FINDING ROBUSTLY FAIR SOLUTIONS IN RESOURCE ALLOCATION

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this study, we consider resource allocation problems where the decisions affect multiple beneficiaries and the decision maker aims to ensure that the effect is distributed to the beneficiaries in an equitable manner. We specifically consider stochastic environments where there is uncertainty in the system and propose a robust programming approach that aims at maximizing system efficiency (measured by the total expected benefit) while guaranteeing an equitable benefit allocation even under the worst scenario. Acknowledging the fact that the robust solution may lead to high efficiency loss and may be over-conservative, we adopt a parametric approach that allows controlling the level of conservatism and present the decision maker alternative solutions that reveal the trade-off between the total expected benefit and the degree of conservatism when incorporating fairness. We obtain tractable formulations, leveraging the results we provide on the properties of highly unfair allocations. We demonstrate the usability of our approach on project selection and shelter allocation applications.

Keywords: Fairness, equity, robust programming, knapsack problems, resource allocation.

ÖZET

KAYNAK PAYLAŞIMINDA GÜRBÜZ ADİL SONUÇLAR BULMAK

İzzet Egemen Elver
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Özlem Karsu
Temmuz 2022

Bu çalışmada kararların birden fazla yararlanıcıyı etkilediği ve karar vericinin etkinin adil bir şekilde dağıtılmasını sağlamayı amaçladığı kaynak paylaşımı problemleri ele alınmaktadır. Özellikle, sistemde belirsizliğin olduğu rassal ortamlar dikkate alınmakta ve en kötü senaryoda bile adil bir fayda paylaşımını garanti ederken sistem verimliliğini (beklenen toplam faydayı) en üst düzeye çıkarmayı hedefleyen gürbüz bir programlama yaklaşımı önerilmektedir. Gürbüz çözümün yüksek verimlilik kaybına yol açabileceği ve aşırı ihtiyatlı olabileceği gerçeğinden yola çıkarak, ihtiyatlılık düzeyini kontrol etmeye izin veren parametrik bir yaklaşım önerilmektedir. Bu yaklaşım sayesinde, karar vericiye, toplam beklenen fayda ile adilliği dahil ederkenki ihtiyatlılık derecesi arasındaki ödünleşmeyi ortaya koyan alternatif çözümler sunulmaktadır. Adilliğin en düşük olduğu paylaşımların özelliklerinden yararlanarak, izlenebilir formülasyonlar elde edilmektedir. Yaklaşımın kullanılabilirliği proje seçimi ve toplanma alanı tahsisi problemlerinde gösterilmektedir.

Anahtar sözcükler: Adillik, eşitlikçilik, gürbüz programlama, sırt çantası problemleri, kaynak paylaşımı.

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Chapter 1

Introduction

There are various real life applications where equity concerns naturally arise and it is important to address these concerns for the proposed solutions to be applicable and acceptable. *Reducing inequalities* is indeed one of the United Nations Sustainable Development goals, which emphasizes the need to adopt policies to progressively achieve greater equality [2]. In line with this effort to integrate fairness concerns into decision making processes, there exist many studies in the operations research (OR) literature that consider classical problems, such as assignment, scheduling or knapsack problems, and extend available models so as to accommodate equity concerns [3, 4, 5, 6, 7, 8, 9, 10, 11]. These models are used across a broad range of applications including but not limited to vehicle routing [12], resource allocation [13], workload allocation [14, 15], disaster relief [16], emergency service facility location [17] and public service provision [18]. This broad range of applications indicates that considering these classical models with an emphasis on equity is practically relevant in addition to being technically interesting.

The equity concept is often studied in an allocation setting, where a resource or good is allocated to a set of entities. The concern for equity involves treating a set of entities in a “fair” manner in the allocation. The received resource or outcome can be a certain good, a bad or be the chance of receiving a good or bad.

The entities can be for example organizations, persons or groups of individuals which are at different locations or are members of different socio-economic classes.

In the general context, there exist many examples involving uncertainty in allocation problems [19, 20, 21]. Additionally, some of these examples investigate these stochastic resource allocation problems integrating fairness concerns such as [22, 23, 24, 25, 26].

In this study we also consider allocation settings in which uncertainty is involved and develop an approach that aims to ensure the desired equity levels even under the worst circumstances. When doing so, we adopt the parametric robust approach proposed by Bertsimas and Sim [27] that controls the degree of conservatism through a parameter. We propose formulations to achieve the most efficient solution that is also robustly fair, when fairness is measured through commonly-used measures such as the sum of absolute pairwise differences, Gini coefficient and Rawlsian approach. The non-linearity of these functions renders a direct implementation of the duality-based approach of Bertsimas and Sim [27] impossible. We, however, provide results on the properties of the mostly inequitable allocations, which helps us to develop tractable formulations for the problem. Our results and formulations would be of use to obtain robustly fair solutions in many applications as demonstrated through examples.

The rest of the work is organized as follows. In Chapter 2 we provide a review of the related literature and elaborate on our contributions. In Chapter 3 we give formulation of the generic model that we aim to solve and discuss two approaches to ensure robustly fair solutions. In Chapter 4, we show the applicability of our controllable robust approach in R&D project selection and shelter allocation problems. In Chapter 5, we give the main results of our work and future research directions.

Chapter 2

Literature Review

Our aim in this work is to handle equity, specifically equitability [28], concerns in a stochastic environment through robust programming and demonstrate the applicability of this approach. Our work lies in the intersection of and extends existing works in two areas: Studies focusing on incorporating fairness in various OR problems and studies that implement robust programming in resource allocation problems.

2.1 Inequity-Averse Resource Allocation

Firstly, we focus on the works in the literature that consider inequity-averse resource allocation problem. Bertsimas et. al. [13, 29] studies the relative system efficiency loss under a "fair" resource allocation, which is the price of fairness. The authors present bounds for this price when max-min and proportional fairness policies are used. Bertsimas et. al. [30] presents a binary optimization framework that incorporates different objective functions, alternative sets of resources and fairness controls for dynamic resource allocation problems.

Kozanidis [4] tries to find the optimal allocation of a resource to disjoint sets of activities while maximizing the profit earned and minimizing the maximum difference between the resource amounts allocated to any two sets of activities in the linear multiple choice knapsack problem setting. Nace and Orlin [3] also shows how lexicographically minimum and maximum load linear programming problem achieves an equitable allocation of resources.

Inequity-averse resource allocation is a critical objective in the humanitarian operations and health delivery. McCoy and Lee [31] proposes a model achieving optimal resource allocations under utilitarian, proportionally fair, and egalitarian objectives in health delivery fleet managements. Arnette and Zobel [32] develops a risk measure addressing the need for generating equitable solutions, and then integrates it into the objective function of a model for reallocating assets to nearby locations after a disaster. Park and Berenguer [33] addresses the location-distribution problem with limited supply that has to be allocated to different demand regions in not-for-profit settings. The authors take this problem into consideration in the form of a bi-objective linear fractional problem, with efficiency and inequity-based objective functions. Stauffer et. al. [34] investigates the mobile pantry programs, additional food bank storage capacity, and improved partner agency capacity to achieve equitable food security while also maximizing food distribution with a limited budget.

