

CAPACITATED ASSORTMENT OPTIMIZATION AND PRICING PROBLEMS UNDER MIXED MULTINOMIAL LOGIT MODEL

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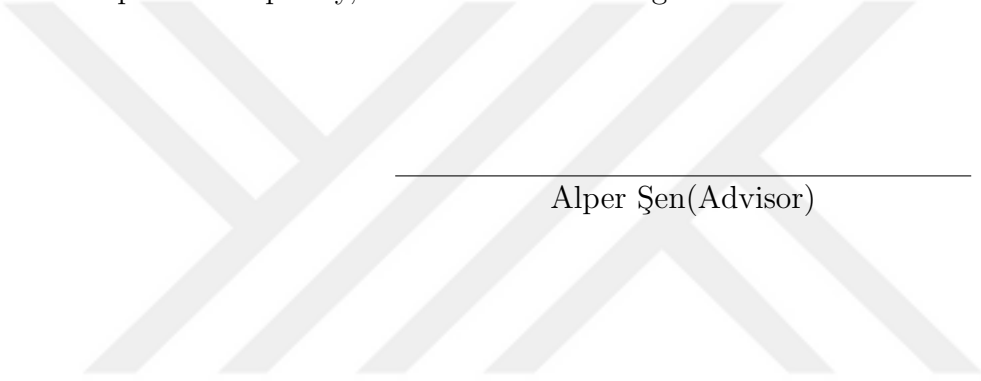
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August 2016

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We certify that we have read this thesis and that in our opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Master of Science.



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ABSTRACT

CAPACITATED ASSORTMENT OPTIMIZATION AND PRICING PROBLEMS UNDER MIXED MULTINOMIAL LOGIT MODEL

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We study capacitated assortment optimization problem under mixed multinomial logit model where a retailer wants to choose the set of products to offer to various customer segments with the goal of maximizing revenue while satisfying different capacity constraints. Each customer segment is identified with a unique purchase behaviour modelled by multinomial logit demand. We consider three general cases of capacity constraints: single resource constraint, multiple resource constraints and multiple cardinality constraints. This problem is NP-hard and there exist two approaches to find exact solutions: formulating the problem as a mixed integer linear program (MILP) or a mixed integer conic quadratic program (CONIC). For each constraint structure, we develop new efficient procedures to derive McCormick valid inequalities. We provide extensive numerical studies the results of which demonstrate that when the CONIC model is accompanied with the McCormick inequalities, the problem can be solved effectively even for large sized instances using a commercial optimization software. We also study joint pricing and assortment optimization problem with a single cardinality constraint and establish a new procedure to construct McCormick inequalities. We then present the related numerical studies which indicate that the CONIC formulation accomplishes the best outcome in the presence of the McCormick inequalities.

Keywords: Assortment optimization, mixed multinomial logit, capacity constraints, pricing, conic programming.

ÖZET

KARIŞIK MULTİNOM LOGİT MODEL ALTINDA KAPASİTE KISITLI ÇEŞİT OPTİMİZASYONU VE FİYATLANDIRMA PROBLEMLERİ

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Bu çalışmada müşterileri her biri farklı multinom tercih modeline göre karar veren farklı segmentlerden gelen ve değişik kapasite kısıtları olan bir perakendecinin ürün çeşidi belirleme ve fiyatlandırma problemleri incelenmiştir. Tek kaynak kısıtı, çoklu kaynak kısıtları ve çoklu kardinalite kısıtları olmak üzere üç farklı genel kapasite kısıtı durumu dikkate alınmıştır. Bu problem NP-zor bir problemdir ve literatürde tam çözümün bulunması için problemi karışık tamsayılı doğrusal program (MILP) ya da karışık tamsayılı konik karesel program (CONIC) olarak modellenmesi önerilmiştir. Bu çalışmada her bir kısıt yapısı için McCormick geçerli eşitsizliklerini elde edecek yeni etkili yöntemler geliştirilmiştir. Sayısal çalışmalarımız CONIC modelin McCormick eşitsizlikleriyle birlikteliği durumunda problemin büyük boyutlarda bile bir ticari optimizasyon yazılımı ile etkin bir şekilde çözülebildiğini göstermektedir. Ayrıca bir adet kardinalite kısıtı altında fiyatlandırma ve çeşit kararlarının birlikte verildiği bir optimizasyon problemi incelenmiş, bu problem için McCormick eşitsizliklerinin bulunması için bir yöntem önerilmiş ve bu yöntemin CONIC formülasyonda etkili sonuçlar verdiğini gösteren sayısal çalışmalar yapılmıştır.

Anahtar sözcükler: Çeşit optimizasyonu, karışık multinom logit, kapasite kısıtları, fiyatlandırma, konik programlama.

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Chapter 1

Introduction

The revenue of a retailer depends to a great extent on the demand of customers for the offered products in the store, whether it is physical or online. Therefore, selecting and displaying the right set of products according to the customer preferences is one of the most important decisions that should be made by retailers. There are also some resource constraints (such as budget or shelf space) involved which can affect the decision making process. This problem of choosing a set of products to offer to the customers while satisfying various resource constraints with the goal of maximizing revenue, is well-known as assortment planning problem. This problem is receiving a growing attention both in practice and academia (see [1], for an extensive review). In this research, we model assortment planning problem as a mathematical program and demonstrate that it can be solved efficiently via common commercial optimization systems.

In practice, when a customer does not find his favourite product in the offered assortment, he will either choose another product in the same category with the highest preference based on its characteristics to substitute with his favourite one, or decide not to buy at all [2]. Therefore, a proper demand model should capture this substitution behaviour of the customer. Multinomial logit (MNL) model is a utility based and one of the most popular methods to model this customer choice process. MNL model is commonly used in operations management, marketing

and economics literature [3, 4, 5, 1].

MNL is a simple model and has been shown to be effective in estimating demand from histories of sales [6, 7] and has also been successfully implemented in real world situations such as revenue management of an airline industry [7]. Moreover, it can be used in modelling demand in assortment planning problem and solved efficiently along with many practical business constraints and applications [8, 9, 10, 11, 12, 13]. In spite of these advantages, MNL bears some inconveniences among which the independence of irrelevant alternatives (IIA) is the most notable one. IIA is an unrealistic property which states that the ratio of purchase probabilities of any two product alternatives is independent of the assortment that includes them. To diminish this inadequacy, researchers have studied an extension of MNL known as the nested logit (NL) model where products with similar characteristics are put in the same set or so-called nest, and a customer first selects a nest he desires to buy from, then he chooses the most favourable product in that nest according to a MNL choice process [5]. NL is also recently used in modelling customer choice process in assortment optimization [14, 15, 16, 17, 18, 19, 20]. Although under NL model the IIA property is no longer a problem among any two products from different nests, it still holds between products of the same nest.

Mixed multinomial logit (MMNL) model is an extension of MNL which completely eradicates IIA property. It assumes the existence of various customer types where each customer type makes a purchase according to a unique MNL model which differs from the choice models of other customer types. This assumption is more realistic than that of MNL which considers only one customer type. Another important benefit of MMNL model is that any random utility maximization choice process can be approximated by MMNL as close as one requires [21]. These advantages make the MMNL a quite interesting choice process to model the demand in assortment optimization problems [22, 23, 24, 25, 26, 27, 28]. Nevertheless, assortment optimization under MMNL is NP-hard even with the existence of only two customer segments [24]. Due to this computational difficulty, most studies in this area focus on developing near optimal heuristics [22, 23, 24, 25] or tight upper bounds in order to be able to examine the accuracy of heuristic

solutions when the optimal solution is not timely obtainable [26, 27]. In the literature, there exist two general exact solution approaches for this problem. First, a mixed integer linear programming (MILP) formulation that can be solved with commercial optimization softwares [22, 23]. However, it can be quite time consuming even for moderately sized problems especially when additional constraints such as an upper bound on the number of products in the assortment are involved in the model [22, 23, 26, 28]. Motivated by this shortcoming of MILP model and a lack of timely and expedient optimal solution procedures, Şen et al. [28] introduce a new mixed integer conic quadratic programming (CONIC) formulation of the problem along with new valid McCormick inequalities for the capacitated version of the problem, and demonstrate that it can be solved quickly even when instances are large sized using standard optimization software. Şen et al. [28] only study the case where there is one cardinality constraint, i.e., an upper bound on the number of products that can be displayed to the customers. However, in practice there can be multiple cardinality constraints in the assortment problem where the set of products is divided into different sets and the retailer wants to restrict the maximum number of products offered in the assortment from each set. Moreover, retailers may be confronted with various resource constraints on the assortment of products they wish to offer such as limited shelf space or budget. In this thesis, we study assortment optimization under MMNL model in the presence of such constraints. We also investigate joint pricing and assortment problem under MMNL demand model subject to a single cardinality constraint, where there is a set of price alternatives for each product, and the retailer wants to determine the optimal set of products and their optimal prices to offer to the customers, at the same time, with the goal of maximizing revenue. We model these problems using MILP and CONIC formulations which exist in the literature and introduce new procedures for obtaining McCormick inequalities for each case to make these models tractable when solving with optimization softwares even for large sized instances. For all cases we present extensive numerical studies, the results of which show that CONIC model performs the best when accompanied with the valid McCormick cuts.

In summary, the contributions of this thesis are as follows: 1) Given the

aforementioned advantages of MMNL over MNL and NL models for representing customer preferences, we establish new efficient procedures to derive valid McCormick inequalities for modelling constrained assortment optimization under mixed multinomial logit demand model; 2) due to the difficulty of the problem, most studies in this area concentrate on developing approximation methods. The existing exact approaches, i.e., MILP and CONIC formulations, are quite time consuming especially when real world constraints are involved. This reveals an important gap in the academic literature to devise timely optimal methods to solve the problem. By using the McCormick inequalities in the CONIC formulation, we close this gap. Another advantage of our approach is that, one can use the available commercial optimization software solutions to solve the problem abating the need for developing customized packages. 3) We specifically provide valid McCormick cuts for three general constraints: single resource constraint, multiple resource constraints and multiple cardinality constraints. 4) We also propose a new method to derive McCormick inequalities for joint pricing and assortment optimization problem under a single cardinality constraint. 5) Finally, we demonstrate the effectiveness of our approach by providing extensive numerical studies.

The rest of the thesis is organized as follows. In the next chapter, we present a review of the relevant literature. In Chapter 3, we describe the assortment optimization under the mixed multinomial logit demand and present the corresponding integer programs including MILP and CONIC formulations. In 3.1, 3.2 and 3.3, we study the assortment problem with a single resource constraint, multiple resource constraints and multiple cardinality constraints, respectively, and establish new approaches to obtain the valid McCormick inequalities associated with each case. We provide the related computational studies in 3.4. In Chapter 4, we study joint pricing and assortment optimization problem and provide a procedure to derive the relevant McCormick inequalities and present the numerical studies. Finally, in Chapter 5 we bring forward the concluding remarks.

Chapter 2

Literature Review

In this chapter, we first present a review of papers in the field of assortment optimization under MMNL. Then, we review the relevant papers under MNL and NL choice models, in which similar constraints to our problem (e.g. cardinality and resource constraints) or joint pricing and assortment optimization are considered. For a review of literature for assortment planning under various choice models we refer to [1] and [29].

Bront et al. [22] and Méndez-Díaz et al. [23] study the mixed integer linear programming (MILP) formulation to obtain the optimal assortment where the customer purchase behaviour is modelled according to mixed multinomial logit model. As solving the MILP model can be quite time consuming even for moderately sized instances, they developed near optimal but fast heuristics to solve the problem. Bront et al. [22] also prove that the problem is NP-hard when the number of customer segments is equal or greater than the total number of products. Rusmevichientong et al. [24] later demonstrate the NP-hardness of the problem even in the presence of two customer segments. Subsequently, they study the case where only the ordered sets of products with respect to the highest revenues or so-called revenue ordered assortments are included in the problem and attain the corresponding approximation guarantees. This heuristic approach reduces the number of possible assortments from 2^N to N where N is the total number

of products from which an assortment is selected to display to the customers. However, the optimal assortment may be excluded throughout this reduction process. They also establish the special cases of the problem where examining only the revenue ordered assortments always yields the optimal solution. Désir and Goyal [25] construct a fully polynomial time approximation scheme (FPTAS) to solve capacitated assortment optimization under MMNL and show that their approach is the best possible approximation for the problem.

The optimality gap of the heuristics developed to solve the problem are difficult to be verified, due to unavailability of optimal solutions for hard problems. Motivated by this, Feldman and Topaloglu [26] established a new approach to derive tight upper bounds on the optimal solution -with 0.15% deviation on average and less than 1% in the worst case- so as to be able to validate the accuracy of heuristic solutions for hard problems. Kunnumkal [27] substantiates that although the Lagrangian relaxation approach proposed by Feldman and Topaloglu [26] constructs the tightest upper bounds among the existing methods in the literature, it still remains intractable. Then he presents a new method which obtains the tightest upper bounds among the existing tractable approaches in the literature. He also provides a property of the optimal solution for the unconstrained assortment optimization problem under MMNL model.

Şen et al. [28] study the problem in the presence of a capacity constraint which restricts the number of products offered to the customers and develop a new mixed integer conic quadratic programming (CONIC) formulation of the problem along with new McCormick inequalities. Then they demonstrate that their approach is much more efficient than the traditional MILP model presented by Bront et al. [22], and can solve even large sized problems optimally in a reasonable amount of time using a commercial optimization software.

When the customer purchase behaviour is modelled according to MNL model, Rusmevichientong et al. [11] study both static and dynamic versions of the assortment optimization with a cardinality constraint. In static settings, the parameters of the MNL model are assumed to be known in advance, however, in the dynamic settings, the parameters of the choice model are unknown and must be

derived from the sales data. Davis et al. [12] consider assortment problem under MNL subject to a collection of totally unimodular constraints and demonstrate that it can be solved equivalently as a linear program. Then they present five applications of their approach in assortment optimization including: multiple cardinality constraints, display location decisions, pricing, quality consistent pricing and product precedence constraints.

Under the nested logit (NL) choice model, Rusmevichientong et al. [14], formulated the assortment planning problem as a sum-of-ratios optimization problem, in the presence of a single resource constraint. They show that the problem is NP-complete and establish polynomial-time approximation scheme (PTAS) to solve it. K  k and Xu [15] study joint pricing assortment optimization under a two-level nested logit model in two different cases: In the first case, customers first select a brand and then a product type of the chosen brand, while in the second case, customers first select a product type and then a brand. For each case, they provide the characteristics of the optimal solution considering the existence of only two brands. Gallego and Topaloglu [16] study constrained assortment optimization under NL model in two cases, a cardinality and a resource constraint in each nest. In the first case, they solve the problem as a linear program. In the second case the problem becomes NP-hard as a result of which they propose an approximation method to achieve a solution with a performance guarantee of 2. They further show that the joint pricing and assortment optimization problem can be solved as a constrained assortment optimization. Feldman and Topaloglu [20] consider assortment optimization problem under NL model subject to a cardinality constraint across all products in all nests and develop a linear program to solve the problem optimally. They also establish a 4-approximation algorithm to solve the problem under a resource constraint across all nests and show that it generates solutions with a 2% optimality gap on average. Rayfield et al. [18] study joint assortment and pricing optimization where the demand is captured by NL model and the price of each product must be assigned from a bounded interval. They establish a fast approximation method to solve the problem, and a linear program to generate upper bounds on the optimal expected revenue so as to examine the quality of the obtained solutions.

Chapter 3

The Capacitated Assortment Optimization Problem

In this chapter, we first introduce assortment optimization problem where the customer choice process is modelled using mixed multinomial logit model. Then we present mixed integer linear programming (MILP) and mixed integer conic quadratic programming (CONIC) formulations of the problem which are developed by Bront et al. [22] and Şen et al. [28], respectively. Finally, we show how to obtain McCormick inequalities to strengthen MILP and CONIC models of the problem in the presence of three different set of constraints: single resource constraint, multiple resource constraints and multiple cardinality constraints.

The notations used in this thesis are similar to those in Şen et al. [28]. Let N represent the set of all products from which assortment S is chosen to offer to the customers. Product 0 is a virtual product that represents the no-purchase option. Let M be the set of all customer segments indexed by i and π_{ij} be the revenue associated with selling product j to a customer from segment i . U_{ij} indicates the utility that an individual customer who belongs to segment i gains from purchasing product j in the set $\{S \cup 0\}$. The utility $U_{ij} = u_{ij} + \epsilon_{ij}$ consists of two parts, the utility u_{ij} is the deterministic component and ϵ_{ij} is a Gumbel random variable with mode zero, scale parameter μ and variance

$\mu^2\pi^2/6$. Let $\nu_{ij} = e^{(u_{ij}-\pi_{ij})/\mu}$ denote the preference of a customer from segment i for product j . Given these definitions and notations, the probability that a customer of type i purchases product j from an offered assortment S is $p_{ij}(S) = \nu_{ij}/(\nu_{i0} + \sum_{k \in S} \nu_{ik})$. The expected revenue associated with a customer from segment i over all products in the given assortment S is $\sum_{j \in S} \pi_{ij} p_{ij}(S)$ which equals $\frac{\sum_{j \in S} \pi_{ij} \nu_{ij}}{\nu_{i0} + \sum_{j \in S} \nu_{ij}}$. We define γ_i as the probability that an arriving customer is from segment i . Hence, the total expected revenue over all customer segments in the set M and a given assortment S is $\sum_{i \in M} \gamma_i \left[\frac{\sum_{j \in S} \pi_{ij} \nu_{ij}}{\nu_{i0} + \sum_{j \in S} \nu_{ij}} \right]$. The problem of maximizing this total expected revenue over all possible assortments of products is called assortment optimization under mixed multinomial logit model and is described mathematically as follows.

$$\text{(AOP)} \quad \max \sum_{i \in M} \gamma_i \left[\frac{\sum_{j \in N} \pi_{ij} \nu_{ij} x_j}{\nu_{i0} + \sum_{j \in N} \nu_{ij} x_j} \right] \quad (3.1)$$

$$\text{s.t.} \quad x_j \in \{0, 1\}, \quad j \in N. \quad (3.2)$$

In the above model, x_j is a binary variable which takes the value 1 if product j is included in the offered assortment and 0, otherwise. (AOP) is a nonlinear integer optimization problem which can be linearized in two steps [22, 23, 28]. First, we reformulate (AOP) as a bilinear integer program by defining the variable $y_i = 1/(\nu_{i0} + \sum_{j \in N} \nu_{ij} x_j)$:

$$\max \sum_{i \in M} \sum_{j \in N} \gamma_i \pi_{ij} \nu_{ij} y_i x_j \quad (3.3)$$

$$\text{(AOP')} \quad \text{s.t.} \quad \nu_{i0} y_i + \sum_{j \in N} \nu_{ij} y_i x_j = 1, \quad i \in M, \quad (3.4)$$

$$y_i \geq 0, \quad i \in M, \quad (3.5)$$

$$x_j \in \{0, 1\}, \quad j \in N. \quad (3.6)$$

The bilinear terms $y_i x_j$ can be linearized using the following technique [30]: Let $z_{ij} = y_i x_j$ where x_j is a binary variable and y_i is a continuous variable, then the bilinear term $y_i x_j$ can be replaced by z_{ij} while adding the following linear inequalities to the model: $y_i - z_{ij} \leq R(1 - x_j)$, $z_{ij} \leq y_i$, $z_{ij} \leq R x_j$ and $0 \leq z_{ij}$, where R is constant and an upper bound on y_i , i.e., $y_i \leq R$. Since the maximum

value that y_i can take is $1/\nu_{i0}$, we let $R = 1/\nu_{i0}$. Applying this approach, (AOP') can be reformulated as the following mixed integer programming (MILP) model:

$$\max \sum_{i \in M} \sum_{j \in N} \gamma_i \pi_{ij} \nu_{ij} z_{ij} \quad (3.7)$$

$$\text{s.t.} \quad \nu_{i0} y_i + \sum_{j \in N} \nu_{ij} z_{ij} = 1, \quad i \in M, \quad (3.8)$$

$$(\text{MILP}) \quad \nu_{i0}(y_i - z_{ij}) \leq 1 - x_j, \quad i \in M, \quad j \in N, \quad (3.9)$$

$$z_{ij} \leq y_i, \quad i \in M \quad j \in N, \quad (3.10)$$

$$\nu_{i0} z_{ij} \leq x_j, \quad i \in M, \quad j \in N, \quad (3.11)$$

$$x_j \in \{0, 1\}, \quad j \in N, \quad (3.12)$$

$$z_{ij} \geq 0, \quad i \in M \quad j \in N, \quad (3.13)$$

$$y_i \geq 0, \quad i \in M. \quad (3.14)$$

Şen et al. [28] reformulated (AOP) as a mixed integer conic quadratic programming problem in the following way. The first step is to change the objective function from maximization to minimization. For each $i \in M$, we let $\bar{\pi}_i = \max_{j \in N} \pi_{ij}$ and write the objective function (3.1) of (AOP) as follows:

$$\max \sum_{i \in M} \gamma_i \bar{\pi}_i - \sum_{i \in M} \gamma_i \left[\frac{\nu_{i0} \bar{\pi}_i + \sum_{j \in N} \nu_{ij} (\bar{\pi}_i - \pi_{ij}) x_j}{\nu_{i0} + \sum_{j \in N} \nu_{ij} x_j} \right]. \quad (3.15)$$

We can omit the first part $\sum_{i \in M} \gamma_i \bar{\pi}_i$ from the formulation above as it is constant, and present the objective function in terms of the second part as a minimization. Moreover, we let $y_i = 1/(\nu_{i0} + \sum_{j \in N} \nu_{ij} x_j)$ and $z_{ij} = y_i x_j$ and add the inequalities $y_i \geq 1/(\nu_{i0} + \sum_{j \in N} \nu_{ij} x_j)$ and $z_{ij} \geq y_i x_j$ to the model. As a result, the following optimization problem can be obtained from reformulating

