

ON COMPLEX CONTINUED FRACTIONS
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ABSTRACT

We study *simple continued fractions* with entries from the ring of Gaussian integers, namely $\mathbb{Z}[\sqrt{-1}]$, and more generally ring of integers of norm-Euclidean imaginary quadratic number fields (i.e. those number fields which admit a Euclidean norm). These number fields are necessarily of class number 1. The aim of this thesis is to demonstrate that from an algebraic and geometric viewpoint the theory of simple continued fractions with entries from integers is parallel to the that of simple continued fractions with entries from aforementioned rings. In order to demonstrate, we interpret the classical theory of simple continued fractions in terms of a one-dimensional cell-complex (in fact, the familiar bipartite Farey tree) admitting an action of the modular group. This free action allows us to use elements of the modular group to label one-cells of the bipartite Farey tree. As a result, we may associate a finite (resp. infinite) path (a subcell-complex) based at the edge labeled I to any finite (resp. infinite) simple continued fraction.

Analogously, we construct a two dimensional cell-complex admitting a free action of a particular subgroup (of index two which is responsible for continued fractions) of the corresponding projective special linear group. Two cells of this cell-complex can be labeled with elements of the mentioned subgroup. As a result, to any finite (resp. infinite) simple continued fraction with coefficients in the corresponding ring of integers, we associate a finite (resp. infinite) 2 dimensional subcell-complex.

Keywords : continued fractions, modular group, imaginary quadratic fields, Picard group, Farey graph, Hurwitz quilt

ÖZET

Bu çalışmada genel olarak *basit sürekli kesirler* incelendi, öyleki buradaki sürekli kesirlerin girdileri Öklityen norma sahip sanal kuadratik sayı cisimlerinin tamsayı halkasından gelmektedir. (Bahsekonu sayı cisimlerinin sınıf sayısı zorunlu olarak 1'dir.) Özel olarak, girdileri Gauss tamsayı halkasından $\mathbb{Z}[\sqrt{-1}]$ gelen sürekli kesirler üzerine çalışıldı. Tezin amacı, girdileri rasyonel tamsayı olan basit sürekli kesirlerin teorisinin bahse konu halkar üzerinde paralellik gösterdiğini, cebirsel ve geometrik bir bakış açısıyla kanıtlamaktır. Geometrik bir yorum olarak, basit sürekli kesirler teorisi, 1 boyutlu bir cell-kompleks (ikili Farey ağacı) yardımıyla anlatıldı. Moduler group, Farey ağacının 1-boyutlu hücreleri üzerine serbest bir etkiye sahip olup bu etki sayesinde Farey ağacının her 1-boyutlu hücresi grubun moduler grubun elemanları ile etiketlenmiştir. Bu sayede, birim eleman I ile etiketlenmiş kenardan başlayan her bir yol (bir alt cell-kompleks) bir basit sürekli kesire karşılık gelmektedir.

Benzer şekilde, girdileri bahse konu tamsayı halkasında gelen basit sürekli kesirler için 2 boyutlu özel bir cell-kompleks inşa edildi öyleki, bu duruma karşılık gelen projektif özel lineer grubun bir altgrubu, bahse konu cell-komplekse serbest bir şekilde etkimektedir. Ayrıca kompleksin 2 boyutlu hücreleri belirtilen bu altgrubun elemanları ile etiketlenebilmektedir. Bu sayede, girdileri sanal kuadratik bir cismin tamsayı halkasından gelen basit sürekli kesirler, 2 boyutlu bir alt cell-komplekse karşılık gelmektedir.

Anahtar Kelimeler : sürekli kesirler, moduler grup, sanal kuadratik cisimler, Picard grubu, Farey çizgesi, Hurwitz yorgani

1 INTRODUCTION

An algebraic number field is a finite extension of the field of rational numbers. The degree of such an extension is defined as the dimension of the extension field as a $\mathbb{Q}$ -vector space. Any degree 2, that is quadratic, number field is of the form $K_d = \mathbb{Q}(\sqrt{d})$; where d is a square-free integer. When $d < 0$, (resp. $d > 0$) the number field is said to be imaginary (resp. real). An element of a number field is called an algebraic integer if it is the root of a monic element of $\mathbb{Z}[X]$. It turns out that the set of algebraic integers in a number field forms a subring called the *ring of integers* of the number field. In the case of quadratic number fields, the ring of integers is of the form $\mathbb{Z}[\sqrt{d}]$ if $d \equiv 2, 3$ modulo 4 and $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1$ modulo 4.

Continued fractions have proven themselves as although ancient, remarkably useful tools in many distinct areas. The topic has a quite rich history. The notion of continued fractions have appeared in many disguises throughout history. For instance, the *Elements* of Euclid, in fact the seventh book, describes his famous algorithm to find the greatest common divisor which may be regarded as the computation of *simple* continued fraction expansion of their ratio. In India, Aryabhata in his work entitled *Aryabhatiya* describes a general method to determine solutions of linear equations in two variables which includes computations related to continued fractions, as well. Coming to the 13th century, the famous book *Liber Abaci* (or *The Book of Calculations*) by *Fibonacci* contains versions of continued fractions.

A more systematic treatment of the theory of continued fractions can be found in the book entitled *L'Algebra Opera* by Bombelli in 16th century which includes the continued fraction expansion of $\sqrt{13}$. The theory has been developed rapidly in 18th and 19th centuries by *Euler*, *Lambert* and *Lagrange*. Many cornerstone results that lead up to more interesting research in the theory have been obtained during this time, such as the Euler-Lagrange Theorem, which states that an irrational number has a periodic continued fraction if and only if it is a quadratic irrational. The theory then expanded in many distinct directions. As is the case with problems related to analysis, the study of continued fractions is treated in two cases : real and complex. In either case, continued fractions are treated mostly from an analysis and/or dynamical systems viewpoint.

The involvement of fractional linear transformations in such a theory is most of the

time overlooked - even though quite apparent. In the case of classical *simple* continued fractions :

$$[a_0; a_1, a_2, \dots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}};$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}_+$, the set of fractional linear transformation that one needs to consider is the modular group $\mathrm{PSL}(2, \mathbb{Z})$. This allows one to express the situation in many different languages. For instance, one can use tools from hyperbolic geometry (e.g. geodesics), or cell-complexes to prove non-trivial theorems. In particular, the group $\mathrm{PSL}(2, \mathbb{Z})$ being isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ implies that it acts freely on the bipartite Farey tree whose topological realization can be obtained (as a one dimensional cell-complex) by considering the $\mathrm{PSL}(2, \mathbb{Z})$ orbit of the geodesic in the upper half plane ($\mathbb{H}$) joining $\sqrt{-1}$ to $\frac{1+\sqrt{-3}}{2}$. In this context, simple continued fractions can also be expressed as paths (i.e. connected one dimensional sub-complex) on the Farey tree.

A continued fraction of the above form $[a_0; a_1, a_2, \dots]$; where $a_i \in \mathbb{Z}[\sqrt{-1}]$ is called a Hurwitz continued fraction. In this thesis, we take on the task of expressing these continued fractions using fractional linear transformations and cell-complexes. More precisely, we describe a subgroup of $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-1}])$ which plays the role of the modular group for Hurwitz continued fractions. Moreover, we propose a two dimensional cell-complex on which the proposed group acts. As in the case of simple continued fractions, 2- cells of this complex are geodesic surfaces in the hyperbolic 3-space. Parallel to the classical case, we express Hurwitz continued fractions as a connected sequence of such cells. In this case, one obtains a sequence that looks like a *quilt*, which we call as a Hurwitz quilt.

The thesis is organized as follows : in the next chapter, we study the theory of complex continued fractions associated with the rings of integers of the imaginary quadratic number fields $\mathbb{Q}(\sqrt{-d})$ where $d = 1, 3, 7, 11$. In this section, it is used nearest integer algorithm defined by A. Hurwitz. So it is sometimes referred to as Hurwitz continued fraction. It is, in fact, a generalization of real continued fraction algorithm. To gain a general idea, we fixed the number field $\mathbb{Q}(i)$ and focus on the continued fraction whose entries lie in the ring of integers $\mathbb{Z}[i]$. After that we presented a geometric interpretation of Gaussian integer continued fractions as in the case real continued fraction. The theory can be immediately generalized to hold over the remaining norm-Euclidean ring of integers.

2 LITERATURE REVIEW

Complex continued fractions have been considered by many authors N. Michelangeli, Adolf Hurwitz, Julius Hurwitz in late 19th century. The topic have been kept its popularity during 20th and moreover still keep interest of many researchers at the beginning of 21th century. An important contribution of Adolf Hurwitz concerned the continued fractions associated with the ring of Gaussian integers using the nearest Gaussian integer (we refer to it as Hurwitz continued fraction). In 1888, Adolf Hurwitz showed, in his article (Hurwitz, 1888), the non-trivial fact that the sequence of moduli of the denominators of convergents is strictly increasing, which implies that such continued fractions is the limit of its convergents. R. Lakein (Lakein, 1973) studied complex continued fractions associated with the rings of integers of the imaginary quadratic number fields $\mathbb{Q}(\sqrt{-d})$ where $d = 1, 3, 7, 11$. About at the same time, in 1975, A. Schmidt proposed an algorithm based on the concept of Farey sets very different from the A. Hurwitz continued fraction algorithm, see (Schmidt, 1984). In 2014, S. G. Dani and A. Nogueira, (Dani and Nogueira, 2014), proposed a general approach to complex continued fractions associated with the ring of Gaussian integers. Their approach has become very popular and been taken up by many authors. Also, to see a theoretical aspects of Hurwitz's algorithm, one may see the PHD thesis (Robert, 2018) which is an almost complete overview of known results as well as many interesting new results.

3 BINARY QUADRATIC FORMS

A binary quadratic form is a homogeneous polynomial of degree 2 in two variables

$$f(x, y) = ax^2 + bxy + cy^2$$

where a, b, c are real numbers. We denote such a binary quadratic form by $f = (a, b, c)$ or in the matrix form $M_f = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ so that $f(x, y) = (x \ y)M_f(x \ y)^t$. A binary quadratic form with integer coefficients is called integral. We will often call such objects binary quadratic form or just *form* throughout this thesis. If f is integral and $\gcd(a, b, c) = 1$ then f is called *primitive*. The discriminant of a binary quadratic form is defined as $\Delta(f) = b^2 - 4ac$. Also, it can be immediately seen that

$$\Delta(f) = b^2 - 4ac = -4(ac - \frac{b^2}{4}) = -4 \det M_f$$

which gives a nice relation between discriminant and determinant of its matrix. This will be useful in calculations.

Proposition 3.1. *The set of discriminants of forms is the set of integers d such that $d \equiv 0, 1 \pmod{4}$.*

Proof If d is the discriminant of a form (a, b, c) , then $d = b^2 - 4ac \equiv b^2 \pmod{4}$, since the quadratic residues modulo 4 are 0 and 1. Conversely, if $d \equiv 0, 1 \pmod{4}$, consider the form

$$\begin{cases} (1, 0, -d/4) & \text{if } d \equiv 0 \pmod{4} \\ (1, 1, (1-d)/4) & \text{if } d \equiv 1 \pmod{4} \end{cases}$$

This form is easily seen to be of discriminant d . □

The form $(1, 0, -d/4)$ for $d \equiv 0 \pmod{4}$ and $(1, 1, (1-d)/4)$ for $d \equiv 1 \pmod{4}$ is called the *principal form* of discriminant d . We say a quadratic form $f = (x, y)$ *represents* an integer n if there exist integers x and y with $f(x, y) = n$. We say that f *properly represents* n if there exist relatively prime integers x and y such that $f(x, y) = n$.

Definition 3.1. *Let $f = (a, b, c)$ be a binary quadratic form. f is called :*

- *positive definite* if $f(x, y) > 0$ for every pair $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$
- *negative definite* if $f(x, y) < 0$ for every pair $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$
- *indefinite* if f is neither positive definite nor negative definite.

A binary quadratic form is *non-degenerate* if $\Delta(f)$ is not a perfect square and we mainly restrict our attention to non-degenerate forms throughout this work.

Example 3.1. *The form $f = (1, 0, 2)$ is a positive definite form. So if we change its sign we get another form $(-1, 0, -2)$ which is negative definite. Also $(1, 0, -2)$ is an indefinite quadratic form.*

Although it is very clear to determine the type of forms in our first examples, this is generally not easy to determine the type of forms at first glance. Fortunately, next proposition gives an explicit characterization by using coefficients of forms.

Proposition 3.2. *Let f be a binary quadratic form. Then*

- *f is positive definite if and only if $\Delta(f) < 0$ and $a > 0$*
- *f is negative definite if and only if $\Delta(f) < 0$ and $a < 0$*
- *f is indefinite if and only if $\Delta(f) > 0$*

Proof Let $f = (a, b, c)$. If $a = c = 0$, then $\Delta(f) = b^2$ and the result is obvious. So we can suppose, by symmetry, that $a \neq 0$. Let us note that, by completing the square,

$$4a(ax^2 + bxy + cy^2) = (2ax + by)^2 - \Delta y^2$$

This expression is always positive if $\Delta < 0$, so f takes only positive or negative values according to the sign of a . On the other hand, if $\Delta > 0$, the right hand side expression represents positive (e.g. take $(x, y) = (1, 0)$) and negative integers (e.g. take $(x, y) = (-b, -2a)$), thus the same holds for f □

3.1 Action of the modular group on binary quadratic forms

Consider the quadratic form $f = (2, 2, 1)$ with discriminant $\Delta = -4$. The identity $f(x, y) = 2x^2 + 2xy + y^2 = (x + y)^2 + y^2$ implies that $f = (2, 2, 1)$ and $f' = (1, 0, 1)$ represent exactly the same integers. In fact, we have $f(x, y) = f'(x + y, y)$ and $f'(x, y) = f(x, y - x)$. Forms that can be transformed into each other by transformations similar to the one above will be called equivalent. More precisely, we define the special linear group over integers as :

$$\text{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mid p, q, r, s \in \mathbb{Z}, ps - qr = +1 \right\}$$

If $f(x, y) = ax^2 + bxy + cy^2$ is an integral binary quadratic form and $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ we define

$$(A \cdot f)(x, y) := f((x, y)A) = f(px + ry, qx + sy)$$

The equation $A \cdot (B \cdot f) = AB \cdot f$ can be checked by brute force. Equivalently, we may think about the action in terms of the matrix representation M_f of f :

$$(A \cdot f)(x, y) = \left[A \begin{pmatrix} x \\ y \end{pmatrix} \right]^t M_f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^t A^t M_f A \begin{pmatrix} x \\ y \end{pmatrix}$$

Remark that the given action is not faithful since $\mp A$ matrices acts on the forms exactly the same way. So the modular group $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm I\}$ acts on the set of quadratic forms faithfully.

Definition 3.2. Two forms f and g are called equivalent if they are in the same orbit, i.e. there exists $U \in \text{SL}(2, \mathbb{Z})$ such that $g = f \cdot U$. The $\text{SL}(2, \mathbb{Z})$ -orbit of a form f is called the equivalence class of f and it is denoted by $[f]$.

3.2 Invariants of equivalence classes of forms

An *invariant* of an equivalence class of forms is a quantity that is the same for all forms in that equivalence class. We will show in this section that the minimum, the content and the discriminant of a form are invariants of its equivalence class.

First we show that equivalent forms represent the same numbers.

Proposition 3.3. Let $A \in \text{SL}(2, \mathbb{Z})$ and let two forms f and g be equivalent. Then f and g represent the same sets of integers.

Proof Let g be a form such that $g = Af$ for some $A \in \text{SL}(2, \mathbb{Z})$. If an integer n is represented by f , there exist $x, y \in \mathbb{Z}$ such that $f(x, y) = n$ and we see that $(x, y)A^{-1} \in \mathbb{Z}^2$ represents n by Af . Conversely, if an integer n represented by Af with $(x, y) \in \mathbb{Z}^2$, then $(x, y)A \in \mathbb{Z}^2$ represents n by f . $\square$

Next, we show that the discriminant is an invariant of the equivalence classes of forms.

Proposition 3.4. The discriminant is an invariant.

Proof Let $U \in \text{SL}(2, \mathbb{Z})$ and $g = fU$. Then

$$\Delta_{fU} = -4 \det M_{fU} = -4 \det U^t M_f U = -4 \det M_f = \Delta_f$$

since $\det U = \det U^t = 1$ □

It follows that all forms in the equivalence class of an indefinite form are indefinite and that all forms in the equivalence class of a positive (resp. negative) definite form are positive (resp. negative) definite.

3.3 Automorphisms of Forms

Let $f = (a, b, c)$ be any integral form of discriminant Δ . The stabilizer of f under the action $\text{SL}(2, \mathbb{Z})$ is called *the automorphism group* of f and denoted by $\text{Aut}(f)$. In other words, $W \in \text{Aut}(f)$ if and only if $Wf = f$ if and only if $W^t M_f W = M_f$.

We will shortly see that automorphisms play an important role in the questions concerning representation of integers by forms. We will furthermore answer the following questions : does a given form have infinitely many automorphisms? Otherwise how many? Can they be parametrized?

Example 3.2. Any form has the two trivial automorphisms

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

Proposition 3.5. If f and g are two equivalent forms, i.e. $f = \sigma g$ with

$$\sigma \in \text{SL}(2, \mathbb{Z}), \text{ then } \text{Aut}(f) = \sigma \text{Aut}(g) \sigma^{-1}$$

Proof An element $\tau \in \text{SL}(2, \mathbb{Z})$ belongs to $\text{Aut}(f)$ if and only if $\tau f = f$, i.e. if and only if $\tau \sigma g = \sigma g$, which is finally equivalent to $\sigma^{-1} \tau \sigma \in \text{Aut}(g)$. The result follows by symmetry. □

There is a connection between the form f and corresponding Pell equation $x^2 - \Delta y^2 = 4$ to construct automorphism group $\text{Aut}(f)$ for the form f .

Proposition 3.6. Let $f = (a, b, c)$ be any primitive integral form of discriminant Δ . Let $(x, y) \in \mathbb{Z}^2$ be a solution of the Pell equation $x^2 - \Delta y^2 = 4$. Then there exists a bijection between the set of those solutions and automorphism group of f . Moreover,

$$\text{Aut}(f) = \left\{ \pm \begin{pmatrix} \frac{u-vb}{2} & -cv \\ av & \frac{u+vb}{2} \end{pmatrix} : u, v \in \mathbb{R}, u^2 - \Delta v^2 = 4 \right\}$$

Proof Let us denote such a matrix $\begin{pmatrix} \frac{u-vb}{2} & -cv \\ av & \frac{u+vb}{2} \end{pmatrix}$ by $W(f, u, v)$ for any solution (u, v) . First we need to show that $W(f, u, v) \in \text{SL}(2, \mathbb{Z})$. Indeed,

$$\det(W(f, u, v)) = \frac{u^2 - v^2 b^2}{4} + acv^2 = \frac{u^2 - \Delta v^2}{4} = 1$$

since (u, v) satisfy the Pell equation. After some algebraic computation, it can also be seen that

$$W(f, u, v)^t M_f W(f, u, v) = M_f,$$

which shows that $W(f, u, v)$ is an automorphism of f for any solution (u, v) .

Now consider the map sending a solution (u, v) of the Pell equation $x^2 - \Delta y^2 = 4$ to $W(f, u, v)$. Let us see that this map is injective. Let $W(f, u, v) = W(f, u', v')$ for any solution $(u, v), (u', v')$. Equalizing corresponding entries, we get

$$av = av' \text{ and } u - u' = (v - v')b \quad (3.1)$$

In the case that $a = 0$, Δ becomes b^2 . Then $(u, v), (u', v')$ satisfy the Pell equation $x^2 - (by)^2 = 4$. But this has only two solutions $(\mp 2, 0)$ because b has to be non zero (otherwise f cannot be primitive). This means that $v = v'$ which also implies $u = u'$ by (3.1). If $a \neq 0$ then we get again $v = v'$ and $u = u'$ by (3.1). As a result, the given map is injective indeed. Let us now prove surjectivity. Let $W = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be an automorphism of f . Then $M_f W = (W^t)^{-1} M_f$. So we have the following equation :

$$\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} s & -r \\ -q & p \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

After matrix multiplication and equalizing the corresponding entries, we get the following equations :

$$aq = -cr \text{ and } br = a(s - p) \text{ and } bq = c(p - s) \quad (3.2)$$

Now let assume that $a \neq 0$. In this case a divide r because $\gcd(a, b, c) = 1$. Then after setting $u = p + s$ and $v = r/a$, we see that $W(f, u, v) = W$. Also, u and v satisfy the

Pell equation because $u^2 - \Delta v^2 = 4 \det W = 4$. If we assume $a = 0$ then equalities in (3.2) becomes

$$\begin{cases} -cr = 0 \\ br = 0 \\ bq = c(p - s) \end{cases}$$

Note that r have to be zero. Because otherwise we get $b = c = 0$ which makes the form f zero. Besides, we know that $ps = 1$ since $\det W = 1$. Hence p and s are the same integers : $p = s = 1$ or $p = s = 0$. This implies $bq = 0$. Hence we get $q = 0$. All these shows that in the case $a = 0$, an arbitrary automorphism W of f have to be I or $-I$ which can be written as $W(f, u, v)$ with $(u, v) = (\mp 2, 0)$ and this proves the result since $(2, 0)$ and $(-2, 0)$ are solutions of the Pell equation $x^2 - \Delta y^2 = 4$.