Our work extends most of the existing literature in the first group as it considers uncertainty in parameters rather than assuming a deterministic setting, which may be unlikely in some real life applications. Instead of assuming that the effects of the resource allocation to the entities are certain, we allow them to change within a given range and aim to provide an efficiency maximizing solution that guarantees a certain degree of fairness whatever the realization is.

2.2 Robust Programming in Resource Allocation

There are a number of works in the literature that consider resource allocation problems through robust programming. As mentioned above, we use the approach of Bertsimas and Sim [27], which provides a way of adjusting the level of conservatism of the robust solutions for linear optimization problems with uncertain parameters in constraint. They also show that their approach generalizes to discrete optimization problems, applying the approach to a knapsack problem.

In another paper, Bertsimas and Sim [35] show that this approach allows controlling the degree of conservatism of the robust solutions for general integer optimization problems, and is computationally tractable. They discuss three different cases in terms of the parameter set in which uncertainty arises and the underlying problem type: 0-1 discrete programming problems in which only the objective function coefficients are uncertain, minimum cost problems where only the objective function coefficients are uncertain and integer programming problems where both the *cost (objective function) coefficients* and the data in the *constraints* are subject to uncertainty. For the first category of problems, they solve the robust counterpart by solving at most $n + 1$ instances of the original problem, where n is the number of decision variables. For the second category, they propose a polynomial time algorithm for the robust counterpart by solving a collection of minimum cost flow problems in a modified network. For the last category of problems they propose ,the approach in [27], a robust integer programming problem of moderately larger size.

There are other robust programming applications in portfolio optimization and project selection. [36, 37, 38, 39]. Liesiö et. al. [36] develop Robust Portfolio Modeling (RPM) to support project portfolio selection under uncertainty and multiple evaluation criteria. Liesiö et. al. [37] extend RPM into a multi-objective zero-one linear programming problem with interval valued objective function coefficients by considering project interdependencies, incomplete cost information and variable budget levels. Fliedner and Liesiö [38] integrate RPM into the robust optimization framework of Bertsimas and Sim [27] for multi-attribute project portfolio selection. They have a single category and multiple attributes with uncertain scores for each project. Additionally, decision maker preferences are based on uncertain attribute weights. As multiple attributes are involved, they propose approaches for establishing dominance between two portfolios and an exact algorithm based on dynamic programming for identifying non-dominated portfolios.

Hassanzadeh et. al. [39] develop a multi-objective binary integer programming model for research and development (R&D) project portfolio selection. They also use the controllable robust optimization framework of Bertsimas and Sim [27] to deal with uncertain problem coefficients in both objective functions and constraints; and generate the so-called *robust non-dominated solutions* by using Tchebycheff scalarization. They then propose an interactive procedure to identify the most preferred solution.

2.3 Fairness and Stochastic Resource Allocation

We can also see some examples integrating fairness into the stochastic resource allocation problems in the literature. Medernach and Sanlaville [22] try to ensure fair resource allocation for different scenarios of demand with a multi-criteria model, one criterion is associated to one scenario. Bateni et. al. [24] propose a stochastic approximation scheme akin to a dynamic market equilibrium for fair resource allocation in a volatile marketplace due to the uncertain, time-varying supply of goods, which comes from a wide family of (possibly non-stationary) Gaussian processes. The scheme frequently resolves an Eisenberg-Gale convex program without requiring any knowledge about how the goods' arrival processes evolve over time. Gutjahr [26] develops a stochastic multistage optimization framework evaluating payoffs by an equitable aggregation function and discusses the differences between the ex-ante approach and the ex-post approach in inequity-averse optimization under uncertainty. Mostajabdaveh et. al. [25] also optimize a weighted mean of the so-called ex-ante and ex-post versions of the inequity-averse shelter location problem under uncertain demand and disruptions in the transportation network. All these works tackle fairness under uncertainty with scenario-based approaches [22, 26, 25] or basic assumptions about the uncertainty [24]. However, we develop a robust approach to guarantee fairness even if the maximum inequity level realizes under uncertainty.

In addition, Hu et. al. [23] work on emergency resource allocation under uncertainty of the demand for relief resource and the unit cost of allocating one unit to affected area with supply shortage. They formulate a bi-objective model to maximize efficiency, which is measured in terms of the total utility among the affected areas, and fairness, measured in terms of the minimum utility satisfaction rate. They consider a scenario-based uncertainty approach for demand and measure solution robustness in the efficiency objective function with the mean value and variance of the overall utility. They, however, do not have robustness consideration in the fairness objective function. As for the other uncertain parameter, cost, they consider an uncertainty set and assume that cost parameters taking values according to a symmetrical distribution in an interval, similar to that of Bertsimas and Sim [35]. They utilize a constraint ensuring that the total cost of allocating the relief resources can not exceed the reserve fund by applying the proposed method by Bertsimas and Sim [27] into the budget constraint with a robustness control parameter Γ. After obtaining the Pareto solutions of the bi-objective programming problem with min-sum and min max utility functions, they search over the Pareto set to find compromise solutions evaluating them with respect to their Gini coefficient values in a heuristic particle swarm optimization algorithm.

Similar to the above works, we acknowledge the importance of obtaining robust solutions under uncertainty. We, however, directly focus on the effect that the uncertainty will have on the resulting benefit allocations and hence formulate a model that ensures robustly fair solutions. As opposed to post-processing solutions with respect to their Gini coefficient values as in [23], we formulate models that directly use the Gini coefficient to quantify fairness. As will be clear in the upcoming chapters, due to the nature of our allocation problems, we address uncertainty by using an approach that relies on both scenario-based and interval based frameworks, leveraging the results we provide on the properties of the mostly unfair solutions with respect to these measures. This makes our approach different than the previous approaches in the literature.

Chapter 3

Developing Mathematical Programming Formulations

In this chapter, we consider a stochastic environment for a resource allocation problem that can be formulated as a binary knapsack problem, in which each item incurs a certain amount of cost and provides benefits to multiple beneficiaries. We assume that the decision maker has equity concerns and hence wants to ensure a certain degree of equitability in the resulting benefit allocation. We take the item costs as certain but benefits of items for each entity as uncertain. Note that in a vast amount of real-life applications including project portfolio selection, the underlying problem is a knapsack problem variant such as [40, 41, 42, 43].

For this setting, we aim to guarantee an equitable benefit allocation even under the worst scenario adopting the robust optimization approach of Bertsimas and Sim [27] while maximizing the total expected benefit. As will be explained later in detail, the non-linearity of the fairness-related functions used in the problem precludes a direct application of the approach proposed in the aforementioned work. We, therefore, provide results on the properties of allocations with minimum fairness, which is defined with respect to various functions such as the sum of pairwise differences, the Gini index and the Rawlsian approach. Using these results we demonstrate that it is possible to find robustly fair solutions to knapsack problems through tractable mixed integer linear programming formulations.