(AOP):

$$\min \sum_{i \in M} \gamma_i \nu_{i0} \bar{\pi}_i y_i + \sum_{i \in M} \sum_{j \in N} \gamma_i \nu_{ij} (\bar{\pi}_i - \pi_{ij}) z_{ij} \quad (3.16)$$

$$\text{s.t.} \quad z_{ij} \geq y_i x_j, \quad i \in M, \quad j \in N, \quad (3.17)$$

$$\text{(AOP'')} \quad y_i \geq \frac{1}{\nu_{i0} + \sum_{j \in N} \nu_{ij} x_j}, \quad i \in M, \quad (3.18)$$

$$x_j \in \{0, 1\}, \quad j \in N, \quad (3.19)$$

$$z_{ij} \geq 0, \quad i \in M, \quad j \in N, \quad (3.20)$$

$$y_i \geq 0, \quad i \in M. \quad (3.21)$$

Using $x_j = x_j^2$ and letting $w_i = 1/y_i = \nu_{i0} + \sum_{j \in N} \nu_{ij} x_j$, we transform constraints (3.17) and (3.18) in conic form:

$$z_{ij} w_i \geq x_j^2, \quad (3.22)$$

$$y_i w_i \geq 1. \quad (3.23)$$

We also apply the reformulation of constraint (3.18) in the following form in order to strengthen the continuous relaxation of the problem:

$$\nu_{i0} y_i + \sum_{j \in N} \nu_{ij} z_{ij} \geq 1, \quad i \in M \quad (3.24)$$

Implementing above transitions to (AOP''), the mixed integer conic quadratic

programming (CONIC) formulation of the problem can be stated as follows:

$$\min \sum_{i \in M} \gamma_i \nu_{i0} \bar{\pi}_i y_i + \sum_{i \in M} \sum_{j \in N} \gamma_i \nu_{ij} (\bar{\pi}_i - \pi_{ij}) z_{ij} \quad (3.25)$$

$$\text{s.t.} \quad w_i = \nu_{i0} + \sum_{j \in N} \nu_{ij} x_j, \quad i \in M, \quad (3.26)$$

$$z_{ij} w_i \geq x_j^2, \quad i \in M, \quad j \in N, \quad (3.27)$$

$$\text{(CONIC)} \quad y_i w_i \geq 1, \quad i \in M, \quad (3.28)$$

$$\nu_{i0} y_i + \sum_{j \in N} \nu_{ij} z_{ij} \geq 1, \quad i \in M, \quad (3.29)$$

$$x_j \in \{0, 1\}, \quad j \in N, \quad (3.30)$$

$$z_{ij} \geq 0, \quad i \in M, \quad j \in N, \quad (3.31)$$

$$y_i \geq 0, \quad i \in M, \quad (3.32)$$

$$w_i \geq 0, \quad i \in M. \quad (3.33)$$

Let $y_{i|x_j=1}^u$ and $y_{i|x_j=1}^\ell$ be the upper and lower bounds on y_i , respectively, conditional on $x_j = 1$. Let $y_{i|x_j=0}^\ell$ be the lower bound on y_i conditional on $x_j = 0$. Also, y_i^u is defined as a global upper bound on y_i . Then, for each bilinear expression $z_{ij} = y_i x_j$, the following McCormick inequalities are valid [31]:

$$z_{ij} \leq y_{i|x_j=1}^u x_j, \quad i \in M, \quad j \in N, \quad (3.34)$$

$$\text{(MC)} \quad z_{ij} \geq y_{i|x_j=1}^\ell x_j, \quad i \in M, \quad j \in N, \quad (3.35)$$

$$z_{ij} \leq y_i - y_{i|x_j=0}^\ell (1 - x_j), \quad i \in M, \quad j \in N, \quad (3.36)$$

$$z_{ij} \geq y_i - y_i^u (1 - x_j), \quad i \in M, \quad j \in N. \quad (3.37)$$

Şen et al. [28] propose corresponding bounds on y_i for a single cardinality constraint case. We introduce new approaches to establish the conditional lower bounds, i.e., $y_{i|x_j=1}^\ell$ and $y_{i|x_j=0}^\ell$, for three different cases: single resource constraint, multiple resource constraints and multiple cardinality constraints. For all of these cases, $y_{i|x_j=1}^u = 1/(\nu_{i0} + \nu_{ij})$ and $y_i^u = 1/\nu_{i0}$ which are also used in Şen et al. [28].

3.1 Single Resource Constraint

In the single resource constraint case, we assume that there is a limitation on the total amount of a single resource (such as space or budget) associated with the offered assortment. Let $r_j \in R$ be the resource requirement of product j if it is offered in the assortment, and K be the available amount of our resource for an offered assortment S . This constraint can be described mathematically as the inequality $\sum_{j \in S} r_j \leq K$. In order to involve this restriction in the assortment optimization problem under MMNL, we can simply add the following constraint to the model:

$$\sum_{j \in N} r_j x_j \leq K. \quad (3.38)$$

We take advantage of the above knapsack constraint to construct lower bounds on y_i . For $i \in M$, let q_i be the optimal objective value of the following one-dimensional knapsack problem:

$$q_i = \max \sum_{j \in N} \nu_{ij} x_j \quad (3.39)$$

$$(KP1) \quad \text{s.t.} \quad \sum_{j \in N} r_j x_j \leq K, \quad (3.40)$$

$$x_j \in \{0, 1\}, \quad j \in N. \quad (3.41)$$

Proposition 1 *In the presence of a single resource constraint, for every $i \in M$, the following lower bound on y_i is valid:*

$$y_i^\ell := \frac{1}{\nu_{i0} + q_i} \leq y_i. \quad (3.42)$$

For $i \in M$ and $j \in N$, let $q_{i|x_j=0}$ and $q_{i|x_j=1}$ be the optimal objective value of the following one-dimensional knapsack problems, respectively:

$$q_{i|x_j=0} = \max \sum_{t \in N \setminus \{j\}} \nu_{it} x_t \quad (3.43)$$

$$(KP2) \quad \text{s.t.} \quad \sum_{t \in N \setminus \{j\}} r_t x_t \leq K, \quad (3.44)$$

$$x_t \in \{0, 1\}, \quad t \in N, \quad (3.45)$$

$$q_{i|x_j=1} = \max \sum_{t \in N \setminus \{j\}} \nu_{it} x_t \quad (3.46)$$

$$(KP3) \quad \text{s.t.} \quad \sum_{t \in N \setminus \{j\}} r_t x_t \leq K - r_j, \quad (3.47)$$

$$x_t \in \{0, 1\}, \quad t \in N. \quad (3.48)$$

Proposition 2 *In the presence of a single resource constraint, for every $i \in M$ and every $j \in N$, the following conditional lower bounds on y_i are valid:*

$$y_{i|x_j=0}^\ell := \frac{1}{\nu_{i0} + q_{i|x_j=0}} \leq y_i, \quad (3.49)$$

$$y_{i|x_j=1}^\ell := \frac{1}{\nu_{i0} + \nu_{ij} + q_{i|x_j=1}} \leq y_i. \quad (3.50)$$

The above global and conditional lower bounds still hold true if we use an upper bound on q_i , $q_{i|x_j=0}$ and $q_{i|x_j=1}$ rather than the optimal solution of the corresponding knapsack problems. Since the one-dimensional knapsack problem is NP-hard [32], we therefore solve the continuous relaxation of knapsack problems to obtain values of $q_{i|x_j=0}$ and $q_{i|x_j=1}$. The continuous knapsack problem with a single resource constraint, can be efficiently solved using the necessary and sufficient characteristic of its optimal solution introduced by Dantzig [33] and can be stated as follows.

Consider the knapsack problem (KP1). Assume that all products are orderly numbered in accordance with the non-increasing values of ν_{ij}/r_j . We define the critical product, c , where $c = \min\{t : \sum_{j=1}^t r_j > K\}$. Then, the optimal solution x^* of the continuous relaxation of the knapsack problem (KP1) is: $x_j^* = 1$, for $j = 1, \dots, c-1$, $x_c^* = \frac{K - \sum_{j=1}^{c-1} r_j}{r_c}$ and $x_j^* = 0$, for $j = c+1, \dots, N$.

We utilize the above optimality property of continuous knapsack problem to derive values of $q_{i|x_j=0}$ and $q_{i|x_j=1}$. This approach does not yield the conditional lower bounds on y_i as tight as solving the integer knapsack problem. Nevertheless, we later show that both approaches lead to very similar computational results for solving the assortment optimization under MMNL. However, using the continuous knapsack problem is more worthwhile as solving the integer one can be quite time consuming due to difficulty of the problem.

3.2 Multiple Resource Constraints

Assortment optimization with multiple resource constraints, corresponds to the case where there are multiple resources required for offering each product in the assortment and the retailer confronts restriction on the available amount of each resource. Let L be the set of all necessary resources, r_{lj} be the consumption of resource l by product j if it is included in the assortment, and K_l be the available amount of resource l for an offered assortment S . For every $l \in L$ the resource constraint can be written mathematically as the inequality $\sum_{j \in S} r_{lj} \leq K_l$. Therefore, the following constraint should be added to the assortment optimization model to satisfy the resource constraints:

$$\sum_{j \in N} r_{lj} x_j \leq K_l, \quad l \in L. \quad (3.51)$$

Similar to the single resource constraint case, we utilize the above knapsack constraint to obtain lower bounds on y_i . For $i \in M$, let p_i be the optimal objective value of the following multi-dimensional knapsack problem:

$$p_i = \max \sum_{j \in N} \nu_{ij} x_j \quad (3.52)$$

$$(MKP1) \quad \text{s.t.} \quad \sum_{j \in N} r_{lj} x_j \leq K_l, \quad l \in L, \quad (3.53)$$

$$x_j \in \{0, 1\}, \quad j \in N. \quad (3.54)$$

Proposition 3 *In the presence of multiple resource constraints, for every $i \in M$, the following lower bound on y_i is valid:*

$$y_i^\ell := \frac{1}{\nu_{i0} + p_i} \leq y_i. \quad (3.55)$$

For $i \in M$ and $j \in N$, let $p_{i|x_j=0}$ and $p_{i|x_j=1}$ be the optimal objective value of

the following multi-dimensional knapsack problems, respectively:

$$p_{i|x_j=0} = \max \sum_{t \in N \setminus \{j\}} \nu_{it} x_t \quad (3.56)$$

$$(MKP2) \quad \text{s.t.} \quad \sum_{t \in N \setminus \{j\}} r_{lt} x_t \leq K_l, \quad l \in L, \quad (3.57)$$

$$x_t \in \{0, 1\}, \quad t \in N, \quad (3.58)$$

$$p_{i|x_j=1} = \max \sum_{t \in N \setminus \{j\}} \nu_{it} x_t \quad (3.59)$$

$$(MKP3) \quad \text{s.t.} \quad \sum_{t \in N \setminus \{j\}} r_{lt} x_t \leq K_l - r_{lj}, \quad l \in L, \quad (3.60)$$

$$x_t \in \{0, 1\}, \quad t \in N. \quad (3.61)$$

Proposition 4 *In the presence of multiple resource constraints, for every $i \in M$ and every $j \in N$, the following conditional lower bounds on y_i are valid:*

$$y_{i|x_j=0}^\ell := \frac{1}{\nu_{i0} + p_{i|x_j=0}} \leq y_i, \quad (3.62)$$

$$y_{i|x_j=1}^\ell := \frac{1}{\nu_{i0} + \nu_{ij} + p_{i|x_j=1}} \leq y_i. \quad (3.63)$$

Similar to the single resource constraint case, the above global and conditional lower bounds still hold true if one uses upper bounds on p_i , $p_{i|x_j=0}$ and $p_{i|x_j=1}$ rather than the optimal solution of the corresponding knapsack problems. Due to NP-hardness of the multi-dimensional knapsack problem [34], we solve their continuous relaxation to attain upper bound for the values of $p_{i|x_j=0}$ and $p_{i|x_j=1}$. Although solving integer knapsack problem provides tighter conditional lower bounds on y_i than the continuous one, our numerical studies demonstrate that both approaches yield similar computational results when we use these bounds in solving our assortment optimization problem.

3.3 Multiple Cardinality Constraints

Suppose that the set of all products in the category is partitioned into disjoint subsets of products, namely nests, and there is a limit on the number of products which can be offered from each nest. Let A be the set of all nests, N_a denote the set of products in nest a where $\cup_{a \in A} N_a = N$ and for any two nests b and c in A , $N_b \cap N_c = \emptyset$, and S_a denote the set of products offered from nest a . For every nest a , we mathematically represent the corresponding cardinality constraint as the inequality $|S_a| \leq K_a$ where K_a indicates the maximum number of products that can be offered from nest a . To satisfy all of the cardinality constraints in the assortment optimization problem, we add the following inequality to the models:

$$\sum_{j \in N_a} x_j \leq K_a, \quad a \in A. \quad (3.64)$$

We make use of the cardinality constraints to attain lower bounds on y_i . For every $i \in M$ and $a \in A$, let $\nu_{i[k]}^a$ be the k th largest of preferences ν_{it} , $t \in N_a$.

Proposition 5 *In the presence of multiple cardinality constraints, for every $i \in M$, the following lower bound on y_i is valid:*

$$y_i^\ell := \frac{1}{\nu_{i0} + \sum_{a \in A} \sum_{k=1}^{K_a} \nu_{i[k]}^a} \leq y_i. \quad (3.65)$$

For every $i \in M$, $a \in A$ and $j \in N_a$, let $\bar{\nu}_{i[k]}^a$ be the k th largest of preferences ν_{it} , $t \in N_a \setminus \{j\}$.

Proposition 6 *In the presence of multiple cardinality constraints, for every $i \in M$, $a \in A$ and $j \in N_a$, the following conditional lower bounds on y_i are valid:*

$$y_{i|x_j=0}^\ell := \frac{1}{\nu_{i0} + \sum_{k=1}^{K_a} \bar{\nu}_{i[k]}^a + \sum_{n \in A \setminus \{a\}} \sum_{k=1}^{K_n} \nu_{i[k]}^n} \leq y_i, \quad (3.66)$$

$$y_{i|x_j=1}^\ell := \frac{1}{\nu_{i0} + \nu_{ij} + \sum_{k=1}^{K_a-1} \bar{\nu}_{i[k]}^a + \sum_{n \in A \setminus \{a\}} \sum_{k=1}^{K_n} \nu_{i[k]}^n} \leq y_i. \quad (3.67)$$

3.4 Numerical Studies

In order to examine the efficacy of our approaches on obtaining McCormick inequalities and the conic formulation for the three aforementioned cases, we carry out several numerical experiments on four sets of mixed integer programs: MILP, MILP+MC, CONIC and CONIC+MC. For each constraint case, the related constraints are added to MILP and CONIC models. Moreover, in MILP+MC and CONIC+MC, the MC constraints are added to MILP and CONIC models, respectively. We use Gurobi 6.5.0 on a personal computer (ASUS X450C Laptop, Intel core i5 processor 1.8 GHz, 6GB RAM, 64-bit Windows 8) to solve the generated instances. The default settings of Gurobi 6.5.0 is used except that it is forced to apply an outer-approximation method to solve the conic model and barrier algorithm for the root node of MIP models. Also for attaining values of $y_{i|x_j=0}^\ell$ and $y_{i|x_j=1}^\ell$ for the multiple resource constraints case, the corresponding continuous knapsack problems are solved via Gurobi 6.5.0 C interface. For all cases, the preferences ν_{ij} are drawn randomly from the uniform distribution on the interval $[0,1]$, the selling price of each product is considered the same for all customer segments ($\pi_{ij} = \pi_j$) and is drawn from the uniform distribution on the interval $[1,3]$, and for all $i \in M$ the parameter γ_i is set equal to $1/|M|$.

Table 3.1 presents the average computational results for the single resource constraint case in the presence of 200 products ($|N| = 200$) and 20 customer segments ($|M| = 20$). Five different values ($\{5, 10, 25, 50, 100\}$) of the capacity parameter K and two values ($\{5, 10\}$) for the no-purchase preference ν_0 (for all $i \in M$, $\nu_{i0} = \nu_0$) are considered, resulting in 10 distinct problems where for each problem 10 instances are generated and a time limit of 600 seconds is used. The computational results of each individual instance are presented in Appendix A.1. For each mixed integer programming model, the column Gap indicates the average integrality gap, i.e., the percentage difference between the optimal integer objective and the corresponding optimal continuous relaxation objective calculated by this formula: $100 \frac{|Optimal\ integer\ objective - optimal\ continuous\ objective|}{Optimal\ integer\ objective}$. The column Nodes represents the average number of explored nodes in the branch and bound algorithm within the given time limit. For the instances that are solved

within the time limit, the first line in the column Time, reports the average time consumed to reach the optimal solution and the number of solved instances is given in the parenthesis. In the second line, the remaining optimality gap is reported for the unsolved instances and the number of such instances is given in the parenthesis. The remaining optimality gap indicates the difference between the best integer objective and the best objective bound as a percentage of the absolute value of the best integer objective obtained by the branch and bound algorithm within the time limit. Under the default settings of Gurobi 6.5.0 the optimal solution is reached when this gap is less than 0.01%.

We can observe that the CONIC formulation is inefficient and cannot solve any of the instances within the time limit. This is due to overabundant branching caused by large integrality gaps.

When the capacity constraint is non-binding ($K \in \{50, 100\}$), all instances are solved by MILP, MILP+MC and CONIC+MC, due to small integrality gaps which lead the branch and bound algorithm to reach the optimal solution quickly. In the presence of binding capacity constraints, out of 60 instances, MILP, MILP+MC and CONIC+MC attain the optimal solution within the time limit in 9, 19 and 58 instances, respectively. By tightening the feasible region and decreasing the root gaps, the McCormick inequalities assist the MILP formulation to obtain the optimal solution of 10 more instances and also the remaining optimality gaps at termination becomes much smaller. However, the CONIC+MC model performs much better as a result of a tighter formulation obtained from joint incorporation of conic constraints and McCormick valid inequalities. Moreover, the remaining optimality gaps for the unsolved instances and the average number of explored nodes required to reach the optimal solution are much smaller than those of MILP+MC. Many instances are solved without any branching.

Table 3.2 reports the average results for 5 instances with 300 products, 30 customer segments, capacities K from the set $\{7, 15, 30, 75, 150\}$, ν_0 is equal to either 5 or 10 and a time limit of 3600 seconds. The individual results are presented in Appendix A.2. As observed earlier, with the CONIC model the time limit is reached for all instances. When the capacity constraint is binding, with MILP

and MILP+MC formulations only 4 and 5 instances are solved, respectively, out of 30 instances. Whereas, CONIC+MC solves 23 instances and results in small remaining gaps for the unsolved ones.

As mentioned previously, due to NP-hardness of the one-dimensional knapsack problem, we solve its continuous relaxation to obtain the conditional lower bounds on y_i used in the McCormick valid inequalities. However, the continuous relaxation approach provides looser bounds on y_i than the integer one. Table 3.3 reports the average computational results of these two approaches for 5 instances with 200 products and 20 customer segments where the columns CONIC+MC 1 and CONIC+MC 2 correspond to the continuous and integer approaches, respectively. The individual results are given in Appendix A.3. With both methods, all instances are solved within the time limit. The integrality gaps, number of explored nodes and solution times, on average are quite close in the two approaches. Therefore, we can conclude that the continuous relaxation approach is in fact a proper alternative to the integer one as one does not need to solve computationally costly integer knapsack problems in this approach.

Table 3.4 reports the average results for the multiple resource constraints case with 200 products ($|N| = 200$), 20 customer segments ($|M| = 20$) and 5 resource requirements ($|L| = 5$). Five different values ($\{5, 10, 25, 50, 100\}$) of the capacity parameter K_l and two values ($\{5, 10\}$) for the no-purchase preference ν_0 (for all $i \in M, \nu_{i0} = \nu_0$) are considered, resulting in 10 problems where for each problem 10 instances are generated and a time limit of 1800 seconds is used. The computational results of all instances are individually presented in Appendix A.4. It is worth mentioning that for each problem with capacity K_l equal to 5, 10, 25, 50 and 100, the corresponding values of $y_{i|x_j=0}^\ell$ and $y_{i|x_j=1}^\ell$ are obtained by solving $2 \times |M| \times |N| = 8000$ continuous knapsack problems as described before, the overall average solution times of which are 13.79, 13.52, 13.61, 13.20 and 9.91 seconds.

As observed in the single resource constraint case, with MILP formulation all instances with non-binding constraints are solved, whereas, only 8 out of 60 problems reach to optimality in the presence of binding constraints. Although

when accompanied by McCormick inequalities the root gaps of MILP model decline considerably, only two more problems are solved with MILP+MC. With the CONIC formulation, although the root gaps of tightly capacitated instances are smaller than those of MILP, still no instances reach the optimal solution within the time limit. However, with the addition of McCormick valid cuts, the continuous relaxation of CONIC model becomes stronger which leads to solving 96 out of 100 problems within the time limit with much smaller remaining optimality gaps than MILP+MC for the unsolved ones.

In this case, we solve continuous relaxation of multi-dimensional knapsack problems to come up with conditional lower bounds on y_i which are used in McCormick inequalities. Table 3.5 reports the average results of five instances with continuous and integer approaches implemented in CONIC+MC 1 and CONIC+MC 2, respectively. The individual results are reported in Appendix A.5. All instances are solved within the time limit and the average results for root gaps, number of nodes and solution times are very similar in the two approaches. Hence, it is worthwhile to use the continuous relaxation approach, as the one-dimensional knapsack problem is NP-hard.

For the multiple cardinality case, the average results of problems with 200 products ($|N| = 200$), 20 customer segments ($|M| = 20$) and 5 cardinality constraints ($|A| = 5$), are reported in Table 3.6. The number of products in each nest is 40. For all $a \in A$ the value of K_a is one of $\{2, 4, 10, 20, 40\}$ and ν_0 (for all $i \in M$, $\nu_{i0} = \nu_0$) is either 5 or 10, resulting in 10 problems where for each problem 10 instances are generated and a time limit of 600 seconds is used. The individual results are provided in Appendix A.6.

MILP fails to solve all of the low capacity instances within the time limit due to large root gaps and reaches the optimal solution for 50 out of 100 problems. Despite the fact that McCormick inequalities notably decrease the root gaps of MILP, MILP+MC does not solve more problems than MILP. Contrary to these two models, with CONIC formulation the optimal solution of 9 tightly capacitated problems are obtained due to small integrality gaps. However, the CONIC model performs poorly for problems with higher capacities. With the

inclusion of McCormick valid cuts, the root gaps of CONIC declines significantly which leads to solving all 100 problems in less than a minute.

