\vec{\lambda}, K, Q} \check{\Pi}(\vec{\lambda}, w, K, Q) = Qs - \sum_{i=1}^N (p_i + \rho \beta_i) \left(\frac{Q}{aK^{\alpha_K}} \right)^{\frac{\lambda_i}{\alpha_i}} - \mathcal{L}(\vec{\lambda}, w, K, Q) \\ - \frac{c_K \gamma'}{\gamma'' + 1} \left(K^{(\gamma'' + 1)} - K_0^{(\gamma'' + 1)} \right), \quad (2.27)$$

$$s.t. \quad K \geq K_0, \quad (2.28)$$

$$Q \in [Q_l, Q_h], \quad (2.29)$$

$$0 \leq x_i \quad for \quad i \in \{1, 2, \dots, N\}, \quad (2.30)$$

where, $\check{\Pi}(\vec{\lambda}, w, K, Q)$ is the firm's profit, $\mathcal{L}(\vec{\lambda}, w, K, Q) = w(1 - \sum_{j=1}^N \lambda_j)$ is the Lagrangean ensuring that $\vec{x}$ is represented by $\vec{\lambda}$, Constraint (2.28) ensures that selected technology level is not less than its initial level, Constraint (2.29) ensures that production quantity is between the maximum and the minimum contract levels, and Constraint (2.30) ensures the non-negativity of the amount of each input.

Let $\check{\Gamma}(\vec{\lambda}, w, K, Q)$ be the total cost function which can be expressed as

$$\check{\Gamma}(\vec{\lambda}, w, K, Q) = c_p(\vec{\lambda}, w) + c_e(\vec{\lambda}, w) + \mathcal{L}(\vec{\lambda}, w, K, Q). \quad (2.31)$$

Then, following lemma provides the structural properties of $\check{\Gamma}(\vec{\lambda}, w, K, Q)$.

Lemma 2.1. $\check{\Gamma}(\vec{\lambda}, w, K, Q)$ is jointly convex in λ_i 's ($i \in \{1, 2, \dots, N\}$) for given w , K , and Q .

Proof. Proved in Appendix A by inspecting the principle minors of the Hessian matrix of $\check{\Gamma}(\vec{\lambda}, w, K, Q)$ with respect to λ_i 's ($i \in \{1, 2, \dots, N\}$). $\square$

Accordingly, the solution procedure can be presented as follows.

STEP (1) $\check{\Gamma}(\vec{\lambda}, w|K, Q)$, which represents $\check{\Gamma}(\vec{\lambda}, w, K, Q)$ for given K and Q , is minimized over λ_i ($i \in \{1, 2, \dots, N\}$). Resulting optimal input level, denoted by $\lambda_i^*(w|K, Q)$,

is substituted for λ_i in $\check{\Gamma}(\vec{\lambda}, w|K, Q)$ to obtain $\check{\Gamma}^*(w|K, Q)$, which is maximized over Lagrange multiplier w to find $w^*(K, Q)$. Eventually, $w^*(K, Q)$ is substituted for w in $\check{\Gamma}^*(w|K, Q)$ and, finding $\lambda_i^*(K, Q)$ and transforming it to $x_i^*(K, Q)$, the minimum cost of producing Q units using $x_i^*(K, Q)$ units of input i under given technology level, $\check{\Gamma}^*(K, Q)$, is found.

STEP (2) For given Q , $\Gamma(K|Q) = \check{\Gamma}^*(K, Q) + c_T(K)$ subjected to Constraint (2.28) is minimized and optimal technology level for given production quantity, $K^*(Q)$, is found. Then, it is substituted for K in $\Gamma(K|Q)$ and the minimum cost of producing Q units of product, $\Gamma^*(Q)$, is obtained.

STEP (3) $\Pi(Q) = Qs - \Gamma^*(Q)$ subjected to Constraint (2.29) is maximized to find optimal production quantity, Q^* , and profit level, Π^* .

2.4.1 Optimal Input Mix

We first consider the optimal input mix through solving the following problem.

$$(P_1) \quad \max_w \min_{\vec{\lambda}} \quad \check{\Gamma}(\vec{\lambda}, w|K, Q) = \sum_{i=1}^N (p_i + \rho\beta_i) \left(\frac{Q}{aK^{\alpha_K}} \right)^{\frac{\lambda_i}{\alpha_i}} + \mathcal{L}(\vec{\lambda}, w|K, Q), \quad (2.32)$$

$$s.t. \quad 0 \leq x_i \quad \text{for} \quad i \in \{1, 2, \dots, N\}, \quad (2.33)$$

where, $\check{\Gamma}(\vec{\lambda}, w|K, Q)$ is the cost of producing Q units of product under given technology level K , $\mathcal{L}(\vec{\lambda}, w|K, Q) = w \left(1 - \sum_{j=1}^N \lambda_j \right)$ is the Lagrangean ensuring that Constraint (2.23) is satisfied, Expression (2.33) ensures that the amount of each input is non-negative.

Note that $\check{\Gamma}(\vec{\lambda}, w|K, Q)$ is jointly convex in λ_i 's for $i \in \{1, 2, \dots, N\}$. Thus, because of the properties of Lagrange Multipliers method, while $\check{\Gamma}(\vec{\lambda}, w|K, Q) = c_p(\vec{\lambda}) + c_e(\vec{\lambda}) + \mathcal{L}(\vec{\lambda}, w|K, Q)$ subjected to (2.30) is minimized over λ_i 's ($i \in \{1, 2, \dots, N\}$), being concave in w , it is maximized over the Lagrange Multiplier, w . Thus, first order conditions of $\check{\Gamma}(\vec{\lambda}, w|K, Q)$ with respect to λ_i yields the minimizing amounts of inputs, $\lambda_i^*(K, Q)$ ($x_i^*(K, Q)$), and its first order conditions with respect to w reveals the maximizing value of the Lagrange Multiplier, $w^*(K, Q)$.

Theorem 2.1. $\check{\Gamma}(\vec{\lambda}^*, w, K, Q)$ is concave in w for given K and Q and the optimal values

of the Lagrange variable, $w^*(K, Q)$, input quantities, $x_i^*(K, Q)$'s, and the emission level, $em^*(K, Q)$, are

$$w^*(K, Q) = \ln\left(\frac{Q}{aK^{\alpha_K}}\right) \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}, \quad (2.34)$$

$$x_i^*(K, Q) = \frac{\alpha_i}{p_i + \rho\beta_i} \left(\frac{Q}{aK^*(Q)^{\alpha_K}}\right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}, \quad (2.35)$$

$$em^*(K, Q) = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho\beta_i} \left(\frac{Q}{aK^*(Q)^{\alpha_K}}\right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}. \quad (2.36)$$

Proof. The first order condition of $\check{\Gamma}(\vec{\lambda}, w|K, Q)$ with respect to λ_i is

$$\frac{\partial \check{\Gamma}(\vec{\lambda}, w|K, Q)}{\partial \lambda_i} = \frac{p_i + \rho\beta_i}{\alpha_i} \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_i}{\alpha_i}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right) - w.$$

Hence, $x_i^*(\vec{\lambda}^*, w|K, Q)$ can be written as

$$x_i^*(\vec{\lambda}^*, w|K, Q) = \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_i^*}{\alpha_i}} = \frac{w}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{\alpha_i}{p_i + \rho\beta_i}. \quad (2.37)$$

Note that natural logarithm of the both sides of Equation (2.37) yields

$$\frac{\lambda_i^*(w|K, Q)}{\alpha_i} \ln\left(\frac{Q}{aK^{\alpha_K}}\right) = \ln\left(\frac{w}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{\alpha_i}{p_i + \rho\beta_i}\right).$$

Hence, we have

$$\lambda_i^*(w|K, Q) = \frac{\alpha_i}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \ln\left(\frac{w}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{\alpha_i}{p_i + \rho\beta_i}\right). \quad (2.38)$$

Then, substituting $\lambda_i^*(w|K, Q)$ for λ_i in (P_1) we get the following optimization

problem:

$$(\check{P}_1) \quad \max_w \quad \check{\Gamma}(\vec{\lambda}^*, w|K, Q) = \frac{rw}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} + w \left(1 - \sum_{j=1}^N \lambda_j^*\right), \quad (2.39)$$

$$s.t. \quad 0 \leq x_i \quad for \quad i \in \{1, 2, \dots, N\}. \quad (2.40)$$

The first and the second order derivatives of $\check{\Gamma}(\vec{\lambda}^*, w|K, Q)$ with respect to w , $\frac{\partial \check{\Gamma}^*(w|K, Q)}{\partial w}$ and $\frac{\partial^2 \check{\Gamma}^*(w|K, Q)}{\partial w^2}$, are

$$\begin{aligned} \frac{\partial \check{\Gamma}^*(w|K, Q)}{\partial w} &= \frac{r}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} + 1 - \sum_{j=1}^N \lambda_j^* - w \sum_{j=1}^N \frac{\partial \lambda_j^*}{\partial w}, \\ \frac{\partial^2 \check{\Gamma}^*(w|K, Q)}{\partial w^2} &= - \sum_{j=1}^N \frac{\partial \lambda_j^*}{\partial w} - \sum_{j=1}^N \frac{\partial \lambda_j^*}{\partial w} - w \sum_{j=1}^N \frac{\partial^2 \lambda_j^*}{\partial w^2}, \end{aligned}$$

Then, since first and second order derivatives of λ_i^* for $i \in \{1, 2, \dots, N\}$ with respect to w , are

$$\frac{\partial \lambda_i^*(w|K, Q)}{\partial w} = \frac{\alpha_i}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{1}{w},$$

and

$$\frac{\partial^2 \lambda_i^*(w|K, Q)}{\partial w^2} = - \frac{\alpha_i}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{1}{w^2},$$

respectively, we have

$$\begin{aligned} \frac{\partial \check{\Gamma}^*(w|K, Q)}{\partial w} &= \frac{r}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} + 1 - \sum_{j=1}^N \lambda_j^* - w \sum_{j=1}^N \frac{\alpha_j}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{1}{w}, \\ &= 1 - \sum_{j=1}^N \lambda_j^*, \end{aligned} \quad (2.41)$$

and

$$\frac{\partial^2 \check{\Gamma}^*(w|K, Q)}{\partial w^2} = - \sum_{j=1}^N \frac{\partial \lambda_j^*}{\partial w} = - \sum_{j=1}^N \frac{\alpha_j}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{1}{w}. \quad (2.42)$$

Note that $\frac{\partial^2 \check{\Gamma}^*(w|K, Q)}{\partial w^2} < 0$ for $((aK^{\alpha_K} < Q) \wedge (w > 0)) \vee ((aK^{\alpha_K} > Q) \wedge (w < 0))$. Furthermore, since $x_i \geq 0$, we know, from Equation (2.37), that if $w > 0$, then $aK^{\alpha_K} < Q$; and if $w < 0$, vice versa. Hence, $\check{\Gamma}^*(w|K, Q)$ is concave in w , as expected from the properties of Lagrange Multipliers method.

Then, equating right hand side of Equation (2.41) to zero, we see that the Cobb Douglas production function constraint holds at optimality as below.

$$\begin{aligned} \prod_{i=1}^N x_i^*(\vec{\lambda}^*, w|K, Q)^{\alpha_i} &= \prod_{i=1}^N \left(\frac{Q}{aK^{\alpha_K}} \right)^{\lambda_i^*} = \left(\frac{Q}{aK^{\alpha_K}} \right)^{\sum_{j=1}^N \lambda_j^*} = \left(\frac{Q}{aK^{\alpha_K}} \right)^1, \\ \Rightarrow \frac{Q}{aK^{\alpha_K}} &= \prod_{i=1}^N \left(\frac{w}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \frac{\alpha_i}{p_i + \rho\beta_i} \right)^{\alpha_i} \\ &= \left(\frac{w}{\ln\left(\frac{Q}{aK^{\alpha_K}}\right)} \right)^r \prod_{i=1}^N \left(\frac{\alpha_i}{p_i + \rho\beta_i} \right)^{\alpha_i}. \end{aligned} \quad (2.43)$$

Then, the optimal value of w , for given K and Q is obtained as

$$w^*(K, Q) = \ln\left(\frac{Q}{aK^{\alpha_K}}\right) \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}.$$

Substituting $w^*(K, Q)$ for w in equation (2.37), we obtain the optimal value of input quantity x_i such that $i \in \{1, 2, \dots, N\}$ for given production quantity, Q , and technology level, K , as

$$x_i^*(K, Q) = \frac{\alpha_i}{p_i + \rho\beta_i} \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}.$$

Accordingly, the total emission with optimal input usage is

$$em^*(K, Q) = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho\beta_i} \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r}}.$$

This completes the proof. $\square$

2.4.2 Optimal Technology Level

Having found the optimal input mix that minimizes the cost of producing Q units with a given technology, we are ready to minimize the total cost of technology acquisition and producing Q units by using the optimized amounts of the production inputs.

Substituting $w^*(K, Q)$ for w in $\check{\Gamma}(\vec{\lambda}^*, w|K, Q)$, we obtain

$$\check{\Gamma}(\vec{\lambda}^*, w^*|K, Q) = r \left(\frac{Q}{aK^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}}. \quad (2.44)$$

Then, the effective cost of producing Q units with the acquired technology level, K , is $\check{\Gamma}(K|Q) = \check{\Gamma}(\vec{\lambda}^*, w^*|K, Q) + c_T(K)$. The optimal technology level, $K^*(Q)$, is then found via solving the following problem.

$$(P_2) \quad \min_K \quad \check{\Gamma}(K|Q) = r \left(\frac{Q}{aK^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} + \frac{\gamma' c_K}{\gamma'' + 1} (K^{\gamma''+1} - K_0^{\gamma''+1}), \quad (2.45)$$

$$s.t. \quad K \geq K_0. \quad (2.46)$$

Lemma 2.2. $\check{\Gamma}(K|Q)$ is convex in technology level, K , $\forall K \geq 0$.

Proof. Convexity of $\check{\Gamma}(K|Q)$ in K is proved in Appendix A. $\square$

Let $K^*(Q)$ be the optimal technology level for a given production quantity, Q , which is produced using the each production input i , for $i \in \{1, 2, \dots, N\}$, as defined by Expression (2.35).

Theorem 2.2. The unique optimal technology level for given Q is

$$K^*(Q) = \begin{cases} \hat{K}(Q), & \text{if } Q \geq Q_{int}, \\ K_0, & \text{otherwise,} \end{cases} \quad (2.47)$$

where

$$\hat{K}(Q) = \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}}, \quad (2.48)$$

is an increasing function of Q , and

$$Q_{int} = \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^r a K_0^{r(\gamma''+1)+\alpha_K} \right] / \left[\prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]. \quad (2.49)$$

Proof. The first order condition of $\check{\Gamma}(K|Q)$, which is defined by Expression (2.45), with respect to K gives the critical technology level for a given production quantity, $\hat{K}(Q)$, as

$$\frac{\partial \check{\Gamma}(K|Q)}{\partial K} = -\alpha_K K^{-\frac{\alpha_K}{r}-1} \left(\frac{Q}{a} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} + \gamma' c_K K^{\gamma''} = 0,$$

$$\hat{K}(Q)^{\gamma''+1+\frac{\alpha_K}{r}} = \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r}},$$

$$\hat{K}(Q) = \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}}.$$

Note that by Lemma 2.2, $\check{\Gamma}(K|Q)$ is convex in K . Hence, $\hat{K}(Q)$ would be the optimal technology level for given Q if technology level were not constrained from below. However, it should be greater than or equal to its initial level, K_0 . Furthermore, since $\check{\Gamma}(K|Q)$ is convex in K , it is convex increasing in K for $\hat{K}(Q) < K_0 < K$. Moreover, first order derivative of $\hat{K}(Q)$ with respect to Q is

$$\frac{\partial \hat{K}(Q)}{\partial Q} = \frac{1}{r(\gamma''+1)+\alpha_K} Q^{-1} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}} > 0, \quad (2.50)$$

which means that $\hat{K}(Q)$ is monotonic increasing in Q . Hence, if $\hat{K}(Q) < K_0$, then $K^*(Q) = K_0$; otherwise, $K^*(Q) = \hat{K}(Q)$.

Letting Q_{int} be the technology level that solves $\hat{K}(Q) = K_0$, it can be expressed as

$$Q_{int} = \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^r K_0^{r(\gamma''+1)+\alpha_K} \right] / \left[\prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right].$$

Then, since, we know that $\hat{K}(Q)$ is monotonic increasing in Q from Inequality (2.50), the optimal technology level for given Q , is K_0 for $Q \leq Q_{int}$, and it is equal to $\hat{K}(Q)$ for $Q_{int} \leq Q$. $\square$

2.4.3 Representation of the Problem

Having found the values of input quantities and technology level that minimize the cost of producing Q units, we are now ready to maximize the total profit. We obtain the objective function of this maximization problem by subtracting the minimized cost for given Q from the firm's revenue as follows.

$$\begin{aligned} \Pi(Q) &= Qs - \Gamma^*(Q) \\ &= Qs - r \left(\frac{Q}{aK^*(Q)^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} - \frac{\gamma' c_K}{\gamma'' + 1} \left(K^*(Q)^{\gamma''+1} - K_0^{\gamma''+1} \right). \end{aligned} \quad (2.51)$$

Note that the conditions for concavity of $\Pi(Q)$ in Q are different for $\Pi(Q|K^*(Q) = K_0)$ and $\Pi(Q|K^*(Q) = \hat{K}(Q))$ since they are different functions. These functions, denoted by $\Pi_0(Q)$ and $\Pi_v(Q)$, can be expressed as

$$\Pi_0(Q) = Qs - r \left(\frac{Q}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} \quad (2.52)$$

and

$$\begin{aligned} \Pi_v(Q) &= Qs - \left(\frac{\gamma' c_K (r(\gamma''+1) + \alpha_K)}{\alpha_K (\gamma''+1)} \right) \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \\ &\quad + \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1}, \end{aligned} \quad (2.53)$$

respectively. Accordingly, the following proposition provides an important relationship between $\Pi_0(Q)$ and $\Pi_v(Q)$.

Proposition 2.2. *The difference, $\Pi_v(Q) - \Pi_0(Q)$, having a unique positive root that is equal to Q_{int} ; is decreasing in Q for $Q < Q_{int}$; is equal to zero for $Q = Q_{int}$; and is increasing in Q for $Q_{int} < Q$.*

Proof. The proof is provided in Appendix A. $\square$

Since $K^*(Q) = K_0$ for $Q < Q_{int}$ and $K^*(Q) = \hat{K}(Q)$ for $Q \geq Q_{int}$ from Theorem 2.2, the optimal production quantity of the problem (P) can be equivalently obtained by solving the following problem:

$$(P_Q) \quad \max_Q \quad \Pi(Q) = \mathbf{1}_{\{Q < Q_{int}\}} \Pi_0(Q) + \mathbf{1}_{\{Q \geq Q_{int}\}} \Pi_v(Q), \quad (2.54)$$

$$s.t. \quad Q \in [Q_l, Q_h]. \quad (2.55)$$

The following Lemma provides the structural properties of the problem (P_Q) and critical values of $\Pi(Q)$.

Lemma 2.3.

1. $\Pi_0(Q)$ is **concave** in Q for $r < 1$, it is **linear** in Q for $r = 1$, and, it is **convex** in Q for $r > 1$.
2. $\Pi_0(Q)$ has a unique critical point, Q_0 , which is given as

$$Q_0 = \left[\frac{1}{s^r a K_0^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r-1}}. \quad (2.56)$$

3. $\Pi_v(Q)$ is **concave** in Q for $r < 1 - \frac{\alpha_K}{\gamma''+1}$, it is **linear** in Q for $r = 1 - \frac{\alpha_K}{\gamma''+1}$, and, it is **convex** in Q for $r > 1 - \frac{\alpha_K}{\gamma''+1}$.
4. $\Pi_v(Q)$ has a unique critical point, Q_v , which is given as

$$Q_v = \left[\left(\frac{1}{s} \right)^{r(\gamma''+1)+\alpha_K} \left(\frac{\gamma' c_K}{\alpha_K} \right)^{\alpha_K} \left[\frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\gamma''+1} \right]^{\frac{1}{(r-1)(\gamma''+1)+\alpha_K}}. \quad (2.57)$$

Proof. The proof is provided in Appendix A. $\square$

Considering that these structural differences resulting from different scale elasticity levels will lead to different optimality results, we define five types of manufacturers as follows.

Definition 2.1. *The manufacturer has a **low** elasticity of scale if $r < 1 - \frac{\alpha_K}{\gamma''+1}$, **mid-borderline** elasticity of scale if $r = 1 - \frac{\alpha_K}{\gamma''+1}$, **mid** elasticity of scale if $1 - \frac{\alpha_K}{\gamma''+1} < r < 1$, **high-borderline** elasticity of scale if $r = 1$, and **high** elasticity level if $r > 1$.*

In order to solve Problem P_Q we need to know the order relations among Q_{int} , Q_0 , and Q_v . To understand this, suppose that $Q_0 < Q_{int} < Q_v$ for $r < 1 - \frac{\alpha_K}{\gamma''+1}$. In this case, since, from Lemma 2.3, $\Pi_0(Q)$ and $\Pi_v(Q)$ are both concave in Q , $\Pi(Q)$ should be increasing in Q up to Q_0 ; decreasing between Q_0 and Q_{int} , increasing between Q_{int} and Q_v , and decreasing afterwards. Therefore, if the relation $Q_0 < Q_{int} < Q_v$ was possible, then both Q_0 and Q_v could be the local maximizers of the problem. However, we found out that if $Q_v > Q_{int}$, there is only one possible order relation, that is, $Q_{int} < Q_0 < Q_v$. Hence, $\Pi(Q)$, being strictly concave in $Q \forall Q > 0$, has a unique global maximizer. Therefore, in the following section we present all of the possible order relations and solve the problem accordingly.

2.4.4 Optimal Policy of the Firm

We introduce a composite entity to ease the analysis of the effect of carbon taxes.

Definition 2.2.

$$Z = \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j}. \quad (2.58)$$

Note that for $Z = 0$, $\rho < 0$. Hence, since carbon tax can not be negative, we define the minimum value of Z under which Z is undefined, as follows.

Definition 2.3.

$$Z_{min} = \prod_{j=1}^N \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j}. \quad (2.59)$$

Lemma 2.4. For $Z \geq Z_{min}$, letting $Q_f \in \{Q_l, Q_h\}$, the behavior of Q_{int} , Q_0 , Q_v , $\hat{K}(Q_v)$ and $\hat{K}(Q_f)$ with respect to ρ is identical to their behavior with respect to Z .

It is found that Q_{int} , Q_0 , and Q_v are equal to each other at a unique Z value which is defined as follows.

Definition 2.4. Z_{int} is the value of Z that solves 1) $Q_{int} = Q_0$ for $r = 1 - \frac{\alpha_K}{\gamma''+1}$; 2) $Q_v = Q_{int}$ for $r = 1$; 3) $Q_{int} = Q_0 = Q_v$ for other positive values of r .

Using Z_{int} as a proxy we provide the order relations among Q_{int} , Q_0 , and Q_v through the following lemma.

Proposition 2.3.

- If $0 < r < 1 - \alpha_K/(\gamma'' + 1)$, then $Q_{int} < Q_0 < Q_v$ for $Z < Z_{int}$; $Q_{int} = Q_0 = Q_v$ for $Z = Z_{int}$; $Q_v < Q_0 < Q_{int}$ for $Z > Z_{int}$.
- If $0 < r = 1 - \alpha_K/(\gamma'' + 1)$, then $Q_{int} < Q_0$ for $Z < Z_{int}$; $Q_{int} = Q_0$ for $Z = Z_{int}$; $Q_0 < Q_{int}$ for $Z > Z_{int}$.
- If $0 < 1 - \alpha_K/(\gamma'' + 1) < r < 1$, then $Q_v < Q_{int} < Q_0$ for $Z < Z_{int}$; $Q_v = Q_{int} = Q_0$ for $Z = Z_{int}$; $Q_0 < Q_{int} < Q_v$ for $Z > Z_{int}$.
- If $r = 1$, then $Q_v < Q_{int}$ for $Z < Z_{int}$; $Q_v = Q_{int}$ for $Z = Z_{int}$; $Q_{int} < Q_v$ for $Z > Z_{int}$.
- If $r > 1$, then $Q_0 < Q_v < Q_{int}$ for $Z < Z_{int}$; $Q_0 = Q_v = Q_{int}$ for $Z = Z_{int}$; $Q_{int} < Q_v < Q_0$ for $Z > Z_{int}$.

Proof. The proof is provided in Appendix A. □

The following corollary based on Lemma 2.3 and Proposition 2.3, provides the structure of $\Pi(Q)$ for each possible order relation among Q_{int} , Q_0 , and Q_v , referred by the location of Z with respect to Z_{int} , for different scale elasticity levels.

Corollary 2.1. The structural properties of $\Pi(Q)$ for each possible order relation among Q_{int} , Q_0 , and Q_v , which can be represented by the order between Z and Z_{int} for different scale elasticity levels are presented below.

- If $0 < r < 1 - \frac{\alpha_K}{\gamma''+1}$, $\Pi(Q)$ is strictly concave in Q and it attains its maximum at Q_v for $Z < Z_{int}$; at Q_{int} for $Z = Z_{int}$; and at Q_0 for $Z > Z_{int}$,
- If $0 < r = 1 - \frac{\alpha_K}{\gamma''+1}$, $\Pi(Q)$ is concave increasing in Q for $Z < Z_{int}$; concave increasing in Q for $Q < Q_{int} = Q_0$ at $Z = Z_{int}$ and constant in Q for $Q \geq Q_{int} = Q_0$ at $Z = Z_{int}$; strictly concave in Q with a local maximum at Q_0 for $Z > Z_{int}$.
- If $0 < 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$, $\Pi(Q)$ is increasing in Q for $Z \leq Z_{int}$; concave-convex in Q with a local maximum at Q_0 for $Z > Z_{int}$.
- If $r = 1$, $\hat{\Pi}_0(Q)$ is convex increasing in Q for $Z < Z_{int}$; constant in Q for $Q < Q_{int} = Q_v$ at $Z = Z_{int}$, strictly convex and increasing in Q for $Q_{int} = Q_v < Q$ at $Z = Z_{int}$; convex in Q with a minimum at Q_v for $Z > Z_{int}$.
- If $r > 1$, $\hat{\Pi}(Q)$ is strictly convex in Q . It is minimized at Q_0 for $Z < Z_{int}$; at Q_{int} for $Z = Z_{int}$; at Q_v for $Z > Z_{int}$.

Knowing the order relations and the structure of the objective function we are ready to provide the optimal policy through the following theorem.

Theorem 2.3. *The optimal production quantity, Q^* , and technology level, K^* , are provided below.*

For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$,

1. if $Q_{int} < Q_0 < Q_v$ and
 - (a) $Q_h \leq Q_v$, then
 - i. for $Q_h \leq Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$;
 - ii. for $Q_h \geq Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - (b) $Q_l \leq Q_v \leq Q_h$, then $Q^* = Q_v$ and $K^* = \hat{K}(Q_v)$,
 - (c) $Q_v \leq Q_l$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$,
2. if $Q_{int} = Q_0 = Q_v$ and
 - (a) $Q_h \leq Q_{int}$, then $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Q_l \leq Q_{int} \leq Q_h$, then $Q^* = Q_{int} = Q_0 = Q_v$ and $K^* = \hat{K}(Q_{int}) = K_0$,
 - (c) $Z_{l,int} \leq Z$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$,

3. if $Q_v < Q_0 < Q_{int}$ and
 - (a) $Q_h \leq Q_0$, then $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Q_l \leq Q_0 \leq Q_h$, then $Q^* = Q_0$ and $K^* = K_0$,
 - (c) $Q_0 \leq Q_l$, then
 - i. for $Q_l \leq Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$;
 - ii. for $Q_l \geq Q_{int}$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $0 < r = 1 - \alpha_K/(\gamma'' + 1)$,

1. if $Q_{int} < Q_0$ and
 - (a) $Q_h \leq Q_{int}$, then $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Q_h \geq Q_{int}$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
2. if $Q_{int} = Q_0$ and
 - (a) $Q_h \leq Q_0$, then $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Q_l \leq Q_0 \leq Q_h$, then $Q^* \in [Q_{int}, Q_h]$ and $K^* = \hat{K}(Q^*)$,
 - (c) $Q_0 \leq Q_l$, then $Q^* \in [Q_l, Q_h]$ and $K^* = \hat{K}(Q^*)$,
3. if $Q_0 < Q_{int}$ and
 - (a) $Q_h \leq Q_0$, then $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Q_l \leq Q_0 \leq Q_h$, then $Q^* = Q_0$ and $K^* = K_0$,
 - (c) $Q_0 \leq Q_l < Q_{int}$, then $Q^* = Q_l$ and $K^* = K_0$,
 - (d) $Q_{int} \leq Q_l$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $0 < 1 - \alpha_K/(\gamma'' + 1) < r < 1$,

1. if $Q_v < Q_{int} < Q_0$, then
 - (a) for $Q_h \leq Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) for $Q_h \geq Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
2. if $Q_v = Q_{int} = Q_0$, then
 - (a) for $Q_h \leq Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,

(b) for $Q_h \geq Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.

3. if $Q_0 < Q_{int} < Q_v$ and

(a) $Q_h \leq Q_{int}$, then

i. for $Q_h \leq Q_0$, $Q^* = Q_h$ and $K^* = K_0$,

ii. for $Q_l \leq Q_0 \leq Q_h$, $Q^* = Q_0$ and $K^* = K_0$,

iii. for $Q_0 \leq Q_l$, $Q^* = Q_l$ and $K^* = K_0$.

(b) $Q_l < Q_0 < Q_{int} \leq Q_h$, then

i. for $\Pi_0(Q_0) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. for $\Pi_0(Q_0) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_0 \text{ and } K^* = K_0)$,

iii. for $\Pi_0(Q_0) > \Pi_v(Q_h)$, $Q^* = Q_0$ and $K^* = K_0$.

(c) $Q_0 \leq Q_l < Q_{int} \leq Q_h$, then

i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.

(d) $Q_{int} \leq Q_l$, then

i. for $\Pi_v(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. for $\Pi_v(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,

iii. for $\Pi_v(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r = 1$,

1. if $Q_v < Q_{int}$, then

(a) for $Q_h \leq Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,

(b) for $Q_h \geq Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.

2. if $Q_v = Q_{int}$ and

(a) if $Q_h \leq Q_{int}$, then $Q^* \in [Q_l, Q_h]$ and $K^* = K_0$,

(b) if $Q_h \geq Q_{int}$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.

3. if $Q_{int} < Q_v$ and

(a) $Q_h < Q_{int}$, then $Q^* = Q_l$ and $K^* = K_0$,

(b) $Q_l < Q_{int} \leq Q_h$, then

i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = K_0$),

iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.

(c) $Q_{int} \leq Q_l$, then

i. for $\Pi_v(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. for $\Pi_v(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$),

iii. for $\Pi_v(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r > 1$,

1. if $Q_0 < Q_v < Q_{int}$ and

(a) if $Q_h < Q_{int}$, then

i. for $\Pi_0(Q_l) < \Pi_0(Q_h)$, $Q^* = Q_h$ and $K^* = K_0$,

ii. for $\Pi_0(Q_l) = \Pi_0(Q_h)$, ($Q^* = Q_h$ and $K^* = K_0$) or ($Q^* = Q_l$ and $K^* = K_0$),

iii. for $\Pi_0(Q_l) > \Pi_0(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.

(b) if $Q_l < Q_{int} \leq Q_h$, then

i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = K_0$),

iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.

- (c) if $Q_{int} \leq Q_l$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
- 2. if $Q_0 = Q_v = Q_{int}$ and
 - (a) if $Q_h < Q_{int}$, then $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Q_l < Q_{int} \leq Q_h$, then
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = K_0$),
 - iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Q_{int} \leq Q_l$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
- 3. if $Q_{int} < Q_v < Q_0$ and
 - (a) if $Q_h < Q_{int}$, then $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Q_l < Q_{int} \leq Q_h$, then
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = K_0$),
 - iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Q_{int} \leq Q_l$, then
 - i. for $\Pi_v(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_v(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$),
 - iii. for $\Pi_v(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

Proof. The proof is provided in Appendix A. □

Accordingly, the optimal policy composed of production quantity-technology level pairs can be presented more compactly in the following tables.

Table 2.2: The Optimal Policy under Conditions on Q Benchmarks for $r < 1$.

$0 < r < 1 - \frac{\alpha K}{\gamma'' + 1}$				
If $Q_{int} < Q_0 < Q_v$, then				
for	$Q_h < Q_{int}$	$Q_{int} \leq Q_h < Q_v$	$Q_l \leq Q_v < Q_h$	$Q_v \leq Q_l$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_v, \hat{K}(Q_v))$	$(Q_l, \hat{K}(Q_l))$
If $Q_{int} = Q_0 = Q_v$, then				
for	$Q_h < Q_{int}$	$Q_l \leq Q_{int} < Q_h$	$Q_{int} \leq Q_l$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	$(Q_l, \hat{K}(Q_l))$	
If $Q_v < Q_0 < Q_{int}$, then				
for	$Q_h < Q_0$	$Q_l \leq Q_0 < Q_h$	$Q_0 \leq Q_l < Q_{int}$	$Q_{int} < Q_l$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$
$0 < r = 1 - \frac{\alpha K}{\gamma'' + 1} < 1$				
If $Q_{int} < Q_0$, then				
for	$Q_h < Q_{int}$	$Q_{int} \leq Q_h$		
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$		
If $Q_{int} = Q_0$, then				
for	$Q_h < Q_{int}$	$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q, \hat{K}(Q))$	$(Q, \hat{K}(Q))$	
		$Q \in [Q_{int}, Q_h]$	$Q \in [Q_l, Q_h]$	
If $Q_{int} > Q_0$, then				
for	$Q_h < Q_0$	$Q_l < Q_0 \leq Q_h$	$Q_0 \leq Q_l < Q_{int}$	$Q_{int} \leq Q_l$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$
$0 < 1 - \frac{\alpha K}{\gamma'' + 1} < r < 1$				
If $Q_v < Q_{int} < Q_0$, then				
for	$Q_h < Q_{int}$	$Q_h = Q_{int}$	$Q_{int} < Q_h$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	
If $Q_v = Q_{int} = Q_0$, then				
for	$Q_h < Q_{int}$	$Q_h = Q_{int}$	$Q_{int} < Q_h$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	$(Q_h, \hat{K}(Q_h))$	
If $Q_0 < Q_{int} < Q_v$, then				
for	$Q_h < Q_{int}$ and	$Q_l < Q_{int} \leq Q_h$	$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
	$Q_h < Q_0$	and $Q_l < Q_0$	and $Q_0 \leq Q_l$	
if		$\Pi_0(Q_0) < \Pi_v(Q_h)$	$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_v(Q_l) < \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for	$Q_h < Q_{int}$ and	$Q_l < Q_{int} \leq Q_h$	$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
	$Q_l < Q_0 \leq Q_h$	and $Q_l < Q_0$	and $Q_0 \leq Q_l$	
if		$\Pi_0(Q_0) = \Pi_v(Q_h)$	$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_v(Q_l) = \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_0, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
		or (Q_0, K_0)	or (Q_l, K_0)	or $(Q_l, \hat{K}(Q_l))$
for	$Q_h < Q_{int}$ and	$Q_l < Q_{int} \leq Q_h$	$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
	$Q_0 \leq Q_l$	and $Q_l < Q_0$	and $Q_0 \leq Q_l$	
if		$\Pi_0(Q_0) > \Pi_v(Q_h)$	$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_v(Q_l) > \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_l, K_0)	(Q_0, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$

Table 2.3: The Optimal Policy under Conditions on Q Benchmarks for $r \geq 1$.

$r = 1$			
If $Q_v < Q_{int}$, then			
for	$Q_h < Q_{int}$	$Q_h = Q_{int}$	$Q_h > Q_{int}$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_h, K_0) or $(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
If $Q_v = Q_{int}$, then			
for	$Q_h < Q_{int}$	$Q_h = Q_{int}$	$Q_h > Q_{int}$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0) or $(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
If $Q_v > Q_{int}$, then			
for	$Q_h < Q_{int}$	$Q_l < Q_{int} \leq Q_h$ and	$Q_{int} \leq Q_l$ and
if		$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_v(Q_l) < \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for		$Q_l < Q_{int} \leq Q_h$ and	$Q_{int} \leq Q_l$ and
if		$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_v(Q_l) = \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$		$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$
for		$Q_l < Q_{int} \leq Q_h$ and	$Q_{int} \leq Q_l$ and
if		$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_v(Q_l) > \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$		(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$
$r > 1$			
If $Q_v < Q_{int}$, then			
for	$Q_h < Q_{int}$ and	$Q_l < Q_{int} \leq Q_h$ and	$Q_{int} \leq Q_l$
if	$\Pi_0(Q_l) < \Pi_0(Q_h)$	$\Pi_0(Q_l) < \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for	$Q_h < Q_{int}$ and	$Q_l < Q_{int} \leq Q_h$ and	
if	$\Pi_0(Q_l) = \Pi_0(Q_h)$	$\Pi_0(Q_l) = \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0) or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	
for	$Q_h < Q_{int}$ and	$Q_l < Q_{int} \leq Q_h$ and	
if	$\Pi_0(Q_l) > \Pi_0(Q_h)$	$\Pi_0(Q_l) > \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	(Q_l, K_0)	
If $Q_v = Q_{int}$, then			
for	$Q_h < Q_{int}$	$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
if		$\Pi_0(Q_l) < \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for		$Q_l < Q_{int} \leq Q_h$	
if		$\Pi_0(Q_l) = \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$		$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	
for		$Q_l < Q_{int} \leq Q_h$	
if		$\Pi_0(Q_l) > \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$		(Q_l, K_0)	
If $Q_v > Q_{int}$, then			
for	$Q_h < Q_{int}$	$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
if		$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_v(Q_l) < \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for		$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
if		$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_v(Q_l) = \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$		$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$
for		$Q_l < Q_{int} \leq Q_h$	$Q_{int} \leq Q_l$
if		$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_v(Q_l) > \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$		(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$

The optimal policy of the manufacturer with low elasticity of scale can be rewritten more compactly in the following corollary.

Corollary 2.2. *The optimal production policy of the manufacturer with **low elasticity** of scale is given by*

$$Q^* = \max\{\max\{Q_l, \min\{Q_0, Q_h\}\}, \max\{Q_l, \min\{Q_v, Q_h\}\}\}; \quad (2.60)$$

$$K^* = \max\{\hat{K}(\min\{Q_{int}, \max\{Q_l, \min\{Q_0, Q_h\}\}\}), \hat{K}(\max\{Q_{int}, \max\{Q_l, \min\{Q_v, Q_h\}\}\})\}. \quad (2.61)$$

2.5 Social Planner's Problem

In this section, we present the parametric analyzes we conduct to obtain the behavior of the firm's response to increasing carbon tax level. In particular, we show the change in the firm's optimal policy in all regions of carbon tax. In order to do so, firstly, we provide optimal policy of the firm under different order conditions among Z Benchmarks, which has one to one correspondence with the optimal policy as presented in Theorem 2.3, and secondly, using the new results we derive the change in optimal policy with respect to carbon tax.

In order to do so, we need to know the behavior of Q_{int} , Q_0 , and Q_v with respect to Z (and ρ from Lemma 2.4), which are provided by the following proposition.

Proposition 2.4. *For given Q , $\hat{K}(Q)$ is **increasing** in Z ; Q_v and $K_v = \hat{K}(Q_v)$ are **decreasing** in Z for $r < 1/\frac{\alpha_K}{\gamma''+1}$ and **increasing** in Z for $r > 1/\frac{\alpha_K}{\gamma''+1}$; Q_0 is **decreasing** in Z for $r < 1$, and it is **increasing** in Z for $r > 1$; Q_{int} is **decreasing** in $Z \forall r > 0$.*

Proof. The proof is provided in Appendix A. □

Remember that depending on the particular order among Q_l , Q_h , Q_{int} , Q_0 , and Q_v and we obtain the optimal policy presented in Theorem 2.3. Note that Q_{int} , Q_0 , and Q_v are monotonic functions of Z (and ρ from Lemma 2.4), and, depending on

the relative location of Z with respect to Z_{int} , there exists a unique order relation among them. Therefore, we can obtain unique Z values by equating Q_{int} , Q_0 , and Q_v with each other and with Q_l and Q_h . Then, letting these values be Z Benchmarks as below, we can obtain the change in the optimal policy with respect to Z (and ρ) in the regions defined by these benchmarks.

Definition 2.5. $Z_{e,g}$ and $Z_{g,e}$ are the values of Z at which $Q_e = Q_g$, where $Q_e \in \{Q_{int}, Q_0, Q_v\}$ and $Q_g \in \{Q_l, Q_h\}$.

Accordingly, following theorem provides the optimal policy under given order relations among Z Benchmarks.

Theorem 2.4. The optimal production quantity, Q^* , and technology level, K^* , in the regions of Z are as follows.

If $0 < r < 1 - \alpha_K/(\gamma'' + 1)$,

1. for $Z < Z_{int}$ and
 - (a) $Z \leq Z_{h,v}$ and
 - i. $Z \leq Z_{h,int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - ii. $Z \geq Z_{h,int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - (b) $Z_{h,v} \leq Z \leq Z_{l,v}$, $Q^* = Q_v$ and $K^* = \hat{K}(Q_v)$,
 - (c) $Z_{l,v} \leq Z$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.
2. at $Z = Z_{int}$
 - (a) if $Z_{int} \leq Z_{h,int}$, then $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Z_{h,int} \leq Z_{int} \leq Z_{l,int}$, then $Q^* = Q_{int} = Q_0 = Q_v$ and $K^* = \hat{K}(Q_{int}) = K_0$,
 - (c) $Z_{l,int} \leq Z_{int}$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.
3. for $Z > Z_{int}$ and
 - (a) $Z \leq Z_{h,0}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Z_{h,0} \leq Z \leq Z_{l,0}$, $Q^* = Q_0$ and $K^* = K_0$,
 - (c) $Z_{l,0} \leq Z$ and
 - i. $Z \leq Z_{l,int}$, $Q^* = Q_l$ and $K^* = K_0$,

$$ii. Z_{l,int} \leq Z, Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l).$$

If $0 < r = 1 - \alpha_K/(\gamma'' + 1)$,

1. for $Z < Z_{int}$ and

$$(a) Z \leq Z_{h,int}, Q^* = Q_h \text{ and } K^* = K_0,$$

$$(b) Z \geq Z_{h,int}, Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h).$$

2. at $Z = Z_{int}$,

$$(a) \text{ if } Z_{int} \leq Z_{h,int}, \text{ then } Q^* = Q_h \text{ and } K^* = K_0,$$

$$(b) \text{ if } Z_{h,int} \leq Z_{int} \leq Z_{l,int}, \text{ then } Q^* \in [Q_0, Q_h] \text{ and } K^* = \hat{K}(Q^*),$$

$$(c) \text{ if } Z_{l,int} \leq Z_{int}, \text{ then } Q^* \in [Q_l, Q_h] \text{ and } K^* = \hat{K}(Q^*).$$

3. for $Z > Z_{int}$ and

$$(a) Z \leq Z_{h,0}, Q^* = Q_h \text{ and } K^* = K_0,$$

$$(b) Z_{h,0} \leq Z \leq Z_{l,0}, Q^* = Q_0 \text{ and } K^* = K_0,$$

$$(c) Z_{l,0} \leq Z \leq Z_{l,int}, Q^* = Q_l \text{ and } K^* = K_0,$$

$$(d) Z_{l,int} \leq Z, Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l).$$

If $0 < 1 - \alpha_K/(\gamma'' + 1) < r < 1$,

1. for $Z < Z_{int}$ and

$$(a) Z \leq Z_{h,int}, Q^* = Q_h \text{ and } K^* = K_0,$$

$$(b) Z \geq Z_{h,int}, Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h).$$

2. at $Z = Z_{int}$,

$$(a) \text{ if } Z_{int} \leq Z_{h,int}, \text{ then } Q^* = Q_h \text{ and } K^* = K_0,$$

$$(b) \text{ if } Z_{int} \geq Z_{h,int}, \text{ then } Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h).$$

3. for $Z > Z_{int}$ and

(a) $Z \leq Z_{h,int}$ and

$$i. Z \leq Z_{h,0}, Q^* = Q_h \text{ and } K^* = K_0,$$

$$ii. Z_{h,0} \leq Z \leq Z_{l,0}, Q^* = Q_0 \text{ and } K^* = K_0,$$

- iii. $Z_{l,0} \leq Z$, $Q^* = Q_l$ and $K^* = K_0$.
- (b) $Z_{h,int} \leq Z \leq Z_{l,int}$ and $Z \leq Z_{l,0}$,
 - i. if $\Pi_0(Q_0) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_0) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_0 \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_0) > \Pi_v(Q_h)$, then $Q^* = Q_0$ and $K^* = K_0$.
- (c) $Z_{h,int} \leq Z \leq Z_{l,int}$, and $Z \geq Z_{l,0}$,
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
- (d) $Z_{l,int} \leq Z$,
 - i. if $\Pi_v(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_v(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,
 - iii. if $\Pi_v(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

If $r = 1$,

1. for $Z < Z_{int}$ and
 - (a) $Z \leq Z_{h,int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) $Z \geq Z_{h,int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
2. at $Z = Z_{int}$,
 - (a) if $Z_{int} \leq Z_{h,int}$, then $Q^* \in [Q_l, Q_h]$ and $K^* = K_0$,
 - (b) if $Z_{int} \geq Z_{h,int}$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
3. for $Z > Z_{int}$ and
 - (a) $Z \leq Z_{h,int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) $Z_{h,int} \leq Z \leq Z_{l,int}$,
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.

(c) $Z_{l,int} \leq Z$,

i. if $\Pi_v(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. if $\Pi_v(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,

iii. if $\Pi_v(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

If $r > 1$,

1. for $Z < Z_{int}$ and

(a) $Z < Z_{h,int}$,

i. if $\Pi_0(Q_l) < \Pi_0(Q_h)$, then $Q^* = Q_h$ and $K^* = K_0$,

ii. if $\Pi_0(Q_l) = \Pi_0(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = K_0)$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

iii. if $\Pi_0(Q_l) > \Pi_0(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.

(b) $Z_{h,int} \leq Z \leq Z_{l,int}$,

i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.

(c) $Z_{l,int} \leq Z$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.

2. at $Z = Z_{int}$,

(a) if $Z_{int} < Z_{h,int}$, then $Q^* = Q_l$ and $K^* = K_0$.

(b) if $Z_{h,int} \leq Z_{int} \leq Z_{l,int}$ and

i. $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

iii. $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.

(c) $Z_{l,int} \leq Z$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.

3. for $Z > Z_{int}$ and

(a) $Z < Z_{h,int}$, $Q^* = Q_l$ and $K^* = K_0$,

(b) $Z_{h,int} \leq Z < Z_{l,int}$,

i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.

(c) $Z_{l,int} \leq Z$,

i. if $\Pi_v(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

ii. if $\Pi_v(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,

iii. if $\Pi_v(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

Accordingly, the optimal policy under conditions on Z Benchmarks can be presented more compactly in the following tables.

Table 2.4: The Optimal Policy under Conditions on Z Benchmarks for $r < 1$.

$0 < r < 1 - \frac{\alpha_K}{\gamma''+1}$				
For $Z \in (0, Z_{int})$				
for $Z \in$	$(0, Z_{h,int})$	$[Z_{h,int}, Z_{h,v})$	$[Z_{h,v}, Z_{l,v})$	$[Z_{l,v}, \infty)$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_v, \hat{K}(Q_v))$	$(Q_l, \hat{K}(Q_l))$
For $Z \in \{Z_{int}\}$				
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	$(Q_l, \hat{K}(Q_l))$	
For $Z \in (Z_{int}, \infty)$				
for $Z \in$	$(0, Z_{h,0})$	$[Z_{h,0}, Z_{l,0})$	$[Z_{l,0}, Z_{l,int})$	$[Z_{l,int}, \infty)$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$
$0 < r = 1 - \frac{\alpha_K}{\gamma''+1} < 1$				
For $Z \in (0, Z_{int})$				
for $Z \in$	$(0, Z_{h,int})$	$[Z_{h,int}, \infty)$		
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$		
For $Z \in \{Z_{int}\}$				
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q, \hat{K}(Q))$	$(Q, \hat{K}(Q))$	
		s.t. $Q \in [Q_0, Q_h]$	s.t. $Q \in [Q_l, Q_h]$	
For $Z \in (Z_{int}, \infty)$				
for $Z \in$	$(0, Z_{h,0})$	$[Z_{h,0}, Z_{l,0})$	$[Z_{l,0}, Z_{l,int})$	$[Z_{l,int}, \infty)$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$
$1 - \frac{\alpha_K}{\gamma''+1} < r < 1$				
For $Z \in (0, Z_{int})$				
for $Z \in$	$(0, Z_{h,int})$	$\{Z_{h,int}\}$	$(Z_{h,int}, \infty)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	
For $Z \in \{Z_{int}\}$				
for $Z \in$	$(0, Z_{h,int})$	$\{Z_{h,int}\}$	$(Z_{h,int}, \infty)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	(Q_0, K_0)	$(Q_h, \hat{K}(Q_h))$	
For $Z \in (Z_{int}, \infty)$				
for $Z \in$	$(0, Z_{h,int}] \cap$	$(Z_{h,int}, Z_{l,int}) \cap$	$(Z_{h,int}, Z_{l,int}) \cap$	$[Z_{l,int}, \infty)$
	$(0, Z_{h,0})$	$(0, Z_{l,0}]$	$(Z_{l,0}, \infty)$	
if		$\Pi_0(Q_0) < \Pi_v(Q_h)$	$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_v(Q_l) < \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for $Z \in$	$(0, Z_{h,int}] \cap$	$(Z_{h,int}, Z_{l,int}) \cap$	$(Z_{h,int}, Z_{l,int}) \cap$	$[Z_{l,int}, \infty)$
	$[Z_{h,0}, Z_{l,0})$	$(0, Z_{l,0}]$	$(Z_{l,0}, \infty)$	
if		$\Pi_0(Q_0) = \Pi_v(Q_h)$	$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_v(Q_l) = \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_0, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
		or (Q_0, K_0)	or (Q_l, K_0)	or $(Q_l, \hat{K}(Q_l))$
for $Z \in$	$(0, Z_{h,int}] \cap$	$(Z_{h,int}, Z_{l,int}) \cap$	$(Z_{h,int}, Z_{l,int}) \cap$	$[Z_{l,int}, \infty)$
	$[Z_{l,0}, \infty)$	$(0, Z_{l,0}]$	$(Z_{l,0}, \infty)$	
if		$\Pi_0(Q_0) > \Pi_v(Q_h)$	$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_v(Q_l) > \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_l, K_0)	(Q_0, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$

Table 2.5: The Optimal Policy under Conditions on Z Benchmarks for $r \geq 1$.