In Table 3.1, we provide a list of the sets, parameters and decision variables used in the models discussed below.

Table 3.1: Problem Structure

Sets	I: set of items J: set of entities S: set of scenarios showing which items have uncertain benefit levels for each entity($\vert S\vert = C(\vert I\vert,\Gamma)^{\vert J\vert}, s \in S$) Ω: set of scenarios showing that whether entities have positive or negative deviation from mean benefit level ($\vert\Omega\vert = 2^{\vert J\vert}, \omega \in \Omega$)
Certain Parameters	c_i: Cost of entity i B: Budget Γ: Control parameter for robustness against the level of conservatism ($\Gamma \in \{0, 1, ..., \vert I\vert\}$) $g^j_{is} = 1$ implies that item i has uncertain benefit for entity j in scenario s, $g^j_{is} = 0$ implies otherwise $v_{j\omega}$= 1 implies that $o_{js\omega} = o^{max}_{js}$, $v_{j\omega}$= 0 implies that $o_{js\omega} = o^{min}_{js}$ for entity j and scenario ω in all scenario pairs (s, ω) G: Inequity threshold
Uncertain Parameter	$\tilde{o}^j_i$: Benefit level of item i for entity j, $\tilde{o}^j_i \in [o^j_i - \hat{o}^j_i, o^j_i + \hat{o}^j_i]$
Decision Variables	o^{max}_{js}: Maximum benefit level enjoyed by entity j in scenario s o^{min}_{js}: Minimum benefit level enjoyed by entity j in scenario s $o_{js\omega}$: Benefit level for entity j at scenario pair (s, ω) $d^{s\omega}_{kl}$: $\vert o_{ks\omega} - o_{ls\omega}\vert$ p_{ij}, λ_j: Dual variables coming from the linearization of Model 3.3 into Model 3.4
Binary Decision Variables	$x_i = 1$ if item i is selected, and $x_i = 0$ otherwise

We let the (uncertain) benefit that entity j enjoys from item i be $\tilde{o}^j_i$ and assume that it takes values according to a symmetric distribution with mean equal to the nominal o^j_i in the interval $[o^j_i - \hat{o}^j_i, o^j_i + \hat{o}^j_i]$. We have the Γ_j parameter to adjust the robustness of proposed method against the level of conservatism of the solution for each entity j and $\Gamma_j \in \{0, 1, ..., |I|\}$. This helps one to avoid the inefficiency of due to over-conservatism of robust optimization and excessive number of scenarios in stochastic optimization. Additionally, the practical issues based on fitting probability distributions over uncertain parameters is eliminated.

Without loss of generality, we take $\Gamma_j = \Gamma, \forall j \in J$ in this work. Specifically, Γ is the number of items which are considered to have uncertain benefit levels for each entity. The remaining items are assumed to have benefit levels equal to the corresponding mean value.

Let $f^\Gamma(x)$ be an inequity function and G be a inequity threshold parameter.

Then, our problem has the following generic form:

$$\text{Formulation } \textbf{Generic Model} \tag{3.1}$$

$$\max \quad o^T x \tag{3.1a}$$

$$\text{s.t. } c^T x \leq B \tag{3.1b}$$

$$f^\Gamma(x) \leq G \tag{3.1c}$$

$$x \in \{0, 1\} \tag{3.1d}$$

The Generic Model is a binary knapsack problem where we try to maximize total expected benefit (3.1a) subject to budget constraints (3.1b) and inequity related constraints (3.1c). Here, constraint (3.1c) is a special type of constraint that ensures that the value of the inequity function is below the threshold G *even under the worst scenario* over all the scenarios in which Γ of the item benefits are considered as uncertain for each entity. As the novelty of suggested approach lies in how robustly fair allocation of benefits is ensured, we do not consider uncertainty in the cost parameters in our model for brevity of discussion. However, when these are also uncertain, belonging to an uncertainty set, it is straightforward to handle it using the method of [27].

We consider three alternative functions to measure inequity: the sum of absolute pairwise differences, the Gini coefficient and the Rawlsian minimum benefit, which are defined below. Let $b_j = \sum_{i \in I} \tilde{o}_i^j x_i$ be the total benefit received by entity j, and n be the number of entities. Then, the sum of absolute pairwise differences is given by the following expression: ([44]).

$$\sum_{j=1}^{n} \sum_{k>j}^{n} |b_j - b_k|$$

The Gini coefficient is given by:

$$\frac{\sum_{j=1}^{n} \sum_{k=1}^{n} |b_j - b_k|}{2n \sum_{j=1}^{n} b_j}$$

([28]) and the Rawlsian approach benefit is given by :

$$\min_{j \in 1,\dots,n} \{b_j\}$$

Below, we provide our theoretical findings and mathematical models that correspond to each one of these functions.

3.1 The Sum of Absolute Pairwise Differences (SOPD)

The generic formulation involves constraint (3.1c), which ensures that the output allocation resulting from decision x is sufficiently fair under any scenario. That is, fairness is guaranteed even in the worst scenario for inequity. When equity is measured in terms of the SOPD between entities, we have to consider the maximum total pairwise difference in entity benefits. As stated in Remark 1, this only occurs when entities receive benefits at extreme levels, i.e. some will take maximum possible benefit while some others take the minimum.

Remark 1. *The maximum value for the SOPD in entity benefits is realized under a scenario where entities receive benefits at extreme levels (maximum/minimum possible benefit).*

Remark 2. *If the number of entities (n) is odd (resp. even), the maximum value for the SOPD in entity benefits is realized under a scenario where* $\frac{n+1}{2} - 1$ *(resp.* $\frac{n}{2}$*) entities take maximum possible benefit while* $\frac{n+1}{2}$ *(resp.* $\frac{n}{2}$*) entities take the minimum or vice versa.*

The proof is provided in Appendix 6.1.

Based on Remark 1, we define *a set of scenarios* Ω*, with each scenario showing whether the entities take their maximum or minimum possible benefit.* We denote the maximum and minimum possible benefits for an entity j as o_j^{max} and o_j^{min}, respectively. Note that these values will be endogenously calculated within the model and the maximum level of SOPD will be observed when some entities receive their minimum (o^{min}) and the others receive their maximum possible benefits (o^{max}), respectively. To enable the correct calculation of the maximum level of SOPD, we define a binary parameter v as follows:

$$v_{j\omega} = \begin{cases} 1 \text{ if entity } j \text{ takes } o_j^{max} \text{ in fairness scenario } \omega \in \Omega \\ 0 \text{ if entity } j \text{ takes } o_j^{min} \text{ in fairness scenario } \omega \in \Omega \end{cases}$$

Note that the number of all possible cases, $|\Omega|$ is $2^{|J|}$.