Table 3.7 reports the average results of problems with 300 products ($|N| = 300$), 30 customer segments ($|M| = 30$) and 5 cardinality constraints ($|A| = 5$). The number of products in each nest is 60. For all $a \in A$ the value of K_a is one of $\{4, 10, 20, 30, 60\}$ and ν_0 (for all $i \in M$, $\nu_{i0} = \nu_0$) is either 5 or 10, and a time limit of 600 seconds is used. The individual results are given in Appendix A.7. Due to large root gaps, none of the problems are solved with the CONIC model, while MILP reach the optimal solution for 56 problems within the time limit. McCormick inequalities tighten the MILP formulation and help it to solve 8 more problems. However, CONIC+MC solves all 100 problems due to joint combination of tight conic constraints and strong McCormick valid cuts. In similar settings, Table 3.8 provides the average computational results of problems with 300 products, 30 customer segments and five values of K_a from the set $\{3, 6, 12, 30, 60\}$. The individual results are reported in Appendix A.8. In this case also no problem is solved via CONIC model. 48 problems are solved using MILP and McCormick inequalities help it to obtain the optimal solution only for 2 more problems. Due to small root gaps, 98 problems are solved with CONIC+MC formulation while the remaining optimality gaps of the two unsolved ones are very small.

Table 3.1: Average results of problems with 200 products and 20 customer segments for the single resource constraint case.

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
5	5	17.45%	6308.90	– 14.17% (10)	6.93%	11369.80	274.91 (1) 3.14% (9)	16.82%	7391.30	– 4.88% (10)	3.46%	2089.80	197.45 (10)
	10	7.42%	15886.00	– 6.76% (10)	3.49%	22302.60	422.44 (2) 2.11% (8)	18.70%	3922.50	– 11.00% (10)	2.89%	2164.80	220.05 (9)
	25	0.19%	18444.30	23.84 (9) 0.11% (1)	0.10%	211.10	4.32 (10)	24.82%	1542.40	– 11.77% (10)	0.09%	39.00	4.86 (10)
	50	0.04%	568.40	2.81 (10)	0.02%	17.40	1.86 (10)	27.26%	935.50	– 18.21% (10)	0.01%	1.80	2.31 (10)
	100	0.04%	568.60	2.91 (10)	0.02%	30.90	1.84 (10)	27.26%	749.00	– 13.03% (10)	0.01%	2.20	2.36 (10)
10	5	23.57%	9038.50	– 16.76% (10)	5.93%	14390.70	308.04 (1) 2.11% (9)	11.37%	8517.60	– 1.96% (10)	1.52%	1118.30	72.80 (10)
	10	15.45%	8330.30	– 15.25% (10)	4.56%	13217.10	– 2.55% (10)	13.51%	4234.80	– 4.57% (10)	1.88%	2303.70	258.00 (9)
	25	2.36%	29700.00	– 2.03% (10)	0.84%	38052.30	406.564 (5) 0.37% (5)	17.87%	1959.30	– 9.29% (10)	0.51%	1766.90	122.29 (10)
	50	0.03%	94.80	1.40 (10)	0.01%	0.00	1.2 (10)	21.51%	759.70	– 9.29% (10)	0.00%	1.20	1.94 (10)
	100	0.03%	100.80	1.36 (10)	0.01%	6.80	1.33 (10)	21.51%	760.20	– 8.57% (10)	0.00%	0.60	1.85 (10)

Table 3.2: Average results of problems with 300 products and 30 customer segments for the single resource constraint case.

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
5	7	12.45%	16474.00	– 12.15% (5)	5.69%	19733.40	– 3.38% (5)	18.17%	12384.00	– 8.61% (5)	3.82%	3378.60	1193.93 (4) 0.53% (1)
	15	4.44%	36895.00	– 4.21% (5)	2.37%	50545.40	– 1.45% (5)	20.76%	3755.60	– 19.33% (5)	2.81%	5827.80	1278.05 (2) 0.13% (3)
	30	0.17%	102262.60	87.82 (4) 0.01% (1)	0.09%	916.00	21.51 (5)	25.70%	2511.40	– 17.32% (5)	0.09%	112.20	24.74 (5)
	75	0.02%	313.60	4.20 (5)	0.01%	49.40	4.15 (5)	29.58%	1013.20	– 24.19% (5)	0.00%	1.20	5.83 (5)
	150	0.02%	287.60	4.01 (5)	0.01%	65.00	4.43 (5)	29.57%	1011.00	– 16.78% (5)	0.00%	2.40	6.57 (5)
10	7	20.94%	13177.80	– 17.42% (5)	5.89%	18084.40	– 3.27% (5)	12.94%	13334.00	– 4.06% (5)	1.98%	2338.80	927.50 (5)
	15	10.09%	16866.20	– 10.28% (5)	3.83%	23082.20	– 2.77% (5)	15.80%	6332.60	– 8.69% (5)	2.50%	6291.60	1823.36 (2) 0.38% (3)
	30	2.15%	54008.40	– 1.89% (5)	0.90%	75478.00	– 0.40% (5)	18.97%	4065.60	– 10.99% (5)	0.62%	1603.40	688.44 (5)
	75	0.02%	93.00	4.39 (5)	0.01%	0.00	3.03 (5)	24.57%	892.40	– 19.79% (5)	0.00%	0.00	4.78 (5)
	150	0.02%	99.80	3.84 (5)	0.01%	3.60	3.26 (5)	24.57%	1082.40	– 12.49% (5)	0.00%	0.00	5.53 (5)

Table 3.3: Average results of problems with 200 products and 20 customer segments for the single resource constraint case.

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.07%	1263.20	140.49 (5)	3.06%	1432.00	159.34 (5)
				—			—
	10	2.43%	1819.20	209.70 (5)	2.43%	2107.80	264.30 (5)
				—			—
	25	0.12%	197.00	11.07 (5)	0.12%	189.20	11.04 (5)
				—			—
10	50	0.01%	0.00	2.15 (5)	0.01%	0.00	2.12 (5)
				—			—
	100	0.01%	2.40	2.48 (5)	0.01%	2.40	2.50 (5)
				—			—
	5	1.19%	461.80	36.04 (5)	1.18%	555.80	43.35 (5)
				—			—
10	10	1.60%	2737.20	232.02 (5)	1.60%	2673.40	235.39 (5)
				—			—
	25	0.49%	2196.80	181.88 (5)	0.49%	1634.00	147.36 (5)
				—			—
	50	0.00%	0.00	1.79 (5)	0.00%	0.00	1.78 (5)
				—			—
10	100	0.00%	0.00	1.82 (5)	0.00%	0.00	1.76 (5)
				—			—

Table 3.4: Average results of problems with 200 products and 20 customer segments for the multiple resource constraints case.

v_0	K	MILP			MLP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
5	5	38.19%	7549.10	– 31.90% (10)	12.52%	17988.20	– 8.69% (10)	11.43%	22763.10	– 5.80% (10)	2.75%	10498.00	498.06 (9) 0.72% (1)
	10	13.11%	25014.70	– 12.27% (10)	5.16%	38228.20	– 3.34% (10)	14.04%	9053.60	– 9.00% (10)	2.08%	6552.90	523.14 (10) –
	25	0.40%	64808.70	263.44 (8) 0.42% (2)	0.17%	3576.20	42.22 (10)	24.03%	6827.20	– 11.12% (10)	0.12%	183.80	10.01 (10) –
	50	0.04%	1095.90	3.80 (10)	0.02%	30.50	2.08 (10)	27.64%	1459.80	– 11.81% (10)	0.01%	4.50	2.51 (10) –
	100	0.04%	1090.00	3.54 (10)	0.02%	52.80	2.19 (10)	27.64%	2071.50	– 11.90% (10)	0.01%	4.00	2.57 (10) –
10	5	45.06%	51910.70	– 22.70% (10)	13.10%	21509.50	– 8.36% (10)	7.30%	40233.20	– 3.88% (10)	1.53%	11095.30	461.64 (10) –
	10	29.78%	10871.30	– 28.90% (10)	8.01%	22674.70	– 6.14% (10)	9.62%	13390.40	– 5.09% (10)	1.54%	15523.00	944.43 (8) 0.30% (2)
	25	4.15%	73468.40	– 3.62% (10)	1.40%	98887.20	– 0.65% (10)	16.46%	9539.20	– 8.74% (10)	0.44%	6769.50	241.76 (9) 0.02% (1)
	50	0.02%	442.20	2.02 (10)	0.01%	0.00	1.12 (10)	21.23%	2321.80	– 9.42% (10)	0.00%	1.90	2.03 (10) –
	100	0.02%	371.40	1.77 (10)	0.01%	7.60	1.15 (10)	21.23%	3717.50	– 8.48% (10)	0.00%	0.00	1.94 (10) –

Table 3.5: Average results of problems with 200 products and 20 customer segments for the multiple resource constraints case.

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	2.81%	9682.80	539.08 (5)	2.74%	9180.40	494.57 (5)
				—			—
	10	1.98%	4882.20	367.29 (5)	1.96%	4874.60	355.69 (5)
				—			—
	25	0.08%	46.60	6.74 (5)	0.08%	75.80	6.16 (5)
10				—			—
	50	0.01%	2.60	2.99 (5)	0.01%	2.60	2.96 (5)
				—			—
	100	0.01%	3.60	3.07 (5)	0.01%	3.60	3.06 (5)
				—			—
10	5	1.62%	7842.20	287.24 (5)	1.58%	8537.80	339.54 (5)
				—			—
	10	1.44%	14747.20	874.39 (5)	1.42%	15027.00	913.98 (5)
				—			—
	25	0.33%	2116.80	140.50 (5)	0.33%	1790.20	104.28 (5)
10				—			—
	50	0.00%	2.00	2.36 (5)	0.00%	3.80	2.41 (5)
				—			—
10	100	0.00%	0.00	2.25 (5)	0.00%	0.00	2.25 (5)
				—			—

Table 3.6: Average results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case.

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
5	2	50.88%	2306.90	–	12.58%	5682.20	–	4.88%	9803.20	387.08 (2)	0.38%	48.80	7.24 (10)
				44.09% (10)			9.12% (10)			5.70% (8)			–
	4	17.42%	10622.10	–	5.61%	13890.70	–	8.83%	3161.40	–	0.64%	196.40	14.61 (10)
				15.60% (10)			3.99% (10)			14.40% (10)			–
10	10	0.87%	25290.50	106.42 (10)	0.30%	317.40	7.00 (10)	21.74%	1239.80	–	0.28%	39.60	7.53 (10)
				–			–			18.95% (10)			–
	20	0.04%	663.30	3.78 (10)	0.02%	7.70	2.36 (10)	26.74%	762.30	–	0.01%	1.80	2.35 (10)
				–			–			15.49% (10)			–
10	40	0.04%	412.90	2.36 (10)	0.02%	18.30	1.73 (10)	26.73%	750.40	–	0.01%	3.80	2.38 (10)
				–			–			12.94% (10)			–
	2	44.09%	66049.40	–	13.57%	7060.80	–	2.80%	12304.60	343.37 (7)	0.12%	10.50	5.40 (10)
				13.92% (10)			8.73% (10)			2.09% (3)			–
10	4	38.70%	2709.50	–	8.67%	7814.00	–	5.25%	5586.00	–	0.27%	98.10	10.85 (10)
				39.11% (10)			7.10% (10)			5.21% (10)			–
	10	5.20%	37164.90	–	1.53%	35330.80	–	13.80%	2625.10	–	0.25%	144.60	13.29 (10)
				4.43% (10)			0.79% (10)			11.58% (10)			–
10	20	0.09%	1373.90	4.39 (10)	0.03%	71.80	2.20 (10)	21.04%	755.80	–	0.02%	25.10	4.96 (10)
				–			–			13.94% (10)			–
	40	0.03%	130.80	1.40 (10)	0.01%	0.00	1.12 (10)	20.98%	763.60	–	0.00%	0.00	1.90 (10)
				–			–			8.52% (10)			–

Table 3.7: Average results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case.

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
5	4	20.03%	1883.00	– 19.63% (10)	6.74%	2470.20	– 5.99% (10)	7.94%	855.40	– 27.79% (10)	0.55%	244.70	70.11 (10)
	10	1.69%	59425.30	403.29 (1) 1.05% (9)	0.64%	10735.50	111.98 (4) 0.32% (6)	21.65%	938.50	– 43.27% (10)	0.45%	283.30	68.56 (10)
	20	0.04%	477.00	5.98 (10)	0.02%	36.00	5.88 (10)	31.55%	869.90	– 40.87% (10)	0.01%	4.60	6.81 (10)
	30	0.03%	456.80	5.61 (10)	0.02%	66.80	5.53 (10)	31.63%	776.50	– 34.86% (10)	0.01%	8.20	7.27 (10)
	60	0.03%	485.90	5.42 (10)	0.02%	95.40	5.98 (10)	31.63%	805.70	– 24.16% (10)	0.01%	8.70	7.90 (10)
10	4	44.53%	801.50	– 69.47% (10)	10.65%	960.90	– 11.01% (10)	4.78%	1132.90	– 17.37% (10)	0.26%	82.90	28.28 (10)
	10	7.83%	5792.60	– 7.65% (10)	2.54%	5431.10	– 2.13% (10)	13.39%	933.40	– 24.42% (10)	0.27%	596.00	160.31 (10)
	20	0.31%	61161.60	74.19 (5) 0.12% (5)	0.11%	1504.70	55.41 (10)	24.54%	870.90	– 34.57% (10)	0.07%	90.50	29.09 (10)
	30	0.02%	321.70	4.71 (10)	0.01%	0.00	2.98 (10)	25.73%	798.70	– 29.68% (10)	0.00%	0.00	5.24 (10)
	60	0.02%	343.20	4.80 (10)	0.01%	14.70	3.49 (10)	25.71%	854.20	– 21.56% (10)	0.00%	0.00	5.61 (10)

Table 3.8: Average results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case.

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
5	3	34.02%	1175.50	–	10.84%	1352.00	–	6.00%	1435.90	–	0.61%	368.60	83.47 (10)
				33.96% (10)			10.06% (10)			25.60% (10)			–
	6	9.79%	6623.20	–	3.63%	5219.60	–	10.86%	845.40	–	0.66%	438.10	65.27 (9)
				9.31% (10)			2.97% (10)			26.32% (10)			0.13% (1)
	12	0.81%	68508.10	297.83 (8)	0.32%	2577.80	95.00 (10)	23.35%	894.70	–	0.45%	283.10	53.90 (10)
				0.27% (2)			–			43.36% (10)			–
	30	0.02%	96.90	3.94 (10)	0.01%	11.20	4.18 (10)	30.15%	814.40	–	0.00%	0.80	5.98 (10)
				–			–			30.57% (10)			–
	60	0.02%	101.60	3.64 (10)	0.01%	16.90	3.68 (10)	30.15%	772.60	–	0.00%	0.80	6.09 (10)
				–			–			23.87% (10)			–
10	3	55.83%	1340.60	–	14.69%	816.20	–	3.66%	1780.10	–	0.27%	143.50	30.03 (10)
				49.64% (10)			15.73% (10)			14.37% (10)			–
	6	24.51%	1512.70	–	6.99%	1837.50	–	6.62%	871.20	–	0.41%	489.60	73.60 (9)
				25.02% (10)			6.67% (10)			22.03% (10)			0.16% (1)
	12	4.76%	12405.20	–	1.56%	8429.80	–	14.86%	938.80	–	0.35%	420.60	96.18 (10)
				4.55% (10)			1.19% (10)			24.73% (10)			–
	30	0.02%	1201.00	6.48 (10)	0.01%	12.40	3.00 (10)	25.07%	779.90	–	0.00%	0.60	5.20 (10)
				–			–			27.27% (10)			–
	60	0.02%	1202.30	6.33 (10)	0.01%	27.70	3.37 (10)	25.07%	809.50	–	0.00%	0.00	5.25 (10)
				–			–			18.09% (10)			–

Chapter 4

Joint Pricing and Assortment Optimization Problem

In previous cases we assumed that the selling price of each product is known and constant. In joint pricing and assortment optimization problem, the selling price of each product is a decision variable and the retailer wants to determine the assortment of products and their prices, at the same time, to offer to the customer with the goal of maximizing revenue. The customer purchase behaviour is captured by mixed multinomial logit model where a higher price results in a lower preference for buying a particular product. The price of each product is selected from a finite and discrete set of values and is the same across all customer segments. Moreover, an upper bound is set on the number of products offered to the customers. We model this problem and establish a new approach to obtain the relevant McCormick valid inequalities. We use a commercial optimization software to conduct the numerical studies, the results of which demonstrate the effectiveness of McCormick inequalities.

Let L_j be the set of price alternatives for product j and π_{jl} be the selling price of product j at price level l . Under the mixed multinomial logit model, each price level corresponds to a different value of preference for a product. Therefore, we define ν_{ijl} as the preference of product j under price level l , which is derived from

the formula $e^{(u_{ij}-\pi_{jl})/\mu}$. Consequently, we can easily transform the assortment optimization problem (AOP) presented in previous chapter, to the joint pricing and assortment optimization problem (PAOP) as follows:

$$\max \sum_{i \in M} \gamma_i \left[\frac{\sum_{j \in N} \sum_{l \in L_j} \pi_{jl} \nu_{ijl} x_{jl}}{\nu_{i0} + \sum_{j \in N} \sum_{l \in L_j} \nu_{ijl} x_{jl}} \right] \quad (4.1)$$

$$\text{(PAOP)} \quad \text{s.t.} \quad \sum_{j \in N} \sum_{l \in L_j} x_{jl} \leq K, \quad (4.2)$$

$$\sum_{l \in L_j} x_{jl} \leq 1, \quad j \in N, \quad (4.3)$$

$$x_{jl} \in \{0, 1\}, \quad j \in N, \quad l \in L_j. \quad (4.4)$$

In above model, x_{jl} is a binary variable which takes the value 1 if product j at price level l is offered to the customers and 0, otherwise. The first constraint limits the number of products in the offered assortment to a constant number K . The second constraint ensures that at most one price level is selected for each product.

Implementing the same procedure described in Chapter 3, it is straightforward to derive the MILP and CONIC formulations from (PAOP) and the corresponding McCormick inequalities, as given below. It is worth mentioning that if we assume the selling price of each product is a continuous variable, then the corresponding MILP and CONIC models cannot be obtained due to nonlinearity of preferences $\nu_{ij} = e^{(u_{ij}-\pi_j)/\mu}$.

The mixed integer linear programming (MILP) formulation of joint pricing

and assortment optimization is:

$$\max \sum_{i \in M} \sum_{j \in N} \sum_{l \in L_j} \gamma_i \pi_{jl} \nu_{ijl} z_{ijl} \quad (4.5)$$

$$\text{s.t.} \quad \sum_{j \in N} \sum_{l \in L_j} x_{jl} \leq K, \quad (4.6)$$

$$\sum_{l \in L_j} x_{jl} \leq 1, \quad j \in N, \quad (4.7)$$

$$\nu_{i0} y_i + \sum_{j \in N} \sum_{l \in L_j} \nu_{ijl} z_{ijl} = 1, \quad i \in M, \quad (4.8)$$

$$(\text{MILP}) \quad \nu_{i0}(y_i - z_{ijl}) \leq 1 - x_{jl}, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.9)$$

$$z_{ijl} \leq y_i, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.10)$$

$$\nu_{i0} z_{ijl} \leq x_{jl}, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.11)$$

$$x_{jl} \in \{0, 1\}, \quad j \in N, \quad l \in L_j, \quad (4.12)$$

$$z_{ijl} \geq 0, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.13)$$

$$y_i \geq 0, \quad i \in M. \quad (4.14)$$

Let $\bar{\pi} = \max_{j \in N, l \in L_j} \pi_{jl}$. The mixed integer conic quadratic program (CONIC) of

the problem is:

$$\min \sum_{i \in M} \gamma_i \nu_{i0} \bar{\pi} y_i + \sum_{i \in M} \sum_{j \in N} \sum_{l \in L_j} \gamma_i \nu_{ijl} (\bar{\pi} - \pi_{jl}) z_{ijl} \quad (4.15)$$

$$\text{s.t.} \quad \sum_{j \in N} \sum_{l \in L_j} x_{jl} \leq K, \quad (4.16)$$

$$\sum_{l \in L_j} x_{jl} \leq 1, \quad j \in N, \quad (4.17)$$

$$w_i = \nu_{i0} + \sum_{j \in N} \sum_{l \in L_j} \nu_{ijl} x_{jl}, \quad i \in M, \quad (4.18)$$

$$z_{ijl} w_i \geq x_{jl}^2, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.19)$$

$$\text{(CONIC)} \quad y_i w_i \geq 1, \quad i \in M, \quad (4.20)$$

$$\nu_{i0} y_i + \sum_{j \in N} \sum_{l \in L_j} \nu_{ijl} z_{ijl} \geq 1, \quad i \in M, \quad (4.21)$$

$$x_{jl} \in \{0, 1\}, \quad j \in N, \quad l \in L_j, \quad (4.22)$$

$$z_{ijl} \geq 0, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.23)$$

$$y_i \geq 0, \quad i \in M, \quad (4.24)$$

$$w_i \geq 0, \quad i \in M. \quad (4.25)$$

The McCormick valid inequalities for the pricing case are as follows:

$$z_{ijl} \leq y_{i|x_{jl}=1}^u x_{jl}, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.26)$$

$$\text{(MC)} \quad z_{ijl} \geq y_{i|x_{jl}=1}^\ell x_{jl}, \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.27)$$

$$z_{ijl} \leq y_i - y_{i|x_{jl}=0}^\ell (1 - x_{jl}), \quad i \in M, \quad j \in N, \quad l \in L_j, \quad (4.28)$$

$$z_{ijl} \geq y_i - y_i^u (1 - x_{jl}), \quad i \in M, \quad j \in N, \quad l \in L_j. \quad (4.29)$$

In this case, $y_{i|x_{jl}=1}^u = 1/(\nu_{i0} + \nu_{ijl})$ and $y_i^u = 1/\nu_{i0}$. We develop new approaches to establish the conditional lower bounds on y_i , i.e., $y_{i|x_{jl}=1}^\ell$ and $y_{i|x_{jl}=0}^\ell$.

We make use of constraints 4.2 and 4.3 to attain lower bounds on y_i . For every $i \in M$ and $j \in N$, let $\bar{\nu}_{ij} = \max_{l \in L_j} \nu_{ijl}$. Also, let $\bar{\nu}_{i[k]}$ be the k th largest of preferences $\bar{\nu}_{it}$, $t \in N$.