$r = 1$			
For $Z \in (0, Z_{int})$			
for $Z \in$	$(0, Z_{h,int})$	$\{Z_{h,int}\}$	$(Z_{h,int}, \infty)$
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
For $Z \in \{Z_{int}\}$			
for $Z \in$	$(0, Z_{h,int})$	$\{Z_{h,int}\}$	$(Z_{h,int}, \infty)$
$(Q^*, K^*) \equiv$	(Q, K_0) s.t. $Q \in [Q_l, Q_h]$	(Q, K_0) s.t. $Q \in [Q_l, Q_h]$	$(Q_h, \hat{K}(Q_h))$
For $Z \in (Z_{int}, \infty)$			
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$(Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) < \Pi_0(Q_h)$	$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_v(Q_l) < \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) = \Pi_0(Q_h)$	$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_v(Q_l) = \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_h, K_0) or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) > \Pi_0(Q_h)$	$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_v(Q_l) > \Pi_v(Q_h)$
$(Q^*, K^*) \equiv$	(Q_l, K_0)	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$
$r > 1$			
For $Z \in (0, Z_{int})$			
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) < \Pi_0(Q_h)$	$\Pi_0(Q_l) < \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	
if	$\Pi_0(Q_l) = \Pi_0(Q_h)$	$\Pi_0(Q_l) = \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_h, K_0) or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	
if	$\Pi_0(Q_l) > \Pi_0(Q_h)$	$\Pi_0(Q_l) > \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	(Q_l, K_0)	
For $Z \in \{Z_{int}\}$			
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_0(Q_l) < \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	
if	$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_0(Q_l) = \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	
if	$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_0(Q_l) > \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	(Q_l, K_0)	
For $Z \in (Z_{int}, \infty)$			
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) < \Pi_v(Q_h)$	$\Pi_v(Q_l) < \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$	$(Q_h, \hat{K}(Q_h))$
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) = \Pi_v(Q_h)$	$\Pi_v(Q_l) = \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	$(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$	
for $Z \in$	$(0, Z_{h,int}]$	$(Z_{h,int}, Z_{l,int})$	$[Z_{l,int}, \infty)$
if	$\Pi_0(Q_l) > \Pi_v(Q_h)$	$\Pi_v(Q_l) > \Pi_v(Q_h)$	
$(Q^*, K^*) \equiv$	(Q_l, K_0)	$(Q_l, \hat{K}(Q_l))$	

Now, using the results on the optimal policy under different conditions of carbon benchmarks, we obtain the variation in the response of the firm due to increasing carbon tax. That is, we obtain the optimal policy in the regions of carbon benchmarks.

Note that minimum value of Z at which $\rho = 0$ is $Z_{min} > 0$. Furthermore, according to the value of the parameters, Z Benchmarks may be greater than zero. For example, for $r < 1 - \alpha_K/(\gamma'' + 1)$ and $Z_{int} < Z_{h,int}$, if we have $Z_{h,int} < Z_{min} < Z_{h,v}$, when there is no carbon tax, that is, $\rho = 0$ ($Z = Z_{min}$), optimal solution is $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$ (have a look at the first row second column of the table 2.4). However, finding the optimal policy for all positive values of Z leads to more general results, since, in this way, all carbon tax values satisfying $\rho \geq -\min\{p_i/\beta_i\}$ s.t. $i \in \{1, 2, \dots, N\}$ can be considered. Note that although $\rho < 0$ ($Z < Z_{min}$) may be considered as an incentive for carbon emission, it may also be considered as an incentive for technology enhancement for some values of parameters (remember that $\hat{K}(Q_v)$ is decreasing in Z from Proposition 2.4). Therefore, we provide the optimal policy for all positive Z values.

Letting Z_s and Z_b be positive infinitesimal and very large numbers such that $Z_s < Z_{int}$ and $Z_{int} < Z_b$, respectively, we provide the order relations among Z Benchmarks for limiting values of Z by the following lemma.

Lemma 2.5. *The following list provides the possible order relations among Q_l , Q_h , Q_{int} , Q_0 , and Q_v at $Z = Z_s$ or at $Z = Z_b$ such that $Z_s < Z_{e,f} < Z_b$ and $Z_s < Z_{e,g} < Z_b$ where $e \neq f$, $e \in \{Q_{int}, Q_0, Q_v\}$, $f \in \{int, 0, v\}$, and $g \in \{Q_l, Q_h\}$.*

- If $0 < r < 1 - \frac{\alpha_K}{\gamma''+1}$, then $Q_l < Q_h < Q_{int} < Q_0 < Q_v$ at $Z = Z_s$; $Q_v < Q_0 < Q_{int} < Q_l < Q_h$ at $Z = Z_b$.
- If $0 < r = 1 - \frac{\alpha_K}{\gamma''+1}$, then $Q_l < Q_h < Q_{int} < Q_0$ at $Z = Z_s$; $Q_0 < Q_{int} < Q_l < Q_h$ at $Z = Z_b$.
- If $0 < 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$, then $Q_v < Q_l < Q_h < Q_{int} < Q_0$ at $Z = Z_s$; $Q_0 < Q_{int} < Q_l < Q_h < Q_v$ at $Z = Z_b$.
- If $r = 1$, then $Q_v < Q_l < Q_h < Q_{int}$ at $Z = Z_s$; $Q_{int} < Q_l < Q_h < Q_v$ at $Z = Z_b$.
- If $r > 1$, then $Q_0 < Q_v < Q_l < Q_h < Q_{int}$ at $Z = Z_s$; $Q_{int} < Q_l < Q_h < Q_v < Q_0$ at $Z = Z_b$.

Proof. The proof is provided in Appendix A. □

Note that, since Q_{int} is decreasing in Z for all positive values of elasticity of scale, r , we have $Z_{h,int} < Z_{l,int}$. Therefore, there are three possible order relation among Z_{int} , $Z_{h,int}$, and $Z_{l,int}$: 1) $Z_{int} \leq Z_{h,int}$, 2) $Z_{h,int} < Z_{int} < Z_{l,int}$, and 3) $Z_{l,int} < Z_{int}$. Depending on each of these conditions, all of the Z Benchmarks that are required to examine the change in the optimal policy with respect to Z are ordered in a particular fashion. Hence, the following proposition provides the relevant order relations among Z Benchmarks for each of these conditions required to examine the change in the optimal policy with respect to Z .

Proposition 2.5. *All of the order relations among Z Benchmarks are presented below.*

1. For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$,

(a) *if $Z_{int} < Z_{h,int}$, then*

$$Z_{int} < Z_{h,v} < Z_{h,0} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,v} < Z_{l,v} < Z_{l,0} \text{ and } Z_{h,v} < Z_{h,0} < Z_{h,int} < Z_{l,int},$$

(b) *if $Z_{int} = Z_{h,int}$, then*

$$Z_{int} = Z_{h,0} = Z_{h,v} < Z_{l,v} < Z_{l,0} < Z_{l,int},$$

(c) *if $Z_{h,int} < Z_{int} < Z_{l,int}$, then*

$$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{int} < Z_{l,v} < Z_{l,0} < Z_{l,int},$$

(d) *if $Z_{l,int} = Z_{int}$, then*

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0},$$

(e) *if $Z_{l,int} < Z_{int}$, then*

$$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{l,v} < Z_{int} \text{ and } Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{l,v} \text{ and } Z_{h,0} < Z_{l,0}.$$

2. For $0 < r = 1 - \alpha_K/(\gamma'' + 1)$,

(a) *if $Z_{int} < Z_{h,int}$, then*

$$Z_{int} < Z_{h,0} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,0} < Z_{h,int} < Z_{l,int}$$

(b) if $Z_{int} = Z_{h,int}$, then

$$Z_{int} = Z_{h,0} < Z_{l,0} < Z_{l,int},$$

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int},$$

(d) if $Z_{int} = Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0},$$

(e) if $Z_{int} > Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ and } Z_{l,int} < Z_{l,0} < Z_{int} \text{ and } Z_{h,int} < Z_{l,int} \text{ and } Z_{h,0} < Z_{l,0}.$$

3. For $0 < 1 - \alpha_K/(\gamma'' + 1) < r < 1$,

(a) if $Z_{int} < Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} < Z_{int} < Z_{h,0} < Z_{h,int} < Z_{l,int} \text{ and } Z_{int} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,0} < Z_{l,0},$$

(b) if $Z_{int} = Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} = Z_{int} = Z_{h,0} = Z_{h,int} < Z_{l,0} < Z_{l,int},$$

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h},$$

(d) if $Z_{int} = Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0} = Z_{v,l} < Z_{v,h},$$

(e) if $Z_{int} > Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} < Z_{l,0} \text{ and } Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{int} < Z_{v,l} < Z_{v,h},$$

4. For $r = 1$,

(a) if $Z_{int} < Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} < Z_{int} < Z_{h,int} < Z_{l,int},$$

(b) If $Z_{int} = Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} = Z_{int} = Z_{h,int} < Z_{l,int},$$

- (c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$$Z_{v,l} < Z_{int} < Z_{v,h} \text{ and } Z_{v,l} < Z_{l,int} \text{ and } Z_{h,int} < Z_{v,h}.$$
- (d) if $Z_{int} = Z_{l,int}$, then

$$Z_{h,int} < Z_{int} = Z_{l,int} = Z_{v,l} < Z_{v,h},$$
- (e) if $Z_{int} > Z_{l,int}$, then

$$Z_{h,int} < Z_{l,int} < Z_{int} < Z_{v,l} < Z_{v,h},$$

5. For $r > 1$,

- (a) if $Z_{int} < Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} < Z_{0,h} < Z_{int} < Z_{h,int} < Z_{l,int} \text{ and } Z_{0,l} < Z_{0,h} < Z_{int} < Z_{h,int} < Z_{l,int}.$$
- (b) if $Z_{int} = Z_{h,int}$, then

$$Z_{v,l} < Z_{0,l} < Z_{0,h} = Z_{v,h} = Z_{int} = Z_{h,int} < Z_{l,int} \text{ and } Z_{0,l} < Z_{0,h} = Z_{v,h} = Z_{int} = Z_{h,int}$$
- (c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$$Z_{v,l} < Z_{0,l} < Z_{int} < Z_{0,h} < Z_{v,h}.$$
- (d) if $Z_{int} = Z_{l,int}$, then

$$Z_{h,int} < Z_{l,int} = Z_{v,l} = Z_{0,l} = Z_{int} < Z_{0,h} < Z_{v,h} \text{ and } Z_{int} < Z_{v,h}.$$
- (e) if $Z_{int} > Z_{l,int}$, then

$$Z_{h,int} < Z_{l,int} < Z_{int} < Z_{0,l} < Z_{v,l} < Z_{v,h} \text{ and } Z_{0,l} < Z_{0,h}.$$

Proof. The proof is provided in Appendix A. □

Note that for some conditions on Z Benchmarks provided in the tables 2.4 and 2.5, we need to compare the optimal profits evaluated at different production quantities and technology levels. However, we see that difference of these functions are increasing in Z in the respective regions of Z Benchmarks. The details are provided by the following lemma.

Lemma 2.6.

1. If $0 < r$, $(\Pi_0(Q_l) - \Pi_0(Q_h))$ is **increasing** in Z for $Z \geq 0$.
2. If $0 < 1 - \frac{\alpha}{\gamma''+1} < r < 1$, $(\Pi_0(Q_0) - \Pi_v(Q_h))$ is **increasing** in Z for $Z \geq Z_{int}$ and $Z \geq Z_{h,0}$ ($Q_0 < Q_{int} < Q_v$ and $Q_0 \leq Q_h$). $(\Pi_0(Q_0) - \Pi_v(Q_h))$, being **increasing** in Z for $Z \geq Z_{int} = Z_{h,0}$ ($Q_0 < Q_{int} < Q_h < Q_v$), attains its root at Z_{int} , for $Z_{int} = Z_{h,0} = Z_{h,int} = Z_{v,h}$.
3. If $0 < r$, $(\Pi_0(Q_l) - \Pi_v(Q_h))$ is **increasing** in Z for $Z < Z_{l,int}$ ($Q_l < Q_{int}$).
4. If $0 < r$, $(\Pi_v(Q_l) - \Pi_v(Q_h))$ is **increasing** in Z for $Z \geq 0$.

Proof. The proof is provided in Appendix A. □

Let $\Phi(Z)$ be a piece-wise function that is defined in the adjacent regions of Z as presented by the following definition.

Definition 2.6. The definitions of the difference function, $\Phi(Z)$, depending on the elasticity level and conditions on the Z Benchmarks, are as follows.

1. If $1 - \frac{\alpha_K}{\gamma''+1} < r < 1$ and
 - (a) if $Z_{int} = Z_{h,int}$, then $\Phi(Z) = \Phi_{h0}^{00}(Z) = (\Pi_0(Q_0) - \Pi_0(Q_h))|_Z$ for $Z = Z_{int} = Z_{h,int}$.
 - (b) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{00}(Z) = (\Pi_0(Q_0) - \Pi_v(Q_h))|_Z$ for $Z_{int} < Z \leq Z_{l,0}$.
 - (c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{l0}(Z) = (\Pi_0(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,0} < Z < Z_{l,int}$.
 - (d) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{lv}(Z) = (\Pi_v(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,int} \leq Z$.
 - (e) if $Z_{l,int} \leq Z_{int}$, then $\Phi(Z) = \Phi_{hv}^{lv}(Z) = (\Pi_v(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,int} \leq Z$.
2. If $r = 1$ and
 - (a) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{l0}(Z) = (\Pi_0(Q_l) - \Pi_v(Q_h))|_Z$ for

$$Z_{h,int} < Z < Z_{l,int}.$$

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{lv}(Z) = (\Pi_v(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,int} \leq Z$.

(c) if $Z_{l,int} \leq Z_{int}$, then $\Phi(Z) = \Phi_{hv}^{lv}(Z) = (\Pi_v(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,int} \leq Z$.

3. If $r > 1$ and

(a) if $Z_{int} \leq Z_{h,int}$, then $\Phi(Z) = \Phi_{h0}^{l0}(Z) = (\Pi_0(Q_l) - \Pi_0(Q_h))|_Z$ for $Z \leq Z_{h,int}$.

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{h0}^{l0}(Z) = (\Pi_0(Q_l) - \Pi_0(Q_h))|_Z$ for $Z \leq Z_{h,int}$.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{l0}(Z) = (\Pi_0(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{h,int} < Z < Z_{l,int}$.

(d) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then $\Phi(Z) = \Phi_{hv}^{lv}(Z) = (\Pi_v(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,int} \leq Z$.

(e) if $Z_{l,int} \leq Z_{int}$, then $\Phi(Z) = \Phi_{hv}^{lv}(Z) = (\Pi_v(Q_l) - \Pi_v(Q_h))|_Z$ for $Z_{l,int} \leq Z$.

Using Definition 2.6 the findings presented in Lemma 2.6 can be rewritten as in the following proposition.

Proposition 2.6.

1. If $1 - \alpha_K/(\gamma'' + 1) < r < 1$, then $\Phi(Z)$ is increasing in $Z \forall Z \geq Z_{int}$ with a unique root, Z_Φ , such that $Z_\Phi = Z_{hv}^{00} = Z_{int}$ for $Z_{int} = Z_{h,int} = Z_{h,0}$; such that $Z_\Phi > Z_{int} > Z_{h,int}$ for $Z_{h,int} < Z_{int} < Z_{l,int}$; such that $Z_\Phi = Z_{hv}^{lv} \geq Z_{l,int}$ for $Z_{l,int} \leq Z_{int}$.
2. If $r = 1$, then $\Phi(Z)$ is increasing in Z with a unique root, Z_Φ , such that $Z_\Phi = Z_{int}$ for $Z_{int} = Z_{h,int} = Z_{h,0}$; such that $Z_\Phi > Z_{int} > Z_{h,int}$ for $Z_{h,int} < Z_{int} < Z_{l,int}$; such that $Z_\Phi = Z_{hv}^{lv} \geq Z_{l,int}$ for $Z_{l,int} \leq Z_{int}$.
3. If $r > 1$, then $\Phi(Z)$ is increasing in Z with a unique root, Z_Φ , such that $Z_\Phi = Z_{h0}^{l0} < Z_{h,int}$ for $Z_{int} \leq Z_{h,int}$; such that $Z_\Phi > 0$ for $Z_{h,int} < Z_{int} < Z_{l,int}$; such that $Z_\Phi = Z_{hv}^{lv} \geq Z_{l,int}$ for $Z_{l,int} \leq Z_{int}$.

Hence, the unique root of the piecewise function, $\Phi(Z)$, constitutes a new Z Benchmark, denoted by Z_Φ , which should satisfy the following conditions.

1. $0 < 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$
 - (a) If $\Phi_{hv}^{00}(Z_{int}) < 0$ and $\Phi_{hv}^{00}(Z_{l,0}) > 0$, then $Z_{int} < Z_\Phi = Z_{hv}^{00} < Z_{l,0}$.
 - (b) If $\Phi_{hv}^{l0}(Z_{l,0}) < 0$ and $\Phi_{hv}^{l0}(Z_{l,int}) > 0$, then $Z_{l,0} < Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$.
 - (c) If $\Phi_{hv}^{lv}(Z_{l,int}) \leq 0$, then $Z_{l,int} \leq Z_\Phi = Z_{hv}^{lv}$.
2. If $r = 1$
 - (a) If $Z_{h,int} < Z_{int} < Z_{l,int}$ and $(\Phi_{hv}^{l0}(Z_{h,int}) \leq 0$ and $\Phi_{hv}^{l0}(Z_{l,int}) \geq 0)$, then $Z_{h,int} \leq Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$.
 - (b) If $Z_{h,int} < Z_{int} < Z_{l,int}$ and $\Phi_{hv}^{lv}(Z_{h,int}) \leq 0$, then $Z_{h,int} \leq Z_\Phi = Z_{hv}^{lv}$.
 - (c) If $Z_{l,int} \leq Z_{int}$ and $\Phi_{hv}^{lv}(Z_{h,int}) \leq 0$, then $Z_{h,int} \leq Z_\Phi = Z_{hv}^{lv}$.
3. $r > 1$
 - (a) If $Z_{h,int} < Z_{int} < Z_{l,int}$ and $\Phi_{h0}^{l0}(Z_{h,int}) \geq 0$, then $Z_\Phi = Z_{h0}^{l0} \leq Z_{h,int}$.
 - (b) If $Z_{h,int} < Z_{int} < Z_{l,int}$ and $\Phi_{hv}^{l0}(Z_{h,int}) = \Phi_{h0}^{l0}(Z_{h,int}) < 0$ and $\Phi_{hv}^{l0}(Z_{l,int}) = \Phi_{hv}^{lv}(Z_{l,int}) > 0$, then $Z_{h,int} < Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$.
 - (c) If $Z_{h,int} < Z_{int} < Z_{l,int}$ and $\Phi_{hv}^{lv}(Z_{l,int}) \leq 0$, then $Z_{l,int} \leq Z_\Phi = Z_{hv}^{lv}$.

Using the order relations among Z Benchmarks provided by Proposition 2.5 and results of Theorem 2.4 we find the change in the optimal solution with respect to Z as in the following theorem.

Theorem 2.5. *Optimal production policy in different regions of Z Benchmarks are as follows.*

For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$,

1. if $Z_{int} < Z_{h,int}$, then
 - (a) for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - (c) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

2. if $Z_{int} = Z_{h,int}$, then

- (a) for $Z \leq Z_{int} = Z_{h,int} = Z_{h,0} = Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- (b) for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
- (c) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
- (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

3. if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- (b) for $Z_{h,int} \leq Z < Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
- (c) for $Z_{h,v} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_v, \hat{K}(Q_v))$,
- (d) at $Z \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
- (e) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
- (f) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

4. if $Z_{int} = Z_{l,int}$, then

- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- (b) for $Z_{h,int} \leq Z < Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
- (c) for $Z_{h,v} \leq Z < Z_{l,v}$, $(Q^*, K^*) \equiv (Q_v, \hat{K}(Q_v))$,
- (d) for $Z_{int} = Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

5. if $Z_{int} > Z_{l,int}$, then

- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- (b) for $Z_{h,int} \leq Z < Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
- (c) for $Z_{h,v} \leq Z < Z_{l,v}$, $(Q^*, K^*) \equiv (Q_v, \hat{K}(Q_v))$,
- (d) for $Z_{l,v} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

For $r = 1 - \alpha_K/(\gamma'' + 1)$,

1. if $Z_{int} < Z_{h,int}$, then

- (a) for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,

- (b) for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - (c) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
2. if $Z_{h,int} = Z_{int}$, then
- (a) for $Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) at $Z_{int} = Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - (c) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
3. if $Z_{h,int} < Z_{int} < Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) at $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in [Q_0, Q_h]$,
 - (d) for $Z_{int} < Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - (e) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (f) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
4. if $Z_{int} = Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}_{Q_h})$;
 - (c) at $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in [Q_l, Q_h]$,
 - (d) for $Z_{int} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
5. if $Z_{l,int} < Z_{int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}_{Q_h})$,
 - (c) at $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in [Q_l, Q_h]$,
 - (d) for $Z_{int} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

For $1 - \alpha_K/(\gamma'' + 1) < r < 1$,

1. if $Z_{int} < Z_{h,int}$, then
 - (a) for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - (c) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
2. if $Z_{int} = Z_{h,int}$, then
 - (a) for $Z < Z_{h,0} = Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z = Z_{h,0} = Z_{h,int} = Z_{hv}^{00} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - (c) for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
3. if $Z_{h,int} < Z_{int} < Z_{l,int}$ and
 - (a) if $Z_{int} < Z_{hv}^{00} < Z_{l,0}$, then
 - i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - ii. for $Z_{h,int} \leq Z < Z_{hv}^{00}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - iii. for $Z = Z_{hv}^{00}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - iv. for $Z_{hv}^{00} < Z \leq Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$,
 - v. for $Z_{l,0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - vi. for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
 - (b) if $Z_{l,0} < Z_{hv}^{l0} < Z_{l,int}$, then
 - i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - ii. for $Z_{h,int} \leq Z < Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - iii. for $Z = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - iv. for $Z_{hv}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - v. for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

- (c) if $Z_{l,int} \leq Z_{hv}^{lv}$, then
- i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - ii. for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - iii. for $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$,
 - iv. for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
4. if $Z_{int} = Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z = Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (d) for $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$,
 - (e) for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
5. if $Z_{int} > Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z = Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) for $Z_{h,int} < Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (d) for $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$,
 - (e) for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

For $r = 1$,

1. if $Z_{int} < Z_{h,int}$, then
- (a) for $Z_{v,h} \leq Z < Z_{int} < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) at $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in [Q_l, Q_h]$,
 - (c) for $Z_{int} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
2. if $Z_{int} = Z_{h,int}$, then
- (a) for $Z < Z_{int} = Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,

- (b) at $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in [Q_l, Q_h]$,
 - (c) for $Z_{int} = Z_{h,int} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
3. if $Z_{h,int} < Z_{int} < Z_{l,int}$ and
- (a) if $Z_{hv}^{l0} < Z_{l,int}$, then
 - i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - ii. for $Z_{h,int} \leq Z < Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - iii. for $Z = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - iv. for $Z_{hv}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - v. for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
 - (b) if $Z_{l,int} \leq Z_{hv}^{lv}$, then
 - i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - ii. for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - iii. for $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$,
 - iv. for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
4. If $Z_{int} = Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) at $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$,
 - (d) for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
5. If $Z_{int} > Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) at $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$,
 - (d) for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

For $r > 1$,

1. if $Z_{int} < Z_{h,int}$, then

- (a) for $Z < Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- (b) for $Z = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in \{Q_l, Q_h\}$,
- (c) for $Z_{h0}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
- (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

2. if $Z_{int} = Z_{h,int}$, then

- (a) for $Z < Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- (b) for $Z = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in \{Q_l, Q_h\}$,
- (c) for $Z_{h0}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
- (d) for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

3. if $Z_{h,int} < Z_{int} < Z_{l,int}$ and

(a) if $Z_{h0}^{l0} \leq Z_{h,int}$, then

- i. for $Z < Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- ii. for $Z = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ or $(Q^*, K^*) \equiv (Q_l, K_0)$,
- iii. for $Z_{h0}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
- iv. for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

(b) if $Z_{h,int} < Z_{hv}^{l0} < Z_{int} < Z_{l,int}$, $Z_{h,int} < Z_{h0}^{l0}$, and $Z_{hv}^{lv} < Z_{l,int}$, then

- i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- ii. for $Z_{h,int} \leq Z < Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
- iii. at $Z = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$,
- iv. for $Z_{hv}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
- v. for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

(c) if $Z_{h,int} < Z_{int} < Z_{hv}^{l0} < Z_{l,int}$, then

- i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
- ii. for $Z_{h,int} < Z < Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,

- iii. at $Z = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - iv. for $Z_{hv}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$,
 - v. for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
- (d) if $Z_{h,int} < Z_{int} < Z_{l,int} \leq Z_{hv}^{lv}$, then
- i. for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - ii. for $Z_{h,int} < Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - iii. at $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$,
 - iv. for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
4. if $Z_{int} = Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) at $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$,
 - (d) for $Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.
5. if $Z_{int} > Z_{l,int}$, then
- (a) for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$,
 - (b) for $Z_{h,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$,
 - (c) at $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$,
 - (d) for $Z > Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$.

Proof. The proof is provided in Appendix A. □

The optimal production quantity and technology level pair, (Q^*, K^*) in different regions of Z Benchmarks can be presented more compactly in the following tables.

Table 2.6: The Optimal Policy in the Regions of Z for $r < 1$.

$0 < r < 1 - \frac{\alpha_K}{\gamma''+1}$					
$Z_{int} \leq Z_{h,int}$					
$Z < Z_{h,0}$ (Q_h, K_0)	$Z_{h,0} \leq Z < Z_{l,0}$ (Q_0, K_0)	$Z_{l,0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)		
$Z_{h,int} < Z_{int} < Z_{l,int}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{h,v}$ ($Q_h, \hat{K}(Q_h)$)	$Z_{h,v} \leq Z < Z_{int}$ ($Q_v, \hat{K}(Q_v)$)	$Z_{int} \leq Z < Z_{l,0}$ (Q_0, K_0)	$Z_{l,0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)
$Z_{l,int} \leq Z_{int}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{h,v}$ ($Q_h, \hat{K}(Q_h)$)	$Z_{h,v} \leq Z < Z_{l,v}$ ($Q_v, \hat{K}(Q_v)$)	$Z_{l,v} \leq Z$ ($Q_l, \hat{K}(Q_l)$)		
$0 < r = 1 - \frac{\alpha_K}{\gamma''+1}$					
$Z_{int} \leq Z_{h,int}$					
$Z < Z_{h,0}$ (Q_h, K_0)	$Z_{h,0} \leq Z < Z_{l,0}$ (Q_0, K_0)	$Z_{l,0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)		
$Z_{h,int} < Z_{int} < Z_{l,int}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{int}$ ($Q_h, \hat{K}(Q_h)$)	$Z = Z_{int}$ ($Q, \hat{K}(Q)$) s.t. $Q \in [Q_0, Q_h]$	$Z_{int} < Z < Z_{l,0}$ (Q_0, K_0)	$Z_{l,0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)
$Z_{l,int} \leq Z_{int}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{int}$ ($Q_h, \hat{K}(Q_h)$)	$Z = Z_{int}$ ($Q, \hat{K}(Q)$) s.t. $Q \in [Q_l, Q_h]$	$Z_{int} < Z$ ($Q_l, \hat{K}(Q_l)$)		
$0 < 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$					
$Z_{int} \leq Z_{h,int}$					
$Z < Z_{h,0}$ (Q_h, K_0)	$Z_{h,0} \leq Z < Z_{l,0}$ (Q_0, K_0)	$Z_{l,0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)		
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{int} < Z_{hv}^{00} \leq Z_{l,0}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{00}$ ($Q_h, \hat{K}(Q_h)$)	$Z = Z_{hv}^{00}$ ($Q_h, \hat{K}(Q_h)$) or (Q_0, K_0)	$Z_{hv}^{00} < Z < Z_{l,0}$ (Q_0, K_0)	$Z_{l,0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,0} < Z_{hv}^{l0} < Z_{l,int}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{l0}$ ($Q_h, \hat{K}(Q_h)$)	$Z = Z_{hv}^{l0}$ ($Q_h, \hat{K}(Q_h)$) or (Q_l, K_0)	$Z_{hv}^{l0} < Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ ($Q_l, \hat{K}(Q_l)$)	
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{lv}$ ($Q_h, \hat{K}(Q_h)$)	$Z = Z_{hv}^{lv}$ ($Q_h, \hat{K}(Q_h)$) or ($Q_l, K(Q_l)$)	$Z_{hv}^{lv} < Z$ ($Q_l, \hat{K}(Q_l)$)		
$Z_{l,int} \leq Z_{int}$					
$Z < Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{lv}$ ($Q_h, \hat{K}(Q_h)$)	$Z = Z_{hv}^{lv}$ ($Q_h, \hat{K}(Q_h)$) or ($Q_l, K(Q_l)$)	$Z_{hv}^{lv} < Z$ ($Q_l, \hat{K}(Q_l)$)		

Table 2.7: The Optimal Policy in the Regions of Z for $r \geq 1$.

$r = 1$				
$Z_{int} \leq Z_{h,int}$				
$Z < Z_{int}$ (Q_h, K_0)	$Z = Z_{int}$ (Q, K_0) s.t. $Q \in [Q_l, Q_h]$	$Z_{int} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ $(Q_l, \hat{K}(Q_l))$	
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{hv}^{lv} < Z_{l,int}$				
$Z \leq Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{l0}$ $(Q_h, \hat{K}(Q_h))$	$Z = Z_{hv}^{l0}$ $(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$Z_{hv}^{l0} < Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ $(Q_l, \hat{K}(Q_l))$
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$				
$Z \leq Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$	$Z = Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$	$Z_{hv}^{lv} < Z$ $(Q_l, \hat{K}(Q_l))$	
$Z_{l,int} \leq Z_{int}$				
$Z \leq Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$	$Z = Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$	$Z_{hv}^{lv} < Z$ $(Q_l, \hat{K}(Q_l))$	
$r > 1$				
$Z_{int} \leq Z_{h,int}$				
$Z < Z_{h0}^{l0}$ (Q_h, K_0)	$Z = Z_{h0}^{l0}$ (Q_h, K_0) or (Q_l, K_0)	$Z_{h0}^{l0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ $(Q_l, \hat{K}(Q_l))$	
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{h0}^{l0} \leq Z_{h,int}$				
$Z < Z_{h0}^{l0}$ (Q_h, K_0)	$Z = Z_{h0}^{l0}$ (Q_h, K_0) or (Q_l, K_0)	$Z_{h0}^{l0} \leq Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ $(Q_l, \hat{K}(Q_l))$	
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{h,int} < Z_{hv}^{l0} < Z_{l,int}$				
$Z \leq Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{l0}$ $(Q_h, \hat{K}(Q_h))$	$Z = Z_{hv}^{l0}$ $(Q_h, \hat{K}(Q_h))$ or (Q_l, K_0)	$Z_{hv}^{l0} < Z < Z_{l,int}$ (Q_l, K_0)	$Z_{l,int} \leq Z$ $(Q_l, \hat{K}(Q_l))$
$Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$				
$Z \leq Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$	$Z = Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$	$Z_{hv}^{lv} < Z$ $(Q_l, \hat{K}(Q_l))$	
$Z_{l,int} \leq Z_{int}$				
$Z \leq Z_{h,int}$ (Q_h, K_0)	$Z_{h,int} \leq Z < Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$	$Z = Z_{hv}^{lv}$ $(Q_h, \hat{K}(Q_h))$ or $(Q_l, \hat{K}(Q_l))$	$Z_{hv}^{lv} < Z$ $(Q_l, \hat{K}(Q_l))$	

After analytical investigations, we see that in some regions of Z , the optimal profit of the firm may be negative. In reality, the manufacturer may decide not to produce when her optimal objective is negative. To that end, the value of Z that leads the firm break-even is defined by Z_{max} as below.

Definition 2.7. Z_{max} is the value of Z at which the firm break-evens.

Then, Z_{max} is derived using the following property of the firm's optimal profit, $\Pi(Q^*, K^*)$.

Lemma 2.7. $\Pi(Q^*, K^*)$ is monotonically decreasing in Z and if $\Pi(Q^*, K^*)|_{Z=0} > 0$, then there exist a unique positive Z_{max} value that satisfies $\Pi(Q^*, K^*)|_{Z=Z_{max}} = 0$.

Proof. The proof is provided in Appendix A. □

There exist some regions of Z that stays above Z_{max} . Hence, since they lead to negative profits, the associated optimal policies are not economically viable. For example, for $r \geq 1$, the set of economically viable optimal solutions reduces to $\{(Q_h, K_0), (Q_h, \hat{K}(Q_h))\}$. Through the following theorem, we provide the expressions for Z_{max} for different elasticity levels and conditions on Z Benchmarks.

Proposition 2.7. The value of carbon price benchmark, Z_{max} , after which the firm has negative optimal profit for each elasticity of scale and condition on Z_{int} and $Z_{f,int}$ such that $f \in \{l, h\}$ can be expressed as below:

1. For $0 < r < 1 - \frac{\alpha_K}{\gamma''+1}$,
 - (a) if $Z_{int} < Z_{l,int}$, then $Z_{max} = Z_{max}(Q_l, K_0)$ for $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi(Q^*)|_{Z=Z_{l,int}} \geq 0$,
 - (b) if $Z_{l,int} \leq Z_{int}$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$,
2. For $0 < r = 1 - \frac{\alpha_K}{\gamma''+1}$,
 - (a) if $Z_{int} < Z_{l,int}$, then $Z_{max} = Z_{max}(Q_l, K_0)$ for $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi(Q^*)|_{Z=Z_{l,int}} \geq 0$,

(b) if $Z_{int} \geq Z_{l,int}$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$.

3. For $0 < 1 - \frac{\alpha_K}{\gamma^{n+1}} < r < 1$,

(a) if $Z_{int} \leq Z_{h,int}$, then $Z_{max} = Z_{max}(Q_l, K_0)$ for $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi(Q^*)|_{Z=Z_{l,int}} > 0$,

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{int} < Z_{hv}^{00} \leq Z_{l,0}$, then $Z_{max} = Z_{max}(Q_l, K_0)$ for $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi(Q^*)|_{Z=Z_{l,int}} > 0$,

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,0} < Z_{hv}^{l0} < Z_{l,int}$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi(Q^*)|_{Z=Z_{hv}^{l0}} < 0$; $Z_{max} = Z_{max}(Q_l, K_0)$ for $\Pi(Q^*)|_{Z=Z_{hv}^{l0}} > 0$ and $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi(Q^*)|_{Z=Z_{l,int}} > 0$.

then $Z_{max} \in \{Z_{max}(Q_h, \hat{K}(Q_h)), Z_{max}(Q_l, K_0), Z_{max}(Q_l, \hat{K}(Q_l))\}$,

(d) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} \geq 0$,

then $Z_{max} \in \{Z_{max}(Q_h, \hat{K}(Q_h)), Z_{max}(Q_l, \hat{K}(Q_l))\}$,

(e) if $Z_{l,int} \leq Z_{int}$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} < 0$; $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$ for $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} \geq 0$,

4. For $r = 1$,

(a) if $Z_{int} \leq Z_{h,int}$, then $Z_{max} = Z_{int}$,

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{hv}^{l0} < Z_{l,int}$, then $Z_{max} = Z_{max}(Q_h, K_0)$ for $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$,

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$, then $Z_{max} = Z_{max}(Q_h, K_0)$ for $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$,

(d) if $Z_{l,int} \leq Z_{int}$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$.

5. For $r > 1$,

(a) if $Z_{int} \leq Z_{h,int}$, then $Z_{max} = Z_{max}(Q_h, K_0)$,

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{h0}^{l0} < Z_{h,int}$, then $Z_{max} = Z_{max}(Q_h, K_0)$,

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{h,int} \leq Z_{hv}^{l0} < Z_{l,int}$, then $Z_{max} = Z_{max}(Q_h, K_0)$ for $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$,

- (d) if $Z_{h,int} < Z_{int} < Z_{l,int}$, and $Z_{l,int} \leq Z_{hv}^{lv}$, then $Z_{max} = Z_{max}(Q_h, K_0)$ for $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$,
- (e) if $Z_{h,int} < Z_{int}$, then $Z_{max} = Z_{max}(Q_h, K_0)$ for $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ for $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$,

where,

$$Z_{max}(Q_f, \hat{K}(Q_f)) =$$

$$\left[\left(Q_f s + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \right) \left(\frac{\alpha_K (\gamma'' + 1)}{\gamma' c_K (r(\gamma'' + 1) + \alpha_K)} \right) \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^r \frac{a}{Q_f} \right], \quad (2.62)$$

$$Z_{max}(Q_f, K_0) = Q_f^{r-1} \left(\frac{s}{r} \right)^r a K_0^{\alpha_K}, \quad (2.63)$$

$$Z_{max}(Q_{int}) = s a \left(\frac{\gamma' c_K}{\alpha_K} \right)^{r-1} K_0^{(r-1)(\gamma'' + 1) + \alpha_K}, \quad (2.64)$$

where $Q_f \in \{Q_l, Q_h\}$.

Proof. The proof is provided in Appendix A. □

2.6 Results and Discussion

The following theorem presents the behavior of the optimal input quantities with respect to carbon tax, ρ .

Theorem 2.6. *Optimal quantity of input i , x_i^* ($i \in \{1, 2, \dots, N\}$), is decreasing in ρ , for*

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{r} \frac{\beta_j}{p_j + \rho \beta_j} \quad \text{if } (Q^*(\rho), K^*(\rho)) = (Q_f, K_0), \quad (2.65)$$

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j (\gamma'' + 1)}{r(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho \beta_j} \quad \text{if } (Q^*(\rho), K^*(\rho)) = (Q_f, \hat{K}(Q_f)), \quad (2.66)$$

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{(r-1)} \frac{\beta_j}{p_j + \rho \beta_j} \quad \text{if } (Q^*(\rho), K^*(\rho)) = (Q_0, \hat{K}(Q_0)), \quad (2.67)$$

$$\frac{\beta_i}{p_i + \rho\beta_i} > \sum_{j=1}^N \frac{\alpha_j(\gamma'' + 1)}{(r-1)(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j} \quad \text{if } (Q^*(\rho), K^*(\rho)) = (Q_v, \hat{K}(Q_v)), \quad (2.68)$$

where, conditions (2.67) and (2.68) hold for all positive values of the carbon tax, ρ , for $r < 1$ and $r < 1 - \frac{\alpha_K}{\gamma''+1}$, respectively.

Proof. The proof is provided in Appendix A. $\square$

If we let $T_j = \frac{\beta_j}{p_j + \rho\beta_j}$ and $WA_1 = \sum_{j=1}^N \frac{\alpha_j}{r} T_j$, then the right hand side of the inequality in Condition (2.65), WA_1 , is the weighted average of T_j 's for $j \in \{1, 2, \dots, N\}$. Therefore, letting $T_{min} = \min_{j \in \{1, 2, \dots, N\}} \{T_j\}|_{\rho=0}$ and $T_{max} = \max_{j \in \{1, 2, \dots, N\}} \{T_j\}|_{\rho=0}$ be the smallest and largest T_j terms when carbon tax is zero, we have $T_{min} < WA_1$ and $T_{max} > WA_1$ for $N \geq 2$ and $\rho = 0$. Furthermore, $\lim_{\rho \rightarrow \infty} T_i = \frac{1}{\rho}$ for $i \in \{1, 2, \dots, N\}$ and $\lim_{\rho \rightarrow \infty} WA_1 = \sum_{j=1}^N \frac{\alpha_j}{r} \frac{1}{\rho}$. Then, we have $\lim_{\rho \rightarrow \infty} T_i = \lim_{\rho \rightarrow \infty} \frac{1}{\rho} = \sum_{j=1}^N \frac{\alpha_j}{r} \frac{1}{\rho} = \lim_{\rho \rightarrow \infty} WA_1$. Therefore, since $T_{min} \leq WA_1$ and $T_{max} \geq WA_1$ the input with the lowest and highest carbon footprint to unit cost ratios are increasing and decreasing in carbon tax, respectively.

Now, let WA_2 be the right hand side of the inequality in Condition (2.66). Furthermore, let $T_{N+1} = 0$ and $\alpha_{N+1} = \frac{\alpha_K}{\gamma''+1}$. Note that $\lim_{\rho \rightarrow \infty} T_j = \frac{1}{\rho}$ for $j \in \{1, 2, \dots, N\}$, $\lim_{\rho \rightarrow \infty} WA_2 = \frac{\frac{\alpha_K}{\gamma''+1}}{r + \frac{\alpha_K}{\gamma''+1}} 0 + \sum_{j=1}^N \frac{\alpha_j}{r + \frac{\alpha_K}{\gamma''+1}} \frac{1}{\rho} = \frac{\alpha_j}{\rho r + \frac{\alpha_K}{\gamma''+1}} 1$. Then, for sufficiently large ρ , we may write $\rho T_i = \rho \frac{1}{\rho} = 1 > \frac{\frac{\alpha_K}{\gamma''+1}}{r + \frac{\alpha_K}{\gamma''+1}} 0 + \rho \sum_{j=1}^N T_j = \frac{\frac{\alpha_K}{\gamma''+1}}{r + \frac{\alpha_K}{\gamma''+1}} 0 + \sum_{j=1}^N \frac{\rho \alpha_j}{\rho r + \frac{\alpha_K}{\gamma''+1}} 1 = \frac{\frac{\alpha_K}{\gamma''+1}}{r + \frac{\alpha_K}{\gamma''+1}} 0 + \sum_{j=1}^N \frac{\alpha_j}{r + \frac{\alpha_K}{\gamma''+1}} 1$ for all $i \in \{1, 2, \dots, N\}$, and $\rho T_{N+1} = 0 < \frac{\frac{\alpha_K}{\gamma''+1}}{r + \frac{\alpha_K}{\gamma''+1}} 0 + \sum_{j=1}^N \frac{\alpha_j}{r + \frac{\alpha_K}{\gamma''+1}} 1$ for all $i \in \{1, 2, \dots, N\}$. Therefore, $\hat{K}(Q_f)$ is increasing in carbon tax for all positive values of ρ and all of the production inputs are decreasing in carbon tax beyond a sufficiently large value of ρ .

Let WA_3 be equal to the right hand side of the inequality in Condition (2.67). Then, if $r < 1$, we have $T_i > WA_3$ for $i \in \{1, 2, \dots, N\}$ and $\forall \rho \geq 0$. Therefore, all of the optimal values of production inputs are decreasing in carbon tax, when optimal production quantity and technology level are Q_0 and K_0 , respectively.

Let WA_4 be equal to the right hand side of the inequality in Condition (2.68). Then, if $r < 1 - \frac{\alpha_K}{\gamma''+1}$, we have $T_i > WA_4$ for $i \in \{1, 2, \dots, N\}$ and $\forall \rho \geq 0$. Therefore,

all of the optimal values of production inputs are decreasing in carbon tax, when optimal production quantity and technology level are Q_v and $\hat{K}(Q_v)$, respectively.

Letting

$$x_{N+1} = K^{\gamma''+1}, x_{N+2} = Q, \alpha_{N+1} = \frac{\alpha_K}{\gamma''+1}, \alpha_{N+2} = -1, p_{N+1} = \frac{\gamma' c_K}{\gamma''+1}, p_{N+2} = -s, \beta_{N+1} = 0, \beta_{N+2} = 0, \text{ and } x_0 = K_0^{\gamma''+1},$$

and letting χ be the set of indices of non-binding decision variables, we can rewrite the finding in Theorem 2.6 as follows.

$$x_i^* \text{ is } \begin{cases} \text{increasing in } \rho, & \text{if } \frac{\beta_i}{p_i + \rho\beta_i} < \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho\beta_j}, \\ \text{nonincreasing in } \rho, & \text{otherwise.} \end{cases} \quad (2.69)$$

where, $r^* = \sum_{j \in \chi} \alpha_j$.

Then economic meaning of Theorem 2.6 is that if the share of the cost of the input i 's due to its carbon footprint ($\rho\beta_i$) in its total cost ($p_i + \rho\beta_i$) is less than the weighted average of that of all non-binding decision variables, the optimal value of x_i is increasing in carbon price, where the weight of the input $i \in \chi$, $\frac{\alpha_i}{r^*}$, is the ratio of x_i 's factor elasticity to the sum of all factor elasticities, including those of Q and K if they are non-binding.

The behavior of the optimal emission of the firm with respect to carbon tax is presented in the following proposition.

Proposition 2.8. *Optimal value of the firm's total emission is monotonic decreasing in carbon tax, ρ .*

Proof. The proof is provided in Appendix A. □

Omitting the economically nonviable regions (the regions which are above Z_{max} for all possible order relations among Z Benchmarks), we provide the following graphs illustrating manufacturer's optimal response to increasing carbon tax.