We will first provide what we call a scenario-based formulation, in which the uncertain items in a scenario is kept through a parameter. We use a binary parameter g_{is}^j, where $s \in S$ is a scenario that shows which items have uncertain benefit levels for each entity. The parameter is defined as follows:

$$g_{is}^j = \begin{cases} 1 \text{ if item } i \text{ has uncertain benefit level for entity } j \text{ in item scenario } s \\ 0 \text{ otherwise.} \end{cases}$$

Note that the number of such scenarios, $|S|$, is $C(|I|, \Gamma)^{|J|}$.

Formulation **Fully Scenario Based Model** (3.2)

$$\max \quad \sum_{i \in I} \sum_{j \in J} o_i^j x_i \tag{3.2a}$$

$$\text{s.t.} \sum_{i \in I} c_i x_i \le B \tag{3.2b}$$

$$o_{js}^{max} = \sum_{i \in I} o_i^j x_i + \sum_{i \in I} \hat{o}_i^j g_{is}^j x_i \qquad \forall j \in J, \forall s \in S \tag{3.2c}$$

$$o_{js}^{min} = \sum_{i \in I} o_i^j x_i - \sum_{i \in I} \hat{o}_i^j g_{is}^j x_i \qquad \forall j \in J, \forall s \in S \tag{3.2d}$$

$$o_{js\omega} = v_{j\omega} o_{js}^{max} + (1 - v_{j\omega}) o_{js}^{min} \qquad \forall j \in J, \forall s \in S, \forall \omega \in \Omega \tag{3.2e}$$

$$o_{ls\omega} - o_{ks\omega} \le d_{kl}^{s\omega} \qquad \forall k \in J, \forall l > k \in J, \forall s \in S, \forall \omega \in \Omega \tag{3.2f}$$

$$o_{ks\omega} - o_{ls\omega} \le d_{kl}^{s\omega} \qquad \forall k \in J, \forall l > k \in J, \forall s \in S, \forall \omega \in \Omega \tag{3.2g}$$

$$\sum_{k\in J}\sum_{l>k\in J} d_{kl}^{s\omega} \leq G \qquad \forall s\in S, \forall \omega\in\Omega \tag{3.2h}$$

$$o_{js\omega} \geq 0 \qquad \forall j\in J, \forall s\in S, \forall\omega\in\Omega \tag{3.2i}$$

$$o_{js}^{max}, o_{js}^{min} \geq 0 \qquad \forall j\in J, \forall s\in S \tag{3.2j}$$

$$d_{kl}^{s\omega} \geq 0 \qquad \forall k\in J, \forall l>k\in J, \forall s\in S, \forall\omega\in\Omega \tag{3.2k}$$

$$x_i \in \{0,1\} \qquad \forall i\in I \tag{3.2l}$$

In (3.2a), we maximize the total expected benefit. (3.2b) is the budget constraint. (3.2c) and (3.2d) define the possible maximum and minimum benefit levels received by each entity $j \in J$, in each scenario $s \in S$, respectively. Specifically, $o^{max}(o^{min})$ occurs when all items with uncertain benefits result in maximum (minimum) benefit for the entity. Constraint (3.2e) determines whether entity j takes maximum or minimum benefit level making use of the scenario-defining parameter v. As we know that an entity will either receive its maximum/minimum possible benefit by Remark 1, we make use of a binary parameter $v_{j\omega}$. We find the absolute pairwise differences of benefit levels among entities with (3.2f)-(3.2g). (3.2h) keeps the sum of absolute pairwise differences of benefit levels among entities under the threshold value G for all scenario pairs $(s,\omega) \in S\times\Omega$. (3.2i)-(3.2k) are non-negativity constraints and (3.2l) defines binary decision variables.

Since the number of scenario pairs $|S\times\Omega|$ is $C(|I|,\Gamma)^{|J|}2^{|J|}$, the number of decision variables is $|I| + 2|J|C(|I|,\Gamma)^{|J|} + |J|C(|I|,\Gamma)^{|J|}2^{|J|} + \frac{C(|J|,2)}{2}C(|I|,\Gamma)^{|J|}2^{|J|}$, and the number of constraints is $1 + 4|J|C(|I|,\Gamma)^{|J|} + 2|J|C(|I|,\Gamma)^{|J|}2^{|J|} + \frac{3}{2}C(|J|,2)C(|I|,\Gamma)^{|J|}2^{|J|} + C(|I|,\Gamma)^{|J|}2^{|J|} + |I|$, Fully Scenario Based Model (FSBM) is computationally expensive. We propose the following formulations to eliminate the excessive number of scenarios in Fully Scenario Based Model using the robust optimization approach introduced by [35]. Note that in the following formulations, decision variables $o_j^{max}, o_j^{min}, o_{j\omega}, d_{kl}^{\omega}$ no longer depend on the item scenario set S because we omit this set and the related parameters g_{is}^{j}. In these models, we directly use the maximum benefit deviation to calculate the possible maximum and minimum benefit levels for each entity j, which gives the maximum SOPD level under the threshold G by Remark 1.

Formulation **Model 2** (3.3)

$$\max \quad \sum_{i\in I}\sum_{j\in J} o_i^j x_i \tag{3.3a}$$

$$\text{s.t.} \sum_{i\in I} c_i x_i \le B \tag{3.3b}$$

$$o_j^{max} = \sum_{i\in I} o_i^j x_i + \max_{H\subseteq I,|H|=\Gamma} \sum_{i\in H} \hat{o}_i^j x_i \quad \forall j \in J \tag{3.3c}$$

$$o_j^{min} = \sum_{i\in I} o_i^j x_i - \max_{H\subseteq I,|H|=\Gamma} \sum_{i\in H} \hat{o}_i^j x_i \quad \forall j \in J \tag{3.3d}$$

$$o_{j\omega} = v_{j\omega} o_j^{max} + (1 - v_{j\omega}) o_j^{min} \quad \forall j \in J, \forall \omega \in \Omega \tag{3.3e}$$

$$o_{k\omega} - o_{l\omega} \le d_{kl}^{\omega} \quad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \tag{3.3f}$$

$$o_{l\omega} - o_{k\omega} \le d_{kl}^{\omega} \quad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \tag{3.3g}$$

$$\sum_{k\in J}\sum_{l>k\in J} d_{kl}^{\omega} \le G \quad \forall \omega \in \Omega \tag{3.3h}$$

$$o_{j\omega} \ge 0 \quad \forall j \in J, \forall \omega \in \Omega \tag{3.3i}$$

$$o_j^{max}, o_j^{min} \ge 0 \quad \forall j \in J \tag{3.3j}$$

$$d_{kl}^{\omega} \ge 0 \quad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \tag{3.3k}$$

$$x_i \in \{0, 1\} \quad \forall i \in I \tag{3.3l}$$

Due to (3.3c)-(3.3d), Model 2 is nonlinear. We propose using Model 3 instead of Model 2.