Proposition 7 *In the pricing and assortment optimization problem, for every $i \in M$, the following lower bound on y_i is valid:*

$$y_i^\ell := \frac{1}{\nu_{i0} + \sum_{k=1}^K \bar{\nu}_{i[k]}} \leq y_i. \quad (4.30)$$

For every $i \in M$, $j \in N$ and $l \in L_j$, let $\bar{\nu}_{ijl} = \max_{m \in L_j \setminus \{l\}} \nu_{ijm}$. We define $\bar{\nu}_{i[k]}$ as the k th largest of preferences from the set $\{\bar{\nu}_{ijl}, \{\bar{\nu}_{it}\}_{\forall t \in N \setminus \{j\}}\}$. Also let $\tilde{\nu}_{i[k]}$ be the k th largest of preferences $\bar{\nu}_{it}$, $t \in N \setminus \{j\}$.

Proposition 8 *In the pricing and assortment optimization problem, for every $i \in M$, $j \in N$ and $l \in L_j$, the following conditional lower bounds on y_i are valid:*

$$y_{i|x_{jl}=0}^\ell := \frac{1}{\nu_{i0} + \sum_{k=1}^K \bar{\nu}_{i[k]}} \leq y_i, \quad (4.31)$$

$$y_{i|x_{jl}=1}^\ell := \frac{1}{\nu_{i0} + \nu_{ijl} + \sum_{k=1}^{K-1} \tilde{\nu}_{i[k]}} \leq y_i. \quad (4.32)$$

4.1 Numerical Studies

In order to examine the effectiveness of conic formulation and McCormick inequalities for the joint pricing and assortment optimization problem, we carry out several numerical experiments on four sets of mixed integer programs: MILP, MILP+MC, CONIC and CONIC+MC. MC constraints are added to MILP and CONIC models forming MILP+MC and CONIC+MC, respectively. We use Gurobi 6.5.0 under the same settings described in section 3.4, to solve the problems.

We perform the numerical experiments with 50 products ($|N| = 50$), 5 customer segments ($|M| = 5$) and 5 price levels ($|L_j| = 5$). Five values ($\{2, 5, 12, 25, 50\}$) of the capacity parameter K and two values ($\{20, 40\}$) for the no-purchase preference ν_0 (for all $i \in M$, $\nu_{i0} = \nu_0$) are considered, resulting in 10 different problems where for each problem 5 instances are generated. For $i \in M$ the parameter γ_i is set equal to $1/5$. For all problems, $\mu = 1.5$, the utilities u_{ij} are

drawn randomly from the uniform distribution on the interval $[1,3]$, the lowest possible price (π_{j1}) for all products is set equal to 1 and the highest possible price of each product (π_{j5}) is drawn from the uniform distribution on the interval $(1,3]$. The other three price levels are drawn from the interval $(1,\pi_{j5})$ in a way that the difference between the values of any two consecutive price levels is the same. Then the expression $e^{(u_{ij}-\pi_{jl})/\mu}$ is employed to calculate the values of preferences ν_{ijl} . We use a time limit of 3600 seconds.

Table 4.1 reports the average computational results. The results of individual instances are presented in Appendix A.9. With MILP model, all 20 low capacitated problems are solved within the time limit, however for none of the problems with medium or high capacities the optimal solution is reached. Even though the McCormick inequalities considerably decrease the root gaps and the number of explored nodes, no more problems are solved with MILP+MC. CONIC formulation attains the optimal solution only for 12 tightly capacitated problems. This meager performance appears to be due to large root gaps leading to excessive number of nodes to be explored. However, when CONIC model is accompanied with McCormick valid cuts, the root gaps decline significantly and 35 out of 50 problems reach to optimality within the time limit. Moreover, the remaining optimality gaps with CONIC+MC is much smaller than those of MILP+MC. We also observe that under our settings, the cheapest price is not selected in any of the optimal solutions, the second price level is rarely optimal and higher prices, especially the two highest ones, are more likely to be optimal.

Table 4.1: Average results of problems with 50 products, 5 customer segments and 5 price levels.

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC		
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time
20	2	17.43%	45.80	0.84 (5)	5.82%	32.40	0.87 (5)	10.07%	23173.20	60.29 (5)	0.02%	0.00	1.22 (5)
	5	38.18%	104626.00	79.25 (5)	21.94%	69814.40	84.60 (5)	19.05%	1536288.40	—	0.42%	284.20	3.69 (5)
	12	65.29%	2149196.40	—	33.93%	590362.40	—	29.62%	692952.00	—	2.08%	111622.40	394.15 (5)
	25	37.74%	850252.40	34.44% (5)	16.90%	680065.80	21.84% (5)	36.93%	479903.00	9.12% (5)	2.97%	323688.00	1756.41 (5)
	50	20.37%	2671383.80	29.97% (5)	9.36%	1111443.60	11.84% (5)	44.54%	326140.00	14.97% (5)	4.29%	565196.60	—
40	2	9.12%	18.40	0.66 (5)	3.17%	4.80	0.79 (5)	5.44%	18352.60	49.40 (5)	0.00%	0.00	0.84 (5)
	5	20.51%	21443.80	23.23 (5)	13.17%	13643.20	16.59 (5)	11.18%	1098178.00	2504.21 (2)	0.09%	53.80	2.09 (5)
	12	41.77%	2887245.60	—	27.23%	1979843.20	—	19.58%	803467.00	0.66% (3)	0.61%	33862.80	78.26 (5)
	25	63.84%	887280.40	16.05% (5)	23.48%	600641.80	13.06% (5)	27.25%	552950.80	6.70% (5)	1.68%	1503376.60	—
	50	35.41%	1581592.20	46.50% (5)	11.16%	745471.00	18.52% (5)	34.31%	530705.20	14.18% (5)	2.89%	900063.00	0.33% (5)
		27.08% (5)			7.52% (5)			17.52% (5)			1.55% (5)		

Chapter 5

Conclusions

We studied assortment optimization problem under mixed multinomial logit model with three different capacity constraint cases: single resource constraint, multiple resource constraints and multiple cardinality constraints. This problem is NP-hard even when there is no constraint and only two optimal exact solution approaches - including a mixed integer linear program (MILP) and a mixed integer conic quadratic program (CONIC) - are suggested in the literature. However these approaches can be quite time consuming even for moderately sized instances. Hence, reviewing the academic literature, reveals an important gap in establishing timely optimal solution procedures to the problem. To cover this gap, for each case we developed efficient approaches to attain McCormick valid inequalities to strengthen the mixed integer programming (MIP) formulations of the problem and solve them using a commercial optimization software. We performed several numerical studies which indicate that the presence of the McCormick inequalities enhances the performance of the MIP models significantly. Specifically, when CONIC formulation is accompanied with McCormick cuts, we were able to solve even large sized problems optimally in a reasonable amount of time.

In addition, we studied joint pricing and assortment optimization problem under mixed multinomial logit model under a single cardinality constraint. For

this case, we established new procedures to construct McCormick valid inequalities. The corresponding numerical studies demonstrated that the incorporation of McCormick inequalities into the CONIC formulation leads to an acceptable performance. An advantage of our approach is that, in practical environments it can model and solve problems optimally using the available commercial optimization software diminishing the need for developing customized solution techniques.

For future research avenues, we suggest establishing new approaches to tighten the McCormick inequalities introduced in this research. Another useful research direction can be incorporation of practical situations such as quality consistent pricing or inventory control into the assortment optimization problem under MMNL and establishing the corresponding CONIC formulation and valid inequalities to solve the problem efficiently.

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Appendix A

Individual Computational Results

A.1 Individual Results of Table 3.1

Table A.1: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 1, $\bar{\pi} = 2.9822$).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Obj.	Gap	Nodes	Time	Obj.	Gap	Nodes	Time	Obj.	
5	5	13.22%	7745	–	1.8839	4.17%	13320	274.91	1.8897	13.65%	9414	–	1.1019	28
				10.72%					2.45%					
	10	5.08%	16583	–	2.0327	1.99%	31803	488.17	2.0360	16.75%	3678	–	1.0023	37
				4.22%					9.32%					
	25	0.09%	6247	10.5	2.1375	0.04%	0	4.36	2.1375	25.69%	790	–	0.9007	58
5				–				–					14.04%	
	50	0.07%	2251	7.88	2.1379	0.04%	94	2.69	2.1379	27.77%	753	–	0.8520	62
				–				–					10.75%	
	100	0.07%	2229	7.8	2.1379	0.04%	107	2.33	2.1379	27.78%	752	–	0.8539	62
				–				–					12.40%	
10	5	20.76%	14638	–	1.4956	4.07%	17718	308.04	1.5056	9.27%	10587	–	1.4772	34
				12.41%				–					0.54%	
	10	12.86%	11805	–	1.6749	3.10%	16993	–	1.6843	11.83%	4100	–	1.3132	45
				12.30%				1.24%					2.45%	
	25	2.04%	36797	–	1.8636	0.71%	50283	–	1.8640	17.72%	937	–	1.1613	66
10				1.71%				0.19%					8.71%	
	50	0.03%	54	1.16	1.9014	0.01%	0	1.02	1.9014	21.22%	756	–	1.0969	85
				–				–					8.29%	
	100	0.03%	87	1.2	1.9014	0.01%	0	1.58	1.9014	21.23%	764	–	1.0812	85
				–				–					8.21%	

Table A.2: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	14.59%	6512	– 14.48%	5.90%	12190	– 1.94%	17.07%	5162	– 5.49%	3.63%	1757	130.85	1.0765	32
	10	6.66%	15700	– 6.10%	3.29%	17774	– 1.89%	19.30%	3730	– 10.15%	3.56%	2570	366.21	0.9343	39
	25	0.04%	2049	8.81	0.02%	0	2.03	24.35%	907	– 9.80%	0.02%	0	2.52	0.7984	62
	50	0.02%	64	1.52	0.01%	0	1	26.39%	748	– 41.51%	0.00%	0	1.86	0.7980	63
	100	0.02%	89	1.63	0.01%	0	1.66	26.38%	750	– 11.72%	0.01%	0	1.92	0.7980	63
10	5	21.02%	9742	– 15.34%	4.79%	19655	– 1.21%	11.06%	5494	– 1.17%	1.22%	794	47.88	1.4221	41
	10	13.31%	7879	– 13.84%	4.16%	11073	– 2.45%	13.72%	3748	– 5.81%	2.24%	2260	397.38	1.2620	50
	25	1.77%	25369	– 1.47%	0.61%	18586	205.51	16.99%	1170	– 9.45%	0.35%	896	52.21	1.0662	69
	50	0.01%	24	1.11	0.01%	0	0.99	20.18%	767	– 8.28%	0.01%	0	2.06	1.0324	89
	100	0.01%	32	0.95	0.01%	0	0.95	20.18%	746	– 7.94%	0.00%	0	1.64	1.0324	89

Table A.3: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	23.30%	5729	– 17.34%	9.60%	8258	– 5.50%	16.57%	8181	– 4.98%	3.52%	3382	294.72	1.2155	26
	10	11.21%	13192	– 11.08%	5.38%	12757	– 3.75%	17.65%	4286	– 9.07%	3.37%	4978	422.32	1.0206	31
	25	0.68%	114926	– 0.11%	0.34%	1450	14.91	20.94%	2040	–	0.18%	165	9.92	0.8142	50
	50	0.01%	3	1.75	0.00%	0	0.94	24.86%	755	–	0.00%	0	1.88	0.7995	63
	100	0.01%	3	1.36	0.00%	0	0.97	24.85%	753	10.16%	0.00%	0	1.8	0.7995	63
10	5	27.41%	6705	– 21.24%	8.37%	10825	– 4.36%	11.48%	10411	–	1.62%	3938	171.46	1.6032	32
	10	21.53%	6702	– 21.14%	6.58%	10976	– 4.38%	12.94%	7791	–	2.04%	3489	310.89	1.3900	41
	25	4.30%	20100	– 4.08%	1.56%	24955	– 0.89%	15.56%	3240	–	0.99%	2772	277.33	1.1248	58
	50	0.02%	53	1.36	0.01%	0	1.09	19.76%	762	–	0.00%	0	1.76	1.0448	90
	100	0.02%	80	1.48	0.01%	0	1.05	19.75%	753	–	0.00%	0	1.84	1.0448	90
		7.08%													

Table A.4: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	13.51%	6602	– 11.53%	5.06%	16862	– 1.13%	15.77%	9021	– 3.22%	2.62%	898	52.42	1.1366	29
	10	4.57%	23741	– 3.40%	1.99%	25254	356.71	18.86%	3377	– 12.91%	1.78%	797	58.82	0.9779	36
	25	0.05%	217	2.19	0.02%	0	2.67	27.92%	1194	– 11.01%	0.02%	6	2.64	0.8871	55
	50	0.05%	288	1.83	0.03%	48	1.95	29.29%	750	– 12.95%	0.02%	6	2.95	0.8871	55
	100	0.05%	745	4.3	0.04%	64	2.09	29.28%	750	– 15.47%	0.02%	6	2.81	0.8871	55
10	5	21.66%	9564	– 13.78%	4.94%	14793	– 1.18%	11.03%	9301	– 1.01%	1.21%	665	43.96	1.5238	33
	10	11.86%	9611	– 12.11%	3.25%	14994	– 1.31%	13.84%	4127	– 5.94%	1.21%	988	70.13	1.3346	43
	25	1.41%	35883	– 1.07%	0.46%	32963	375.19	20.27%	1208	– 11.87%	0.28%	880	51.03	1.1638	68
	50	0.03%	116	1.58	0.01%	0	1.03	23.37%	759	– 10.65%	0.00%	0	1.86	1.1387	89
	100	0.03%	156	1.67	0.01%	0	0.98	23.36%	752	– 10.15%	0.00%	0	1.72	1.1387	89

Table A.5: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	14.72%	7027	– 10.90%	5.39%	14564	– 1.67%	17.09%	6344	– 3.78%	3.03%	1864	94.82 –	1.1563	27
	10	5.88%	19398	– 4.66%	2.48%	30477	– 0.94%	19.55%	4147	– 11.69%	1.14%	627	52.64 –	1.0043	32
	25	0.14%	12714	27.58	0.06%	76	3.33	27.85%	848	– 15.12%	0.02%	4	3.33 –	0.8916	57
	50	0.03%	256	2.22	0.02%	0	2.47	29.23%	744	– 14.41%	0.01%	0	2.03 –	0.8893	61
	100	0.03%	300	2.2	0.02%	0	3.25	29.23%	741	– 13.45%	0.01%	0	2.09 –	0.8893	61
10	5	21.65%	14682	– 14.31%	5.46%	14475	– 2.05%	11.63%	11449	– 2.69%	1.23%	559	34.31 –	1.5286	33
	10	14.96%	7792	– 14.19%	4.51%	13695	– 2.89%	14.69%	3324	– 6.33%	1.96%	6545	– 0.06%	1.3652	40
	25	2.37%	29430	– 2.00%	0.89%	49053	– 0.28%	19.76%	2466	– 9.67%	0.50%	3777	202.67 –	1.1672	62
	50	0.04%	164	1.64	0.01%	0	1.59	22.32%	761	– 10.69%	0.01%	0	1.92 –	1.1251	85
	100	0.04%	139	1.56	0.02%	0	2.13	22.32%	776	– 9.55%	0.01%	0	1.81 –	1.1251	85

Table A.6: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 6).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	18.94%	7341	— 15.89%	7.68%	9579	— 3.52%	16.02%	6448	— 2.99%	3.57%	1205	183.34 —	1.1029	27
	10	9.27%	15736	— 8.92%	4.58%	14719	— 3.08%	17.76%	4628	— 5.78%	4.51%	3496	— 0.81%	0.9358	34
	25	0.20%	3408	10.58	0.12%	85	2.94	21.12%	3505	— 9.16%	0.25%	45	6.53 —	0.7498	52
	50	0.04%	219	1.83	0.02%	0	2.2	24.65%	2603	— 11.46%	0.00%	0	2.33 —	0.7463	57
	100	0.04%	301	1.7	0.02%	21	1.66	24.65%	749	— 10.58%	0.00%	0	2.23 —	0.7463	57
10	5	24.07%	6541	— 18.36%	6.40%	12951	— 2.28%	10.85%	8908	— 2.19%	1.39%	761	53.32 —	1.4722	33
	10	16.24%	9413	— 15.85%	5.38%	13346	— 3.36%	12.82%	3357	— 3.04%	2.53%	3298	439.98 —	1.2861	43
	25	2.12%	35593	— 1.63%	0.80%	42910	396.5 —	15.77%	2694	— 7.99%	0.70%	1437	73.63 —	1.0506	62
	50	0.04%	180	1.55	0.02%	0	1.58	20.78%	756	— 7.68%	0.01%	12	2.44 —	1.0102	78
	100	0.04%	207	1.58	0.02%	24	1.86	20.77%	770	— 6.98%	0.01%	6	2.48 —	1.0102	78

Table A.7: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 7).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	21.50%	5940	– 17.39%	8.88%	10850	– 3.65%	17.76%	4704	– 4.21%	3.52%	2175	146.34 –	1.2277	26
	10	9.84%	15275	– 8.90%	4.83%	14642	– 3.00%	19.58%	2807	– 13.01%	3.88%	3119	419.57 –	1.0402	33
	25	0.25%	30280	91.25	0.14%	330	4.23	24.36%	1426	– 11.85%	0.11%	53	5.88 –	0.8535	52
	50	0.02%	80	1.56	0.01%	0	0.92	27.89%	754	–	0.00%	0	2.11 –	0.8486	57
	100	0.02%	44	1.33	0.01%	0	0.95	27.89%	747	– 13.01%	0.00%	0	1.92 –	0.8486	57
10	5	25.52%	8073	– 17.64%	6.88%	14620	– 1.87%	12.25%	7229	– 2.77%	1.68%	811	56.17 –	1.6061	31
	10	19.04%	6044	– 18.72%	5.94%	10341	– 3.60%	14.09%	4552	– 7.45%	2.05%	1467	239.95 –	1.4020	40
	25	3.09%	29357	– 2.79%	1.10%	37866	– 0.34%	17.74%	1917	– 10.64%	0.47%	1380	93.58 –	1.1561	59
	50	0.02%	26	1.22	0.00%	0	1.02	22.22%	752	– 9.12%	0.00%	0	1.83 –	1.0996	89
	100	0.02%	9	1.14	0.00%	0	0.97	22.22%	763	– 8.52%	0.00%	0	1.72 –	1.0996	89

Table A.8: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 8).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	16.31%	6497	— 11.83%	7.08%	9514	— 2.53%	18.94%	7220	— 5.44%	4.76%	1623	181.57 —	1.1765	28
	10	5.99%	13372	— 4.68%	3.04%	30156	— 0.55%	20.23%	3607	— 19.18%	2.96%	1780	191.01 —	0.9989	31
	25	0.06%	312	1.95 —	0.02%	0	2.13 —	27.68%	1216	— 13.40%	0.01%	0	2 —	0.8803	50
	50	0.06%	1400	4.05 —	0.03%	32	1.92 —	29.11%	749	— 11.71%	0.01%	0	2.25 —	0.8803	50
	100	0.06%	1183	3.83 —	0.04%	57	1.98 —	29.11%	744	— 14.48%	0.02%	2	2.88 —	0.8803	50
10	5	22.41%	6013	— 15.29%	5.91%	14654	— 1.16%	13.32%	6082	— 3.35%	2.06%	1143	127.38 —	1.5361	38
	10	13.30%	10720	— 12.60%	4.03%	14676	— 1.77%	15.08%	3500	— 4.85%	1.85%	1204	288.98 —	1.3530	45
	25	1.64%	34292	— 1.35%	0.62%	38307	498.92 —	20.33%	793	— 11.62%	0.30%	963	62.82 —	1.1641	67
	50	0.03%	237	1.78 —	0.01%	0	1.56 —	23.18%	765	— 10.02%	0.01%	0	1.86 —	1.1347	87
	100	0.03%	188	1.66 —	0.02%	44	1.77 —	23.18%	750	— 10.71%	0.00%	0	1.83 —	1.1347	87

Table A.9: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 9).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	19.37%	4455	– 17.22%	7.33%	9091	– 4.10%	18.33%	9278	– 11.82%	3.81%	4674	463.88	1.2155	29
	10	7.32%	12967	– 7.27%	3.16%	29423	– 1.16%	19.94%	4592	– 9.50%	2.36%	1341	114.69	1.0177	37
	25	0.22%	11107	51.78	0.09%	104	4.22	26.86%	792	– 14.53%	0.10%	69	7.11	0.8789	58
	50	0.05%	786	3.28	0.02%	0	3.59	29.01%	747	– 13.05%	0.01%	6	3.08	0.8753	62
	100	0.05%	736	3.33	0.03%	60	2.53	29.01%	756	– 14.94%	0.01%	8	2.95	0.8753	62
10	5	27.12%	7030	– 21.92%	5.77%	10993	– 2.66%	11.15%	7954	– 2.16%	1.56%	318	49.77	1.5639	37
	10	15.53%	6221	– 15.57%	3.87%	14662	– 1.77%	13.35%	4330	– 3.30%	1.51%	1798	223.81	1.3556	46
	25	2.44%	22538	– 2.12%	0.81%	41454	– 0.13%	18.42%	2468	– 8.41%	0.44%	2625	214.84	1.1485	67
	50	0.02%	55	1.31	0.01%	0	1.06	21.89%	762	– 10.33%	0.00%	0	1.95	1.1042	96
	100	0.02%	73	1.16	0.01%	0	1.06	21.89%	748	– 9.24%	0.00%	0	1.86	1.1042	96