□

4 REAL CONTINUED FRACTIONS

4.1 The continued fraction expansion of a real number

For any given real number x , the *floor* $\lfloor x \rfloor \in \mathbb{Z}$ of x is defined to be the greatest integer so that $\lfloor x \rfloor \leq x$. The remainder part of x is defined as the difference $x - \lfloor x \rfloor$ and denoted by $\{x\}$. Hence we have the following :

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, and
- $0 \leq \{x\} < 1$

This is the first step in the process of describing x as a continued fraction. It is a recursive process. Given x , one starts with a unique decomposing x as

$$x = \lfloor x \rfloor + \{x\},$$

where $a_0 = \lfloor x \rfloor$ is an integer and $t_0 = \{x\}$ with $0 \leq t_0 < 1$.

If $t_0 = 0$ if and only if x is an integer. In this case the recursive process terminates at this first step. So assume $t_0 \neq 0$. Note that the reciprocal $1/t_0$ of t_0 satisfies $\frac{1}{t_0} > 1$ since $t_0 \in (0, 1)$. For the second step, one applies the same decomposition to $\frac{1}{t_0}$. One writes

$$\frac{1}{t_0} = a_1 + t_1,$$

where $a_1 = \lfloor \frac{1}{t_0} \rfloor$ is an integer and $t_1 = \{\frac{1}{t_0}\}$ with $0 \leq t_1 < 1$. Combining the equations that represent the first two steps, one may write

$$x = a_0 + \frac{1}{a_1 + t_1}.$$

Either $t_1 = 0$, in which case the process ends with the expansion

$$x = a_0 + \frac{1}{a_1},$$

or $t_1 > 0$. In the latter case one does to t_1 what had just been done to t_0 above under the assumption $t_0 > 0$. One writes

$$\frac{1}{t_1} = a_2 + t_2,$$

where a_2 is an integer and $0 \leq t_2 < 1$. Then combining the equations that represent the first three steps, one may write

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + t_2}}$$

After k steps, one has a sequence of integers $a_0, a_1, \dots, a_{k-1}$ together with real numbers $t_0, t_1, \dots, t_{k-1}$ that are members of $(0, 1)$ if the process has not terminated. Hence it is written as

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_{k-1} + t_{k-1}}}}}$$

or it may be written as

$$x = [a_0, a_1, a_2, \dots, a_{k-1} + t_{k-1}].$$

If $t_{k-1} = 0$, the process ends after k steps. Or else, the process continues at least one more step with

$$\frac{1}{t_{k-1}} = a_k + t_k.$$

Applying this process, one associates with any real number x a sequence, which may be either finite or infinite, $a_0, a_1, a_2, \dots$ of integers. This sequence is called the *simple continued fraction expansion* of x . For brevity, we call just continued fraction for these expressions throughout the thesis. The integers are called the *entries of continued fraction*.

Remark 4.1. If $[a_0, a_1, \dots]$ represents a real number, it follows that all a_j 's have to be positive integers except for a_0 . If the continued fraction ends up with a_n , to get a unique expansion, it is assumed $a_n > 1$.

Briefly, the process of finding the continued fraction expansion of a real number x as follows : After decomposition

$$x = [x] + \{x\} = a_0 + t_0,$$

we define

$$T : [0, 1) \rightarrow [0, 1) \text{ by } t \mapsto \{1/t\} \text{ and } T(0) = 0$$

and if applying T to t_0 we get two sequences

$$t_{n+1} = \{1/t_n\} = T^{n+1}(t_0) \text{ and } a_{n+1} = [1/t_n] = \left\lfloor \frac{1}{T^n(t_0)} \right\rfloor$$

then it gives the continued fraction expansion

$$x = [a_0, a_1, a_2, \dots].$$

If $t_n = 0$ for some n , then we stop generating new terms, which makes the expansion finite, or else it will be infinite.

Example 4.1.

$$\begin{aligned}
 \sqrt{10} &= 3 + \frac{1}{\sqrt{10} - 3} \\
 &= 3 + \frac{1}{\sqrt{10} + 3} \\
 &= 3 + \frac{1}{6 + \frac{1}{\frac{1}{\sqrt{10} - 3}}} \\
 &= 3 + \frac{1}{6 + \frac{1}{\sqrt{10} + 3}} \\
 &= 3 + \frac{1}{6 + \frac{1}{6 + \dots}} \\
 &= [3, 6, 6, 6, \dots].
 \end{aligned}$$

Now let us compute what exactly the following number is :

$$x = -2 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2 + \dots}}}}}$$

So one finds that $x = -2 + \frac{1}{y}$, where

$$y = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2 + \dots}}}}$$

As y has self-similar structure :

$$y = 2 + \frac{1}{3 + y}$$

This simplifies to

$$y = \frac{7y + 2}{3y + 1}$$

and give the following quadratic equation

$$3y^2 - 6y - 2 = 0$$

As $y > 2$ and satisfy above quadratic equation,

$$y = \frac{3 + \sqrt{15}}{3}.$$

As a result,

$$x = \frac{\sqrt{15} - 7}{2}.$$

The idea of the calculation above leads to the conclusion that any continued fraction $[n_1, n_2, \dots]$ that eventually periodic is the solution of a quadratic equation with positive discriminant and integer coefficients. The converse of this statement is also true, but a proof requires further consideration.

4.2 The case of a rational number

Given any rational number $x = \frac{a}{b}$, the procedure of computing its continued fraction expansion is essentially identical to applying the Euclidean algorithm to the pair of a, b . Indeed, decomposing x into its integral and remainder parts, we see that

$$\frac{a}{b} = n_0 + t_0 \quad \text{and} \quad t_0 = \frac{a - n_0 b}{b}$$

implies $t_0 = \frac{r_0}{b}$, where r_0 is the remainder for division of a by b . The case where $t_0 = 0$ which happens the case where x is an integer. Or else $t_0 > 0$, and we have the decomposition of $\frac{1}{t_0}$ which gives

$$\frac{b}{r_0} = n_1 + t_1, \quad t_1 = \frac{b - n_1 r_0}{r_0}$$

This implies that $t_1 = \frac{r_1}{r_0}$, where r_1 is the remainder for division of b by r_0 . Hence, the sequence of quotients in Euclid's algorithm are the integers $n_0, n_1, \dots$ occurring in the continued fraction. The algorithm stops after a finite number of steps if the remainder equals zero. Hence, the continued fraction expansion of every rational number is finite.

Theorem 4.1. *Let $x \in \mathbb{R}$. The continued fraction expansion of x number is finite if and only if it is rational.*

Proof We have just already shown that if $x \in \mathbb{Q}$ then the continued fraction expansion of x is finite because it is computed by applying the Euclidean algorithm to the numerator and denominator of x . The converse statement is obvious. $\square$

Definition 4.1. *For any $n \in \mathbb{N}$, the n -th convergent of the (finite or infinite) continued fraction $[a_0, a_1, \dots]$ is $[a_0, \dots, a_n]$. For each n with $-2 \leq n$, define integers p_n and q_n as follows :*

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_0 = a_0, \quad \dots \quad p_n = a_n p_{n-1} + p_{n-2} \quad \dots$$

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_0 = 1, \quad \dots \quad q_n = a_n q_{n-1} + q_{n-2} \quad \dots$$

In terms of matrices, we can restate the relations as

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix}$$

Proposition 4.1. *If a_i, p_i, q_i are defined in the definition above then*

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

holds for all $n \in \mathbb{N}$.

Proof We apply induction over n . For the base case, it is true as

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{pmatrix}$$

holds by definition. Now suppose the proposition holds for $n - 1$. Then we may write

$$\begin{aligned} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_n p_{n-1} + p_{n-2} & p_{n-1} \\ a_n q_{n-1} + q_{n-2} & q_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}, \end{aligned}$$

which proves our proposition. □

Proposition 4.2. *For $n \in \mathbb{N}$, we have*

$$[a_0, \dots, a_n] = \frac{p_n}{q_n}.$$

Proof We apply induction for the assertion. It is obvious when $n = 0, 1$. Suppose

$$\begin{aligned} [a_0, \dots, a_{n-2}, a_{n-1}] &= \frac{p_{n-1}}{q_{n-1}} \\ &= \frac{a_{n-1} p_{n-2} + p_{n-3}}{a_{n-1} q_{n-2} + q_{n-3}} \end{aligned}$$

then it follows that

$$\begin{aligned}
[a_0, \dots, a_n] &= [a_0, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}] \\
&= \frac{(a_{n-1} + \frac{1}{a_n})p_{n-2} + p_{n-3}}{(a_{n-1} + \frac{1}{a_n})q_{n-2} + q_{n-3}} \\
&= \frac{(a_{n-1}a_n + 1)p_{n-2} + a_n p_{n-3}}{(a_{n-1}a_n + 1)q_{n-2} + a_n q_{n-3}} \\
&= \frac{a_n(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1}q_{n-2} + q_{n-3}) + q_{n-2}} \\
&= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \\
&= \frac{p_n}{q_n}.
\end{aligned}$$

□

Proposition 4.3. *For $n \in \mathbb{N}$, we have*

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1} \text{ and } p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n \quad (4.1)$$

Equivalently,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \frac{1}{q_n q_{n-1}} \text{ and } \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-1)^n \frac{a_n}{q_n q_{n-2}} \quad (4.2)$$

Proof The case for $n = 0$ is obvious from the definitions. Now suppose $n > 0$ and the statement is true for $n - 1$. Then

$$\begin{aligned}
p_n q_{n-1} - q_n p_{n-1} &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - (a_n q_{n-1} + q_{n-2}) p_{n-1} \\
&= p_{n-2} q_{n-1} - q_{n-2} p_{n-1} \\
&= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \\
&= -(-1)^{n-2} = (-1)^{n-1}.
\end{aligned}$$

This completes the proof of (4.1). For (4.2) we have

$$\begin{aligned}
 p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_n q_{n-1} + q_{n-2}) \\
 &= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \\
 &= (-1)^n a_n
 \end{aligned}$$

□

Remark 4.2. *If we express the proposition in terms of matrices, it asserts that the determinant of $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ is $(-1)^{n-1}$, and of $\begin{pmatrix} p_n & p_{n-2} \\ q_n & q_{n-2} \end{pmatrix}$ is $(-1)^n a_n$. So this implies the following corollary.*

Corollary 4.1. *If $[a_0, a_1, \dots, a_m]$ is a continued fraction, so each a_i is an integer, then the p_n and q_n are integers and they are coprime.*

Proof It is clear that the p_n and q_n are integers since a_n 's are integers. If $d > 0$ divides both p_n and q_n , then $d \mid (-1)^{n-1}$, so $d = 1$. □

4.3 The sequence of convergents

Let $[a_0, a_1, \dots]$ be a (finite or infinite) continued fraction, and for $n \in \mathbb{N}$, let

$$c_n = [a_0, \dots, a_n] = \frac{p_n}{q_n}$$

denote the n the convergent. We know that $a_n > 0$ for $n > 0$ by definition of continued fraction. This gives the convergents of a continued fraction an additional structure. For instance, the convergents of $[2, 2, 1, 1, 5, 1, 2, 1, 1, 2]$ are

$$2, 5/2, 7/3, 12/5, 67/28, 79/33, 225/94, 304/127, 529/221, 1362/569$$

Let us approximate them using decimals to make the size of these numbers clearer, We also underline odd indexed convergents, to see some extra structure.

$$2, \underline{2.5}, 2.33, \underline{2.4}, 2.392, \underline{2.3939}, 2.39361, \underline{2.3937}, 2.39366, \underline{2.39367}$$

Note that the underlined numbers are smaller than all of the non-underlined numbers. Moreover, the sequence of underlined numbers is strictly increasing, whereas the non-underlined numbers strictly decrease.

Proposition 4.4. *(How convergents converge) The sequence of even indexed convergents c_{2n} strictly increases, and the sequence of odd indexed convergents c_{2n+1} strictly decreases. Also, $c_{2n+1} > c_{2n}$ for all $n \in \mathbb{N}$.*

Proof The a_n are positive for $n \geq 1$, so the q_n are positive. By Proposition 4.3, for $n \geq 2$,

$$c_n - c_{n-2} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}},$$

which proves the first claim.

Suppose for the sake of contradiction that there exist integers r, m such that $c_{2m+1} < c_{2r}$. Proposition 4.3 implies that for $n \geq 1$,

$$c_n - c_{n-1} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}}$$

has sign $(-1)^{n-1}$, so for all $s \geq 0$ we have $c_{2s+1} > c_{2s}$. Thus it is impossible that $r = m$. If $r < m$, then by what we proved in the first paragraph, $c_{2m+1} < c_{2r} < c_{2m}$, a contradiction (with $s = m$). If $r > m$, then $c_{2r+1} < c_{2m+1} < c_{2r}$, which is also a contradiction (with $s = r$). $\square$

Proposition 4.5. *Every nonzero rational number can be represented by a simple continued fraction.*

Proof Without loss of generality we may assume that the rational number is a/b with $b \geq 1$ and $\gcd(a, b) = 1$. Euclid Algorithm gives :

$$a = b \cdot a_0 + r_1, \quad 0 < r_1 < b$$

$$b = r_1 \cdot a_1 + r_2, \quad 0 < r_2 < r_1$$

...

$$r_{n-2} = r_{n-1} \cdot a_{n-1} + r_n, \quad 0 < r_n < r_{n-1}$$

$$r_{n-1} = r_n \cdot a_n + 0$$

Note that $a_i > 0$ for $i > 0$ (also $r_n = 1$ since $\gcd(a, b) = 1$). Rewrite the equations as

follows :

$$\begin{aligned} a/b &= a_0 + r_1/b = a_0 + 1/(b/r_1) \\ b/r_1 &= a_1 + r_2/r_1 = a_1 + 1/(r_1/r_2) \\ r_1/r_2 &= a_2 + r_3/r_2 = a_2 + 1/(r_2/r_3) \\ &\vdots \\ r_{n-1}/r_n &= a_n \end{aligned}$$

It follows that

$$\frac{a}{b} = [a_0, a_1, \dots, a_n]$$

□

Lemma 4.1. *The sequence of denominators of convergents satisfies $n \leq q_n$ for all n .*

Proof $n \leq q_n$ holds for $n = 0, 1$. Also we see that

$$q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1} + 1 \geq (n-1) + 1 = n,$$

since $a_n, q_n \geq 1$ for $n > 1$. The last inequality follows from induction hypothesis. □

Remark 4.3. *It is seen that q_n forms an increasing sequence of natural numbers.*

Theorem 4.2. *Suppose that $(a_n)_{n \in \mathbb{N}}$ is an infinite sequence of integers such that $a_n > 0$ for all $n \geq 1$. If $c_n = [a_0, a_1, \dots, a_n]$ for each $n \geq 0$ then $\lim_{n \rightarrow \infty} c_n$ exists.*

Proof We know that the even convergents c_{2n} form a strictly increasing sequence and the odd convergents c_{2n+1} form a strictly decreasing sequence by Proposition 4.4. Besides, for all $n \in \mathbb{N}$, we have $c_{2n} \leq c_1$ and $c_{2n+1} \geq c_0$. Hence $\alpha_0 = \lim_{n \rightarrow \infty} c_{2n}$ and $\alpha_1 = \lim_{n \rightarrow \infty} c_{2n+1}$ both exist and $\alpha_0 \leq \alpha_1$. Finally, by Proposition 4.3 and Lemma 4.3

$$|c_{2n} - c_{2n-1}| = \frac{1}{q_{2n} \cdot q_{2n-1}} \leq \frac{1}{2n(2n-1)} \rightarrow 0,$$

which implies $\alpha_0 = \alpha_1$. □

We define

$$[a_0, a_1, \dots] = \lim_{n \rightarrow \infty} c_n.$$

Theorem 4.3. *Let $x \in \mathbb{R}$. Then x is the value of some finite or infinite continued fraction $[a_0, a_1, a_2, \dots]$*

Proof We know that x is a rational number if and only if it is the value of a finite continued fraction. So suppose x is an irrational number and corresponding sequence is $[a_0, a_1, \dots]$. We will show that $x = \lim_{n \rightarrow \infty} c_n$. Write

$$x = [a_0, a_1, \dots, a_n + t_n] = [a_0, a_1, \dots, a_n, \frac{1}{t_n}]$$

where $0 < t_n < 1$.

By Proposition 4.2 (note that terms of continued fraction do not need to be integers!), we have

$$x = \frac{\frac{1}{t_n} \cdot p_n + p_{n-1}}{\frac{1}{t_n} \cdot q_n + q_{n-1}}.$$

Thus if $c_n = [a_0, a_1, \dots, a_n]$, then

$$\begin{aligned} x - c_n &= x - \frac{p_n}{q_n} = \frac{\frac{1}{t_n} p_n q_n + p_{n-1} q_n - \frac{1}{t_n} p_n q_n - p_n q_{n-1}}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)} \\ &= \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)} \\ &= \frac{(-1)^n}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)}. \end{aligned}$$

Thus

$$|x - c_n| = \frac{1}{q_n \left(\frac{1}{t_n} q_n + q_{n-1} \right)} < \frac{1}{q_n (a_{n+1} q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} \leq \frac{1}{n(n+1)} \rightarrow 0.$$

Note that a_{n+1} is the integer part of $\frac{1}{t_n}$, and a_{n+1} is lesser than $\frac{1}{t_n}$. □

This corollary follows from the proof of the above theorem.

Corollary 4.2. *Let $x \in \mathbb{R}$ be any number with $x = [a_0, a_1, \dots]$. Then for all n ,*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n \cdot q_{n+1}}.$$

4.4 Group theoretical approach

The group $\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ is generated by two matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ having infinite order and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ having order 4.

Lemma 4.2. $\text{SL}_2(\mathbb{Z}) = \langle S, T \rangle$

Proof We will show that all elements of $\text{SL}_2(\mathbb{Z})$ can be generated by S and T . Observe that

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad (4.3)$$

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. We want to show that M is generated by S and T . First, we suppose $c = 0$. In this case, we get $ad = 1$. So $a = d = \pm 1$. It means

$$M = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = T^b \text{ or } S^2 T^b,$$

which shows the matrix M is in the desired form. So we study the case for $c \neq 0$. Without loss of generality, we may assume $|a| > |c|$ in view of the first observation (4.3).

Step 1. Applying the division algorithm for a and c , find the integers q and r such that $a = cq + r$ and $0 \leq r < |c|$.

Step 2. Multiply M by ST^{-q} as follows :

$$\begin{aligned} S(T^{-q}M) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a - cq & b - dq \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ r & b - qd \end{pmatrix}. \end{aligned}$$

If $r = 0$ then it means that we are in the first case. So we are done. Otherwise, going to the step 1 and apply the division algorithm for the integers c and r . We continue

this iterative procedure. Note that we get lesser remainders whenever we apply step 1. So we produce a strictly decreasing sequence of remainders. Since the terms are non negative integers, the algorithm will stop after finitely many step. (when the last remainder equals zero.) This proves $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle$. $\square$

Now, consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane $\mathcal{H}_2$ defined by fractional linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \alpha = \frac{a\alpha + b}{c\alpha + d}.$$

Algebraic operations by ∞ are defined as follows :

$$g(\infty) = a/c, g(-d/c) = \infty \text{ if } c \neq 0$$

$$\infty + \infty = \infty, \infty \times \infty = \infty$$

and also

$$\text{if } \alpha \neq 0 \text{ then } \frac{\alpha}{0} = \infty \text{ and if } \alpha \neq \infty \text{ then } \frac{\alpha}{\infty} = 0.$$

Remark 4.4. $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$ and $0 \times \infty$ are left undefined.

Note that this action also leaves the boundary $\partial\mathcal{H}_2 = \mathbb{R} \cup \{\infty\}$ invariant and the corresponding group homomorphism

$$\phi : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{Sym}(\partial\mathcal{H}_2), \quad g \mapsto \tilde{g}(\alpha) = \frac{a\alpha + b}{c\alpha + d}$$

has $\ker(\phi) = \{\pm I\}$, which means that it is not faithful. To make it faithful, we always consider the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$, denoted by Γ . In this case, each element of Γ is characterized by how move the points in the real line and so we may view the elements of Γ as a fractional linear transformations which are some composition of

$$T : \alpha \mapsto \alpha + 1 \text{ and } S : \alpha \mapsto \frac{-1}{\alpha}$$

We now want to see that the elements of the modular group suffice to express a continued fraction expansion of a real number.

Let us define

$$L := TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

which corresponds to the transformation $\alpha \mapsto \frac{\alpha-1}{\alpha}$. As L is an element of order 3, and there is no any algebraic relation between L and S , it follows that

$$\Gamma = \langle L, S | L^3 = S^2 = 1 \rangle,$$

which is isomorphic to the free product of two cyclic groups of order 2 and 3.