For $0 < r < 1 - \frac{\alpha_K}{\gamma''+1}$

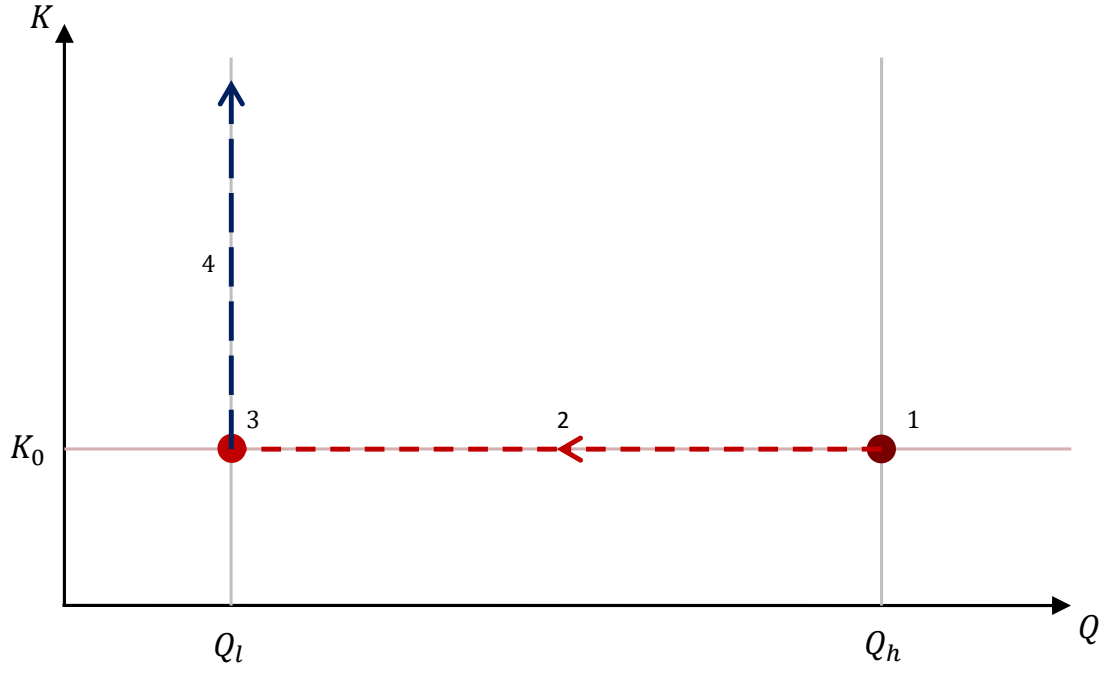


Figure 2.1: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{int} \leq Z_{h,int}$

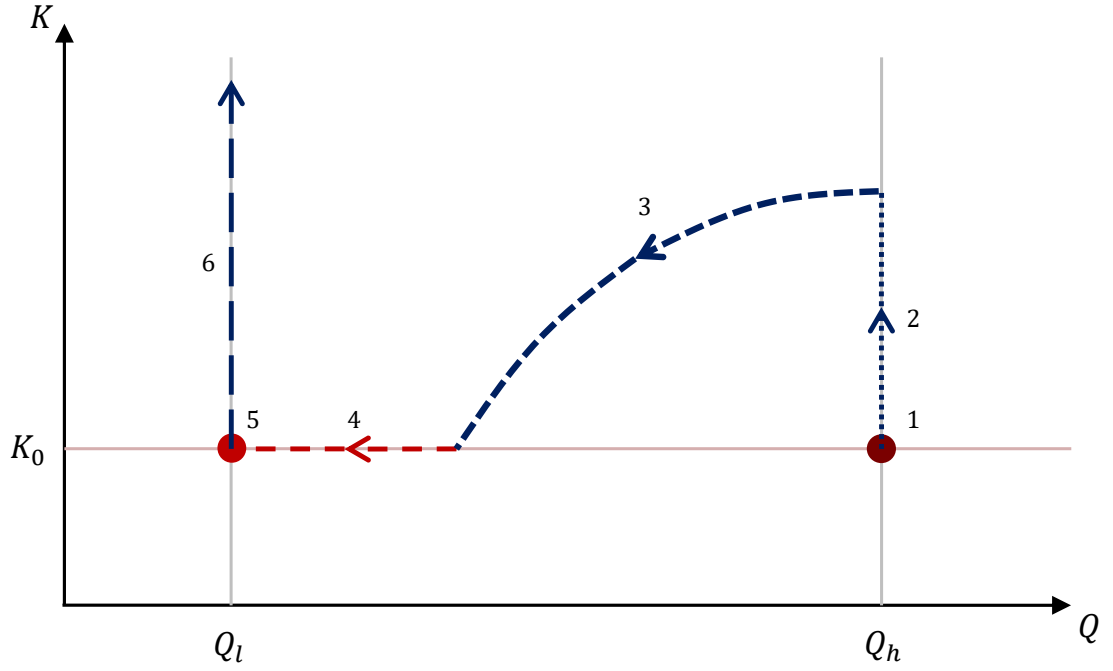


Figure 2.2: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{h,int} < Z_{int} < Z_{l,int}$

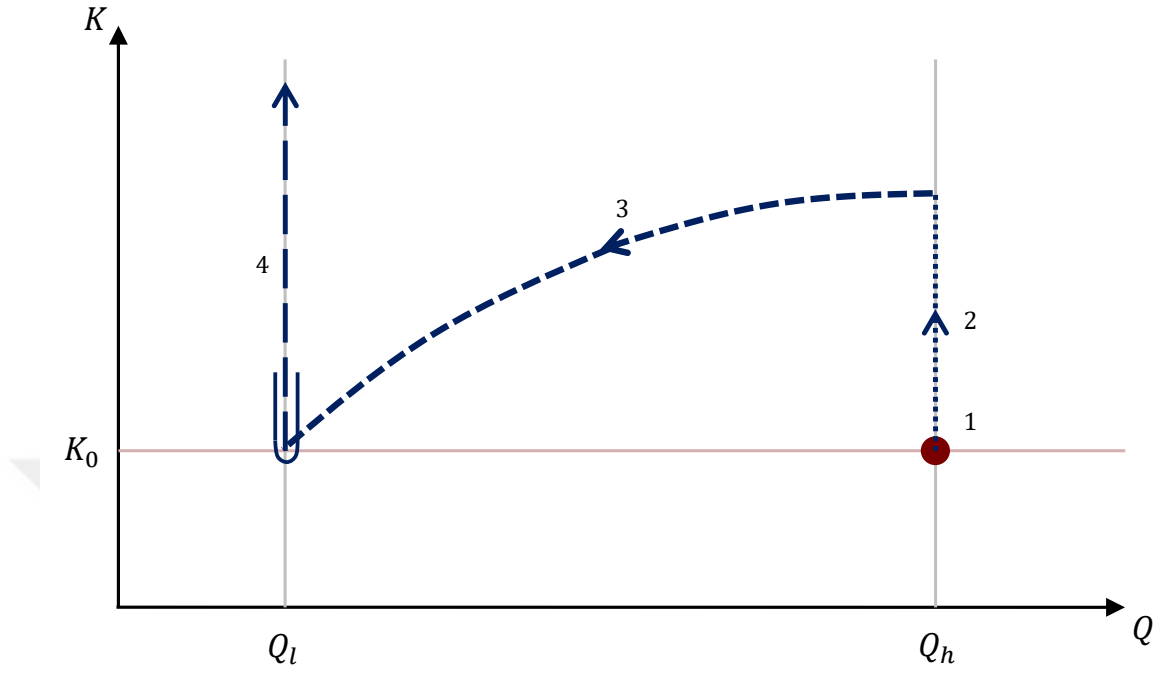


Figure 2.3: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{l,int} \leq Z_{int}$

For $0 < r = 1 - \frac{\alpha_K}{\gamma''+1}$

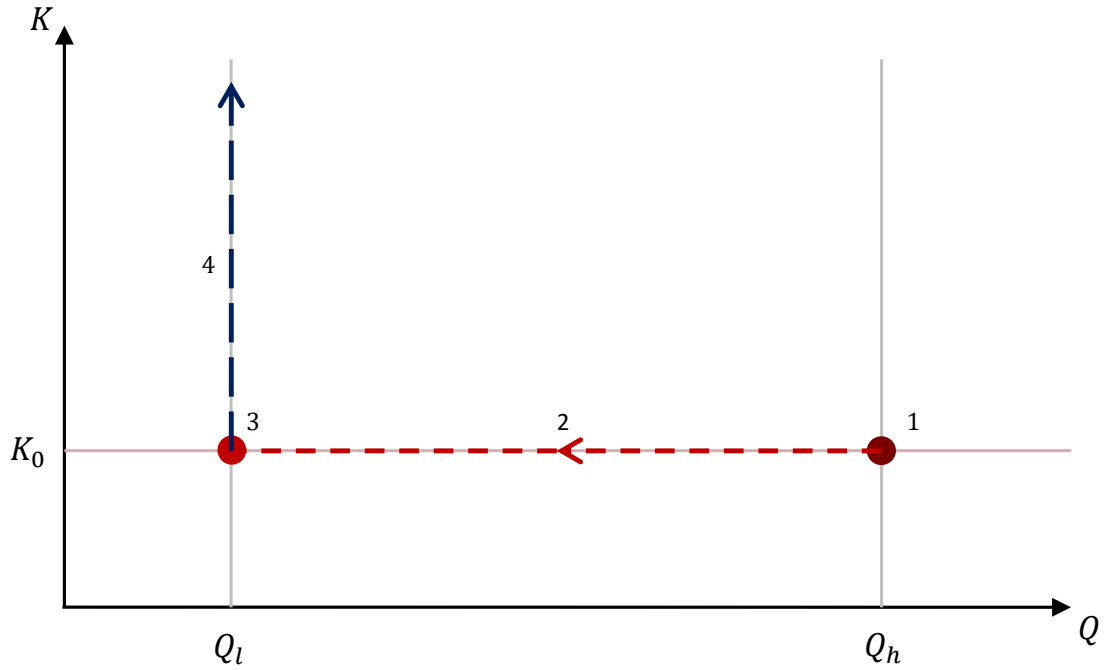


Figure 2.4: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{int} \leq Z_{h,int}$

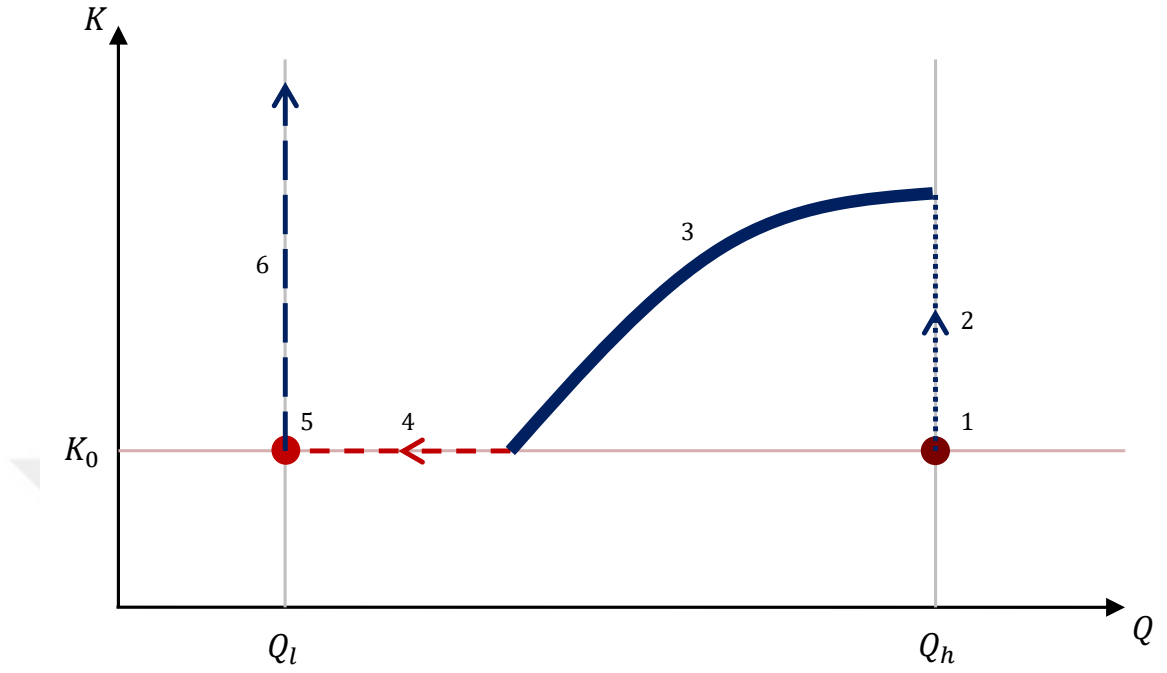


Figure 2.5: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{h,int} < Z_{int} < Z_{l,int}$

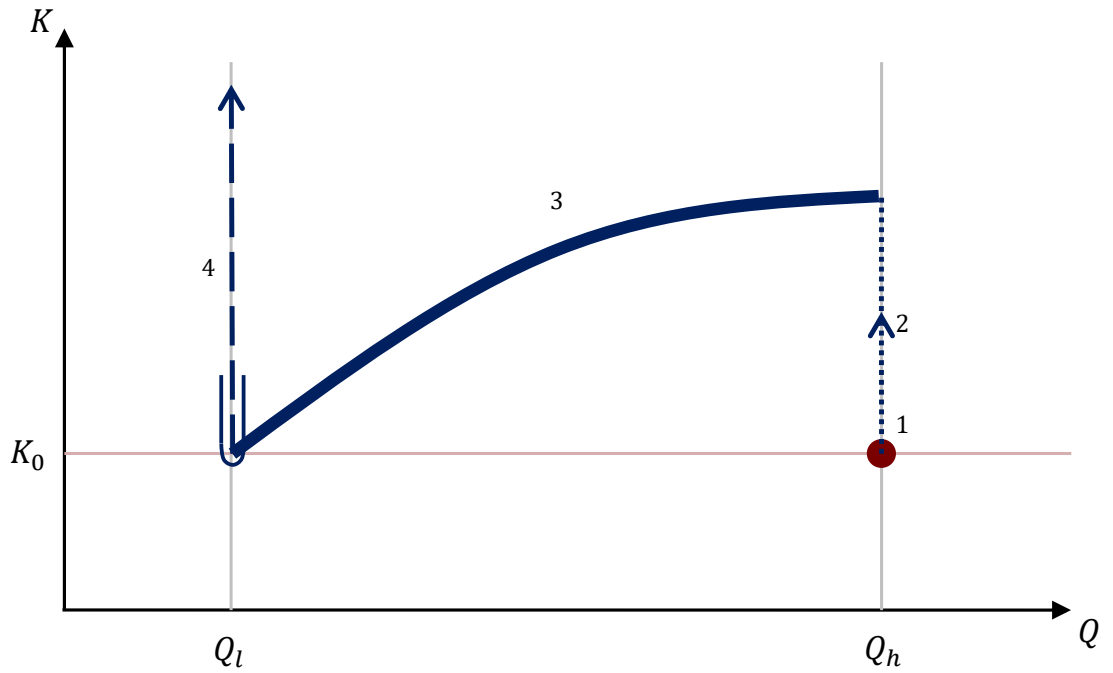


Figure 2.6: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{l,int} \leq Z_{int}$

For $0 < 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$

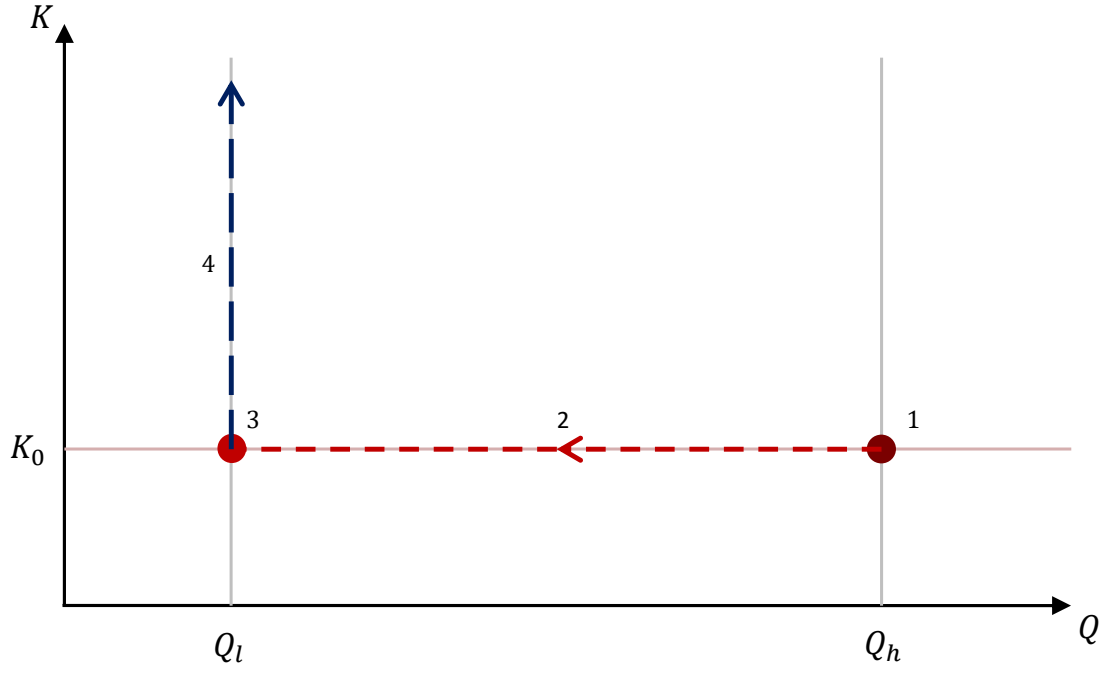


Figure 2.7: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{int} \leq Z_{h,int}$

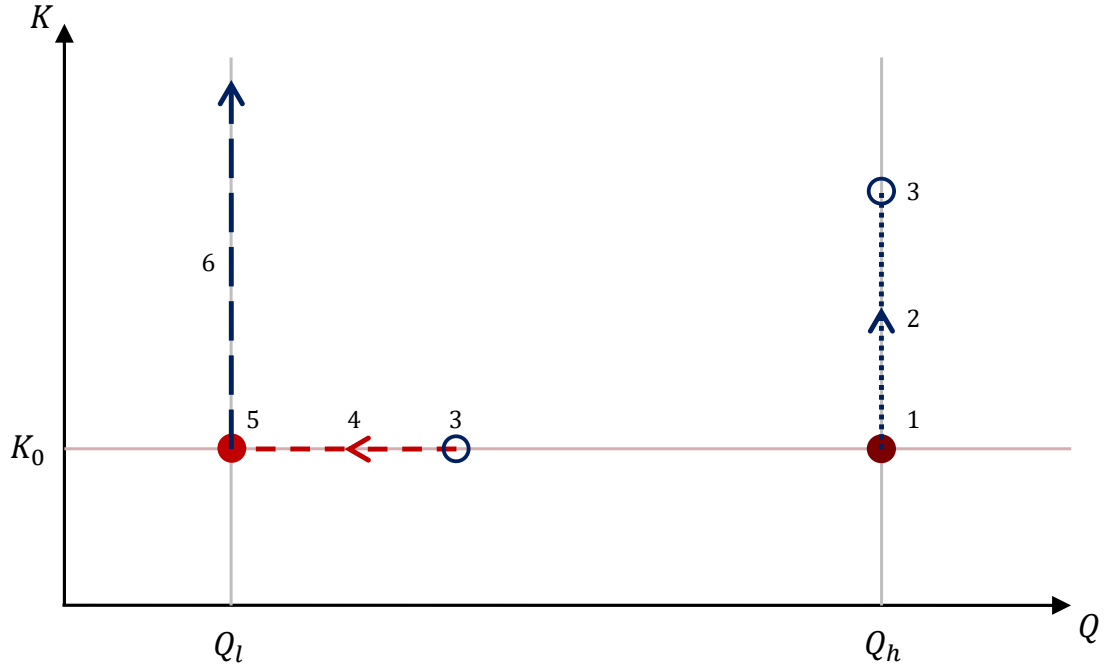


Figure 2.8: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{int} < Z_{hv}^{00} \leq Z_{l,0}$

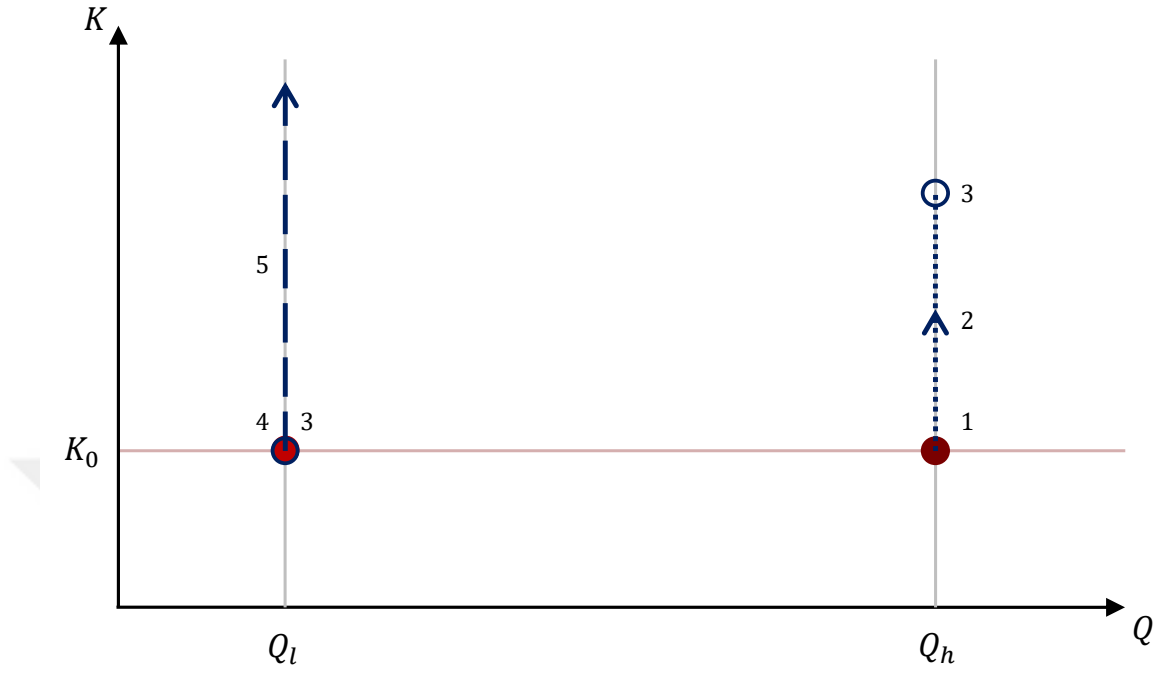


Figure 2.9: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,0} < Z_{hv}^{l0} < Z_{l,int}$

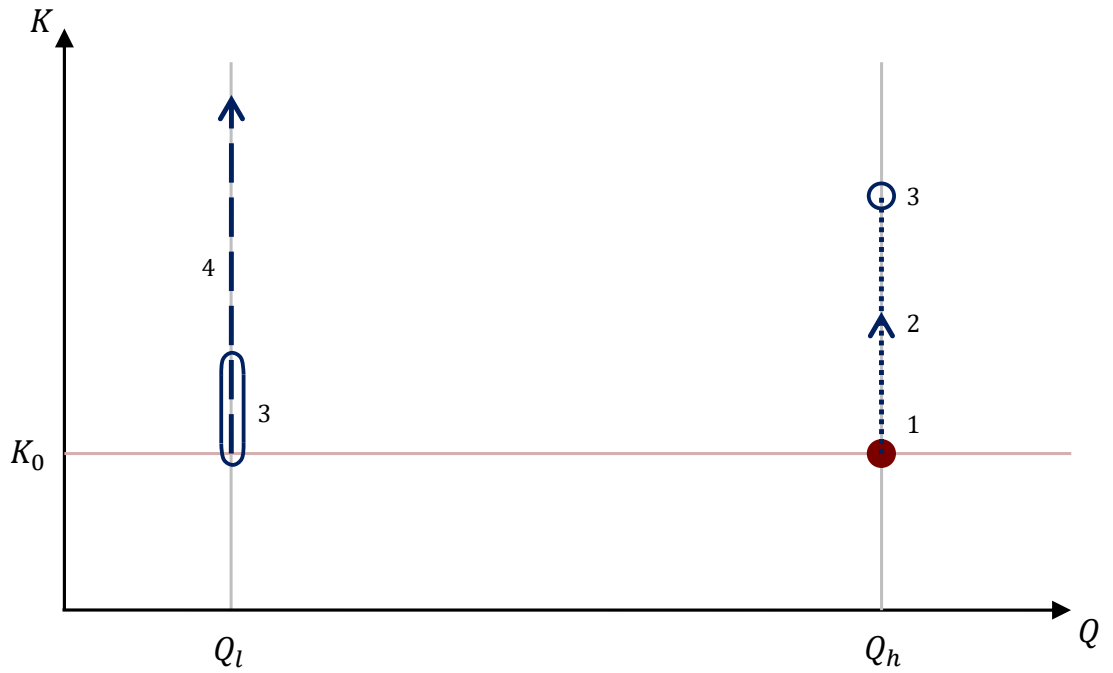


Figure 2.10: Change in the Optimal Policy w.r.t. Carbon Tax for $(Z_{h,int} < Z_{int} < Z_{l,int} \text{ and } Z_{l,int} \leq Z_{hv}^{lv}) \text{ or } Z_{l,int} \leq Z_{int}$

For $r = 1$

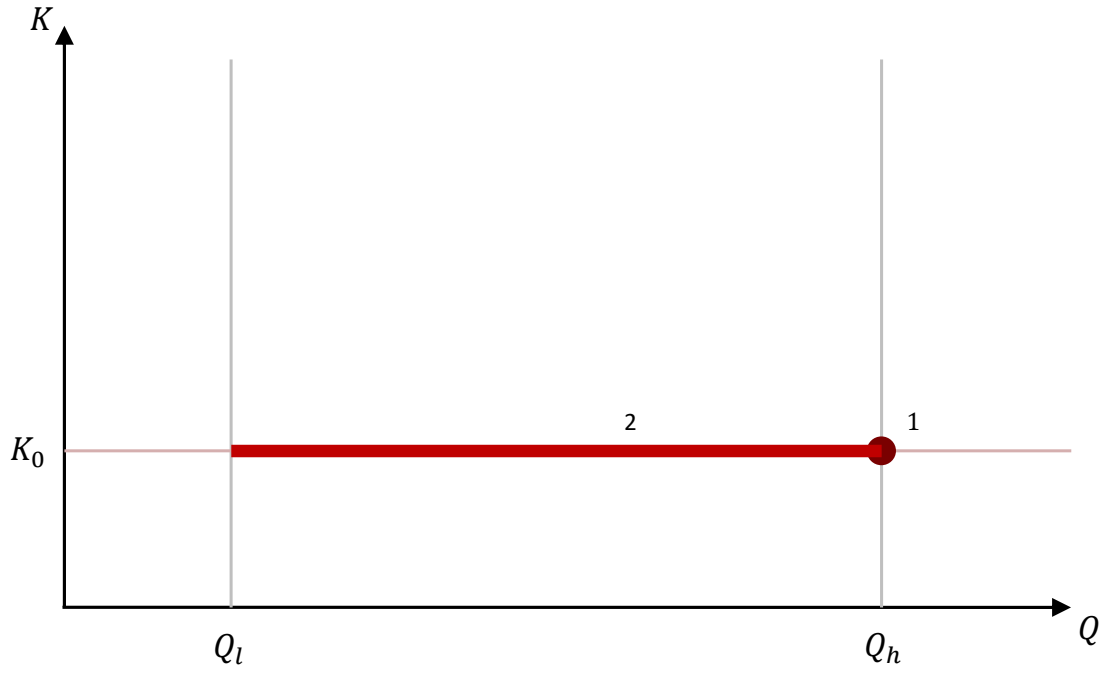


Figure 2.11: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{int} \leq Z_{h,int}$

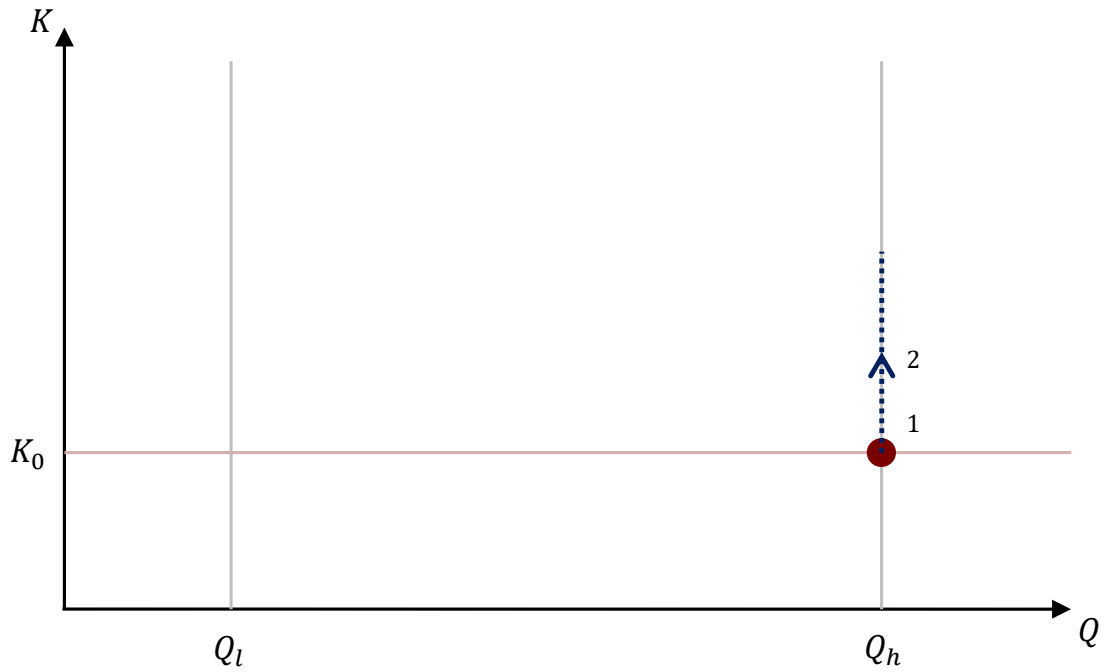


Figure 2.12: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{h,int} < Z_{int}$

For $r > 1$

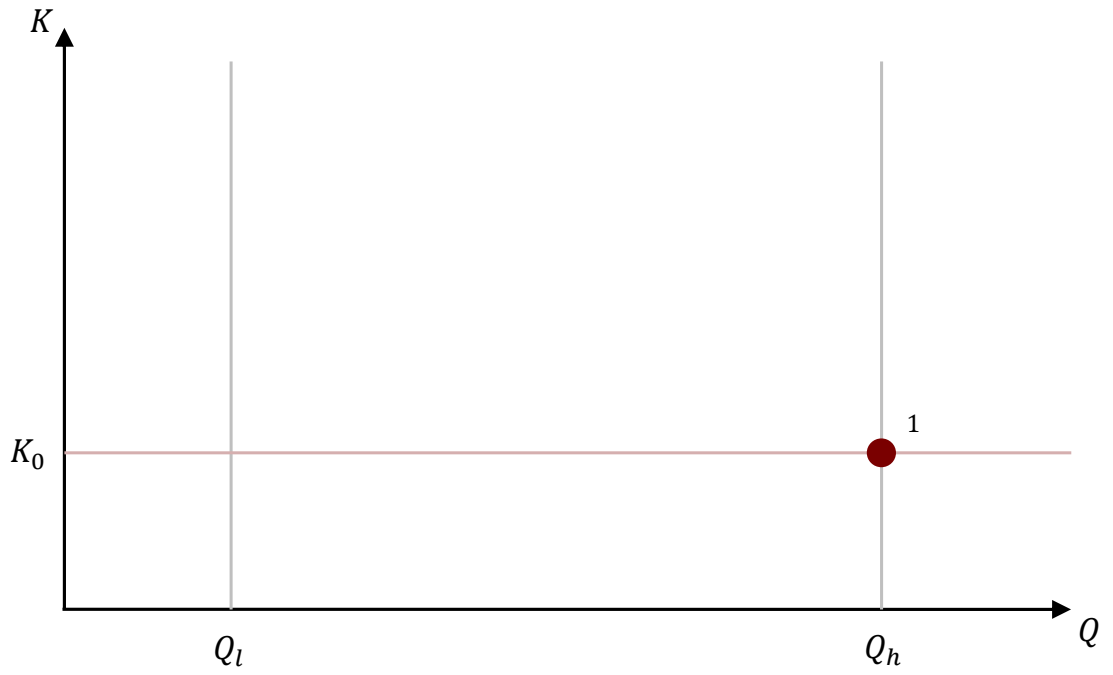


Figure 2.13: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{int} \leq Z_{h,int}$

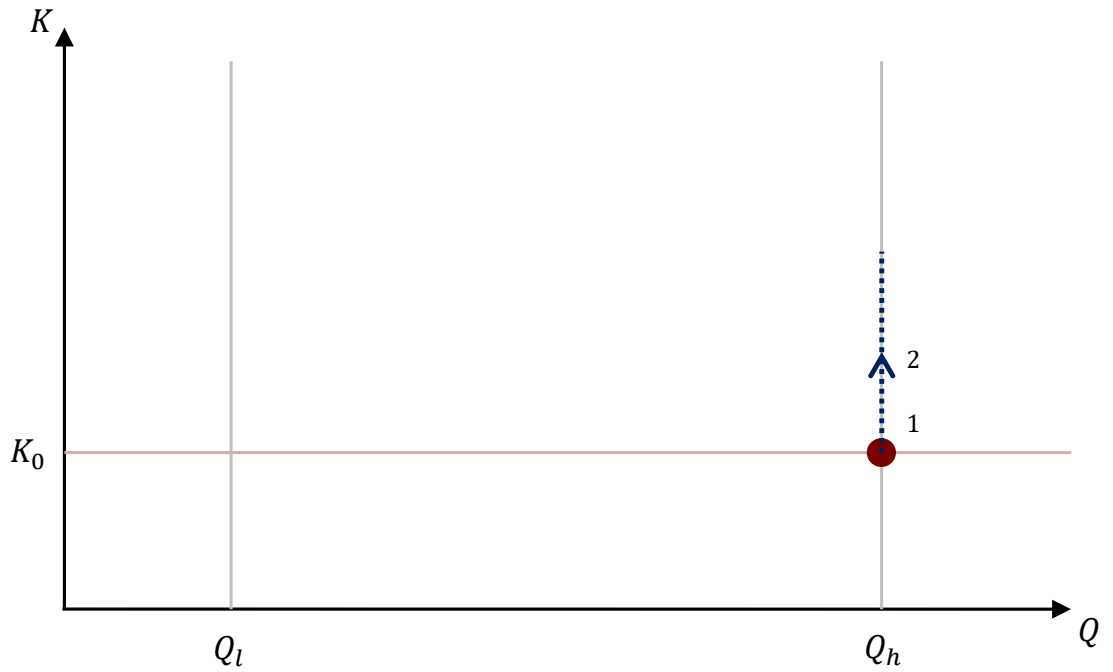


Figure 2.14: Change in the Optimal Policy w.r.t. Carbon Tax for $Z_{h,int} < Z_{int}$

The interpretations derived from Theorem 2.5 and from above graphs are as follows.

1. If $r < 1 - \alpha_K/(\gamma'' + 1)$, then, as we increase carbon tax from $-\min\{p_i/\beta_i\}$ s.t. $i \in \{1, 2, \dots, N\}$,
 - (a) the graph 2.1 illustrates the change in the optimal policy when $Z_{int} \leq Z_{h,int}$ and $Z_{min} < Z_{h,0}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{l,int}$ (in the regions 1, 2, and 3); and if $Z_{max} < Z_{l,int}$, technology enhancement is not economically viable; if $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 4), and thus, technology level is maximized at $\hat{K}(Q_l)|_{Z_{max}}$. On the other hand, at optimality, until Z reaches $Z_{h,0}$ (in the region 1), firm produce at her maximum amount, Q_h ; production quantity reduces gradually from Q_h to Q_l in between $Z_{h,0}$ and $Z_{l,0}$ (in the region 2); and firm produces at her minimum contract level, Q_l , between $Z_{l,0}$ and Z_{max} (in the regions 3 and 4). Therefore, any carbon tax value that satisfies $Z < Z_{h,0}$ leads to maximum output.
 - (b) the graph 2.2 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{min} < Z_{h,int}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1); the firm's technology level is increasing in carbon tax between $Z_{h,int}$ and $Z_{h,v}$ (in the region 2); she acquires new technology but her technology level is decreasing in carbon price between $Z_{h,v}$ and Z_{int} (in the region 3); the firm does not acquire new technology between Z_{int} and $Z_{l,int}$ (in the regions 4 and 5); and given that $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 6). On the other hand, at optimality, firm produces at her maximum amount, Q_h , until Z reaches $Z_{h,v}$ (in the regions 1 and 2); production quantity reduces gradually from Q_h to Q_l between $Z_{h,v}$ and $Z_{l,0}$ (in the regions 3 and 4); and firm produces at her minimum contract level, Q_l , between $Z_{l,0}$ and Z_{max} (in the regions 5 and 6).
 - (c) the graph 2.3 illustrates the change in the optimal policy when $Z_{l,int} \leq Z_{int}$ and $Z_{min} < Z_{h,int}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1); the firm's technology

level is increasing in carbon tax between $Z_{h,int}$ and $Z_{h,v}$ (in the region 2); the firm acquires new technology but her technology level is decreasing in carbon price between $Z_{h,v}$ and $Z_{l,v}$ (in the region 3); and given that $Z_{l,v} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,v}$ and Z_{max} (in the region 4). On the other hand, at optimality, the firm produces at her maximum amount, Q_h , until Z reaches $Z_{h,v}$ (in the regions 1 and 2); her production quantity reduces gradually from Q_h to Q_l between $Z_{h,v}$ and $Z_{l,v}$ (in the region 3); and firm produces at her minimum contract level, Q_l , between $Z_{l,v}$ and Z_{max} (in the region 4).

2. If $r = 1 - \alpha_K/(\gamma'' + 1)$, then, as we increase carbon tax from $-\min\{p_i/\beta_i\}$ s.t. $i \in \{1, 2, \dots, N\}$,

(a) the graph 2.4 illustrates the change in the optimal policy when $Z_{int} \leq Z_{h,int}$ and $Z_{min} < Z_{h,0}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{l,int}$ (in the regions 1, 2, and 3); and given that $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 4). On the other hand, at optimality, firm produce at her maximum amount, Q_h , until Z reaches $Z_{h,0}$ (in the region 1), optimal production quantity reduces gradually from Q_h to Q_l between $Z_{h,0}$ and $Z_{l,0}$ (in the region 2); and firm produces at her minimum contract level, Q_l , between $Z_{l,0}$ and Z_{max} (in the regions 3 and 4).

(b) the graph 2.5 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{min} < Z_{h,int}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1); the firm's technology level is increasing in carbon tax between $Z_{h,int}$ and Z_{int} (in the region 2); the firm's technology level can be increased to any value in between K_0 and $\hat{K}(Q_h, Z_{int})$ depending on her production quantity at $Z = Z_{int}$ (technology level is decreasing in Q at this Z value) (in the region 3); the firm does not acquire new technology between Z_{int} and $Z_{l,int}$ (in the regions 4 and 5); and given that $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 6). On the other hand, at optimality, until Z reaches Z_{int} , firm produces at her maximum amount, Q_h , (in the regions 1 and 2); she may produce an amount in between Q_0 and Q_h at Z_{int} (technology

level is decreasing in Q at this Z value) (in the region 3); her production quantity reduces gradually from $Q_0(Z_{int})$ to $Q_0(Z_{l,0})$ as Z increases from Z_{int} to $Z_{l,0}$ (in the region 4); and firm produces at her minimum contract level, Q_l , between $Z_{l,0}$ and Z_{max} (in the regions 5 and 6).

- (c) the graph 2.6 illustrates the change in the optimal policy when $Z_{l,int} \leq Z_{int}$ and $Z_{min} < Z_{h,int}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1), the firm's technology level is increasing in carbon tax between $Z_{h,int}$ and Z_{int} (in the region 2), at Z_{int} she may increase her technology level to some level in between $\hat{K}(Q_l, Z_{int})$ and $\hat{K}(Q_h, Z_{int})$ (technology level is decreasing in Q at this Z value) (in the region 3); and given that $Z_{int} < Z_{max}$, her technology level is increasing in carbon tax between Z_{int} and Z_{max} (in the region 4). On the other hand, at optimality, firm produces at her maximum amount, Q_h , until Z reaches Z_{int} (in the regions 1 and 2); she may produce an amount in between Q_l and Q_h at Z_{int} (technology level is decreasing in Q at this Z value) (in the region 3); and firm produces at her minimum contract level, Q_l , between Z_{int} and Z_{max} (in the region 4).

3. If $1 - \alpha_K/(\gamma'' + 1) < r < 1$, then, as we increase carbon tax from $-\min\{p_i/\beta_i\}$ s.t. $i \in \{1, 2, \dots, N\}$,

- (a) the graph 2.7 illustrates the change in the optimal policy when $Z_{int} \leq Z_{h,int}$ and $Z_{min} < Z_{h,0}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{l,int}$ (in the regions 1, 2, and 3); and given that $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 4). On the other hand, at optimality, firm produce at her maximum amount, Q_h , until Z reaches $Z_{h,0}$ (in the region 1), optimal production quantity reduces gradually from Q_h to Q_l between $Z_{h,0}$ and $Z_{l,0}$ (in the region 2); and firm produces at her minimum contract level, Q_l , between $Z_{l,0}$ and Z_{max} (in the regions 3 and 4).

- (b) the graph 2.8 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{min} < Z_{h,int}$, and $Z_{int} < Z_{hv}^{00} \leq Z_{l,0}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1); firm increases her technology level between $Z_{h,int}$ and Z_{hv}^{00} (in

the region 2); the firm may either increase her technology level to $\hat{K}(Q_h)$ or stay at K_0 at Z_{hv}^{00} (in the region 3 shown by two empty circles); the firm does not acquire new technology between Z_{hv}^{00} and $Z_{l,int}$ (in the regions 4 and 5); and given that $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 6). On the other hand, at optimality, until Z reaches Z_{hv}^{00} , firm produces at her maximum amount (in the regions 1 and 2); she may produce an amount that is equal to either Q_0 or Q_h at Z_{hv}^{00} (in the regions 4 and 5); her optimal production quantity reduces gradually from $Q_0(Z_{int})$ to $Q_0(Z_{l,0})$ between Z_{hv}^{00} and $Z_{l,0}$ (in the region 4); and firm produces at her minimum contract level, Q_l , between $Z_{l,0}$ and Z_{max} (in the regions 5 and 6).

(c) the graph 2.9 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{min} < Z_{h,int}$, and $Z_{l,0} < Z_{hv}^{l0} \leq Z_{l,int}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1), between $Z_{h,int}$ and Z_{hv}^{l0} firm increases her technology level (in the region 2), at Z_{hv}^{l0} firm may either increase her technology level to $\hat{K}(Q_h)$ or stay at K_0 (in the region 3 shown by two circles); the firm does not acquire new technology between Z_{hv}^{l0} and $Z_{l,int}$ (in the region 4); and given that $Z_{l,int} < Z_{max}$, her technology level is increasing in carbon tax between $Z_{l,int}$ and Z_{max} (in the region 5). On the other hand, at optimality, firm produces at her maximum amount until Z reaches Z_{hv}^{l0} (in the regions 1 and 2), she may produce an amount that is equal to either Q_l or Q_h at Z_{hv}^{l0} (in the region 3 shown by two circles); and firm produces at her minimum contract level, Q_l , between Z_{hv}^{l0} and Z_{max} (in the regions 4 and 5).

(d) the graph 2.10 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{min} < Z_{h,int}$, and $Z_{l,int} \leq Z_{hv}^{lv}$ or for $Z_{l,int} \leq Z_{int}$. At optimality, the firm does not acquire new technology until Z reaches $Z_{h,int}$ (in the region 1), the firm increases her technology level between $Z_{h,int}$ and Z_{hv}^{lv} (in the region 2), firm may increase her technology level either to $\hat{K}(Q_h)$ or to $\hat{K}(Q_l)$ at Z_{hv}^{lv} (in the region 3 shown by two empty circles); and given that $Z_{hv}^{lv} < Z_{max}$, her technology level is increasing in carbon tax between Z_{hv}^{lv} and Z_{max} (in the region 4). On the other hand, at optimality, firm produces at her maximum amount, Q_h , until

Z reaches Z_{hv}^{lv} (in the regions 1 and 2); she may produce an amount that is equal to either Q_l or Q_h at Z_{hv}^{lv} (in the region 3 shown by two empty circles); and firm produces at her minimum contract level, Q_l , between Z_{hv}^{lv} and Z_{max} (in the region 4).

4. If $r = 1$, then, as we increase carbon tax from $-\min\{p_i/\beta_i\}$ s.t. $i \in \{1, 2, \dots, N\}$,
 - (a) the graph 2.11 illustrates the change in the optimal policy when $Z_{int} \leq Z_{h,int}$. At optimality, the firm does not enhance her technology (in the regions 1 and 2). On the other hand, at optimality, until carbon benchmark Z reaches Z_{int} her optimal production quantity stays at Q_h (in the region 1), and at Z_{int} she may produce an amount in between Q_l and Q_h (since $Z_{max} = Z_{int}$ from Proposition 2.7) (in the region 2).
 - (b) the graph 2.12 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int}$. At optimality, the firm does not enhance her technology until carbon benchmark Z reaches $Z_{h,int}$ (in the region 1); and her optimal technology level is increasing in carbon price between $Z_{h,int}$ and Z_{max} (in the region 2). On the other hand, at optimality, her production quantity is Q_h for all values of carbon price that fall below Z_{max} (in the regions 1 and 2).
5. If $r > 1$, then, as we increase carbon tax from $-\min\{p_i/\beta_i\}$ s.t. $i \in \{1, 2, \dots, N\}$,
 - (a) the graph 2.13 illustrates the change in the optimal policy when $Z_{int} \leq Z_{h,int}$. At optimality, the firm neither enhances her technology nor changes her production level and her production quantity is Q_h (in the region 1).
 - (b) the graph 2.14 illustrates the change in the optimal policy when $Z_{h,int} < Z_{int}$. At optimality, the firm does not enhance her technology level until Z reaches $Z_{h,int}$ (in the region 1); and her technology level is increasing in carbon price between $Z_{h,int}$ and Z_{max} (in the region 2). On the other hand, at optimality, her optimal production quantity is Q_h for all values of carbon price that fall below Z_{max} (in the regions 1 and 2).

Therefore, there is a significant trade-off between technology enhancement and consumer welfare.

Now, suppose that the social planner has a emission target, κ , and that the total emission of the firm should be less than or equal to this target. Assuming that this target will not lead to negative profits, we will make a discussion that summarizes our recommendations regarding the problem of finding the carbon tax intervals which will lead the firm to reduce her emission to/below this target value, while considering the trade-offs between technology enhancement and consumers' welfare. Now, remember that the optimal emission level, which is expressed as

$$em^*(\rho) = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \left(\frac{Q^*}{aK^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}},$$

where Q^* is the optimal production quantity, K^* is the optimal technology level, and r^* is equivalent to the elasticity of scale, that is, $r^* = r$, when production quantity and technology level are binding; the elasticity of scale minus the elasticity of the production quantity, that is, $r^* = r - 1$, when production quantity is non-binding and technology level is binding; the elasticity of scale plus the elasticity of technology minus the elasticity of production quantity, that is, $r^* = r + \frac{\alpha_K}{\gamma''+1} - 1$ when production quantity and technology level are non-binding, at optimality.

Let ρ_κ be the carbon tax value at which the optimal emission level of the firm is equal to the carbon emission target, κ , of the social planner. Then, the carbon tax value, ρ_κ , that leads the optimal emission level of the firm be equal to the social planner's target can be found through the following equality:

$$em^*(\rho_\kappa) = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho_\kappa \beta_i} \left(\frac{Q^*}{aK^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho_\kappa \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} = \kappa.$$

Note that for $1 - \frac{\alpha_K}{\gamma''+1} < r < 1$, there exists an alternate optima for $Z_{int} > Z_{h,int}$ after which the optimal production quantity and technology level pair falls from $(Q_h, \hat{K}(Q_h))$ to (Q_0, K_0) or (Q_l, K_0) or $(Q_l, \hat{K}(Q_l))$ for $Z_\Phi = Z_{hv}^{00}$ or $Z_\Phi = Z_{hv}^{l0}$ or $Z_\Phi = Z_{hv}^{lv}$, respectively, depending on the order relations among $Z_{l,0}$, $Z_{l,int}$, Z_{hv}^{00} , Z_{hv}^{l0} , and Z_{hv}^{lv} .

If we let $E(\rho) = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \prod_{j=1}^N \frac{p_j + \rho \beta_j}{\alpha_j}$, then we can represent optimal emission levels at different critical values of the production quantity and technology level as $em_v(Q_f) = \frac{Q_h}{a\hat{K}(Q_h)^{\alpha_K}} E(\rho)$, $em_0(Q_0) = \frac{Q_0}{aK_0^{\alpha_K}} E(\rho)$, $em_0(Q_l) = \frac{Q_l}{aK_0^{\alpha_K}} E(\rho)$, and $em_v(Q_l) = \frac{Q_l}{a\hat{K}(Q_l)^{\alpha_K}} E(\rho)$.

We proved earlier that $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ is increasing in Q . Let ϵ be an infinitesimally small positive real number. Suppose that $Z_\Phi = Z_{hv}^{00}$. Then, for a given Z such that $Z = Z_\Phi + \epsilon > Z_{int} > Z_{h,int}$, since $Q_0 < Q_{int} \leq Q_h$ and $\hat{K}(Q_0) < \hat{K}(Q_{int}) = K_0$, we should have

$$\frac{Q_h}{\hat{K}(Q_h)^{\alpha_K}} > \frac{Q_{int}}{\hat{K}(Q_{int})^{\alpha_K}} = \frac{Q_{int}}{K_0^{\alpha_K}} > \frac{Q_0}{K_0^{\alpha_K}}, \text{ that is, } em_v(Q_h) > em_0(Q_0).$$

Suppose that $Z_\Phi = Z_{hv}^{l0}$. Then, for a given Z such that $Z = Z_\Phi + \epsilon > Z_{l,0} > Z_{int} > Z_{h,int}$, since $Q_0 < Q_l < Q_{int} \leq Q_h$ and $\hat{K}(Q_l) < \hat{K}(Q_{int}) = K_0$, we should have

$$\frac{Q_h}{\hat{K}(Q_h)^{\alpha_K}} > \frac{Q_{int}}{\hat{K}(Q_{int})^{\alpha_K}} = \frac{Q_{int}}{K_0^{\alpha_K}} > \frac{Q_l}{K_0^{\alpha_K}}, \text{ that is, } em_v(Q_h) > em_0(Q_l).$$

Suppose that $Z_\Phi = Z_{hv}^{lv}$. Then, for a given Z such that $Z = Z_\Phi + \epsilon > Z_{l,int} > Z_{l,0} > Z_{int} > Z_{h,int}$, since $Q_0 < Q_{int} \leq Q_l < Q_h$, we should have

$$\frac{Q_h}{\hat{K}(Q_h)^{\alpha_K}} > \frac{Q_l}{\hat{K}(Q_l)^{\alpha_K}}, \text{ that is, } em_v(Q_h) > em_v(Q_l).$$

Therefore, there exists a discontinuity (downward) in the optimal emission level, production quantity, and technology level of the manufacturer with respect to carbon price, ρ , (Z). This also means that, if $Z_\Phi = Z_{hv}^{00}$, carbon emission targets that are in the interval $(em_0(Q_0), em_v(Q_h))|_{Z_\Phi}$, if $Z_\Phi = Z_{hv}^{l0}$, carbon emission targets that are in the interval $(em_0(Q_l), em_v(Q_h))|_{Z_\Phi}$; if $Z_\Phi = Z_{hv}^{lv}$, carbon emission targets that are in the interval $(em_v(Q_l), em_v(Q_h))|_{Z_\Phi}$ are not attainable at optimality.

Suppose that the social planner's emission target is within one of these intervals but far from the upper and lower extremes of the considered interval, then, assuming that the emission target is inflexible, and considering that optimal profit is continuous in carbon tax, social planner's only option is to set a carbon tax that will reduce the emission level of the producer to the lower extreme of the emission interval. In this case, while, at optimality, the emission reduces far below the emission target, indicating higher ecological and societal gains, since both technology level and output will also reduce sharply, a significant loss in consumers' welfare will be observed.

Now, suppose that the difference is large but the emission target of the social planner is very close to the higher extreme of the interval containing sub-optimal emission levels. In this case, there may be two scenarios: 1) There is no flexibility in choosing the emission target, and thus, social planner sets the carbon tax

infinitesimally larger than the value leading to region of sub-optimal emission levels. 2) There is some flexibility in deciding emission target, and thus, social planner sets the carbon tax infinitesimally smaller than this value.

In the first scenario, the firm reduces her emission level sharply below the target level of the social planner in the short term, at optimality. In the second scenario, the firm reduces her emission level just a few BTU's lower than the normal target of the social planner, but the optimal service and technology levels will be significantly higher than the values that will be observed in the first scenario.

Note that, since the loss in the consumers' welfare may turn out to be much higher than the ecological and societal gains, choosing the higher extreme of the sub-optimal emission range as the new emission target and setting the carbon tax infinitesimally lower than the value leading to region of sub-optimal emission levels as in the second scenario may be considered as more rational. Furthermore, in this thesis we studied a single period problem. If a multi-period production planning and tax setting problem is considered, since a small compromise on the emission reduction level in the first period, may lead to huge gains in terms of technological enhancements and the firm will operate with a much higher productivity level with no additional cost starting from the next planning horizon, total emission reduction of the second scenario may turn out to be much more than that of the first scenario in the long run. Therefore, a multi-period problem can be considered as an important extension of this thesis.

Now, let $em_{\Phi,l}$ and $em_{\Phi,h}$ be the lower and higher extremes of the set of sub-optimal emission levels defined by the carbon tax value at which there are alternative optimal values for the decision variables, respectively. Then, if the emission target, κ , of the social planner is not flexible, then the total carbon emission of the firm should be less than or equal to this target (i.e., $em^* \leq \kappa$).

Accordingly, considering the discussion in the last five paragraphs, it can be assumed that the social planner should set an emission target outside the range of suboptimal emission levels. This means we may assume that $\kappa \notin [em_{\Phi,l}, em_{\Phi,h}]$. In this context, considering the trade-offs between technology enhancement, consumer welfare, and emission level and supposing that the social planner's emission target is inflexible and does not lead to negative profits ($Z_{\kappa} \leq Z_{max}$), since the optimal emission level is a decreasing function of Z , following statements wrap

up our findings.