Table A.10: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 10).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	18.99%	5241	– 14.42%	8.19%	9470	– 4.21%	17.03%	8141	– 4.39%	4.60%	2818	392.97	1.1151	27
	10	8.37%	12896	– 8.35%	4.15%	16021	– 2.50%	17.37%	4373	– 9.38%	3.72%	1799	271.42	0.9323	33
	25	0.17%	3183	9.92	0.10%	66	2.36	21.42%	2706	– 10.58%	0.16%	42	6.03	0.7646	52
	50	0.01%	337	2.16	0.00%	0	0.94	24.41%	752	–	0.00%	0	1.83	0.7611	56
	100	0.01%	56	1.59	0.01%	0	0.97	24.41%	748	– 10.78%	0.00%	0	1.94	0.7611	56
10	5	24.05%	7397	– 17.33%	6.67%	13223	– 2.25%	11.67%	7761	– 1.37%	2.03%	1589	112.43	1.4844	34
	10	15.83%	7116	– 16.16%	4.80%	11415	– 2.76%	12.70%	3519	– 2.77%	1.88%	848	218.04	1.2864	45
	25	2.39%	27641	– 2.05%	0.89%	44146	556.7	16.13%	2700	– 8.54%	0.66%	897	53.52	1.0640	62
	50	0.02%	39	1.33	0.00%	0	1.03	20.19%	757	– 7.90%	0.00%	0	2	1.0186	88
	100	0.02%	37	1.19	0.01%	0	0.98	20.18%	780	– 7.28%	0.00%	0	1.89	1.0186	88

A.2 Individual Results of Table 3.2



Table A.11: Results for problems with 300 products and 30 customer segments for the single resource constraint case (Instance 1).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	7	13.85%	15931	– 13.38%	6.80%	16002	– 4.27%	18.92%	10821	– 9.49%	5.26%	8196	– 0.53%	0.9612	38
	15	4.98%	30544	– 5.28%	2.74%	52025	– 1.56%	20.03%	3942	– 16.65%	3.43%	2889	1569.27 –	0.7890	46
	30	0.21%	11273	115.49 –	0.12%	249	9.03 –	24.44%	2249	– 14.27%	0.17%	225	35.72 –	0.6839	62
10	75	0.01%	0	1.77 –	0.00%	0	2.66 –	28.77%	1967	– 16.58%	0.00%	0	5.06 –	0.6793	68
	150	0.01%	36	3.2 –	0.00%	0	2.7 –	28.77%	957	– 15.45%	0.00%	0	5.34 –	0.6793	68
	7	21.46%	12056	– 18.01%	6.35%	18669	– 3.77%	13.10%	15026	– 2.94%	2.31%	2012	873.69 –	1.2993	44
10	15	10.35%	15305	– 10.89%	3.88%	23826	– 2.73%	15.05%	7500	– 6.44%	2.18%	3575	1710.34 –	1.1057	59
	30	2.37%	50524	– 2.06%	1.02%	63993	– 0.48%	18.65%	5091	– 9.56%	0.72%	1416	659.10 –	0.9582	76
	75	0.01%	3	4.34 –	0.00%	0	3.06 –	24.55%	1114	– 12.86%	0.00%	0	4.41 –	0.9100	106
	150	0.01%	68	4.69 –	0.00%	0	2.94 –	24.55%	761	– 11.58%	0.00%	0	5.00 –	0.9100	106

Table A.12: Results for problems with 300 products and 30 customer segments for the single resource constraint case (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	7	11.31%	15436	– 11.31%	4.61%	27452	– 2.29%	15.27%	14544	– 4.49%	1.85%	621	143.77 –	0.9047	35
	15	4.01%	38588	– 3.71%	2.02%	63878	– 1.10%	19.44%	3608	– 21.53%	1.89%	2256	986.82 –	0.7589	50
	30	0.17%	39418	196.98	0.09%	427	12.66	24.19%	2190	– 12.58%	0.10%	77	28.78 –	0.6737	71
	75	0.01%	17	3.45	0.00%	0	2.77	27.49%	793	– 11.88%	0.00%	0	4.48 –	0.6699	77
	150	0.01%	18	2.59	0.00%	0	2.69	27.47%	924	– 19.18%	0.00%	0	4.61 –	0.6699	77
10	7	21.38%	12152	– 17.42%	5.64%	21443	– 3.17%	11.67%	18429	– 2.58%	1.53%	2211	504.15 –	1.2768	42
	15	9.83%	18222	– 10.04%	3.79%	26659	– 2.58%	15.05%	6681	– 7.95%	2.12%	6304	1936.37 –	1.0755	58
	30	2.27%	53975	– 2.05%	0.97%	122552	– 0.47%	18.14%	3874	– 10.78%	0.65%	1829	739.29 –	0.9345	77
	75	0.02%	79	3.92	0.01%	0	3.14	22.67%	863	– 11.65%	0.00%	0	4.64 –	0.8883	107
	150	0.02%	74	3.08	0.01%	0	2.86	22.67%	1080	– 11.99%	0.00%	0	5.36 –	0.8883	107

Table A.13: Results for problems with 300 products and 30 customer segments for the single resource constraint case (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	7	12.37%	14418	— 12.36%	4.94%	18160	— 2.52%	16.29%	13588	— 5.30%	2.91%	1447	651.53 —	0.9566	37
	15	4.94%	31923	— 4.77%	2.49%	32353	— 1.79%	20.38%	3137	— 16.26%	2.89%	9033	— 0.19%	0.8122	46
	30	0.35%	451465	— 0.01%	0.19%	3733	71.1 —	25.18%	1613	— 25.22%	0.15%	235	43.72 —	0.7122	68
	75	0.01%	0	1.63 —	0.00%	0	2.75 —	29.46%	765	— 19.07%	0.00%	0	4.55 —	0.7043	77
10	150	0.01%	0	1.47 —	0.00%	0	2.69 —	29.46%	1435	— 16.03%	0.00%	0	4.41 —	0.7043	77
	7	20.61%	14753	— 16.56%	4.78%	17385	— 1.93%	11.12%	11287	— 2.61%	1.26%	1843	380.66 —	1.3024	46
	15	10.61%	16207	— 10.65%	3.75%	18218	— 2.76%	14.99%	4301	— 9.93%	2.32%	9067	— 0.23%	1.1222	60
	30	2.58%	31866	— 2.35%	1.04%	55732	— 0.52%	18.27%	3618	— 10.08%	0.75%	1856	799.84 —	0.9754	76
	75	0.02%	157	5.23 —	0.01%	0	3.19 —	24.31%	917	— 37.89%	0.00%	0	4.84 —	0.9236	109
	150	0.02%	142	3.86 —	0.01%	0	2.83 —	24.33%	956	— 13.05%	0.00%	0	5.19 —	0.9236	109

Table A.14: Results for problems with 300 products and 30 customer segments for the single resource constraint case (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	7	11.29%	19974	— 11.34%	5.30%	19726	— 3.30%	19.78%	12470	— 11.27%	4.05%	2576	1506.33	0.9639	38
	15	3.67%	44338	— 3.17%	1.95%	54619	— 1.12%	22.10%	3634	— 19.64%	2.50%	8653	— 0.12%	0.8150	48
	30	0.05%	6026	18.08	0.02%	129	8.55	28.06%	2741	— 18.48%	0.01%	24	9.14	0.7363	72
5	75	0.05%	1436	9.78	0.03%	204	7.5	31.33%	754	— 21.96%	0.01%	6	8.88	0.7362	74
	150	0.05%	1273	8.52	0.03%	282	8.36	31.31%	898	— 15.12%	0.01%	6	9.84	0.7362	74
10	7	19.57%	14494	— 17.11%	5.36%	16791	— 2.70%	13.33%	13664	— 4.01%	1.67%	1344	670.02	1.2909	50
	15	8.97%	21214	— 8.87%	3.31%	27527	— 2.46%	16.20%	6577	— 9.62%	2.17%	7477	— 0.13%	1.1166	63
	30	1.51%	82203	— 1.29%	0.60%	84966	— 0.14%	19.43%	3536	— 11.59%	0.34%	1481	455.38	0.9791	83
10	75	0.02%	119	3.97	0.01%	0	2.91	25.15%	798	— 12.19%	0.00%	0	4.69	0.9491	109
	150	0.02%	92	3.09	0.01%	0	2.83	25.14%	843	— 11.76%	0.00%	0	5.61	0.9491	109
				—			—						—		

Table A.15: Results for problems with 300 products and 30 customer segments for the single resource constraint case (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	7	13.45%	16611	– 12.38%	6.82%	17327	– 4.50%	20.59%	10497	– 12.50%	5.05%	4053	2474.10 –	0.9844	33
	15	4.63%	39082	– 4.10%	2.63%	49852	– 1.68%	21.85%	4457	– 22.57%	3.37%	6308	– 0.09%	0.8155	43
	30	0.05%	3131	20.71	0.03%	42	6.22	26.63%	3764	– 16.05%	0.02%	0	6.34 –	0.7162	62
10	75	0.03%	115	4.39 –	0.02%	43	5.08 –	30.83%	787	– 51.46%	0.01%	0	6.19 –	0.7158	65
	150	0.03%	111	4.27 –	0.02%	43	5.69 –	30.83%	841	– 18.14%	0.01%	6	8.67 –	0.7158	65
	7	21.66%	12434	– 17.97%	7.31%	16134	– 4.77%	15.46%	8264	– 8.14%	3.12%	4284	2208.97 –	1.3333	47
10	15	10.70%	13383	– 10.95%	4.44%	19181	– 3.31%	17.68%	6604	– 9.50%	3.69%	5035	– 0.79%	1.1465	55
	30	2.01%	51474	– 1.72%	0.86%	50147	– 0.38%	20.36%	4209	– 12.92%	0.64%	1435	788.60 –	0.9898	72
	75	0.02%	107	4.47 –	0.01%	0	2.84 –	26.16%	770	– 24.34%	0.00%	0	5.34 –	0.9501	104
	150	0.02%	123	4.5 –	0.01%	18	4.84 –	26.15%	1772	– 14.08%	0.00%	0	6.50 –	0.9501	104

A.3 Individual Results of Table 3.3

Table A.16: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 1).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	2.48%	483	32.99	2.48%	516	34.77
				—			—
	10	1.27%	957	69.36	1.27%	1034	139.59
				—			—
	25	0.05%	96	8.39	0.05%	64	7.08
10				—			—
	50	0.01%	0	1.80	0.01%	0	1.84
				—			—
	100	0.01%	0	1.91	0.01%	0	1.98
				—			—
10	5	0.78%	104	18.33	0.78%	92	17.16
				—			—
	10	1.22%	1793	152.74	1.22%	1701	165.82
				—			—
	25	0.43%	3944	315.02	0.43%	1865	173.04
10				—			—
	50	0.00%	0	1.83	0.00%	0	1.80
				—			—
	100	0.00%	0	2.03	0.00%	0	1.78
				—			—

Table A.17: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 2).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.35%	2054	269.09	3.35%	2071	276.2
				—			—
	10	2.75%	1645	259.61	2.74%	1724	287.25
				—			—
	25	0.27%	635	28.91	0.27%	619	29.86
10				—			—
	50	0.01%	0	2.02	0.01%	0	2.02
				—			—
	100	0.01%	0	1.94	0.01%	0	1.95
				—			—
10	5	1.37%	927	66.46	1.37%	1246	93.16
				—			—
	10	1.11%	1095	77.46	1.10%	1237	76.19
				—			—
	25	0.58%	2708	189.95	0.58%	2054	157.71
10				—			—
	50	0.00%	0	1.86	0.00%	0	1.77
				—			—
	100	0.00%	0	1.70	0.00%	0	1.67
				—			—

Table A.18: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 3).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	4.40%	2415	297.97	4.39%	3116	378.11
				—			—
	10	3.66%	1741	260.33	3.66%	3699	466.88
				—			—
	25	0.18%	97	7.72	0.18%	91	7.53
				—			—
10	50	0.01%	0	2.19	0.01%	0	2.19
				—			—
	100	0.01%	0	2.50	0.01%	0	2.53
				—			—
	5	1.67%	565	41.47	1.66%	669	49.03
				—			—
10	10	2.50%	4000	411.74	2.50%	4153	440.18
				—			—
	25	0.81%	2273	195.73	0.80%	2412	228.89
				—			—
	50	0.00%	0	1.67	0.00%	0	1.69
				—			—
10	100	0.00%	0	1.84	0.00%	0	1.86
				—			—

Table A.19: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 4).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.18%	761	54.05	3.17%	774	56.8
				—			—
	10	2.60%	1725	136.76	2.58%	1429	115.69
				—			—
	25	0.07%	157	7.81	0.07%	172	8.19
10				—			—
	50	0.01%	0	2.20	0.01%	0	2.16
				—			—
	100	0.01%	6	3.00	0.01%	6	3.00
				—			—
10	5	1.12%	316	28.56	1.12%	342	32.49
				—			—
	10	1.73%	2849	173.01	1.73%	2470	150.85
				—			—
	25	0.27%	1028	84.79	0.27%	901	67.10
10				—			—
	50	0.00%	0	1.80	0.00%	0	1.81
				—			—
	100	0.00%	0	1.75	0.00%	0	1.73
				—			—

Table A.20: Results for problems with 200 products and 20 customer segments for the single resource constraint case (Instance 5).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	1.94%	603	48.36	1.93%	683	50.83
				—			—
	10	1.89%	3028	322.44	1.89%	2653	312.09
				—			—
	25	0.02%	0	2.50	0.03%	0	2.55
				—			—
10	50	0.01%	0	2.53	0.01%	0	2.39
				—			—
	100	0.01%	6	3.03	0.01%	6	3.06
				—			—
	5	0.99%	397	25.36	0.99%	430	24.89
				—			—
10	10	1.46%	3949	345.14	1.46%	3806	343.89
				—			—
	25	0.35%	1031	123.90	0.35%	938	110.05
				—			—
	50	0.00%	0	1.80	0.00%	0	1.81
				—			—
10	100	0.00%	0	1.78	0.00%	0	1.77
				—			—

A.4 Individual Results of Table 3.4

Table A.21: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 1).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	44.05%	8649	– 38.73%	16.05%	16361	– 12.48%	14.25%	17510	– 6.37%	3.71%	23563	– 0.72%	1.5058	13
	10	14.79%	15336	– 15.26%	6.07%	30265	– 4.31%	15.06%	8879	– 11.21%	2.74%	6458	571.78 –	1.1254	22
	25	0.46%	59883	801.09	0.19%	1447	16.06	24.60%	7743	– 16.33%	0.12%	111	7.48 –	0.8584	51
	50	0.09%	4658	7.84	0.04%	176	3.25	28.91%	1115	– 14.39%	0.03%	30	3.91 –	0.8505	62
	100	0.09%	4228	7.11	0.04%	320	3.38	28.92%	3072	– 11.56%	0.03%	25	3.94 –	0.8505	62
10	5	49.67%	45784	– 29.29%	16.08%	22380	– 11.97%	8.97%	33313	– 8.73%	1.78%	11140	627.67 –	1.9278	16
	10	33.64%	11934	– 32.88%	9.49%	20784	– 8.36%	10.61%	13783	– 5.40%	2.19%	20326	– 0.27%	1.5547	26
	25	5.00%	69411	– 4.51%	1.65%	93164	– 0.88%	16.48%	9798	– 8.94%	0.48%	3535	245.81 –	1.1613	56
	50	0.02%	33	1.3	0.00%	0	1.11	21.04%	4173	– 9.59%	0.00%	0	2.05 –	1.0699	95
	100	0.02%	48	1.14	0.01%	0	0.97	21.04%	782	– 7.93%	0.00%	0	1.88 –	1.0699	95

Table A.22: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	35.99%	7045	–	11.30%	18240	–	9.34%	28653	–	2.52%	5295	253.14	1.3948	14
				33.00%			6.84%			3.85%					
	10	12.39%	30158	–	4.79%	43036	–	12.81%	8319	–	1.85%	6177	474.41	1.0578	23
				11.01%			2.83%			7.17%					
	25	0.37%	20860	88.25	0.17%	530	6.59	23.87%	6944	–	0.16%	189	11.84	0.8256	52
50				–			–			10.46%					
	50	0.02%	312	2.84	0.01%	0	1.03	27.09%	4021	–	0.00%	0	2.59	0.8180	58
				–			–			9.90%					
	100	0.02%	98	1.48	0.01%	0	0.92	27.09%	1073	–	0.00%	0	2.30	0.8180	58
				–			–			12.47%					
10	5	44.31%	53143	–	12.17%	21530	–	6.18%	72976	–	1.61%	10246	368.07	1.8618	14
				20.37%			6.78%			5.92%					
	10	28.89%	9664	–	7.66%	23315	–	8.84%	11375	–	1.48%	20811	1647.48	1.4924	25
				28.97%			6.07%			6.83%					
	25	3.78%	82681	–	1.26%	127584	–	16.13%	9169	–	0.46%	2995	226.39	1.1281	55
50				3.24%			0.44%			8.16%					
	50	0.02%	44	1.33	0.00%	0	1.02	20.94%	794	–	0.00%	0	2.09	1.0578	85
				–			–			9.57%					
	100	0.02%	51	1.33	0.01%	0	1	20.94%	3659	–	0.00%	0	2.34	1.0578	85
				–			–			7.70%					

Table A.23: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	39.65%	7180	–	12.23%	17834	–	9.81%	19121	–	2.71%	14373	674.19	1.4170	15
				31.30%			8.16%			3.44%					
	10	14.27%	28463	–	5.46%	38938	–	11.71%	10643	–	2.18%	3893	350.1	1.0675	25
				13.15%			3.52%			4.10%					
	25	0.53%	18282	98.1	0.24%	593	7.08	20.50%	8937	–	0.18%	108	8.17	0.8054	47
50				–			–			8.37%					
	50	0.00%	0	0.58	0.00%	0	0.91	25.58%	871	–	0.00%	0	1.33	0.7938	58
				–			–			8.51%					
	100	0.00%	0	0.48	0.00%	0	0.88	25.58%	787	–	0.00%	0	1.44	0.7938	58
				–			–			10.27%					
10	5	43.33%	52026	–	11.88%	19939	–	5.84%	53549	–	1.24%	2294	106.18	1.8578	16
				19.72%			7.50%			1.47%					
	10	29.90%	8019	–	7.94%	25917	–	7.97%	15143	–	1.61%	16282	1121.14	1.4963	27
				29.04%			5.62%			2.68%					
	25	4.10%	79979	–	1.45%	122789	–	14.45%	11792	–	0.57%	3707	205.81	1.1274	53
50				3.40%			0.45%			6.31%					
	50	0.01%	6	1.14	0.00%	0	1	20.89%	3960	–	0.00%	0	1.47	1.0514	80
				–			–			9.93%					
	100	0.01%	48	1.17	0.00%	0	0.89	20.89%	787	–	0.00%	0	1.53	1.0514	80
				–			–			7.63%					

Table A.24: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	37.07%	7333	– 32.19%	13.11%	14655	– 8.24%	11.50%	22592	– 8.04%	2.63%	9361	593.87	1.4128	13
	10	11.97%	24101	– 10.63%	4.88%	44139	– 2.61%	13.42%	8522	– 6.92%	1.91%	3344	231.54	1.0570	23
	25	0.06%	2356	11.7	0.02%	0	2.92	23.32%	6033	– 9.93%	0.03%	0	2.39	0.8258	49
	50	0.03%	292	2.48	0.01%	0	2.41	27.49%	811	– 11.45%	0.00%	0	2.20	0.8252	52
	100	0.03%	541	3.78	0.02%	0	3.48	27.49%	5957	– 12.35%	0.01%	0	2.25	0.8252	52
10	5	46.45%	47787	– 22.34%	14.16%	19827	– 9.02%	7.63%	31641	– 3.01%	1.67%	15764	658.27	1.8744	14
	10	27.66%	9819	– 25.95%	7.42%	20112	– 5.01%	9.47%	13700	– 4.21%	1.54%	12409	584.84	1.4944	24
	25	3.03%	46397	– 2.54%	1.00%	55517	– 0.33%	15.98%	8019	– 8.07%	0.32%	1668	105.76	1.1346	53
	50	0.01%	32	1.27	0.00%	0	1.03	22.06%	3020	– 9.66%	0.00%	0	1.80	1.0783	88
	100	0.01%	50	1.27	0.00%	0	0.98	22.05%	5416	– 9.54%	0.00%	0	1.94	1.0783	88

Table A.25: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	30.44%	8177	— 23.84%	8.91%	27951	— 4.93%	8.06%	24038	— 1.74%	2.43%	3126	124.6	1.3093	15
	10	9.90%	32792	— 8.89%	3.57%	52938	— 1.61%	11.36%	14489	— 5.49%	1.57%	1686	118.88	0.9985	26
	25	0.07%	3335	9.44	0.03%	0	3.08	22.87%	4910	— 6.75%	0.03%	0	2.58	0.8046	51
	50	0.03%	375	3.31	0.01%	0	1.52	26.76%	3685	— 10.44%	0.00%	0	2.31	0.8037	54
	100	0.03%	294	1.89	0.01%	0	1.48	26.76%	1652	— 9.59%	0.00%	0	2.36	0.8037	54
10	5	41.10%	33218	— 19.87%	9.13%	26376	— 4.76%	5.25%	40644	— 1.15%	1.45%	2400	76.72	1.7673	16
	10	24.14%	12594	— 22.52%	5.91%	30891	— 3.79%	7.93%	17469	— 3.27%	1.49%	16155	923.77	1.4245	28
	25	3.03%	92022	— 2.45%	0.93%	143625	— 0.19%	15.65%	11569	— 7.59%	0.30%	1783	97.49	1.1074	57
	50	0.03%	86	1.59	0.01%	0	1.03	20.99%	789	— 12.35%	0.01%	0	2.25	1.0515	86
	100	0.04%	139	1.56	0.02%	0	2.13	22.32%	776	— 7.05%	0.01%	0	2.22	1.0515	86

Table A.26: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 6).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	41.64%	6475	– 35.23%	13.45%	14662	– 9.76%	13.43%	26053	– 16.65%	2.53%	12078	774.92 –	1.4988	14
	10	15.64%	16836	– 15.34%	6.39%	31438	– 5.13%	17.42%	7249	– 11.46%	2.23%	17269	1410.26	1.1643	24
	25	0.76%	213179	– 0.36%	0.33%	3254	33.39	25.30%	5773	– 13.15%	0.24%	400	15.74 –	0.8954	51
	50	0.07%	2170	6.69 –	0.03%	68	2.39 –	28.27%	784	– 14.78%	0.02%	15	3.16 –	0.8809	63
	100	0.07%	1484	4.47 –	0.04%	124	2.3 –	28.26%	2728	– 12.77%	0.02%	15	3.09 –	0.8809	63
10	5	45.86%	70183	– 23.12%	13.81%	20000	– 9.76%	8.07%	41178	– 5.03%	1.19%	7079	288.70	1.9280	15
	10	34.15%	11491	– 34.26%	9.54%	17637	– 7.93%	11.42%	10927	– 7.06%	1.42%	21062	– 0.33%	1.5786	26
	25	5.37%	61005	– 4.89%	1.75%	86012	– 0.91%	17.31%	7520	– 8.92%	0.54%	11752	713.12	1.1944	55
	50	0.02%	39	1.31 –	0.00%	0	1 –	20.90%	3525	– 8.87%	0.00%	0	2.05 –	1.0986	96
	100	0.02%	32	1.05 –	0.01%	0	1.06 –	20.90%	2757	– 9.35%	0.00%	0	1.83 –	1.0986	96