For observation, let us consider a rational number $\alpha \in \mathbb{Q}$. We know that it has a finite continued fraction expansion, say :

$$\alpha = [a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\dots a_{n-1} + \frac{1}{a_n}}}$$

So we only need translations and multiplicative inverse to construct a continued fraction. Hence we have to consider the following transformations :

$$T^a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : \alpha \mapsto \alpha + a, \quad \text{for some } a \in \mathbb{Z}$$

$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \alpha \mapsto \frac{1}{\alpha} \text{ so that } U^2 = I.$$

Rewriting α in terms of the above transformations, we have

$$\alpha = T^{a_0} U T^{a_1} U \dots U T^{a_{n-1}} U T^{a_n} U(\infty) \quad (4.4)$$

At first glance, the matrix U may cause a problem since it does not lie in the modular group Γ (note $\det U \neq 1$). Nevertheless, we may handle that issue by using the following relations :

$$L^2 S = U T U \text{ and } (U T U)^a = U T^a U \quad (4.5)$$

Remark that second equality holds as U is an element of order 2. We now have two cases depending on the length of continued fraction.

Case-1 (EVEN) : If the length of the continued fraction is even, then we may write α

in terms of L and S by the equalities (4.5)

$$\begin{aligned}
\alpha &= T^{a_0}(UT^{a_1}U) \cdots UT^{a_{n-1}}(UT^{a_n}U)(\infty) \\
&= T^{a_0}(UTU)^{a_1} \cdots UT^{a_{n-1}}(UTU)^{a_n}(\infty) \\
&= (LS)^{a_0}(L^2S)^{a_1} \cdots (LS)^{a_{n-1}}(L^2S)^{a_n}(\infty)
\end{aligned}$$

Case-2 (ODD) : If the length of the continued fraction is odd, then we may write

$$\begin{aligned}
\alpha &= T^{a_0}(UT^{a_1}U) \cdots (UT^{a_{n-1}}U)T^{a_n}U(\infty) \\
&= T^{a_0}(UTU)^{a_1} \cdots (UTU)^{a_{n-1}}T^{a_n}U(\infty) \\
&= (LS)^{a_0}(L^2S)^{a_1} \cdots (L^2S)^{a_{n-1}}(LS)^{a_n}(0)
\end{aligned}$$

Corollary 4.3. *If $\alpha \in \mathbb{Q}$ and $\alpha = [a_0, a_1, \dots, a_n]$ then $\alpha = A(0)$ or $A(\infty)$ where*

$$A := (LS)^{a_0}(L^2S)^{a_1} \cdots (L^{(i)}S)^{a_n}$$

so that $i = 1$ or 2 if n is even or odd number respectively.

The matrix A defined in Corollary 4.3 is called the *matrix of a finite continued fraction*. If the finite continued fraction is n -th convergent of a real number then A is called the *matrix of n -th convergent*.

Suppose that α be an irrational number and $\alpha = [a_0, a_1, \dots]$ with sequence of convergents $(c_n)_{n \in \mathbb{N}}$. Let us consider the strictly decreasing subsequence of $(c_{2n+1})_n$ which still converges to α . Then each convergent c_{2n+1} has even number of coefficients in its continued fraction expansion, so we may write, for all $n \in \mathbb{N}$,

$$\begin{aligned}
c_{2n+1} &= [a_0, a_1, \dots, a_{2n+1}] \\
&= (LS)^{a_0}(L^2S)^{a_1} \cdots (L^2S)^{a_{2n+1}}(\infty) \\
&= A_{2n+1}(\infty)
\end{aligned}$$

Then we have

$$\alpha = \lim_{n \rightarrow \infty} A_{2n+1}(\infty).$$

Similary, the strictly increasing subsequence $(c_{2n})_n$ satisfies

$$c_{2n} = (LS)^{a_0}(L^2S)^{a_1} \dots (L^2S)^{a_{n-1}}(LS)^{a_{2n}}(0) = A_{2n}(0) \text{ and}$$

we see that

$$\alpha = \lim_{n \rightarrow \infty} A_{2n}(0).$$

Corollary 4.4. *Let α be an irrational number whose continued fraction expansion is $[a_0, a_1 \dots]$ and $(c_n)_n$ be the sequence of the convergents of α . If $A_n \in \Gamma$ be the matrix of c_n . Then*

$$\alpha = \lim_{n \rightarrow \infty} A_{2n}(0) = \lim_{n \rightarrow \infty} A_{2n+1}(\infty)$$

and explicitly $A_n = \pm \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ or $\pm \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$ if n is odd or even respectively.

Proof It is already discussed why those limits above hold. So we only show what A_n is exactly. Let $c_n = [a_0, a_1, \dots, a_n]$. By proposition (4.1), we have

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad (4.6)$$

Set $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_{a_k} = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$. After writing $M_{a_{2k}}UU$ in place of $M_{a_{2k}}$ for each k , we may rewrite (4.6) in two ways depending on n . Assuming n is even,

$$\begin{aligned} \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} &= (M_{a_0}U)(UM_{a_1})(M_{a_2}U)(UM_{a_3}) \dots (UM_{a_{n-1}})(M_{a_n}U)U \\ &= \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & 0 \\ a_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} U \\ &= (LS)^{a_0}(L^2S)^{a_1}(LS)^{a_2} \dots (L^2S)^{a_{n-1}}(LS)^{a_n}U \\ &= A_nU \end{aligned}$$

Multiplying both sides by U , we have $A_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \in \Gamma$. Now assuming n is odd,

we see that

$$\begin{aligned}
\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} &= (M_{a_0}U)(UM_{a_1})(M_{a_2}U)(UM_{a_3}) \cdots (M_{a_{n-1}}U)(UM_{a_n}) \\
&= \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_{n-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix} \\
&= (LS)^{a_0}(L^2S)^{a_1}(LS)^{a_2} \cdots (L^2S)^{a_{n-1}}(LS)^{a_n} \\
&= A_n,
\end{aligned}$$

which proves our proposition. $\square$

So, our discussion in this section make us feel that we need to explore more the modular group in order to understand continued fractions for a real number.

Example 4.2. Let $\alpha = -\frac{34}{19} = [-2, 4, 1, 3] = (LS)^{-2}(L^2S)^4(LS)^1(L^2S)^3(\infty)$. Then the matrices of convergents :

$$A_0 = (LS)^{-2} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \text{ and } c_0 = A_0(0) = -2,$$

$$A_1 = (LS)^{-2}(L^2S)^4 = \begin{pmatrix} -7 & -2 \\ 4 & 1 \end{pmatrix} \text{ and } c_1 = A_1(\infty) = -\frac{7}{4},$$

$$A_2 = (LS)^{-2}(L^2S)^4(LS)^1 = -\begin{pmatrix} -7 & -9 \\ 4 & 5 \end{pmatrix} \text{ and } c_2 = A_2(0) = -\frac{9}{5},$$

$$A_3 = (LS)^{-2}(L^2S)^4(LS)^1(L^2S)^3 = -\begin{pmatrix} -34 & -9 \\ 19 & 5 \end{pmatrix} \text{ and } c_3 = A_3(\infty) = -\frac{34}{19}.$$

4.5 Geometric interpretation of continued fractions

Let define the *hyperbolic metric*

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

on the upper half-plane $\mathcal{H}_2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. In this case, it is called *upper half plane model of the hyperbolic plane*. Any line here is called *geodesic* which is either a semi-circle orthogonal to the real axis or a vertical ray.

First, we now consider the action of

$$\text{PSL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} p & q \\ q & s \end{pmatrix} : p, q, r, s \in \mathbb{R} \text{ with } ps - qr = 1 \right\} / \{\pm I\}$$

on $\mathcal{H}_2$ defined by Möbius transformations

$$g \cdot \alpha = \frac{p\alpha + q}{r\alpha + s}$$

where $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$ and $\alpha \in \mathcal{H}_2$. Note that this action also leaves the boundary $\partial\mathcal{H}_2 = \mathbb{R} \cup \{\infty\}$ invariant. Algebraic operations by ∞ are defined as follows :

$$g(\infty) = p/r, g(-s/r) = \infty \text{ if } r \neq 0$$

$$\infty + \infty = \infty, \infty \times \infty = \infty$$

and also

$$\text{if } \alpha \neq 0 \text{ then } \frac{\alpha}{0} = \infty \text{ and if } \alpha \neq \infty \text{ then } \frac{\alpha}{\infty} = 0.$$

Note that $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$ and $0 \times \infty$ are left undefined. As the action is faithful, the elements of $\text{PSL}(2, \mathbb{R})$ are identified as Möbius transformations. Now we want to classify these elements by considering their fixed points.

4.5.1 Classification of elements of $\text{PSL}_2(\mathbb{R})$

Definition 4.2. Let $g \in \text{PSL}_2(\mathbb{R})$ be a non-identity element. Then

- i. g is hyperbolic if it has two fixed points in $\partial\mathcal{H}_2$ and none in $\mathcal{H}_2$;
- ii. g is parabolic if it has one fixed point in $\partial\mathcal{H}_2$ and none in $\mathcal{H}_2$;
- iii. g is elliptic if it has one fixed point in $\mathcal{H}_2$ and none in $\partial\mathcal{H}_2$.

Proposition 4.6. Let $g \in \text{PSL}_2(\mathbb{R})$ be a non-identity element. Then

- i. $\text{tr}(g)^2 > 4$ if and only if g is hyperbolic
- ii. $\text{tr}(g)^2 < 4$ if and only if g is elliptic
- iii. $\text{tr}(g)^2 = 4$ if and only if g is parabolic

Proof Let $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$ be any non-identity element. Assume $r \neq 0$. Then we get

$$\alpha \in \text{Fix}(g) \iff \frac{p\alpha + q}{r\alpha + s} = \alpha$$

$$\iff r\alpha^2 + (s - p)\alpha - q = 0$$

$$\iff \alpha_{1,2} = \frac{p - s \pm \sqrt{\text{tr}(g)^2 - 4}}{2r}$$

Considering the discriminant $\text{tr}^2 - 4$, i, ii and iii are verified. Now let us suppose $r = 0$. Then $p \neq 0$ and $s = 1/p$ since $\det(g) = 1$. If $s = p$ then we have $q \neq 0$ and $p = \pm 1$. Hence $g(\alpha) = \alpha$ if and only if $\alpha \pm q = \alpha$. So in this case, ∞ is the only fixed point and $\text{tr}^2(g) = 4$. So now assume $p \neq s$. In this case

$$g(\alpha) = p^2\alpha + pq \text{ with } \text{tr}^2(g) = p^2 + 1/p^2 + 2.$$

Note that $g(\alpha) = \alpha$ if and only if $\alpha = \frac{pq}{1-p^2}$. So if $p = \pm 1$ then ∞ is the unique fixed point of g with $\text{tr}^2(g) = 4$ which verifies (iii). If $p \neq \pm 1$ then g has only two fixed points $\frac{pq}{1-p^2}$ and ∞ on the boundary with $\text{tr}^2(g) > 4$, which verifies (i).

□

Proposition 4.7. *Let $g \in \text{PSL}_2(\mathbb{R})$ be a non-identity element. Then*

(i) *g is hyperbolic if and only if it is conjugate to $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ for some $p \in \mathbb{R}$.*

(ii) *g is parabolic if and only if it is conjugate to $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$ for some $q \in \mathbb{R}$.*

(iii) *g is elliptic if and only if it is conjugate to $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ for some $\theta \in (-\pi, \pi)$.*

Proof i. Suppose $g \in \text{PSL}_2(\mathbb{R})$ be a hyperbolic element with fixed points α and β . Let $h \in \text{PSL}_2(\mathbb{R})$ with $h(\alpha) = 0$ and $h(\beta) = \infty$. Then we have a hyperbolic element $T = hgh^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ having fixed points 0 and ∞ . $T(0) = 0$ implies $q = 0$. $T(\infty) = \infty$

implies $s = 0$. Finally, since $\det(T) = 1$, it follows that $T = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$. Converse of the assertion is also true since the number of fixed points is invariant under conjugation.

ii. Suppose g is a parabolic element having a unique fixed point $\alpha \in \partial\mathcal{H}_2$. Let $h \in \text{PSL}_2(\mathbb{R})$ be an element such that $h(\alpha) = \infty$. Then we get a parabolic element $T = hgh^{-1}$ having a unique fixed point at ∞ as the number of fixed points is invariant under conjugation. Note that if $T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ then r must be zero as $T(\infty) = \infty$. Hence $T(z) = \frac{pz+q}{s}$ and $T(z) = z$ holds if and only if $z = \frac{q}{s-p}$. As we know that T has no other fixed points, we conclude $p = s$. But as $\det(T) = 1$ we have $p = \pm 1$, which implies $T = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$.

iii. Suppose g is an elliptic element whose fixed point $\alpha \in \mathcal{H}_2$. Let h be an element with $h(\alpha) = i$. Then $T = hgh^{-1}$ fixes only $i \in \mathcal{H}_2$. If $T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ then $T(i) = i$ implies $T = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}$. Since $\det(T) = 1$, it follows that $p^2 + q^2 = 1$, which proves the assertion. $\square$

Remark 4.5. *As an elliptic element corresponds a rotation about its fixed point in $\mathcal{H}_2$, it is determined by its fixed point, and its angle of rotation. Notice that angle of rotation is invariant under conjugation by the elements of $\text{PSL}(2, \mathbb{R})$. Given any elliptic element M , then we have $\text{tr}(M)^2 = (2\cos\theta)^2$ so $\theta = \pm \cos^{-1}(\frac{|\text{tr}(M)|}{2})$.*

An hyperbolic element is conjugate to $z \mapsto kz$ for some $k > 0$, having fixed points 0 and ∞ . So it can be thought of as a translation along the geodesic joining the two fixed points in $\partial\mathcal{H}_2$.

4.5.2 Farey Graph

Let us now consider modular group $\Gamma = \text{PSL}_2(\mathbb{Z})$ which is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$. Then Γ still acts faithfully on the upper half plane $\mathcal{H}_2$ by Möbius transformation. Recall that $\Gamma = \langle S, L \mid S^2 = L^3 = I \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. So every element of Γ can be expressed as a word in terms of three letters S, L and L^2 . Note that the generators S and L of Γ are elliptic elements whose unique fixed points in $\mathcal{H}_2$ are $\zeta_4 = i$ and $\zeta_6 = e^{i\frac{\pi}{3}}$ respectively. So geometrically, the action of S corresponds to a rotation around its fixed point i through π while L rotating the points around ζ through $\frac{2\pi}{3}$.

We now start with the arc γ on the unit circle joining the points ζ_4 and ζ_6 in $\mathcal{H}_2$, that

is exactly a segment of the geodesic joining these points. The images of γ under the elements of modular group Γ give rise to a graph $\mathcal{F}$ (that is called *Farey graph*) whose vertices consists of disjoint union of two Γ -orbits of ζ_4 and ζ_6 ,

$$V(\mathcal{F}) = \{A(\zeta_4) : A \in \Gamma\} \sqcup \{A(\zeta_6) : A \in \Gamma\},$$

and the set of edges is

$$E(\mathcal{F}) = \{\{A(\zeta_4), A(\zeta_6)\} : A \in \Gamma\}.$$

For any $A \in \Gamma$, notice that $A \cdot \gamma$ is still a circular arc or a line segment joining the points $A(\zeta_4)$ and $A(\zeta_6)$ as any Möbius transformation maps a geodesic (semicircle/vertical ray) to a geodesic. In fact, $A \cdot \gamma$ is a concrete realization of the edge $\{A(\zeta_4), A(\zeta_6)\}$ in the plane $\mathcal{H}_2$, and it is uniquely determined by its endpoints $A(\zeta_4)$ and $A(\zeta_6)$. Hence we may write

$$A \cdot \gamma = \{A(\zeta_4), A(\zeta_6)\}.$$

As S and L are elements of order 2 and 3 respectively which rotate the points around their fixed points, it follows that vertices in the orbit of ζ_4 have degree of 2 and vertices in the orbit of ζ_6 have degree of 3. To distinguish this fact, vertices in $\Gamma \cdot \zeta_4$ are marked with $\bullet$, and those in $\Gamma \cdot \zeta_6$ are marked with $\circ$.

Notice that, for each $A \in \Gamma$, we get distinct edge $\{A(\zeta_4), A(\zeta_6)\}$ which implies the following proposition.

Proposition 4.8. *There exists one-to-one correspondence between the edges of Farey graph and the elements of modular group by*

$$A \leftrightarrow \{A(\zeta_4), A(\zeta_6)\}.$$

Proof For any $A, B \in \Gamma$, we have

$$\{A(\zeta_4), A(\zeta_6)\} = \{B(\zeta_4), B(\zeta_6)\} \iff A(\zeta_4) = B(\zeta_4) \text{ and } A(\zeta_6) = B(\zeta_6)$$

$$\iff B^{-1}A \in \text{Stab}(\zeta_4) \cap \text{Stab}(\zeta_6)$$

$$\iff B^{-1}A \in \{I, S\} \cap \{I, L, L^2\}$$

$$\iff A = B$$

□

Hence we can label each edge by different element of Γ , see Figure 4.1

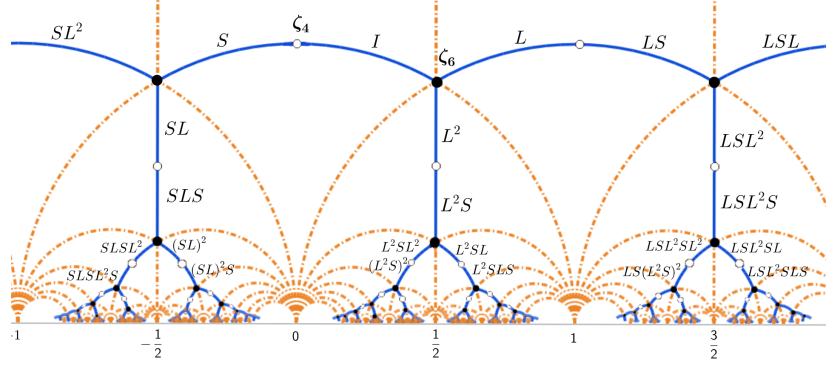


FIGURE 4.1 – Farey graph in the upper half-plane is shown in blue color and the geodesics on which the edges lie are shown in orange color.

Proposition 4.9. *Every element Γ can be written uniquely as*

$$(LS)^{a_0}(L^2S)^{a_1}(LS)^{a_2} \dots (L^{(i)}S)^{a_n} L^j$$

where $j \in \{0, 1, 2\}$, $i \in \{1, 2\}$ and $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}^{>0}$ with $k > 0$ and $n \in \mathbb{N}$.

Proof I, S, L, L^2 can be written as $I = (LS)^0$, $S = (LS)^{-1}L$, $L = (LS)^0L$ and $L^2 = (LS)^0L^2$. Let $W \in \Gamma$ be a (reduced) word starting with LS or L^2S . If it has even number of letters then it can be written in the desired form by grouping powers of (LS) and (L^2S) . If it has odd number of letters then in this case W ends up with letter L or L^2 , which is still in the desired form. Now, let W be a word starting with S then by reading W from the left, substitute $S = (LS)^{-1}L$ finitely many times until getting a word starting with $(LS)^{a_0}(L^2S)^{a_1} \dots$ where a_0 is a negative integer. $\square$

Remark 4.6. *After expressing any element $A \in \Gamma$ in the form as above, if it is seen $j = 0$ then A is a matrix of a finite continued fraction .*

4.5.3 Hyperbolic geodesics and continued fractions

Any geodesic in upper half plane $\mathcal{H}_2$ is determined by two distinct points in $\partial\mathcal{H}_2$. These two boundary points are called *vertices* of the geodesic. If we order the pair of vertices of a geodesic then we get a *oriented geodesic* denoted by ordered pair $\langle \alpha : \beta \rangle$ where $\alpha, \beta \in \partial\mathcal{H}_2$ and we call it *oriented geodesic from α to β* . Now, let us consider the oriented geodesic $\mathbb{I} = \langle \infty : 0 \rangle$ that is a vertical ray meeting the edge $\gamma = \{\zeta_4, \zeta_6\}$ at its vertex ζ_4 at a right angle. For any $A \in \Gamma$, note that $A \cdot \mathbb{I}$ is still an oriented geodesic from $A(\infty)$ to $A(0)$ and also intersects the edge $A \cdot \gamma$ at the point $A(\zeta_4)$ at right angle

since A acts as an orientation-preserving isometry. So any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ moves the geodesic $\mathbb{I}$ as follows

$$A \cdot \mathbb{I} = \langle A(\infty) : A(0) \rangle = \left\langle \frac{a}{c} : \frac{b}{d} \right\rangle.$$

Moreover, if $A \cdot \mathbb{I} = \mathbb{I}$ then it follows that

$$\left\langle \frac{a}{c} : \frac{b}{d} \right\rangle = \langle \infty : 0 \rangle$$

and so $\frac{a}{c} = \infty$ and $\frac{b}{d} = 0$ with $ad - bc = 1$, which implies $A = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Γ . This means that for each $A \in \Gamma$, we get different oriented geodesic $A \cdot \mathbb{I}$. Hence we conclude the following proposition.