1. For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$,

(a) when $Z_{int} \leq Z_{h,int}$, since $Z_{l,0} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,0} < Z_{l,int} \leq Z_K < Z_{max}$, then increasing carbon price satisfying $Z_K \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases technology level and decreases the emission below the target level, at optimality.
- ii. if $Z_{l,0} \leq Z_K < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level; satisfying $Z_K < Z < Z_{l,int}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_K$, decreases the emission below the target level, at optimality.
- iii. if $Z_{l,0} < Z_K \leq Z_{max} < Z_{l,int}$, then increasing carbon price satisfying $Z_K < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_K$, decreases the emission level below the target level, at optimality.
- iv. if $Z_{h,0} < Z_K \leq Z_{l,0} < Z_{max}$, then a carbon tax satisfying $Z = Z_K$ may be selected since, for $Z_{h,0} < Z < Z_{l,int}$, output is decreasing in carbon tax and technology level is constant at its initial level, K_0 , at optimality.
- v. if $Z_K < Z_{h,0} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_K \leq Z < Z_{h,0}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

(b) when $Z_{h,int} < Z_{int} < Z_{l,int}$, since $Z_{l,0} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,0} < Z_{l,int} \leq Z_K < Z_{max}$, then increasing carbon price satisfying $Z_K \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality
- ii. if $Z_{l,0} \leq Z_K < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , in-

creases the technology level and decreases the emission below the target level; satisfying $Z_\kappa < Z < Z_{l,int}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_\kappa$, decreases the emission below the target level, at optimality.

- iii. if $Z_{l,0} < Z_\kappa \leq Z_{max} < Z_{l,int}$, then increasing carbon price satisfying $Z_\kappa < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_\kappa$, decreases the emission level below the target level, at optimality.
- iv. if $Z_{h,v} < Z_\kappa \leq Z_{l,0} < Z_{max}$, a carbon tax satisfying $Z = Z_\kappa$ may be selected since, for $Z_{h,v} < Z < Z_{l,int}$, output is decreasing and technology level is non-increasing in carbon tax, at optimality.
- v. if $Z_{h,int} < Z_\kappa \leq Z_{h,v} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z \leq Z_{h,v}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level, at optimality.
- vi. if $Z_\kappa \leq Z_{h,int} < Z_{h,v} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_{h,int} \leq Z \leq Z_{h,v}$, while maintaining maximum output, Q_h , increases the technology level and decreases the emission below the target level; satisfying $Z_\kappa \leq Z \leq Z_{h,int}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

(c) when $Z_{l,int} \leq Z_{int}$, since $Z_{l,v} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,v} \leq Z_\kappa < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality.
- ii. if $Z_{h,v} < Z_\kappa \leq Z_{l,v} < Z_{max}$, a carbon tax satisfying $Z = Z_\kappa$ may be selected since, for $Z_{h,v} < Z < Z_{l,v}$, the output and the technology level are decreasing in carbon tax, at optimality.
- iii. if $Z_{h,int} < Z_\kappa \leq Z_{h,v} < Z_{l,v} < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z \leq Z_{h,v}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level, at optimality.

- iv. if $Z_\kappa \leq Z_{h,int} < Z_{h,v} < Z_{l,v} < Z_{max}$, then increasing carbon price satisfying $Z_{h,int} \leq Z \leq Z_{h,v}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level; satisfying $Z_\kappa \leq Z \leq Z_{h,int}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

2. For $0 < r = 1 - \alpha_K/(\gamma'' + 1)$,

(a) when $Z_{int} \leq Z_{h,int}$, since $Z_{l,0} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,0} < Z_{l,int} \leq Z_\kappa < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases technology level and decreases emission below the target level, at optimality.
- ii. if $Z_{l,0} \leq Z_\kappa < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level; satisfying $Z_\kappa \leq Z < Z_{l,int}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_\kappa$, decreases the emission below the target level, at optimality.
- iii. if $Z_{l,0} < Z_\kappa \leq Z_{max} < Z_{l,int}$, then increasing carbon price satisfying $Z_\kappa < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_\kappa$, decreases the emission level below the target level, at optimality
- iv. if $Z_{h,0} < Z_\kappa \leq Z_{l,0} < Z_{max}$, then a carbon tax satisfying $Z = Z_\kappa$ may be selected since, for $Z_{h,0} < Z < Z_{l,int}$, output is decreasing in carbon tax and technology level is constant at its initial level, at optimality.
- v. if $Z_\kappa < Z_{h,0} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_1$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

(b) when $Z_{h,int} < Z_{int} < Z_{l,int}$, since $Z_{l,0} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,0} < Z_{l,int} \leq Z_\kappa < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases

technology level and decreases the emission below the target level, at optimality.

- ii. if $Z_{l,0} \leq Z_\kappa < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, increases the technology level and decreases the emission below the target level; satisfying $Z_\kappa \leq Z < Z_{l,int}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_\kappa$, decreases the emission below the target level, at optimality.
 - iii. if $Z_{l,0} < Z_\kappa \leq Z_{max} < Z_{l,int}$, then increasing carbon price satisfying $Z_\kappa < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , as for $Z = Z_\kappa$, decreases the emission level below the target level, at optimality.
 - iv. if $Z_{int} < Z_\kappa \leq Z_{l,0} < Z_{max}$, a carbon price satisfying $Z = Z_\kappa$ may be selected since, for $Z_{int} < Z < Z_{l,0}$, output is decreasing and technology level is non-increasing in carbon tax, at optimality.
 - v. if $Z_{h,int} < Z_\kappa \leq Z_{int} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z \leq Z_{int}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level, at optimality.
 - vi. if $Z_\kappa \leq Z_{h,int} < Z_{int} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_{h,int} \leq Z \leq Z_{int}$, while maintaining maximum output, Q_h , increases the technology level and decreases the emission below the target level; satisfying $Z_\kappa \leq Z \leq Z_{h,int}$, while maintaining maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.
- (c) when $Z_{l,int} \leq Z_{int}$, since $Z_{int} < Z_{max}$ by Theorem 2.7,
- i. if $Z_{int} \leq Z_\kappa < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases technology level and decreases the emission below the target level, at optimality.
 - ii. if $Z_{h,int} < Z_\kappa \leq Z_{int} < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{int}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level, at

optimality.

- iii. if $Z_\kappa \leq Z_{h,int} < Z_{int} < Z_{max}$, then increasing carbon price satisfying $Z_{h,int} \leq Z \leq Z_{int}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level; satisfying $Z_\kappa \leq Z \leq Z_{h,int}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality

3. For $0 < 1 - \alpha_K/(\gamma'' + 1) < r < 1$,

(a) when $Z_{int} \leq Z_{h,int}$, since $Z_{l,0} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,0} < Z_{l,int} \leq Z_\kappa < Z_{max}$, then increasing carbon price while satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality.
- ii. if $Z_{l,0} \leq Z_\kappa < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality.
- iii. if $Z_{l,0} < Z_\kappa \leq Z_{max} < Z_{l,int}$, then increasing carbon price while satisfying $Z_\kappa < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , decreases the emission below the target level, at optimality.
- iv. if $Z_{h,0} < Z_\kappa \leq Z_2$, a carbon tax satisfying $Z = Z_\kappa$ may be selected since, for $Z_{h,0} < Z < Z_{l,int}$, output and technology level are decreasing in carbon tax, at optimality.
- v. if $Z_\kappa < Z_{h,0}$, then increasing carbon price while satisfying $Z_\kappa \leq Z < Z_1$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

(b) when $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{int} < Z_\Phi = Z_{hv}^{00} < Z_{l,0}$, since $Z_{l,0} < Z_{max}$ by Theorem 2.7,

- i. if $Z_{l,0} < Z_{l,int} \leq Z_\kappa < Z_{max}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases

- the technology level and decreases the emission below the target level, at optimality.
- ii. if $Z_{l,0} \leq Z_{\kappa} < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality.
 - iii. if $Z_{l,0} < Z_{\kappa} \leq Z_{max} < Z_{l,int}$, then increasing carbon price satisfying $Z_{\kappa} < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , decreases the optimal emission below the target level, at optimality.
 - iv. if $Z_{\Phi} = Z_{hv}^{00} < Z_{\kappa} \leq Z_{l,0} < Z_{max}$, then $Z = Z_{\kappa}$ may be selected since, for $Z_{h,0} < Z < Z_{l,int}$, output and technology level are decreasing in carbon tax, at optimality.
 - v. if $Z_{h,int} < Z_{\kappa} < Z_{\Phi} = Z_{hv}^{00} < Z_{l,0} < Z_{max}$, then increasing carbon price while satisfying $Z_{\kappa} \leq Z < Z_{hv}^{00}$, maintains maximum output, Q_h , increases the technology level, and decreases the optimal emission below the target level, at optimality.
 - vi. if $Z_{\kappa} \leq Z_{h,int} < Z_{\Phi} = Z_{hv}^{00} < Z_{l,0} < Z_{max}$, then increasing carbon price satisfying $Z_{h,int} \leq Z < Z_{hv}^{00}$, maintains maximum output, Q_h , increases the technology level, and decreases the optimal emission below the target level; satisfying $Z_{\kappa} \leq Z < Z_{h,int}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.
- (c) when $Z_{h,int} < Z_{int} < Z_{l,int}$ & $Z_{l,0} < Z_{\Phi} = Z_{hv}^{l0} < Z_{l,int}$, or, when $Z_{l,int} \leq Z_{int}$, since $Z_{h,int} < Z_{max}$ by Theorem 2.7,
- i. if $Z_{\Phi} = Z_{hv}^{l0} < Z_{l,int} \leq Z_{\kappa} < Z_{max}$, then increasing carbon price satisfying $Z_{\kappa} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality.
 - ii. if $Z_{\Phi} = Z_{hv}^{l0} \leq Z_{\kappa} < Z_{l,int} < Z_{max}$, then increasing carbon price satisfying $Z_{l,int} \leq Z < Z_{max}$, while maintaining minimum output, Q_l , increases the technology level and decreases the emission below the target level, at optimality.

- iii. if $Z_\Phi = Z_{hv}^{l0} < Z_\kappa \leq Z_{max} < Z_{l,int}$, then increasing carbon price satisfying $Z_\kappa < Z < Z_{max}$, while maintaining minimum output and technology level, Q_l and K_0 , decreases the optimal emission below the target level.
 - iv. if $Z_{h,int} < Z_\kappa < Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$, then increasing carbon price while satisfying $Z_\kappa \leq Z < \min\{Z_{hv}^{l0}, Z_{max}\}$, maintains maximum output, Q_h , increases the optimal technology level and decreases the optimal emission below the target level.
 - v. if $Z_\kappa \leq Z_{h,int} < Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$, then increasing carbon price satisfying $Z_{h,int} \leq Z < \min\{Z_{hv}^{l0}, Z_{max}\}$, while maintaining maximum output, Q_h , increases the optimal technology level and decreases the optimal emission below the target level; satisfying $Z_\kappa \leq Z < Z_{h,int}$, maintains maximum output, Q_h , and decreases the optimal emission below the target level, but does not lead to a technology enhancement, at optimality.
- (d) when $Z_{h,int} < Z_{int} < Z_{l,int}$ & $Z_{l,int} < Z_\Phi = Z_{hv}^{lv}$, or, when $Z_{l,int} \leq Z_{int}$, since $Z_{h,int} < Z_{max}$ by Theorem 2.7,
- i. if $Z_\Phi = Z_{hv}^{lv} \leq Z_\kappa < Z_{max}$ increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, while maintaining minimum output, increases technology level and decreases the emission below the target level, at optimality.
 - ii. if $Z_{h,int} < Z_\kappa \leq Z_\Phi = Z_{hv}^{lv}$, then increasing carbon price satisfying $Z_\kappa \leq Z < \min\{Z_{hv}^{lv}, Z_{max}\}$, maintains maximum output, Q_h , increases the technology level and decreases the emission below the target level, at optimality.
 - iii. if $Z_\kappa < Z_{h,int} \leq Z_\Phi = Z_{hv}^{lv}$, then increasing carbon price satisfying $Z_{h,int} \leq Z < \min\{Z_{hv}^{lv}, Z_{max}\}$, maintains maximum output, Q_h , increases the optimal technology level and decreases the optimal emission below the target level; satisfying $Z_\kappa \leq Z < Z_{h,int}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

4. For $r = 1$,

- (a) when $Z_{int} \leq Z_{h,int}$, since $Z_\kappa < Z_\Phi = Z_{int} = Z_{max}$ by Theorem 2.7, increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.
- (b) when $Z_{h,int} < Z_{int}$, since $Z_{max} < Z_\Phi < Z_{int}$ by Theorem 2.7,
 - i. if $Z_{h,int} < Z_\kappa < Z_{max} < Z_\Phi < Z_{int}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level at optimality.
 - ii. if $Z_\kappa \leq Z_{h,int} < Z_{max} < Z_\Phi < Z_{int}$, then increasing carbon price satisfying $Z_{h,int} \leq Z < Z_{max}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target; satisfying $Z_\kappa \leq Z < Z_{h,int}$, maintaining maximum output, does not lead to a technology enhancement, but decreases the emission below the target level, at optimality.
 - iii. if $Z_\kappa < Z_{max} \leq Z_{h,int} < Z_\Phi < Z_{int}$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

5. For $r > 1$,

- (a) when $Z_{int} \leq Z_{h,int}$, since $Z_\kappa < Z_{max} < Z_\Phi < Z_{h,int}$ by Theorem 2.7, increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.
- (b) when $Z_{h,int} < Z_{int}$, since $Z_{h,int} < Z_\Phi$ and $Z_{max} < Z_\Phi$ by Theorem 2.7,
 - i. if $Z_{h,int} < Z_\kappa < Z_{max} < Z_\Phi$, then increasing carbon price satisfying $Z_\kappa \leq Z < Z_{max}$, maintains maximum output, Q_h , increases the technology level, and decreases the emission below the target level at optimality.
 - ii. if $Z_\kappa < Z_{h,int} < Z_{max} < Z_\Phi$, then increasing carbon price satisfying $Z_{h,int} \leq Z < Z_{max}$, maintains maximum output, Q_h , increases

the technology level, and decreases the emission below the target; satisfying $Z_{\kappa} \leq Z < Z_{h,int}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.

- iii. if $Z_{\kappa} < Z_{max} \leq Z_{h,int} < Z_{\Phi}$, then increasing carbon price satisfying $Z_{\kappa} \leq Z < Z_{max}$, maintains maximum output, Q_h , and decreases the emission below the target level, but does not lead to a technology enhancement, at optimality.



CHAPTER 3

OPTIMAL PROCESS IMPROVEMENT EFFORTS FOR SUPPLY CHAIN OPERATIONS UNDER DISRUPTIONS

Disruptions like a pandemic probe the resiliency of supply chains that mainly relate to inventory management under demand and delivery lead time uncertainties. Supply chain operations are rife with fixed costs. This study proposes a new inventory management model that explicitly integrates process improvement efforts to enhance supply chain sustainability through better use of capital (materials, assets, technology) and labor (workforce and know-how). In the case of a manufacturing environment, we study reducing setup costs when they are functions of two (input) decision variables: the level of capital (e.g., process change, technology investments) and the level of labor required. To that end, this research is motivated by lean manufacturing practices, which rely on shaping the operating environment and operating optimally within that business environment. Based on mathematical modeling and analysis, we provide closed-form optimality expressions and structural results that lend themselves to decision insights.

3.1 Literature Review

Supply Chains (SCs) are rife with fixed (or, sunk) costs, whether in the form of overhead in management, setup in manufacturing, ordering costs in retailing logistics, or the acquisition of data for a specific market. For instance, a buyer may commit to a fixed transport capacity (a sunk cost), the utilization of which may be wavering due to uncertainties in demand and supply. Often, fixed costs dictate operational decisions (e.g., inventory management). Investments in reducing fixed costs (e.g., choosing a more reliable and less costly third-party logistics

provider) may reduce overall costs and decrease environmental damage because of increased capacity utilization (see, Ülkü (2012)). Therefore, it is plausible to state that fixed costs impact the sustainability of SCs.

Although the literature is well-established in “economies of scale” (whereby, for instance, given the fixed dispatch cost of a truck, lower per unit transportation cost could be attained if the truck’s carrying capacity is better utilized), more research is needed in understanding the operational factors of and investing in reducing fixed costs. In this study, we focus on reducing fixed inventory replenishment costs when they are functions of two (input) decision variables: the level of capital (e.g., process change, technology investments) and the level of labor required. This issue was first addressed in the context of lean manufacturing practices, which rely on shaping the operating environment and operating optimally within that business environment. As such, it is conventional to refer to fixed cost improvements as ‘setup reductions.’ We shall retain this usage herein as shorthand, although our analysis is applicable to both manufacturing and retail settings under the given assumptions.

The seminal work on the impact of investing in reduced setups is by Porteus (1985) where he investigated the economic justification of lean practices with stylized cost structures representing improvement efforts. In particular, he considered the linear, power function, and logarithmic costs for reducing setups. Billington (1987) extended the setup reduction problem to the Economic Production Quantity (EPQ) model with deterministic demand and finite production rate when the setup reduction cost is a linear or exponential function of capital investment. Kim, Hayya and Hong (1992) investigated the problem from the perspective of setup reduction cost curves and considered linear, concave-parabolic, convex-parabolic, logarithmic, logistic, and exponential functions. Trevino et al. (1993) illustrated that setup reduction costs may be step functions corresponding to a fixed number of reduction opportunities (levels).

Although these structures have been widely accepted and utilized in various succeeding various versions of the basic model, empirical studies must be conducted to verify or justify the use of such analytically convenient functional forms. For example, all the studies in setup reduction literature that use the logarithmic investment function justify it by citing Hall (1983) and referring to the Japanese process improvements discussed therein. Yet, interestingly, Hall does not provide

or allude to any structural form for setup reduction costs in Japan. To the best of our knowledge, Trevino, Hurley and Friedrich (1993) provide the only empirical comparison of setup reduction efforts in practice based on four case studies. Data for only one of these case studies are readily available in Trevino et al. (1993). We employ these as base-case data to motivate our research further and highlight the value of functional forms in setup cost reduction. To that end, let Δ denote a non-negative scalar for the reduction effort. Our analysis of that data reveals that the (continuous) functional relationship between setup reduction cost $\Gamma(\Delta)$ and $(1 - \Delta) \times 100\%$ reduction in setup ($0 \leq \Delta \leq 1$) is best described by a power function $\Gamma(\Delta) = 73.03\Delta^{-3.326} - 8349.91$ with an R-squared = 0.999, whereas the best logarithmic function fitted to the data ($\Gamma(\Delta) = 73.03\Delta^{-3.326} - 8349.91$) has a corresponding R-squared = 0.853. (See Figure 3.1.) Note that $\Gamma(\Delta = 1) = 0$ because no action would incur no cost. The logarithmic form has been suggested and widely studied in the literature for analytical purposes. However, the power function form of reduction costs arises naturally from fundamental economic activities, as we elucidate below in this work. Thus, there is a gap in the literature for a bottom-up approach to constructing setup reduction cost functions aside from mere analytical convenience. We attempt to do this by using the so-called economic production functions.

Aside from a theoretical justification of a specific cost function, such an approach is also beneficial from an organizational resources perspective. Currently, there is no study in the literature that addresses the resource needs explicitly for setup reduction efforts. In this study we assume that setup reduction activity is essentially an input conversion process: capital (equipment, tools, and automation) and labor are combined to achieve the desired reduction. For example, a manager may purchase a new press that allows for faster changeovers (reducing the setup time and related cost) by utilizing more capital or assigning more workers to a changeover operation to achieve the same result. A product design change that would result in faster changeover times might require the firm's capital and labor investments. We take this bottom-up approach in this work to represent the setup reduction efforts. Our approach to utilizing economic production functions to represent setup reduction efforts justifies models in the literature that use convex power functions. We provide closed-form solutions for the resulting setup levels and the required inputs (resources) to be allocated to that purpose. As such, our findings bridge the decision areas of process improvement and resource

planning.

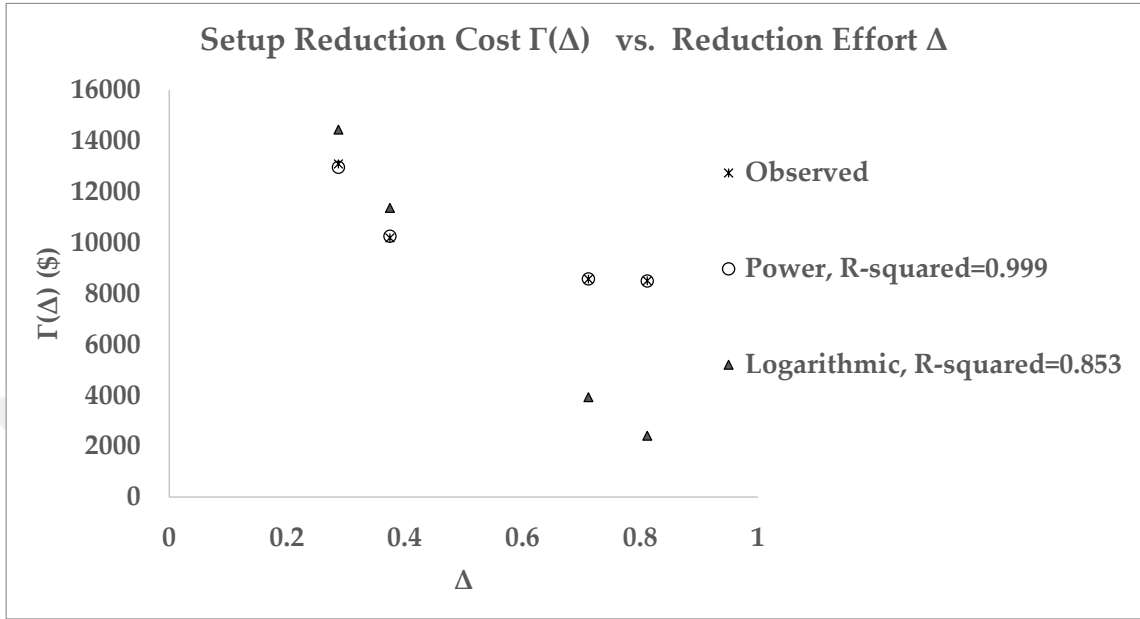


Figure 3.1: Comparison of setup cost reduction functions with data in (Trevino et al. (1993)).

There is a vast literature on the setup reduction problem. Below, we review the literature focusing on the works that are most relevant in terms of demand stochasticity accounting for shortages and functional forms for setup reduction costs.

In the presence of randomness in demands during lead times, the lot size-reorder point (Q, r) control policy is employed for inventory management in systems that are continuously monitored. (Periodic review policies and corresponding models are outside the scope of our work and, hence, are omitted in this review.) The setup reduction problem for the lot size-reorder point model with stochastic demand during lead time was first addressed by Keller and Noori (1988). In their model, explicit shortage costs are charged per unit short, and setup reduction costs are described by logarithmic and power functions of capital investment. Although the formulation is for general demands, the analysis and optimality results are restricted to the special cases of uniform and exponential demand distributions. Nasri and Paknejad (1992) analyze the special cases of geometric and exponential demands during lead time when shortage costs are explicitly accounted for by charging them per stockout instance per unit time and per unit short. Setup reduction is achieved by employing a logarithmic investment function. Follow-

ing these early works, research has evolved in two directions: random yields in replenished quantities (defective deliveries) starting with (Paknejad, Nasri and Afflsco (1995)), and lead time reduction option in conjunction with setup reductions following the deterministic model proposed by (Hill and Khosla (1992)). Models developed in either direction directly reduce to the basic setting as special cases: when the yield is 100% (expressed with a Dirac Delta function) and/or the lead time reduction option is not executed.

Paknejad et al. (1995) study the case when delivered orders contain random defectives. They account for shortage costs explicitly in the objective function, and the assumed setup reduction cost function is assumed logarithmic. Closed-form optimality results are derived for exponential and uniform demand distributions. Ouyang and Chang (2000) revisit the problem with a service constraint. However, they reformulate the service level constraint through an approximation proposed by Gallego and Moon (1993) and work with the worst-case scenario (upper bound) instead of the exact constraint. Ouyang and Chang (2002) extend this work by considering normally distributed demands and joint reductions in setups and lead times. Shortage costs are accounted for explicitly in the objective function. The authors use both logarithmic and power investment functions for setup reductions. Our work differs from theirs in that we impose a hard service level constraint. Along the lines of (Trevino et al. (1993)), Ouyang, Wu and Ho (2007) provide an approximation to this model when setup reduction costs are of logarithmic form. Lin and Hou (2003) analyze generally distributed random yields and normally distributed demands. Shortage costs are explicitly accounted for in the objective function. Capital investment in setup reduction is assumed to be logarithmic in the reduction amount. Uthayakumar and Parvathi (2009) extend the model to allow for lead time reductions (jointly with setup reductions) and lead time-dependent backorder rates. The investment function for setup reductions is logarithmic. Shortage costs are explicitly included in the optimization but then re-formulated by considering an upper bound as suggested in (Gallego and Moon (1993)). The resulting approximate model holds for any demand distribution.

Annadurai and Uthayakumar (2010) consider defective deliveries with setup and lead time reductions in conjunction with Normally-distributed demands. Shortages are explicitly costed in the optimization, and setup reductions incur logarithmic costs. Finally, Cheng, Huang and Chen (2004) [21] and Uthayakumar and

Parvathi (2008) consider random demand environments for the lot size-reorder point model when setup costs are step functions corresponding to discrete levels of reduction possibilities. All of these works take lead times to be independent of replenishment quantities. Ben-Daya and Hariga (2003) consider the case when (manufacturing) lead time depends on the lot size. Specifically, they assume that it is the sum of the setup time and the time to produce the lot, as proposed by Kim and Benton (1995). They assume that a service level constraint is binding and that setup reduction is achieved through shortening the setup time. Aside from convexity, no specific functional form is assumed for time reduction cost, but an inversely proportional structure is used in the illustrative numerical example. The direct interaction between lead times and setup times implies that reduction efforts in setup times also decrease lead times: This renders their analysis fundamentally different from and not applicable to settings with lot size-independent (delivery) lead times. Thus, we see that the setup reduction problem for random demand environments with delivery lead times has yet to be analyzed in an exact manner under a hard service level constraint. We attempt to fill this gap in the literature, as well. Next, we focus on positioning our study in the big picture of sustainable SC management.

In the face of the pressing climate crisis, outbreaks, social instabilities, and changing demographics and needs, sustainability is at the front and center of the business world, which runs on supply chains (c.f., Mefford (2011)). Hence, the sustainable supply chain management literature has expanded (e.g., Khan, Yu, Golpira, Sharif and Mardani (2021)). Parallel to advances in intelligent technologies and data storing and processing capacities, supply chain analytics has emerged as an enabler for better decision-making with sustainability imperatives (e.g., Ülkü and Engau (2021)). We note here that the risks arising from natural disasters, labor strikes, outbreaks, and political conflicts, unlike those attributable to coordination problems, disrupt and break supply chains (e.g., Kleindorfer and Saad (2005)) and require integrated management systems to cope with (e.g., Sodhi, Tang and Willenson (2023)). A recent case, COVID-19, has wreaked havoc on many SCs and exposed their fragilities (Altay and Pal (2023)). This has propelled researchers to integrate resiliency and technological advances into models taking on the challenging task of synchronizing production, location/allocation, distribution, transportation, and inventory management. For example, Goodarzian, Ghasemi, Gunasekaren, Taleizadeh and Abraham (2021) proposed a resilient health-

care SC network while minimizing the design and operating, environmental, and social costs. Several papers have highlighted the use of intelligent technologies in supply chain planning and execution under disruptions (e.g., Nabipour and Ülkü (2021); Kamran, Kia, Goodarzian and Ghasemi (2023); Oguntola and Ülkü (2023), Oguntola, Ülkü, Saif and Engau (2023)).

As regulations press firms to disclose their environmental, social, and governance (ESG) impacts (e.g., Dai and Tang (2022)) disclosures, SCs need to be better coordinated, transparent, engaged with their customers, and resilient and intelligent in minimizing “waste”. According to Shingo (1988), industrial waste, a sign of inefficiency, surfaces in various ways, such as overproduction, unnecessary processing-stocking-transportation, and the making of defective products. Waste in both capital (e.g., materials, energy, the use of technology, the cost of unproductive time, landfills) and in labor (e.g., unnecessary human errors, lack of adequate level of automation, the cost of time to train workers) causes not only additional SC costs but also hurts sustainability. In short, the ill-operated processes (how and where products are produced, services rendered) add additional costs. To that end, “leaning” the logistics systems (e.g., Bookbinder (2021)), such as minimizing setup times for discrete manufacturing or maximizing transport capacity utilization for retailers or carriers, becomes a critical goal.

The analysis herein has implications in two such directions, as well. From the perspective of response to disruptions, our study provides explicit conditions on when and how much investments are desirable in setup reduction efforts in the face of demand regime changes in terms of overall demand rates and lead time demand variability. To the best of our knowledge, such conditions have yet to be theoretically investigated for setup reduction efforts in the literature and hence, this study. Coupled with resource allocation decisions, our findings provide valuable guidelines for managers for research and development (R&D) activities in times of supply and demand uncertainties. From a sustainability perspective, our models offer a framework to investigate the impact of setup reductions on sustainability.

As greenhouse gas emissions have become a pressing global issue, inventory literature has seen the emergence of sustainable operations. (See Agi, Faramarzi-Oghani and Hazır (2021) and references therein for a recent comprehensive review on green operations.) A particularly relevant stream of research regarding

sustainability is on the celebrated EOQ models where setup and/or stock-keeping operations result in carbon emissions. Among these works, we cite the following: Hua, Cheng and Wang (2011) study an EOQ model under a cap-and-trade mechanism and discuss managing the carbon footprints in inventory control. Benjaafar, Li and Daskin (2012) investigate how to incorporate carbon emissions concerns into the operational decision-making models. Chen, Benjaafar and Elomri (2013) revisit the EOQ model with a cap on carbon emissions and analytically support the observations in (Benjaafar, Li and Daskin (2012)). In a similar setting, Topal, Özlü and Konur (2014) jointly address the inventory replenishment problem and carbon emission reduction investment with varying carbon regulation settings. We provide a brief discussion on how our model can be used to incorporate carbon emissions in Section 3.4.

The contributions of this work can be summarized as follows. This study is the first in the literature to adopt a resource-based approach to investigate process improvement efforts. This provides a justification for specific functional forms used in the literature. In particular, we model setup reductions via (economic) production functions considering factor efficiencies. For the least-cost process improvement level, we offer optimal levels of labor and capital investments and the corresponding inventory system's optimal control variables: optimal reordering and replenishment quantity. This enables managers to plan for and allocate labor and capital inputs for R&D activities while optimizing inventory decisions under demand and lead time uncertainty. The specific formulation of the stochastic inventory problem under a hard service constraint fills a gap in the inventory literature with setup reduction efforts. Our results apply to any situation with fixed replenishment opportunities in manufacturing and retail settings. In so doing, as another contribution, we reaffirm and extend the results in the seminal work by Porteus (1985) using a bottom-up, nested optimization approach. Figure 3.2 displays a pictorial of the problem at hand.

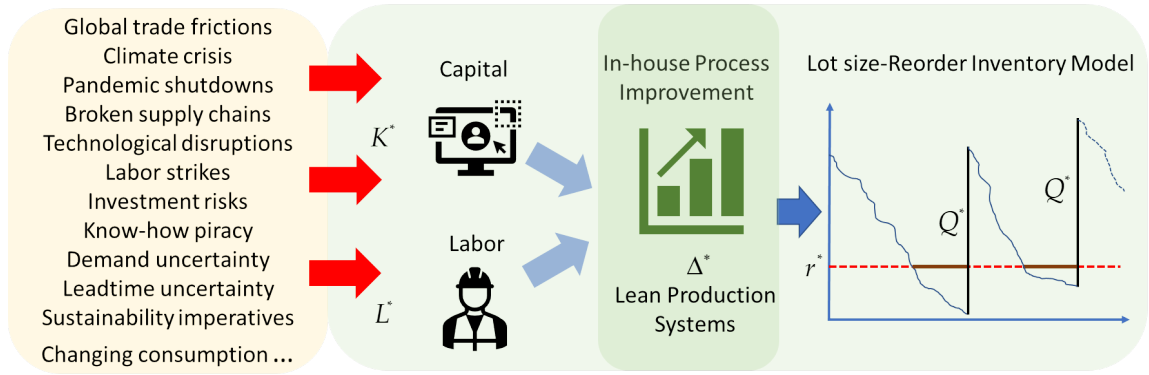


Figure 3.2: Pictorial of the Problem of Optimizing Process Improvement

The layout of this study follows: In Section 3.2 , we construct the cost functions and relate them to the existing literature, present the mathematical model of the problem, and cast the generic optimization model. Section 3.3 provides all the related optimality results and structural insights. Section 3.4 describes some extensions on the generic models and provides additional insights into how process improvements may impact resource planning and allocation, guard against disruptions, and impact sustainability for SC operations. Then, in Section 3.5, we support our analytical results with a small numerical example and sensitivity analyses. Finally, Section 3.6 concludes with some remarks and future research venues.

3.2 The Generic Model Formulation

Any process improvement incurs fixed costs. In times of technological disruptions and to enhance their resiliencies, supply chains must invest in process improvement costs. Inventory management is a fundamental supply chain function in which one of the goals is to benefit from economies of scale so that the risk of sunk cost is minimized. To that end, we build on an inventory model to optimize process-improvement costs. Before presenting the details of this generic model we list the key notations in the following table.

Table 3.1: Notations

D	Mean demand rate (units per unit time, say, per year)
μ_{DL}	Mean demand during lead time
σ_{DL}	The standard deviation of demand during lead time
S	Fixed cost per setup (or order)
H	Inventory cost rate per unit per unit time held in stock
π	Backordering cost rate per unit
η	Desired service level (fraction of immediately unmet demand per year)
P	Unit acquisition cost
Q	Lot size
r	Reorder point
z	Safety stock factor
$L[z]$	The loss function of a standard Normal variable
$\Phi(z)$	Cumulative distribution function (cdf) of a standard Normal variable
$OC(Q, r)$	Expected total operating cost per unit time for lot size-reorder (Q, r) policy
S_0	Setup level before reduction efforts
Δ	A nonnegative scalar (≤ 1) denoting the reduction effort
$\Gamma(\Delta)$	Cost of achieving a $(1 - \Delta) \times 100\%$ fixed replenishment cost reduction
$\Gamma^*(\Delta)$	Least-cost of achieving a reduction of Δ in fixed replenishment cost
K	Capital input level devoted to the reduction activity
L	Labor input level devoted to the reduction activity
K_0	Inherent technical capital stock
L_0	Inherent human know-how
K'	Additional capital investment
L'	Additional labor investment
$K^*(\Delta^*)$	Optimal allocation of capital input to achieve that reduction
$L^*(\Delta^*)$	Optimal allocation of labor input to achieve that reduction
$\theta(K, L)$	Production function describing reduction activity
A	Productivity or technological efficiency for the overall conversion process
A_K	Capital productivity or technological efficiency in Leontief function
A_L	Labor productivity or technological efficiency in Leontief function
β_K	Capital input elasticity in Leontief production function (< 0)
β_L	Labor input elasticity in Leontief production function (< 0)
ρ	The elasticity of scale for Cobb-Douglas production function
$\hat{j}$	Solution type when the least cost is achieved
$\hat{j}^*(\Delta)$	Optimal input usage policy

Any other less-used notation is explained when needed in the remainder of the thesis.

Consider a product to be manufactured (or replenished by a retailer) whose demand is random with a mean rate D (units per unit time, say, per year). We assume that all the uncertainties in demand and delivery can be represented through demand during lead time. Demand during the lead time has a Normal distribution with mean demand during lead time denoted by μ_{DL} and the standard deviation of demand during lead time denoted by σ_{DL} . There is a fixed cost S per setup (or order) associated with every order; units on hand inventory cost at a rate of H per unit per unit time held in stock. All unmet demand is backordered at a rate of π with a desired annual service level (fraction of immediately unmet demand per year), η . Each unit is acquired at the cost of P . The inventory system employs a lotsize-reorder point (Q, r) policy. Making the conventional assumption of at most one outstanding order at any time, the reorder point $r = \mu_{DL} + z\sigma_{DL}$ with the safety stock factor z selected such that the annual service level target is achieved: $\eta = [(D/Q)L(z)\sigma_{DL}]/D$ where $L[z]$ denotes the loss function of a standard Normal variable.

The expected total operating cost per unit time can be written as follows.

$$OC(Q, r) = SD/Q + H[Q/2 + (r - \mu_{DL})] + \pi\eta D + PD. \quad (3.1)$$

For the given system parameters, the decision variables are the values of the inventory control policy variables Q and r .

Proposition 3.1. *The cost-minimizing replenishment policy is to manufacture (or order) Q^* whenever the inventory level hits*

$$r^* = \mu_{DL} + z^* \sigma_{DL},$$

where

$$Q^* = \sqrt{SD/H(1/2 - \eta/(1 - \Phi(z^*)))}$$

and z^* uniquely solves

$$[L(z^*)]^2 = [SD\eta^2] / [H(1/2 - \eta/(1 - \Phi(z^*)))\sigma_{DL}^2].$$

Then, the ensuing optimal cost per unit time, OC^* is given by:

$$\begin{aligned}
 OC^* &= OC(Q = Q^*, z = z^*) \\
 &= \sqrt{2SDH} \left(\frac{\sqrt{(1-\Phi(z^*)-2\eta)}}{\sqrt{(1-\Phi(z^*))}} + \frac{\sqrt{(1-\Phi(z^*))}}{\sqrt{(1-\Phi(z^*)-2\eta)}} \right) + Hz^* \sigma_{DL} + (\pi\eta + P)D \\
 &= \sqrt{2SDH\eta(1 + \eta(1 - \Phi(z^*))/2\eta(1 - \Phi(z^*)))} + Hz^* \sigma_{DL} + (\pi\eta + P)D.
 \end{aligned} \tag{3.2}$$

Proof. The necessary first order condition (FOC) for optimality on Q gives

$$\partial OC(Q)/\partial Q = \partial/\partial Q\{SD/Q + HQ/2\} + \partial/\partial Q\{Hz\sigma_{DL}\} = 0$$

where $\partial z/\partial Q$ is obtained by differentiating both sides of the desired annual service level condition $\eta = [(D/Q)L(z)\sigma_{DL}]/D$. Also,

$$\partial L(z)/\partial z = -(1 - \Phi(z)), \quad \partial z/\partial Q = -\eta/[(1 - \Phi(z))\sigma_{DL}].$$

Then,

$$\partial OC(Q)/\partial Q = -SD/Q^2 + H/2 - H\sigma_{DL}\eta/[(1 - \Phi(z))\sigma_{DL}] = -SD/Q^2 + H/2 - H\eta/(1 - \Phi(z)) = 0.$$

Solving for the optimal replenishment quantity for a given z value, we obtain

$$Q^*|z = \sqrt{SD/H(1/2 - \eta/(1 - \Phi(z)))}.$$

Note that the second order condition (SOC)

$$\partial^2 z/\partial Q^2 = \left[-\eta\phi(z)\sigma_{DL}/[(1 - \Phi(z))\sigma_{DL}]^2 \right] [-\eta/[(1 - \Phi(z))\sigma_{DL}]] > 0$$

guarantees that the optimal quantity obtained minimizes the expected total cost rate. Substituting the conditional optimal quantity expression into the desired annual service level expression and squaring both sides, the corresponding optimal safety stock factor z^* is found to solve

$$[L(z^*)]^2 = SD\eta^2/H(1/2 - \eta/(1 - \Phi(z^*))) / \sigma_{DL}^2.$$

This completes the proof. □

We see that the optimal batching decision and the resulting expected cost rate depend on both the fixed cost S and, indirectly, the uncertainties faced by the operations as captured by σ_{DL} . To alleviate the negative impact of uncertainty,

a firm may engage in improvement efforts in two ways. (i) *Externally* by trying to reduce σ_{DL} directly through improving on variability in demand, lead times *etc.* (ii) *Internally* by reducing the fixed cost S . *Extra-firm* improvements typically involve other stakeholders who may show organizational resistance to change. It may also not be physically possible to eliminate the sources of uncertainties in times of crises as were experienced globally during the recent COVID-19 pandemic which disrupted all supply chains. *Intra-firm* improvements, on the other hand, require less managerial burden due to direct control over operations and provide faster results. Therefore, we consider the latter option and focus on fixed replenishment (setup or order) cost reduction efforts.

With process improvement efforts, the fixed cost S associated with each replenishment essentially becomes a managerial decision variable. By employing capital (in terms of automation, equipment, *etc.*) and labor, a firm is able to reduce fixed costs through process improvements. Such process improvements have been very well documented. We represent the *effective setup level* as $S \doteq S_0 \Delta$, where S_0 denotes the current setup level prior to reduction efforts and $\Delta \leq 1$ is a scalar denoting the reduction effort. Let $\Gamma(\Delta)$ denote the cost of achieving a $(1 - \Delta) \times 100\%$ fixed replenishment cost reduction.

We depart from the existing literature on setup reduction efforts in a fundamental way. Instead of envisioning ‘acquisition’ of available reduction technologies from outside at a specific cost, we consider an *intra-firm* technology development capability. We assume that the internal technology development can be modeled via economic production functions. That is, Δ is an output of capital K and labor L input levels devoted to the reduction activity in the firm; $\Delta = \theta(K, L)$. In the economics parlance, such input conversion relationships have been called ‘production functions’. (See Heathfield and Wibe (1987) for a basic introduction to economic production functions to describe input conversion processes.) Currently, the firm is assumed to possess some inherent human know-how L_0 and technical capital stock K_0 so that $\theta(K_0, L_0) = 1$. That is, the firm is able to have achieved its current process capabilities and, now, desires to improve on them with additional labor L' and capital K' investments for process research and development (R&D). Thus, $\Delta = \theta(K' + K_0, L' + L_0)$. Additional labor and capital are acquired at unit variable costs c_K and c_L , respectively. We assume that acquisition costs of inputs are (annually) amortized values but suppress the explicit cost of capital notation. Noting that current assets K_0 and L_0 are sunk costs, $\Gamma(\Delta) \doteq c_K K' + c_L L'$.

In the presence of fixed replenishment cost reduction efforts, a cost-minimizing decision maker's optimization problem involves the determination of (i) the optimal values of the inventory control policy parameters, Q^* and r^* or, equivalently, $z^* (= (r^* - \mu_{DL})/\sigma_{DL})$, (ii) the optimal setup reduction rate, Δ^* and (iii) the optimal allocation of capital and labor inputs to achieve that reduction, $K'^*(\Delta^*)$ and $L'^*(\Delta^*)$. Thus, we have a nested optimization problem.

$$P^0 : \min_{Q,r} \left\{ \min_{\Delta} \left\{ \min_{K',L'} Z_0 \left\{ \begin{array}{l} = SD\Delta/Q + H[Q/2 + (r - \mu_{DL})] + \pi\eta D + PD \\ + [c_K K' + c_L L'] \\ \Delta = \theta(K' + K_0, L' + L_0), z = (r - \mu_{DL})/\sigma_{DL}, \\ [L(z)]^2(1 - \Phi(z)) = 2\eta SD\Delta/h\sigma_{DL}^2 \end{array} \right\} \right\} \right\}. \quad (3.3)$$

This nested optimization problem can be posed equivalently in different ways. The research problem at hand emerged in the literature as selecting the optimal setup level with a corresponding cost of setup reduction (Porteus (1985)). For consistency with the literature and elucidation of the impact of economic inputs, we also recast the optimization in this equivalent canonical form and state it as P^1 .

$$P^1 : \min_{K',L'} Z_1 \left\{ \begin{array}{l} = \sqrt{2SDH\eta(1 + \eta\Phi(z^*)/2\eta\Phi(z^*))} \sqrt{\Delta} + \Gamma^*(\Delta) \\ + [Hz^*\sigma_{DL} + (\pi\eta + P)D]|\Delta \leq 0, \\ [L(z^*)]^2(1 - \Phi(z^*)) = 2\eta SD\Delta/h\sigma_{DL}^2 \end{array} \right\}. \quad (3.4)$$

where Δ is now the non-negative decision variable and $\Gamma^*(\Delta)$ denotes the least-cost of achieving a reduction of in fixed replenishment cost. That is, $\Gamma^*(\Delta)$ is the solution of problem P^2 presented below.

$$P^2 : \min_{K',L'} \Gamma(\Delta) \left\{ = c_K K' + c_L L' \mid \Delta = \theta(K' + K_0, L' + L_0) \right\}. \quad (3.5)$$

The functional form of $\Gamma^*(\Delta)$ depends on the the adopted specific functional form of $\theta(\bullet)$ and the solution type ($\hat{j}$) when the least-cost is achieved. The type of a solution refers to it being a corner solution ($\hat{j} = 0; L'^* = K'^* = 0$), a boundary solution ($\hat{j} = 1; L'^* > 0, K'^* = 0$ or $\hat{j} = 2; L'^* = 0, K'^* > 0$) or an interior solution ($\hat{j} = 3; L'^* > 0, K'^* > 0$). Aside from their mathematical interpretation, the solution types also correspond, respectively, to the following input usage policies: The no-change policy, the labor-reliant single input policy, the capital-reliant single input policy, and the dual input policy. Such specific input usage policies may be of interest or even dictated to managers due either to organizational factors

such as freezes on new labor recruits or new capital expenditures, or to business environment conditions such as labor shortages or inability to access new capital in times of crises.

To facilitate our exposition, let $\theta_j(\bullet)$ and $\hat{\Gamma}_j(\Delta)$ denote, respectively, the particular structure of the setup-reduction rate function and the corresponding least-cost when a particular solution type $\hat{j}$ is imposed. By definition, $\Delta = 1$ and $\hat{\Gamma}_j(\Delta) := 0$ for $\hat{j} = 0$. Then, $\Gamma^*(\Delta) = \min_{\hat{j}} \hat{\Gamma}_j(\Delta)$. Thus, the minimal cost of setup reduction in achieving a non-zero $(1 - \Delta) \times 100\%$ setup cost reduction is obtained by the best input usage policy $\hat{j}^*(\Delta) = \arg \min \{\Gamma_j^*(\Delta) : \hat{j} = 1, 2, 3\}$. There may be multiple optima for $\hat{j}^*(\Delta)$ for a given reduction rate, corresponding to intersection points of least-costs for different input usage policies. In such cases, we will interpret $\hat{j}^*(\Delta)$ as the smallest of the set. For example, $\hat{j}^*(\Delta)=1$ if labor-reliant and capital-reliant single input policies give the same reduction costs for Δ . We emphasize that The model herein enables one to determine the value (or cost) of imposed input usage policies and may be used to study R&D budget or labor constraints.

Next, we delineate the functional form of the setup reduction rate function $\theta(\bullet)$. For some forms of $\theta(\bullet)$, one is able *a priori* to determine the ranges of system parameters for which a particular solution type would be the least-cost solution type for any given reduction Δ . The notion is identical to that of determining the ranges of demand for which a particular process alternative is best in process selection decisions. We discuss some examples later. However, in general, delineating such regions analytically seems practically intractable. Instead, the base problem is best solved by treating each solution type as a separate problem and choose the feasible solution as the global optimal. We present an algorithm later.

Economic production functions are mathematical constructs invented to describe the existing characteristics of an economic activity in a particular setting. Suitability of a specific functional form can only be tested empirically for individual firms. As such, there is no one functional form that can be viewed as the best which is applicable to all firms across all industries. That is, managers are not at liberty to select a productin function to describe their own process improvement efforts. Data on past projects would indicate an organization's own input conversion capabilities to generate process improvements. Therefore, it is vital importance that optimal decisions be investigated within different formulations of process improvement efforts via economic production functions.

In our analysis, we consider the Cobb-Douglas production function which is the most commonly used functional form in the literature : As demonstrated below, it is amenable to obtaining closed expressions for optimality results. Moreover, it results in a *piecewise* power function. Thereby, it provides a bottom-up justification of the commonly assumed technology acquisition cost structure in the operations management literature. However, the piecewise nature introduces novelties in optimality results heretofore not investigated in the literature. In the following section, we also discuss the Leontief and the Constant Elasticity of Substitution (CES) production functions and their modeling and optimization implications.

The Cobb-Douglas production function given by

$$\theta_1 = \theta(K, L) = AK^{\beta_K}L^{\beta_L} = A(K' + K_0)^{\beta_K}(L' + L_0)^{\beta_L}, \quad (3.6)$$

where A is interpreted as the productivity or technological efficiency for the overall conversion process, and the parameters $\beta_K(< 0)$ and $\beta_L(< 0)$ determine the elasticities of capital and labor inputs, respectively. The surrogate parameter $\rho \doteq \beta_K + \beta_L$ denotes the elasticity of scale, which shows the extent of the effect of changing inputs on the output. It was first introduced by Cobb and Douglas (1928) to relate capital stock, labor force and GNP for the US manufacturing industries for the period 1899-1922. Since then, it has been widely applied to energy generation (Komiya (1962), Wei (2007)), for several labor-intensive sectors in Japan (Ozaki (1976)), for applications in paper, steel and oil industries (Shad-begian and Gray (2005)), for regional agricultural production in Türkiye (Hatirli, Ozkan and Fert (2006)), for supply chain operations (Kogan and Tapiero (2009)), for energy sector (Yuan, Liu and Wu (2009)) and for radia frequency identification (RFID) productivity in retail supply chains (Shin and Eksioglu (2015)).

3.3 Optimality and Structural Results for Different Production Functions

In this section, we derive analytical results for the optimal cost-reduction values (and their corresponding optimal values of capital and labor levels for each reduction-rate function. We begin our analysis with the Cobb-Douglas function.