Table A.27: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 7).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	33.28%	8257	–	9.97%	18410	–	12.17%	21362	–	2.23%	5728	252.26	1.3999	16
			30.32%		6.47%			4.25%							
	10	11.25%	29923	–	4.07%	35698	–	15.21%	6699	–	1.86%	5505	447.52	1.0846	26
			10.45%		2.45%			13.69%							
	25	0.36%	83740	645.44	0.15%	1082	13.24	25.48%	5966	–	0.11%	182	10.67	0.8778	55
				–			–			12.68%					
	50	0.03%	102	1.86	0.01%	0	1.53	27.82%	775	–	0.01%	0	2.63	0.8708	62
				–			–			12.24%					
	100	0.03%	261	1.88	0.01%	0	1.53	27.82%	3094	–	0.01%	0	2.59	0.8708	62
				–			–			12.98%					
10	5	43.62%	35542	–	10.78%	19354	–	7.60%	31268	–	1.40%	6668	203.89	1.8398	17
			23.87%		6.58%			3.82%							
	10	26.05%	13552	–	6.52%	26546	–	9.76%	12105	–	1.26%	7153	504.57	1.4869	29
			24.87%		4.81%			5.06%							
	25	3.97%	105852	–	1.27%	106179	–	17.46%	8718	–	0.36%	4480	261.97	1.1675	57
			3.51%		0.70%			9.22%							
	50	0.01%	0	0.59	0.00%	0	1	20.80%	1518	–	0.00%	0	1.92	1.0952	94
				–			–	6.98%							
	100	0.01%	33	0.97	0.00%	0	0.97	20.81%	6299	–	0.00%	0	1.97	1.0952	94
				–			–	11.04%							

Table A.28: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 8).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	39.18%	7269	— 32.80%	13.86%	15722	— 10.50%	13.82%	15932	— 4.77%	3.62%	19414	1267.95	1.4648	14
	10	13.17%	23391	— 12.00%	5.57%	33308	— 3.73%	16.23%	7021	— 11.79%	2.85%	11298	943.64	1.1205	25
	25	0.21%	47830	242.5	0.08%	144	3.77	26.72%	6035	— 11.04%	0.04%	11	4.12	0.8823	53
	50	0.08%	1191	4.41	0.04%	61	2.69	29.58%	781	— 12.64%	0.01%	0	2.46	0.8795	59
	100	0.08%	1765	4.86	0.04%	84	2.61	29.58%	774	— 13.12%	0.01%	0	2.68	0.8795	59
10	5	46.12%	43834	— 27.47%	13.67%	19956	— 9.52%	8.68%	23981	— 3.77%	1.85%	17572	1058.42	1.8832	17
	10	29.09%	10719	— 28.55%	8.22%	21561	— 6.16%	10.91%	13232	— 5.95%	1.75%	16009	1310.62	1.5317	27
	25	3.49%	60367	— 3.00%	1.19%	87263	— 0.55%	18.30%	9475	— 11.38%	0.29%	1460	111.51	1.1777	56
	50	0.07%	3982	8.84	0.02%	0	1.98	22.75%	1444	— 9.56%	0.01%	19	3.13	1.1168	93
	100	0.07%	3213	7	0.02%	76	2.2	22.75%	5405	— 10.30%	0.01%	0	2.02	1.1168	93

Table A.29: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 9).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	44.74%	9280	– 32.61%	15.02%	16069	– 12.34%	11.71%	26980	– 4.95%	2.91%	9085	403.92	1.5168	12
	10	14.91%	21709	– 14.38%	5.85%	27029	– 4.11%	13.67%	8784	– 10.82%	1.64%	4320	269.67	1.1297	22
	25	0.83%	151333	– 0.49%	0.36%	28063	328.27	25.20%	8400	– 10.20%	0.21%	695	28.95	0.8690	50
	50	0.05%	1259	4.3	0.02%	0	1.8	28.74%	957	– 14.13%	0.00%	0	2.34	0.8524	67
	100	0.05%	1513	6.2	0.02%	0	2.75	28.74%	788	– 13.98%	0.01%	0	2.92	0.8524	67
10	5	47.98%	79348	– 21.93%	17.37%	21906	– 11.55%	7.93%	45100	– 3.78%	1.75%	34646	1142.81	1.9789	13
	10	35.84%	10154	– 34.03%	9.74%	18942	– 8.17%	9.75%	14187	– 5.42%	1.34%	17222	996.65	1.5813	23
	25	6.17%	51610	– 5.79%	2.22%	78407	– 1.56%	17.13%	8466	– 10.73%	0.56%	32510	– 0.02%	1.1857	53
	50	0.03%	155	1.42	0.01%	0	1.08	20.89%	2892	– 8.90%	0.00%	0	1.95	1.0744	95
	100	0.03%	117	1.17	0.01%	0	0.98	20.89%	3555	– 8.40%	0.00%	0	1.97	1.0744	95

Table A.30: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 10).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	5	35.87%	5826	— 28.94%	11.28%	19978	— 7.21%	10.23%	25390	— 3.91%	2.21%	2957	137.71 —	1.3559	14
	10	12.81%	27438	— 11.57%	5.01%	45493	— 3.15%	13.48%	9931	— 7.32%	1.97%	5579	413.55 —	1.0284	25
	25	0.29%	47289	210.96	0.14%	649	7.75	22.41%	7531	— 12.27%	0.12%	142	8.16 —	0.7877	50
	50	0.04%	600	3.67	0.02%	0	3.28	26.19%	798	— 9.60%	0.01%	0	2.20 —	0.7823	57
	100	0.04%	716	3.27	0.02%	0	2.52	26.19%	790	— 9.87%	0.01%	0	2.17 —	0.7823	57
10	5	42.11%	58242	— 19.05%	11.95%	23827	— 6.16%	6.89%	28682	— 2.16%	1.35%	3144	85.66 —	1.8184	15
	10	28.45%	10767	— 27.92%	7.68%	21042	— 5.52%	9.52%	11983	— 5.03%	1.31%	7801	466.37 —	1.4574	26
	25	3.52%	85360	— 2.92%	1.25%	88332	— 0.50%	15.72%	10866	— 8.08%	0.46%	3805	208.01 —	1.0963	54
	50	0.02%	45	1.44	0.00%	0	0.97	21.03%	1103	— 8.74%	0.00%	0	1.55 —	1.0311	82
	100	0.02%	33	1.23	0.00%	0	0.94	21.02%	3544	— 5.90%	0.00%	0	1.70 —	1.0311	82

A.5 Individual Results of Table 3.5

Table A.31: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 1).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	1.58%	582	32.94	1.53%	534	24.85
				—			—
	10	2.07%	7059	479.57	2.05%	7972	457.84
				—			—
	25	0.11%	98	7.48	0.10%	48	5.72
				—			—
10	50	0.01%	1	2.72	0.01%	1	2.81
				—			—
	100	0.01%	6	3.08	0.01%	6	3.09
				—			—
	5	1.46%	3205	95.65	1.45%	3170	131.99
				—			—
10	10	1.29%	12820	683.68	1.28%	16092	917.11
				—			—
	25	0.42%	1767	125.49	0.42%	1367	75.89
				—			—
	50	0.01%	0	1.77	0.01%	0	1.80
				—			—
10	100	0.00%	0	1.78	0.00%	0	1.81
				—			—

Table A.32: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 2).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.02%	3660	152.71	2.97%	4724	237.86
				—			—
	10	1.67%	1132	52.55	1.66%	1023	50.88
				—			—
	25	0.05%	12	4.08	0.05%	8	3.97
10				—			—
	50	0.00%	0	2.66	0.00%	0	2.64
				—			—
	100	0.00%	0	2.77	0.00%	0	2.77
				—			—
10	5	1.50%	4137	162.82	1.47%	5606	204.98
				—			—
	10	1.16%	5445	235.82	1.14%	6513	278.19
				—			—
	25	0.24%	699	33.66	0.24%	649	35.00
10				—			—
	50	0.00%	0	2.31	0.00%	0	2.28
				—			—
	100	0.00%	0	2.31	0.00%	0	2.31
				—			—

Table A.33: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 3).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.15%	26455	1554.00	3.07%	23456	1314.68
				—			—
	10	2.22%	8982	766.23	2.19%	7926	687.43
				—			—
	25	0.07%	34	5.47	0.07%	178	8.34
10				—			—
	50	0.02%	6	3.58	0.02%	6	3.50
				—			—
	100	0.02%	6	3.67	0.02%	6	3.64
				—			—
10	5	1.83%	17828	628.86	1.79%	18609	739.87
				—			—
	10	1.85%	25896	1553.46	1.82%	26328	1455.23
				—			—
	25	0.41%	1428	122.30	0.41%	1404	117.55
10				—			—
	50	0.00%	0	2.39	0.00%	0	2.36
				—			—
10	100	0.00%	0	2.52	0.00%	0	2.52
				—			—

Table A.34: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 4).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.04%	11368	652.36	2.96%	11093	633.63
				—			—
	10	2.13%	4539	353.99	2.10%	4454	338.1
				—			—
	25	0.10%	43	11.02	0.10%	114	7.20
10				—			—
	50	0.00%	0	2.67	0.00%	0	2.63
				—			—
	100	0.00%	0	2.59	0.00%	0	2.56
				—			—
10	5	1.50%	5596	205.71	1.47%	5826	249.23
				—			—
	10	1.55%	20688	1258.47	1.53%	18985	1448.08
				—			—
	25	0.36%	5400	306.84	0.36%	4456	225.93
10				—			—
	50	0.00%	0	2.11	0.00%	0	2.09
				—			—
10	100	0.00%	0	2.19	0.00%	0	2.14
				—			—

Table A.35: Results of problems with 200 products and 20 customer segments for the multiple resource constraints case (Instance 5).

v_0	K	CONIC+MC 1			CONIC+MC 2		
		Gap	Nodes	Time	Gap	Nodes	Time
5	5	3.26%	6349	303.38	3.15%	6095	261.81
				—			—
	10	1.83%	2699	184.09	1.81%	2998	244.22
				—			—
	25	0.09%	46	5.63	0.09%	31	5.58
				—			—
10	50	0.01%	6	3.30	0.01%	6	3.23
				—			—
	100	0.02%	6	3.23	0.02%	6	3.22
				—			—
	5	1.79%	8445	343.17	1.74%	9478	371.61
				—			—
10	10	1.35%	8887	640.52	1.33%	7217	471.29
				—			—
	25	0.25%	1290	114.19	0.25%	1075	67.03
				—			—
	50	0.01%	10	3.22	0.01%	19	3.50
				—			—
10	100	0.01%	0	2.47	0.01%	0	2.48
				—			—

A.6 Individual Results of Table 3.6

Table A.36: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 1).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	52.01%	2284	– 43.60%	13.29%	5438	– 9.54%	4.28%	10693	– 1.80%	0.37%	14	5.28 –	1.5385	10
	4	18.06%	9292	– 16.39%	5.91%	14179	– 4.42%	7.48%	2940	– 10.98%	0.78%	204	14.27 –	1.1171	20
	10	0.96%	9957	33.39	0.34%	335	6.41	20.17%	2106	– 15.53%	0.51%	59	7.84 –	0.7999	50
	20	0.02%	72	1.84 –	0.01%	0	0.94 –	25.63%	767	– 11.13%	0.01%	6	2.77 –	0.7793	60
10	40	0.02%	78	1.67 –	0.01%	0	0.92 –	25.64%	752	– 13.91%	0.01%	0	1.95 –	0.7793	60
	2	43.38%	81860	– 11.96%	14.06%	6193	– 9.77%	2.53%	6287	214.87 –	0.07%	0	5.72 –	2.0122	10
	4	39.12%	2200	– 43.75%	8.75%	6672	– 7.48%	4.56%	6124	– 3.65%	0.21%	0	9.80 –	1.5849	20
	10	5.17%	51581	– 4.41%	1.57%	35760	– 0.81%	12.84%	2936	– 8.45%	0.39%	215	16.70 –	1.1305	50
40	20	0.05%	517	2.48 –	0.02%	0	1.59 –	20.83%	766	– 14.94%	0.01%	0	2.53 –	1.0355	86
	40	0.02%	81	1.36 –	0.01%	0	0.94 –	20.81%	760	– 7.62%	0.00%	0	1.94 –	1.0348	86

Table A.37: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	49.64%	2432	– 38.73%	12.02%	5365	– 8.64%	5.04%	9681	– 5.57%	0.32%	59	5.00 –	1.5679	10
	4	16.21%	10459	– 14.57%	5.18%	14329	– 3.61%	9.53%	2944	– 33.47%	0.45%	60	7.03 –	1.1555	20
	10	0.38%	17075	36.53	0.12%	0	6.58	23.11%	933	– 17.23%	0.02%	0	4.86 –	0.8648	50
	20	0.04%	1437	5.58	0.02%	0	4.63	27.85%	751	– 15.16%	0.01%	0	2.27 –	0.8575	60
10	40	0.04%	557	2.13	0.02%	42	1.91	27.85%	743	– 13.74%	0.01%	8	2.98 –	0.8575	60
	2	45.19%	28434	– 18.89%	13.42%	6507	– 8.96%	2.95%	13217	432.3 –	0.10%	0	5.27 –	2.0386	10
	4	37.19%	2799	– 37.96%	7.95%	7297	– 6.51%	5.38%	5636	– 5.17%	0.18%	0	6.88 –	1.6118	20
	10	4.65%	27876	– 4.23%	1.38%	34178	– 0.80%	14.72%	1010	– 14.22%	0.07%	0	8.66 –	1.1802	50
40	20	0.03%	200	1.77	0.01%	0	1.03	21.67%	750	– 13.86%	0.00%	0	2.00 –	1.0963	87
	40	0.03%	180	1.48	0.01%	0	1.05	21.68%	768	– 9.18%	0.00%	0	1.81 –	1.0963	87

Table A.38: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	45.07%	2440	– 36.36%	9.66%	6745	– 5.97%	3.39%	9174	306.13	0.28%	0	5.72	1.5193	10
	4	13.33%	11683	– 10.94%	3.64%	14570	– 1.52%	7.38%	3171	– 10.45%	0.21%	0	7.47	1.1072	20
	10	0.34%	2175	9.92	0.11%	0	4.84	24.62%	774	–	0.06%	0	7.19	0.8636	50
	20	0.02%	111	1.84	0.01%	0	0.95	28.91%	770	–	0.00%	0	2.25	0.8568	55
	40	0.02%	190	1.69	0.01%	0	0.91	28.90%	743	–	0.00%	0	2.03	0.8568	55
	2	43.85%	76573	– 9.74%	10.52%	9632	– 4.41%	1.93%	13628	256.08	0.15%	0	4.48	1.9963	10
10	4	32.81%	2788	– 31.80%	5.83%	9811	– 3.71%	4.13%	6437	– 3.41%	0.04%	0	4.72	1.5683	20
	10	4.04%	33003	– 3.29%	1.08%	36057	– 0.38%	15.60%	2754	– 8.62%	0.08%	0	8.77	1.1754	50
	20	0.03%	87	1.45	0.01%	0	1	22.55%	760	–	0.00%	0	2.02	1.1026	88
	40	0.03%	103	1.48	0.01%	0	0.95	22.55%	752	–	0.00%	0	1.81	1.1026	88
	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	4	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Table A.39: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	52.44%	2495	— 50.23%	13.92%	4703	— 10.59%	6.70%	9031	— 7.77%	0.73%	86	10.56 —	1.6036	10
	4	18.19%	10483	— 16.14%	6.06%	11506	— 4.49%	11.55%	2617	— 32.93%	0.68%	121	10.39 —	1.1950	20
	10	1.12%	38775	299.78 —	0.39%	730	13.16 —	24.17%	1158	— 29.14%	0.25%	53	5.61 —	0.8924	48
	20	0.05%	1865	8.92 —	0.02%	4	4.14 —	28.17%	768	— 13.77%	0.01%	6	2.77 —	0.8699	62
10	40	0.05%	991	3.98 —	0.02%	44	2.41 —	28.17%	746	— 14.17%	0.01%	6	2.75 —	0.8699	62
	2	44.70%	43177	— 16.51%	15.31%	6812	— 9.73%	3.92%	16554	— 1.26%	0.24%	60	7.70 —	2.0629	10
	4	40.73%	2232	— 43.06%	9.84%	6500	— 8.47%	6.96%	4582	— 7.00%	0.34%	94	16.25 —	1.6512	20
	10	6.18%	28809	— 5.45%	1.87%	37290	— 1.14%	15.48%	2715	— 19.79%	0.29%	214	15.41 —	1.2078	50
40	20	0.46%	9658	24.09 —	0.17%	718	10.17 —	21.54%	755	— 13.87%	0.16%	251	22.55 —	1.1065	85
	40	0.02%	57	1.31 —	0.00%	0	0.94 —	20.96%	755	— 7.61%	0.00%	0	1.81 —	1.0980	89

Table A.40: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	51.58%	3217	– 39.08%	11.99%	5858	– 8.87%	3.69%	12691	468.02	0.30%	13	6.59	1.5008	10
	4	19.17%	7836	– 17.73%	6.15%	12218	– 4.63%	7.53%	3938	– 5.80%	0.69%	213	16.95	1.0984	20
	10	1.11%	94527	361.58	0.38%	641	10.28	17.76%	1896	– 18.25%	0.04%	0	7.08	0.7656	50
	20	0.03%	629	3.3	0.01%	2	2.78	24.53%	766	–	0.01%	0	2.20	0.7420	66
	40	0.03%	461	2.67	0.02%	0	1.91	24.53%	754	– 10.14%	0.01%	0	2.33	0.7420	66
10	2	42.99%	98672	– 10.22%	12.07%	5752	– 7.58%	2.06%	11028	346.95	0.07%	0	4.64	1.9726	10
	4	41.98%	2597	– 45.93%	9.36%	7424	– 7.97%	4.69%	5599	– 5.26%	0.39%	214	13.88	1.5608	20
	10	6.63%	29870	– 5.94%	1.97%	25799	– 1.30%	11.15%	2786	– 7.05%	0.03%	0	7.64	1.0959	50
	20	0.10%	822	3.61	0.03%	0	1.78	18.61%	761	– 12.81%	0.03%	0	6.13	0.9741	94
	40	0.04%	265	1.88	0.01%	0	1.17	18.60%	753	– 6.51%	0.01%	0	1.88	0.9729	97

Table A.41: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 6).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	51.60%	1759	– 46.46%	13.31%	5241	– 10.03%	4.96%	8507	– 5.00%	0.25%	26	5.52 –	1.5684	10
	4	17.17%	11457	– 15.34%	5.45%	13181	– 4.17%	8.72%	2758	– 12.10%	0.60%	49	11.81 –	1.1462	20
	10	0.81%	9349	32.78	0.27%	161	4.17	21.84%	958	– 11.58%	0.19%	0	12.64 –	0.8459	48
	20	0.02%	312	2.84	0.01%	0	1	26.75%	764	– 14.14%	0.00%	0	2.05 –	0.8290	59
10	40	0.02%	110	1.58	0.01%	0	1.45	26.75%	749	– 13.24%	0.01%	0	1.89 –	0.8290	59
	2	45.94%	37936	– 19.17%	14.62%	6724	– 9.78%	2.73%	15382	– 0.90%	0.07%	0	5.11 –	2.0365	10
	4	38.41%	2901	– 37.14%	8.75%	7149	– 7.43%	5.05%	4951	– 9.49%	0.24%	64	10.14 –	1.6069	20
	10	5.09%	37673	– 4.11%	1.50%	37852	– 0.70%	13.77%	2501	– 16.82%	0.29%	306	18.41 –	1.1656	50
40	20	0.01%	2	0.95	0.00%	0	1.03	20.76%	758	– 12.41%	0.00%	0	1.88 –	1.0726	82
	40	0.01%	0	0.47	0.00%	0	0.92	20.78%	765	– 7.70%	0.00%	0	1.84 –	1.0725	83

Table A.42: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 7).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	48.93%	1814	— 48.99%	11.91%	6877	— 7.97%	4.18%	12341	— 3.97%	0.21%	0	6.08 —	1.5206	10
	4	16.37%	14876	— 14.29%	5.06%	15802	— 3.29%	7.12%	3503	— 4.37%	0.34%	78	8.95 —	1.1003	20
	10	1.13%	8253	28.41	0.38%	232	6.2	20.97%	1389	— 13.09%	0.43%	48	8.81 —	0.8142	45
	20	0.02%	165	1.98	0.00%	0	0.97	25.39%	779	—	0.00%	0	2.06 —	0.7899	56
10	40	0.02%	126	1.66	0.01%	0	0.91	25.39%	758	— 11.59%	0.00%	0	2.17 —	0.7899	56
	2	44.28%	84798	— 11.51%	12.75%	7475	— 8.52%	2.28%	10934	332.67 —	0.03%	0	4.53 —	1.9953	10
	4	36.36%	3571	— 33.13%	7.94%	9429	— 6.03%	4.28%	5637	— 2.08%	0.26%	122	9.09 —	1.5670	20
	10	4.72%	43182	— 3.68%	1.26%	38687	— 0.39%	13.34%	3022	— 10.91%	0.40%	126	11.19 —	1.1339	50
40	20	0.02%	24	1.25	0.00%	0	1.03	20.61%	758	— 11.72%	0.00%	0	1.84 —	1.0462	82
	40	0.02%	69	1.36	0.01%	0	0.91	20.66%	779	— 6.50%	0.00%	0	2.02 —	1.0462	82