Proposition 4.10. *The map*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left\langle \frac{a}{c} : \frac{b}{d} \right\rangle$$

defines a bijection from Γ to the set of oriented geodesics

$$\mathcal{G} = \left\{ \left\langle \frac{a}{c} : \frac{b}{d} \right\rangle : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}$$

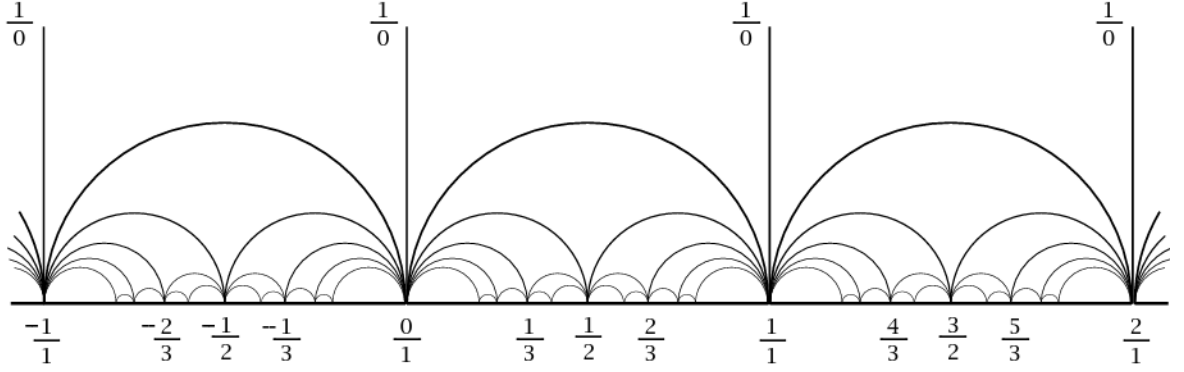
.

Example 4.3. *Translation matrix $T^b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ corresponds to the geodesic $\langle \infty, b \rangle$*

that is vertical ray from ∞ to the integer b while $T^b S = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to $\langle b : \infty \rangle$. All other matrices in Γ corresponds to a geodesic which is a semi circle centered at the boundary.

Remark 4.7. *Some basic properties of elements of $\mathcal{G}$ are following :*

- i. The set of vertices of geodesics in $\mathcal{G}$ consists of all rational numbers together with ∞ .*
- ii. Two geodesics in $\mathcal{G}$ are either disjoint or meet at only one of their vertices.*
- iii. Two rational numbers r/s and r'/s' are connected by a geodesic in $\mathcal{G}$ if and only if $|rs' - r's| = 1$.*

FIGURE 4.2 – Γ -orbit of the geodesic $\mathbb{I}$.

Theorem 4.4. *There exists one-to-one correspondences among the elements of modular group Γ , the edges of Farey graph $\mathcal{F}$ and the oriented geodesics in $\mathcal{G}$ by*

$$A \cdot \gamma \leftrightarrow A \leftrightarrow A \cdot \mathbb{I}$$

Besides, for each $A \in \Gamma$, the edge $A \cdot \gamma$ of Farey graph and the geodesic $A \cdot \mathbb{I}$ meets at the point $A(\zeta_4)$ at right angle.

4.5.4 Geometric meaning of n -th convergent

Suppose $\alpha \in \mathbb{R}$ with $\alpha = [a_0, a_1, \dots]$ and $A_n \in \Gamma$ is the matrix of n -th convergent $c_n = \frac{p_n}{q_n}$. Explicitly, we know

$$A_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \text{ or } \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$$

if n is odd or even respectively. Then the corresponding geodesic of A_n is

$$\langle A_n(\infty) : A_n(0) \rangle = \begin{cases} \langle c_n : c_{n-1} \rangle & \text{if } n \text{ is odd} \\ \langle c_{n-1} : c_n \rangle & \text{if } n \text{ is even} \end{cases}$$

where $A_{-1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to the geodesic $\mathbb{I}$ and $c_{-1} = \infty$. So the sequence of the matrices A_n gives the sequence of geodesics

$$\langle c_{-1} : c_0 \rangle, \langle c_1 : c_0 \rangle, \langle c_1 : c_2 \rangle, \langle c_3 : c_2 \rangle, \dots$$

Except for the initial geodesic $\langle c_{-1}, c_0 \rangle$, all geodesics here are semi-circles. This shows that consecutive convergents of a real number are connected to each others by semi-circles. As n is increasing, these semi-circles are getting smaller and their endpoints

approaching α since convergents satisfy the following relation :

$$c_0 < c_2 < c_4 < \cdots < \alpha < \cdots < c_3 < c_1 < c_{-1} = \infty.$$

In sum, we can find the n -th convergent of a continued fraction geometrically as follows :

- Find the position of the edge labelled by the matrix A_n in Farey graph.
- Draw the semi-circle meeting that edge perpendicularly at its $\circ$ -vertex.
- One of the end points of the semi-circle gives n -th convergent c_n .

Remark 4.8. *If $(A_n)_n$ is a sequence of the convergent matrices of an irrational number α then it forms an infinite path on Farey graph which starts from the edge γ and proceeds to the boundary. Moreover, the path hits exactly the irrational number α at the infinity.*

4.5.5 Fundamental Region for $\text{PSL}(2, \mathbb{Z})$

In this subsection, we will recall the well-known fundamental region of the action of $\text{PSL}(2, \mathbb{Z})$ on the upper half plane. The computation of such a region plays a crucial role in our considerations.

Definition 4.3. *Let G be a group acting on a topological space X . A closed connected subset $\mathcal{D}$ is called a fundamental region for the action G on X if its interior contains exactly one point from each orbit.*

Theorem 4.5. *The set $D = \{z \in \mathcal{H}_2 \mid -\frac{1}{2} \leq \text{Re}(z) \leq \frac{1}{2} \text{ and } |z| \geq 1\}$ is a fundamental region for the action Γ on $\mathcal{H}_2$.*

Proof Fix $z \in \mathcal{H}_2 - \mathcal{D}$. We want show that there exists some $g \in \Gamma$ with $gz \in \mathcal{D}$. We may assume $|\text{Re}(z)| \leq \frac{1}{2}$ by using a suitable translation T^n . Hence it is enough to show that there exists some g so that $|gz| \geq 1$. To do it, we will choose some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that the function $(c, d) \mapsto |cz + d|^2$ attains minimum. There exists such a g since the set $\{(c, d) \in \mathbb{Z}^2 : |cz + d| \leq 1\}$ is a non-empty discrete subset in $\mathbb{C}$. Then $\text{Im}(gz) = \frac{\text{Im}(z)}{|cz + d|^2}$ attains maximum value for such g . We claim that $|gz| \geq 1$. Suppose not : $|gz| < 1$. Then we see that

$$\text{Im}((Sg)z) = \frac{\text{Im}(gz)}{|gz|^2} > \text{Im}(gz),$$

which contradicts to the choice of g .

Now we show that any two interior points of $\mathcal{D}$ cannot be in the same orbit. Fix $z_1, z_2 \in \mathcal{D}$. We may assume $\text{Im}(z_2) \geq \text{Im}(z_1)$ without loss of generality. Suppose that there exists some $g \in \Gamma$ so that $z_2 = g \cdot z_1$. We want to show that z_1 and z_2 lie in the boundary of D . Under these assumptions, it follows that

$$\text{Im}(z_2) = \text{Im}(g \cdot z_1) = \frac{\text{Im}(z_1)}{|cz_1 + d|^2} \geq \text{Im}(z_1).$$

So $|cz_1 + d|^2 \leq 1$. As $\text{Im}(z_1) \geq \frac{\sqrt{3}}{2}$ and the entries in g are integers, it follows that $|c| < 2$ because otherwise

$$1 \leq \text{Im}(cz_1) = \text{Im}(cz_1 + d) \leq |cz_1 + d|.$$

Besides, we have

$$|d| - \frac{1}{2} \leq |d + \text{Re}(z)c| \leq |cz + d| \leq 1.$$

This reduces the possibilities for g and we study them case by case.

CASE 1 : $c = 0, d = \pm 1$. By the restriction $ad - bc = 1$, we see that $g = T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. But for gz_1 to lie in $\mathcal{D}$ again only if $n \in \{-1, 0, 1\}$. For $n = 0$, we have $z_1 = z_2$. For $n = \pm 1$, we see that z_2 and z_1 must lie on the boundary lines $\text{Re}(z) = \pm \frac{1}{2}$ of $\mathcal{D}$ and hence not in the interior of $\mathcal{D}$.

CASE 2 : $c = \pm 1, d = 0$. Then z_1 lies on the unit circle since we have $|cz_1| \leq 1$ and $|z_1| \geq 1$. To satisfy $ad - bc = 1$, g has to be of the form

$$g = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} = T^a S$$

for some $a \in \mathbb{Z}$. For $|a| > 2$ we see that $T^a S.z_1 \notin \mathcal{D}$, thus $a \in \{-1, 0, 1\}$.

For $a = 0$, we get $g = S$ and $|z_1| = |z_2| = 1$ so that $z_1, z_2 \in \mathcal{D}$ are symmetric with respect to imaginary axis.

For $a = 1$, we get $g = TS$, but then $z_1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$. The case $a = -1$ is similar : We get $g = ST$ and $z_1 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

CASE 3 : Suppose $c = d = \pm 1$. As $|cz + d| \leq 1$, we see that $z_1 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, as for any other $z_1 \in \mathcal{D}$ it seen that $|\pm z_1 \pm 1| > 1$.

Since $\det(g) = 1$, it follows that : $a - b = 1 \implies b = -1 + a$. Hence $g = \begin{pmatrix} a & -1 + a \\ 1 & 1 \end{pmatrix} =$

$$T^a \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

By assumption we want $gz_1 \in \mathcal{D}$. One solution is $a = 0$ but then $z_1 = z_2 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. For the case $a = 1$ we see that $z_2 = z_1 + 1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$.

CASE 4 : $c = -d = \pm 1$ and $z_1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ Here we proceed in the same way as in case 3 and see the same result - viable pairs z_1, z_2 must lie in the boundary of F .

So we conclude that z_1 and z_2 does not belong to the interior unless $g = I$ and $z_1 = z_2$.
□

4.6 Quadratic irrationals

The main result of this section is that the continued fraction expansion of a number is eventually periodic if and only if the number is a quadratic irrational. The proof that continued fractions of quadratic irrationals are eventually periodic is surprisingly difficult and involves an interesting finiteness argument.

Definition 4.4. (*Quadratic Irrational*) A real number $\alpha \in \mathbb{R}$ is a quadratic irrational if it is irrational and satisfies a quadratic polynomial with coefficients in $\mathbb{Q}$.

Thus, e.g., $(1 + \sqrt{5})/2$ is a quadratic irrational. Recall that $\frac{1 + \sqrt{5}}{2} = [1, 1, 1, \dots]$. The continued fraction of $\sqrt{2}$ is $[1, 2, 2, 2, 2, 2, \dots]$, and the continued fraction of $\sqrt{389}$ is $[19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \dots]$.

4.7 Periodic Continued Fractions

Definition 4.5. (*Periodic Continued Fraction*) A periodic continued fraction is a continued fraction $[a_0, a_1, \dots, a_n, \dots]$ such that $a_n = a_{n+h}$ for some fixed positive integer h and all sufficiently large n . We call the minimal such h the period of the continued fraction.

Example 4.4. Consider the periodic continued fraction $[1, 2, 1, 2, \dots] = \overline{[1, 2]}$. What

does it converge to ? We have

$$[\overline{1, 2}] = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \dots}}}}$$

so if $\alpha = [\overline{1, 2}]$ then

$$\alpha = 1 + \frac{1}{2 + \frac{1}{\alpha}} = 1 + \frac{1}{\frac{2\alpha + 1}{\alpha}} = 1 + \frac{\alpha}{2\alpha + 1} = \frac{3\alpha + 1}{2\alpha + 1}.$$

Thus $2\alpha^2 - 2\alpha - 1 = 0$, so

$$\alpha = \frac{1 + \sqrt{3}}{2}.$$

Theorem 4.6 (Periodicity Characterization). *An infinite simple continued fraction is periodic if and only if it represents a quadratic irrational.*

Proof ($\implies$) First suppose that $[a_0, a_1, \dots, a_n, \overline{a_{n+1}, \dots, a_{n+h}}]$ is a periodic continued fraction. Set $\alpha = [a_{n+1}, a_{n+2}, \dots]$. Then $\alpha = [a_{n+1}, \dots, a_{n+h}, \alpha]$, so by Proposition 4.2

$$\alpha = \frac{\alpha p_{n+h} + p_{n+h-1}}{\alpha q_{n+h} + q_{n+h-1}}.$$

Thus, α satisfies a quadratic equation with coefficients in $\mathbb{Q}$. Computing as in previous example and rationalizing the denominators, and using that the a_i are all integers, shows that

$$[a_0, a_1, \dots] = [a_0, a_1, \dots, a_n, \alpha] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{\alpha}}}$$

is of the form $c + d\alpha$, with $c, d \in \mathbb{Q}$, so $[a_0, a_1, \dots]$ also satisfies a quadratic polynomial over $\mathbb{Q}$. By Proposition 5.2.12, $\alpha \notin \mathbb{Q}$ because it is the value of an infinite continued fraction.

($\impliedby$) Suppose $\alpha \in \mathbb{R}$ is a quadratic irrational satisfying

$$a\alpha^2 + b\alpha + c = 0 \tag{4.7}$$

with $a, b, c \in \mathbb{Z}$ and $a \neq 0$ and let $[a_0, a_1, \dots]$ be its continued fraction expansion. If we define $r_n = [a_n, a_{n+1}, \dots]$ for each n then $\alpha = [a_0, a_1, \dots, a_{n-1}, r_n]$. We will prove periodicity by showing that there are finitely many distinct r_n 's. We have

$$\alpha = \frac{r_n p_{n-1} + p_{n-2}}{r_n q_{n-1} + q_{n-2}}.$$

Substituting this expression for α into the quadratic equation (4.7), we see that

$$A_n r_n^2 + B_n r_n + C_n = 0,$$

where

$$A_n = ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2,$$

$$B_n = 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + 2cq_{n-1}q_{n-2},$$

$$C_n = ap_{n-2}^2 + bp_{n-2}q_{n-2} + cp_{n-2}^2.$$

Note that $A_n, B_n, C_n \in \mathbb{Z}$, that $C_n = A_{n-1}$, and that

$$B_n^2 - 4A_n C_n = (b^2 - 4ac)(p_{n-1}q_{n-2} - q_{n-1}p_{n-2})^2 = b^2 - 4ac.$$

By Corollary 4.2, we know that

$$\left| \alpha - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{q_n q_{n-1}}.$$

Thus $|\alpha q_{n-1} - p_{n-1}| < \frac{1}{q_n} < \frac{1}{q_{n-1}}$, so $p_{n-1} = \alpha q_{n-1} + \frac{\delta}{q_{n-1}}$ with $|\delta| < 1$. Hence

$$\begin{aligned} A_n &= a \left(\alpha q_{n-1} + \frac{\delta}{q_{n-1}} \right)^2 + b \left(\alpha q_{n-1} + \frac{\delta}{q_{n-1}} \right) q_{n-1} + cq_{n-1}^2 \\ &= (a\alpha^2 + b\alpha + c)q_{n-1}^2 + 2a\alpha\delta + a\frac{\delta^2}{q_{n-1}^2} + b\delta \\ &= 2a\alpha\delta + a\frac{\delta^2}{q_{n-1}^2} + b\delta. \end{aligned}$$

Thus

$$|A_n| = \left| 2a\alpha\delta + a\frac{\delta^2}{q_{n-1}^2} + b\delta \right| < 2|a\alpha| + |a| + |b|.$$

So there are only finitely many possibilities for the integer A_n . Also,

$$|C_n| = |A_{n-1}|$$

and

$$|B_n| = \sqrt{b^2 - 4(ac - A_n C_n)},$$

hence there are only finitely many triples (A_n, B_n, C_n) , and so only finitely many possibilities for r_n as n varies, which completes the proof. $\square$

Definition 4.6. A continued fraction which is periodic from the first term is called purely periodic. If the period starts with the second term, the continued fraction is called simply periodic.

Definition 4.7. A quadratic irrational α is said to be reduced if $\alpha > 1$ is the root of a quadratic equation with integral coefficients whose conjugate root α' lies between -1 and 0. A reduced quadratic irrational associated to D can be written as $\frac{P+\sqrt{D}}{Q}$, where P, D, Q are integers with $D, Q > 0$.

Lemma 4.3. For any given positive integer D , there is only a finite number of reduced quadratic irrational associated to D .

Proof

If α is a reduced quadratic irrational, then

$$\alpha = \frac{P + \sqrt{D}}{Q} > 1 \text{ with } Q > 0,$$

and its conjugate satisfies

$$-1 < \alpha' = \frac{P - \sqrt{D}}{Q} < 0.$$

It suffices that there are only finitely many possibilities for P and Q since D is fixed. The conditions $\alpha > 1$ and $\alpha' > -1$ imply that $\alpha + \alpha' > 0$, or $\frac{2P}{Q} > 0$, and since $Q > 0$ we conclude that $P > 0$. Also from $\alpha' < 0$ and $Q > 0$ it follows that $0 < P < \sqrt{D}$. On the other hand, the inequality $\alpha > 1$ implies that $P + \sqrt{D} > Q$ and, thus $Q < 2\sqrt{D}$, which proves the lemma. $\square$

Theorem 4.7. If α is a reduced quadratic irrational if and only if its continued fraction is purely periodic.

Proof Suppose first that α is represented by a purely periodic continued fraction such as

$$\alpha = [\overline{a_0, a_1, \dots, a_{r-1}}].$$

By the construction of the continued fraction for any real number, all terms must be precisely positive except first term. Then $a_0 \geq 1$ since it appears again in the continued fraction. Thus $\alpha > 1$. As

$$\alpha = [\overline{a_0, a_1, \dots, a_{r-1}}, \alpha],$$

we have

$$\alpha = \frac{\alpha p_{r-1} + p_{r-2}}{\alpha q_{r-1} + q_{r-2}}$$

so that

$$q_{r-1}\alpha^2 + (q_{r-2} - p_{r-1})\alpha - p_{r-2} = 0$$

Then α is a root of the polynomial

$$f(x) = q_{r-1}x^2 + (q_{r-2} - p_{r-1})x - p_{r-2},$$

so the other root of $f(x)$ is the conjugate β of α . But

$$f(-1) = (p_{r-1} - p_{r-2}) + (q_{r-1} - q_{r-2}) > 0$$

since the p_i and the q_i are both strictly increasing sequences, while

$$f(0) = -p_{r-2} < 0.$$

So $f(x)$ has a root between -1 and 0 . This cannot be α because it is positive. So β lies between -1 and 0 and we are done.

Now suppose that $\alpha > 1$ and $-1 < \beta < 0$, and let the continued fraction for α be $[a_0, a_1, \dots]$. Let α_n be the n^{th} complete convergent of α , and $\beta_n = \alpha'_n$. Thus $\beta_0 = \beta$. Then

$$\alpha = \alpha_0 = a_0 + \frac{1}{\alpha_1}$$

so that

$$\beta_0 = \alpha'_0 = a_0 + \frac{1}{\alpha'_1} = a_0 + \frac{1}{\beta_1}$$

and thus

$$\frac{1}{\beta_1} = -a_0 + \beta_0 < -a_0 \leq -1$$

so that $-1 < \beta_1 < 0$. Inductively, we have $-1 < \beta_n = \alpha'_n < 0$ for all $n \geq 0$. Suppose now that the continued fraction for α is not purely periodic, but rather has the form

$$\alpha = [a_0, a_1, \dots, a_{k-1}, \overline{a_k, a_{k+1}, \dots, a_{k+j-1}}]$$

for $k \geq 1$. Then $\alpha_k = \alpha_{k+j}$ and so

$$\alpha_{k-1} - \alpha_{k+j-1} = \left(a_{k-1} + \frac{1}{\alpha_k}\right) - \left(a_{k+j-1} + \frac{1}{\alpha_{k+j}}\right) = a_{k-1} - a_{k+j-1}$$

But $a_{k-1} \neq a_{k+j-1}$, otherwise a_{k-1} would have been the first element of the repeating period. Thus $\alpha_{k-1} - \alpha_{k+j-1}$ is a nonzero integer and thus $\beta_{k-1} - \beta_{k+j-1}$ is as well. But $-1 < \beta_{k-1} - \beta_{k+j-1} < 1$, which is a contradiction. Thus $k = 0$ and the continued fraction is purely periodic. $\square$

For example, $\alpha = \frac{1+\sqrt{3}}{2}$ satisfies $\alpha > 1$ and $-1 < \alpha' < 0$. Its continued fraction is $\alpha = [1, 2]$ On the other hand, $\sqrt{3} > 1$, but $-\sqrt{3}$ does not lie between -1 and 0 . Its continued fraction is $\sqrt{3} = [1, \overline{1, 2}]$.