Formulation **Model 3** (3.4)

$$\max \quad \sum_{i\in I}\sum_{j\in J} o_i^j x_i \tag{3.4a}$$

$$\text{s.t.} \sum_{i\in I} c_i x_i \le B \tag{3.4b}$$

$$o_j^{max} = \sum_{i\in I} o_i^j x_i + \sum_{i\in I} p_{ij} + \Gamma\lambda_j \quad \forall j \in J \tag{3.4c}$$

$$o_j^{min} = \sum_{i \in I} o_i^j x_i - \sum_{i \in I} p_{ij} - \Gamma \lambda_j \qquad \forall j \in J \quad (3.4d)$$

$$\lambda_j + p_{ij} \geq \hat{o}_i^j x_i, \qquad \forall i \in I, \forall j \in J \quad (3.4e)$$

$$o_{j\omega} = v_{j\omega} o_j^{max} + (1 - v_{j\omega}) o_j^{min} \qquad \forall j \in J, \forall \omega \in \Omega \quad (3.4f)$$

$$o_{l\omega} - o_{k\omega} \leq d_{kl}^{\omega} \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \quad (3.4g)$$

$$o_{k\omega} - o_{l\omega} \leq d_{kl}^{\omega} \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \quad (3.4h)$$

$$\sum_{k \in J} \sum_{l > k \in J} d_{kl}^{\omega} \leq G \qquad \forall \omega \in \Omega \quad (3.4i)$$

$$p_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J \quad (3.4j)$$

$$\lambda_j \geq 0 \qquad \forall j \in J \quad (3.4k)$$

$$o_{j\omega} \geq 0 \qquad \forall j \in J, \forall \omega \in \Omega \quad (3.4l)$$

$$o_j^{max}, o_j^{min} \geq 0 \qquad \forall j \in J \quad (3.4m)$$

$$d_{kl}^{\omega} \geq 0 \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \quad (3.4n)$$

$$x_i \in \{0, 1\} \qquad \forall i \in I \quad (3.4o)$$

Note that the number of decision variables is $|I| + 3|J| + |I||J| + |J|2^{|J|} + \frac{C(|J|,2)}{2}2^{|J|}$, and the number of constraints is $1 + 5|J| + 2|I||J| + 2|J|2^{|J|} + \frac{3}{2}C(|J|,2)2^{|J|} + 2^{|J|} + |I|$.

3.2 The Gini Index

We now consider the Gini index as another measure to quantify equity. This measure is scaled between 0 and 1, hence it allows a structured parametric analysis over its entire range. Similar to the case with the SOPD, we make use of the following result that we proved so as to be able obtain a tractable formulation for our problem.

Remark 3. *The maximum value for the Gini index of the entity benefit distribution is realized under a scenario where entities receive benefits at extreme levels (maximum/minimum possible benefit).*

The proof is provided in Appendix 6.3. Based on Remark 3, the generic formulation for the robust programming problem that uses the Gini index can be written as follows:

Formulation **Gini (Nonlinear)**

$$\begin{aligned} \max \quad & (3.3a) \\ \text{s.t.} \quad & (3.3b)-(3.3g),(3.3i)-(3.3l) \\ & \sum_{k\in J}\sum_{l>k\in J} d_{kl}^{\omega} \leq G|J|\sum_{j\in J} o_{j\omega} \quad \forall \omega \in \Omega \end{aligned} \tag{3.5a}$$

Formulation **Gini (Linear)**

$$\begin{aligned} \max \quad & (3.4a) \\ \text{s.t.} \quad & (3.4b)-(3.4h),(3.4j)-(3.4o) \\ & (3.5a) \end{aligned}$$

Proposition 1. *The optimal solution of Model 3 (resp. linear Gini model) is equivalent to the optimal solution of Model 2 (resp. nonlinear Gini model).*

The proof is provided in Appendix 6.2.

3.3 The Rawlsian Approach

Since the Rawlsian approach is focused on the minimum benefit level in a distribution, we keep the possible minimum benefit for all entities above a given threshold G.

Formulation **Rawlsian (Nonlinear)**

$$\begin{aligned} \max \quad & (3.3a) \\ & \text{s.t.} (3.3b), (3.3d), (3.3l) \\ & o_j^{min} \geq G \quad \forall j \in J \end{aligned} \tag{3.7a}$$

Formulation **Rawlsian (Linear)**

$$\begin{aligned} \max \quad & (3.4a) \\ & \text{s.t.} (3.4b), (3.4d), (3.4e), (3.4j), (3.4k), (3.4o) \\ & (3.7a) \end{aligned}$$

We can explain the equivalence of nonlinear and linear optimal solutions in Rawlsian model with a similar approach to the one given in the proof of Proposition 1, coming from weak duality.

Chapter 4

Application Examples

In this chapter, we illustrate our approach on three example problems. The first one is a small knapsack problem while the second one is a real-life based project selection application. Finally, we demonstrate how our approach would be used in a humanitarian application.

We coded all mixed-integer linear models in Optimization Programming Language (OPL) and solved using IBM CPLEX 20.1.0 with a quad core computer (Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz 2.10 GHz) that has 16 GB RAM.

4.1 A Knapsack Problem

Consider a decision maker who has to choose the ones to invest in among 10 projects given a budget of 42 units. We assume that the project costs are certain while the project benefits on 3 different population groups are uncertain. These output values have symmetric distribution with mean and deviation values shown in Table 4.1. Such problems could be observed, e.g. in healthcare project selection, where each health technology is supposed to benefit different population groups to various extents (see [45] for more discussion).

Table 4.1: Data for The Knapsack Problem

Project (i)	c_i	o_{i1}	o_{i2}	o_{i3}	$\hat{o}_{i1}$	$\hat{o}_{i2}$	$\hat{o}_{i3}$
1	5	3	4	5	2	3	4
2	8	8	5	7	6	4	3
3	10	5	6	7	1	1	6
4	12	7	8	6	5	3	3
5	6	4	2	5	2	1	3
6	4	2	5	3	1	2	2
7	7	3	9	7	2	7	4
8	9	5	6	6	2	4	3
9	10	8	7	8	3	4	5
10	11	8	4	6	4	1	1

In Table 4.2, we provide the results of a parametric analysis over difference levels of uncertainty and inequity aversion for problems using SOPD, Gini and Rawlsian functions to incorporate fairness. Overall, one can observe the effect of uncertainty level and the fairness constraint on the total expected output. For the same inequity threshold (resp. Γ), the total expected output is non-increasing with Γ (resp. threshold).