Table A.43: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 8).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	54.22%	2329	– 47.74%	13.99%	5436	– 10.51%	8.03%	6649	– 12.88%	1.03%	244	16.69	1.5911	10
	4	19.01%	10542	– 17.71%	6.50%	13889	– 4.99%	12.78%	3019	– 19.25%	1.14%	833	43.63	1.1848	20
	10	1.02%	47021	186.32	0.36%	587	7.97	24.16%	1146	– 18.95%	0.42%	78	7.78	0.8688	50
	20	0.08%	1453	6.23	0.04%	69	2.23	28.31%	749	– 42.11%	0.02%	6	3.16	0.8491	59
10	40	0.08%	1338	4.89	0.04%	97	2.52	28.31%	744	– 14.77%	0.03%	24	3.53	0.8491	59
	2	45.60%	65882	– 15.86%	15.64%	6329	– 11.79%	4.57%	12199	– 4.12%	0.33%	45	6.58	2.0397	10
	4	41.25%	3131	– 40.04%	9.92%	7217	– 8.32%	7.62%	4321	– 7.28%	0.52%	286	18.31	1.6300	20
	10	5.62%	40019	– 4.92%	1.63%	39494	– 0.87%	15.44%	2229	– 9.96%	0.31%	270	14.06	1.1807	50
40	20	0.09%	718	2.86	0.02%	0	1.75	21.82%	753	– 17.22%	0.01%	0	4.44	1.0824	87
	40	0.04%	235	1.64	0.01%	0	1.88	21.79%	755	– 9.56%	0.01%	0	2.20	1.0816	88

Table A.44: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 9).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	54.54%	1906	– 50.23%	14.20%	5493	– 11.27%	4.08%	9047	– 3.91%	0.20%	46	6.3 –	1.5519	10
	4	19.21%	9624	– 17.05%	6.41%	14653	– 4.75%	6.67%	3658	– 6.09%	0.62%	117	9.69 –	1.1183	20
	10	1.11%	14142	42.27	0.41%	317	6.31	18.76%	1085	–	0.63%	109	8.5	0.7842	50
	20	0.05%	119	1.8	0.02%	2	3.75	24.77%	753	–	0.01%	0	2.02	0.7609	59
10	40	0.05%	133	1.5	0.02%	0	2.22	24.77%	769	–	0.01%	0	2.06	0.7609	59
	2	43.20%	52946	– 15.73%	15.48%	5942	– 10.78%	2.49%	10731	337.27 –	0.06%	0	5.11	2.0256	10
	4	40.86%	2476	– 41.24%	9.92%	8254	– 8.30%	4.34%	8047	– 2.16%	0.28%	90	8.97 –	1.5933	20
	10	5.05%	34100	– 4.10%	1.65%	36438	– 0.88%	11.82%	3201	– 8.49%	0.43%	315	18.83 –	1.1190	50
40	20	0.03%	32	1.16	0.01%	0	0.94	20.50%	754	–	0.01%	0	1.66	1.0252	79
	40	0.02%	99	1.3	0.01%	0	0.91	20.55%	801	– 8.53%	0.00%	0	1.61 –	1.0252	79

Table A.45: Results of problems with 200 products and 20 customer segments for the multiple cardinality constraints case (Instance 10).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	2	48.78%	2393	– 39.50%	11.54%	5666	– 7.82%	4.46%	10218	– 4.68%	0.13%	0	4.63	1.5139	10
	4	17.49%	9969	– 15.81%	5.72%	14580	– 4.00%	9.58%	3066	– 8.63%	0.91%	289	15.94	1.1287	20
	10	0.75%	11631	33.17	0.26%	171	4.06	21.89%	953	– 16.74%	0.25%	49	5.02	0.8272	49
	20	0.04%	470	3.45	0.01%	0	2.16	27.05%	756	– 12.32%	0.00%	0	1.97	0.8122	60
10	40	0.04%	145	1.86	0.02%	0	2.09	27.04%	746	– 12.72%	0.01%	0	2.09	0.8122	60
	2	41.76%	90216	– 9.59%	11.83%	9242	– 6.00%	2.52%	13086	483.43	0.02%	0	4.86	1.9778	10
	4	38.27%	2400	– 37.10%	8.43%	8387	– 6.76%	5.53%	4526	– 6.56%	0.27%	111	10.45	1.5775	20
	10	4.87%	45536	– 4.17%	1.35%	31753	– 0.66%	13.81%	3097	– 11.54%	0.18%	0	13.2	1.1416	50
40	20	0.09%	1679	4.28	0.03%	0	1.66	21.55%	743	– 14.80%	0.03%	0	4.53	1.0556	89
	40	0.04%	219	1.67	0.01%	0	1.56	21.47%	748	– 8.93%	0.00%	0	2.06	1.0546	90

A.7 Individual Results of Table 3.7



Table A.46: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 1).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	24.31%	2020	– 23.89%	8.67%	2208	– 8.07%	7.22%	862	– 34.05%	1.05%	911	251.75	1.1126	20
	10	2.80%	56059	– 1.96%	1.10%	15352	– 0.48%	17.03%	952	– 30.90%	1.06%	358	88.69	0.7191	50
	20	0.01%	29	3.77	0.00%	0	2.56	27.91%	855	– 38.07%	0.00%	0	5.03	0.6555	71
	30	0.01%	29	3.59	0.00%	0	2.77	28.31%	750	– 30.06%	0.00%	0	4.92	0.6555	71
	60	0.01%	31	3.48	0.00%	0	2.53	28.31%	736	– 17.93%	0.00%	0	4.81	0.6555	71
10	4	49.46%	850	– 66.18%	12.95%	974	– 13.77%	4.20%	1012	– 17.04%	0.32%	73	32.30	1.5843	20
	10	8.91%	6603	– 8.79%	2.89%	5830	– 2.34%	10.31%	850	– 21.66%	0.54%	414	160.26	1.0582	50
	20	0.32%	138809	– 0.08%	0.11%	599	28.27	21.69%	848	– 32.77%	0.04%	41	15.44	0.8923	94
	30	0.01%	8	3.44	0.00%	0	2.67	23.81%	822	– 18.58%	0.00%	0	4.94	0.8859	109
	60	0.01%	56	3.64	0.01%	0	2.61	23.77%	913	– 16.56%	0.00%	0	4.91	0.8859	109

Table A.47: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	17.07%	2150	– 16.55%	5.46%	3758	– 4.48%	6.31%	835	– 27.06%	0.31%	20	30.75	1.0795	20
	10	0.91%	60361	403.29	0.32%	671	33.2	22.62%	1005	– 34.00%	0.42%	134	28.53	0.7726	50
	20	0.05%	982	7.72	0.03%	122	6.36	32.98%	830	– 43.68%	0.01%	16	8.73	0.7536	64
60	30	0.05%	1091	8.22	0.03%	162	6.91	32.96%	740	– 31.98%	0.01%	24	9.52	0.7536	64
	60	0.05%	1185	7.81	0.03%	210	7.56	32.98%	761	– 21.29%	0.02%	27	12.38	0.7536	64
10	4	40.08%	797	– 52.04%	9.06%	1012	– 9.16%	3.93%	1188	– 0.123935	0.29%	62	24.88	1.5552	20
	10	6.30%	7028	– 6.01%	2.01%	3451	– 1.63%	13.84%	915	– 24.08%	0.25%	414	96.07	1.0975	50
	20	0.11%	3671	18.61	0.04%	0	6.72	26.03%	880	– 37.95%	0.01%	0	13.49	0.9801	95
60	30	0.02%	33	3.61	0.00%	0	2.8	27.01%	764	– 27.31%	0.00%	0	5.06	0.9782	101
	60	0.01%	56	3.69	0.01%	0	2.63	26.99%	845	– 21.09%	0.00%	0	5.17	0.9782	101

Table A.48: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	19.78%	1416	– 19.72%	6.53%	1835	– 5.84%	9.94%	831	– 20.02%	0.69%	178	48.46 –	1.1361	20
	10	1.63%	32539	– 1.20%	0.62%	14647	– 0.26%	24.51%	976	– 44.40%	0.34%	47	44.66 –	0.8092	50
	20	0.04%	540	6.7 –	0.01%	5	8.7 –	33.49%	921	– 40.59%	0.00%	0	6.42 –	0.7750	77
	30	0.04%	621	6.27 –	0.02%	98	6.47 –	33.51%	732	– 30.39%	0.01%	0	6.38 –	0.7750	77
10	60	0.04%	753	6.75 –	0.02%	148	6.11 –	33.51%	744	– 30.34%	0.00%	0	7.64 –	0.7750	77
	4	45.46%	827	– 77.74%	10.87%	755	– 12.83%	6.02%	986	– 19.79%	0.34%	140	39.33 –	1.5966	20
	10	8.55%	4876	– 8.39%	2.85%	4620	– 2.53%	15.31%	845	– 30.05%	0.28%	641	140.69 –	1.1289	50
	20	0.57%	78670	– 0.32%	0.20%	12023	385.69 –	25.45%	813	– 33.02%	0.08%	459	105.33 –	0.9831	97
60	30	0.01%	55	3.41 –	0.01%	0	2.7 –	25.98%	754	– 28.01%	0.00%	0	4.67 –	0.9721	111
	60	0.01%	50	3.09 –	0.00%	0	2.56 –	25.98%	733	– 16.82%	0.00%	0	5.42 –	0.9721	111

Table A.49: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	18.74%	2029	– 18.12%	6.05%	2285	– 5.22%	7.50%	899	– 27.44%	0.51%	121	29.69 –	1.1144	20
	10	1.46%	65564	– 0.56%	0.48%	1047	90.85	22.83%	932	– 50.03%	0.22%	0	33.47 –	0.7940	50
	20	0.02%	55	4.17	0.01%	0	4.41	32.51%	918	– 41.00%	0.00%	0	5.69 –	0.7623	69
	30	0.02%	53	3.98	0.01%	3	8.13	32.60%	933	– 33.57%	0.01%	0	5.78 –	0.7623	69
	60	0.02%	53	3.89	0.01%	25	4.56	32.60%	741	– 23.50%	0.00%	0	5.86 –	0.7623	69
	4	42.46%	784	– 65.63%	9.71%	939	– 10.19%	4.44%	1290	– 20.97%	0.23%	39	26.17 –	1.5854	20
10	10	7.29%	6099	– 6.98%	2.27%	6699	– 1.78%	14.30%	960	– 23.09%	0.10%	0	26.20 –	1.1226	50
	20	0.27%	8943	51.74	0.10%	111	8.52	25.49%	796	– 35.08%	0.12%	78	28.00 –	0.9914	97
	30	0.03%	359	5.09	0.01%	0	2.81	26.57%	742	– 23.74%	0.00%	0	5.17 –	0.9866	103
	60	0.03%	435	5.81	0.01%	0	2.64	26.57%	733	– 21.17%	0.00%	0	6.19 –	0.9866	103

Table A.50: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	18.07%	1809	– 17.57%	6.09%	2683	– 5.24%	9.63%	851	– 26.64%	0.46%	65	37.39 –	1.1486	20
	10	1.00%	62566	– 0.39%	0.39%	5968	264.34 –	25.64%	896	– 50.77%	0.28%	140	24.85 –	0.8362	50
	20	0.04%	361	6.03 –	0.02%	66	6.88 –	34.68%	859	– 39.13%	0.01%	6	7.73 –	0.8154	63
	30	0.04%	363	5.13 –	0.02%	62	5.95 –	34.67%	735	– 32.16%	0.01%	6	9.14 –	0.8154	63
	60	0.04%	450	5.95 –	0.03%	115	6.22 –	34.67%	759	– 23.48%	0.01%	6	8.91 –	0.8154	63
	4	42.37%	740	– 66.26%	9.93%	1041	– 9.52%	5.76%	909	– 22.11%	0.23%	67	23.00 –	1.6126	20
10	10	7.02%	5079	– 6.84%	2.31%	4265	– 2.00%	16.12%	1004	– 25.36%	0.13%	0	43.19 –	1.1547	50
	20	0.18%	22467	117.37 –	0.06%	79	7.83 –	27.00%	892	– 37.12%	0.00%	0	12.91 –	1.0291	99
	30	0.01%	0	1.55 –	0.00%	0	2.77 –	27.66%	732	– 26.87%	0.00%	0	5.09 –	1.0256	110
	60	0.01%	49	2.84 –	0.00%	0	2.56 –	27.66%	806	– 31.52%	0.00%	0	5.33 –	1.0256	110

Table A.51: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 6).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	21.61%	1757	– 21.35%	7.54%	2371	– 6.96%	8.83%	811	– 20.50%	0.36%	34	45.00 –	1.1404	20
	10	1.72%	51851	– 1.13%	0.67%	21620	– 0.13%	20.89%	943	– 46.44%	0.22%	0	25.30 –	0.7774	50
	20	0.05%	1298	9.16	0.02%	99	6.03	31.09%	778	– 40.89%	0.01%	18	8.50 –	0.7403	68
	30	0.05%	1135	8.75	0.03%	175	7.23	31.11%	879	– 30.95%	0.01%	23	9.02 –	0.7403	68
10	60	0.05%	1229	7.98	0.03%	229	7.75	31.11%	902	– 25.16%	0.01%	28	11.34 –	0.7403	68
	4	46.86%	813	– 68.38%	11.44%	927	– 11.91%	5.13%	1035	– 19.69%	0.14%	33	23.33 –	1.6052	20
	10	8.28%	4666	– 8.14%	2.62%	6395	– 2.19%	13.04%	955	– 23.00%	0.17%	76	26.67 –	1.1092	50
	20	0.29%	108425	– 0.02%	0.09%	419	22.58	23.93%	976	– 34.39%	0.05%	91	20.03 –	0.9587	100
10	30	0.03%	234	4.97	0.01%	0	2.81	25.34%	839	– 25.16%	0.00%	0	5.14 –	0.9533	114
	60	0.03%	448	4.73	0.01%	30	4.45	25.33%	1002	– 26.93%	0.00%	0	5.53 –	0.9533	114

Table A.52: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 7).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	18.25%	1458	– 17.76%	5.71%	1978	– 4.79%	9.83%	880	– 27.74%	0.23%	0	28.83	1.1315	20
	10	1.79%	42106	– 1.20%	0.67%	12601	– 0.28%	26.54%	981	– 48.50%	0.35%	107	30.20	0.8355	50
	20	0.03%	174	4.91	0.01%	0	5.33	34.24%	916	– 43.39%	0.00%	0	6.16	0.7980	76
	30	0.03%	171	4.47	0.01%	49	5.84	34.24%	788	– 41.09%	0.00%	0	6.98	0.7980	76
10	60	0.03%	206	4.33	0.02%	46	4.81	34.24%	924	– 37.18%	0.00%	0	6.47	0.7980	76
	4	43.94%	816	– 113.78%	9.93%	793	– 10.67%	6.14%	947	– 16.56%	0.16%	40	20.41	1.5951	20
	10	8.59%	4688	– 8.59%	2.88%	4942	– 2.50%	16.74%	950	– 27.83%	0.28%	2031	467.91	1.1502	50
	20	0.39%	87517	– 0.18%	0.13%	882	36.2	25.92%	847	– 33.82%	0.08%	132	23.63	1.0022	97
60	30	0.03%	1601	10.06	0.01%	0	2.95	26.48%	811	– 46.89%	0.00%	0	6.30	0.9952	113
	60	0.03%	1582	10.06	0.01%	34	5.06	26.45%	913	– 19.99%	0.00%	0	6.42	0.9952	113

Table A.53: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 8).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	22.06%	2089	– 21.73%	7.70%	2420	– 7.12%	6.02%	891	– 29.63%	0.68%	754	139.30 –	1.0796	20
	10	2.13%	72841	– 1.56%	0.82%	23968	– 0.37%	16.38%	881	– 48.56%	0.41%	585	97.51 –	0.7056	50
	20	0.02%	33	3.8 –	0.01%	0	2.63 –	27.50%	878	– 43.16%	0.00%	0	5.42 –	0.6572	71
	30	0.02%	33	3.86 –	0.01%	0	2.72 –	27.54%	736	– 59.55%	0.00%	0	5.41 –	0.6572	71
10	60	0.02%	17	3.56 –	0.01%	2	7.09 –	27.53%	736	– 18.35%	0.00%	0	5.34 –	0.6572	71
	4	46.11%	751	– 57.28%	11.75%	986	– 11.60%	3.58%	1620	– 15.40%	0.35%	155	36.13 –	1.5542	20
	10	7.79%	6027	– 7.77%	2.49%	5341	– 2.11%	9.70%	989	– 20.90%	0.27%	417	105.55 –	1.0414	50
	20	0.10%	1096	15.7 –	0.03%	0	5.36 –	21.69%	878	– 34.45%	0.03%	0	20.44 –	0.8913	98
60	30	0.03%	571	6.38 –	0.01%	0	4.75 –	23.90%	877	– 50.47%	0.00%	0	5.95 –	0.8898	101
	60	0.03%	432	6.02 –	0.01%	77	4.91 –	23.91%	954	– 15.05%	0.00%	0	5.78 –	0.8898	101

Table A.54: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 9).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	21.90%	1935	— 21.58%	7.56%	2460	— 6.91%	6.95%	857	— 31.77%	0.73%	169	43.85 —	1.0911	20
	10	2.33%	47141	— 1.43%	0.93%	10327	— 0.43%	18.48%	898	— 33.68%	0.84%	1376	286.75 —	0.7323	50
	20	0.04%	194	5.16 —	0.02%	0	9.36 —	28.56%	893	— 42.60%	0.01%	0	6.61 —	0.6810	68
10	30	0.02%	45	4.02 —	0.01%	0	2.84 —	28.75%	732	— 28.63%	0.00%	0	5.55 —	0.6806	69
	60	0.02%	42	3.41 —	0.01%	0	6.67 —	28.76%	780	— 19.12%	0.00%	0	5.36 —	0.6806	69
	4	45.59%	798	— 70.63%	11.17%	1256	— 11.05%	4.05%	1327	— 10.86%	0.23%	10	22.75 —	1.5557	20
10	10	8.19%	7263	— 7.87%	2.66%	7304	— 2.21%	11.16%	942	— 18.39%	0.52%	1914	513.2 —	1.0615	50
	20	0.55%	14720	167.52 —	0.18%	472	35.11 —	22.92%	920	— 34.27%	0.21%	70	35.83 —	0.9166	95
	30	0.02%	143	3.61 —	0.01%	0	2.75 —	24.32%	836	— 26.83%	0.00%	0	4.92 —	0.9058	104
	60	0.02%	89	2.98 —	0.01%	6	4.33 —	24.28%	836	— 18.93%	0.00%	0	5.28 —	0.9058	104

Table A.55: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 10).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	4	18.54%	2167	— 18.07%	6.09%	2704	— 5.29%	7.12%	837	— 33.06%	0.44%	195	46.08 —	1.1029	20
	10	1.13%	103225	— 0.01%	0.42%	1154	59.53 —	21.56%	921	— 45.39%	0.31%	86	25.63 —	0.7763	50
	20	0.06%	1104	8.38 —	0.02%	68	6.53 —	32.56%	851	— 36.15%	0.01%	6	7.83 —	0.7525	65
	30	0.06%	1027	7.78 —	0.03%	119	6.47 —	32.56%	740	— 30.27%	0.02%	29	10.02 —	0.7525	65
	60	0.06%	893	7.05 —	0.03%	179	6.52 —	32.57%	974	— 25.21%	0.02%	26	10.84 —	0.7525	65
	4	42.95%	839	— 56.74%	9.68%	926	— 9.38%	4.57%	1015	— 18.94%	0.34%	210	34.53 —	1.5808	20
10	10	7.43%	5597	— 7.16%	2.44%	5464	— 2.02%	13.42%	924	— 29.83%	0.19%	53	23.36 —	1.1105	50
	20	0.37%	147298	— 0.03%	0.12%	462	17.84 —	25.29%	859	— 32.86%	0.08%	34	15.75 —	0.9775	98
	30	0.03%	213	4.97 —	0.01%	0	2.78 —	26.20%	810	— 22.92%	0.00%	0	5.17 —	0.9707	111
	60	0.03%	235	5.09 —	0.01%	0	3.17 —	26.18%	807	— 27.50%	0.00%	0	6.11 —	0.9707	111

A.8 Individual Results of Table 3.8



Table A.56: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 1).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	34.73%	1128	— 34.47%	11.35%	1520	— 10.29%	5.57%	987	— 30.55%	0.56%	163	45.50 —	1.2786	15
	6	10.00%	6356	— 9.54%	3.84%	6176	— 3.15%	9.64%	890	— 21.20%	0.46%	276	77.61 —	0.8925	30
	12	0.82%	52713	374.66 —	0.34%	1950	47.07 —	21.99%	925	— 30.35%	0.40%	128	29.36 —	0.7010	56
	30	0.01%	31	3.72 —	0.00%	0	2.64 —	28.81%	738	— 31.03%	0.00%	0	3.56 —	0.6822	66
10	60	0.01%	0	1.45 —	0.00%	0	2.39 —	28.81%	883	— 17.68%	0.00%	0	4.47 —	0.6822	66
	3	56.34%	1318	— 56.86%	15.23%	759	— 15.18%	3.51%	1345	— 10.80%	0.27%	159	31.66 —	1.7678	15
	6	24.63%	1610	— 24.56%	7.17%	1958	— 6.91%	6.04%	906	— 27.08%	0.39%	626	98.63 —	1.3246	30
	12	4.65%	11897	— 4.39%	1.52%	8791	— 1.13%	14.02%	919	— 28.58%	0.43%	798	178.28 —	1.0055	60
105	30	0.02%	104	4.33 —	0.01%	0	2.67 —	24.64%	731	— 45.35%	0.00%	0	6.02 —	0.9134	105
	60	0.02%	108	3.33 —	0.01%	0	2.64 —	24.64%	728	— 17.23%	0.00%	0	6.42 —	0.9134	105