Corollary 4.5. *For any N , positive integer which is not a perfect square, the continued fraction of $\sqrt{N}$ is simply periodic, more precisely*

$$\sqrt{N} = [a_0, \overline{a_1, a_2, \dots, a_n, 2a_0}], \text{ for some } n \geq 0.$$

Proof Let a_0 be the greatest integer less than $\sqrt{N}$. Then $\sqrt{N} + a_0 > 1$ and its conjugate, $-\sqrt{N} + a_0$ lies between -1 and 0 . Thus, $\sqrt{N} + a_0$ is a reduced quadratic irrational with the greatest integer less than it equal to $2a_0$. We can apply Theorem 4.7 :

$$\sqrt{N} + a_0 = [\overline{2a_0, a_1, \dots, a_n}] \text{ for some } n.$$

which is equivalent to

$$\sqrt{N} + a_0 = [2a_0, \overline{a_1, a_2, \dots, a_n, 2a_0}].$$

Consequently,

$$\sqrt{N} = [a_0, \overline{a_1, a_2, \dots, a_n, 2a_0}] \text{ where } a_0 \geq 1.$$

□

Theorem 4.8. *Let $x = [\overline{a_0, a_1, \dots, a_{n-1}, a_n}]$ be a purely periodic quadratic irrational. Then $-\frac{1}{x'}$ is purely periodic, and*

$$-\frac{1}{x'} = [\overline{a_n, a_{n-1}, \dots, a_1, a_0}]$$

Proof

Let $y = [\overline{a_n, a_{n-1}, \dots, a_0}]$. It suffices to show x and $\frac{-1}{y}$ are the roots of the same quadratic equation.

Case 1. Suppose that the length of the period of x is EVEN :

$$\begin{aligned} x = [a_0, a_1, \dots, a_n, x] &= T^{a_0}(UT^{a_1}U)T^{a_2}(UT^{a_3}U) \dots T^{a_{n-1}}(UT^{a_n}U)(x) \\ &= (LS)^{a_0}(L^2S)^{a_1}(LS)^{a_2}(L^2S)^{a_3} \dots (LS)^{a_{n-1}}(L^2S)^{a_n}(x) \\ &= A \cdot x \end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{-1}{y} &= -[0, a_n, a_{n-1}, \dots, a_0, y] \\
&= -T^0 U T^{a_n} (U T^{a_{n-1}} U) T^{a_{n-2}} \dots (U T^{a_2} U) T^{a_1} (U T^{a_0} U)(y) \\
&= V U (LS)^{a_n} (L^2 S)^{a_{n-1}} (LS)^{a_{n-2}} \dots (L^2 S)^{a_2} (LS)^{a_1} (L^2 S)^{a_0} (y) \\
&= S U U (LS)^{a_n} (L^2 S)^{a_{n-1}} (LS)^{a_{n-2}} \dots (L^2 S)^{a_2} (LS)^{a_1} (L^2 S)^{a_0} (y) \\
&= S (LS)^{a_n} (L^2 S)^{a_{n-1}} (LS)^{a_{n-2}} \dots (L^2 S)^{a_2} (LS)^{a_1} (L^2 S)^{a_0} (y) \\
&= (SL)^{a_n} (SL^2)^{a_{n-1}} (SL)^{a_{n-2}} \dots (SL^2)^{a_2} (SL)^{a_1} (SL^2)^{a_0} \left(\frac{-1}{y}\right) \\
&= A^{-1} \left(\frac{-1}{y}\right)
\end{aligned}$$

where $V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $V = SU$. Since A fixes $\frac{-1}{y}$, it follows that both x and $\frac{-1}{y}$ are the roots of the same quadratic equation.

Case 2. Suppose that the length of the period of x is ODD :

$$\begin{aligned}
x = [a_0, a_1, \dots, a_n, x] &= T^{a_0} (U T^{a_1} U) T^{a_2} (U T^{a_3} U) \dots (U T^{a_{n-1}} U) T^{a_n} U \cdot x \\
&= (LS)^{a_0} (L^2 S)^{a_1} (LS)^{a_2} (L^2 S)^{a_3} \dots (L^2 S)^{a_{n-1}} (LS)^{a_n} U \cdot x \\
&= B \cdot x
\end{aligned}$$

Also we may write

$$\begin{aligned}
-\frac{1}{y} &= -[0, a_n, a_{n-1}, \dots, a_0, y] \\
&= -T^0(UT^{a_n}U)T^{a_{n-1}}(UT^{a_{n-2}}U) \dots T^{a_1}(UT^{a_0}U) \cdot y \\
&= V(L^2S)^{a_n}(LS)^{a_{n-1}}(L^2S)^{a_{n-2}} \dots (LS)^{a_1}(L^2S)^{a_0-1}L^2S \cdot y \\
&= US(L^2S)^{a_n}(LS)^{a_{n-1}}(L^2S)^{a_{n-2}} \dots (LS)^{a_1}(L^2S)^{a_0-1}L^2\left(\frac{-1}{y}\right) \\
&= U(SL^2)^{a_n}(SL)^{a_{n-1}}(SL^2)^{a_{n-2}} \dots (SL)^{a_1}(SL^2)^{a_0-1}SL^2\left(\frac{-1}{y}\right) \\
&= B^{-1}\left(\frac{-1}{y}\right)
\end{aligned}$$

Hence $\frac{-1}{y}$ is fixed by B just like x , which means that both $\frac{-1}{y}$ and x satisfy the same quadratic equation.

□

Lemma 4.4. *For any square free positive integer N , let $\sqrt{N} = [a_0, \overline{a_1, a_2, \dots, a_n, 2a_0}]$. Except for the term $2a_0$, the periodic part of the continued fraction of $\sqrt{N}$ is symmetrical*

Proof From the continued fraction of $\sqrt{N}$ we get that

$$\sqrt{N} - a_0 = [0, \overline{a_1, a_2, \dots, a_n, 2a_0}].$$

We can easily get now that

$$\frac{1}{\sqrt{N} - a_1} = [\overline{a_1, a_2, \dots, a_n, 2a_0}] \quad (4.8)$$

On the other hand, $a_0 + \sqrt{N} = [2a_0, \overline{a_1, a_2, \dots, a_n}]$ and from Theorem 4.8, we get

$$\frac{1}{\sqrt{N} - a_0} = [\overline{a_n, a_{n-1}, \dots, a_1, 2a_0}] \quad (4.9)$$

However, we know that the continued fraction expansions are unique. Comparing (4.8) and (4.9) we conclude that

$$a_n = a_1, a_{n-1} = a_2, \dots, a_2 = a_{n-1}, a_1 = a_n$$

Thus, except for the term $2a_0$ the periodic part of the continued fraction of $\sqrt{N}$ is symmetrical. $\square$

Theorem 4.9. *Let $\omega = \frac{\sigma + \sqrt{\Delta}}{2}$ be a quadratic irrational and $\Delta = 4D + \sigma$, where $D \in \mathbb{N}$ and $\sigma \in \{0, 1\}$. Let the continued fraction expansion of ω be $[a_0, a_1, \dots]$ with period k . Then $a_k = 2a_0 - \sigma \neq a_0$, $a_{k-i} = a_i$ for all $i \in [1, \dots, k-1]$, and*

$$\omega = [a_0, \overline{a_1, a_2, \dots, a_2, a_1}, 2a_0 - \sigma].$$

Proof Let $\omega = [a_0, a_1, a_2, \dots]$. By definition $\omega = \frac{\sigma + \sqrt{\Delta}}{2} > 1$ and so $\lfloor \omega \rfloor = a_0 \geq 1$. Consider the following quadratic irrational $\alpha := \frac{1}{\omega - a_0} = [a_1, a_2, \dots]$ and its conjugate $\alpha' = \frac{1}{\omega' - a_0} = \frac{1}{\sigma - \omega - a_0}$. Note that $\alpha > 1$ and $-1 < \alpha' < 0$. This means that α is a reduced quadratic irrational. Hence it can be written as $\alpha = [\overline{a_1, a_2, \dots, a_k}]$ for some $k \in \mathbb{N} \setminus \{0\}$. By the Theorem 4.8,

$$\frac{-1}{\alpha'} = [\overline{a_k, a_{k-1}, \dots, a_2, a_1}]. \quad (4.10)$$

On the other hand, it also equals

$$\frac{-1}{\alpha'} = \omega + a_0 - \sigma = [2a_0 - \sigma, a_1, a_2, \dots]. \quad (4.11)$$

Comparing the (4.10) and (4.11), it follows that

$$\omega = [a_0, \overline{a_1, a_2, \dots, a_2, a_1}, 2a_0 - \sigma].$$

$\square$

5 COMPLEX CONTINUED FRACTIONS

Throughout this chapter, K will denote an imaginary quadratic field whose ring of integers is Euclidean domain with respect to usual norm $|z| = z\bar{z}$. Such number fields are fully classified and they are of the form $K = \mathbb{Q}(\sqrt{-d})$ where $d = 1, 2, 3, 7, 11$. Consider the additive action $\mathcal{O}_d$ on the complex plane $\mathbb{C}$ and the set of complex numbers closer to zero than to other algebraic integers, which is a fundamental region of the additive action :

$$\mathcal{V} = \mathcal{V}_d = \{z \in \mathbb{C} : |z| < |z - a|, a \in \mathcal{O}_d\} \cup \mathcal{E}$$

where $\mathcal{E}$ is a suitable subset of the boundary $\partial\mathcal{V}$ so that $\mathcal{V}$ contains a *unique* representative from each orbit. Hence the fundamental region depends on the choice of $\mathcal{E}$ as well as the integer d . After some computations, we see that $\mathcal{V}_d$'s are contained in the unit ball $\mathcal{B}(0, 1)$ for all $d = 1, 2, 3, 7, 11$ and the shapes of $\mathcal{V}_1$ and $\mathcal{V}_2$ are rectangle while $\mathcal{V}_3, \mathcal{V}_7$ and $\mathcal{V}_{11}$ are hexagons. For any $z \in \mathbb{C}$, we denote the nearest algebraic integer

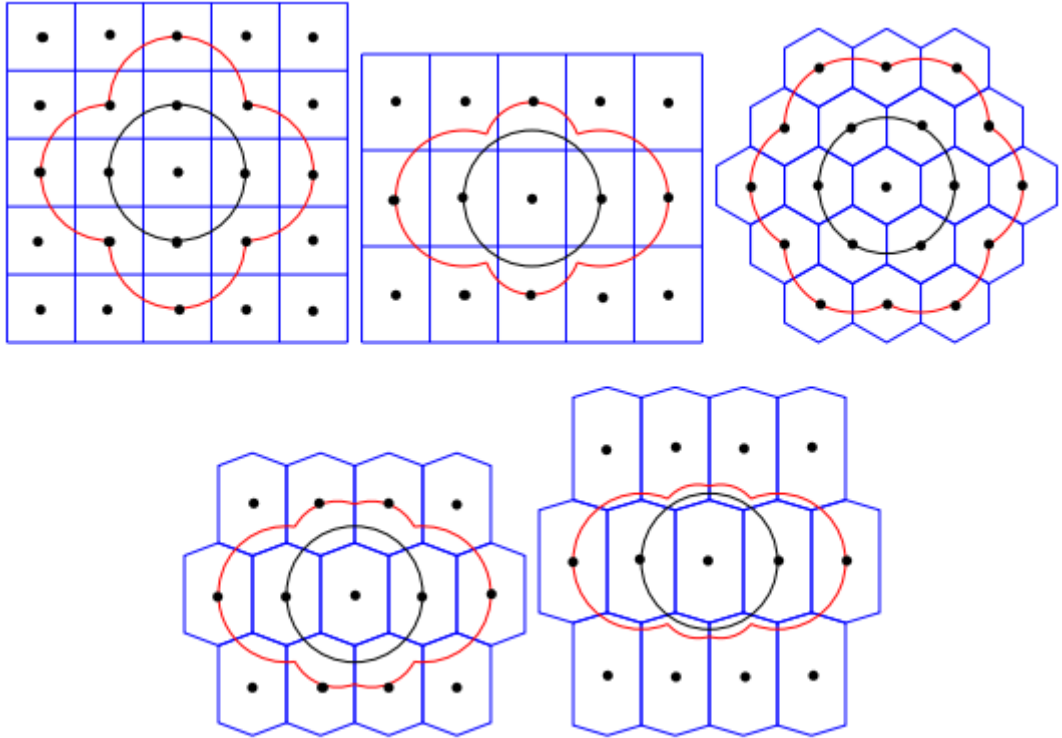


FIGURE 5.1 – $\partial\mathcal{V}$ and its translates in blue color, unit circle in black, $\partial\mathcal{V}^{-1}$ in red color

to z by $[z] \in \mathcal{O}$ and the remainder by $\{z\} \in \mathcal{V}$ so that z can be written uniquely as the sum of its integral part and remainder part :

$$z = [z] + \{z\}.$$

Remark that we have to make some choices for $\lfloor z \rfloor$ depending on the position of z in the complex plane since there may exist more than one number of integers having the same distance to z . Nevertheless, if we fix the rule of that choice then we obtain a well-defined process to find a continued fraction expression for complex numbers. In fact, that is the reason we add some points from the boundary to the fundamental region $\mathcal{V}$, and the rule of the choice exactly determines the subset $\mathcal{E}$. It can also be easily seen that the region $\mathcal{V}$ consists of exactly the remainder parts $\{z\}$ of for all $z \in \mathbb{C}$.

Proposition 5.1. $\mathcal{V} = \{z - \lfloor z \rfloor : z \in \mathbb{C}\}.$

Proof By the definition of the nearest integer function $\lfloor \cdot \rfloor : \mathbb{C} \rightarrow \mathcal{O}$, $z - \lfloor z \rfloor$ lies in $\mathcal{V}$ for all z . Conversely let ζ be a number so that $|\zeta| < |\zeta - a|$ for all $a \in \mathcal{O}$. This means that $\lfloor \zeta \rfloor = 0$. Hence by choosing $z = \zeta$, we may write $\zeta = z - \lfloor z \rfloor$. $\square$

Similar to the real case, a continued fraction expression for a complex number $z \in \mathbb{C}$ is obtained from the following sequences : $a_n \in \mathcal{O}, z_n \in \mathcal{V}$, for $n \geq 0$:

$$a_0 = \lfloor z \rfloor \text{ and } z_0 = z - a_0 = \{z\}$$

$$a_n = \lfloor 1/z_{n-1} \rfloor \text{ and } z_n = \{1/z_{n-1}\} = \frac{1}{z_{n-1}} - a_n = T^n(z_0).$$

where $T : \mathcal{V} \rightarrow \mathcal{V}$ defined by $z \mapsto \{1/z\}$ and $T(0) = 0$ while $z = 0$. In this way, we get a complex continued fraction expression for z :

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} =: [a_0; a_1, a_2, \dots]$$

and it is seen that $|a_n| > 1$ for all $n > 0$ since $\mathcal{V} \subset \mathcal{B}(0, 1)$. The expansion is finite for $z \in K$. By construction, the continued fraction expansion for $1/z_n$ is $[a_{n+1}, a_{n+2}, \dots]$ and

$$[a_0, a_1, \dots] = [a_0, \dots, a_n, 1/z_n] \text{ for all } n \in \mathbb{N}.$$

The n -th convergent for z will be denoted by $[a_0, \dots, a_n]$ and it can be computed just like the real case :

$$\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$$

so that (p_n) and (q_n) satisfy the following equations :

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_0 &= a_0, & \dots & p_n &= a_n p_{n-1} + p_{n-2} & \dots \\ q_{-2} &= 1, & q_{-1} &= 0, & q_0 &= 1, & \dots & q_n &= a_n q_{n-1} + q_{n-2} & \dots \end{aligned}$$

Several classical identities of continued fractions holds since most of them depend on the relations between the sequences $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$. For brevity, we call these **$\mathcal{Q}$ -pair sequences** of z . We do not restate the other properties for $(p_n)_n$ and $(q_n)_n$, see Proposition 4.3.

Definition 5.1. *If the continued fraction of a complex number z is obtained by applying the nearest integer algorithm then it is called **Hurwitz continued fraction (HCF)** of z over K .*

Proposition 5.2. *The radius of the smallest circle centered at zero containing $\mathcal{V}_d$ is*

$$r_d = \begin{cases} \frac{\sqrt{1+d}}{2} & \text{if } d = 1, 2, \\ \frac{1+d}{4\sqrt{d}}, & \text{if } d = 3, 7, 11. \end{cases}$$

which is always strictly lesser than 1 and so $\mathcal{V}_d$ is contained in the unit disk.

Remark 5.1. *By the proposition (5.2), we have $r_d < 1$ for each $d = 1, 2, 3, 7, 11$ and $|z_n| \leq r_d$ so*

- $|1/z_n| \geq 1/r_d > 1 \quad \forall n \geq 0$
- $|a_n| \geq 1/r_d > 1 \quad \forall n \geq 1$. (Note that a_0 may be zero.)

It is going to be seen that the containment $\mathcal{V} \subset \mathcal{B}(0, 1)$ plays crucial role for proving convergence of convergents and also for other proofs. In fact, a key ingredient in the following proofs is that $|z_n| = |1/z_{n-1} - a_n|$ is bounded by a fixed constant $r_d < 1$. Such a constant exists precisely because open unit discs centered on the lattice points of $\mathcal{O}_d$ cover the complex plane for $d = 1, 2, 3, 7, 11$ but no others. This explains why the application and study of continued fractions over imaginary quadratic fields has been restricted to these five cases.

Proposition 5.3. *For any $z \in \mathbb{C}$, the sequence $(|q_n|)_{n \geq 0}$ is strictly increasing for all $d = 1, 2, 3, 7, 11$.*

The monocity of the denominators of convergents was proved by Lunz for $d = 2$, see (Lunz, 1937) and by Hurwitz for $d = 1, 3$, see (Hurwitz, 1888). Robert Hines outlined the proof for the cases $d = 7, 11$, see (Hines, 2019).

Proposition 5.4. *For $z \in \mathbb{C} \setminus K$, the convergents $\frac{p_n}{q_n}$ satisfy*

$$\left| z - \frac{p_n}{q_n} \right| \leq \frac{1}{|q_n|^2 \left(\frac{1}{r_d} - 1 \right)}.$$

Proof Using simple properties of the $\mathcal{Q}$ -pair sequences $(p_n), (q_n)$ of z and $|q_{n-1}/q_n| < 1$, $|z_n^{-1}| > 1$ we have

$$\begin{aligned}
\left| z - \frac{p_n}{q_n} \right| &= \left| \frac{p_n z_n^{-1} + p_{n-1}}{q_n z_n^{-1} + q_{n-1}} - \frac{p_n}{q_n} \right| \\
&= \left| \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n^2 z_n^{-1} + q_n q_{n-1}} \right| \\
&= \frac{1}{|q_n|^2 |z_n^{-1} + \frac{q_{n-1}}{q_n}|} \\
&\leq \frac{1}{|q_n|^2 (|z_n^{-1}| - |q_{n-1}/q_n|)} \\
&\leq \frac{1}{|q_n|^2 (1/r_d - 1)}
\end{aligned}$$

□

Proposition 5.5. *For all $n \in \mathbb{N}$, $q_n z - p_n = (-1)^n z_0 \cdots z_n$.*

Proof We apply induction. It is obvious that the statement holds for $n = 0$. Suppose it holds for all $m \leq n \in \mathbb{N}$.

$$\begin{aligned}
q_{n+1} z - p_{n+1} &= (a_{n+1} q_n + q_{n-1}) z - (a_{n+1} p_n + p_{n-1}) \\
&= a_{n+1} (q_n z - p_n) + q_{n-1} z - p_{n-1} \\
&= a_{n+1} (-1)^n z_0 \cdots z_n + (-1)^{n-1} z_0 \cdots z_{n-1} \\
&= (-1)^{n-1} z_0 \cdots z_{n-1} (a_{n+1} (-1) z_n + 1) \\
&= (-1)^{n-1} z_0 \cdots z_{n-1} \left(\left(\frac{1}{z_n} - z_{n+1} \right) (-1) z_n + 1 \right) \\
&= (-1)^{n-1} z_0 \cdots z_{n-1} (z_n z_{n+1})
\end{aligned}$$

which proves the proposition. □

Corollary 5.1. *Let $z \in \mathbb{C}$ be a complex number with infinite continued fraction expansion. Then z is the limit of its convergents*

Proof

$$\left| z - \frac{p_n}{q_n} \right| = \left| \frac{(-1)^n z_0 \cdots z_n}{q_n} \right| \rightarrow 0 \text{ as } n \rightarrow \infty$$

since $|q_n|$ is strictly increasing sequence and $|z_0 \cdots z_n| < r^{n+1} < 1$ for all n where $r := r_d$. $\square$

Theorem 5.1. *Let z be a complex number whose HCF over $\mathcal{O}_d$ is $[a_0, a_1, \dots]$.*

i. *If $z \in K$, then $(a_n)_{n \geq 0}$ is finite,*

ii. *If $z \in \mathbb{C} \setminus K$, then $(a_n)_{n \geq 0}$ is infinite and $z = \lim_{n \rightarrow \infty} \frac{p_n}{q_n}$*