Proposition 3.2. *When the fixed replenishment cost reduction rate is of the Cobb-Douglas form, the minimal cost of setup reduction in achieving a non-zero $(1 - \Delta) \times 100\%$ fixed replenishment cost reduction is obtained by the best input usage policy $\hat{j}^*(\Delta) = \arg\min\{\Gamma_j^*(\Delta) : \hat{j} = 1, 2, 3\}$ with the optimal capital input level $\hat{K}^*(\Delta|\hat{j}^*) = \hat{K}'(\Delta|\hat{j}^*) + K_0$, the optimal labor input level $\hat{L}^*(\Delta|\hat{j}^*) = \hat{L}'(\Delta|\hat{j}^*) + L_0$, the corresponding minimal cost of setup reduction $\Gamma^*(\Delta) = \min_{\hat{j}}\{\hat{\Gamma}_{\hat{j}}^*(\Delta) : \hat{j} = 1, 2, 3\}$ where each input usage policy $\hat{j}$ has the following properties.*

1. *Under the labor-reliant (single input) policy ($\hat{j} = 1$), the (least-cost) additional capital input is $\hat{K}'(\Delta|\hat{j}) = 0$, the additional labor input is*

$$\hat{L}'(\Delta|\hat{j}) = (\Delta/A)^{1/\beta_L} K_0^{-\beta_K/\beta_L} - L_0$$

and the corresponding minimal cost of fixed cost reduction is

$$\Gamma_1^*(\Delta) = \Psi_1 \Delta^{1/\beta_L} - \psi_1,$$

where $\Psi_1 = c_L(\Delta/A)^{\omega_1} K_0^{-\beta_K/\beta_L}$ and $\psi_1 = c_L L_0$.

2. *Under the capital-reliant (single input) policy ($\hat{j} = 2$), the (least-cost) additional labor input is $\hat{L}'(\Delta|\hat{j}) = 0$, the additional capital input is*

$$\hat{K}'(\Delta|\hat{j}) = (\Delta/A)^{1/\beta_K} L_0^{-\beta_L/\beta_K} - K_0$$

and the corresponding minimal cost of fixed cost reduction is

$$\Gamma_2^*(\Delta) = \Psi_2 \Delta^{1/\beta_K} - \psi_2,$$

where $\Psi_2 = c_K(\Delta/A)^{\omega_2} L_0^{-\beta_L/\beta_K}$ and $\psi_2 = c_K K_0$.

3. *Under the dual input policy ($\hat{j} = 3$), the least-cost additional labor input $\hat{L}'(\Delta|\hat{j})$, the least cost additional capital input $\hat{K}'(\Delta|\hat{j})$ and the corresponding minimal cost of fixed cost reduction $\Gamma_3^*(\Delta)$ are as follows.*

$$\begin{aligned} \hat{K}'^*(\Delta|\hat{j}^*) &= (\Omega^{-\beta_L} \Delta/A)^{1/\rho} - K_0, \\ \hat{L}'^*(\Delta|\hat{j}^*) &= (\Omega^{\beta_K} \Delta/A)^{1/\rho} - L_0, \\ \hat{\Gamma}_3^*(\Delta) &= \Psi_3 \Delta^{1/\rho} - \psi_3, \end{aligned} \tag{3.7}$$

where

$$\Omega = c_K \beta_L / c_L \beta_K = \hat{L}'^*(\Delta|\hat{j}^*) / \hat{K}'^*(\Delta|\hat{j}^*),$$

$$\rho = (\beta_K + \beta_L),$$

$$\Psi_3 = k_1 c_K + k_2 c_L,$$

$$k_1 = (\Omega^{\beta_L} A)^{-1/\rho},$$

$$k_2 = (\Omega^{-\beta_K} A)^{-1/\rho},$$

and

$$\psi = c_K K_0 + c_L L_0.$$

Proof. Setting $\theta = \Delta$ in Equation (3.6) and solving for labor input, we get

$$L' + L_0 = \left(\Delta (K' + K_0)^{-\beta_K} / A \right)^{1/\beta_L}$$

and accordingly from Equation (3.5), we can derive

$$\Gamma(\Delta) = c_K K' + c_L L' = c_K K' + c_L \left(\Delta (K' + K_0)^{-\beta_K} / A \right)^{1/\beta_L} - (c_K K_0 + c_L L_0).$$

For the boundary solutions, the least-cost value of the non-zero input can be obtained directly. For the boundary solution type $\hat{j} = 1$, suppressing the solution type notation for inputs, we get

$$L'^* + L_0 = \left(\Delta K_0^{-\beta_K} / A \right)^{1/\beta_L}$$

and

$$\Gamma_j^*(\Delta) = c_L \left(\Delta K_0^{-\beta_K} / A \right)^{1/\beta_L} - c_L L_0 = \left[c_L \left(K_0^{-\beta_K} / A \right)^{1/\beta_L} \right] \Delta^{1/\beta_L} - c_L L_0.$$

For the boundary solution type $\hat{j} = 2$, we get

$$K'^* + K_0 = \left(\Delta L_0^{-\beta_L} / A \right)^{1/\beta_K} \text{ and } \Gamma_j^*(\Delta) = \left[c_K \left(L_0^{-\beta_L} / A \right)^{1/\beta_K} \right] \Delta^{1/\beta_K} - c_K K_0.$$

For the interior solutions ($\hat{j} = 3$), via standard multivariable calculus; from first-order conditions, obtaining K'^* such that

$$\partial \Gamma_3(\Delta) / \partial K'^* = 0$$

and rewriting L' in terms of K'^* , we derive the following optimal levels of capital

and labor levels. Once the expressions K^* and L^* are plugged into $\Gamma(\Delta)$, after some algebra, we find the closed-form optimality expressions as in Equation (3.7). $\square$

Next, we can get the sufficiency conditions on the system parameters for the best input usage policy given as the following corollary.

Corollary 3.1. *When the setup cost reduction rate is of the Cobb-Douglas form, the best input usage policy giving the minimal cost of fixed replenishment cost reduction in achieving a non-zero $(1 - \Delta) \times 100\%$ fixed cost reduction follows.*

1. *If $\Delta > \max\{A\Omega^{\beta_L}K_0^\rho, A\Omega^{-\beta_K}L_0^\rho\}$, the best input usage policy achieving least cost for Δ is the dual input usage policy ($\hat{j}^*(\Delta) = 3$) and $\Gamma^*(\Delta) = \hat{\Gamma}_3(\Delta)$.*
2. *If $\Delta \leq \max\{A\Omega^{\beta_L}K_0^\rho, A\Omega^{-\beta_K}L_0^\rho\}$ and*

$$c_L(\Delta/A)^{1/\beta_L}K_0^{-\beta_K/\beta_L} - c_L L_0 \leq c_K(\Delta/A)^{1/\beta_K}L_0^{-\beta_L/\beta_K} - c_K K_0,$$
then the best input usage policy achieving least cost for Δ is the labor-reliant single input usage policy ($\hat{j}^*(\Delta) = 1$) and $\Gamma^*(\Delta) = \hat{\Gamma}_1(\Delta)$.
3. *If $\Delta \leq \max\{A\Omega^{\beta_L}K_0^\rho, A\Omega^{-\beta_K}L_0^\rho\}$ and*

$$c_L(\Delta/A)^{1/\beta_L}K_0^{-\beta_K/\beta_L} - c_L L_0 > c_K(\Delta/A)^{1/\beta_K}L_0^{-\beta_L/\beta_K} - c_K K_0,$$
then the best input usage policy achieving least cost for Δ is the capital-reliant single input usage policy ($\hat{j}^*(\Delta) = 2$) and $\Gamma^*(\Delta) = \hat{\Gamma}_2(\Delta)$.
4. *If the initial capital and labor endowments K_0 and L_0 have been chosen in accordance the marginal cost condition that $c_K \partial \Gamma(\Delta) / \partial L_0 = c_L \partial \Gamma(\Delta) / \partial K_0$, then it is always optimal to use the dual input policy.*

Proof. The conditions (1)-(3) on the parameters follow immediately from solving for positive additional inputs. The condition (4) is the same logic behind the relationship obtained between the additional inputs in the preceding Proposition 3.2, $c_K \beta_L / c_L \beta_K = \hat{L}^*(\Delta | \hat{j}^*) / \hat{K}^*(\Delta | \hat{j}^*)$. This condition also holds for total inputs, as well under the dual input usage policy. $\square$

Note that when $\theta(\bullet)$ is of Cobb-Douglas form, the least-cost of achieving Δ is a power function under each input usage policy. Hence, the technology acquisition

costs of the power function form used in the literature has a theoretical foundation in economic production functions. But the overall least-cost $\Gamma^*(\Delta)$, which is the minimum of all reduction costs arising from the three different input usage policies for any given Δ , is a *piecewise* power function and is expressed via power functions which are valid only over specific intervals. Therefore, when technology is produced in-house, there is not a single power function that represents reduction cost for all reduction rates. In this way, our work differs from the research in the literature.

Next, we consider the Constant Elasticity of Substitution (CES) production function of the form:

$$\theta_3 = \theta(K, L) = \left(\alpha_1 (K' + K_0)^{\beta_K} + \alpha_2 (L' + L_0)^{\beta_L} \right)^\gamma, \quad (3.8)$$

where γ determines the elasticities of the inputs and of the overall transformation and the other parameters are the same as in Eqns. (3.3) and (3.4). To ensure decreasing rates of return on capital and labor inputs, we assume $\gamma\beta_L < 0$ and $\gamma\beta_K < 0$. The CES production function has been originally proposed by Arrow *et al.* (1961) in a special format where $\beta_K = \beta_L = \beta$ and $\gamma = -\beta^{-1}$. Since then, it has been generalized to allow for different elasticities and has been successfully used in energy generation (Dhrymes and Kurz (1964)) and automation studies (Acemoglu and Restrepo (2018)).

Proposition 3.3. *When the fixed replenishment cost reduction rate is of a special case CES form ($\beta_K = \beta_L = \beta$, $\gamma = -1/\beta$), the minimal cost of fixed cost reduction in achieving a non-zero $(1 - \Delta) \times 100\%$ fixed cost reduction is obtained by the best input usage policy $\hat{j}^*(\Delta) = \arg \min \{\Gamma_j^*(\Delta) : \hat{j} = 1, 2, 3\}$ with the optimal capital input level $\hat{K}^*(\Delta|\hat{j}^*) = \hat{K}'(\Delta|\hat{j}^*) + K_0$, the optimal labor input level $\hat{L}^*(\Delta|\hat{j}^*) = \hat{L}'(\Delta|\hat{j}^*) + L_0$, the corresponding minimal cost of setup reduction $\Gamma^*(\Delta) = \min_j \{\hat{\Gamma}_j^*(\Delta) : \hat{j} = 1, 2, 3\}$ where each input usage policy $\hat{j}$ has the following properties.*

1. *Under the labor-reliant (single input) policy ($\hat{j} = 1$), the (least-cost) additional capital input is $\hat{K}'(\Delta|\hat{j}) = 0$, the additional labor input is $\hat{L}'(\Delta|\hat{j}) = [(1/\alpha_L)\Delta^{-\beta} - (\alpha_K/\alpha_L)K_0^\beta]^{1/\beta} - L_0$ and the corresponding minimal cost of setup reduction is $\hat{\Gamma}_1^*(\Delta) = c_L[(1/\alpha_L)\Delta^{-\beta} - (\alpha_K/\alpha_L)K_0^\beta]^{1/\beta} - c_L L_0$.*
2. *Under the capital-reliant (single input) policy ($\hat{j} = 2$), the (least-cost) additional*

labor input is $\hat{L}'(\Delta|\hat{j}) = 0$, the additional capital input is $\hat{K}'(\Delta|\hat{j}) = [(1/\alpha_K)\Delta^{-\beta} - (\alpha_L/\alpha_K)L_0^\beta]^{1/\beta} - K_0$ and the corresponding minimal cost of setup reduction $\hat{\Gamma}_2(\Delta) = c_K[(1/\alpha_K)\Delta^{-\beta} - (\alpha_L/\alpha_K)L_0^\beta]^{1/\beta} - c_K K_0$.

3. Under the dual input policy ($\hat{j} = 3$), the least-cost additional labor input $\hat{L}'(\Delta|\hat{j})$, the least cost additional capital input $\hat{K}'(\Delta|\hat{j})$ and the corresponding minimal cost of fixed cost reduction $\hat{\Gamma}_3(\Delta)$ are as follows.

$$\begin{aligned} (\hat{K}'(\Delta|\hat{j}) + K_0)/(\hat{L}'(\Delta|\hat{j}) + L_0) &= (c_L\alpha_K/c_K\alpha_L)^{1/(\beta-1)}, \\ \hat{K}'(\Delta|\hat{j}) &= \Delta^{-1} \left[\alpha_K + \alpha_L (c_L\alpha_K/c_K\alpha_L)^{\beta/(\beta-1)} \right]^{-1/\beta} - K_0, \\ \hat{L}'(\Delta|\hat{j}) &= \Delta^{-1} \left[\alpha_L + \alpha_K (c_K\alpha_L/c_L\alpha_K)^{\beta/(\beta-1)} \right]^{-1/\beta} - L_0, \\ \hat{\Gamma}_3^*(\Delta) &= \Psi_3\Delta^{-1} - \psi_3, \end{aligned} \quad (3.9)$$

where $\Psi_3 = c_K \left[\alpha_K + \alpha_L (c_L\alpha_K/c_K\alpha_L)^{\beta/(\beta-1)} \right]^{-1/\beta}$ and the final term $\psi_3 = c_K K_0$. Equivalently, $\Psi_3 = c_L \left[\alpha_L + \alpha_K (c_K\alpha_L/c_L\alpha_K)^{\beta/(\beta-1)} \right]^{-1/\beta}$ and the final term $\psi_3 = c_L L_0$.

Proof. For the boundary solutions, the least-cost value of the non-zero input can be obtained directly. For the interior solutions ($\hat{j} = 3$), via standard multivariable calculus; we obtain $\partial K'/\partial L' = -(\alpha_L/\alpha_K)(L/K)^{\beta-1}$ and $\partial L'/\partial K' = -(\alpha_K/\alpha_L)(K/L)^{\beta-1}$. From first-order condition by applying the chain rule, we obtain K'^* , which is found to be inversely proportional to Δ and vice versa. Similar relationships hold for L'^* . Substitution in into $\Gamma(\Delta)$ gives the closed-form optimality expressions as in Equation (3.9). $\square$

For the CES production function, the least-cost $\Gamma^*(\Delta)$ is inversely proportional to the fixed cost reduction rate Δ when the the best input usage policy employs both additional labor and capital for improvement efforts. The sufficiency conditions on the system parameters for this case are given below.

Corollary 3.2. *When the setup cost reduction rate is of the CES form, the best input usage policy giving the minimal cost of setup reduction in achieving a non-zero $(1-\Delta) \times 100\%$ setup cost reduction is the dual input usage policy ($\hat{j}^*(\Delta) = 3$) and $\Gamma^*(\Delta) = \hat{\Gamma}_3(\Delta)$ if*

$$1. \Delta > \max \left\{ K_0 \left[\alpha_K + \alpha_L (c_L \alpha_K / c_K \alpha_L)^{\beta/(\beta-1)} \right]^{1/\beta}, L_0 \left[\alpha_L + \alpha_K (c_K \alpha_L / c_L \alpha_K)^{\beta/(\beta-1)} \right]^{1/\beta} \right\},$$

or

$$2. \text{ the initial capital and labor endowments } K_0 \text{ and } L_0 \text{ have been chosen in accordance with the marginal cost condition that } K_0/L_0 = (c_L \alpha_K / c_K \alpha_L)^{1/(\beta-1)}.$$

Proof. The condition on the initial endowments K_0 and L_0 satisfies the marginal cost balance $c_K \partial \Delta / \partial L_0 = c_L \partial \Delta / \partial K_0$ and, thereby, reasonably precludes suboptimal decision-making prior to current process improvement efforts. $\square$

Proposition 3.4. *When the reduction cost rate is of CES form, the optimal reduction rate Δ^* , optimal replenishment quantity Q^* and reorder point r^* are given by the simultaneous solution of the following relationships:*

$$[(\alpha_L / c_L) S_0 D / Q^*]^{\beta/(1-\beta)} \Delta^{*2\beta} + \Delta^{*\beta} \alpha_K K_0^\beta / \alpha_L - 1 / \alpha_L = 0,$$

$$Q^* = \sqrt{2 D S_0 (1 - \Phi(z^*)) / H \eta} \sqrt{\Delta^*},$$

and

$$r^* = \mu_{DL} + z^* \sigma_{DL},$$

$$\text{where } z^* \text{ solves } [L(z^*)]^2 (1 - \Phi(z^*)) = 2 \eta S_0 D \Delta^* / h \sigma_{DL}^2.$$

Proof. The proof of the proposition rests on standard multivariate calculus and the necessary FOC conditions. $\square$

Aside from this peculiar case, CES is not amenable for obtaining closed form expressions for optimal reduction efforts and require numerical methods. For completeness, we provide the following result on the optimal decision variables.

Proposition 3.5. *The optimal reduction rate, replenishment quantity, and the reorder*

point for CES production function under the dual input usage policy are

$$\Delta^* = \left[S_0 D \Psi_3^2 / [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))] \right]^{1/3},$$

$$Q^* = \left[S_0 D \Psi_3 / [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))]^2 \right]^{1/3},$$

and

$$r^* = \mu_{DL} + z^* \sigma_{DL},$$

respectively, where z^* solves $\eta Q^* = L(z) \sigma_{DL}$.

Proof. Having already obtained $\hat{\Gamma}_j(\Delta)$, the nested optimization is carried out by first optimizing on Δ for given inventory control policy parameters and then optimizing on the policy parameters themselves. First note that $TC(\Delta|Q, r) = S_0 D \Delta / Q + H Q / 2 + H(r - \mu_{DL}) \sigma_{DL} + P D + \pi \eta D + \Psi_3 \Delta^{-1} - \psi_3$. From the SOC condition, $\partial^2 TC / \partial \Delta^2 = 2 \Psi_3 \Delta^{-3} > 0$ meaning TC is strictly convex in Δ . Therefore, the FOC gives the unique minimizer $\Delta^* | Q, r = \sqrt{\Psi_3 Q / S_0 D}$. The expected total cost rate can now be written in terms of only the inventory control policy parameters:

$$\begin{aligned} TC(Q, r) &= \sqrt{S_0 D \Psi_3} Q^{-1+1/2} + H Q / 2 + H(r - \mu_{DL}) \sigma_{DL} \\ &\quad + P D + \pi \eta D + \Psi_3 \sqrt{S_0 D / \Psi_3} Q^{-1/2} - \psi_3 \\ &= 2 \sqrt{S_0 D \Psi_3} Q^{-1/2} + H Q / 2 + H(r - \mu_{DL}) \sigma_{DL} + P D + \pi \eta D - \psi_3. \end{aligned}$$

FOC on Q given z is $\partial TC / \partial Q = -\sqrt{S_0 D \Psi_3} Q^{-3/2} + H(1/2 - \eta \sigma_{DL} / ((1 - \Phi(z)) \sigma_{DL})) = 0$. Solving for Q , we get $Q^* = \left[S_0 D \Psi_3 / [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))]^2 \right]^{1/3}$ with z^* satisfying the service level condition. Noting that the SOC

$$\partial^2 TC / \partial Q = (3/2) \sqrt{S_0 D \Psi_3} Q^{-5/2} - H \eta / ((1 - \Phi(z))^2) (-\eta / (1 - \Phi(z)) \sigma_{DL}) > 0$$

guarantees optimality. Substitution of all entities gives the result:

$$\begin{aligned} \Delta^* &= \sqrt{\Psi_3 / S_0 D} \left[S_0 D \Psi_3 / [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))]^2 \right]^{1/6} \\ &= (S_0 D)^{1/6-1/2} \Psi_3^{1/6+1/2} [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))]^{-1/3} \\ &= \left[S_0 D \Psi_3^2 / [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))] \right]^{1/3}, \\ Q^* &= \left[S_0 D \Psi_3 / [H(1/2 - \eta / ((1 - \Phi(z)) \sigma_{DL}))]^2 \right]^{1/3}, \end{aligned}$$

and

$$r^* = \mu_{DL} + z^* \sigma_{DL}$$

where z^* solves $\eta Q^* = L(z) \sigma_{DL}$. This completes the proof. $\square$

Corollary 3.3. *When the reduction cost rate is of the general CES form, one cannot obtain a closed-expression minimal cost of setup reduction $\Gamma_3^*(\Delta)$ in achieving a $(1 - \Delta) \times 100\%$ setup cost reduction.*

Proof. Similar to the proof of Proposition 3.1, but this time with the setup cost reduction function being in the form of CES production function. From Equation (3.5), setting $\Delta = \theta_3$, we obtain

$$L_3 = \left[\alpha_L^{-1} (\Delta/A)^{1/\gamma} - \alpha_L^{-1} \alpha_K K_3^{\beta_K} \right]^{1/\beta_L}.$$

Next, from the FOCs, we invoke

$$\partial L_3 / \partial K_3 = K_3^{\beta_K-1} L_3^{1-\beta_L} (-\alpha_K \beta_K / \alpha_L \beta_L) = 0$$

from which the solution of L_3 is plugged into $\Gamma_3(\Delta)$ yielding (via some algebra)

$$K_3^*(\Delta) = (c_K \alpha_L \beta_L / c_L \alpha_K \beta_K)^{1/(\beta_K-1)} L_3^*(\Delta)^{(\beta_L-1)/(\beta_K-1)}.$$

Also, $K_3^*(\Delta)$ holds for

$$(\Delta/A)^{1/\gamma} = (c_L \alpha_K \beta_K / c_K \alpha_L \beta_L)^\beta K_3^*(\Delta)^{\beta_L(\beta_K-1)/(\beta_L-1)} + \alpha_K K_3^*(\Delta)^{\beta_K},$$

which creates circularity and intractability. Therefore, for the general CES function, one cannot obtain a closed-form expression for $\Gamma_3^*(\Delta)$. $\square$

Finally, we consider the Leontief production function which is of the form:

$$\theta_2 = \theta(K, L) = \min \left(A_K (K' + K_0)^{\beta_K}, A_L (L' + L_0)^{\beta_L} \right), \quad (3.10)$$

where A_K and A_L denote the productivity or technological efficiencies of the capital and labor inputs, respectively, and the parameters β_K and β_L are the corresponding input elasticities, as in Equation (3.3). This particular functional form

was introduced by Leontief (1947). The Leontief production function assumes that the inputs are complementary to each other. The Leontief production function may also represent the settings where, typically in the short term, changes in the production process and/or product characteristics are not possible. Among the applications, we may cite Komiya (1962) for the steam power industry in the U.S., Haldi and Whitcomb (1967) for various industries such as petroleum refining, primary metals, and electric power, Ozaki (1976) for sectors with large quantity processing, large-scale assembly production and capital-intensive technology in Japan, and Nakamura (1990) for the iron and steel industry in Japan.

Proposition 3.6. *When the reduction cost rate is of Leontief form, the best input usage policy is always a dual input policy and the optimal capital input level ($K^*(\Delta) = \hat{K}'_3(\Delta) + K_0$), the optimal labor input level ($L^*(\Delta) = \hat{L}'_3(\Delta) + L_0$), the corresponding minimal cost of setup reduction ($\Gamma^*(\Delta)$) in achieving a $(1 - \Delta) \times 100\%$ setup cost reduction are as follows.*

$$\begin{aligned} K^*(\Delta) &= (\Delta/A_K)^{1/\beta_K}, \\ L^*(\Delta) &= (\Delta/A_L)^{1/\beta_L}, \\ \Gamma^*(\Delta) &= c_K \left[(\Delta/A_K)^{1/\beta_K} - K_0 \right] + c_L \left[(\Delta/A_L)^{1/\beta_L} - L_0 \right]. \end{aligned} \tag{3.11}$$

Proof. Similar to the proof of Proposition 3.1, but this time with the setup cost reduction rate being in the form of Leontief production function, i.e., θ_2 in Equation (3.10). $\square$

Thus, when improvement efforts are represented via the Leontief production function, the firm must devote additional labor and capital to achieve fixed cost reductions. The resulting least-cost is a polynomial in Δ and does not lend itself to closed form expressions for the optimal fixed cost reduction rate. However, the optimality results for P^1 can be established from first order conditions as stated below.

Corollary 3.4. *When the reduction cost rate is of Leontief form, the optimal reduction rate Δ^* , optimal replenishment quantity Q^* and reorder point r^* are given by the*

simultaneous solution of the following relationships:

$$S_0 D/Q + (c_K/\beta_K) \Delta^{*(1-\beta_K)/\beta_K} (1/A_K)^{1/\beta_K} + (c_L/\beta_L) \Delta^{*(1-\beta_L)/\beta_L} (1/A_L)^{1/\beta_L} = 0,$$

$$Q^* = \sqrt{2DS_0(1-\Phi(z^*))}/H\eta \sqrt{\Delta^*}, \text{ and } r^* = \mu_{DL} + z^* \sigma_{DL}, \text{ where } z^* \text{ solves } [L(z^*)]^2(1 - \Phi(z^*)) = 2\eta S_0 D \Delta^* / h \sigma_{DL}^2.$$

For both the Leontief and general CES functional forms, one can also recommend very versatile conic quadratic solution techniques in practice. (See (Alizadeh and Goldfarb (2003)) and (Kian, Berk and Gürlér (2019)).)

Having derived the optimality results for the input levels and the total setup cost reduction cost, we are now ready to determine the conditions for and find the optimal level of reduction effort, Δ^* . We encapsulate these results in the next Proposition.

When setup reduction efforts are expressed by Cobb Douglas functions in general and equal input elasticity CES functions under the dual input usage policy (c.f., Propositions 3.1 and 3.3), the optimal setup reduction cost, $\Gamma^*(\Delta)$ is a power function of the form $\kappa \Delta^{-\lambda} - \varphi$, where $\Delta \in [0, 1]$, $\lambda > 0$, and $\varphi > 0$. This is important because this is the cost structure assumed by Porteus (1985) and the majority of the following research stream. Thus, we can establish that the power function form of setup reduction cost directly follows from the fundamental economic production functions and may be verified in empirical settings. Conversely, an approach utilizing economic production functions to represent setup reduction efforts justifies models in the literature that use convex power functions. The optimal setup reduction level can be obtained directly from the result in (Porteus (1985)) which we recast in the following proposition.

Proposition 3.7. *Suppose that $\Gamma^*(\Delta) = \kappa \Delta^{-\lambda} - \varphi$ with $\Delta \in [0, 1]$, $\lambda > 0$, and $\varphi > 0$. Then, the optimal reduction rate Δ^* , replenishment quantity Q^* , and reorder point r^* are as follows.*

$$\Delta^* = \left[\frac{\lambda \kappa}{S_0 D} \right]^{1/(1+\lambda)} Q^{*1/(1+\lambda)},$$

$$Q^* = \left[\frac{\lambda \left[S_0 D \left[\frac{\lambda \kappa}{S_0 D} \right]^{1/(1+\lambda)} + \kappa \left[\frac{\lambda \kappa}{S_0 D} \right]^{-\lambda/(1+\lambda)} \right]}{(1+\lambda)H(1/2 - \frac{\eta}{1-\Phi(z^*)})} \right]^{(1+\lambda)/(1+2\lambda)},$$

and

$$r^* = \mu_{DL} + z^* \sigma_{DL},$$

where z^* solves $Q^* \eta = L(z^*) \sigma_{DL}$.

Proof. Recall

$$TC(\Delta, Q, z) = S_0 \Delta D / Q + H(Q/2 + z \sigma_{DL}) + \pi \eta D + PD + \Gamma^*(\Delta).$$

The FOC on Δ gives $\Delta^*|Q, z = [\lambda \kappa / S_0 D]^{1/(1+\lambda)} Q^{1/(1+\lambda)}$. Substituting the conditional best reduction rate, the expected total cost rate can be cast as

$$\begin{aligned} TC(\Delta, Q, z) = & S_0 D \left(\frac{\lambda \kappa}{S_0 D} \right)^{\frac{1}{1+\lambda}} Q^{1/(1+\lambda)-1} + H\left(\frac{Q}{2} + z \sigma_{DL}\right) + \pi \eta D + PD \\ & + \kappa \left(\frac{\lambda \kappa}{S_0 D} \right)^{\frac{-\lambda}{1+\lambda}} Q^{-\lambda/(1+\lambda)} - \varphi. \end{aligned}$$

Then, collecting the terms, we get

$$TC(\Delta, Q, z) = \left[S_0 D \left(\frac{\lambda \kappa}{S_0 D} \right)^{\frac{1}{1+\lambda}} + \kappa \left(\frac{\lambda \kappa}{S_0 D} \right)^{\frac{-\lambda}{1+\lambda}} \right] Q^{-\lambda/(1+\lambda)} + H\left(\frac{Q}{2} + z \sigma_{DL}\right) + \pi \eta D + PD - \varphi.$$

The FOC on Q gives

$$\begin{aligned} \partial TC(Q, z) / \partial Q = & \left[S_0 D [\lambda \kappa / S_0 D]^{1/(1+\lambda)} + \kappa [\lambda \kappa / S_0 D]^{-\lambda/(1+\lambda)} \right] (-\lambda/(1+\lambda)) Q^{-\lambda/(1+\lambda)-1} \\ & + H(1/2 + \sigma_{DL} \partial z / \partial Q) \\ = & 0. \end{aligned}$$

Solving for Q gives $Q^*|z$ as follows:

$$Q^*|z = \left[\frac{\lambda \left[S_0 D [\lambda \kappa / S_0 D]^{1/(1+\lambda)} + \kappa [\lambda \kappa / S_0 D]^{-\lambda/(1+\lambda)} \right]}{(1+\lambda)H(1/2 + \sigma_{DL} \partial z / \partial Q)} \right]^{(1+\lambda)/(1+2\lambda)},$$

and rearranging terms and expanding over the term with the derivative, we obtain

$$Q^*|z = \left[\frac{\lambda \left[S_0 D \left[\frac{\lambda \kappa}{S_0 D} \right]^{1/(1+\lambda)} + \kappa \left[\frac{\lambda \kappa}{S_0 D} \right]^{-\lambda/(1+\lambda)} \right]}{(1+\lambda)H(1/2 - \frac{\eta}{1-\Phi(z^*)})} \right]^{(1+\lambda)/(1+2\lambda)},$$

where, at optimality, z^* satisfies $Q^* \eta = L(z^*) \sigma_{DL}$.

Noting that

$$\partial^2 TC(Q, z) / \partial Q^2 = \tilde{A}(-\lambda/(1+\lambda))(-1+2\lambda)/(1+\lambda) Q^{-(1+2\lambda)/(1+\lambda)-1} + H \sigma_{DL} \partial^2 z / \partial Q^2 > 0,$$

$$\text{where } \tilde{A} = \left[S_0 D \left[\lambda \kappa / S_0 D \right]^{1/(1+\lambda)} + \kappa \left[\lambda \kappa / S_0 D \right]^{-\lambda/(1+\lambda)} \right] > 0,$$

we show the convexity and the joint optimality. $\square$

Finally, we consider a special case of Proposition 3.7 in which the parameters are now considered rather deterministic and the lead times are insignificant (i.e., assumed to be 0). When there are no uncertainties in the system (in terms of demand and/or supply deliveries) and negligible lead times, the inventory setting reduces to that studied in Porteus (1985). Demand is now constant with the rate D (units per unit time), and delivery lead time is negligible. In our notation, we set μ_{DL} , σ_{DL} and to zero.; we retain the rest of the setting and notation. In this setting, the optimal continuous control policy is known to be of the EOQ-type, the total cost per unit time for an order size of Q is $TC(Q) = QS/D + QH/2 + PD$. Then, $Q^* = \arg \min\{TC(Q) : Q > 0\} = \sqrt{2DS/H}$, i.e., the EOQ. Thus, the cost-minimizing ordering policy is to manufacture (or order) Q^* whenever inventory level hits zero. Then, the ensuing optimal cost per unit time, is given by $TC(Q = Q^*) = \sqrt{2DSH} + PD$. The setup reduction optimization problem now becomes $P^D : \min_{\Delta} \{ \sqrt{2DHS_0\Delta} + PD + \Gamma^*(\Delta) \mid \Delta \geq 0 \}$, where Δ and $\Gamma^*(\Delta)$ are as defined before.

Corollary 3.5. Suppose $\Gamma_j^*(\Delta) = \Psi_j \Delta^{a_j} - \psi_j$. Then, the optimal setup reduction rate is given by $\Delta^* = -(1/2\Psi_j a_j)[2S_0 DH]^{1/(2a_j-1)}$. If this much reduction were to be implemented under the input usage policy j , then the resulting setup reduction cost would be $\hat{\Gamma}_j(\Delta_j^*) = \Psi_j(-(1/2\Psi_j a_j)[2S_0 DH]^{1/(2a_j-1)})^{a_j} - \psi_j$.

3.4 Some Insights on Process Improvements for Sustainable Supply Chains

In the face of the pressing climate crisis, outbreaks, social instabilities, and changing demographics and needs, sustainability is to be at the front and center of the business world, which in essence runs on supply chains (c.f., Mefford (2011)). Hence, the literature on sustainable supply chain management has been expanding (e.g., Khan, Yu, Golpira, Sharif and Mardani (2021)). Parallel to advances in intelligent technologies and data storing and processing capacities, supply chain analytics has emerged as an enabler for better decision-making with sustainability imperatives (e.g., Ülkü and Engau (2021)). As regulations press firms to disclose their environmental, social and governance (ESG) impacts (e.g., Dai and Tang (2022)) disclosures, SCs need to be better coordinated, transparent, engaged with their customers, and be resilient and smart in minimizing “waste.”

According to Shingo (1988), industrial waste, a sign of inefficiency, surfaces in various ways, such as overproduction, unnecessary processing-stocking- transportation, and the making of defective products. Waste in both capital (e.g., materials, energy, the use of technology, the cost of unproductive time, sunk costs, landfills) and in labor (e.g., unnecessary human errs, lack of adequate level of automation, the cost of time train workers) causes not only additional supply chain costs but also hurts sustainability. In short, the ill-operated processes (how and where products are produced, services rendered) add additional costs. To that end, “leaning” the logistics systems (e.g., Bookbinder (2021)), such as minimizing setup times for discrete manufacturing or maximizing transport capacity utilization for retailers or carriers, becomes a critical goal. Our models studied in the preceding section help achieve this goal: How to achieve a desired process improvement with minimal cost, such that capital and labor waste is minimized.

We build on and extend the setup cost reduction model developed by Porteus (1985). Exposed in the forms of propositions and corollaries, our structural results provide neat, closed-form solutions for optimal input levels (capital and labor) and inventory control variables (reorder point and replenishment quantity) to achieve the optimal cost of (in-house) process improvement level (i.e., setup cost reduction effort) for a given inventory service (i.e., safety stock) level, all while the uncertainties in both demand and the delivery/manufacturing lead time are accounted for. At the end, in Proposition 3.7, we bring together all the

production functions considered (Cobb-Douglas, CES, and Leontief) and unify them in the form of a general power function, which is a crucial step toward implementable analytical results in developing decision tools for lean inventory systems in supply chain management.

As discussed earlier, reducing waste requires continuous process improvement. Most changes in processes face resistance and incur fixed costs. Our models consider in-house process improvement efforts, which reduces the likelihood of “change resistance” because the incumbent labor and management undertake a collective goal. A good example of such effective process improvements is the concept a “quality circle” in which a designated group of employees work toward a solution to a specific, smaller problem in the company. While emerging intelligent technologies such as Blockchain and IoT seem promising in enhancing sustainability performance (a process improvement), caution needs to be taken when assessing the fixed cost of establishing their infrastructure and implementation. At that stage, our models may also be employed as tools for cost-benefit analysis. Consider, for instance, a dyadic supply chain with a retailer and a manufacturer. The retailer may optimize its scheduled deliveries from the manufacturer while the manufacturer synchronizes its production lots to be loaded on the retailer’s own fleet of trucks. In such a case, a logistics service improvement could entail better coordinating the order-production-delivery process, which may be achieved by investing in an advanced transport management system. Then, both the manufacturer and the retailer can assess their current capital and labor constraints and decide on whether it will be more economical for them to invest in reducing their common fixed costs compared to the cost of purchasing and implementing alternative new technologies (c.f., Saberi, Kouhizadeh, Sarkis and Shen (2019)). Proposition 3.7 could be utilized in such a trade-off scenario.

CHAPTER 4

CONCLUSION

In this thesis, we studied two problems: 1) Studying the problem of optimizing the level of technology enhancement of a firm through knowledge capital acquisition in the presence of carbon tax and constraints on output and technology level, we conducted parametric analysis to derive insight for social planners. 2) We studied the problem of optimizing process improvement efforts to enhance supply chain sustainability through better use of capital and labor, in the presence of disruptive events.

For the first problem, a manufacturer producing a single product by using multiple inputs under carbon taxation is considered in a news-vendor setting where input-output relations are integrated into the model via the Cobb Douglas production function. It is supposed that the firm's output level is bounded below and above and that she may increase her knowledge capital, which in turn enhances her productivity and decreases carbon emissions. We solved the problem using representation theorem, Lagrange Multipliers method and decomposition. Via sensitivity analysis we examine the behavior of optimal policy with respect to carbon tax and obtained the following results at optimality: 1) when technology level is increasing in carbon tax, production quantity is always constant and amount of an input is increasing in carbon tax only if share of the tax paid due to its carbon footprint of its total variable cost is less than the weighted average of that of all production inputs; 2) when technology level is constant, if the production quantity is also constant, amount of an input shows the same behavior we state in the first finding; 3) if the production quantity is decreasing in carbon tax, amounts of all inputs are decreasing in carbon tax; 4) if the technology level is decreasing in carbon tax, the output, as well as the inputs, are decreasing in carbon tax. Furthermore, while the technology level and the output of a firm

with elasticity of scale less than unity may decrease, a firm with elasticity of scale greater than or equal to unity always produces at her higher limit and an increase in carbon tax may lead to a status quo or an increase in their technology levels, without leading to negative profits, at optimality. Furthermore, we find out that optimal emission level of a firm is always decreasing in carbon tax, and for some sub-types of mid elasticity firms, there exists a discontinuity (downward) in the optimal emission level, production quantity, and technology level with respect to carbon tax, which indicates an interval of sub-optimal emission levels, and thus, we discussed that the social planner may be flexible in deciding the emission target. Moreover, in the thesis we studied a single period problem. If a multi-period production planning and tax setting problem is considered, since a small compromise on the emission reduction level in the first period, may lead to huge gains in terms of technological enhancements and the firm may begin to operate with a much higher productivity level with no additional cost starting from the next planning horizon, a flexible emission target setting decision may turn out to be much more effective in decreasing total emissions in the long term. Therefore, a multi-period problem can be considered as an important extension of the first problem of the thesis.

For the second problem, considering the disruptive effects of the COVID-19 pandemic on society, we develop a model for inventory management that is closely related to the resiliency of supply chains. We explicitly integrate process improvement efforts leading to better use of capital (materials, assets, technology) and labor (workforce and know-how) into the model, which may be devised to enhance the resiliency, and thus, the sustainability of supply chains. More specifically, we incorporated the capital and labor levels as endogenous decision variables in optimizing in-house process improvement efforts embedded in the classical (Q, r) inventory system that is widely used when the demand and lead time are probabilistic. To do so, we utilized production functions that relate two inputs (capital and labor) to the efficiency of output (production level). We found closed-form solutions when possible and proved non-existence of such solutions, otherwise. Particularly, for each of the production functions considered (Cobb-Douglas, CES, and Leontief), we provided the conditions under which optimal interior or boundary solutions exist, ascertaining uniqueness. These structural analytical results were obtained via multivariate calculus (first- and second-order conditions), and we showed joint convexity. Albeit stylized, our models enable tractable solutions

and general, robust insights. Thus, an imminent research could be to test the mechanics of these models through a numeric case study. It would also be curious to analyze industry-specific data and see how each production function would be the best fit for which type of supply chain strategy (e.g., responsive vs. efficient). Last but not least, a future extension of this study could explore the impact of advanced big data-processing capabilities such as machine learning (c.f., Oguntola, Ülkü, Saif and Engau (2023)) in better prediction of the uncertain parameters (demand and lead time) in the models we proposed.



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APPENDIX A

APPENDIX

A.1 Proof of Lemma 2.1

The joint convexity of $\check{\Gamma}(\vec{\lambda}, w|K, Q)$, in λ_i for $i \in \{1, 2, \dots, N\}$, is proven through examining its Hessian as follows.

$$\begin{vmatrix} \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_1}{\alpha_1}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right)^2 \frac{p_1 + \rho\beta_1}{\alpha_1^2} & 0 & \dots & 0 \\ 0 & \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_2}{\alpha_2}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right)^2 \frac{p_2 + \rho\beta_2}{\alpha_2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_N}{\alpha_N}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right)^2 \frac{p_N + \rho\beta_N}{\alpha_N^2} \end{vmatrix}$$

Note that the determinant of the first principal minor of the hessian matrix is

$$\det[P_1] = \left(\frac{Q}{aK^{\alpha_K}}\right)^{\frac{\lambda_1}{\alpha_1}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right)^2 \frac{p_1 + \rho\beta_1}{\alpha_1^2};$$

N^{th} principal minor is

$$\det[P_k] = \left(\frac{Q}{aK^{\alpha_K}}\right)^{\sum_{j=1}^k \frac{\lambda_j}{\alpha_j}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right)^{2k} \prod_{j=1}^k \frac{p_j + \rho\beta_j}{\alpha_j^2};$$

and $N + 1^{th}$ principal minor is

$$\det[P_{k+1}] = \left(\frac{Q}{aK^{\alpha_K}}\right)^{\sum_{j=1}^{k+1} \frac{\lambda_j}{\alpha_j}} \ln\left(\frac{Q}{aK^{\alpha_K}}\right)^{2(k+1)} \prod_{j=1}^{k+1} \frac{p_j + \rho\beta_j}{\alpha_j^2},$$

where k is an integer. Then, letting $k = N - 1$ and $k + 1 = N$, we observe that signs of the determinants of the first N principal minors of the Hessian Matrix are positive, for $\rho > \max_{i \in \{1, 2, \dots, N\}} \{-\frac{p_i}{\beta_i}\}$, $\check{\Gamma}(\vec{x}, w|K, Q)$ is jointly convex in x_i for $i \in \{1, 2, \dots, N\}$ for given K and Q . $\square$

A.2 Proof of Lemma 2.2

Remember that $\check{\Gamma}(K|Q)$ is expressed as follows.

$$\check{\Gamma}(K|Q) = r \left(\frac{Q}{aK^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} + \frac{\gamma' c_K}{\gamma'' + 1} (K^{\gamma''+1} - K_0^{\gamma''+1}).$$

Then, first and second derivatives of this function respect to K as follows.

$$\begin{aligned} \frac{\partial \check{\Gamma}(K|Q)}{\partial K} &= -\alpha_K K^{-\frac{\alpha_K}{r}-1} \left(\frac{Q}{a} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} + \gamma' c_K K^{\gamma''}, \\ \frac{\partial^2 \check{\Gamma}(K|Q)}{\partial K^2} &= \alpha_K \left(\frac{\alpha_K}{r} + 1 \right) K^{-\frac{\alpha_K}{r}-2} \left(\frac{Q}{a} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} + \gamma'' \gamma' c_K K^{\gamma''-1} > 0, \end{aligned}$$

which shows that $\check{\Gamma}(K|Q)$ is convex in K . $\square$

A.3 Proof of Proposition 2.2

From Theorem 2.2 we have

$$Q_{int} = a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r \left[\prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{-1} K_0^{r(\gamma''+1)+\alpha_K}.$$

Then substituting Q_{int} for Q in functions (2.52) and (2.53), one obtains

$$\Pi_0(Q_{int}) = a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r \left[\prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{-1} K_0^{r(\gamma''+1)+\alpha_K} s - r \left(\frac{\gamma' c_K}{\alpha_K} \right) K_0^{\gamma''+1}.$$

and

$$\begin{aligned}\Pi_v(Q_{int}) &= a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r \left[\prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{-1} K_0^{r(\gamma''+1)+\alpha_K} s \\ &\quad - \left(\frac{\gamma' c_K (r(\gamma''+1) + \alpha_K)}{\alpha_K (\gamma''+1)} \right) K_0^{\gamma''+1} \\ &\quad + \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1} \\ &= a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r \left[\prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{-1} K_0^{r(\gamma''+1)+\alpha_K} s - r \left(\frac{\gamma' c_K}{\alpha_K} \right) K_0^{\gamma''+1},\end{aligned}$$

respectively. Hence, $\Pi_0(Q_{int}) = \Pi_v(Q_{int})$.

Now, note that first derivatives of $\Pi_0(Q)$ and $\Pi_v(Q)$ defined by expressions (2.52) and (2.53) with respect to Q are

$$\frac{\partial \Pi_0(Q)}{\partial Q} = s - Q^{-1} \left(\frac{Q}{a K_0^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{r}}$$

and

$$\begin{aligned}\frac{\partial \Pi_v(Q)}{\partial Q} &= s - Q^{-1} \frac{\gamma' c_K}{\alpha_K} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \\ &= s - Q^{-1} \left(\frac{Q}{a \hat{K}(Q)^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{r}},\end{aligned}$$

respectively.

Note that

$$\begin{aligned}
 \Pi_v(Q) - \Pi_0(Q) &= r \left(\frac{Q}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \\
 &\quad - \left(\frac{\gamma' c_K (r(\gamma'' + 1) + \alpha_K)}{\alpha_K (\gamma'' + 1)} \right) \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} \\
 &= \left(\frac{1}{K_0^{\alpha_K}} - \frac{1}{\hat{K}(Q)^{\alpha_K}} \right)^{\frac{1}{r}} r \left(\frac{Q}{a} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} \\
 &\quad - \frac{\gamma' c_K}{\gamma'' + 1} (\hat{K}(Q)^{\gamma'' + 1} - K_0^{\gamma'' + 1}).
 \end{aligned}$$

Thus, we have

$$\frac{\partial \Pi_v(Q)}{\partial Q} - \frac{\partial \Pi_0(Q)}{\partial Q} = \left(\frac{1}{K_0^{\alpha_K}} - \frac{1}{\hat{K}(Q)^{\alpha_K}} \right)^{\frac{1}{r}} Q^{-1} \left(\frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{r}}.$$

Furthermore, since $\hat{K}(Q)$ is a monotonic increasing function of Q , by definition of Q_{int} , $\hat{K}(Q) < K_0$ for $Q < Q_{int}$; $K_0 \leq \hat{K}(Q)$ for $Q_{int} \leq Q$. Hence, $\frac{\partial \Pi_v(Q)}{\partial Q} - \frac{\partial \Pi_0(Q)}{\partial Q} < 0$ for $Q < Q_{int}$; $\frac{\partial \Pi_v(Q)}{\partial Q} - \frac{\partial \Pi_0(Q)}{\partial Q} \geq 0$ for $Q_{int} \leq Q$. Moreover, optimal value of the objective function of an unconstrained maximization problem is greater than that of its constrained instances. Thus, since $\Pi_0(Q)$ is a constrained instance of $\Pi_v(Q)$, we should have $\Pi_v(Q) > \Pi_0(Q) \forall Q$. Therefore, $\Pi_v(Q) - \Pi_0(Q)$ is decreasing in Q for $Q < Q_{int}$, $\Pi_v(Q) - \Pi_0(Q) = 0$ for $Q = Q_{int}$, and $\Pi_v(Q) - \Pi_0(Q)$ is increasing in Q for $Q_{int} < Q$, which means that $\Pi_v(Q) - \Pi_0(Q)$ has a unique positive root that is equal to Q_{int} .

Hence, if $\hat{K}(Q) < K_0$ ($Q < Q_{int}$), then $\Pi'_v(Q) < \Pi'_0(Q)$ and $\Pi_v(Q) > \Pi_0(Q)$; if $\hat{K}(Q) = K_0$ ($Q = Q_{int}$), $\Pi'_v(Q) = \Pi'_0(Q)$ and $\Pi_v(Q) = \Pi_0(Q)$; and if $\hat{K}(Q) > K_0$ ($Q > Q_{int}$), then $\Pi'_v(Q) > \Pi'_0(Q)$ and $\Pi_v(Q) > \Pi_0(Q)$. $\square$

A.4 Proof of Lemma 2.3

Substituting K_0 and $\hat{K}(Q)$ for K in $\check{\Gamma}(K, Q)$, we get

$$\Pi_0(Q) = Q^s - r \left(\frac{Q}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}}$$

and

$$\begin{aligned} \Pi_v(Q) = & Q^s - \left(\frac{\gamma'c_K(r(\gamma''+1) + \alpha_K)}{\alpha_K(\gamma''+1)} \right) \left[\left(\frac{\alpha_K}{\gamma'c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma''+1}{r(\gamma''+1) + \alpha_K}} \\ & + \frac{\gamma'c_K}{\gamma''+1} K_0^{\gamma''+1}, \end{aligned}$$

respectively.

The second derivatives

$$\frac{\partial^2 \Pi_0(Q)}{\partial Q^2} = \left(1 - \frac{1}{r} \right) Q^{-2} \left(\frac{Q}{aK_0^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{r}}$$

and

$$\frac{\partial^2 \Pi_v(Q)}{\partial Q^2} = \left(1 - \frac{\gamma''+1}{r(\gamma''+1) + \alpha_K} \right) Q^{-2} \frac{\gamma'c_K}{\alpha_K} \left[\left(\frac{\alpha_K}{\gamma'c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma''+1}{r(\gamma''+1) + \alpha_K}}$$

are negative for $r < 1$, and $r < 1 - \alpha_K/(\gamma''+1)$, respectively. Furthermore, for $r = 1$ and $r = 1 - \alpha_K/(\gamma''+1)$, $\frac{\partial^2 \Pi_0(Q)}{\partial Q^2} = 0$ and $\frac{\partial^2 \Pi_v(Q)}{\partial Q^2} = 0$, respectively, and for $r > 1$ and $r > 1 - \alpha_K/(\gamma''+1)$, $\frac{\partial^2 \Pi_0(Q)}{\partial Q^2} > 0$ and $\frac{\partial^2 \Pi_v(Q)}{\partial Q^2} > 0$, respectively. Hence, $\Pi_0(Q)$ is **concave** in Q for $r < 1$, it is **linear** in Q for $r = 1$, and it is **convex** in Q if $r > 1$; $\Pi_v(Q)$ is **concave** in Q for $r < 1 - \frac{\alpha_K}{\gamma''+1}$, it is **linear** in Q for $r = 1 - \frac{\alpha_K}{\gamma''+1}$, and it is **convex** in Q for $r > 1 - \frac{\alpha_K}{\gamma''+1}$.