Table 4.2: Total Expected Benefits for Different Parameters (Knapsack Problem)

Metric	SOPD						Gini						Rawlsian					
Γ / G	∞	76	69	62	55	48	0.40	0.35	0.30	0.25	0.20	0	0	12	13	14	15	16
0						95					95	84						95
1						95					95	0						95
2					95	93					95	0						95
3				95	87	84			95	93	0	0						95
4		**95**	**90**	**85**	**78**	**77**		**95**	**93**	**80**	**0**	**0**				**95**	**93**	**93**
5	95	90	88	84	77	69	95	93	88	63	0	0		95	93	88	Inf.	Inf.
6	95	90	88	84	77	69	95	90	88	63	0	0	95	92	88	83	Inf.	Inf.
7	95	90	88	84	77	69	95	90	88	63	0	0	95	92	88	83	Inf.	Inf.
8	95	90	88	84	77	69	95	90	88	63	0	0	95	92	88	83	Inf.	Inf.
9	95	90	88	84	77	69	95	90	88	63	0	0	95	92	88	83	Inf.	Inf.
10	95	90	88	84	77	69	95	90	88	63	0	0	95	92	88	83	Inf.	Inf.

We now discuss detailed results for a moderate level of conservatism with $\Gamma = 4$ for all population groups. If we solve this problem ignoring the robust fairness constraint ($G = \infty$), projects 1, 2, 4, 7, 9 are chosen, resulting in the total expected output of 95 units. The maximum SOPD that can be observed is 76. Population group 1 takes total output values between 13 and 45 while Population groups 2 and 3 take output values between 15 and 51; and between 17 and 49, respectively.

When we solve the same problem with $G = 48$, projects 4, 5, 6, 8, 10 are chosen, with a total expected output of 77 and maximum SOPD of 48. Thus, we compromise the total expected output against the fairness of solution. Under these circumstances, population groups 1, 2 and 3 takes total output values between 13 and 39; 15 and 35; and between 15 and 37, respectively. In this problem, there are $2^3 = 8$ different scenarios where each population group takes either minimum or maximum values (for m population groups, 2^m different scenarios) and we do not exceed the inequity threshold in any scenario.

In the Figures 4.1 and 4.2, we can see the output distributions of some illustrative realizations based on the solutions of problem with $G = 48$ and $G = \infty$, respectively. For $G = 48$, the solution of the problem provides 20 different realizations (cases) giving the maximum inequity, SOPD=48. We separated these 20 cases in 4 groups according to the similarity of their output distributions. For example two allocations such as (13,15,37) and (13,18,37) belong to a group; while (39,15,37) and (39,15,36) belong to a different group. We show the cases giving the largest total output of each group in Figure 4.1.

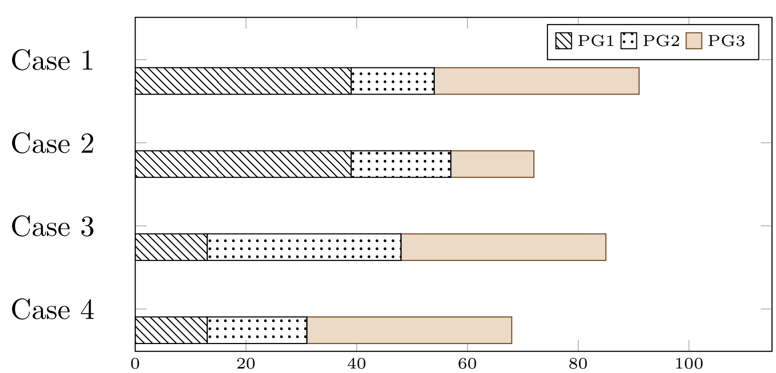


Figure 4.1: Output Distributions of Cases with Maximum SOPD=48 ($G = 48$)

It is seen that due to the uncertainty in the benefit values, the policy maker may end up with different levels of the total benefit: it could be as high as 91 or as low as 68.

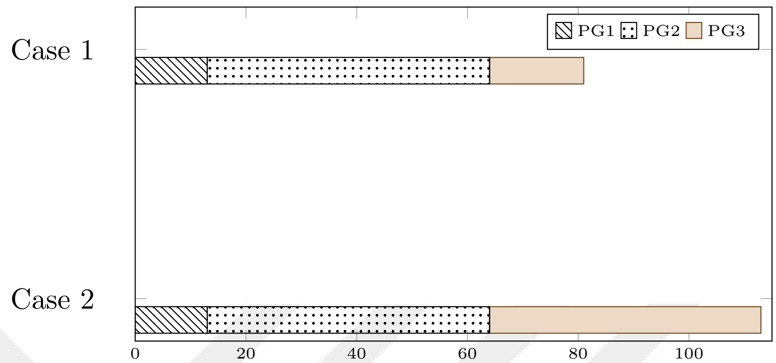


Figure 4.2: Output Distributions of Cases with Maximum SOPD=76 ($G = \infty$)

We observe that in cases leading to similar level of total outputs, including the SOPD threshold leads to solutions in which the total is more evenly distributed. To illustrate, consider two solutions of found using $G = \infty$ and $G = 48$, with total outputs of 81 and 85, respectively. These two solutions are shown as Case 1 in Figure 4.2 and Case 3 in Figure 4.1, respectively. We observe that the threshold on the SOPD controls the total envy in the distributions: when $G = \infty$, outputs are distributed as (13,51,17) while when (G=48) the outputs are distributed as (13,35,37).

Table 4.3 and Table 4.4 report the solution times in central processing unit (CPU) seconds for Model 3 and FSBM with the same data in Table 4.1, respectively. Since the number of scenarios in S is greater than or equal to 91125 for $\Gamma \in \{2, 3, ..., 8\}$, we need to set a 2 hours of time limit in FSBM. When $\Gamma = 2$, $G = 1$ the solver stopped with a 17.99% gap in FSBM after 2 hours. In Table 4.3, we can observe that Model 3 gives optimal solutions in negligible times. Although solution times of FSBM in Table 4.4 are also negligible for $\Gamma = 0$ and $\Gamma = 10$, which is because $|S| = C(10, 0)^3 = C(10, 10)^3 = 1$, we cannot find optimal solutions in 2 hours for $\Gamma \in \{2, 3, ..., 8\}$. This situation shows the superiority of our controllable robust programming approach with its computational inexpensiveness against the FSBM.