Proof i. Write $z = \frac{a}{b}$ with $a, b \in \mathcal{O}_d$ coprime. First, note that

$$\frac{1}{|b|} < \left| p_n \frac{a}{b} - q_n \right|$$

since $1 < |ap_n - bq_n|$. By Proposition 5.5, for any $n \in \mathbb{N}$ so that $\frac{p_n}{q_n} \neq z$, we then have

$$0 < \frac{1}{|b|} < \left| p_n \frac{a}{b} - q_n \right| = |z_0 \cdot z_1 \cdots z_n| < r^{n+1}$$

where $0 < r = r_d < 1$. Hence we conclude that n has to be bounded, which implies $(a_n)_n$ is finite. (ii) follows from (i) and Corollary 5.1. $\square$

Definition 5.2. *If $z \in \mathbb{C} \setminus K$ then it is called **irrational** complex number over K . If $K(z)$ is a quadratic extension of K then it is called a **quadratic irrational** over K .*

Theorem 5.2. *A complex number z is an quadratic irrational over K if and only if its HCF expansion is eventually periodic.*

Proof Let $aX^2 + bX + c$ be minimal polynomial of z where $a, b, c \in \mathcal{O}_d$ and $a \neq 0$. As $z = [a_0, \dots, a_{n-1}, \omega_n] = \frac{p_{n-1}w_n + p_{n-2}}{q_{n-1}w_n + q_{n-2}}$ where $\omega_n = 1/z_{n-1}$, we may write

$$a \left(\frac{p_{n-1}w_n + p_{n-2}}{q_{n-1}w_n + q_{n-2}} \right)^2 + b \left(\frac{p_{n-1}w_n + p_{n-2}}{q_{n-1}w_n + q_{n-2}} \right) + c = 0.$$

Expanding and simplifying the previous expression, we obtain that

$$\forall n \in \mathbb{N} \quad A_n w_n^2 + B_n w_n + C_n = 0, \quad (5.1)$$

where

$$A_n = ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2$$

$$B_n = 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + q_{n-1}p_{n-2}) + 2cq_{n-1}q_{n-2}$$

$$C_n = ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2 = A_{n-1}.$$

Notice that $A_n \neq 0$, or else ω_n would be in K and so is z . After rewriting,

$$\begin{aligned} A_n &= (ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2) - q_{n-1}^2 (az^2 + bz + c) \\ &= a(p_{n-1}^2 - q_{n-1}^2 z^2) + b(p_{n-1} - q_{n-1}z)q_{n-1} \\ &= (p_{n-1} - zq_{n-1})[a(p_{n-1} + zq_{n-1}) + bq_{n-1}] \\ &= (p_{n-1} - zq_{n-1})[a(p_{n-1} - zq_{n-1}) + (2az + b)q_{n-1}] \\ &= a(p_{n-1} - zq_{n-1})^2 + (p_{n-1} - zq_{n-1})(2az + b)q_{n-1} \end{aligned}$$

By proposition (5.5), there exists some $n_0 \in \mathbb{N}$ such that $|a(p_{n-1} - zq_{n-1})^2| < 1$ holds for all $n > n_0$ since $|p_{n-1} - zq_{n-1}| \rightarrow 0$. By theorem (5.4), we also know

$$|p_{n-1} - zq_{n-1}| < \frac{1}{|q_{n-1}|(1/r - 1)}.$$

Hence we have the following inequalities

$$|A_n| \leq |a(p_{n-1} - zq_{n-1})^2| + |p_{n-1} - zq_{n-1}||2az + b||q_{n-1}| < 1 + \frac{|2az + b|}{1/r - 1}$$

for all $n > n_0$. Same bound works for C_n since $C_n = A_{n-1}$. On the other hand, Since the discriminant is invariant (note that $B_n^2 - 4A_nC_n = b^2 - 4ac$ for all n), each B_n with $n > n_0$ is bounded. Therefore, the sequences $(A_n)_{n>n_0}, (B_n)_{n>n_0}, (C_n)_{n>n_0}$ take finitely many values, so the set of polynomials

$$\{P_n(X) = A_nX^2 + B_nX + C_n : n > n_0\}$$

and hence the set of their roots

$$F = \{w \in \mathbb{C} : \exists n > n_0 \quad P_n(w) = 0\}$$

are finite and $\{w_n : n > n_0\} \subseteq F$. Then there exists two distinct natural numbers $m > n$ such that $\omega_n = \omega_m$. Hence, their HCF coincide :

$$w_n = [a_n; a_{n+1}, a_{n+2}, \dots], \quad w_m = [a_m; a_{m+1}, a_{m+2}, \dots]$$

and if $k = m - n$ then $a_j = a_{j+k}$ for all $j \geq n$. This shows $z = [a_0, a_1, \dots]$ is eventually periodic.

Suppose now that z has periodic HCF expansion :

$$z = [a_0, \dots, a_{n-1}, \overline{a_n, a_{n+1}, \dots, a_{n+m}}] = [a_0, \dots, a_{n-1}, \omega]. \quad (5.2)$$

where $m > n \geq 0$ and $\omega = [\overline{a_n, a_{n+1}, \dots, a_{n+m}}]$. Then we have

$$z = \frac{p_{n-1}\omega + p_{n-2}}{q_{n-1}\omega + q_{n-2}} \text{ and so } \omega = \frac{q_{n-2}z - p_{n-2}}{-q_{n-1}z + p_{n-1}}.$$

Note that if ω is a quadratic irrational then so is z . Hence it is enough to prove the argument only for purely periodic HCF's. But this is obvious. Indeed, after resetting ω :

$$\omega = [\overline{a_0, \dots, a_{n-1}}] = [a_0, \dots, a_{n-1}, \omega] = \frac{p_{n-1}\omega + p_{n-2}}{q_{n-1}\omega + q_{n-2}}$$

($q_{n-1} \neq 0$) which implies that ω satisfies a quadratic polynomial over K .

□

5.1 Gaussian Integer Continued Fractions

In this case, we fix our number field as $K = \mathbb{Q}(i)$. Then its ring of integers is $\mathcal{O}_1 = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. In particular, elements of $\mathbb{Z}[i]$ are called *Gaussian integers* so that these integers will be coefficients of Hurwitz continued fractions throughout the section. Here, the nearest integer function $\lfloor \cdot \rfloor : \mathbb{C} \rightarrow \mathbb{Z}[i]$ is given by

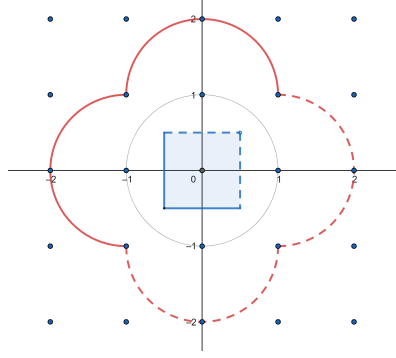
$$\lfloor z \rfloor = \lfloor \operatorname{Re}(z) + 1/2 \rfloor + i \lfloor \operatorname{Im}(z) + 1/2 \rfloor.$$

Note that we made a choice here by using the notation "less or equal". After computing $z - \lfloor z \rfloor$ for four cases above, it leads to

$$\begin{aligned} \mathcal{V} &= \{z - \lfloor z \rfloor : z \in \mathbb{C}\} \\ &= \{u + vi \in \mathbb{C} : -1/2 \leq u < 1/2 \text{ and } -1/2 \leq v < 1/2\} \end{aligned}$$

which is a unit square centered at the origin.

Definition 5.3. *If the entries of HCF lie in the ring $\mathbb{Z}[i]$ then it is called Gaussian integer continued fraction. Not every sequence in $\mathbb{Z}[i]$ gives a HCF of a complex number. An infinite sequence of Gaussian integers is called **valid** if it is the HCF of some complex number and we will denote the set of valid sequences by Ω^{HCF} .*

FIGURE 5.2 – V in blue color, $\partial\mathcal{V}^{-1}$ in red color

Proposition 5.6. *If $(a_n)_{n \geq 0} \in \Omega^{\text{HCF}}$ is the HCF of z , then for any $n \geq 1$,*

$$|a_n| \geq \sqrt{2}.$$

We also have

$$\left\{ (a_n)_{n \geq 0} \in \mathbb{Z}[i]^{\mathbb{N}} : \forall n \in \mathbb{Z}_+ \quad |a_n| \geq \sqrt{8} \right\} \subseteq \Omega^{\text{HCF}}.$$

Remark 5.2. $[0, i, i, i, \dots]$ does not converge any complex number. Indeed, it is seen that the same terms in the sequence of convergents are repeated over and over :

$$[0] = 0$$

$$[0, i] = -i$$

$$[0, i, i] = \infty$$

$$[0, i, i, i] = 0$$

$$[0, i, i, i, i] = -i$$

$$[0, i, i, i, i, i] = \infty.$$

Example 5.1. Let $z = \frac{4+5i}{3} \in \mathbb{Q}(i)$. Then

$$a_0 = \lfloor z \rfloor = 1 + 2i \in \mathbb{Z}[i] \text{ and } z_0 = z - a_0 = \frac{1-i}{3} \in \mathcal{V},$$

$$a_1 = \left\lfloor \frac{1}{z_0} \right\rfloor = \left\lfloor \frac{3+3i}{2} \right\rfloor = 2 + 2i \text{ and } z_1 = \left\{ \frac{1}{z_0} \right\} = \frac{1}{z_0} - a_1 = \frac{-1-i}{2} \in \mathcal{V},$$

$$a_2 = \left\lfloor \frac{-2}{1+i} \right\rfloor = -1 + i \text{ and } z_2 = 0.$$

Therefore the continued fraction expansion is

$$z = [1 + 2i, 2 + 2i, -1 + i]$$

with convergents $c_0 = 1 + 2i$, $c_1 = \frac{5+7i}{4}$, $c_2 = z$.

Example 5.2. $\sqrt{2} + i\sqrt{5} = [1 + 2i, 2 - i, -5 + i, -2 - i, -4, -2 + i, -4, -2, -1 + i, -2i, \dots]$

Example 5.3. $\sqrt{2} + i\sqrt{2} = [1 + i, \overline{1 - i}, 2 + 2i]$

It is mentioned that the sequence of denominators strictly increasing in absolute value, which implies $\left| \frac{q_{n-1}}{q_n} \right| < 1$. The following proposition shows that this phenomena occur in a special way.

Proposition 5.7. *Let $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ with $z = [a_0, a_1, a_2, a_3, \dots]$ and let $(p_n)_{n \geq 0}, (q_n)_{n \geq 0}$ be its Q -pair. Set $\alpha = \frac{\sqrt{5}-1}{2} < 1$. Then*

$$\forall n \geq 1 \quad \frac{|q_{n-1}|}{|q_n|} \leq \alpha \quad \text{or} \quad \frac{|q_n|}{|q_{n+1}|} \leq \alpha.$$

Proof See (Dani and Nogueira, 2014), corollary 5.3. □

Even though there are some similarities between complex and real continued fractions, these have significant differences. For instance, we know two real numbers belong to the same $\text{PSL}_2(\mathbb{Z})$ -orbit if and only if they are both rational or if they are irrational numbers so that their continued fraction expansion eventually coincide. The analogue does not work in $\mathbb{C}$ with Hurwitz continued fractions. Richard Lakein (Lakein, 1974) gave a counter example a pair of $\text{PGL}_2(\mathbb{Z}[i])$ -equivalent complex numbers such that the tails of their Hurwitz continued fraction never coincide.

5.2 Group theoretical approach for complex continued fractions

We introduced the action of $\text{PSL}_2(\mathbb{C})$ on the extended upper half 3-space $\hat{\mathcal{H}}_3 = \mathcal{H}_3 \cup \partial\mathcal{H}_3$ by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, t) \mapsto (z^*, t^*)$$

where

$$z^* = \frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2t^2}$$

$$t^* = \frac{t}{|cz + d|^2 + |c|^2t^2}$$

In this section, we restrict the action to a special subgroup

$$\text{PSL}_2(\mathbb{Z}[i]) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{Z}[i] \right\}$$

which is called *Picard group*, and we see how complex continued fractions are expressed in terms of the elements of Picard group. For brevity, let us denote it by G . Since it acts on extended upper half 3-space $\hat{\mathcal{H}}_3$ faithfully, we can identify the elements of G as transformations of $\hat{\mathcal{H}}_3$. G is generated by the following matrices :

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, W = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

with the relations

$$TW = WT \text{ and } L^3 = S^2 = (RV)^3 = V^2 = (TV)^2 = (WV)^2 = (SV)^2 = I,$$

where

$$L := TS \text{ and } R := WS. \quad (5.3)$$

These generators are viewed as transformations :

$$T(z, t) = (z + 1, t), \quad S(z, t) = \left(\frac{-\bar{z}}{|z|^2 + t^2}, \frac{t}{|z|^2 + t^2} \right)$$

$$W(z, t) = (z + i, t), \quad V(z, t) = (-z, t)$$

Note that G also leaves the boundary of $\mathcal{H}_3$ invariant. While $t = 0$, the action on $\partial\mathcal{H}_3 = \hat{\mathbb{C}}$ is given by standard fractional linear transformation of $\hat{\mathbb{C}}$:

$$g(z) = \frac{az + b}{cz + d} \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \text{ and } z \in \hat{\mathbb{C}}.$$

In this section, we fix our space as $\hat{\mathbb{C}}$ since we want to study complex numbers. Moreover, we want understand how to describe the continued fractions of complex numbers in terms of the generators of G . Notice that we can express any Gauss integer $z = a + bi$ as $z = T^a W^b(0)$ since

$$T^a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : z \mapsto z + a, \quad W^b = \begin{pmatrix} 1 & bi \\ 0 & 1 \end{pmatrix} : z \mapsto z + bi \quad \text{for all } a, b, \in \mathbb{Z}.$$

Now, consider the following elements

$$U := SV = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} : z \mapsto \frac{1}{z} \text{ and } R = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto \frac{iz - 1}{z}$$

Suppose $z \in \mathbb{Q}(i)$ with $z = [a_0 + b_0i, a_1 + b_1i, \dots, a_n + b_ni]$. Then we may express it in terms of T, W and U so that :

$$z = T^{a_0} W^{b_0} U T^{a_1} W^{b_1} U \dots U T^{a_n} W^{b_n} U(\infty) \quad (5.4)$$

However, we want to use the matrices L, S, R in the last expression. To do it, now reset $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Remark that the equation (5.4) still holds even after replacing $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ since both matrices correspond to the same transformation $z \mapsto \frac{1}{z}$ on the extended complex plane $\hat{\mathbb{C}}$.

We also have following relations : $UTU = L^2S$ and $UWU = R^{-1}S$. Then by using the fact $U^2 = I$, we obtain, for all $a, b \in \mathbb{Z}$,

$$\begin{cases} UT^aU = (UTU)^a = (L^2S)^a \\ UW^bU = (UWU)^b = (R^{-1}S)^b \\ T = LS \text{ and } W = RS \end{cases}$$

If we write $T^{a_k}UUW^{b_k}$ in place of $T^{a_k}W^{b_k}$ for all $k \in 2\mathbb{N} + 1$ then the expression (5.4) can be rewritten in two ways depending on the length of the continued fraction.

Case 1 (EVEN) : Suppose that the length of the continued fraction is EVEN : Then $z \in \mathbb{Q}(i)$ can be written as

$$\begin{aligned} z &= T^{a_0}W^{b_0}(UT^{a_1}U)(UW^{b_1}U)T^{a_2}W^{b_2} \dots UT^{a_n}UUW^{b_n}U(\infty) \\ &= T^{a_0}W^{b_0}(UTU)^{a_1}(UWU)^{b_1}T^{a_2}W^{b_2} \dots (UTU)^{a_n}(UWU)^{b_n}(\infty) \\ &= (LS)^{a_0}(RS)^{b_0}(L^2S)^{a_1}(R^{-1}S)^{b_1}(LS)^{a_2}(RS)^{b_2} \dots (L^2S)^{a_n}(R^{-1}S)^{b_n}(\infty) \end{aligned}$$

Case 2 (ODD) : Suppose that the length of the continued fraction of z is ODD :

$$\begin{aligned} z &= T^{a_0}W^{b_0}(UT^{a_1}U)(UW^{b_1}U)T^{a_2}W^{b_2} \dots T^{a_n}W^{b_n}(0) \\ &= (LS)^{a_0}(RS)^{b_0}(L^2S)^{a_1}(R^{-1}S)^{b_1}(LS)^{a_2}(RS)^{b_2} \dots (LS)^{a_n}(RS)^{b_n}(0) \end{aligned}$$

Remark that it suffices to use three generators T, W, S of Picard group except for $V = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ to describe the numbers in $\mathbb{Q}(i)$. This means that these numbers are the images of 0 or ∞ under the elements of a proper subgroup

$$\Gamma' := \langle L, R, S \rangle = \langle T, W, S \rangle$$

of Picard group.

Corollary 5.2. *Any number $z \in \mathbb{Q}(i)$ is the image of 0 or ∞ under the elements of the subgroup $\Gamma' = \langle L, R, S \rangle$. More explicitly, if*

$$z = [a_0 + b_0i, a_1 + b_1i, \dots, a_n + b_ni]$$

then $z = A(0)$ or $A(\infty)$ where

$$A := (LS)^{a_0}(RS)^{b_0}(L^2S)^{a_1}(R^{-1}S)^{b_1} \dots (L^{(j)}S)^{a_n}(R^{(3-2j)}S)^{b_n}$$

so that $j = 1$ or 2 while n is even or odd number respectively.

Definition 5.4. *The matrix A defined in corollary (5.2) is called the matrix of a finite continued fraction. If it is the matrix of n -th convergent of a (finite or infinite) continued fraction then it is called the matrix of n -th convergent and denoted by A_n .*

We know that any complex number is the limit of the sequence of its convergents, by Corollary 5.1. Therefore we can approximate to any complex number by a sequence in $\mathbb{Q}(i)$.

Corollary 5.3. *If z an irrational complex number then it can be approximated by applying infinitely many transformations in Γ' to the points 0 (or ∞). More explicitly,*

$$z = [a_0 + b_0i, a_1 + b_1i, \dots]$$

and $c_n = \frac{p_n}{q_n}$ be the n -th convergent of z . If A_n is the matrix of c_n then

$$z = \lim_{n \rightarrow \infty} A_{2n}(0) = \lim_{n \rightarrow \infty} A_{2n+1}(\infty)$$

and $A_n = \pm \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ or $\pm \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$ if n is odd or even respectively.

Proof We already discussed how the limits converge and a similar proof of Corollary 4.4 shows exactly what the entries of matrix A_n has to be. $\square$

Example 5.4. $z = \frac{-10}{9} + \frac{11i}{2} = [-1 + 6i, 2i, -2 + i, -1 + 2i, 1 + 2i]$

$$A_0 = - \begin{pmatrix} 1 & -1 + 6i \\ 0 & 1 \end{pmatrix} \text{ and } c_0 = A_0(0) = -1 + 6i,$$

$$A_1 = - \begin{pmatrix} -11 - 2i & -1 + 6i \\ 2i & 1 \end{pmatrix} \text{ and } c_1 = A_1(\infty) = \frac{-11-2i}{2i} = -1 + \frac{11}{2}i,$$

$$A_2 = \begin{pmatrix} -11 - 2i & 23 - i \\ 2i & -1 - 4i \end{pmatrix} \text{ and } c_2 = A_2(0) = \frac{23-i}{-1-4i} = \frac{-19+93i}{17},$$

$$A_3 = \begin{pmatrix} -32 + 45i & 23 - i \\ 9 + 4i & -1 - 4i \end{pmatrix} \text{ and } c_3 = A_3(\infty) = \frac{-32+45i}{9+4i} = \frac{-108+533i}{97},$$

$$A_4 = - \begin{pmatrix} -32 + 45i & -99 - 20i \\ 9 + 4i & 18i \end{pmatrix} \text{ and } c_4 = A_4(0) = \frac{-99-20i}{18i} = z$$

5.3 The action of $\text{PSL}_2(\mathbb{C})$ on the Hyperbolic 3-space

The group $\text{PSL}_2(\mathbb{C})$ acts on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by fractional linear transformations :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

We now want to describe the action of $\text{PSL}_2(\mathbb{C})$ on the upper half 3-space. In order to do so we will use quaternions. Let us denote this space by $\mathcal{H}_3$ and write its points as $z + tj$ where $z \in \mathbb{C}$ and $t \in \mathbb{R}^{>0}$:

$$\mathcal{H}_3 = \{z + tj : z \in \mathbb{C}, t \in \mathbb{R}^{>0}\} \cong \mathbb{C} \times (0, \infty)$$

Quaternions are generally represented in the form $x_1 + x_2 \mathbf{i} + x_3 \mathbf{j} + x_4 \mathbf{k}$ where x_i 's are real numbers, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$, are symbols that can be interpreted as unit-vectors pointing along the three spatial axes. So we may think the coefficient of $\mathbf{k}$ is always zero in expression of any point of $\mathcal{H}_3$.