Moreover, equating first derivatives of $\Pi_0(Q)$ and $\Pi_v(Q)$ with respect to Q to zero

as

$$\frac{\partial \Pi_0(Q)}{\partial Q} = s - Q^{-1} \left(\frac{Q}{aK_0^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{r}} = 0$$

and

$$\frac{\partial \hat{\Pi}_v(Q)}{\partial Q} = s - Q^{-1} \frac{\gamma' c_K}{\alpha_K} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} = 0,$$

respectively, we obtain the critical values of the production quantities of $\Pi_0(Q)$ and $\Pi_v(Q)$, Q_0 and Q_v , as

$$Q_0 = \left[\frac{1}{s^r a K_0^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r-1}}$$

and

$$\begin{aligned} Q_v &= \frac{\gamma' c_K d}{s \alpha_K} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^{r-1} \frac{1}{s a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{\gamma''+1}{(r-1)(\gamma''+1)+\alpha_K}} \\ &= \left[\left(\frac{1}{s} \right)^{r(\gamma''+1)+\alpha_K} \left(\frac{\gamma' c_K}{\alpha_K} \right)^{\alpha_K} \left[\frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\gamma''+1} \right]^{\frac{1}{(r-1)(\gamma''+1)+\alpha_K}}, \end{aligned}$$

respectively, which completes the proof. $\square$

A.5 Proof of Lemma 2.4

From the chain rule we have

$$\frac{\partial f(Z)}{\partial \rho} = \frac{\partial f(Z)}{\partial Z} \frac{\partial Z}{\partial \rho}.$$

Then, since

$$\frac{\partial Z}{\partial \rho} = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} > 0,$$

we have

$$\text{sgn} \left[\frac{\partial f(Z)}{\partial \rho} \right] = \text{sgn} \left[\frac{\partial f(Z)}{\partial Z} \right].$$

Hence, for $Z \geq 0$, optimal and critical values of the decision variables of the problem (P) show the same behavior as with respect to Z as with respect to ρ . $\square$

A.6 Proof of Proposition 2.3

Now, we show that if $Q_0 = Q_{int}$, then $Q_{int} = Q_0 = Q_v$, as follows.

Remember that, from Theorem 2.2 and Lemma 2.3, we have

$$Q_{int} = a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r Z^{-1} K_0^{r(\gamma''+1)+\alpha_K},$$

$$Q_0 = \left[\frac{Z}{s^r a K_0^{\alpha_K}} \right]^{\frac{1}{r-1}},$$

and

$$Q_v = \frac{\gamma' c_K}{s \alpha_K} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^{r-1} \frac{1}{sa} Z \right]^{\frac{\gamma''+1}{(r-1)(\gamma''+1)+\alpha_K}},$$

Thus, equating Q_0 and Q_{int} we get the value of Z at which $Q_0 = Q_{int}$, Z_{int} , as

$$Z_{int} = sa \left(\frac{\gamma' c_K}{\alpha_K} \right)^{r-1} K_0^{(r-1)(\gamma''+1)+\alpha_K}.$$

Substituting Z_{int} for Z in Q_{int} , Q_0 , and Q_v , we have

$$Q_{int}(Z_{int}) = Q_0(Z_{int}) = Q_v(Z_{int}) = \frac{1}{s} \left(\frac{\gamma' c_K}{\alpha_K} \right) K_0^{\gamma''+1}.$$

Now, let

$$E_{int} = a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r K_0^{r(\gamma''+1)+\alpha_K},$$

$$E_0 = \left[\frac{1}{s^r a K_0^{\alpha_K}} \right]^{\frac{1}{r-1}},$$

and

$$E_v = \frac{\gamma' c_K d}{s \alpha_K} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^{r-1} \frac{1}{s a} \right]^{\frac{\gamma''+1}{(r-1)(\gamma''+1)+\alpha_K}}.$$

Then, all of the conditions for a given order relation among Q_{int} , Q_0 , and Q_v are given below.

1. $Q_{int} \leq (\geq) Q_0$ holds if $Z^{\frac{r}{1-r}} \leq (\geq) [E_0/E_{int}]$, that is, if
 - (a) $Z \leq (\geq) [E_0/E_{int}]^{\frac{1-r}{r}} = Z_{int}$ for $r < 1$,
 - (b) $Z \geq (\leq) [E_0/E_{int}]^{\frac{1-r}{r}} = Z_{int}$ for $r > 1$.
2. $Q_{int} \leq (\geq) Q_v$ holds if $Z^{\frac{-r(\gamma''+1)-\alpha_K}{(r-1)(\gamma''+1)+\alpha_K}} \leq (\geq) [E_0/E_{int}]$, that is, if
 - (a) $Z \leq (\geq) [E_0/E_{int}]^{\frac{(r-1)(\gamma''+1)+\alpha_K}{-r(\gamma''+1)-\alpha_K}} = Z_{int}$ for $r < 1 - \alpha_K/(\gamma''+1)$,
 - (b) $Z \geq (\leq) [E_0/E_{int}]^{\frac{(r-1)(\gamma''+1)+\alpha_K}{-r(\gamma''+1)-\alpha_K}} = Z_{int}$ for $r > 1 - \alpha_K/(\gamma''+1)$.
3. $Q_0 \leq (\geq) Q_v$ holds if $Z^{\frac{\alpha_K}{(r-1)[(r-1)(\gamma''+1)+\alpha_K]}} \leq (\geq) [E_0/E_{int}]$, that is, if
 - (a) $Z \leq (\geq) [E_0/E_v]^{\frac{(r-1)[(r-1)(\gamma''+1)+\alpha_K]}{\alpha_K}} = Z_{int}$ for $r < 1 - \alpha_K/(\gamma''+1)$,
 - (b) $Z \geq (\leq) [E_0/E_v]^{\frac{(r-1)[(r-1)(\gamma''+1)+\alpha_K]}{\alpha_K}} = Z_{int}$ for $1 - \alpha_K/(\gamma''+1) < r < 1$,
 - (c) $Z \leq (\geq) [E_0/E_v]^{\frac{(r-1)[(r-1)(\gamma''+1)+\alpha_K]}{\alpha_K}} = Z_{int}$ for $1 < r$.

Hence, all of the possible order relations among Q_{int} , Q_0 , and Q_v are as follows.

- From (1.a) and (3.a), if $0 < r < 1 - \alpha_K/(\gamma''+1)$, $Q_{int} < Q_0 < Q_v$ for $Z < Z_{int}$;
 $Q_{int} = Q_0 = Q_v$ for $Z = Z_{int}$; $Q_v < Q_0 < Q_{int}$ for $Z > Z_{int}$.

- From (1.a), if $r = 1 - \alpha_K/(\gamma'' + 1)$, $Q_{int} < Q_0$ for $Z < Z_{int}$; $Q_{int} = Q_0$ for $Z = Z_{int}$; $Q_0 < Q_{int}$ for $Z > Z_{int}$.
- From (2.b) and (1.b), if $1 - \alpha_K/(\gamma'' + 1) < r < 1$, $Q_v < Q_{int} < Q_0$ for $Z < Z_{int}$; $Q_v = Q_{int} = Q_0$ for $Z = Z_{int}$; $Q_0 < Q_{int} < Q_v$ for $Z > Z_{int}$.
- From (2.b), if $r = 1$, $Q_v < Q_{int}$ for $Z < Z_{int}$; $Q_v = Q_{int}$ for $Z = Z_{int}$; $Q_{int} < Q_v$ for $Z > Z_{int}$.
- From (3.c) and (2.b), if $r > 1$, $Q_0 < Q_v < Q_{int}$ for $Z < Z_{int}$; $Q_0 = Q_v = Q_{int}$ for $Z = Z_{int}$; $Q_{int} < Q_v < Q_0$ for $Z > Z_{int}$. $\square$

A.7 Proof of Theorem 2.3

For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$, $\Pi(Q)$ is concave in Q from Lemma 2.3. Then, by Lemma 2.5,

1. if $Q_{int} < Q_0 < Q_v$, $\Pi(Q) = \Pi_0(Q)$ is increasing concave up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is increasing concave between Q_{int} and Q_v , decreasing concave afterwards. Hence,
 - (a) if $Q_h < Q_v$, then,
 - i. for $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - ii. for $Q_h \geq Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - (b) if $Q_l < Q_v \leq Q_h$, $Q^* = Q_v$ and $K^* = \hat{K}(Q_v)$,
 - (c) if $Q_v \leq Q_l$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$,
2. if $Q_{int} = Q_0 = Q_v$, $\Pi(Q) = \Pi_0(Q)$ is increasing concave up to Q_{int} ; $\Pi(Q) = \Pi_0(Q)$ is decreasing concave after Q_{int} . Hence,
 - (a) if $Q_h \leq Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Q_l < Q_{int} \leq Q_h$, $Q^* = Q_{int} = Q_0 = Q_v$ and $K^* = \hat{K}(Q_{int}) = K_0$,
 - (c) if $Z_{l,int} \leq Z$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$,
3. if $Q_v < Q_0 < Q_{int}$, $\Pi(Q) = \Pi_0(Q)$ is increasing concave up to Q_0 , decreasing concave between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is decreasing concave after-

wards. Hence,

- (a) if $Q_h \leq Q_0$, $Q^* = Q_h$ and $K^* = K_0$,
- (b) if $Q_l < Q_0 \leq Q_h$, $Q^* = Q_0$ and $K^* = K_0$,
- (c) if $Q_0 \leq Q_l$, then
 - i. for $Q_l \leq Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$;
 - ii. for $Q_l \geq Q_{int}$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r = 1 - \alpha_K/(\gamma'' + 1)$, from Lemma 2.3, $\Pi(Q)$ is concave in Q . Then, by Lemma 2.5,

1. if $Q_{int} < Q_0$, $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is linear increasing afterwards. Hence,
 - (a) if $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Q_h \geq Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
2. if $Q_{int} = Q_0$, $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is constant afterwards. Hence
 - (a) if $Q_h < Q_0$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Q_l < Q_0 \leq Q_h$, $Q^* \in [Q_{int}, Q_h]$ and $K^* = \hat{K}(Q^*)$,
 - (c) if $Q_0 \leq Q_l$, $Q^* \in [Q_l, Q_h]$ and $K^* = \hat{K}(Q^*)$,
3. if $Q_0 < Q_{int}$, $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_0 , concave decreasing between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is linear decreasing afterwards. Hence,
 - (a) if $Q_h < Q_0$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Q_l < Q_0 \leq Q_h$, $Q^* = Q_0$ and $K^* = K_0$,
 - (c) if $Q_0 \leq Q_l < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (d) if $Q_{int} \leq Q_l$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $1 - \alpha_K/(\gamma'' + 1) < r < 1$, from Lemma 2.3, $\Pi(Q)$ is concave-convex in Q . Then, by Lemma 2.5,

1. if $Q_v < Q_{int} < Q_0$, $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) =$

$\Pi_v(Q)$ is convex increasing afterwards. Hence, $Q^* = Q_h$ and

- (a) if $Q_h < Q_{int}$, $K^* = K_0$,
- (b) if $Q_h \geq Q_{int}$, $K^* = \hat{K}(Q_h)$.

2. if $Q_v = Q_{int} = Q_0$, $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence, $Q^* = Q_h$ and

- (a) if $Q_h < Q_{int}$, $K^* = K_0$,
- (b) if $Q_h \geq Q_{int}$, $K^* = \hat{K}(Q_h)$.

3. if $Q_0 < Q_{int} < Q_v$, $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_0 , concave decreasing between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex decreasing between Q_{int} and Q_v , convex increasing afterwards. Hence,

- (a) if $Q_h < Q_{int}$,
 - i. if $Q_h < Q_0$, $Q^* = Q_h$ and $K^* = K_0$,
 - ii. if $Q_l < Q_0 \leq Q_h$, $Q^* = Q_0$ and $K^* = K_0$,
 - iii. if $Q_0 \leq Q_l$, $Q^* = Q_l$ and $K^* = K_0$.
- (b) if $Q_l < Q_0 < Q_{int} \leq Q_h$,
 - i. for $\Pi_0(Q_0) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_0) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_0 \text{ and } K^* = K_0)$,
 - iii. for $\Pi_0(Q_0) > \Pi_v(Q_h)$, $Q^* = Q_0$ and $K^* = K_0$.
- (c) if $Q_0 \leq Q_l < Q_{int} \leq Q_h$,
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
- (d) if $Q_{int} \leq Q_l$,
 - i. for $\Pi_v(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_v(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,

iii. for $\Pi_v(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r = 1$, from Lemma 2.3, $\Pi(Q)$ is convex in Q . Then, by Lemma 2.5

1. if $Q_v < Q_{int}$, $\Pi(Q) = \Pi_0(Q)$ is linear increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Q_{int} \leq Q_h$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
2. if $Q_v = Q_{int}$, $\Pi(Q) = \Pi_0(Q)$ is constant up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Q_h \leq Q_{int}$, $Q^* \in [Q_l, Q_h]$ and $K^* = K_0$,
 - (b) if $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
3. if $Q_{int} < Q_v$, $\Pi(Q) = \Pi_0(Q)$ is linear decreasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex decreasing between Q_{int} and Q_v , convex increasing afterwards. Hence,
 - (a) if $Q_h < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Q_l < Q_{int} \leq Q_h$,
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = K_0$),
 - iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Q_{int} \leq Q_l$,
 - i. for $\Pi_v(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_v(Q_l) = \Pi_v(Q_h)$, ($Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$) or ($Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$),
 - iii. for $\Pi_v(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r > 1$, from Lemma 2.3, $\Pi(Q)$ is convex in Q . Then, by Lemma 2.5,

1. if $Q_0 < Q_v < Q_{int}$, $\Pi(Q) = \Pi_0(Q)$ is convex decreasing up to Q_0 , convex increasing between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,

- (a) if $Q_h < Q_{int}$,
 - i. for $\Pi_0(Q_l) < \Pi_0(Q_h)$, $Q^* = Q_h$ and $K^* = K_0$,
 - ii. for $\Pi_0(Q_l) = \Pi_0(Q_h)$, $(Q^* = Q_h \text{ and } K^* = K_0)$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. for $\Pi_0(Q_l) > \Pi_0(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
 - (b) if $Q_l < Q_{int} \leq Q_h$,
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Q_{int} \leq Q_l$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
2. if $Q_0 = Q_v = Q_{int}$, $\Pi(Q) = \Pi_0(Q)$ is convex decreasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
- (a) if $Q_h < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Q_l < Q_{int} \leq Q_h$,
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Q_{int} \leq Q_l$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
3. if $Q_{int} < Q_v < Q_0$, $\Pi(Q) = \Pi_0(Q)$ is convex decreasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex decreasing between Q_{int} and Q_v , convex increasing afterwards. Hence,
- (a) if $Q_h < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Q_l < Q_{int} \leq Q_h$,
 - i. for $\Pi_0(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_0(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

- iii. for $\Pi_0(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = K_0$.
- (c) if $Q_{int} \leq Q_l$,
 - i. for $\Pi_v(Q_l) < \Pi_v(Q_h)$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. for $\Pi_v(Q_l) = \Pi_v(Q_h)$, $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,
 - iii. for $\Pi_v(Q_l) > \Pi_v(Q_h)$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

A.8 Proof of Proposition 2.4

Let $x_{N+1} = K^{\gamma''+1}$, $x_{N+2} = Q$, $\alpha_{N+1} = \frac{\alpha_K}{\gamma''+1}$, $\alpha_{N+2} = -1$, $p_{N+1} = \frac{\gamma'c_K}{\gamma''+1}$, $p_{N+2} = -s$, $\beta_{N+1} = 0$, $\beta_{N+2} = 0$,

$$x_{N+1} \begin{cases} = K_0^{\gamma''+1} & \text{if } (N+1) \in \phi; \\ > K_0^{\gamma''+1} & \text{if } (N+1) \in \chi, \end{cases}$$

$$r = \sum_{i=1}^N \alpha_i,$$

and

$$r^* = \sum_{i \in \chi} \alpha_i,$$

where χ and ϕ are the set of indices of non-binding and binding decision variables, respectively. Then, critical values of the Lagrange variable w , the decision variables of the problem and the emission level can be rewritten as

$$\begin{aligned} w^*(x_{N+1}^*, x_{N+2}^*) &= \ln \left(\frac{x_{N+2}^*}{a x_{N+1}^{*\alpha_K}} \right) \left(\frac{x_{N+2}^*}{a x_{N+1}^{*\alpha_{N+1}}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} \\ &= \left[\frac{1}{a} \prod_{j \in \phi} x_j^{-\alpha_j} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r^*}}, \end{aligned}$$

$$x_i^* = \frac{\alpha_i}{p_i + \rho \beta_i} \left[\frac{1}{a} \prod_{j \in \phi} x_j^{-\alpha_j} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r^*}},$$

and

$$em^* = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \left[\frac{1}{a} \prod_{j \in \phi} x_j^{-\alpha_j} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r^*}},$$

respectively.

Note that first derivative of x_i^* and em^* with respect to ρ are

$$\begin{aligned} \frac{\partial x_i^*}{\partial \rho} = & - \frac{\beta_i \alpha_i}{(p_i + \rho \beta_i)^2} \left(\frac{1}{a} \right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r^*}} \\ & + \frac{\alpha_i}{p_i + \rho \beta_i} \left(\frac{1}{a} \right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r^*}} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial em^*}{\partial \rho} = & - \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \left(\frac{1}{a} \right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r^*}} \\ & + \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \left(\frac{1}{a} \right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r^*}}. \end{aligned}$$

Hence

$$\frac{\partial x_i^*}{\partial \rho} \begin{cases} < 0 & \text{if } \frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j}; \\ \geq 0 & \text{otherwise,} \end{cases}$$

and

$$\frac{\partial em^*}{\partial \rho} \begin{cases} < 0, & \text{if } \sum_{i=1}^N \frac{\beta_i^2}{(p_i + \rho \beta_i)^2} \frac{\alpha_i}{r^*} > (<) \sum_{i=1}^N \frac{\alpha_i}{r^*} \frac{\beta_i}{p_i + \rho \beta_i} \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j} \text{ for } r^* < (>) 0, \\ \geq 0, & \text{otherwise.} \end{cases}$$

Then, since, for given Q , if $r^* = r + \alpha_K/(\gamma'' + 1) > 0$, since

$$\frac{\partial \hat{K}(Q)}{\partial \rho} = \frac{\partial \left(x_{N+1}^{\frac{1}{\gamma''+1}} \right)}{\partial \rho} = \frac{1}{\gamma'' + 1} x_{N+1}^{\frac{-\gamma''}{\gamma''+1}} \frac{\partial x_{N+1}}{\partial \rho} > 0,$$

$\hat{K}(Q)$ is **increasing** in ρ .

Furthermore, since $r^* = r + \frac{\alpha_K}{\gamma''+1} - 1 < 0$ for $r < 1 - \frac{\alpha_K}{\gamma''+1}$; $r^* = r + \frac{\alpha_K}{\gamma''+1} - 1 = 0$ for $r = 1 - \frac{\alpha_K}{\gamma''+1}$; and $r^* = r + \frac{\alpha_K}{\gamma''+1} - 1 > 0$ for $r > 1 - \frac{\alpha_K}{\gamma''+1}$, Q_v and $\hat{K}(Q_v)$ are **decreasing** in ρ for $r < 1 - \frac{\alpha_K}{\gamma''+1}$ and **increasing** in ρ for $r > 1 - \frac{\alpha_K}{\gamma''+1}$.

Since $\frac{1}{r-1} < 0$ for $r < 1$; $\frac{1}{r-1} = \infty$ for $r = 1$; and $\frac{1}{r-1} > 0$ for $r > 1$, Q_0 is **decreasing** in ρ for $r < 1$, does not exist for $r = 1$, and is **increasing** in ρ for $r > 1$.

Since $\frac{\partial Q_{int}}{\partial \rho} < 0$ for $r > 0$, Q_{int} is **decreasing** in $\rho \forall r \in R^+$. □

A.9 Proof of Theorem 2.4

Proof rests on the results of the Theorem 2.3 and the finding of the Proposition 2.4; that is, Q_{int} , Q_0 , and Q_v are monotonic in Z , which is presented below:

For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$, $\Pi(Q)$ is concave in Q from Lemma 2.3, and Q_{int} , Q_0 , and Q_v are monotonic decreasing in Z from Proposition 2.4. Then, by Lemma 2.5,

1. if $Z < Z_{int}$, then $Q_{int} < Q_0 < Q_v$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is increasing concave up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is increasing concave between Q_{int} and Q_v ; $\Pi(Q) = \Pi_v(Q)$ is decreasing concave after Q_v . Hence,
 - (a) if $Z < Z_{h,v}$, since $Q_h < Q_v$, $Q^* = Q_h$,
 - i. if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $K^* = K_0$,
 - ii. if $Z \geq Z_{h,int}$, since $Q_h \geq Q_{int}$, $K^* = \hat{K}(Q_h)$,
 - (b) if $Z_{h,v} \leq Z < Z_{l,v}$, since $Q_v < Q_h$ and $Q_l < Q_v$, $Q^* = Q_v$ and $K^* = \hat{K}(Q_v)$,
 - (c) if $Z_{l,v} \leq Z$, since $Q_v \leq Q_l$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$,
2. if $Z = Z_{int}$, then $Q_{int} = Q_0 = Q_v$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is increasing concave up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is decreasing concave after Q_{int} . Hence,
 - (a) if $Z \leq Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z_{h,int} \leq Z < Z_{l,int}$, since $Q_{int} \leq Q_h$ and $Q_l < Q_{int}$, $Q^* = Q_{int} = Q_0 = Q_v$ and $K^* = \hat{K}(Q_{int}) = K_0$,

- (c) if $Z_{l,int} \leq Z$, since $Q_{int} \leq Q_l$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.
- 3. if $Z_{int} < Z$, then $Q_v < Q_0 < Q_{int}$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is increasing concave up to Q_0 , decreasing concave between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is decreasing concave afterwards. Hence,
 - (a) if $Z < Z_{h,0}$, since $Q_h < Q_0$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z_{h,0} \leq Z < Z_{l,0}$, since $Q_l < Q_0$ and $Q_0 < Q_h$, $Q^* = Q_0$ and $K^* = K_0$,
 - (c) $Z_{l,0} \leq Z$ and
 - i. $Z \leq Z_{l,int}$, since $Q_0 \leq Q_l \leq Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - ii. $Z_{l,int} \leq Z$, since $Q_0 \leq Q_l$ and $Q_{int} \leq Q_l$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r = 1 - \alpha_K/(\gamma'' + 1)$, from Lemma 2.3, $\Pi(Q)$ is concave in Q , and Q_{int} and Q_0 are monotonic decreasing in Z from Proposition 2.4. Then, by Lemma 2.5,

- 1. if $Z < Z_{int}$, we have $Q_{int} < Q_0$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is linear increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z \geq Z_{h,int}$, since $Q_{int} \leq Q_h$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
- 2. if $Z = Z_{int}$, then $Q_{int} = Q_0$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is constant afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z_{h,int} \leq Z < Z_{l,int}$, since $Q_{int} \leq Q_h$ and $Q_l < Q_{int}$, $Q^* \in [Q_0, Q_h]$ and $K^* = \hat{K}(Q^*)$,
 - (c) if $Z_{l,int} \leq Z$, since $Q_{int} \leq Q_l$, $Q^* \in [Q_l, Q_h]$ and $K^* = \hat{K}(Q^*)$,
- 3. if $Z > Z_{int}$, then $Z_0 < Z_{int}$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_0 , concave decreasing between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is linear decreasing afterwards. Hence,
 - (a) if $Z < Z_{h,0}$, since $Q_h < Q_0$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z_{h,0} \leq Z < Z_{l,0}$, since $Q_0 \leq Q_h$ and $Q_l < Q_0$, $Q^* = Q_0$ and $K^* = K_0$,
 - (c) if $Z_{l,0} \leq Z < Z_{l,int}$, since $Q_0 < Q_l$ and $Q_l < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (d) if $Z_{l,int} \leq Z$, since $Q_{int} \leq Q_l$, $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $1 - \alpha_K/(\gamma'' + 1) < r < 1$, from Lemma 2.3, $\Pi(Q)$ is concave-convex in Q , and Q_{int} and Q_0 are monotonic decreasing, but Q_v is monotonic increasing in Z from Proposition 2.4. Then, by Lemma 2.5,

1. if $Z < Z_{int}$, then $Q_v < Q_{int} < Q_0$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence, $Q^* = Q_h$ and
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $K^* = K_0$,
 - (b) if $Z \geq Z_{h,int}$, since $Q_{int} \leq Q_h$; $K^* = \hat{K}(Q_h)$,
2. if $Z = Z_{int}$, then $Q_v = Q_{int} = Q_0$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Z = Z_{int} < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z = Z_{int} \geq Z_{h,int}$, since $Q_{int} \leq Q_h$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
3. if $Z > Z_{int}$, then $Q_0 < Q_{int} < Q_v$ by Lemma 2.5, and $\Pi(Q) = \Pi_0(Q)$ is concave increasing up to Q_0 , concave decreasing between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex decreasing between Q_{int} and Q_v , convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, we have $Q_h < Q_{int}$ by Lemma 2.5, then
 - i. if $Z < Z_{h,0}$, since $Q_h < Q_0$, $Q^* = Q_h$; $K^* = K_0$,
 - ii. if $Z_{h,0} \leq Z < Z_{l,0}$, since $Q_0 \leq Q_h$ and $Q_l < Q_0$, $Q^* = Q_0$; $K^* = K_0$,
 - iii. if $Z_{l,0} \leq Z$, since $Q_0 \leq Q_l$, $Q^* = Q_l$; $K^* = K_0$,
 - (b) If $Z_{h,int} \leq Z \leq Z_{l,int}$ and $Z \leq Z_{l,0}$, since we have $Q_l < Q_{int} \leq Q_h$ and $Q_0 \geq Q_l$ by Lemma 2.5, then
 - i. if $\Pi_0(Q_0) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_0) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_0 \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_0) > \Pi_v(Q_h)$, then $Q^* = Q_0$ and $K^* = K_0$.
 - (c) If $Z_{h,int} \leq Z \leq Z_{l,int}$, and $Z \geq Z_{l,0}$, since we have $Q_l < Q_{int} \leq Q_h$ and $Q_0 \leq Q_l$ by Lemma 2.5, then
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,

- ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
- (d) if $Z_{l,int} \leq Z$, we have $Q_{int} \leq Q_l$, then,
- i. if $\Pi_v(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_v(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,
 - iii. if $\Pi_v(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r = 1$, from Lemma 2.3, $\Pi(Q)$ is convex in Q , and Q_{int} is monotonic decreasing, but Q_v is monotonic increasing in Z from Proposition 2.4. Then, by Lemma 2.5,

1. if $Z < Z_{int}$, then $Q_v = Q_{int}$ and $\Pi(Q) = \Pi_0(Q)$ is linear increasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_h$ and $K^* = K_0$,
 - (b) if $Z \geq Z_{h,int}$, since $Q_{int} \leq Q_h$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$.
2. if $Z = Z_{int}$, then $Q_v = Q_{int}$ and $\Pi(Q) = \Pi_0(Q)$ is constant up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* \in [Q_l, Q_h]$ and $K^* = K_0$,
 - (b) if $Z \geq Z_{h,int}$, since $Q_{int} \leq Q_h$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
3. if $Z > Z_{int}$, then $Q_{int} < Q_v$ and $\Pi(Q) = \Pi_0(Q)$ is linear decreasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex decreasing between Q_{int} and Q_v , convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Z_{h,int} \leq Z < Z_{l,int}$, since we have $Q_l < Q_{int} \leq Q_h$, then
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Z_{l,int} \leq Z$, we have $Q_{int} \leq Q_l$, then

- i. if $\Pi_v(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
- ii. if $\Pi_v(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,
- iii. if $\Pi_v(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$.

For $r > 1$, from Lemma 2.3, $\Pi(Q)$ is convex in Q , and Q_{int} is monotonic decreasing, but Q_0 and Q_v are monotonic increasing in Z from Proposition 2.4. Then, by Lemma 2.5,

1. if $Z < Z_{int}$, then $Q_0 < Q_v < Q_{int}$ and $\Pi(Q) = \Pi_0(Q)$ is convex decreasing up to Q_0 , convex increasing between Q_0 and Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$,
 - i. if $\Pi_0(Q_l) < \Pi_0(Q_h)$, then $Q^* = Q_h$ and $K^* = K_0$,
 - ii. if $\Pi_0(Q_l) = \Pi_0(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = K_0)$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_l) > \Pi_0(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
 - (b) if $Z_{h,int} \leq Z < Z_{l,int}$, since $Q_l < Q_{int} \leq Q_h$,
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Z_{l,int} \leq Z$, since $Q_{int} \leq Q_l$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
2. if $Z = Z_{int}$, then $Q_0 = Q_v = Q_{int}$ and $\Pi(Q) = \Pi_0(Q)$ is convex decreasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Z_{h,int} \leq Z < Z_{l,int}$, since $Q_l < Q_{int} \leq Q_h$,
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,

- iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
- (c) if $Z_{l,int} \leq Z$, since $Q_{int} \leq Q_l$, $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
- 3. if $Z > Z_{int}$, then $Q_{int} < Q_v < Q_0$ and $\Pi(Q) = \Pi_0(Q)$ is convex decreasing up to Q_{int} ; $\Pi(Q) = \Pi_v(Q)$ is convex decreasing between Q_{int} and Q_v , convex increasing afterwards. Hence,
 - (a) if $Z < Z_{h,int}$, since $Q_h < Q_{int}$, $Q^* = Q_l$ and $K^* = K_0$,
 - (b) if $Z_{h,int} \leq Z < Z_{l,int}$, we have $Q_l < Q_{int} \leq Q_h$, then
 - i. if $\Pi_0(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_0(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = K_0)$,
 - iii. if $\Pi_0(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = K_0$.
 - (c) if $Z_{l,int} \leq Z$, since we have $Q_{int} \leq Q_l$, then
 - i. if $\Pi_v(Q_l) < \Pi_v(Q_h)$, then $Q^* = Q_h$ and $K^* = \hat{K}(Q_h)$,
 - ii. if $\Pi_v(Q_l) = \Pi_v(Q_h)$, then $(Q^* = Q_h \text{ and } K^* = \hat{K}(Q_h))$ or $(Q^* = Q_l \text{ and } K^* = \hat{K}(Q_l))$,
 - iii. if $\Pi_v(Q_l) > \Pi_v(Q_h)$, then $Q^* = Q_l$ and $K^* = \hat{K}(Q_l)$. □

A.10 Proof of Lemma 2.5

We will show the order relations among Q_l , Q_h , Q_{int} , Q_0 , and Q_v as Z approaches 0^+ from right and as Z approaches ∞^+ from the left. Furthermore, we will show that for $Z = Z_{int}$, the existing ones of the expressions Q_{int} , Q_0 , and Q_v are equal to each other, as follows.

- If $0 < r < 1 - \alpha_K/(\gamma'' + 1)$, Q_{int} , Q_0 , and Q_v are decreasing in Z from Proposition 2.4. Then, as $Z \rightarrow 0^+$, $Q_{int} \rightarrow +\infty$, $Q_0 \rightarrow +\infty$, and $Q_v \rightarrow +\infty$, as $Z \rightarrow +\infty$, $Q_{int} \rightarrow 0$, $Q_0 \rightarrow 0$, and $Q_v \rightarrow 0$. Hence, from Lemma 2.3, for $Z = Z_s < Z_{int}$, $Q_l < Q_h < Q_{int} < Q_0 < Q_v$; for $Z = Z_{int}$, $Q_{int} = Q_0 = Q_v$; for $Z = Z_b > Z_{int}$, $Q_v < Q_0 < Q_{int} < Q_l < Q_h$.

- If $r = 1 - \alpha_K/(\gamma'' + 1)$, Q_{int} and Q_0 are decreasing in Z from Proposition 2.4. Then, as $Z \rightarrow 0^+$, $Q_{int} \rightarrow +\infty$, and $Q_0 \rightarrow +\infty$, as $Z \rightarrow +\infty$, $Q_{int} \rightarrow 0$ and $Q_0 \rightarrow 0$. Hence, from Lemma 2.3, for $Z = Z_s < Z_{int}$, $Q_l < Q_h < Q_{int} < Q_0$; for $Z = Z_{int}$, $Q_{int} = Q_0$; for $Z = Z_b > Z_{int}$, $Q_0 < Q_{int} < Q_l < Q_h$.
- If $1 - \alpha_K/(\gamma'' + 1) < r < 1$, Q_{int} and Q_0 are decreasing, but Q_v is increasing in Z from Proposition 2.4. Then, as $Z \rightarrow 0^+$, $Q_{int} \rightarrow +\infty$, $Q_0 \rightarrow +\infty$, and $Q_v \rightarrow 0$, as $Z \rightarrow +\infty$, $Q_{int} \rightarrow 0$, $Q_0 \rightarrow 0$, and $Q_v \rightarrow +\infty$. Hence, from Lemma 2.3, for $Z = Z_s < Z_{int}$, $Q_v < Q_l < Q_h < Q_{int} < Q_0$; for $Z = Z_{int}$, $Q_v = Q_{int} = Q_0$; for $Z = Z_b > Z_{int}$, $Q_0 < Q_{int} < Q_l < Q_h < Q_v$.
- If $r = 1$, Q_{int} is decreasing, but Q_v is increasing in Z from Proposition 2.4. Then, as $Z \rightarrow 0^+$, $Q_{int} \rightarrow +\infty$, and $Q_v \rightarrow 0$, as $Z \rightarrow +\infty$, $Q_{int} \rightarrow 0$, $Q_v \rightarrow +\infty$. Hence, from Lemma 2.3, for $Z = Z_s < Z_{int}$, $Q_v < Q_l < Q_h < Q_{int}$; for $Z = Z_{int}$, $Q_v = Q_{int}$; for $Z = Z_b > Z_{int}$, $Q_{int} < Q_l < Q_h < Q_v$.
- If $r > 1$, Q_{int} is decreasing, but Q_0 and Q_v are increasing in Z from Proposition 2.4. Then, as $Z \rightarrow 0^+$, $Q_{int} \rightarrow +\infty$, $Q_0 \rightarrow 0$, and $Q_v \rightarrow 0$, as $Z \rightarrow +\infty$, $Q_{int} \rightarrow 0$, $Q_0 \rightarrow +\infty$, and $Q_v \rightarrow +\infty$. Hence, from Lemma 2.3, for $Z = Z_s < Z_{int}$, $Q_0 < Q_v < Q_l < Q_h < Q_{int}$; for $Z = Z_{int}$, $Q_0 = Q_v = Q_{int}$; for $Z = Z_b > Z_{int}$, $Q_{int} < Q_l < Q_h < Q_v < Q_0$. $\square$

A.11 Proof of Proposition 2.5

We will show the order relations among Z_{int} and the other Z Benchmarks given in Definition 2.5 as follows.

Remember that $\Pi(Q) = \Pi_0(Q)$ for $Q < Q_{int}$ and $\Pi(Q) = \Pi_v(Q)$ for $Q \geq Q_{int}$, which can be expressed as

$$\Pi_0(Q) = Q_s - r \left(\frac{Q}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}},$$

$$\Pi_v(Q) = Q_s - \left(\frac{\gamma' c_K (r(\gamma'' + 1) + \alpha_K)}{\alpha_K (\gamma'' + 1)} \right) \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} Z \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1},$$

through which $\Pi_0(Q_0)|_Z$ and $\Pi_v(Q_v)|_Z$ are obtained by substituting Q_0 and Q_v

for Q in the functions $\Pi_0(Q)$ and $\Pi_v(Q)$, as below:

$$\Pi_0(Q_0)|_Z = (1-r) \left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}},$$

and

$$\Pi_v(Q_v)|_Z = \frac{(1-r)(\gamma''+1) - \alpha_K}{\gamma''+1} \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\alpha_K} \left[\frac{1}{as} Z \right]^{\gamma''+1} \right]^{\frac{1}{(r-1)(\gamma''+1) + \alpha_K}} + \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1},$$

respectively.

From Proposition 2.3, we know that $Q_{int} = Q_0 = Q_v$ for $Z = Z_{int}$. Hence, since Q_{int} and Q_0 are expressed as

$$Q_{int} = a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r Z^{-1} K_0^{r(\gamma''+1) + \alpha_K},$$

$$Q_0 = \left[\frac{Z}{s^r a K_0^{\alpha_K}} \right]^{\frac{1}{r-1}},$$

respectively, equating Q_{int} and Q_0 we get the value of Z at which $Q_{int} = Q_0 = Q_v$, Z_{int} , as

$$Z_{int} = sa \left(\frac{\gamma' c_K}{\alpha_K} \right)^{r-1} K_0^{(r-1)(\gamma''+1) + \alpha_K}.$$

Furthermore, letting $f \in \{Q_l, Q_h\}$ and equating Q_f to Q_{int} , Q_0 , and Q_v we obtain

$$Z_{f,int} = a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r K_0^{r(\gamma''+1) + \alpha_K} Q_f^{-1},$$

$$Z_{f,0} = s^r a K_0^{\alpha_K} Q_f^{r-1},$$

and

$$Z_{f,v} = a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1) + \alpha_K}{\gamma''+1}} Q_f^{\frac{(r-1)(\gamma''+1) + \alpha_K}{\gamma''+1}},$$

respectively.

Note that, if $Z_{int} < (\geq) Z_{f,int}$, then $Q_f < (\geq) \frac{1}{s} \left(\frac{\gamma' c_K}{\alpha_K} \right) K_0^{\gamma''+1}$.

Therefore, if $Z_{int} < (\geq) Z_{f,int}$, then

1. for $r < 1 - \alpha_K/(\gamma'' + 1)$,

(a) $s^r a K_0^{\alpha_K} Q_l^{r-1} > s^r a K_0^{\alpha_K} Q_h^{r-1}$. Hence, we have $Z_{l,0} > Z_{h,0}$.

(b) $a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_l^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}}$
 $> a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_h^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}}.$

Hence, we have $Z_{l,v} > Z_{h,v}$.

(c) $Q_f^{r-1} > (\leq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{1-r} K_0^{(r-1)(\gamma''+1)}$. Hence, we have $Z_{int} < (\geq) Z_{f,0}$.

(d) $Q_f^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}} > (\leq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{\frac{(1-r)(\gamma''+1)-\alpha_K}{\gamma''+1}} K_0^{(r-1)(\gamma''+1)+\alpha_K}.$

Hence, we have $Z_{int} < (\geq) Z_{f,v}$.

2. for $r = 1 - \alpha_K/(\gamma'' + 1)$,

(a) $s^r a K_0^{\alpha_K} Q_l^{r-1} > s^r a K_0^{\alpha_K} Q_h^{r-1}$. Hence, we have $Z_{l,0} > Z_{h,0}$

(b) $Z_{f,v}$ is undefined since Q_v is undefined.

(c) $Q_f^{r-1} > (\leq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{1-r} K_0^{(r-1)(\gamma''+1)}$. Hence, we have $Z_{int} < (\geq) Z_{f,0}$.

3. for $1 - \alpha_K/(\gamma'' + 1) < r < 1$,

(a) $s^r a K_0^{\alpha_K} Q_l^{r-1} > s^r a K_0^{\alpha_K} Q_h^{r-1}$. Hence, we have $Z_{l,0} > Z_{h,0}$

(b) $a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_l^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}}$
 $< a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_h^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}}.$

Hence, we have $Z_{l,v} < Z_{h,v}$.

(c) $Q_f^{r-1} > (\leq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{1-r} K_0^{(r-1)(\gamma''+1)}$. Hence, we have $Z_{int} < (\geq) Z_{f,0}$.

(d) $Q_f^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}} < (\geq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{\frac{(1-r)(\gamma''+1)-\alpha_K}{\gamma''+1}} K_0^{(r-1)(\gamma''+1)+\alpha_K}.$

Hence, we have $Z_{int} > (\leq) Z_{f,v}$.

4. for $r = 1$

(a) $Z_{f,0}$ is undefined since Q_0 is undefined.

$$(b) \ a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_l^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}} \\ < a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_h^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}}. \text{ Hence, we have } Z_{l,v} < Z_{h,v}.$$

$$(c) \ Q_f^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}} < (\geq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{\frac{(1-r)(\gamma''+1)-\alpha_K}{\gamma''+1}} K_0^{(r-1)(\gamma''+1)+\alpha_K}.$$

Hence, we have $Z_{int} > (\leq) Z_{f,v}$.

5. for $r > 1$

$$(a) \ s^r a K_0^{\alpha_K} Q_l^{r-1} < s^r a K_0^{\alpha_K} Q_h^{r-1}.$$

Hence, we have $Z_{l,0} < Z_{h,0}$.

$$(b) \ a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_l^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}} \\ < a \left(\frac{\alpha_K}{\gamma' c_{Kd}} \right)^{\frac{\alpha_K}{\gamma''+1}} s^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} Q_h^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}}.$$

Hence, we have $Z_{l,v} < Z_{h,v}$.

$$(c) \ Q_f^{r-1} < (\geq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{1-r} K_0^{(r-1)(\gamma''+1)}. \text{ Hence, we have } Z_{int} > (\leq) Z_{f,0}.$$

$$(d) \ Q_f^{\frac{(r-1)(\gamma''+1)+\alpha_K}{\gamma''+1}} < (\geq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{\frac{(1-r)(\gamma''+1)-\alpha_K}{\gamma''+1}} K_0^{(r-1)(\gamma''+1)+\alpha_K}.$$

Hence, we have $Z_{int} > (\leq) Z_{f,v}$.

6. $\forall r$ such that $r > 0$,

$$(a) \ a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r K_0^{r(\gamma''+1)+\alpha_K} Q_h^{-1} < a \left(\frac{\gamma' c_K}{\alpha_K} \right)^r K_0^{r(\gamma''+1)+\alpha_K} Q_l^{-1}.$$

Hence, we have $Z_{h,int} < Z_{l,int}$.

$$7. \ (a) \ Q_f^{\frac{\alpha_K}{\gamma''+1}} < (\geq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{\frac{-\alpha_K}{\gamma''+1}} K_0^{\alpha_K}.$$

Hence, we have $Z_{f,v} < (\geq) Z_{f,0}$ for $r \notin \{(1 - \frac{\alpha_K}{\gamma''+1}), 1\}$.

$$(b) \ Q_f^r < (\geq) \left(s \frac{\alpha_K}{\gamma' c_K} \right)^{-r} K_0^{r(\gamma''+1)}. \text{ Hence, we have } Z_{f,0} < (\geq) Z_{f,int} \text{ for } r \neq 1.$$

$$(c) \quad Q_f^{\frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} < (\geq) \left(s_{\frac{\alpha_K}{\gamma'c_K}} \right)^{\frac{-r(\gamma''+1)-\alpha_K}{\gamma''+1}} K_0^{r(\gamma''+1)+\alpha_K}.$$

Hence, we have $Z_{f,v} < (\geq) Z_{f,int}$ for $r \neq \frac{\alpha_K}{\gamma''+1}$.

Thus, all of the order relationships among carbon benchmarks, which are required to find the change in the optimal solution with respect to Z are given below:

- For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$:

- If $Z_{int} < Z_{h,int}$, then

$Z_{int} < Z_{h,v}$ by 1.(d); $Z_{h,v} < Z_{h,0}$ by 7.(a); $Z_{h,0} < Z_{l,0}$ by 1.(a); $Z_{l,0} < Z_{l,int}$ by 7.(b) and $Z_{h,v} < Z_{l,v}$ by 1.(b); $Z_{l,v} < Z_{l,0}$ by 7.(a) and $Z_{h,v} < Z_{h,0}$ by 7.(a); $Z_{h,0} < Z_{h,int}$ by 7.(b); $Z_{h,int} < Z_{l,int}$ by 6.(a). Thus,

$Z_{int} < Z_{h,v} < Z_{h,0} < Z_{l,0} < Z_{l,int}$ and $Z_{h,v} < Z_{l,v} < Z_{l,0}$ and $Z_{h,v} < Z_{h,0} < Z_{h,int} < Z_{l,int}$,

- If $Z_{int} = Z_{h,int}$, then

$Z_{int} = Z_{h,v}$ by 1.(d); $Z_{h,v} = Z_{h,0}$ by 7.(a); $Z_{h,v} < Z_{l,v}$ by 1.(b); $Z_{l,v} < Z_{l,0}$ by 7.(a); $Z_{l,0} < Z_{l,int}$ by 7.(b). Thus,

$Z_{int} = Z_{h,0} = Z_{h,v} < Z_{l,v} < Z_{l,0} < Z_{l,int}$,

- If $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$Z_{h,int} < Z_{h,0}$ by 7.(b); $Z_{h,0} < Z_{h,v}$ by 7.(a); $Z_{h,v} < Z_{int}$ by 1.(d); $Z_{int} < Z_{l,v}$ by 1.(d); $Z_{l,v} < Z_{l,0}$ by 7.(a); $Z_{l,0} < Z_{l,int}$ by 7.(b). Thus,

$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{int} < Z_{l,v} < Z_{l,0} < Z_{l,int}$,

- If $Z_{l,int} = Z_{int}$, then

$Z_{h,int} < Z_{h,0}$ by 7.(b); $Z_{h,0} < Z_{int}$ by 1.(d); $Z_{int} = Z_{l,int}$ by definition; $Z_{l,int} = Z_{l,0}$ by 7.(b). Thus,

$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0}$,

- If $Z_{l,int} < Z_{int}$, then

$Z_{h,int} < Z_{h,0}$ by 7.(b); $Z_{h,0} < Z_{h,v}$ by 7.(a); $Z_{h,v} < Z_{l,v}$ by 1.(b); $Z_{l,v} < Z_{int}$ by 1.(d) and $Z_{h,int} < Z_{l,int}$ by 6.(a); $Z_{l,int} < Z_{l,0}$ by 7.(b); $Z_{l,0} < Z_{l,v}$ by 7.(a) and $Z_{h,0} < Z_{l,0}$ by 1.(a). Thus,

$$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{l,v} < Z_{int} \text{ and } Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{l,v} \text{ and } Z_{h,0} < Z_{l,0}.$$

- For $0 < r = 1 - \alpha_K/(\gamma'' + 1)$:

- If $Z_{int} < Z_{h,int}$, then

$$Z_{int} < Z_{h,0} \text{ by 2.(c); } Z_{h,0} < Z_{l,0} \text{ by 2.(a); } Z_{l,0} < Z_{l,int} \text{ by 7.(b) and } Z_{h,0} < Z_{h,int} \text{ by 7.(b); } Z_{h,int} < Z_{l,int} \text{ by 6.(a). Thus,}$$

$$Z_{int} < Z_{h,0} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,0} < Z_{h,int} < Z_{l,int}$$

- If $Z_{int} = Z_{h,int}$, then

$$Z_{int} = Z_{h,0} \text{ by 2.(c); } Z_{h,0} < Z_{l,0} \text{ by 2.(a); } Z_{l,0} < Z_{l,int} \text{ by 7.(b). Thus,}$$

$$Z_{int} = Z_{h,0} < Z_{l,0} < Z_{l,int},$$

- If $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ by 7.(b); } Z_{h,0} < Z_{int} \text{ by 2.(c); } Z_{int} < Z_{l,0} \text{ by 2.(c); } Z_{l,0} < Z_{l,int} \text{ by 7.(b). Thus,}$$

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int},$$

- If $Z_{int} = Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ by 7.(b); } Z_{h,0} < Z_{int} \text{ by 1.(d); } Z_{int} = Z_{l,int} \text{ by definition; } Z_{l,int} = Z_{l,0} \text{ by 7.(b). Thus,}$$

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0},$$

- If $Z_{int} > Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ by 7.(b) and } Z_{l,int} < Z_{l,0} \text{ by 7.(b); } Z_{l,0} < Z_{int} \text{ by 2.(c) and } Z_{h,int} < Z_{l,int} \text{ by 6.(a) and } Z_{h,0} < Z_{l,0} \text{ by 1.(a). Thus,}$$

$$Z_{h,int} < Z_{h,0} \text{ and } Z_{l,int} < Z_{l,0} < Z_{int} \text{ and } Z_{h,int} < Z_{l,int} \text{ and } Z_{h,0} < Z_{l,0}.$$

- For $0 < 1 - \alpha_K/(\gamma'' + 1) < r < 1$:

- If $Z_{int} < Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} \text{ by 3.(b); } Z_{v,h} < Z_{int} \text{ by 3.(d); } Z_{int} < Z_{h,0} \text{ by 3.(c); } Z_{h,0} < Z_{h,int} \text{ by 7.(b); } Z_{h,int} < Z_{l,int} \text{ by 6.(a) and } Z_{int} < Z_{l,0} \text{ by 3.(c); } Z_{l,0} < Z_{l,int} \text{ by 7.(b) and } Z_{h,0} < Z_{l,0} \text{ by 3.(a). Thus,}$$

$$Z_{v,l} < Z_{v,h} < Z_{int} < Z_{h,0} < Z_{h,int} < Z_{l,int} \text{ and } Z_{int} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,0} < Z_{l,0},$$

– If $Z_{int} = Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} \text{ by 3.(b); } Z_{v,h} = Z_{int} \text{ by 3.(d); } Z_{int} = Z_{h,0} \text{ by 3.(c); } Z_{h,0} = Z_{h,int} \text{ by 7.(b); } Z_{h,int} = Z_{h,0} < Z_{l,0} \text{ by 3.(a); } Z_{l,0} < Z_{l,int} \text{ by 7.(b). Thus,}$$

$$Z_{v,l} < Z_{v,h} = Z_{int} = Z_{h,0} = Z_{h,int} < Z_{l,0} < Z_{l,int},$$

– If $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ by 7.(b); } Z_{h,0} < Z_{int} \text{ by 3.(c); } Z_{int} < Z_{l,0} \text{ by 3.(c); } Z_{l,0} < Z_{l,int} \text{ by 7.(b) and } Z_{v,l} < Z_{int} \text{ by 3.(d); } Z_{int} < Z_{v,h} \text{ by 3.(d). Thus,}$$

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h},$$

– If $Z_{int} = Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ by 7.(b); } Z_{h,0} < Z_{int} \text{ by 3.(c); } Z_{int} = Z_{l,int} = Z_{l,0} = Z_{v,l} \text{ by definition; } Z_{v,l} < Z_{v,h} \text{ by 3.(b). Thus,}$$

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0} = Z_{v,l} < Z_{v,h},$$

– If $Z_{int} > Z_{l,int}$, then

$$Z_{h,int} < Z_{h,0} \text{ by 7.(b); } Z_{h,0} < Z_{l,0} \text{ by 3.(a); } Z_{h,int} < Z_{l,int} \text{ by 6.(a); } Z_{l,int} < Z_{l,0} \text{ by 7.(b); } Z_{l,0} < Z_{int} \text{ by 3.(c); } Z_{int} < Z_{v,l} \text{ by 3.(d); } Z_{v,l} < Z_{v,h} \text{ by 3.(b). Thus,}$$

$$Z_{h,int} < Z_{h,0} < Z_{l,0} \text{ and } Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{int} < Z_{v,l} < Z_{v,h},$$

• For $r = 1$:

– If $Z_{int} < Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} \text{ by 4.(b); } Z_{v,h} < Z_{int} \text{ by 4.(c); } Z_{int} < Z_{h,int} \text{ by definition; } Z_{h,int} < Z_{l,int} \text{ by 6.(a). Thus,}$$

$$Z_{v,l} < Z_{v,h} < Z_{int} < Z_{h,int} < Z_{l,int},$$

– If $Z_{int} = Z_{h,int}$, then

$$Z_{v,l} < Z_{v,h} \text{ by 4.(b); } Z_{v,h} = Z_{int} = Z_{h,int} \text{ by definition; } Z_{h,int} < Z_{l,int} \text{ by 6.(a). Thus,}$$

$$Z_{v,l} < Z_{v,h} = Z_{int} = Z_{h,int} < Z_{l,int},$$

- If $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$Z_{v,l} < Z_{int}$ by 4.(c); $Z_{int} < Z_{v,h}$ by 4.(c) and $Z_{v,l} < Z_{l,int}$ by 7.(c) and $Z_{h,int} < Z_{v,h}$ by 7.(c). Thus,

$Z_{v,l} < Z_{int} < Z_{v,h}$ and $Z_{v,l} < Z_{l,int}$ and $Z_{h,int} < Z_{v,h}$.
- If $Z_{int} = Z_{l,int}$, then

$Z_{h,int} < Z_{l,int} = Z_{int}$ by 6.(a); $Z_{int} = Z_{l,int} = Z_{v,l}$ by definition; $Z_{v,l} < Z_{v,h}$ by 4.(b). Thus,

$Z_{h,int} < Z_{int} = Z_{l,int} = Z_{v,l} < Z_{v,h}$,
- If $Z_{int} > Z_{l,int}$, then

$Z_{h,int} < Z_{l,int}$ by 6.(a); $Z_{l,int} < Z_{int}$ by definition; $Z_{int} < Z_{v,l}$ by 4.(d); $Z_{v,l} < Z_{v,h}$ by 4.(b). Thus,

$Z_{h,int} < Z_{l,int} < Z_{int} < Z_{v,l} < Z_{v,h}$,
- For $r > 1$:
 - If $Z_{int} < Z_{h,int}$, then

$Z_{v,l} < Z_{v,h}$ by 5.(b); $Z_{v,h} < Z_{0,h}$ by 7.(a); $Z_{0,h} < Z_{int}$ by 5.(c); $Z_{int} < Z_{h,int}$ by definition; $Z_{h,int} < Z_{l,int}$ by 6.(a) and $Z_{0,l} < Z_{0,h}$ by 5.(a); $Z_{0,h} < Z_{int}$ by 5.(c); $Z_{int} < Z_{h,int}$ by definition; $Z_{h,int} < Z_{l,int}$ by 6.(a). Thus,

$Z_{v,l} < Z_{v,h} < Z_{0,h} < Z_{int} < Z_{h,int} < Z_{l,int}$ and $Z_{0,l} < Z_{0,h} < Z_{int} < Z_{h,int} < Z_{l,int}$.
 - If $Z_{int} = Z_{h,int}$, then

$Z_{v,l} < Z_{0,l}$ by 7.(a); $Z_{0,l} < Z_{0,h}$ by 5.(a); $Z_{0,h} = Z_{v,h} = Z_{int} = Z_{h,int}$ by definition; $Z_{h,int} < Z_{l,int}$ by 6.(a) and $Z_{0,l} < Z_{0,h}$ by 5.(a). Thus,

$Z_{v,l} < Z_{0,l} < Z_{0,h} = Z_{v,h} = Z_{int} = Z_{h,int} < Z_{l,int}$ and $Z_{0,l} < Z_{0,h} = Z_{v,h} = Z_{int} = Z_{h,int}$
 - If $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$Z_{v,l} < Z_{0,l}$ by 7.(a); $Z_{0,l} < Z_{int}$ by 5.(c); $Z_{int} < Z_{0,h}$ by 7.(b); $Z_{0,h} < Z_{v,h}$ by 7.(a). Thus,

$Z_{v,l} < Z_{0,l} < Z_{int} < Z_{0,h} < Z_{v,h}$.