Table 4.3: Solution Times of Model 3 (Knapsack Problem)

Metric	SOPD						Gini							Rawlsian					
Γ / G	∞	76	69	62	55	48	1	0.40	0.35	0.30	0.25	0.20	0	0	12	13	14	15	16
0	0.065	0.082	0.063	0.063	0.092	0.082	0.056	0.057	0.066	0.081	0.096	0.067	0.054	0.062	0.067	0.047	0.087	0.12	0.144
1	0.055	0.047	0.082	0.059	0.131	0.153	0.068	0.128	0.058	0.055	0.058	0.133	0.044	0.101	0.116	0.065	0.071	0.066	0.078
2	0.055	0.133	0.067	0.071	0.066	0.087	0.067	0.058	0.058	0.057	0.075	0.062	0.039	0.139	0.113	0.059	0.062	0.059	0.071
3	0.101	0.059	0.061	0.064	0.11	0.082	0.058	0.122	0.076	0.119	0.072	0.088	0.082	0.054	0.072	0.066	0.069	0.124	0.178
4	0.079	0.072	0.06	0.071	0.115	0.073	0.097	0.047	0.059	0.059	0.115	0.073	0.044	0.051	0.067	0.057	0.123	0.069	0.09
5	0.117	0.063	0.084	0.156	0.121	0.108	0.057	0.059	0.087	0.111	0.056	0.065	0.039	0.061	0.133	0.091	0.087	0.13	0.12
6	0.059	0.06	0.074	0.065	0.099	0.166	0.056	0.064	0.083	0.062	0.115	0.055	0.06	0.054	0.1	0.055	0.058	0.11	0.12
7	0.054	0.064	0.06	0.101	0.067	0.075	0.065	0.067	0.074	0.107	0.099	0.044	0.041	0.055	0.051	0.054	0.106	0.08	0.10
8	0.097	0.063	0.096	0.085	0.058	0.113	0.058	0.058	0.07	0.066	0.066	0.059	0.043	0.074	0.09	0.054	0.064	0.09	0.11
9	0.095	0.092	0.066	0.068	0.118	0.112	0.08	0.12	0.069	0.08	0.052	0.224	0.111	0.058	0.065	0.087	0.165	0.10	0.09
10	0.056	0.074	0.064	0.096	0.078	0.063	0.091	0.085	0.059	0.063	0.062	0.069	0.04	0.053	0.063	0.124	0.056	0.08	0.11

Table 4.4: Solution Times of FSBM (Knapsack Problem)

Metric	Gini						
Γ / G	1	0.40	0.35	0.30	0.25	0.20	0
0	0.125	0.13	0.12	0.122	0.106	0.061	0.119
1	11.375	11.531	11.354	13.888	9.853	10.129	1.978
9	10.438	9.744	38.364	29.805	15.234	4.753	2.494
10	0.064	0.073	0.074	0.063	0.063	0.064	0.043

4.2 R & D Project Selection

In this example, we consider the real-life based R & D project selection problem described in [1], in which projects from three categories are to be selected for investment with the aim of maximizing the total output of the project portfolio while ensuring a balanced allocation of the output across the three categories. We extend the problem and allow for uncertainty in the output values of the projects. When each category is treated as an entity, the resulting formulation becomes a special case of the one described in the previous section, where each project has benefit for only one entity. The data of the problem is given in Appendix 6.4. We assume that project benefits may have up to 10% deviation from their mean values, and take costs and the overall budget of 9.31 units as certain. The results are provided in Figure 4.3 for the Gini model. The results for SOPD and Rawlsian models in Appendix 6.5 are similar to the Gini results. We also report the solution times in Appendix 6.5. Note that we cannot take any solution or gap in 2 hours of time limit for $\Gamma \in 1, 2, ..., 38$ due to the excessive number of scenarios.

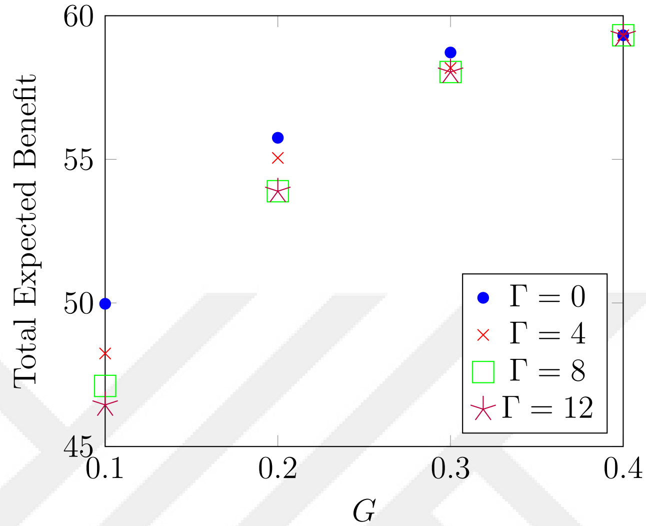


Figure 4.3: Total Expected Benefits in Gini Model (R&D Project Selection)

Note that when there is no uncertainty in the system , i.e. $\Gamma = 0$ or when there is no concern for equity, i.e. $G = \infty$, $G = 1$, $G = 0$ in SOPD, Gini and Rawlsian functions, respectively, the problem reduces to an optimization problem whose solution is the portfolio maximizing the total expected benefit. For a fixed level of conservatism (Γ), as the inequity aversion increases, the solution changes and the expected total benefit is sacrificed for the sake of ensuring equity. Similarly, for a fixed level of inequity aversion, the expected benefit tends to decrease as the decision maker becomes more conservative. We investigate these results in detail.

When the problem is treated as a deterministic problem, where each project is assumed to have a certain mean benefit ($\Gamma = 0$), we can observe that changing the inequity threshold G may affect the benefit distributions sharply in Figure 4.4. When $G = 0.1$, the benefit distribution vector is [11.72, 19.07, 19.18] for categories 1,2 and 3 respectively. This distribution leads to a Gini coefficient value of 0.10 and a total expected benefit of 49.97. When $G = 1$, the benefit distribution vector becomes [7.8, 14.59, 36.93] with a realized Gini coefficient value of 0.33. Although the total benefit increases to 59.32, we can observe the increasing differences between category benefits compared to the previous case.

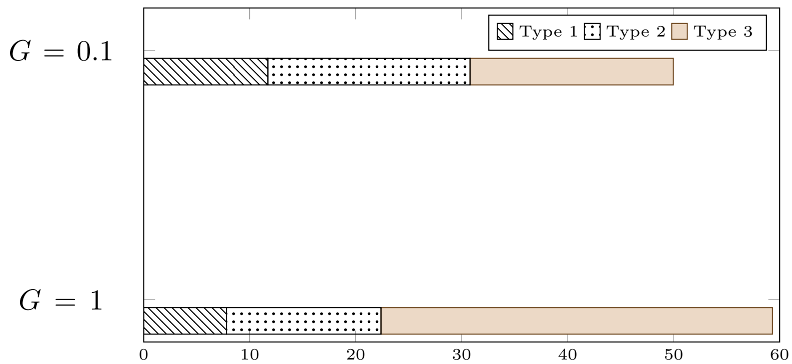


Figure 4.4: Benefit Distributions of Categories ($\Gamma = 0$)

We compare the results of ($G = 0.1$) and ($G = 0.3$) for $\Gamma = 4$ in Figure 4.5 to see the effect of inequity threshold at the same level of conservatism. Since we allow uncertainty in item benefits, we can talk about minimum and maximum benefit distribution for a specific Γ. When $G = 0.1$, the minimum benefit distribution vector is [12.48, 16.57, 16.02] and the maximum benefit distribution vector is [13.84, 18.91, 18.66]. If we increase the inequity threshold to 0.3, i.e. $G = 0.3$, the minimum benefit distribution vector is [7.13, 18.14, 29.59] and the maximum benefit distribution vector is [8.47, 20.48, 32.55]. Relaxing the threshold increases the total expected benefit from 48.24 to 58.18, but the resulting solution may lead to high inequity. For example , let us assume a specific scenario where all categories have minimum benefit levels ($v = [0, 0, 0]$) with the solution obtained when $G = 0.3$, i.e. the benefit distribution vector is [7.13, 18.14, 29.59] in this scenario. Thus, the realized Gini level becomes 0.28, which makes the solution obtained when $G = 0.3$ infeasible for $G = 0.1$.