We are now interested in how an element of $\text{PSL}_2(\mathbb{C})$ acts on $z + tj$ so that it will be an extension of the action on $\hat{\mathbb{C}}$. So elements act on the points by Mobius transformations. But we have to be careful in which order we multiply numbers involving both i and j . Indeed, the imaginary unit j , satisfies $ij = -ji$ and $j^2 = -1$. Therefore we choose to write our action as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z + tj) = (a(z + tj) + b)(c(z + tj) + d)^{-1} \quad (5.5)$$

Let us now see what $(c(z + tj) + d)^{-1}$ has to be. It should be a number so that when multiplied by $(c(z + tj) + d)$ gives 1 . We claim that it is a real multiple of $(\bar{z} - tj)\bar{c} + \bar{d}$. In order to see this, observe first that, if $z = x + iy$ then

$$j\bar{z} = j(x - yi) = xj - yji = xj + yij = (x + yi)j = zj$$

Therefore

$$\begin{aligned}
& (c(z + tj) + d)((\bar{z} - tj)\bar{c} + \bar{d}) \\
&= c(z + tj)(\bar{z} - tj)\bar{c} + c(z + tj)\bar{d} + d(\bar{z} - tj)\bar{c} + d\bar{d} \\
&= c(|z|^2 - ztj + ztj + t^2)\bar{c} + cz\bar{d} + cdtj + d\bar{z}\bar{c} - d\bar{c}tj + |d|^2 \\
&= |cz|^2 + |c|^2t^2 + cz\bar{d} + d\bar{z}\bar{c} + |d|^2 \\
&= |cz + d|^2 + |c|^2t^2
\end{aligned}$$

This implies

$$(c(z + tj) + d)^{-1} = \frac{(\bar{z} - tj)\bar{c} + \bar{d}}{|cz + d|^2 + |c|^2t^2}$$

(We are allowed to divide by non-zero real numbers since they commute with i and j and that is why we used fraction notation recklessly at the end.)

Proposition 5.8. *The group $\text{PSL}_2(\mathbb{C})$ acts on $\mathcal{H}_3 = \{z + tj : z \in \mathbb{C}, t \in \mathbb{R}^{>0}\}$ by the rule shown in (5.5)*

Proof Let $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. We see

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z + tj) &= (a(z + tj) + b)(c(z + tj) + d)^{-1} \\
&= \frac{(a(z + tj) + b)((\bar{z} - tj)\bar{c} + \bar{d})}{|cz + d|^2 + |c|^2t^2} \\
&= \frac{a(z + tj)(\bar{z} - tj)\bar{c} + a(z + tj)\bar{d} + b(\bar{z} - tj)\bar{c} + b\bar{d}}{|cz + d|^2 + |c|^2t^2} \\
&= \frac{a(|z|^2 - ztj + ztj + t^2)\bar{c} + az\bar{d} + adtj + b\bar{z}\bar{c} - b\bar{c}tj + b\bar{d}}{|cz + d|^2 + |c|^2t^2} \\
&= \frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2 + tj}{|cz + d|^2 + |c|^2t^2} \in \mathcal{H}_3
\end{aligned}$$

□

Briefly, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$ and $(z, t) \in \mathcal{H}_3$ then we may write

$$A : (z, t) \mapsto (z^*, t^*)$$

where

Note that if we set $t = 0$, it is seen that the elements of $\text{PSL}_2(\mathbb{C})$ still acts on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as fractional linear transformations. This shows that it is indeed an extension of Möbius transformation.

Corollary 5.4. $\text{PSL}_2(\mathbb{C})$ acts on the extended upper half 3-space

$$\hat{\mathcal{H}}_3 = \mathcal{H}_3 \cup \partial\mathcal{H}_3 \text{ where } \partial\mathcal{H}_3 = \mathbb{C} \cup \{\infty\}$$

by

$$A \cdot (z, t) = \left(\frac{(az + b)(\overline{cz + d}) + a\bar{c}t^2}{|cz + d|^2 + |c|^2t^2}, \frac{t}{|cz + d|^2 + |c|^2t^2} \right)$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C}) \text{ and } (z, t) \in \mathcal{H}_3.$$

5.4 Classification of elements of $\text{PSL}_2(\mathbb{C})$

Theorem 5.3. A non-identity element $T \in \text{PSL}_2(\mathbb{C})$ has either 1 or 2 fixed points in $\partial\mathcal{H}_3$. If it has 1, then it is conjugate to $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If it has 2, then it is conjugate

to $M_\lambda = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$ for some $\lambda \in \mathbb{C}$ with $\lambda \neq 0, 1$.

Proof Suppose that T has only two fixed points α and β in $\hat{\mathbb{C}}$. We may choose an element $A \in \text{PSL}_2(\mathbb{C})$ with $A(\alpha) = 0$ and $A(\beta) = \infty$ since the action of $\text{PSL}_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ is doubly transitive (in fact it is sharply 3-transitive). Then the conjugate ATA^{-1} fixes *only* 0 and ∞ . (Note that the number of fixed points are invariant under conjugation.

So ATA^{-1} cannot fix other points of $\hat{\mathbb{C}}$. This implies that $ATA^{-1} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for some complex number $a \neq -1, 0, 1$. Since a is a square root of complex number λ with $\lambda \neq 0, 1$, it has exactly the desired form. Suppose that T has a single fixed point α in $\hat{\mathbb{C}}$. If we choose a matrix $A \in \text{PSL}_2(\mathbb{C})$ such that $A(\alpha) = \infty$, $A(z_0) = 0$ and $A(T(z_0)) = 1$ for some z_0 , then ATA^{-1} has to fix ∞ and also $ATA^{-1}(0) = 1$. This means that $ATA^{-1} = \begin{pmatrix} a & \frac{1}{a} \\ 0 & \frac{1}{a} \end{pmatrix}$ for some non-zero complex number a . Since ATA^{-1} cannot fix any point in $\hat{\mathbb{C}}$ other than ∞ , it follows that $a = \mp 1$, which completes the proof. $\square$

Briefly, every element of $\mathrm{PSL}_2(\mathbb{C})$ is conjugate to some M_λ for a suitable non-zero complex number λ . Remark that the converse of the theorem still works due to the fact that the number of fixed points is invariant under conjugation. More explicitly, for any group action, if two elements conjugate to each other and they have non empty fixed points, then we have a bijection between their sets of fixed points. Hence, if $\lambda = 1$ then an element which is conjugate to M_λ has to fix only one point, or else exactly two points in $\partial\mathcal{H}_3$. In addition, $\mathrm{tr}^2(M_\lambda) = \lambda + \frac{1}{\lambda} + 2$ and this still holds true for $\lambda = 1$. In fact, in the next theorem, we see that the trace-square parametrizes the conjugacy classes. Moreover, it will play one of the main role for classifying the elements of $\mathrm{PSL}_2(\mathbb{C})$.

Theorem 5.4. *Let $A, B \in \mathrm{PSL}_2(\mathbb{C})$ be non-identity matrices. Then A and B are conjugate if and only if $\mathrm{tr}^2(A) = \mathrm{tr}^2(B)$.*

Proof For brevity, we use $\sim$ to denote conjugacy in $\mathrm{PSL}_2(\mathbb{C})$. Suppose that $A \sim B$. So $A = WBW^{-1}$ for some W . Hence, we get immediately

$$\mathrm{tr}(A) = \mathrm{tr}((WB)W^{-1}) = \mathrm{tr}(W^{-1}(WB)) = \mathrm{tr}(B),$$

using the following fact : $\mathrm{tr}(UV) = \mathrm{tr}(VU)$ for all matrices $U, V \in M_n(R)$ where R is a commutative ring. Thus $\mathrm{tr}^2(A) = \mathrm{tr}^2(B)$. Conversely, we now assume that $\mathrm{tr}^2(A) = \mathrm{tr}^2(B)$. By the previous Theorem 5.3, we know that A and B are each conjugate to one of the matrices M_λ , say $A \sim M_p$ and $B \sim M_q$ for some $p, q \in \mathbb{C}^*$. Thus

$$p + \frac{1}{p} + 2 = \mathrm{tr}^2(M_p) = \mathrm{tr}^2(A) = \mathrm{tr}^2(B) = \mathrm{tr}^2(M_q) = q + \frac{1}{q} + 2$$

and this implies that $p = q$ or $p = \frac{1}{q}$. Now note that $M_p \sim M_{\frac{1}{p}}$: this is trivial if $p = 1$ while if $p \neq 1$, we see

$$SM_pS^{-1} = M_{1/p} \quad \text{with } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hence we now have $A \sim M_p \sim M_q \sim B$ and this implies that $A \sim B$ since conjugacy is an equivalence relation, which proves our argument. $\square$

Definition 5.5. *A non-identity element $A \in \mathrm{PSL}_2(\mathbb{C})$ is called parabolic if it satisfies one of the following equivalent statements :*

- $\mathrm{tr}^2(A) = 4$
- it is conjugate to M_1

- it has a single fixed point in $\partial\mathcal{H}_3$.

Remark 5.3. If we consider the action of the element M_1 on the upper half-space $\mathcal{H}_3$,

$$M_1 : (z, t) \mapsto (z + 1, t)$$

we see that it has no fixed point in $\mathcal{H}_3$ since it translates all points of the space. Hence parabolic elements necessarily cannot fix any point in $\mathcal{H}_3$ while having a single fixed point in $\partial\mathcal{H}_3$.

Definition 5.6. A non-identity element $A \in \text{PSL}_2(\mathbb{C})$ is called elliptic if it is conjugate to M_λ for some $\lambda \neq 1$ with $|\lambda| = 1$. More explicitly, if $\lambda = e^{i\theta}$, an elliptic element is conjugate to $\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$ where $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$.

If $\lambda \neq 1$ and $|\lambda| = 1$, we have

$$\text{tr}^2(M_\lambda) = (e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}})^2 = e^{i\theta} + e^{-i\theta} + 2 = 2\cos\theta + 2$$

which implies $\text{tr}^2(M_\lambda) \in [0, 4)$, and therefore $\text{tr}^2(A) \in [0, 4)$ for all elliptic elements A . Conversely, for any $A \in \text{PSL}_2(\mathbb{C})$ with $\text{tr}^2(A) \in [0, 4)$, we can find some M_λ (by choosing suitable θ) such that $\text{tr}^2(M_\lambda) = \text{tr}^2(A)$. This means that A is conjugate to some M_λ where $\lambda \neq 1$ and $|\lambda| = 1$, i.e. A has to be an elliptic element. Then we get another characterization of elliptic elements :

$$A \text{ is an elliptic element} \iff \text{tr}^2(A) \in [0, 4).$$

If we consider the action of an elliptic element on the upper half-space $\mathcal{H}_3$, we see that

$$M_\lambda : (z, t) \mapsto (e^{i\theta}z, t)$$

which rotates the whole space $\mathcal{H}_3$ around vertical axis $\mathbb{R}j$. Note that the motion still fixes two points in the boundary $\partial\mathcal{H}_3$, now at the origin and infinity. But also, M_λ fixes infinitely many points in $\mathcal{H}_3$. This shows any elliptic element fixes infinitely many points in $\mathcal{H}_3$ while fixing only two points in the boundary. In fact this is another characterization for elliptic transformations.

Corollary 5.5. Let $A \in \text{PSL}_2(\mathbb{C})$ be any non-identity element. The following statements are equivalent :

- $\text{tr}^2(A) \in [0, 4)$
- A is an elliptic element

- A fixes only two points in $\partial\mathcal{H}_3$ and fixes infinitely many points in $\mathcal{H}_3$.

Definition 5.7. If a non-identity element $A \in \text{PSL}_2(\mathbb{C})$ is conjugate to M_λ for some $|\lambda| \neq 1$ then it is called *loxodromic*. In addition, if λ is a positive real number then it is called *hyperbolic*, or else *strictly loxodromic*.

Note that if $|\lambda| \neq 1$, $\text{tr}^2(M_\lambda) = \lambda + \frac{1}{\lambda} + 2 \in \mathbb{C} - [0, 4]$, so it follows that

$$A \text{ is loxodromic} \iff \text{tr}^2(A) \in \mathbb{C} - [0, 4].$$

In addition, if $\lambda \in \mathbb{R}^{>0}$ then $\text{tr}^2(M_\lambda) \in (4, \infty)$. So this implies

$$A \text{ is hyperbolic} \iff \text{tr}^2(A) \in (4, \infty)$$

The action of M_λ on $\mathcal{H}_3$ as follows :

$$M_\lambda : (z, t) \mapsto (\lambda z, |\lambda|t)$$

As $|\lambda| \neq 1$, M_λ will change the height of all points of the space. This means that there is no any fixed point in $\mathcal{H}_3$. Then any loxodromic element does not fix any point in the upper half-space $\mathcal{H}_3$ while fixing exactly two points on the boundary.

Corollary 5.6. Let $A \in \text{PSL}_2(\mathbb{C})$ be a non-identity element. Then the following are equivalent :

- $\text{tr}^2(A) \in \mathbb{C} - [0, 4]$
- A is loxodromic
- A has only two fixed points in $\partial\mathcal{H}_3$ and no fixed point in $\mathcal{H}_3$.

Example 5.5.

$$L = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} : (z, t) \mapsto \left(1 - \frac{\bar{z}}{|z|^2 + t^2}, \frac{t}{|z|^2 + t^2}\right)$$

is an elliptic element since $\text{tr}^2(L) = 1 \in [0, 4]$. So we know that L is conjugate to

$$M_\lambda : (z, t) \mapsto (e^{i\theta}z, t)$$

for some $\theta \in [0, 2\pi)$ and L rotates points in $\mathcal{H}_3$ about the geodesic joining two points $\frac{1-\sqrt{3}i}{2}$ and $\frac{1+\sqrt{3}i}{2}$ at the boundary. For computing the angle of the rotation, it suffices to equalize trace-squares of L and M :

$$1 = \text{tr}^2(L) = \text{tr}^2(M_\lambda) = \lambda + \frac{1}{\lambda} + 2$$

which implies $\lambda = e^{i\frac{2\pi}{3}}$ or $e^{i\frac{4\pi}{3}}$. Hence this implies that L makes a clockwise rotation through the angle $\frac{2\pi}{3}$.

Example 5.6.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : (z, t) \mapsto \left(\frac{-\bar{z}}{|z|^2 + t^2}, \frac{t}{|z|^2 + t^2} \right)$$

is an elliptic element as $\text{tr}(S)^2 = 0 \in [0, 4)$. So it is conjugate to

$$M_\lambda : (z, t) \mapsto (e^{i\theta} z, t)$$

for some $\theta \in [0, 2\pi)$ and it rotates the points of space $\mathcal{H}_3$ about the geodesic joining the points $-i$ and i at the boundary $\partial\mathcal{H}_3$. As trace-squares of S and M_λ are the same,

$$0 = \text{tr}(S)^2 = \text{tr}^2(M_\lambda) = \lambda + \frac{1}{\lambda} + 2$$

implies $\lambda = e^{i\pi}$, and so the angle of rotation of S is π .

Example 5.7.

$$R = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} : (z, t) \mapsto \left(i - \frac{\bar{z}}{|z|^2 + t^2}, \frac{t}{|z|^2 + t^2} \right)$$

is a hyperbolic element since $\text{tr}(R)^2 = -1 \in \mathbb{R} - [0, 4]$. So it translates the points of the space $\mathcal{H}_3$ along the geodesic joining its fixed points $\frac{1 \pm \sqrt{5}}{2}i$ in $\partial\mathcal{H}_3$.

5.5 Geometric interpretation of complex continued fractions

In previous chapter, we presented a geometrical interpretation to continued fractions for real numbers and showed how modular group $\Gamma = \text{PSL}_2(\mathbb{Z})$ governs the theory of real continued fractions. We now know that every rational number can be expressed by a matrix in Γ , and so any irrational number can be expressed by the sequence of its convergent matrices in Γ . We also have shown that each element of modular group corresponds to a finite path which starts from the edge $\gamma = \{\zeta_4, \zeta_6\}$ of Farey graph $\mathcal{F}$, and so the sequence of the convergent matrices of an irrational number forms an infinite path starting from the edge γ and that path goes down to given irrational in the boundary.

No doubt, the choice of a suitable γ is quite important, which is in fact a 1-dimensional cell complex in the upper half plane $\mathcal{H}_2$. Geometrically, γ is an arc on the unit circle joining the fixed points of two elliptic elements S and L that are generators of $\text{PSL}_2(\mathbb{Z})$ and $\text{PSL}_2(\mathbb{Z})$ moves γ freely.

In this chapter, we generalize our geometric approach for the complex numbers by choosing a suitable 2-dimensional cell complex

$$\gamma' = \left\{ (z, t) \in \mathcal{H}_3 : |z|^2 + t^2 = 1, \quad 0 \leq \operatorname{Re}(z) \leq \frac{1}{2} \text{ and } |\operatorname{Im}(z)| \leq \frac{1}{2} \right\},$$

which is infact a geodesic surface. Indeed, in real case, γ was obtained by joining the fixed points of L and S in $\mathcal{H}_2$. Similary these matrices now fix geodesics pointwise in $\mathcal{H}_3$ as shown in the examples 5.6 and 5.7. Hence it is seen that γ' is obtained by joining these two geodesics which are parallel to each others with the constraint $|\operatorname{Im}(z)| \leq \frac{1}{2}$ in $\mathcal{H}_3$. So we may call it a geodesic surface. As $\Gamma' = \langle S, L, R \rangle$ acts on the γ' freely, we may label each which makes our 2-cell suitable.

Definition 5.8. Γ' -orbit of γ' is called the **Hurwitz quilt** and any element of Hurwitz quilt is called **leaf**.

Remark 5.4. As Γ' acts freely on Hurwitz quilt, we can label each element of Hurwitz quilt by a distinct element of Γ' , see Figure 5.3.

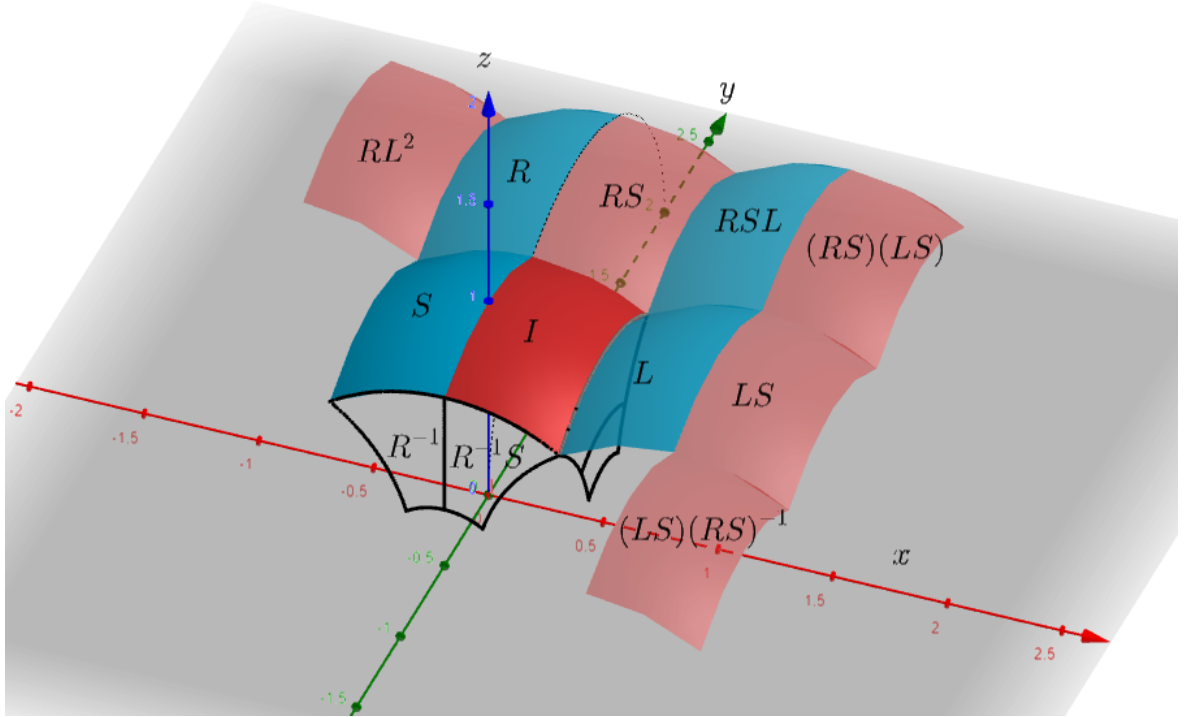


FIGURE 5.3 – γ' is shown in red color. Some elements of Γ' -orbit of γ' , that is a part of Hurwitz quilt.

5.5.1 The action of convergent matrices

Any leaf of Hurwitz quilt is geometrically a connected subset of a hemisphere or a vertical plane which is orthogonal to the boundary. In fact, each leaf labelled by a

convergent matrix of a complex number (except for $1/a, i/b$ with $a, b \in \mathbb{Z}$) sits on a hemisphere. Here, we associate each convergent of a complex number to a distinct disk in the complex plane which is obtained by projecting the corresponding hemisphere vertically onto the complex plane. In this section, we will investigate some properties of these disks and relations with the convergents.