– If $Z_{int} = Z_{l,int}$, then

$Z_{h,int} < Z_{l,int}$ by 6.(a); $Z_{l,int} = Z_{v,l} = Z_{0,l} = Z_{int}$ by definition; $Z_{int} < Z_{0,l}$ by 5.(c); $Z_{0,l} < Z_{v,l}$ by 7.(a) and $Z_{int} < Z_{v,h}$, since $Z_{int} = Z_{l,int} = Z_{v,l} < Z_{v,h}$ by 5.(b). Thus,

$$Z_{h,int} < Z_{l,int} = Z_{v,l} = Z_{0,l} = Z_{int} < Z_{0,l} < Z_{v,l} \text{ and } Z_{int} < Z_{v,h}.$$

– If $Z_{int} > Z_{l,int}$, then

$Z_{h,int} < Z_{l,int}$ by 6.(a); $Z_{l,int} < Z_{int}$ by definition; $Z_{int} < Z_{0,l}$ by 5.(c); $Z_{0,l} < Z_{v,l}$ by 7.(a); $Z_{v,l} < Z_{v,h}$ by 5.(b) and $Z_{0,l} < Z_{0,h}$ by 5.(a). Thus,

$$Z_{h,int} < Z_{l,int} < Z_{int} < Z_{0,l} < Z_{v,l} < Z_{v,h} \text{ and } Z_{0,l} < Z_{0,h}. \quad \square$$

A.12 Proof of Lemma 2.6

1. The difference of the candidate optimal profits evaluated at (Q_l, K_0) and (Q_h, K_0) is

$$\Pi_0(Q_l) - \Pi_0(Q_h) = (Q_l - Q_h)s + r \left(\frac{1}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} (Q_h^{\frac{1}{r}} - Q_l^{\frac{1}{r}})$$

and its first derivative with respect to Z is positive; that is,

$$\frac{\partial(\Pi_0(Q_l) - \Pi_0(Q_h))}{\partial \rho} = \sum_{j=1}^N \frac{\beta_j \alpha_j}{p_j + \rho \beta_j} \left(\frac{1}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} (Q_h^{\frac{1}{r}} - Q_l^{\frac{1}{r}}) > 0,$$

since $Q_l < Q_h$. Hence, $(\Pi_0(Q_l) - \Pi_0(Q_h))$ is **increasing** in Z .

2. The difference of the candidate optimal profits evaluated at (Q_0, K_0) and $(Q_h, \hat{K}(Q_h))$ can be written as:

$$\begin{aligned} \Pi_0(Q_0) - \Pi_v(Q_h) &= Q_0 s - r \left[\frac{Q_0}{aK_0^{\alpha_K}} Z \right]^{\frac{1}{r}} - Q_h s - \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \\ &\quad + \frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1} \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma'' + 1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma'' + 1}{r(86579 + \gamma'' + 1) + \alpha_K}}. \end{aligned}$$

Note that from the Theorem 2.2, we have

$$\hat{K}(Q) = \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}}$$

Then, since

$$\hat{K}(Q)^{\frac{-\alpha_K}{r}} = \left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{r(\gamma''+1)+\alpha_K}} \left(\frac{a}{Q} \right)^{\frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} Z^{\frac{-\alpha_K}{r^2(\gamma''+1)+r\alpha_K}},$$

we have

$$\begin{aligned} Q^{\frac{1}{r}} a^{\frac{-1}{r}} \hat{K}(Q)^{\frac{-\alpha_K}{r}} Z^{\frac{1}{r}} &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} a^{\frac{-1}{r} + \frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} Q^{\frac{1}{r} - \frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} \\ &\quad Z^{\frac{1}{r} - \frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} \\ &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} a^{\frac{-r(\gamma''+1)}{r^2(\gamma''+1)+r\alpha_K}} Q^{\frac{r(\gamma''+1)}{r^2(\gamma''+1)+r\alpha_K}} Z^{\frac{r(\gamma''+1)}{r^2(\gamma''+1)+r\alpha_K}} \\ &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} a^{\frac{-(\gamma''+1)}{r(\gamma''+1)+\alpha_K}} Q^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} Z^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \\ &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \frac{Q}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}}. \end{aligned}$$

Furthermore, by Lemma 2.3, we have

$$Q_0 = \left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}}.$$

Hence, we may rewrite $\Pi_0(Q_0) - \Pi_v(Q_h)$ as

$$\begin{aligned}\Pi_0(Q_0) - \Pi_v(Q_h) &= (1-r)Q_0s + \frac{r(\gamma''+1) + \alpha_K}{\gamma''+1} \left[\left(\frac{\gamma'c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1) + \alpha_K}} \\ &= (1-r) \left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}} s - Q_h s - \frac{\gamma'c_K}{\gamma''+1} K_0^{\gamma''+1} \\ &\quad + \frac{r(\gamma''+1) + \alpha_K}{\gamma''+1} \left[\left(\frac{\gamma'c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1) + \alpha_K}},\end{aligned}$$

and its first derivative with respect to Z is

$$\frac{\partial(\Pi_0(Q_0) - \Pi_v(Q_h))}{\partial Z} = -Z^{-1} s \left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}} + Z^{-1} \left[\left(\frac{\gamma'c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1) + \alpha_K}}.$$

Now, remember that, by Lemma 2.3, we have

$$Q_0 = \left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}}.$$

Then, we may write

$$Q_0 s = s \left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}} = \left[\frac{1}{s a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}},$$

and thus,

$$\begin{aligned}\left(\frac{Q_0}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} &= \left(\frac{\left[\frac{1}{s^r a K_0^{\alpha_K}} Z \right]^{\frac{1}{r-1}}}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} = \left(\left[\frac{Z}{s^r a K_0^{\alpha_K}} \right]^{\frac{1}{r-1}} \frac{Z}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} = \left[\frac{Z}{s a K_0^{\alpha_K}} \right]^{\frac{1}{r-1}} \\ &= Q_0 s.\end{aligned}$$

Therefore, we may rewrite $\frac{\partial(\Pi_0(Q_0) - \Pi_v(Q_h))}{\partial Z}$ as

$$\begin{aligned} \frac{\partial(\Pi_0(Q_0) - \Pi_v(Q_h))}{\partial Z} &= -Z^{-1}sQ_0 + Z^{-1}\left(\frac{Q_h}{a\hat{K}(Q_h)^{\alpha_K}}\right)^{\frac{1}{r}}Z^{\frac{1}{r}} \\ &= -Z^{-1}\left(\frac{Q_0}{aK_0^{\alpha_K}}\right)^{\frac{1}{r}}Z^{\frac{1}{r}} + Z^{-1}\left(\frac{Q_h}{a\hat{K}(Q_h)^{\alpha_K}}\right)^{\frac{1}{r}}Z^{\frac{1}{r}} \\ &= -\left(\frac{Q_0}{aK_0^{\alpha_K}}\right)^{\frac{1}{r}}Z^{\frac{1-r}{r}} + \left(\frac{Q_h}{a\hat{K}(Q_h)^{\alpha_K}}\right)^{\frac{1}{r}}Z^{\frac{1-r}{r}}. \end{aligned}$$

Then, $(\Pi_0(Q_0) - \Pi_v(Q_h))$ is increasing in Z if $Q_0\hat{K}(Q_h)^{\alpha_K} \leq Q_hK_0^{\alpha_K}$, and it is decreasing in Z otherwise.

Now, we will prove that if $Q_0 \leq Q_h$ ($Z \geq Z_{h,0}$) and $Q_0 \leq Q_{int} \leq Q_v$ ($Z \geq Z_{int}$), then $Q_0\hat{K}(Q_h)^{\alpha_K} < Q_hK_0^{\alpha_K}$, and thus, $(\Pi_0(Q_0) - \Pi_v(Q_h))$ is increasing in Z .

Note that we have

$$\hat{K}(Q) = \left[\left(\frac{\alpha_K}{\gamma'c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}}$$

and

$$\frac{Q}{\hat{K}(Q)^{\alpha_K}} = Q^{\frac{r(\gamma''+1)}{r(\gamma''+1)+\alpha_K}} \left[\left(\frac{\alpha_K}{\gamma'c_K} \right)^r \frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{-\alpha_K}{r(\gamma''+1)+\alpha_K}}$$

from Theorem 2.2. Then, first derivatives of $\hat{K}(Q)$ and $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ with respect to Q are

$$\frac{\partial \hat{K}(Q)}{\partial Q} = \frac{1}{r(\gamma''+1)+\alpha_K} Q^{-1} \left[\left(\frac{\alpha_K}{\gamma'c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right] > 0$$

and

$$\frac{\partial \frac{Q}{\hat{K}(Q)^{\alpha_K}}}{\partial Q} = \frac{r(\gamma''+1)}{r(\gamma''+1)+\alpha_K} Q^{\frac{-\alpha_K}{r(\gamma''+1)+\alpha_K}} \left[\left(\frac{\alpha_K}{\gamma'c_K} \right)^r \frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{-\alpha_K}{r(\gamma''+1)+\alpha_K}} > 0,$$

respectively. Hence, $\hat{K}(Q)$ and $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ are increasing in Q .

Suppose that $Q_0 \leq Q_h$ ($Z \geq Z_{h,0}$) and $Q_0 \leq Q_{int} \leq Q_v$ ($Z \geq Z_{int}$). Then, since

$\hat{K}(Q)$ and $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ are increasing in Q , respectively, we have

$$\frac{Q_0}{K_0^{\alpha_K}} = \frac{Q_{int}}{\hat{K}(Q_{int})^{\alpha_K}} \leq \frac{Q_0}{\hat{K}(Q_{int})^{\alpha_K}} \leq \frac{Q_h}{\hat{K}(Q_h)^{\alpha_K}}.$$

Hence $(\Pi_0(Q_l) - \Pi_v(Q_h))$ is **increasing** in Z for $Q_0 \leq Q_h$ ($Z \geq Z_{h,0}$) and $Q_0 \leq Q_{int} \leq Q_v$ ($Z \geq Z_{int}$).

Now suppose that $Z_{int} = Z_{h,0}$. Then, since we have $Q_h = Q_0 = Q_{int} = Q_v$, $Q_0 \hat{K}(Q_h)^{\alpha_K} = Q_h K_0^{\alpha_K}$, and thus, first derivative of $(\Pi_0(Q_0) - \Pi_v(Q_h))$ with respect to Z is equal to zero, which means that $(\Pi_0(Q_0) - \Pi_v(Q_h))$ should have a unique positive root that is equal to Z_{int} for $Z_{int} = Z_{h,0}$.

3. The difference of the candidate optimal profits evaluated at (Q_l, K_0) and $(Q_h, \hat{K}(Q_h))$ is

$$\begin{aligned} \Pi_0(Q_l) - \Pi_v(Q_h) = & (Q_l - Q_h)s - r \left(\frac{Q_l}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} \\ & + \frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1} \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma'' + 1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} \\ & - \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1}, \end{aligned}$$

and its first derivative with respect to Z is

$$\frac{\partial(\Pi_0(Q_l) - \Pi_v(Q_h))}{\partial Z} = -Z^{-1} \left(\frac{Q_l}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} + Z^{-1} \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma'' + 1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}}.$$

Note that from the Theorem 2.2, we have

$$\hat{K}(Q) = \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma'' + 1) + \alpha_K}}.$$

Then, since

$$\hat{K}(Q_h)^{\frac{-\alpha_K}{r}} = \left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{r(\gamma'' + 1) + \alpha_K}} \left(\frac{a}{Q_h} \right)^{\frac{\alpha_K}{r^2(\gamma'' + 1) + r\alpha_K}} Z^{\frac{-\alpha_K}{r^2(\gamma'' + 1) + r\alpha_K}},$$

we have

$$\begin{aligned}
 Q_h^{\frac{1}{r}} a^{\frac{-1}{r}} \hat{K}(Q_h)^{\frac{-\alpha_K}{r}} Z^{\frac{1}{r}} &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} a^{\frac{-1}{r} + \frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} Q_h^{\frac{1}{r} - \frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} \\
 &\quad \cdot Z^{\frac{1}{r} - \frac{\alpha_K}{r^2(\gamma''+1)+r\alpha_K}} \\
 &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} a^{\frac{-r(\gamma''+1)}{r^2(\gamma''+1)+r\alpha_K}} Q_h^{\frac{r(\gamma''+1)}{r^2(\gamma''+1)+r\alpha_K}} \\
 &\quad \cdot Z^{\frac{r(\gamma''+1)}{r^2(\gamma''+1)+r\alpha_K}} \\
 &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} a^{\frac{-(\gamma''+1)}{r(\gamma''+1)+\alpha_K}} Q_h^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \\
 &\quad \cdot Z^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \\
 &= \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^{\frac{\alpha_K}{\gamma''+1}} \frac{Q_h}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}}.
 \end{aligned}$$

We can also write:

$$\frac{\partial(\Pi_0(Q_l) - \Pi_v(Q_h))}{\partial Z} = -Z^{-1} \left(\frac{Q_l}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} + Z^{-1} \left(\frac{Q_h}{a \hat{K}(Q_h)^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}}.$$

Therefore, if $Q_l \hat{K}(Q_h)^{\alpha_K} < Q_h K_0^{\alpha_K}$, $(\Pi_0(Q_l) - \Pi_v(Q_h))$ is increasing in Z , and it is decreasing in Z otherwise.

Now, we will prove that if $Q_l < Q_{int}$ ($Z < Z_{l,int}$), then $Q_l \hat{K}(Q_h)^{\alpha_K} < Q_h K_0^{\alpha_K}$, and thus, $(\Pi_0(Q_l) - \Pi_v(Q_h))$ is increasing in Z .

Note that we have

$$\hat{K}(Q) = \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}}$$

and

$$\frac{Q}{\hat{K}(Q)^{\alpha_K}} = Q^{\frac{r(\gamma''+1)}{r(\gamma''+1)+\alpha_K}} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{-\alpha_K}{r(\gamma''+1)+\alpha_K}}$$

from Theorem 2.2. Then, first derivatives of $\hat{K}(Q)$ and $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ with respect to Q are

$$\frac{\partial \hat{K}(Q)}{\partial Q} = \frac{1}{r(\gamma''+1)+\alpha_K} Q^{-1} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right] > 0$$

and

$$\frac{\partial \frac{Q}{\hat{K}(Q)^{\alpha_K}}}{\partial Q} = \frac{r(\gamma''+1)}{r(\gamma''+1)+\alpha_K} Q^{\frac{-\alpha_K}{r(\gamma''+1)+\alpha_K}} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{-\alpha_K}{r(\gamma''+1)+\alpha_K}} > 0,$$

respectively. Hence, $\hat{K}(Q)$ and $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ are increasing in Q .

Suppose that $Q_l < Q_{int}$ ($Z < Z_{l,int}$). Then, since $\hat{K}(Q)$ and $\frac{Q}{\hat{K}(Q)^{\alpha_K}}$ are increasing in Q , respectively, we have

$$\frac{Q_l}{K_0^{\alpha_K}} = \frac{Q_l}{\hat{K}(Q_{int})^{\alpha_K}} < \frac{Q_l}{\hat{K}(Q_l)^{\alpha_K}} < \frac{Q_h}{\hat{K}(Q_h)^{\alpha_K}}.$$

Hence $(\Pi_0(Q_l) - \Pi_v(Q_h))$ is **increasing** in Z for $Q_l < Q_{int}$.

4. The difference of the candidate optimal profits evaluated at $(Q_l, \hat{K}(Q_l))$ and $(Q_h, \hat{K}(Q_h))$ is

$$\begin{aligned} \Pi_v(Q_l) - \Pi_v(Q_h) = & (Q_l - Q_h)s + \left[\left(\frac{\gamma' c_K d (r(\gamma''+1) + \alpha_K)}{\alpha_K (\gamma''+1)} \right) \right. \\ & \left. \cdot \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{1}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \left(Q_h^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} - Q_l^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \right) \right], \end{aligned}$$

and its first derivative with respect to Z is positive; that is,

$$\begin{aligned} \frac{\partial (\Pi_v(Q_l) - \Pi_v(Q_h))}{\partial Z} = & \left[\left(\frac{\gamma' c_K d}{\alpha_K} \right) Z^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}-1} \right] \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{1}{a} \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \\ & \cdot \left(Q_h^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} - Q_l^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} \right) > 0, \end{aligned}$$

since $Q_h > Q_l$. Hence, $(\Pi_v(Q_l) - \Pi_v(Q_h))$ is **increasing** in Z . □

A.13 Proof of Proposition 2.6

Using Definition 2.6 we will show that $\Phi(Z)$ is increasing in Z and it has a unique root, Z_Φ , as follows.

1. For $1 - \alpha_K/(\gamma'' + 1) < r < 1$,

(a) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

$Z_{l,0} < Z_{l,int}$ by Proposition 2.5. Hence, we have

- 1) $(Z_{int}, Z_{l,int}) \cap [0, Z_{l,0}] = (Z_{int}, Z_{l,0})$,
- 2) $(Z_{int}, Z_{l,int}) \cap (Z_{l,0}, \infty) = [Z_{l,0}, Z_{l,int})$, and
- 3) $(Z_{int}, Z_{l,int}) \cap (Z_{l,int}, \infty) = [Z_{l,int}, \infty)$.

Considering above result together with the Lemma 2.6,

$\Phi(Z) = \Pi_0(Q_0) - \Pi_v(Q_h)$ is increasing in Z for $Z \in (Z_{int}, Z_{l,0})$; $\Phi(Z) = \Pi_0(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in [Z_{l,0}, Z_{l,int})$; $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in [Z_{l,int}, \infty)$.

Now to see that $\Phi(Z)$ is continuous, we make the following observations:

At $Z = Z_{l,0}$, we have $Q_l = Q_0$, and hence, $\Pi_0(Q_0) = \Pi_0(Q_l)$. Therefore, $\Phi(Z) = \Pi_0(Q_0) - \Pi_v(Q_h) = \Pi_0(Q_l) - \Pi_v(Q_h)$ at $Z = Z_{l,0}$.

At $Z = Z_{l,int}$, we have $Q_l = Q_{int}$, and hence, $\Pi_0(Q_l) = \Pi_v(Q_l)$ since $\hat{K}(Q_l) = \hat{K}(Q_{int}) = K_0$. Therefore, $\Phi(Z) = \Pi_0(Q_l) - \Pi_v(Q_h) = \Pi_v(Q_l) - \Pi_v(Q_h)$ at $Z = Z_{l,int}$.

Hence, $\Phi(Z)$ is continuously increasing in Z .

(b) if $Z_{l,int} \leq Z_{int}$, then

from Lemma 2.6, $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in [Z_{l,int}, \infty)$. Hence, $\Phi(Z)$ is increasing in Z .

2. If $r = 1$ and

(a) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

by Lemma 2.6, $\Phi(Z) = \Pi_0(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in (0, Z_{l,int})$; $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in [Z_{l,int}, \infty)$.

Hence, $\Phi(Z)$ is increasing in Z .

Now to see that $\Phi(Z)$ is continuous, we make the following observations:

At $Z = Z_{l,int}$, we have $Q_l = Q_{int}$, and hence, $\Pi_0(Q_l) = \Pi_v(Q_l)$ since $\hat{K}(Q_l) = \hat{K}(Q_{int}) = K_0$. Therefore, $\Phi(Z) = \Pi_0(Q_l) - \Pi_v(Q_h) = \Pi_v(Q_l) - \Pi_v(Q_h)$ at $Z = Z_{l,int}$.

Hence, $\Phi(Z)$ is continuously increasing in Z .

(b) if $Z_{l,int} \leq Z_{int}$, then

from Lemma 2.6, $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in [Z_{l,int}, \infty)$. Hence, $\Phi(Z)$ is increasing in Z .

3. If $r > 1$ and

(a) if $Z_{int} \leq Z_{h,int}$, then

from Lemma 2.6, $\Phi(Z) = \Pi_0(Q_l) - \Pi_0(Q_h)$ is increasing in $Z \forall Z > 0$. Hence, $\Phi(Z)$ is increasing in Z for $Z \in [0, Z_{h,int})$.

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then

from Lemma 2.6, $\Phi(Z) = \Pi_0(Q_l) - \Pi_0(Q_h)$ is increasing in Z for $Z \in (0, Z_{h,int}]$; $\Phi(Z) = \Pi_0(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in (Z_{h,int}, Z_{l,int})$; $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ is increasing in Z for $Z \in [Z_{l,int}, \infty)$. Hence, $\Phi(Z)$ is increasing in Z .

Now to see that $\Phi(Z)$ is continuous, we make the following observations:

At $Z = Z_{h,int}$, we have $Q_h = Q_{int}$, and hence, $\Pi_0(Q_h) = \Pi_v(Q_h)$ since $\hat{K}(Q_h) = \hat{K}(Q_{int}) = K_0$. Therefore, $\Phi(Z) = \Pi_0(Q_l) - \Pi_0(Q_h) = \Pi_0(Q_l) - \Pi_v(Q_h)$ at $Z = Z_{h,int}$.

At $Z = Z_{l,int}$, we have $Q_l = Q_{int}$, and hence, $\Pi_0(Q_l) = \Pi_v(Q_l)$ since $\hat{K}(Q_l) = \hat{K}(Q_{int}) = K_0$. Therefore, $\Phi(Z) = \Pi_0(Q_l) - \Pi_v(Q_h) = \Pi_v(Q_l) - \Pi_v(Q_h)$ at $Z = Z_{l,int}$.

Hence, $\Phi(Z)$ is continuously increasing in Z .

(c) if $Z_{l,int} \leq Z_{int}$, then

from Lemma 2.6, $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ is increasing in $Z \forall Z \geq 0$.
Hence, $\Phi(Z)$ is increasing in $Z \in [Z_{l,int}, \infty)$.

Proving that each $\Phi(Z)$ is continuous increasing in Z , by showing its negativity at its first boundary point and its divergence to $+\infty$ as Z goes to $+\infty$, we will prove that each $\Phi(Z)$ has a unique positive root as below:

Firstly, we will show the negativity of each $\Phi(Z)$ at its first boundary point as follows.

$$1. \ 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$$

The proof rests on the values attained by and the shape of the difference function $\Phi(Z)$ within the intervals of interest: $(0, Z_{int}]$, $(Z_{int}, Z_{l,0}]$, $(Z_{l,0}, Z_{l,int})$, and $[Z_{l,int}, \infty)$. In particular, in the following we investigate the negativity of $\Phi(Z)$ at the boundary value, $Z = Z_{int}$:

Suppose that $Z_{int} < Z_{h,int}$. At $Z = Z_{int} < Z_{h,int}$, we have $Q_l < Q_h < Q_v = Q_{int} = Q_0$ by definition. Then, since $\Pi(Q) = \Pi_0(Q)$ is strictly increasing in Q for $Q_l \leq Q < Q_h < Q_v = Q_{int} = Q_0$, we have $\Phi(Z) = \Phi_{h0}^{l0}(Z) = \Pi_0(Q_l) - \Pi_0(Q_h) < 0$ at $Z = Z_{int}$.

Suppose that $Z_{h,int} = Z_{int}$. At $Z = Z_{int} = Z_{h,int}$, we have $Q_l < Q_h = Q_v = Q_{int} = Q_0$ by definition. Then, since $\Pi(Q) = \Pi_0(Q)$ is increasing in Q for $Q_l \leq Q < Q_h = Q_v = Q_{int} = Q_0$ and $\Pi(Q) = \Pi_v(Q)$ is increasing in Q for $Q_l < Q_h = Q_v = Q_{int} = Q_0 \leq Q$, we have $\Phi(Z) = \Phi_{h0}^{l0}(Z) = \Pi_0(Q_l) - \Pi_0(Q_h) < 0$ at $Z = Z_{int}$ (Note that $\Phi(Z) = \Phi_{h0}^{l0}(Z) = \Phi_{hv}^{lv}(Z)$ for $Z = Z_{h,int} = Z_{int}$).

Suppose that $Z_{h,int} < Z_{int} < Z_{l,int}$. At $Z = Z_{int}$, we have $Q_l < Q_v = Q_{int} = Q_0 < Q_h$ by Lemma 2.3. Then, since $\Pi(Q) = \Pi_0(Q)$ is increasing in Q for $Q_l \leq Q < Q_v = Q_{int} = Q_0 < Q_h$ and $\Pi(Q) = \Pi_v(Q)$ is increasing in Q for $Q_l < Q_v = Q_{int} = Q_0 \leq Q < Q_h$; we have $\Phi(Z) = \Phi_{h0}^{l0}(Z) = \Pi_0(Q_l) - \Pi_0(Q_h) < 0$ at $Z = Z_{int}$.

Suppose that $Z_{l,int} \leq Z_{int}$. We will next prove the negativity of $\Phi(Z) = \Pi_v(Q_l) - \Pi_v(Q_h)$ at the boundary value $Z = Z_{int}$. At $Z = Z_{int}$, we have $Q_v = Q_{int} = Q_0 \leq Q_l < Q_h$ by Lemma 2.3. Then, since $\Pi(Q) = \Pi_v(Q)$ is increasing in Q for $Q_v = Q_{int} = Q_0 \leq Q_l \leq Q < Q_h$; we have $\Phi(Z) = \Phi_{hv}^{lv}(Z) = \Pi_v(Q_l) - \Pi_v(Q_h) < 0$ at $Z = Z_{int}$.

2. $r = 1$

The proof rests on the values attained by and the shape of the function $\Phi(Z)$ within the interval of interest: $(0, Z_{int})$ and $[Z_{l,int}, \infty)$. In particular, in the following we investigate the negativity of $\Phi(Z)$ at $Z = Z_s$ where Z_s corresponds to an infinitesimally small positive number.

At $Z = Z_s$, we have $Q_v < Q_l < Q_h < Q_{int}$ by Lemma 2.5. Then, since $\Pi(Q) = \Pi_0(Q)$ is linear increasing in Q for $Q_l \leq Q < Q_h < Q_{int}$; constant in Q for $Q_l < Q = Q_{int} < Q_h$; linear decreasing in Q for $Q_{int} < Q < Q_h$, we have $\Phi(Z) = \Phi_{0h}^{0l}(Z) = \Pi_0(Q_l) - \Pi_0(Q_h) < 0$ at $Z = Z_s$.

3. $r > 1$

The proof rests on the values attained by and the shape of the function $\Phi(Z)$ within the interval of interest: $[0, Z_{h,int})$, $[Z_{h,int}, Z_{l,int})$, $[Z_{l,int}, \infty)$. In particular, in the following we investigate the negativity of $\Phi(Z)$ at $Z = Z_s$ where Z_s corresponds to an infinitesimally small positive number.

At $Z = Z_s$, we have $Q_0 < Q_v < Q_l < Q_h < Q_{int}$ by Lemma 2.5. Then, since $\Pi(Q) = \Pi_0(Q)$ is increasing in Q for $Q_0 < Q_v < Q_l \leq Q < Q_h < Q_{int}$; we have $\Phi(Z) = \Phi_{0h}^{0l}(Z) = \Pi_0(Q_l) - \Pi_0(Q_h) < 0$ at $Z = Z_s$.

Secondly, we will show the divergence of each $\Phi(Z)$ to $+\infty$ as Z goes to $+\infty$ as follows.

1. For $0 < 1 - \frac{\alpha_K}{\gamma''+1} < r$,

Suppose that $Z_{h,int} < Z_{int} < Z_{l,int}$. The proof rests on the values attained by and the shape of the function $\Phi(Z)$ within the last interval in which $\Phi(Z)$'s are defined: $[Z_{l,int}, \infty)$. In particular, in the following we investigate that $\Phi(Z)$ is positive at limiting conditions, that is, as $Z \rightarrow +\infty$:

Note that, if $Z \geq Z_{l,int}$, $\Pi(Q)$ is solely defined by $\Pi_v(Q)$ for $Q_l \leq Q \leq Q_h$, we should have $\Phi(Z) = \Phi_{hv}^{lv} = (\Pi_v(Q_l) - \Pi_v(Q_h))$. Therefore, it is sufficient to examine $\Pi_\Phi = (\Pi_v(Q_l) - \Pi_v(Q_h))$ as Z goes to ∞ . Hence, remember that $\Phi_{hv}^{lv} = (\Pi_v(Q_l) - \Pi_v(Q_h))$ is of the form

$$\begin{aligned} \Pi_v(Q_l) - \Pi_v(Q_h) = & (Q_l - Q_h)s + \left(\frac{\gamma' c_K d(r(\gamma'' + 1) + \alpha_K)}{\alpha_K(\gamma'' + 1)} \right) \\ & \cdot \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{1}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} (Q_h^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} - Q_l^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}}). \end{aligned}$$

Then, since $\lim_{Z \rightarrow +\infty} Z^{\frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}} = +\infty$, limit of $\Pi_v(Q_l) - \Pi_v(Q_h)$ as Z goes to $+\infty$ is equal to the sum of a negative constant term and $+\infty$. Hence, $\lim_{Z \rightarrow +\infty} \Pi_v(Q_l) - \Pi_v(Q_h) = +\infty$. Thus, $\lim_{Z \rightarrow +\infty} \Phi(Z) = +\infty$.

Therefore, for each possible order relation among Z_{int} , $Z_{h,int}$, and $Z_{l,int}$, since $\Phi(Z)$ is continuous increasing in Z in the interval in which $\Phi(Z)$ is negative and positive at the interval's lower and higher boundary values, respectively, $\Phi(Z)$ has a unique positive root. $\square$

A.14 Proof of Theorem 2.5

We will find the optimal solution as Z increases in the region $(0, \infty)$. To do so, since optimal decision differs in different regions of Z Benchmarks, which can be observed by having a look at the Theorem 2.4, we will use the order relations given by the Proposition 2.5. Note that, there are three possible order relations among Z_{int} and $Z_{f,int}$ for $f \in \{l, h\}$: 1) $0 < Z_{int} < Z_{h,int}$, 2) $Z_{h,int} \leq Z_{int} < Z_{l,int}$, and 3) $Z_{l,int} \leq Z_{int} < \infty$. For each of these conditions, it is presented by the Proposition 2.5 that relevant Z Benchmarks together with Z_{int} , $Z_{l,int}$, and $Z_{h,int}$, are ordered in a particular fashion, which allows us to find the optimal policy in the adjacent regions defined by these benchmarks. Furthermore, by Proposition 2.6, since the difference $\Phi(Z)$ is increasing in Z with a unique root, after the optimal policy becomes alternate optima, that is, after the optimal production quantity becomes Q_0 or Q_l , it can not turn back to Q_h . Hence, following statements are true.

1. For $0 < r < 1 - \alpha_K/(\gamma'' + 1)$,

(a) if $Z_{int} < Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{int} < Z_{h,v} < Z_{h,0} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,v} < Z_{l,v} < Z_{l,0} \text{ and } Z_{h,v} < Z_{h,0} <$$

$$Z_{h,int} < Z_{l,int}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^1)$ since $Z < Z_{int} < Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{2,a}^1)$ since $Z = Z_{int} < Z_{h,int}$.

If $Z > Z_{int}$, for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{3,a}^1)$; for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b}^1)$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,i}^1)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^1)$.

(b) if $Z_{int} = Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{int} = Z_{h,int} = Z_{h,0} = Z_{h,v} < Z_{l,v} < Z_{l,0} < Z_{l,int}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^1)$ since $Z_{int} = Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{2,a}^1)$ since $Z_{int} = Z_{h,int}$.

If $Z > Z_{int}$, for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b}^1)$ since $Z_{h,0} = Z_{int}$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,i}^1)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^1)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 1.(a) and 1.(b), one gets the variation of the optimal solution with respect to Z as given by the first item of the Theorem 2.5 for low elasticity level.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{int} < Z_{l,v} < Z_{l,0} < Z_{l,int}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^1)$; for $Z_{h,int} \leq Z < Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,a,ii}^1)$; for $Z_{h,v} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_v, \hat{K}(Q_v))$ from $(B_{1,b}^1)$ since $Z_{int} < Z_{l,v}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{2,b}^1)$ since $Z_{h,int} < Z_{h,v} < Z_{int} < Z_{l,v} < Z_{l,int}$.

If $Z > Z_{int}$, for $Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b}^1)$ since $Z_{h,0} < Z_{int} < Z_{l,0}$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,i}^1)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^1)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 1.(c), one gets the variation of the optimal solution with respect to Z as given by the second item of the Theorem 2.5 for low elasticity level.

(d) if $Z_{int} = Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{int} = Z_{l,int} = Z_{l,0} = Z_{l,v}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^1)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,a,ii}^1)$ since $Z_{h,v} < Z_{int}$; for $Z_{h,v} \leq Z < Z_{l,v}$, $(Q^*, K^*) \equiv (Q_v, \hat{K}(Q_v))$ from $(B_{1,b}^1)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{2,c}^1)$ since $Z_{int} = Z_{l,int} = Z_{l,v}$.

If $Z > Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^1)$ since $Z_{int} = Z_{l,0} = Z_{l,int}$.

(e) if $Z_{int} > Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{h,v} < Z_{l,v} < Z_{int} \text{ and } Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{l,v} \text{ and } Z_{h,0} < Z_{l,0}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^1)$ since $Z_{h,int} < Z_{h,v} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{h,v}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,a,ii}^1)$ since $Z_{h,v} < Z_{int}$; for $Z_{h,v} \leq Z < Z_{l,v}$, $(Q^*, K^*) \equiv (Q_v, \hat{K}(Q_v))$ from $(B_{1,b}^1)$ since $Z_{l,v} < Z_{int}$; for $Z_{l,v} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{1,c}^1)$ since $Z_{l,v} < Z_{int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{2,c}^1)$ since $Z_{l,int} < Z_{int}$.

If $Z > Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^1)$ since $Z_{l,int} < Z_{l,0} < Z_{int}$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 1.(d) and 1.(e), one gets the variation of the optimal solution with respect to Z as given by the third item of the Theorem 2.5 for low elasticity level.

2. For $r = 1 - \alpha_K / (\gamma'' + 1)$,

(a) if $Z_{int} < Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{int} < Z_{h,0} < Z_{l,0} < Z_{l,int} \text{ and } Z_{h,0} < Z_{h,int} < Z_{l,int} \text{ and } Z_{h,int} < Z_{l,int}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^2)$ since $Z_{int} < Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{2,a}^2)$ since $Z_{int} < Z_{h,int}$.

If $Z > Z_{int}$, since $Z_{int} < Z_{h,0} < Z_{l,0} < Z_{l,int}$, for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{3,a}^2)$; for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b}^2)$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c}^2)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d}^2)$.

(b) if $Z_{h,int} = Z_{int}$, then, by proposition 2.5, we have

$$Z_{int} = Z_{h,int} = Z_{h,0} < Z_{l,0} < Z_{l,int}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^2)$ since $Z_{int} = Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{2,a}^2)$ since $Z_{int} = Z_{h,int}$.

If $Z > Z_{int}$, for $Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b}^2)$ since $Z_{int} = Z_{h,0} < Z_{l,0}$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c}^2)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d}^2)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 2.(a) and 2.(b), one gets the variation of the optimal solution with respect to Z as given by the first item of the Theorem 2.5 for mid-borderline elasticity level.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^2)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^2)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in [Q_0, Q_h]$ from $(B_{2,b}^2)$ since $Z_{h,int} < Z_{int} < Z_{l,int}$.

If $Z > Z_{int}$, for $Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b}^2)$ since $Z_{h,0} < Z_{int}$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c}^2)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d}^2)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 2.(c), one gets the variation of the optimal solution with respect to Z as given by the second item of the Theorem 2.5 for mid-borderline elasticity level.

(d) if $Z_{int} = Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0},$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^2)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}_{Q_h})$ from $(B_{1,b}^2)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in [Q_l, Q_h]$ from $(B_{2,c}^2)$ since $Z_{int} = Z_{l,int}$.

If $Z > Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d}^2)$ since $Z_{int} = Z_{l,int} = Z_{l,0}$.

(e) if $Z_{l,int} < Z_{int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} \text{ and } Z_{l,int} < Z_{l,0} < Z_{int} \text{ and } Z_{h,int} < Z_{l,int} \text{ and } Z_{h,0} < Z_{l,0}.$$

Thus, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^2)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}_{Q_h})$ from $(B_{1,2}^2)$ since $Z_{h,int} < Z_{int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in [Q_l, Q_h]$ from $(B_{2,c}^2)$ since $Z_{l,int} < Z_{int}$.

If $Z > Z_{int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d}^2)$ since $Z_{l,int} < Z_{l,0} < Z_{int}$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 2.(d) and 2.(e), one gets the variation of the optimal solution with respect to Z as given by the third item of the Theorem 2.5 for mid-borderline elasticity level.

3. For $1 - \alpha_K/(\gamma'' + 1) < r < 1$,

(a) if $Z_{int} < Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{v,l} < Z_{v,h} < Z_{int} < Z_{h,0} < Z_{h,int} < Z_{l,int} \text{ and } Z_{int} < Z_{h,0} < Z_{l,0} < Z_{l,int}.$$

Hence, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, then $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^3)$ since $Z_{int} < Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{2,a}^3)$ since $Z_{int} < Z_{h,int}$.

If $Z > Z_{int}$, for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{3,a,i}^3)$ since $Z_{int} < Z_{h,0} < Z_{h,int}$.

Now, we may have $Z_{l,0} \leq Z_{h,int}$ or $Z_{l,0} > Z_{h,int}$.

- i. If $Z_{l,0} \leq Z_{h,int}$, since $Z_{int} < Z_{h,0} < Z_{l,0} \leq Z_{h,int} < Z_{l,int}$, for $Z_{h,0} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,a,ii}^3)$; for $Z_{l,0} \leq Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,a,iii}^3)$; for $Z_{h,int} \leq Z < Z_{l,int}$, since $\Phi(Z) > 0$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,iii}^3)$ since $Z_{l,0} \leq Z_{h,int}$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$ since $Z_{l,0} \leq Z_{h,int} < Z_{l,int}$.
 - ii. If $Z_{l,0} > Z_{h,int}$, since $Z_{h,0} < Z_{h,int} < Z_{l,0} < Z_{l,int}$ by proposition 2.5, for $Z_{h,0} \leq Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,a,ii}^3)$; for $Z_{h,int} \leq Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b,iii}^3)$ since $\Phi(Z) > 0$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,iii}^3)$ since $\Phi(Z) > 0$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$ since $\Phi(Z) > 0$.
- (b) if $Z_{int} = Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{v,l} < Z_{v,h} = Z_{int} = Z_{h,0} = Z_{h,int} < Z_{l,0} < Z_{l,int} \text{ and } Z_{hv}^{00} = Z_{int}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z \geq Z_{int}$ by proposition 2.6. Hence, $\Phi(Z) \leq 0 \forall Z \in [Z_{int}, Z_\Phi]$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$.

Then, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^3)$ since $Z_{h,0} = Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{2,a}^3)$ since $Z_{int} = Z_{h,int}$.

If $Z > Z_{int}$, for $Z < Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,a,ii}^3)$ since $Z > Z_\Phi = Z_{hv}^{00} = Z_{int} = Z_{h,0}$ and $\Phi(Z) > 0$; for $Z_{l,0} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,iii}^3)$ since $\Phi(Z) = \Phi_{hv}^{00}(Z) = \Phi_{hv}^{l0}(Z)$ at $Z = Z_{l,0}$ ($Q_l = Q_0$) and $\Phi(Z) > 0$ afterwards; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$ since $\Phi(Z) = \Phi_{hv}^{l0}(Z) = \Phi_{hv}^{lv}(Z)$ at $Z = Z_{l,int}$ ($Q_l = Q_{int}$) and $\Phi(Z) > 0$ afterwards.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 3.(a) and 3.(b), one gets the

variation of the optimal solution with respect to Z as given by the second row of the table 2.6 for mid elasticity level.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z \geq Z_{int}$ by proposition 2.6. Hence, $\Phi(Z) \leq 0 \forall Z \in [Z_{int}, Z_\Phi]$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$.

Then, by Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^3)$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^3)$ since $Z_{h,int} < Z_{int}$,

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b}^3)$ since $Z_{h,int} < Z_{int}$;

If $Z > Z_{int}$, then, considering the Lemma 2.6, we examine the following conditions:

i. If $Z_{hv}^{00} < Z_{l,0}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} \leq Z_\Phi = Z_{hv}^{00} \leq Z_{l,0} < Z_{l,int} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h}.$$

Then, since $\Phi(Z) \leq 0 \forall Z \in [Z_{int}, Z_\Phi]$, for $Z < Z_\Phi = Z_{hv}^{00}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^3)$.

At $Z = Z_\Phi = Z_{hv}^{00}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b,ii}^3)$.

Since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi = Z_{hv}^{00} < Z \leq Z_{l,0}$, $(Q^*, K^*) \equiv (Q_0, K_0)$ from $(B_{3,b,iii}^3)$; for $Z_{l,0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,iii}^3)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 3,c,i, one gets the variation of the optimal solution with respect to Z as given by the second row of the table 2.6 for mid elasticity level.

ii. $Z_{l,0} < Z_{hv}^{l0} < Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_\Phi = Z_{hv}^{l0} < Z_{l,int} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z \geq Z_{int}$ by proposition 2.6. Hence, $\Phi(Z) \leq 0 \forall Z \in [Z_{int}, Z_\Phi]$ and

$$\Phi(Z) > 0 \quad \forall Z > Z_\Phi.$$

Then, since $\Phi(Z) \leq 0 \quad \forall Z \in [Z_{int}, Z_\Phi]$, for $Z_{int} < Z \leq Z_{l,0} < Z_\Phi$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^3)$; for $Z_{l,0} < Z < Z_\Phi = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^3)$;

At $Z = Z_\Phi = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,ii}^3)$,

Since $\Phi(Z) > 0 \quad \forall Z > Z_\Phi$, for $\Phi(Z) = Z_{hv}^{l0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,c,iii}^3)$, for $\Phi(Z) < Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 3,c,ii, one gets the variation of the optimal solution with respect to Z as given by the third row of the table 2.6 for mid elasticity level.

iii. $Z_{l,int} \leq Z_{hv}^{lv}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} < Z_{l,0} < Z_{l,int} \leq Z_\Phi = Z_{hv}^{l0} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z \geq Z_{int}$ by proposition 2.6. Hence, $\Phi(Z) \leq 0 \quad \forall Z \in [Z_{int}, Z_\Phi]$ and $\Phi(Z) > 0 \quad \forall Z > Z_\Phi$. Hence, considering Theorem 2.4, the following statements are true:

Since $\Phi(Z) \leq 0 \quad \forall Z \in [Z_{int}, Z_\Phi]$, for $Z_{int} < Z \leq Z_{l,0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^3)$; for $Z_{l,0} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^3)$; for $Z_{l,int} < Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,d,i}^3)$;

At $Z = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,ii}^3)$.

Since $\Phi(Z) > 0 \quad \forall Z > Z_\Phi$, for $Z_\Phi = Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 3,c,iii, one gets the variation of the optimal solution with respect to Z as given by the fourth row of the table 2.6 for mid elasticity level.

(d) if $Z_{int} = Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} = Z_{l,0} = Z_{v,l} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z \geq Z_{l,int} = Z_{int}$ by proposition 2.6. Hence, $\Phi(Z) \leq 0 \forall Z \in [Z_{l,int}, Z_\Phi]$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_{h,int} < Z_{h,0} < Z_{int} = Z_{l,int} < Z_{hv}^{lv}$. Hence, considering Theorem 2.4, the following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) \leq 0 \forall Z \in [Z_{l,int}, Z_\Phi]$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^3)$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^3)$ since $Z_{l,int} = Z_{int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b}^3)$ since $Z_{int} = Z_{l,int} > Z_{h,int}$.

If $Z > Z_{int}$, for $Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,d,i}^3)$.

At $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,ii}^3)$.

Since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi = Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$.

(e) if $Z_{int} > Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{h,0} < Z_{l,0} \text{ and } Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{int} < Z_{v,l} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{hv}^{lv}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) \leq 0 \forall Z \in [Z_{int}, Z_\Phi]$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{int} < Z_\Phi = Z_{hv}^{lv}$. Hence, considering Theorem 2.4, the following statements are true:

For $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^3)$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^3)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b}^3)$.

Since $\Phi(Z) \leq 0 \forall Z \in (0, Z_\Phi]$, for $Z_{int} < Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,d,i}^3)$ since $Z_{l,int} < Z_{int} < Z < Z_{hv}^{lv}$.

At $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,ii}^3)$.

Since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi = Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,d,iii}^3)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 3.(d) and 3.(e), one gets the variation of the optimal solution with respect to Z as given by the fifth row of the table 2.6 for mid elasticity level.

4. For $r = 1$,

(a) if $Z_{int} < Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{v,l} < Z_{v,h} < Z_{int} < Z_{h,int} < Z_{l,int}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{h0}^{l0} = Z_{int}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_{h,int} < Z_{l,int} < Z_{l,0} < Z_{int} = Z_\Phi = Z_{h0}^{l0}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, for $Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^4)$ since $Z_{int} < Z_{h,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in [Q_l, Q_h]$ from $(B_{2,a}^4)$.