Table A.57: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	30.95%	1042	– 30.23%	8.84%	1803	– 8.08%	3.74%	5265	– 22.14%	0.15%	0	18.83	1.2358	15
	6	8.66%	6529	– 8.19%	3.11%	7496	– 2.35%	9.26%	872	– 20.08%	0.37%	171	50.80	0.8759	30
	12	0.59%	24191	201.81	0.23%	599	30.72	23.04%	899	– 46.87%	0.26%	197	36.33	0.7063	56
	30	0.04%	256	5.3	0.02%	38	5.33	30.36%	918	– 30.96%	0.00%	0	6.83	0.6938	67
	60	0.04%	200	4.31	0.02%	56	4.92	30.32%	805	– 22.13%	0.00%	0	6.34	0.6938	67
	3	51.36%	1735	– 40.30%	11.65%	1149	– 11.05%	2.18%	3208	– 6.63%	0.03%	0	16.11	1.7253	15
10	6	22.67%	1348	– 23.20%	6.07%	2109	– 5.60%	5.63%	898	– 20.69%	0.18%	16	28.36	1.3027	30
	12	4.34%	11381	– 4.02%	1.35%	5858	– 1.04%	14.67%	977	– 25.95%	0.25%	68	34.08	1.0062	60
	30	0.05%	11378	33.35	0.02%	124	5.3	25.58%	737	– 25.95%	0.01%	6	7.92	0.9212	114
	60	0.05%	11226	31.2	0.02%	246	8.2	25.57%	881	– 21.20%	0.00%	0	6.30	0.9212	114
				–			–								

Table A.58: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	35.26%	1087	– 35.24%	11.25%	1162	– 10.76%	5.66%	1205	– 27.48%	0.43%	53	29.59	1.2889	15
	6	10.44%	5581	– 9.90%	3.82%	5169	– 3.20%	10.17%	862	– 22.76%	0.65%	277	58.75	0.9045	30
	12	0.95%	92632	487.88 –	0.37%	2101	71.57 –	21.74%	921	– 49.19%	0.37%	96	29.75	0.7075	60
	30	0.01%	40	3.27 –	0.01%	0	2.78 –	28.95%	777	– 33.87%	0.00%	0	5.08	0.6860	74
10	60	0.01%	33	3.38 –	0.01%	0	2.5 –	28.95%	736	– 27.22%	0.00%	0	5.14	0.6860	74
	3	57.13%	1407	– 63.13%	15.45%	746	– 15.09%	3.45%	1641	– 13.25%	0.29%	181	31.64	1.7759	15
	6	25.58%	1510	– 26.45%	7.37%	1679	– 7.28%	6.17%	870	– 18.26%	0.45%	600	99.83	1.3325	30
	12	5.17%	13362	– 5.01%	1.71%	8868	– 1.29%	13.69%	956	– 23.02%	0.30%	206	51.55	1.0089	60
30	30	0.01%	0	1.5 –	0.00%	0	2.69 –	24.01%	854	– 23.74%	0.00%	0	4.28	0.9063	112
	60	0.01%	58	4.16 –	0.00%	0	2.63 –	24.01%	904	– 14.34%	0.00%	0	3.86	0.9063	112

Table A.59: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	33.91%	1216	— 33.36%	11.07%	1415	— 10.48%	6.11%	1036	— 27.91%	0.53%	239	49.29 —	1.2886	15
	6	9.28%	6229	— 8.85%	3.48%	5713	— 2.70%	10.18%	818	— 27.15%	0.16%	41	23.16 —	0.9043	30
	12	0.66%	15325	77.49 —	0.27%	489	17.67 —	23.32%	863	— 43.66%	0.53%	132	33.37 —	0.7255	58
	30	0.03%	153	4.67 —	0.02%	10	4.66 —	30.25%	805	— 26.13%	0.00%	0	6.34 —	0.7111	67
10	60	0.03%	153	4.08 —	0.02%	25	4.59 —	30.29%	736	— 24.78%	0.01%	0	6.36 —	0.7111	67
	3	55.33%	1490	— 45.61%	14.86%	805	— 14.06%	3.75%	1103	— 18.80%	0.23%	122	26.50 —	1.7755	15
	6	23.87%	1474	— 24.10%	6.80%	2110	— 6.51%	6.61%	815	— 20.66%	0.33%	126	38.96 —	1.3377	30
	12	4.39%	14668	— 4.12%	1.46%	9395	— 1.05%	14.96%	935	— 23.88%	0.43%	660	102.41 —	1.0287	60
60	30	0.02%	101	3.25 —	0.01%	0	2.73 —	25.70%	785	— 27.89%	0.00%	0	4.78 —	0.9428	106
	60	0.02%	106	3.19 —	0.01%	0	2.64 —	25.71%	739	— 18.08%	0.00%	0	5.00 —	0.9428	106

Table A.60: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	33.43%	1091	– 34.10%	10.53%	1134	– 9.98%	5.54%	954	– 26.61%	0.54%	84	30.83	1.2665	15
	6	9.73%	5994	– 9.31%	3.60%	4657	– 3.00%	10.41%	886	– 22.65%	0.71%	392	95.08	0.8945	30
	12	0.44%	12359	66.85	0.16%	101	11.39	21.88%	842	– 39.02%	0.16%	41	14.89	0.7008	59
30	30	0.02%	31	3.58	0.01%	0	2.84	29.85%	787	– 31.25%	0.00%	0	5.47	0.6911	67
	60	0.02%	56	3.67	0.01%	0	2.5	29.85%	764	– 24.23%	0.00%	0	5.08	0.6911	67
10	3	53.38%	1151	– 42.35%	13.82%	736	– 19.41%	3.18%	2086	– 16.64%	0.11%	0	20.42	1.7495	15
	6	24.24%	1422	– 24.87%	6.76%	1465	– 6.54%	6.41%	871	– 25.14%	0.37%	205	71.60	1.3207	30
	12	4.54%	8430	– 4.55%	1.49%	8784	– 1.15%	14.16%	866	– 25.40%	0.17%	129	32.00	1.0065	60
30	30	0.02%	210	3.98	0.01%	0	2.8	25.03%	769	– 24.92%	0.00%	0	4.92	0.9169	111
	60	0.02%	246	3.95	0.01%	31	4.47	25.03%	937	– 16.79%	0.00%	0	5.56	0.9169	111

Table A.61: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 6).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	37.11%	1145	— 36.88%	12.14%	1184	— 11.31%	8.17%	907	— 18.38%	1.23%	2627	506.29 —	1.3279	15
	6	11.43%	4021	— 11.16%	4.34%	3298	— 3.74%	13.08%	829	— 30.61%	1.14%	2118	— 0.13%	0.9429	30
	12	1.20%	120901	— 0.29%	0.51%	6660	289.08	23.50%	901	— 44.95%	0.54%	171	48.66 —	0.7351	55
	30	0.01%	3	3.03 —	0.00%	0	2.72 —	29.99%	734	— 30.08%	0.00%	0	5.56 —	0.7082	72
	60	0.01%	38	3.66 —	0.01%	0	2.53 —	29.99%	729	— 25.06%	0.00%	0	5.09 —	0.7082	72
10	3	58.87%	1229	— 49.21%	16.44%	747	— 21.80%	4.77%	1008	— 19.80%	0.46%	412	48.75 —	1.8043	15
	6	26.75%	1305	— 27.70%	7.85%	1233	— 7.71%	7.89%	845	— 18.94%	0.66%	2109	— 0.16%	1.3654	30
	12	5.48%	7555	— 5.35%	1.87%	5834	— 1.47%	15.18%	957	— 23.16%	0.44%	214	85.99 —	1.0360	60
	30	0.01%	35	3.86 —	0.00%	0	2.77 —	24.80%	766	— 22.43%	0.00%	0	4.84 —	0.9288	110
	60	0.01%	31	3.92 —	0.00%	0	2.61 —	24.78%	733	— 14.16%	0.00%	0	4.98 —	0.9288	110

Table A.62: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 7).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	32.89%	1229	— 32.50%	10.32%	1195	— 9.54%	7.17%	1033	— 31.22%	0.62%	67	26.33 —	1.2986	15
	6	9.32%	6891	— 8.86%	3.34%	5140	— 2.74%	12.90%	885	— 25.55%	0.50%	136	37.81 —	0.9321	30
	12	1.08%	138270	— 0.25%	0.43%	9377	298.89	26.47%	841	— 44.95%	0.69%	1389	200.76 —	0.7638	60
	30	0.02%	29	3.78 —	0.01%	0	2.7 —	31.43%	921	— 29.92%	0.00%	0	5.80 —	0.7401	72
	60	0.02%	40	3.56 —	0.01%	3	4.58 —	31.43%	794	— 21.67%	0.00%	0	6.39 —	0.7401	72
10	3	55.61%	1317	— 47.02%	14.36%	743	— 14.33%	4.36%	1510	— 18.12%	0.33%	140	30.81 —	1.7823	15
	6	24.18%	1515	— 23.98%	6.97%	1939	— 6.51%	7.83%	846	— 22.60%	0.49%	566	135.08 —	1.3543	30
	12	4.89%	14039	— 4.53%	1.53%	7247	— 1.18%	16.68%	957	— 26.34%	0.38%	734	159.79 —	1.0520	60
	30	0.02%	30	3.67 —	0.00%	0	2.83 —	25.53%	765	— 20.52%	0.00%	0	4.78 —	0.9572	113
	60	0.02%	66	3.13 —	0.01%	0	2.67 —	25.56%	730	— 16.26%	0.00%	0	5.20 —	0.9572	113

Table A.63: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 8).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	34.03%	1277	— 34.00%	10.94%	1672	— 10.06%	5.27%	1080	— 21.84%	0.59%	115	28.02 —	1.2697	15
	6	9.75%	9029	— 9.25%	3.61%	5325	— 2.95%	9.36%	774	— 34.30%	0.74%	138	41.39 —	0.8873	30
	12	0.90%	119983	568.7 —	0.37%	1663	53.77 —	22.10%	948	— 50.26%	0.68%	297	46.21 —	0.7020	56
	30	0.03%	73	3.53 —	0.01%	2	9.22 —	29.37%	864	— 30.20%	0.00%	2	8.19 —	0.6820	65
10	60	0.02%	117	3.47 —	0.01%	15	4.69 —	29.37%	808	— 24.35%	0.00%	0	6.44 —	0.6820	65
	3	56.02%	1260	— 52.32%	14.54%	951	— 16.26%	3.18%	2814	— 17.77%	0.33%	140	28.97 —	1.7568	15
	6	23.56%	1634	— 23.39%	6.56%	2023	— 6.09%	5.50%	844	— 26.90%	0.32%	35	36.58 —	1.3129	30
	12	4.37%	14565	— 4.14%	1.41%	10535	— 1.01%	13.94%	960	— 22.45%	0.56%	1091	207.88 —	1.0029	60
30	30	0.02%	99	4.06 —	0.01%	0	2.77 —	25.16%	798	— 21.95%	0.00%	0	4.59 —	0.9161	103
	60	0.02%	125	3.97 —	0.01%	0	2.66 —	25.16%	837	— 16.32%	0.00%	0	5.06 —	0.9161	103

Table A.64: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 9).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	35.47%	1211	— 36.61%	11.77%	1260	— 10.91%	6.63%	929	— 28.16%	0.86%	270	69.82 —	1.2991	15
	6	10.37%	8353	— 9.84%	3.98%	4084	— 3.30%	11.17%	832	— 26.08%	1.10%	513	139.48 —	0.9164	30
	12	1.04%	73002	407.97 —	0.43%	2611	115.79 —	23.90%	866	— 36.17%	0.77%	340	76.79 —	0.7256	57
10	30	0.04%	334	4.83 —	0.02%	62	6.08 —	30.79%	736	— 29.68%	0.01%	6	7.52 —	0.7031	68
	60	0.04%	328	5.22 —	0.02%	70	5.59 —	30.78%	741	— 28.05%	0.01%	8	9.97 —	0.7031	68
	3	57.48%	1286	— 50.31%	16.09%	755	— 16.37%	4.23%	1358	— 10.19%	0.37%	181	38.61 —	1.7849	15
10	6	26.05%	1559	— 26.41%	7.73%	1583	— 7.37%	6.91%	915	— 19.10%	0.56%	482	118.13 —	1.3449	30
	12	5.40%	14192	— 5.19%	1.82%	10128	— 1.48%	15.08%	901	— 25.68%	0.47%	306	84.47 —	1.0242	60
	30	0.01%	3	2.94 —	0.00%	0	2.73 —	24.76%	782	— 36.52%	0.00%	0	4.92 —	0.9187	108
	60	0.01%	26	2.78 —	0.00%	0	2.53 —	24.74%	730	— 18.77%	0.00%	0	5.28 —	0.9187	108

Table A.65: Results of problems with 300 products and 30 customer segments for the multiple cardinality constraints case (Instance 10).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Objective	$\sum_{j \in N} x_j$
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time		
5	3	32.44%	1329	– 32.17%	10.17%	1175	– 9.17%	6.16%	963	– 21.70%	0.55%	68	30.2 –	1.2881	15
	6	8.90%	7249	– 8.18%	3.23%	5138	– 2.56%	12.38%	806	– 32.78%	0.77%	319	63.36 –	0.9197	30
	12	0.39%	35705	197.28	0.15%	227	14.06	25.50%	941	– 48.15%	0.06%	40	22.83 –	0.7443	60
30	30	0.02%	19	3.69 –	0.01%	0	2.83 –	31.75%	864	– 32.57%	0.00%	0	5.44 –	0.7357	74
	60	0.02%	51	3.58 –	0.01%	0	2.48 –	31.74%	730	– 23.51%	0.00%	0	5.59 –	0.7357	74
10	3	56.81%	1213	– 49.29%	14.42%	771	– 13.77%	3.95%	1728	– 11.74%	0.22%	100	26.81 –	1.7772	15
	6	23.54%	1750	– 25.53%	6.60%	2276	– 6.19%	7.20%	902	– 20.89%	0.37%	131	35.22 –	1.3394	30
	12	4.39%	13963	– 4.20%	1.41%	8858	– 1.05%	16.19%	960	– 22.88%	0.04%	0	25.39 –	1.0362	60
30	30	0.01%	50	3.84 –	0.01%	0	2.72 –	25.52%	812	– 23.48%	0.00%	0	4.92 –	0.9507	110
	60	0.01%	31	3.66 –	0.01%	0	2.64 –	25.50%	876	– 27.78%	0.00%	0	4.8 –	0.9507	110

A.9 Individual Results of Table 4.1



Table A.66: Results of problems with 50 products, 5 customer segments and 5 price levels (Instance 1).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Obj.	$\sum_{j \in N} x_j$	Price levels
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time			
20	2	17.98%	32	0.50	—	—	1.02	9.55%	13662	44.11	0.01%	0	0.98	2.7391	2	3
	5	37.50%	43805	37.8	—	—	43.55	18.53%	1221098	—	0.39%	316	2.66	2.5071	5	3, 4
	12	64.59%	2147689	—	—	—	—	29.34%	583750	2.63%	1.77%	33476	98.63	2.1667	12	4, 5
	25	37.38%	831026	32.40%	—	—	20.70%	36.85%	441744	8.48%	2.83%	164787	880.69	1.8197	25	4, 5
	50	20.37%	2694866	29.51%	—	—	11.82%	44.69%	275802	15.06%	4.29%	520772	—	1.6543	44	5
40	2	9.00%	12	0.28	—	—	0.61	5.13%	12435	20.50%	0.00%	0	0.58	2.8542	2	3
	5	20.21%	16913	17.74	—	—	14.14	10.87%	356272	2252.58	0.09%	211	1.98	2.7136	5	3, 4
	12	40.07%	3057510	—	—	—	—	19.41%	723750	—	0.51%	27799	68.08	2.4786	12	3, 4, 5
	25	64.16%	1036317	14.42%	—	—	12.03%	27.26%	537565	7.47%	1.59%	1399732	—	2.1996	25	4, 5
	50	35.82%	1545048	44.57%	—	—	19.17%	34.45%	519748	13.49%	2.94%	731573	0.42%	1.9709	50	4, 5
				27.69%	—	—	7.74%	18.03%					1.72%			

Table A.67: Results of problems with 50 products, 5 customer segments and 5 price levels (Instance 2).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Obj.	$\sum_{j \in N} x_j$	Price levels
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time			
20	2	16.41%	53	0.86	—	—	0.86	9.99%	24528	54.78	0.00%	0	1.45	2.7474	2	3, 4
	5	37.62%	174529	120.41	—	—	134.13	18.88%	1654339	—	0.26%	62	4.61	2.5100	5	4, 5
	12	63.18%	2395068	—	—	—	—	28.91%	797612	—	1.98%	26240	80.85	2.1573	12	4, 5
	25	40.27%	900951	32.58%	—	—	22.02%	35.80%	487893	—	3.16%	603677	3213.35	1.8090	25	4, 5
	50	19.73%	3072980	—	—	—	12.85%	42.50%	307132	—	3.92%	544900	—	1.6081	46	5
40	2	8.61%	14	0.77	—	—	0.80	5.40%	20568	43.78	0.00%	0	1.28	2.8552	2	3, 4
	5	20.03%	32108	41.16	—	—	20.89	11.06%	1700913	—	0.06%	2	2.58	2.7102	5	3, 4, 5
	12	40.86%	3039546	—	—	—	—	19.07%	781667	—	0.50%	9969	34.44	2.4653	12	3, 4, 5
	25	61.46%	1114645	—	16.16%	13.45%	—	26.27%	547017	—	1.54%	1628388	—	2.1793	25	4, 5
	50	35.77%	1577849	—	46.93%	18.57%	—	32.59%	647545	—	2.76%	902729	—	1.9383	50	4, 5
				28.08%		7.92%				14.71%			1.30%			

Table A.68: Results of problems with 50 products, 5 customer segments and 5 price levels (Instance 3).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Obj.	$\sum_{j \in N} x_j$	Price levels
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time			
20	2	16.33%	53	1.34	5.99%	35	1.06	10.03%	30672	92.48	0.00%	0	1.23	2.7324	2	3
	5	37.33%	145909	105.22	22.01%	90329	90.91	18.99%	1271824	—	0.49%	686	3.34	2.4957	5	3, 4
	12	65.67%	2066243	—	35.28%	569425	—	29.67%	602331	2.47%	2.41%	373071	1331.90	2.1440	12	4, 5
	25	37.13%	832801	35.55%	16.75%	728152	22.11%	36.82%	529353	9.87%	2.92%	234269	1190.75	1.7871	25	4, 5
	50	22.70%	2581153	28.32%	10.96%	1147312	10.95%	45.56%	340305	14.83%	5.17%	389083	—	1.6492	42	5
40	2	9.03%	27	0.66	3.59%	10	1.08	5.43%	13574	21.71%	0.01%	0	0.56	2.8384	2	3
	5	20.13%	25672	19.89	12.83%	16830	21.25	11.09%	1064130	—	0.10%	10	1.66	2.6938	5	3, 4
	12	41.99%	3066012	—	27.15%	1959990	—	19.60%	730651	0.41%	0.75%	57192	130.80	2.4516	12	3, 4, 5
	25	64.77%	720633	16.45%	23.79%	597874	12.98%	27.38%	505584	5.53%	1.96%	1411098	—	2.1676	25	4, 5
	50	36.51%	1393141	44.58%	11.28%	737090	18.04%	35.06%	587802	15.20%	3.19%	988220	0.47%	1.9536	50	4, 5
				27.07%			7.39%			17.82%			1.73%			

Table A.69: Results of problems with 50 products, 5 customer segments and 5 price levels (Instance 4).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Obj.	$\sum_{j \in N} x_j$	Price levels
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time			
20	2	18.32%	35	0.53	—	—	0.70	10.36%	18589	51.47	0.03%	0	0.91	2.7277	2	3
	5	40.49%	47233	41.08	—	—	44.58	19.34%	1768409	—	0.47%	218	3.53	2.4801	5	4, 5
	12	66.17%	2084999	—	—	—	—	32.04%	565298	—	1.86%	—	—	2.1366	12	4, 5
	25	36.07%	838071	34.13%	—	—	20.63%	30.21%	807737	—	9.49%	—	—	1.7991	25	4, 5
	50	20.76%	2174668	28.60%	—	—	11.13%	37.97%	388808	—	15.77%	—	—	1.6494	40	5
40	2	9.18%	22	0.32	—	—	0.59	5.62%	15198	54.78	0.00%	0	0.56	2.8432	2	3
	5	21.76%	12559	13.61	—	—	8.34	11.39%	630027	2755.85	—	2	1.83	2.6904	5	3, 4, 5
	12	43.16%	2445710	—	—	—	—	19.98%	1028066	—	5.41%	9585	25.25	2.4473	12	3, 4, 5
	25	64.15%	660095	15.61%	—	—	12.25%	28.03%	587381	—	—	1515458	—	2.1688	25	4, 5
	50	35.18%	1737165	48.05%	—	—	17.57%	35.48%	433997	14.95%	—	879583	0.28%	1.9571	50	4, 5
				26.57%	—	—	7.18%			19.63%			1.84%			

Table A.70: Results of problems with 50 products, 5 customer segments and 5 price levels (Instance 5).

v_0	K	MILP			MILP+MC			CONIC			CONIC+MC			Obj.	$\sum_{j \in N} x_j$	Price levels
		Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes	Time			
	2	18.10%	56	0.98	6.15%	36	0.73	10.42%	28415	58.61	0.04%	0	1.55	2.7435	2	3
	5	37.97%	111654	91.76	21.99%	85750	109.82	19.48%	1765772	—	0.46%	139	4.33	2.5016	5	3, 4, 5
	12	66.82%	2051983	—	35.36%	508099	—	29.96%	673330	3.29%	2.30%	99707	394.25	2.1515	12	4, 5
	25	37.85%	848413	37.55%	17.21%	600260	23.77%	37.23%	551717	9.74%	3.13%	454246	2792.66	1.8046	25	4, 5
	50	18.27%	2833252	30.63%	8.06%	1001754	12.44%	44.08%	388207	14.16%	3.65%	794404	—	1.6100	44	5
	2	9.79%	17	1.28	3.33%	8	0.88	5.62%	29988	20.19%	0.00%	0	1.20	2.8523	2	2
	5	20.44%	19967	23.74	13.10%	14183	18.33	11.46%	1739548	—	0.09%	44	2.42	2.7049	5	3, 5
	12	42.77%	2827450	—	28.61%	1983836	—	19.86%	753201	1.10%	0.72%	64769	132.71	2.4607	12	3, 4, 5
	25	64.64%	904712	17.59%	23.74%	496021	14.60%	27.29%	587207	8.48%	1.69%	1562207	—	2.1721	25	4, 5
	50	33.75%	1654758	48.38%	10.71%	898888	19.26%	33.97%	464434	12.56%	2.56%	998210	0.26%	1.9302	48	4, 5
				26.00%			7.37%			17.39%			1.14%			