Proposition 5.9. *Any circle in the complex plane is a locus of points satisfying*

$$|z - u| = \lambda|z - v|, \quad (5.6)$$

where u, v are distinct complex numbers and $0 < \lambda < 1$. That circle is centered at the point

$$\zeta = \frac{u - \lambda^2 v}{1 - \lambda^2}$$

of radius

$$\rho = \frac{\lambda|u - v|}{1 - \lambda^2}.$$

Proof If we let $z = w + v$ and $c = u - v$; then the equation (5.6) becomes $|w - c| = \lambda|w|$. Upon squaring and transposing terms, this can be written as

$$|w|^2 (1 - \lambda^2) - 2 \operatorname{Re}(w\bar{c}) + |c|^2 = 0.$$

Dividing by $1 - \lambda^2$, completing the square of the left side, and taking the square root will yield that

$$\left| w - \frac{c}{1 - \lambda^2} \right| = |c| \frac{\lambda}{1 - \rho^2}$$

Therefore (5.6) is equivalent to

$$\left| z - v - \frac{u - v}{1 - \lambda^2} \right| = |u - v| \frac{\lambda}{1 - \lambda^2}.$$

which is a circle centered at $\zeta = \frac{u - \lambda^2 v}{1 - \lambda^2}$ with radius $r = \frac{\lambda|u - v|}{1 - \lambda^2}$. $\square$

Hence this yields the following result.

Proposition 5.10. *If A_n be the n -th convergent matrix of a Hurwitz expansion then it moves the unit circle in the complex plane to a circle which is centered at the point*

$$\zeta_n = \frac{p_n \bar{q}_n - p_{n-1} \bar{q}_{n-1}}{|q_n|^2 - |q_{n-1}|^2} \text{ of radius } \rho_n = \frac{1}{|q_n|^2 - |q_{n-1}|^2}.$$

Proof For an odd natural number n , suppose that $A_n \cdot z = w$ where $A_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ and $|z| = 1$. Then

$$z = A_n^{-1} \cdot w = \frac{q_{n-1}w - p_{n-1}}{-q_n w + p_n}.$$

Taking absolute value of both sides yields,

$$\begin{aligned}
| -q_n w + p_n | &= | q_{n-1} w - p_{n-1} |, \\
|q_n| \left| w - \frac{p_n}{q_n} \right| &= |q_{n-1}| \left| w - \frac{p_{n-1}}{q_{n-1}} \right|, \\
\left| w - \frac{p_n}{q_n} \right| &= \left| \frac{q_{n-1}}{q_n} \right| \left| w - \frac{p_{n-1}}{q_{n-1}} \right|
\end{aligned} \tag{5.7}$$

which gives an equation of circle. After setting $u = p_n/q_n, v = p_{n-1}/q_{n-1}$ and $0 < \lambda = |q_{n-1}/q_n| < 1$ as in the proposition (5.9), we can compute its center and radius as follows :

$$\begin{aligned}
\zeta_n &= \frac{u - \lambda^2 v}{1 - \lambda^2} = \frac{\frac{p_n}{q_n} - \left| \frac{q_{n-1}}{q_n} \right|^2 \frac{p_{n-1}}{q_{n-1}}}{1 - \left| \frac{q_{n-1}}{q_n} \right|^2} = \frac{p_n \overline{q_n} - p_{n-1} \overline{q_{n-1}}}{|q_n|^2 - |q_{n-1}|^2} \\
\rho_n &= \frac{\lambda |u - v|}{1 - \lambda^2} = \frac{\left| \frac{q_{n-1}}{q_n} \right| \left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right|}{1 - \left| \frac{q_{n-1}}{q_n} \right|^2} = \frac{1}{|q_n|^2 - |q_{n-1}|^2}
\end{aligned}$$

Note that for an even natural number n , A_n gives rise to the same equation as in (5.7), hence the same formulas for center and radius are still valid. $\square$

Proposition 5.11. *Let A_n be the n -th convergent matrix of a complex number z . Suppose that $B_n := B(\zeta_n, \rho_n)$ is an open disk in the complex plane where ζ_n and ρ_n are defined as in (5.10). Then*

1. $c_n = \frac{p_n}{q_n} \in B_n$ for all $n \in \mathbb{N}$.
2. $z \in B_n$ for all $n \in \mathbb{N}$.
3. $\lim_{n \rightarrow \infty} \rho_n = 0$ if $z \in \mathbb{C} - \mathbb{Q}(i)$.

Proof 1)

$$\begin{aligned}
|c_n - \zeta_n| &= \left| \frac{p_n}{q_n} - \frac{p_n \overline{q_n} - p_{n-1} \overline{q_{n-1}}}{|q_n|^2 - |q_{n-1}|^2} \right| = \left| \frac{p_n |q_n|^2 - p_n q_{n-1} \overline{q_{n-1}} - p_n |q_n|^2 + p_{n-1} \overline{q_{n-1}} q_n}{q_n (|q_n|^2 - |q_{n-1}|^2)} \right| \\
&= \frac{| -\overline{q_{n-1}} | |p_n q_{n-1} - p_{n-1} q_n|}{|q_n| (|q_n|^2 - |q_{n-1}|^2)} \\
&< \frac{1}{|q_n|^2 - |q_{n-1}|^2} = \rho_n.
\end{aligned}$$

2)

$$\begin{aligned}
z - \zeta_n &= \frac{p_n + z_n p_{n-1}}{q_n + z_n q_{n-1}} - \frac{p_n \overline{q_n} - p_{n-1} \overline{q_{n-1}}}{|q_n|^2 - |q_{n-1}|^2} \\
&= \frac{-p_n |q_{n-1}|^2 + z_n p_{n-1} |q_n|^2 - p_n \overline{q_n} z_n q_{n-1} + p_{n-1} \overline{q_{n-1}} q_n}{(q_n + z_n q_{n-1}) (|q_n|^2 - |q_{n-1}|^2)} \\
&= \frac{\overline{q_{n-1}} (-p_n q_{n-1} + p_{n-1} q_n) + z_n \overline{q_n} (p_{n-1} q_n - p_n q_{n-1})}{(q_n + z_n q_{n-1}) (|q_n|^2 - |q_{n-1}|^2)} \\
&= \frac{(\overline{q_{n-1}} + z_n \overline{q_n}) (p_{n-1} q_n - p_n q_{n-1})}{(q_n + z_n q_{n-1}) (|q_n|^2 - |q_{n-1}|^2)}
\end{aligned}$$

Hence we get

$$|z - \zeta_n| = \frac{|\overline{q_{n-1}} + z_n \overline{q_n}|}{|q_n + z_n q_{n-1}| (|q_n|^2 - |q_{n-1}|^2)}.$$

This means that to prove the inequality

$$|z - \zeta_n| < \frac{1}{|q_n|^2 - |q_{n-1}|^2}$$

it suffices to show the following inequality.

$$|\overline{q_{n-1}} + z_n \overline{q_n}| < |q_n + z_n q_{n-1}| \quad (5.8)$$

3) As $(|q_n|)_n$ is a strictly increasing sequence, we have $0 < \left| \frac{q_{n-1}}{q_n} \right| < 1$ and by proposition 5.7, we know $\left| \frac{q_{n-1}}{q_n} \right| \not\rightarrow 1$, so it follows that

$$|q_n|^2 \left(1 - \left| \frac{q_{n-1}}{q_n} \right|^2 \right) \rightarrow \infty$$

This implies

$$\rho_n = \frac{1}{|q_n|^2 - |q_{n-1}|^2} = \frac{1}{|q_n|^2 \left(1 - \left| \frac{q_{n-1}}{q_n} \right|^2 \right)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

which proves the desired result. $\square$

Remark 5.5. If $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ and $(A_n)_n$ is the sequence of its convergent matrices, then as a result of the fact (1) and (3) in proposition 5.11, one conclude that the leaves labelled by the convergent matrices A_n are getting smaller and closing to the boundary as n goes to infinity. Moreover, the distance of these leaves to the number z approaches to zero.

Remark 5.6. After finding a suitable subgroup of Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$ and suitable 2-cell complex in $\mathcal{H}_3$, the propositions 5.10 and 5.11 in fact can be used for other norm Euclidean fields.

Example 5.8. Let $z = \frac{1-i}{3} \in \mathbb{Q}(i)$. Then its Hurwitz expansion is $[0, 2 + 2i, -1 + i]$ with convergents matrices

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ 2 + 2i & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -1 + i \\ 2 + 2i & -3 \end{pmatrix}$$

and its convergents $c_0 = 0, c_1 = \frac{1-i}{4}, c_2 = z$. These convergent matrices moves the unit disk to the disks $B_0 = B(0, 1), B_1 = B(\frac{2-2i}{7}, \frac{1}{7})$ and $B_2 = B(1-i, 1)$ respectively. Each leaf in Hurwitz quilt labelled by A_0, A_1 and A_2 is in fact a part of some hemisphere. So these three disks can be obtained by projecting of hemispheres corresponding to these convergent matrices.

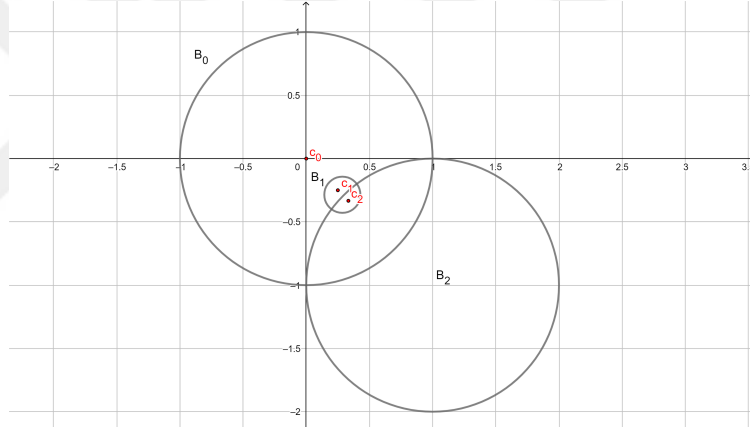


FIGURE 5.4 – All three disks B_0, B_1, B_2 correspond to the convergent matrices A_0, A_1, A_2 for $z = \frac{1-i}{3}$.

5.5.2 Geodesics and Convergent Matrices

Now, let us consider the oriented geodesic $\mathbb{I} = \langle \infty : 0 \rangle$ from ∞ to 0 that is a vertical ray. For any $A \in \mathrm{PSL}_2(\mathbb{Z}[i])$, note that $A \cdot \mathbb{I}$ is still an oriented geodesic from $A(\infty)$ to $A(0)$ as $\mathrm{PSL}_2(\mathbb{Z}[i])$ preserves the boundary $\partial\mathcal{H}_3$. So any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Picard group moves the geodesic $\mathbb{I}$ as follows

$$A \cdot \mathbb{I} = \langle A(\infty) : A(0) \rangle = \langle \frac{a}{c} : \frac{b}{d} \rangle.$$

Moreover, if $A \cdot \mathbb{I} = \mathbb{I}$ then it follows that

$$\langle \frac{a}{c} : \frac{b}{d} \rangle = \langle \infty : 0 \rangle$$

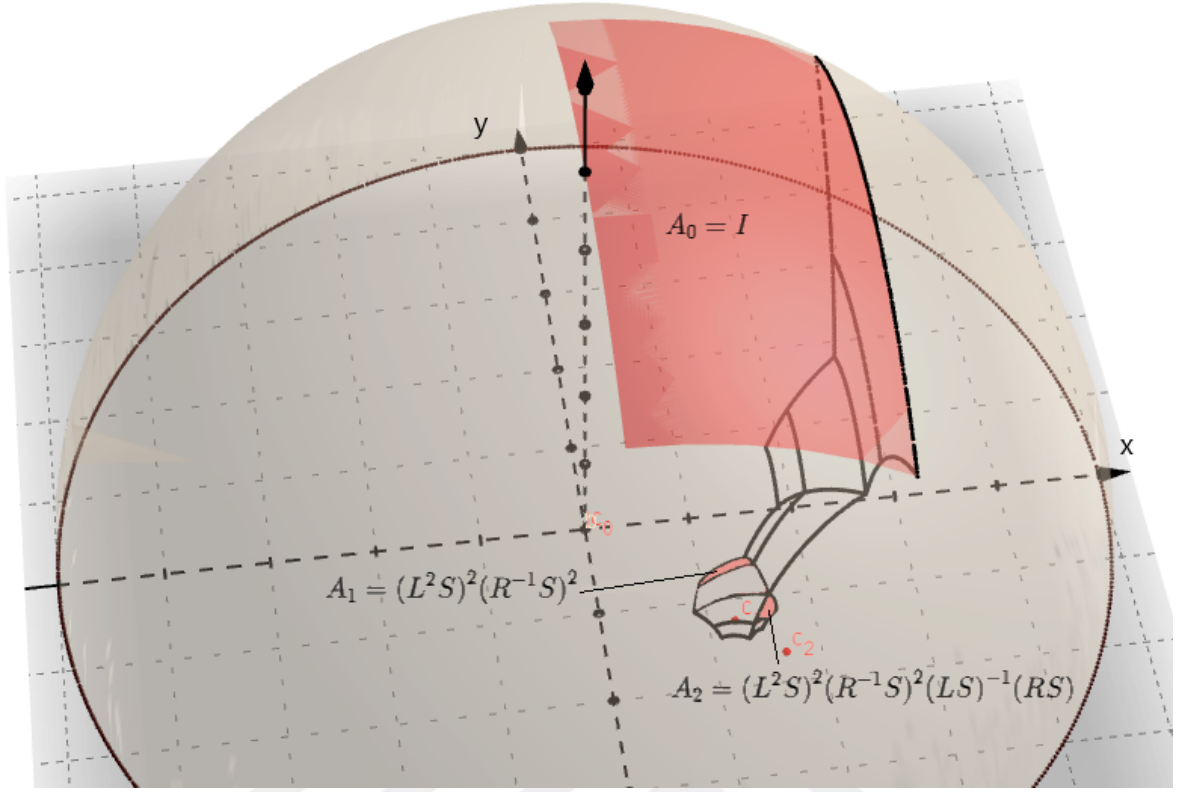


FIGURE 5.5 – This is a view of the sub-cell complex of Hurwitz quilt determined by $z = \frac{1-i}{3}$. The leaves labelled by convergent matrices are colored in pink. As z is a rational complex number, the sub-cell complex consists of only finitely many leaves.

and so $\frac{a}{c} = \infty$ and $\frac{b}{d} = 0$ with $ad - bc = 1$, which implies $A = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Picard group. This means that for each A , we get different oriented geodesic $A \cdot \mathbb{I}$. Hence we conclude the following proposition.

Proposition 5.12. *The map*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \langle \frac{a}{c} : \frac{b}{d} \rangle$$

defines a injection from $\text{PSL}_2(\mathbb{Z}[i])$ to the set of oriented geodesics

$$\mathcal{G} = \{ \langle \frac{a}{c} : \frac{b}{d} \rangle : a, b, c, d \in \mathbb{Z}[i] \text{ and } ad - bc = 1 \}$$

.

Example 5.9. *For $a, b \in \mathbb{Z}$, the matrices $T^a W^b = \begin{pmatrix} 1 & a+bi \\ 0 & 1 \end{pmatrix}$ and $T^a W^b S = \begin{pmatrix} a+bi & -1 \\ 1 & 0 \end{pmatrix}$ correspond to the geodesics $\langle \infty, a+bi \rangle$ and $\langle a+bi : \infty \rangle$ respectively, that are vertical rays while all other matrices in $\text{PSL}_2(\mathbb{Z}[i])$ correspond to a geodesic which is a semi circle orthogonal to the boundary.*

- Remark 5.7.** 1. The set of vertices of geodesics in $\mathcal{G}$ consists of all numbers in $\mathbb{Q}(i)$ together with ∞ .
2. Two rational numbers r/s and r'/s' are connected by a geodesic in $\mathcal{G}$ if and only if $|rs' - r's| = 1$.

If z is any complex number and A_n is the matrix of n -th convergent $c_n = \frac{p_n}{q_n}$ then

$$A_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \text{ or } \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$$

the corresponding geodesic of A_n is

$$\langle A_n(\infty) : A_n(0) \rangle = \begin{cases} \langle c_n : c_{n-1} \rangle & \text{if } n \text{ is odd} \\ \langle c_{n-1} : c_n \rangle & \text{if } n \text{ is even} \end{cases}$$

where $A_{-1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to the geodesic $\mathbb{I}$ and $c_{-1} = \infty$. Hence the sequence of the matrices A_n gives the sequence of geodesics

$$\langle c_{-1} : c_0 \rangle, \langle c_1 : c_0 \rangle, \langle c_1 : c_2 \rangle, \langle c_3 : c_2 \rangle, \dots$$

Except for the initial geodesic $\langle c_{-1}, c_0 \rangle$, all geodesics here are semi-circles. This shows that consecutive convergents of a complex number z are connected to each others by semi-circles othogonal the complex plane. Hence Hurwitz expansion of z gives a path from ∞ to z in the hyperbolic 3-space.

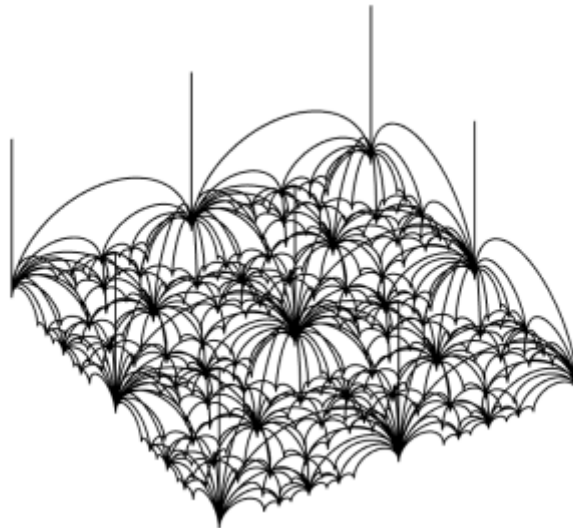


FIGURE 5.6 – Γ' -orbit of $\mathbb{I} = \langle \infty : 0 \rangle$

Example 5.10. *The convergent matrices A_0, A_1, A_2 for $z = \frac{1-i}{3}$ in example 5.8 corresponds the geodesics $\langle \infty : 0 \rangle, \langle \frac{1-i}{4} : 0 \rangle, \langle \frac{1-i}{4} : z \rangle$, see Figure 5.7.*

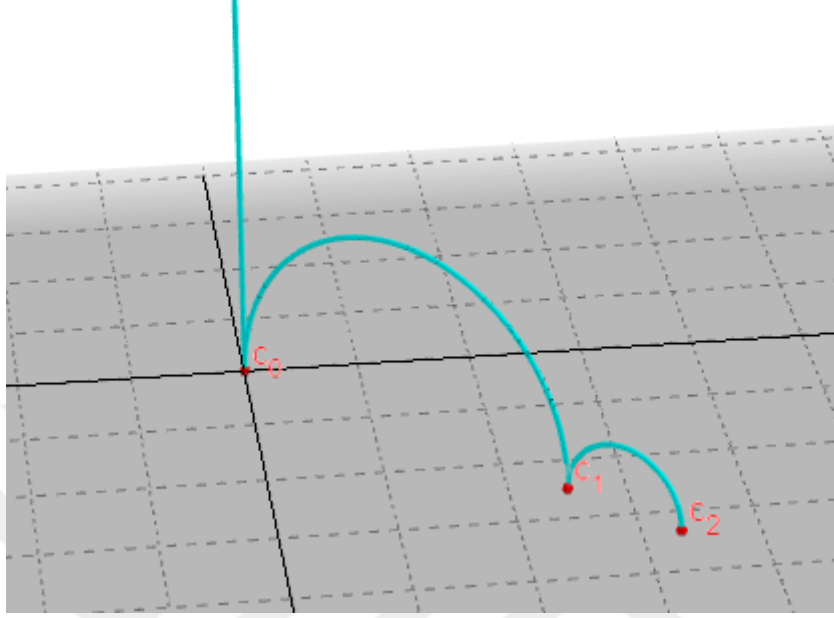


FIGURE 5.7 – Geodesics correspond to the convergent matrices A_0, A_1, A_2 of $z = \frac{1-i}{3}$.

5.6 Conclusion

In this thesis we first interpreted the classical theory of simple continued fractions for real numbers in terms of one dimensional cell-complex (Farey graph) admitting an action of the modular group. Thanks to this free action we labelled one the elements of the modular group to label one-cells of the bipartite Farey tree and we associated a finite (resp. infinite) path (a subcell-complex) based at the edge labeled I to any finite (resp. infinite) simple continued fraction.

To generalize the theory for complex numbers, we first fixed the number field $\mathbb{Q}(i)$ and focus on the continued fraction whose entries lie in the ring of integers $\mathbb{Z}[i]$. This time, we constructed a two dimensional cell-complex which is called Hurwitz quilt. It admits a free action of a particular subgroup Γ' of Picard group $\text{PSL}_2(\mathbb{Z}[i])$ and two cells of this cell-complex is labelled with elements of Γ' . Thanks to this construction, to any finite (resp. infinite) simple continued fraction with Gauss integer coefficients, we associate a finite (resp. infinite) 2 dimensional subcell-complex of Hurwitz quilt.

This geometric interpretation of Gaussian integer continued fraction shows that if one find a suitable subgroup of corresponding Bianchi group, the theory of continued frac-

tions for complex numbers can be generalized to hold over the other norm-Euclidean ring of integers.



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