If $Z > Z_{int}$, since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_{int} < Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,a}^4)$ since $Z_\Phi = Z_{int}$; for $Z_{h,int} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^4)$ since $Z_\Phi = Z_{h0}^{l0} = Z_{int} < Z_{h,int}$; for $Z_\Phi < Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^4)$ since $Z_\Phi = Z_{h0}^{l0} < Z_{l,int}$.

(b) if $Z_{int} = Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{v,l} < Z_{v,h} = Z_{int} = Z_{h,int} < Z_{l,int}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{h0}^{l0} = Z_{int}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_{v,l} < Z_{v,h} = Z_{h,int} = Z_{int} = Z_\Phi = Z_{h0}^{l0}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, for $Z < Z_{int} = Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^4)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in [Q_l, Q_h]$ from $(B_{2,a}^4)$ since $Z_\Phi = Z_{int} = Z_{h,int}$.

If $Z > Z_{int}$, since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^4)$ since $Z_\Phi = Z_{int} = Z_{h,int} < Z_{l,int}$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^4)$ since $Z_\Phi < Z_{l,int}$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 4, *a* and 4, *b*, one gets the variation of the optimal solution with respect to Z as given by the first row of the table 2.7 for high-borderline elasticity level.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{int} < Z_{l,int} \text{ and } Z_{v,l} < Z_{int} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^4)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^4)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b}^4)$.

If $Z > Z_{int}$, then, considering the Lemma 2.6, we examine the following conditions:

i. If $Z_{hv}^{l0} < Z_{l,int}$, then, by proposition 2.5, we have

$Z_{h,int} < Z_{int} < Z_{hv}^{l0} < Z_{l,int}$ and $Z_{v,l} < Z_{int} < Z_{hv}^{l0} < Z_{v,h}$. Hence, considering the Theorem 2.4, following statements are true:

Since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^4)$.

At $Z = Z_\Phi = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,ii}^4)$.

Since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^4)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^4)$ since $Z_\Phi < Z_{l,int}$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 4, *c, i*, one gets the variation of the optimal solution with respect to Z as given by the second row of the table 2.7 for high-borderline elasticity level.

ii. If $Z_{l,int} \leq Z_{hv}^{lv}$, then, by proposition 2.5, we have

$Z_{h,int} < Z_{int} < Z_{l,int} \leq Z_{hv}^{lv}$ and $Z_{v,l} < Z_{int} < Z_{v,h}$. Hence, considering the Theorem 2.4, following statements are true:

Since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^4)$; for $Z_{l,int} \leq Z < Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^4)$.

At $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^4)$;

Since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^4)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 4,c,ii, one gets the variation of the optimal solution with respect to Z as given by the third row of the table 2.7 for high-borderline elasticity level.

(d) if $Z_{int} = Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{int} = Z_{l,int} = Z_{v,l} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{hv}^{lv}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_{h,int} < Z_{v,l} = Z_{l,int} = Z_{int} < Z_\Phi = Z_{hv}^{lv} < Z_{v,h}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^4)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^4)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b}^4)$ since $Z_{int} = Z_{l,int} > Z_{h,int}$.

If $Z > Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{l,int} = Z_{int}$, for $Z < Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^4)$; at $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$ from $(B_{3,c,ii}^4)$; since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^4)$.

(e) if $Z_{int} > Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{l,int} < Z_{int} < Z_{v,l} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{hv}^{lv}$, is increasing in Z for

$Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_{h,int} < Z_{l,int} < Z_{int} < Z_\Phi = Z_{hv}^{lv}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^4)$ since $Z_{h,int} < Z_{int}$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b}^4)$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b}^4)$ since $Z_{h,int} < Z_{l,int} < Z_{int}$.

If $Z > Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{l,int} < Z_{int}$, for $Z < Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^4)$; at $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$ from $(B_{3,c,ii}^4)$. since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^4)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 4,d and 4,e, one gets the variation of the optimal solution with respect to Z as given by the fourth row of the table 2.7 for high-borderline elasticity level.

5. For $r > 1$,

(a) if $Z_{int} < Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{v,l} < Z_{v,h} < Z_{0,h} < Z_{int} < Z_{h,int} < Z_{l,int} \text{ and } Z_{v,l} < Z_{0,l} < Z_{0,h} < Z_{int} < Z_{h,int} < Z_{l,int}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{h0}^{l0}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_\Phi = Z_{h0}^{l0} < Z_{h,int} < Z_{l,int}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, for $Z < Z_\Phi = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a}^5)$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$; at $Z = Z_\Phi = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in \{Q_l, Q_h\}$ from $(B_{1,a,ii}^5)$ since $Z_\Phi = Z_{h0}^{l0} < Z_{h,int}$; for $Z_\Phi < Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,a,iii}^5)$ since $Z_{int} < Z_{h,int}$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{2,a}^5)$.

If $Z > Z_{int}$, since $Z_{int} > \Phi(Z)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z < Z_{h,int}$,

$(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,a}^5)$; for $Z_{h,int} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^5)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$.

(b) if $Z_{int} = Z_{h,int}$, then, by proposition 2.5, we have

$$Z_{v,l} < Z_{0,l} < Z_{v,h} = Z_{0,h} = Z_{int} = Z_{h,int} < Z_{l,int}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{h0}^{l0} < Z_{h,int}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then we may write

$Z_\Phi = Z_{h0}^{l0} < Z_{h,int} < Z_{l,int}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $Z_\Phi = Z_{h0}^{l0} < Z_{h,int}$ and $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_\Phi = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; at $Z = Z_\Phi = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q, K_0)$ s.t. $Q \in \{Q_l, Q_h\}$ is the alternate optima from $(B_{1,a,ii}^5)$; since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi < Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,a,iii}^5)$;

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{2,a}^5)$ since $Z_{int} < Z_{h,int}$;

If $Z > Z_{int}$, since $\Phi(Z) > 0 \forall Z > Z_\Phi$ and $Z_{int} = Z_{h,int} > Z_\Phi$, for $Z_{h,int} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^5)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 5,a and 5,b, one gets the variation of the optimal solution with respect to Z as given by the first row of the table 2.7 for high elasticity level.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{int} < Z_{l,int} \text{ and } Z_{v,l} < Z_{0,l} < Z_{int} < Z_{0,h} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, Z_Φ , is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$.

Note also that from Lemma 2.6, we may have $Z_\Phi = Z_{h0}^{l0} \leq Z_{h,int} < Z_{int} < Z_{l,int}$ or $Z_{h,int} < Z_\Phi = Z_{hv}^{l0} < Z_{int} < Z_{l,int}$ or $Z_{h,int} < Z_{int} \leq Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$ or $Z_{h,int} < Z_{int} < Z_{l,int} \leq Z_\Phi = Z_{hv}^{lv}$. Hence,

i. if $Z_\Phi = Z_{h0}^{l0} \leq Z_{h,int}$, then we may write

$$Z_\Phi = Z_{h0}^{l0} \leq Z_{h,int} < Z_{int} < Z_{l,int} \text{ and } Z_{v,l} < Z_{0,l} < Z_{int} < Z_{0,h} < Z_{v,h}.$$

Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) \leq Z_{h,int} < Z_{int}$ and $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_\Phi = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; at $Z = Z_\Phi = Z_{h0}^{l0}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ or $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,a,ii}^5)$; since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi = Z_{h0}^{l0} < Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,a,iii}^5)$; for $Z_{h,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,b,iii}^5)$ since $Z_{int} < Z_{l,int}$.

At $Z = Z_{int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{2,b,iii}^5)$ since $Z_{int} < Z_{l,int}$;

If $Z > Z_{int}$, since $\Phi(Z) > 0 \forall Z > Z_\Phi$ and $\Phi(Z) \leq Z_{h,int} < Z_{int} < Z_{l,int}$, for $Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^5)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$;

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 5, c, i, one gets the variation of the optimal solution with respect to Z as given by the second row of the table 2.7 for high elasticity level.

ii. if $Z_{h,int} < Z_\Phi = Z_{hv}^{l0} < Z_{int} < Z_{l,int}$, then we may write

$Z_{h,int} < Z_\Phi = Z_{hv}^{l0} \leq Z_{int} < Z_{l,int}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{h,int} < Z_\Phi$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; for $Z_{h,int} \leq Z < Z_\Phi = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b,i}^5)$; at $Z = Z_\Phi = Z_{hv}^{l0}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,b,ii}^5)$; since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi = Z_{hv}^{l0} < Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{1,b,iii}^5)$.

At $Z = Z_{int}$, since $Z_\Phi < Z_{int}$, $Z_{h,int} < Z_{int} < Z_{l,int}$, and $\Phi(Z) > 0 \forall Z > Z_\Phi$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{2,b,iii}^5)$.

If $Z > Z_{int}$, since $Z_{h,int} < Z_{int} < Z_{l,int}$, $Z_{int} > Z_\Phi$, and $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_{int} < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^5)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$.

iii. if $Z_{h,int} < Z_{int} < Z_{hv}^{l0} < Z_{l,int}$, then we may write

$Z_{h,int} < Z_{int} < Z_\Phi = Z_{hv}^{l0} < Z_{l,int}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{int} < Z_\Phi$, for $Z <$

$Z_{h,int}, (Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; for $Z_{h,int} < Z < Z_{int}, (Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b,i}^5)$;

At $Z = Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{int} < Z_\Phi$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b,i}^5)$,

If $Z > Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_\Phi$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^5)$; at $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,ii}^5)$; since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z_\Phi < Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, K_0)$ from $(B_{3,b,iii}^5)$; for $Z_{l,int} \leq Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$ since $Z_\Phi < Z_{l,int}$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 5,c,ii and 5,c,iii, one gets the variation of the optimal solution with respect to Z as given by the third row of the table 2.7 for high elasticity level.

iv. if $Z_{h,int} < Z_{int} < Z_{l,int} \leq Z_{hv}^{lv}$, then we may write

$Z_{h,int} < Z_{int} < Z_{l,int} \leq Z_\Phi = Z_{hv}^{lv}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{int} < Z_\Phi$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; for $Z_{h,int} < Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b,i}^5)$.

At $Z = Z_{int}$, since $Z_{h,int} < Z_{int} < Z_{l,int}$, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{int} < Z_\Phi$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b,i}^5)$.

If $Z > Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{l,int} < Z_\Phi$, for $Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^5)$; for $Z_{l,int} \leq Z < Z_\Phi$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^5)$; at $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ or $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,ii}^5)$; for $Z_\Phi = Z_{hv}^{lv} < Z$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$ since $\Phi(Z) > 0 \forall Z > Z_\Phi$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 5,c,iv, one gets the variation of the optimal solution with respect to Z as given by the fourth row of the table 2.7 for high elasticity level.

(d) if $Z_{int} = Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{l,int} = Z_{v,l} = Z_{0,l} = Z_{int} < Z_{v,l}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{hv}^{lv} > Z_{int} > Z_{l,int}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then, we may write

$Z_{h,int} < Z_{l,int} = Z_{v,l} = Z_{0,l} = Z_{int} < Z_\Phi = Z_{hv}^{lv}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{int} < Z_\Phi$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; for $Z_{h,int} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b,i}^5)$;

At $Z = Z_{int}$, since $Z_{int} = Z_{l,int}$ and $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{int} < Z_\Phi$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,b,i}^5)$.

If $Z > Z_{int}$, since $Z_{l,int} = Z_{int}$ and $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$, for $Z < Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,b,i}^5)$; at $Z = Z_\Phi = Z_{hv}^{lv}$, there is an alternate optima; namely: $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$ from $(B_{3,b,ii}^5)$;

since $\Phi(Z) > 0 \forall Z > Z_\Phi$, for $Z > Z_\Phi = Z_{hv}^{lv} > Z_{l,int}$, $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$.

(e) if $Z_{int} > Z_{l,int}$, then, by proposition 2.5, we have

$$Z_{h,int} < Z_{l,int} < Z_{int} < Z_{0,l} < Z_{v,l} < Z_{v,h} \text{ and } Z_{0,l} < Z_{0,h} < Z_{v,h}.$$

Note that, $\Phi(Z)$, having a unique root, $Z_\Phi = Z_{hv}^{lv} > Z_{int} > Z_{l,int}$, is increasing in Z for $Z > 0$ by proposition 2.6. Hence, $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $\Phi(Z) > 0 \forall Z > Z_\Phi$. Then, we may write

$Z_{h,int} < Z_{l,int} < Z_{int} < Z_\Phi = Z_{hv}^{lv}$. Hence, considering the Theorem 2.4, following statements are true:

If $Z < Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and $Z_{h,int} < Z_{l,int} < Z_{int} < Z_\Phi$, for $Z < Z_{h,int}$, $(Q^*, K^*) \equiv (Q_h, K_0)$ from $(B_{1,a,i}^5)$; for $Z_{h,int} \leq Z < Z_{l,int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,b,i}^5)$; for $Z_{l,int} \leq Z < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{1,c}^5)$.

At $Z = Z_{int}$, since $Z_{l,int} < Z_{int}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{2,c}^5)$.

If $Z > Z_{int}$, since $\Phi(Z) < 0 \forall Z \in (0, Z_\Phi)$ and since $Z_{l,int} < Z_{int}$, for $Z < Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q_h, \hat{K}(Q_h))$ from $(B_{3,c,i}^5)$; since $Z_\Phi > Z_{int} > Z_{l,int}$, at $Z = Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv (Q, \hat{K}(Q))$ s.t. $Q \in \{Q_l, Q_h\}$ from $(B_{3,c,ii}^5)$; since $\Phi(Z) > 0 \forall Z > Z_\Phi$ and $Z_\Phi > Z_{int} > Z_{l,int}$, for $Z > Z_\Phi = Z_{hv}^{lv}$, $(Q^*, K^*) \equiv$

$(Q_l, \hat{K}(Q_l))$ from $(B_{3,c,iii}^5)$.

Then, by merging the adjacent regions of Z that correspond to same optimal solutions, which are mentioned in 5,*d* and 5,*e*, one gets the variation of the optimal solution with respect to Z as given by the fifth row of the table 2.7 for high elasticity level. $\square$

A.15 Proof of Lemma 2.7

Let k_{max} be the total number of Z Benchmarks that defines the regions in a given row of the tables 2.6 and 2.7. By definition, if $Z_k = Z_\Phi$, we have $\Pi(Q_k^*)|_{Z=Z_k} = \Pi(Q_{k+1}^*)|_{Z=Z_k}$, where $Z_\Phi \in \{Z_{h0}^{l0}, Z_{hv}^{00}, Z_{hv}^{l0}, Z_{hv}^{lv}\}$. On the other hand, if $Z_k \neq Z_\Phi$, then $(Q_k^*, K_k^*) = (Q_{k+1}^*, K_{k+1}^*)$, and thus, $\Pi(Q_k^*)|_{Z=Z_k} = \Pi(Q_{k+1}^*)|_{Z=Z_k}$. Hence, we have $\Pi(Q_k^*)|_{Z=Z_k} = \Pi(Q_{k+1}^*)|_{Z=Z_k} \forall k \text{ s.t. } k \in \{1, 2, \dots, k_{max}\}$.

Furthermore, $\frac{\partial \Pi_0(Q_0)}{\partial Z} < 0$ for $r < 1$; $\frac{\partial \Pi_v(Q_v)}{\partial Z} < 0$ for $r < 1 - \frac{\alpha_K}{\gamma''+1}$. Moreover, $\frac{\partial \Pi_0(Q)}{\partial Z} < 0$ and $\frac{\partial \Pi_v(Q)}{\partial Z} < 0 \forall r > 0$, where $Q \in \{Q_l, Q_h\}$. Hence, the firm's optimal profit, say $\Pi^*(Z)$ ($\Pi^*(Z) = \Pi(Q^*, K^*)$), is a continuous and decreasing function of Z . Thus, letting $\Pi^*(Z_{max}) = 0$, if $\Pi(Q^*, K^*)|_{Z=0} > 0$, there exists a unique positive value for Z_{max} , which should be in a unique interval of Z .

Note that at $Z = 0$, $(Q^*, K^*) \equiv (Q_h, K_0)$, for all firm types and r values. Then, since

$$\Pi_0(Q_h)|_{Z=\epsilon} = Q_h s - r \left(\frac{Q_h}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}} \cong Q_h s,$$

where, $\epsilon \rightarrow 0$, $\Pi^*(Z)$ is positive for $Z = \epsilon$.

Furthermore, we know that when $(Q^*, K^*) \equiv (Q_l, \hat{K}(Q_l))$, optimal profit can be expressed as

$$\Pi_v(Q_l) = Q_l s - \left(\frac{\gamma' c_K (r(\gamma''+1) + \alpha_K)}{\alpha_K (\gamma''+1)} \right) \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q_l}{a} Z \right]^{\frac{\gamma''+1}{r(\gamma''+1) + \alpha_K}} + \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1}.$$

Then, since $\Pi_v(Q_l)$ is composed of a positive fixed term and a negative term diverging to $-\infty$ as Z goes to $+\infty$, $\lim_{Z \rightarrow +\infty} \Pi_v(Q_l) = -\infty$.

Hence, since $\Pi^*(Z)$ continuously decreases from a positive value and diverges to $-\infty$ as Z goes to $+\infty$, it has a unique positive root which is denoted by Z_{max} . $\square$

A.16 Proof of Proposition 2.7

Substituting $Q_f \in \{Q_l, Q_h\}$ in $\Pi_0(Q)$ and $\Pi_v(Q)$, we get

$$\Pi_0(Q_f) = Q_f s - r \left(\frac{Q_f}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} Z^{\frac{1}{r}}$$

and

$$\begin{aligned} \Pi_v(Q_f) &= Q_f s - \left(\frac{\gamma' c_K (r(\gamma'' + 1) + \alpha_K)}{\alpha_K (\gamma'' + 1)} \right) \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q_f}{a} Z \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} \\ &\quad + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \\ &= Q_f s - \left(\frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1} \right) \frac{\gamma' c_K}{\alpha_K} \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q_f}{a} Z \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} \\ &\quad + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \\ &= Q_f s - \frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma'' + 1} \\ &\quad + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1}, \end{aligned}$$

respectively.

We assume that the firm operates with positive profits under zero carbon tax. Then, since $Z = 0$ means that firm receives incentives for emitting carbon credits, it should operate with positive profits for $Z = \epsilon$. Furthermore, $\hat{K}(Q_f)$ is increasing in Z , and diverges to $+\infty$ as Z goes to $+\infty$. Hence, since $\Pi_v(Q_f)$ is composed of a positive term that is constant in Z and a negative term diverging to $-\infty$ as Z goes to infinity, $\lim_{Z \rightarrow +\infty} \Pi_v(Q_f) = -\infty$.

Then, since $\Pi(Q^*)$, being positive when $(Q^*, K^*) \equiv (Q_h, K_0)$, is a decreasing one-to-one function of Z from Lemma 2.7, and diverges to $-\infty$ as Z goes to $+\infty$, $\Pi^*(Z)$ has a unique root, Z_Φ .

Substituting Q_0 and Q_v in $\Pi_0(Q)$ and $\Pi_v(Q)$, we get

$$\Pi_0(Q_0) = Q_0 s - r Q_0 s = (1 - r) Q_0 s$$

and

$$\begin{aligned}\Pi_v(Q_v) &= Q_v s - \left(\frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1} \right) Q_v s + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \\ &= \left(1 - \frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1} \right) Q_v s + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1},\end{aligned}$$

respectively.

Hence,

1. for $0 < r < 1 - \frac{\alpha_K}{\gamma'' + 1}$,

$\Pi_0(Q_0) > 0$ since $0 < r < 1 - \frac{\alpha_K}{\gamma'' + 1} < 1$, and $\Pi_v(Q_v) > 0$ since $0 < 1 - \frac{r\gamma'' + 1 + \alpha_K}{\gamma'' + 1}$, respectively. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7. Therefore,

- (a) if $Z_{int} < Z_{l,int}$, since $(Q^*, K^*) = (Q_0, K_0)$, we have $\Pi_0(Q_0) > 0$ at $Z = Z_{l,0}$. Hence $Z_{max} > Z_{l,0}$. Furthermore, $\Pi(Q^*)$ is decreasing in Z .

Hence,

$$Z_{max} = Z_{max}(Q_l, K_0) \text{ for } \Pi(Q^*)|_{Z=Z_{l,int}} < 0; \quad Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l)) \text{ for } \Pi(Q^*)|_{Z=Z_{l,int}} \geq 0.$$

- (b) if $Z_{l,int} \leq Z_{int}$, then $Q_l = Q_{int} < Q_0 < Q_v$ at $Z = Z_{l,int} \leq Z_{int}$. Note that $\Pi_0(Q)$ and $\Pi_v(Q)$ are concave increasing in Q .

Therefore, $0 < \Pi(Q_{l,int})$, and thus,

$$Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l)).$$

2. for $0 < r = 1 - \frac{\alpha_K}{\gamma'' + 1}$,

$\Pi_0(Q_0) > 0$ since $0 < r = 1 - \frac{\alpha_K}{\gamma'' + 1} < 1$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

- (a) if $Z_{int} < Z_{l,int}$, since $Z_{int} < Z_{l,0}$ and $\Pi(Q) = \Pi_0(Q_0) > 0$ for $Z_{int} < Z \leq Z_{l,0}$, $Z_{max} \geq Z_{l,0}$ from Theorem 2.5.

Hence,

$$Z_{max} = Z_{max}(Q_l, K_0) \text{ for } \Pi(Q^*)|_{Z=Z_{l,int}} < 0; \quad Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l)) \text{ for } \Pi(Q^*)|_{Z=Z_{l,int}} \geq 0.$$

- (b) if $Z_{int} \geq Z_{l,int}$, we know that $\Pi_v(Q_v) = \Pi_0(Q_v) = \Pi_v(Q_{int}) = \Pi_0(Q_{int}) > 0$ at $Z = Z_{int}$. Furthermore, at $Z = Z_{int}$, since $\Pi(Q) = \Pi_v(Q) = \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1}$ is constant in Q , optimal solution between Q_l and Q_h is $(Q, \hat{K}(Q))$ s.t. $Q \in [Q_l, Q_h]$. Then, since $\Pi_v(Q_h) > 0$ at $Z = Z_{int}$, we have $Z_{max} \geq Z_{int}$ from Theorem 2.5.

Hence,

$$Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l)) \text{ for } Z_{int} \geq Z_{l,int}.$$

3. for $0 < 1 - \frac{\alpha_K}{\gamma''+1} < r < 1$,

$\Pi_0(Q_0) > 0$ since $0 < r = 1 - \frac{\alpha_K}{\gamma''+1} < 1$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

- (a) if $Z_{int} \leq Z_{h,int}$, then, by table 2.6, (Q_0, K_0) can be optimal and when (Q_0, K_0) is optimal, $\Pi(Q^*) > 0$. Therefore, $Z_{max} \geq Z_{l,0}$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Thus,

if $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$, then $Z_{max} = Z_{max}(Q_l, K_0)$; if $\Pi(Q^*)|_{Z=Z_{l,int}} > 0$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$.

- (b) if $Z_{h,int} < Z_{int} < Z_{l,int} \wedge Z_{int} < Z_{hv}^{00} \leq Z_{l,0}$, then, by table 2.6, (Q_0, K_0) can be optimal and when (Q_0, K_0) is optimal, $\Pi(Q^*) > 0$. Therefore, $Z_{max} \geq Z_{l,0}$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Thus,

if $\Pi(Q^*)|_{Z=Z_{l,int}} < 0$, then $Z_{max} = Z_{max}(Q_l, K_0)$; if $\Pi(Q^*)|_{Z=Z_{l,int}} > 0$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$.

- (c) if $Z_{h,int} < Z_{int} < Z_{l,int} \wedge Z_{l,0} < Z_{hv}^{l0} < Z_{l,int}$, at $Z = Z_{int}$, we know, by Corollary 2.1, that $\Pi_v(Q_v) = \Pi_0(Q_v) = \Pi_v(Q_{int}) = \Pi_0(Q_{int}) > 0$. Furthermore, at $Z = Z_{int}$, since $\Pi(Q) = \Pi_0(Q)$ is concave increasing in Q for $Q \in [Q_l, Q_{int}]$; $\Pi(Q) = \Pi_v(Q)$ is convex increasing in Q for $Q \in (Q_{int}, Q_h]$, optimal solution is $(Q_h, \hat{K}(Q_h))$. Then, $\Pi_v(Q_h) > 0$ for $Z = Z_{int}$, $\Pi_0(Q_l) > 0$ for $Z = Z_{l,0}$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, by table 2.6,

if $\Pi(Q^*)|_{Z=Z_{hv}^{l0}} < 0$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$; if $\Pi(Q^*)|_{Z=Z_{hv}^{l0}} > 0$ and

$\Pi(Q^*)|_{Z=Z_{l,int}} < 0$, then $Z_{max} = Z_{max}(Q_l, K_0)$; if $\Pi(Q^*)|_{Z=Z_{l,int}} > 0$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$.

- (d) if $Z_{h,int} < Z_{int} < Z_{l,int} \wedge Z_{l,int} \leq Z_{hv}^{lv}$, at $Z = Z_{int}$, we know, by Corollary 2.1, that $\Pi_v(Q_v) = \Pi_0(Q_v) = \Pi_v(Q_{int}) = \Pi_0(Q_{int}) > 0$. Furthermore, at $Z = Z_{int}$, since $\Pi(Q) = \Pi_0(Q)$ is concave increasing in Q for $Q \in [Q_l, Q_{int}]$; $\Pi(Q) = \Pi_v(Q)$ is convex increasing in Q for $Q \in (Q_{int}, Q_l]$, optimal solution is $(Q_h, \hat{K}(Q_h))$. Then, at $Z = Z_{int}$, $\Pi_v(Q_h) > 0$. Therefore, $Z_{max} \geq Z_{int}$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, by table 2.6,

if $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} < 0$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$; if $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} \geq 0$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$.

- (e) if $Z_{l,int} \leq Z_{int}$, then $0 < Q_v = Q_{int} = Q_0 < Q_l$ for $Z_{l,int} < Z = Z_{int}$. Furthermore, since $Q_{int} < Q_l$ and $\Pi_v(Q)$ is increasing in Q for $Z_{l,int} < Z = Z_{int}$, we have $\Pi(Q^*) = \Pi_v(Q_h) > 0$. Then, $Z_{max} > Z_{int}$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, by table 2.6,

if $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} < 0$, then $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$; if $\Pi_v(Q_h)|_{Z=Z_{hv}^{lv}} \geq 0$, then $Z_{max} = Z_{max}(Q_l, \hat{K}(Q_l))$.

4. for $r = 1$,

- (a) if $Z_{int} \leq Z_{h,int}$, then,

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{h0}^{l0} = Z_{int}$. Note that by Corollary 2.1, $\Pi(Q)$ is linear decreasing for $Q \in (0, Q_{int})$, convex decreasing for $Q \in (Q_{int}, Q_v)$ and convex increasing for $Q \in (Q_v, \infty)$. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{int}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{int}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{int}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$$Z_{max} = Z_{max}(Q_h, K_0).$$

(b) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{hv}^{l0} < Z_{l,int}$, then,

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{hv}^{l0}$. Note that by Corollary 2.1, $\Pi(Q)$ is linear decreasing for $Q \in (0, Q_{int})$, convex decreasing for $Q \in (Q_{int}, Q_v)$ and convex increasing for $Q \in (Q_v, \infty)$. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{hv}^{lv}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$Z_{max} = Z_{max}(Q_h, K_0)$ if $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ if $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$.

(c) if $Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$, then,

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{hv}^{lv}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{hv}^{lv}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$Z_{max} = Z_{max}(Q_h, K_0)$ if $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ if $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$.

(d) if $Z_{l,int} \leq Z_{int}$, then,

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{hv}^{lv}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{hv}^{lv}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$$Z_{max} = Z_{max}(Q_h, K_0) \text{ if } \Pi(Q^*)|_{Z=Z_{h,int}} < 0 \text{ and } Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h)) \text{ if } \Pi(Q^*)|_{Z=Z_{h,int}} > 0.$$

5. for $r > 1$,

(a) if $Z_{int} \leq Z_{h,int}$,

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{h0}^{l0}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{h0}^{l0}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{h0}^{l0}} < 0$. Furthermore, $\Pi(Q_l)$ is decreasing in Z . Then, $\Pi(Q_l)|_{Z>Z_\Phi=Z_{h0}^{l0}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$$Z_{max} = Z_{max}(Q_h, K_0).$$

(b) if $(Z_{h,int} < Z_{int} < Z_{l,int} \text{ and } Z_{h0}^{l0} < Z_{h,int})$, then

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{h0}^{l0}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{h0}^{l0}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{h0}^{l0}} < 0$. Furthermore, $\Pi(Q_l)$ is decreasing in Z . Then, $\Pi(Q_l)|_{Z>Z_\Phi=Z_{h0}^{l0}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$$Z_{max} = Z_{max}(Q_h, K_0).$$

(c) if $(Z_{h,int} < Z_{int} < Z_{l,int} \text{ and } Z_{h,int} \leq Z_{hv}^{l0} < Z_{l,int})$, then

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{hv}^{l0}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{hv}^{l0}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{hv}^{l0}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{hv}^{l0}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$Z_{max} = Z_{max}(Q_h, K_0)$ if $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ if $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$.

(d) if $(Z_{h,int} < Z_{int} < Z_{l,int}$ and $Z_{l,int} \leq Z_{hv}^{lv}$), then

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{hv}^{lv}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{hv}^{lv}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$Z_{max} = Z_{max}(Q_h, K_0)$ if $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ if $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$.

(e) if $(Z_{l,int} \leq Z_{int})$, then

by table 2.7, Q_l can be optimal only if $Z \geq Z_\Phi = Z_{hv}^{lv}$. Note that by Corollary 2.1, $\Pi(Q)$ is convex. Furthermore, Q_l can be optimal only if $\Pi(Q)$ is decreasing in Q at $Q = Q_l$.

Then, since $\Pi(0)|_{Z=Z_{hv}^{lv}} = 0$, we have $\Pi(Q_l)|_{Z=Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, since $\Pi(Q_l)$ is decreasing in Z , we have $\Pi(Q_l)|_{Z>Z_\Phi=Z_{hv}^{lv}} < 0$. Furthermore, $\Pi(Q^*)$ is decreasing in Z from Lemma 2.7.

Hence, assuming that $\Pi(Q^*)|_{Z=0} = \Pi_0(Q_h)|_{Z=0} > 0$, we have

$Z_{max} = Z_{max}(Q_h, K_0)$ if $\Pi(Q^*)|_{Z=Z_{h,int}} < 0$ and $Z_{max} = Z_{max}(Q_h, \hat{K}(Q_h))$ if $\Pi(Q^*)|_{Z=Z_{h,int}} > 0$.

where,

$$Z_{max}(Q_f, \hat{K}(Q_f)) = \left[\left(Q_f s + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1} \right) \left(\frac{\alpha_K (\gamma'' + 1)}{\gamma' c_K (r(\gamma'' + 1) + \alpha_K)} \right) \right]^{\frac{\gamma'' + 1}{r(\gamma'' + 1) + \alpha_K}} \left[\left(\frac{\gamma' c_K}{\alpha_K} \right)^r \frac{a}{Q_f} \right],$$

$$Z_{\max}(Q_f, K_0) = Q_f^{r-1} \left(\frac{s}{r} \right)^r a K_0^{\alpha_K},$$

$$Z_{\max}(Q_{int}) = sa \left(\frac{\gamma' c_K}{\alpha_K} \right)^{r-1} K_0^{(r-1)(\gamma''+1)+\alpha_K},$$

such that $Q_f \in \{Q_l, Q_h\}$. □

A.17 Proof of Theorem 2.6

In order to prove Theorem 2.6, we will have a look at the behaviors of the optimal values of x_i , $x_i^*(Q^*, K^*)$, for all optimal production quantity - technology level pairs.

Let Q_f be the boundary value of the production quantity, that is, $Q_f \in \{Q_l, Q_h\}$.

Then, when the optimal solution is (Q_f, K_0) , optimal quantity of input i is

$$x_i^*(Q_f, K_0) = \frac{\alpha_i}{p_i + \rho \beta_i} \left(\frac{Q_f}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}}.$$

The first derivative of the above function with respect to ρ is

$$\begin{aligned} \frac{\partial x_i^*(Q_f, K_0)}{\partial \rho} = & - \frac{\beta_i \alpha_i}{(p_i + \rho \beta_i)^2} \left(\frac{Q_f}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} \\ & + \frac{\alpha_i}{p_i + \rho \beta_i} \left(\frac{Q_f}{a K_0^{\alpha_K}} \right)^{\frac{1}{r}} \sum_{j=1}^N \frac{\alpha_j}{r} \frac{\beta_j}{p_j + \rho \beta_j} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}}, \end{aligned}$$

which is negative if

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{r} \frac{\beta_j}{p_j + \rho \beta_j},$$

and positive otherwise. Therefore, $x_i^*(K_0, Q_f)$ is decreasing in ρ if

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{r} \frac{\beta_j}{p_j + \rho \beta_j},$$

and increasing in ρ otherwise when $(Q^*(\rho), K^*(\rho)) \equiv (Q_f, K_0)$.

When the optimal solution is $(Q_f, \hat{K}(Q_f))$, optimal amount of input i is

$$\begin{aligned} x_i^*(Q_f, \hat{K}(Q_f)) &= \frac{\alpha_i}{p_i + \rho\beta_i} \left(\frac{Q_f}{a\hat{K}(Q_f)^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} \\ &= \frac{\alpha_i}{p_i + \rho\beta_i} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1}, \end{aligned}$$

whose first derivative with respect to ρ is

$$\begin{aligned} \frac{x_i^*(Q_f, \hat{K}(Q_f))}{\partial \rho} &= - \frac{\beta_i \alpha_i}{(p_i + \rho\beta_i)^2} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\ &\quad + \frac{\alpha_i}{p_i + \rho\beta_i} (\gamma'' + 1) \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''} \frac{\partial \hat{K}(Q_f)}{\partial \rho} \\ &= - \frac{\beta_i \alpha_i}{(p_i + \rho\beta_i)^2} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\ &\quad + \frac{\alpha_i}{p_i + \rho\beta_i} \sum_{j=1}^N \frac{\alpha_j (\gamma'' + 1)}{r(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''} \\ &\quad \cdot \left[\left(\frac{\alpha_K}{\gamma' c_K} \right)^r \frac{Q}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r(\gamma''+1)+\alpha_K}} \\ &= \frac{\alpha_i}{p_i + \rho\beta_i} \left[- \frac{\beta_i}{p_i + \rho\beta_i} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \right. \\ &\quad \left. + \sum_{j=1}^N \frac{\alpha_j (\gamma'' + 1)}{r(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \right], \end{aligned}$$

which is negative if

$$\frac{\beta_i}{p_i + \rho\beta_i} > \sum_{j=1}^N \frac{\alpha_j (\gamma'' + 1)}{r(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j},$$

and positive otherwise. Therefore, $x_i^*(Q_f, \hat{K}(Q_f))$ is decreasing in ρ if

$$\frac{\beta_i}{p_i + \rho\beta_i} > \sum_{j=1}^N \frac{\alpha_j (\gamma'' + 1)}{r(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j},$$

and increasing in ρ otherwise when $(Q^*(\rho), K^*(\rho)) \equiv (Q_f, \hat{K}(Q_f))$.

When the optimal solution is $(Q_v, \hat{K}(Q_v))$, the optimal quantity of input i is

$$\begin{aligned} x_i^*(Q_v, \hat{K}(Q_v)) &= \frac{\alpha_i}{p_i + \rho\beta_i} \left(\frac{Q_v}{a\hat{K}(Q_v)^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}} \\ &= \frac{\alpha_i}{p_i + \rho\beta_i} Q_v s, \end{aligned}$$

whose first derivative with respect to ρ is

$$\begin{aligned} \frac{\partial x_i^*(Q_v, \hat{K}(Q_v))}{\partial \rho} &= -\frac{\beta_i \alpha_i}{(p_i + \rho\beta_i)^2} Q_v s + \frac{\alpha_i}{p_i + \rho\beta_i} s \frac{\partial(Q_v)}{\partial \rho} \\ &= -\frac{\beta_i \alpha_i}{(p_i + \rho\beta_i)^2} Q_v s \\ &\quad + \frac{\alpha_i}{p_i + \rho\beta_i} \sum_{j=1}^N \frac{\alpha_j(\gamma'' + 1)}{(r-1)(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j} s \\ &\quad \cdot \left[\left(\frac{1}{s} \right)^{r(\gamma'' + 1) + \alpha_K} \left(\frac{\gamma' c_K}{\alpha_K} \right)^{\alpha_K} \left[\frac{1}{a} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\gamma'' + 1} \right]^{\frac{1}{(r-1)(\gamma'' + 1) + \alpha_K}} \\ &= -\frac{\beta_i \alpha_i}{(p_i + \rho\beta_i)^2} Q_v s + \frac{\alpha_i}{p_i + \rho\beta_i} \sum_{j=1}^N \frac{\alpha_j(\gamma'' + 1)}{(r-1)(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j} s Q_v, \end{aligned}$$

which is negative for

$$\frac{\beta_i}{p_i + \rho\beta_i} > \sum_{j=1}^N \frac{\alpha_j(\gamma'' + 1)}{(r-1)(\gamma'' + 1) + \alpha_K} \frac{\beta_j}{p_j + \rho\beta_j},$$

which always holds for $r < 1 - \alpha_K/(\gamma'' + 1)$ (always holds when $(Q_v, \hat{K}(Q_v))$ is optimal). Therefore, $x_i^*(Q_v, \hat{K}(Q_v))$ is **decreasing** in ρ if $(Q^*(\rho), K^*(\rho)) \equiv (Q_v, \hat{K}(Q_v))$.

When the optimal solution is (Q_0, K_0) , the optimal quantity of input i is

$$\begin{aligned} x_i^*(Q_0, K_0) &= \frac{\alpha_i}{p_i + \rho\beta_i} \left(\frac{Q_0}{aK_0^{\alpha_K}} \right)^{\frac{1}{r}} \prod_{j=1}^N \left(\frac{p_j + \rho\beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r}}, \\ &= \frac{\alpha_i}{p_i + \rho\beta_i} Q_0 s, \end{aligned}$$

whose first derivative with respect to ρ is

$$\begin{aligned} \frac{\partial x_i^*(Q_0, K_0)}{\partial \rho} &= -\frac{\beta_i \alpha_i}{(p_i + \rho \beta_i)^2} Q_0 s + \frac{\alpha_i}{p_i + \rho \beta_i} s \frac{\partial(Q_0)}{\partial \rho} \\ &= -\frac{\beta_i \alpha_i}{(p_i + \rho \beta_i)^2} Q_0 s + \frac{\alpha_i}{p_i + \rho \beta_i} \sum_{j=1}^N \frac{\alpha_j}{(r-1)} \frac{\beta_j}{p_j + \rho \beta_j} s \\ &\quad \cdot \left[\frac{1}{s^r a K_0^{\alpha_K}} \prod_{j=1}^N \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\alpha_j} \right]^{\frac{1}{r-1}} \\ &= -\frac{\beta_i \alpha_i}{(p_i + \rho \beta_i)^2} Q_0 s + \frac{\alpha_i}{p_i + \rho \beta_i} \sum_{j=1}^N \frac{\alpha_j}{(r-1)} \frac{\beta_j}{p_j + \rho \beta_j} s Q_0, \end{aligned}$$

which is negative for

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{(r-1)} \frac{\beta_j}{p_j + \rho \beta_j},$$

which always holds for $r < 1$ (always holds when (Q_0, K_0) is optimal). Therefore, $x_i^*(Q_0, K_0)$ is **decreasing** in ρ if $(Q^*(\rho), K^*(\rho)) \equiv (Q_0, K_0)$. $\square$

A.18 Proof of Theorem 2.6 (Compact Version)

Letting $x_{N+1} = K^{\gamma''+1}$, $x_{N+2} = Q$, $\alpha_{N+1} = \frac{\alpha_K}{\gamma''+1}$, $\alpha_{N+2} = -1$, $p_{N+1} = \frac{\gamma' c_K}{\gamma''+1}$, $p_{N+2} = -s$, $\beta_{N+1} = 0$, $\beta_{N+2} = 0$, and $x_0 = K_0^{\gamma''+1}$,

and letting χ and ϕ be the set of indices of the non-binding and the binding decision variables, respectively, the optimal value of the decision variable i can be rewritten as

$$x_i^* = \frac{\alpha_i}{p_i + \rho \beta_i} \left(\frac{1}{a} \right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j} \right)^{\frac{\alpha_j}{r^*}},$$

where

$$r^* = \sum_{j \in \chi} \alpha_j.$$

Then, first derivative of x_i^* with respect to ρ can be written as

$$\begin{aligned} \frac{\partial x_i^*}{\partial \rho} = & -\frac{\beta_i \alpha_i}{(p_i + \rho \beta_i)^2} \left(\frac{1}{a}\right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r^*}} \\ & + \frac{\alpha_i}{p_i + \rho \beta_i} \left(\frac{1}{a}\right)^{\frac{1}{r^*}} \prod_{j \in \phi} x_j^{\frac{-\alpha_j}{r^*}} \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j} \prod_{j \in \chi} \left(\frac{p_j + \rho \beta_j}{\alpha_j}\right)^{\frac{\alpha_j}{r^*}}, \end{aligned}$$

which is negative for

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j},$$

and positive otherwise. Therefore, x_i^* is decreasing in ρ if

$$\frac{\beta_i}{p_i + \rho \beta_i} > \sum_{j=1}^N \frac{\alpha_j}{r^*} \frac{\beta_j}{p_j + \rho \beta_j},$$

and it is increasing in ρ otherwise. □

A.19 Proof of Proposition 2.8

Note that $\Pi_0(Q_f) = Q_f s - r w^*(Q_f, K_0) / \ln\left(\frac{Q_f}{a K_0^{\alpha_K}}\right)$ for $Q_f \in \{Q_l, Q_h\}$. Hence,

$$\frac{\partial \Pi_0(Q_f)}{\partial \rho} = -em(Q_f, K_0),$$

and thus, we have

$$\begin{aligned}
 \frac{\partial^2 \Pi_0(Q_f)}{\partial \rho^2} &= -\frac{\partial em(Q_f, K_0)}{\partial \rho} \\
 &= \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_f, K_0)}{\ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right)} - \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\partial \left(w^*(Q_f, K_0) / \ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right) \right)}{\partial \rho} \\
 &= \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_f, K_0)}{\ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right)} - \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{em^*(Q_f, K_0)}{r}.
 \end{aligned}$$

Then,

$$\begin{aligned}
 \frac{1}{r} \frac{\partial^2 \Pi_0(Q_f)}{\partial \rho^2} &= \frac{1}{r} \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_f, K_0)}{\ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right)} \\
 &\quad - \frac{1}{r^2} \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{w^*(Q_f, K_0)}{\ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right)} \\
 &= \sum_{i=1}^N \frac{\beta_i^2}{(p_i + \rho \beta_i)^2} \frac{\alpha_i}{r} \frac{w^*(Q_f, K_0)}{\ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right)} - \left(\sum_{i=1}^N \frac{\beta_i}{p_i + \rho \beta_i} \frac{\alpha_i}{r} \right)^2 \frac{w^*(Q_f, K_0)}{\ln\left(\frac{Q_f}{aK_0^{\alpha_K}}\right)} > 0,
 \end{aligned}$$

since variance of a positive random variable is positive ($\frac{\alpha_i}{r}$ can be considered as the probability of a positive random variable taking the value $\frac{\beta_i}{p_i + \rho \beta_i}$). Then, $em(Q_f, K_0)$ is **decreasing** in ρ and $\Pi_0(Q_f)$ **convex decreasing** in $\rho \forall r$.

Note that

$$\Pi_0(Q_0) = Q_0 s - r w^*(Q_0, K_0) / \ln\left(\frac{Q_0}{aK_0^{\alpha_K}}\right) = Q_0 s - r Q_0 s = (1 - r) Q_0 s.$$

Furthermore,

$$\frac{\partial em(Q_0, K_0)}{\partial \rho} = - \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} Q_0 s + \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\partial Q_0}{\partial \rho} s.$$

Then, for $r < 1$, since $\frac{\partial Q_0}{\partial \rho} < 0$ by proposition 2.4, $\frac{\partial em(Q_0, K_0)}{\partial \rho} < 0$. Moreover,

$$\begin{aligned} \frac{\partial^2 \Pi_0(Q_0)}{\partial \rho^2} &= - \frac{\partial em(Q_0, K_0)}{\partial \rho} \\ &= \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_0, K_0)}{\ln\left(\frac{Q_0}{aK_0^{\alpha_K}}\right)} - \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\partial\left(w^*(Q_0, K_0)/\ln\left(\frac{Q_0}{aK_0^{\alpha_K}}\right)\right)}{\partial \rho} \\ &= \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_0, K_0)}{\ln\left(\frac{Q_0}{aK_0^{\alpha_K}}\right)} + \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{em^*(Q_0, K_0)}{1-r}, \end{aligned}$$

which is, for $r < 1$, greater than zero. Therefore, for $r < 1$, $em(Q_0, K_0)$ is **decreasing** and $\Pi_0(Q_0)$ is **convex decreasing** in ρ .

Note that

$$\Pi_v(Q_v) = \left(1 - \frac{r(\gamma'' + 1) + \alpha_K}{\gamma'' + 1}\right) Q_v s + \frac{\gamma' c_K}{\gamma'' + 1} K_0^{\gamma'' + 1}.$$

Furthermore,

$$\frac{\partial em(Q_v, \hat{K}(Q_v))}{\partial \rho} = - \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} Q_v s + \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\partial Q_v}{\partial \rho}.$$

Then, for $r < 1 - \frac{\alpha_K}{\gamma'' + 1}$, we have $\frac{\partial Q_v}{\partial \rho} < 0$ from Proposition 2.4. Hence,

$$\frac{\partial em(Q_v, \hat{K}(Q_v))}{\partial \rho} < 0.$$

Moreover,

$$\begin{aligned}
 \frac{\partial^2 \Pi_v(Q_v)}{\partial \rho^2} &= - \frac{\partial em(Q_v, K_v)}{\partial \rho} \\
 &= + \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_v, K_v)}{\ln\left(\frac{Q_v}{aK_v^{\alpha_K}}\right)} - \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\partial \left(w^*(Q_v, K_v) / \ln\left(\frac{Q_v}{aK_v^{\alpha_K}}\right) \right)}{\partial \rho} \\
 &= \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_v, K_v)}{\ln\left(\frac{Q_v}{aK_v^{\alpha_K}}\right)} + \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{em^*(Q_v, K_v)}{1 - \frac{r(\gamma''+1)+\alpha_K}{\gamma''+1}} \\
 &= \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{w^*(Q_v, K_v)}{\ln\left(\frac{Q_v}{aK_v^{\alpha_K}}\right)} + \left(\sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \right)^2 \frac{1}{1 - \frac{\alpha_K}{\gamma''+1} - r} \frac{w^*(Q_v, K_v)}{\ln\left(\frac{Q_v}{aK_v^{\alpha_K}}\right)},
 \end{aligned}$$

which is greater than zero for $r < 1 - \frac{\alpha_K}{\gamma''+1}$. Therefore, for $r < 1 - \frac{\alpha_K}{\gamma''+1}$, $em(Q_v, K_v)$ is **decreasing** and $\Pi_v(Q_v)$ is **convex decreasing** in ρ .

Note that

$$\begin{aligned}
 \Pi_v(Q_f) &= Q_f s - \frac{r(\gamma''+1)+\alpha_K}{\gamma''+1} \frac{w^*(Q_f, \hat{K}(Q_f))}{\ln\left(\frac{Q_f}{a\hat{K}(Q_f)^{\alpha_K}}\right)} + \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1} \\
 &= \left(1 - \frac{r(\gamma''+1)+\alpha_K}{\gamma''+1} \right) \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} + \frac{\gamma' c_K}{\gamma''+1} K_0^{\gamma''+1},
 \end{aligned}$$

for $Q_f \in \{Q_l, Q_h\}$. Furthermore, since

$$em(Q_f, \hat{K}(Q_f)) = \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1},$$

we have

$$\begin{aligned}
 \frac{\partial em(Q_f, \hat{K}(Q_f))}{\partial \rho} &= - \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\
 &\quad + \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \frac{\gamma''+1}{r(\gamma''+1)+\alpha_K}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \frac{r(\gamma''+1)+\alpha_K}{r^2(\gamma''+1)} \frac{\partial^2 \Pi_v(Q_f)}{\partial \rho^2} &= \frac{r(\gamma''+1)+\alpha_K}{r^2(\gamma''+1)} \frac{\partial em(Q_f, \hat{K}(Q_f))}{\partial \rho} \\
 &= \frac{r(\gamma''+1)+\alpha_K}{r^2(\gamma''+1)} \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\
 &\quad - \frac{1}{r^2} \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\
 &= \frac{r(\gamma''+1)+\alpha_K}{r^2(\gamma''+1)} \sum_{i=1}^N \frac{\beta_i^2 \alpha_i}{(p_i + \rho \beta_i)^2} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\
 &\quad - \frac{1}{r^2} \left(\sum_{i=1}^N \frac{\beta_i \alpha_i}{p_i + \rho \beta_i} \right)^2 \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\
 &> \sum_{i=1}^N \frac{\beta_i^2}{(p_i + \rho \beta_i)^2} \frac{\alpha_i}{r} \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} \\
 &\quad - \left(\sum_{i=1}^N \frac{\beta_i}{p_i + \rho \beta_i} \frac{\alpha_i}{r} \right)^2 \frac{\gamma' c_K}{\alpha_K} \hat{K}(Q_f)^{\gamma''+1} > 0,
 \end{aligned}$$

since $\frac{r(\gamma''+1)+\alpha_K}{r(\gamma''+1)} > 1$ and variance of a positive random variable is positive ($\frac{\alpha_i}{r}$ can be considered as the probability of a positive random variable to take the value $\frac{\beta_i}{p_i + \rho \beta_i}$). Hence, $em(Q_f, \hat{K}(Q_f))$ is **decreasing** in ρ and $\Pi_v(Q_f)$ is **convex decreasing** in $\rho \forall r$. The proof is completed. $\square$