To see the effect of different levels of conservatism (different Γ parameters) for the same level of fairness (G level), we compare the results of ($\Gamma = 4$) and ($\Gamma = 12$) for $G = 0.1$ in Figure 4.6. As expected, decreasing the conservatism parameter Γ leads to an improvement in the total expected benefit (It increases from 46.45 to 48.24). When $\Gamma = 12$, the minimum benefit distribution vector is [11.84, 14.76, 15.20] and the maximum benefit distribution vector is [14.48, 18.04, 18.58]. We can observe that the range between minimum and maximum benefit levels for each category increases. If we keep the project selection decisions of $\Gamma = 4$ the same and apply the uncertainty based on $\Gamma = 12$, the minimum benefit

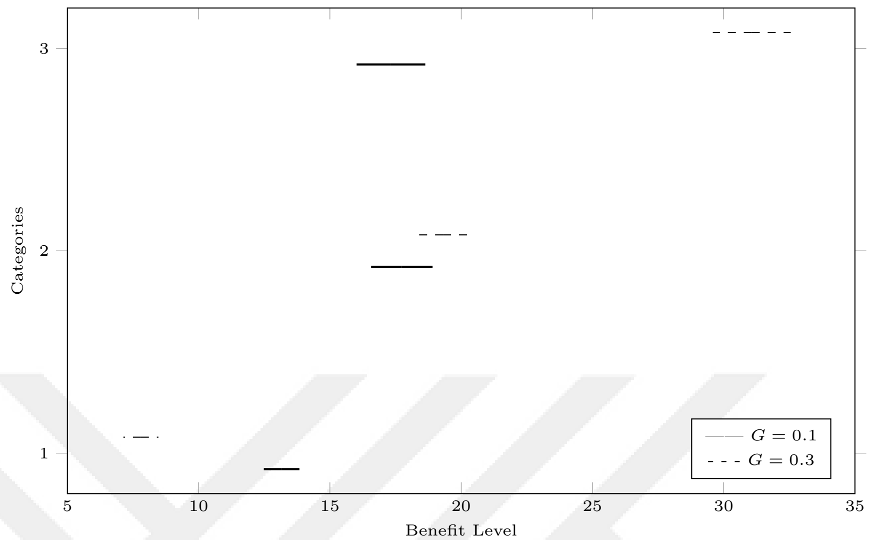


Figure 4.5: Benefit Levels of Categories ($\Gamma = 4$)

distribution vector is [11.84, 15.97, 15.61] and the maximum benefit distribution vector is [14.48, 19.51, 19.07]. In a specific scenario where categories 1 and 2 have the minimum benefit level and category 3 has the maximum benefit level (v=[0, 0, 1]), the realized Gini level is 0.103, which makes the solution obtained when $\Gamma = 4$ infeasible for $\Gamma = 12$ and $G = 0.1$.

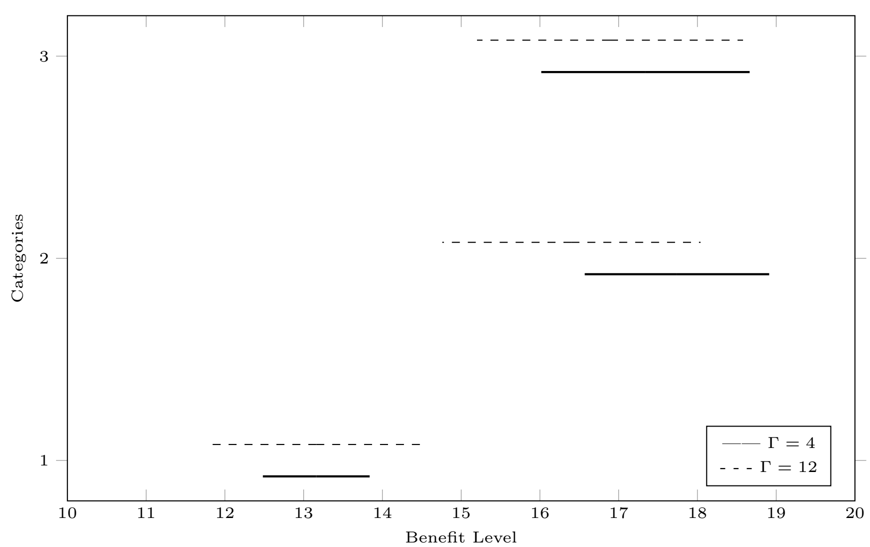


Figure 4.6: Benefit Levels of Categories ($G = 0.1$)

4.3 Shelter Allocation

We now demonstrate our methodology on a real-life motivated disaster planning problem. In this case, we consider a region that will possibly be affected by an earthquake, where a set of shelter sites is determined for the victims to gather after a disaster occurs.

Table 4.5: Shelter Allocation Problem Structure

Sets	I: set of population nodes M: subset of I having shelters J: set of shelters Ω: set of scenarios showing which shelters have maximum or minimum usage ($\mid \Omega \mid = 2^{\mid J \mid}$)
Certain Parameters	cap_j: Capacity of shelter j t_{ij}: Distance between node i and shelter j Γ: Control parameter for the robustness $v_{j\omega}$= 1 implies that $n_{j\omega} = n_j^{max}$, $v_{j\omega}$= 0 implies that $n_{j\omega} = n_j^{min}$ $a_{ij} = 1$ if shelter j is located in population node i, and $a_{ij} = 0$ otherwise G: Inequity threshold
Uncertain Parameter	$\tilde{n}_i$: Number of people affected in node i $\tilde{n}_i \in [n_i - \hat{n}_i, n_i + \hat{n}_i]$
Decision Variables	n_j^{max}: Maximum density in shelter j n_j^{min}: Minimum density in shelter j p_{ij}, λ_j: Dual variables coming from the linearization $n_{j\omega}$: Density in shelter j at scenario ω d_{kl}^{ω}: $\mid n_{k\omega} - n_{l\omega} \mid$
Binary Decision Variables	$y_{ij} = 1$ if population node i is assigned to location j, and $y_{ij} = 0$ otherwise

The assignment of population in affected areas to shelter sites is important to ensure the safety of disaster victims without wasting time. Hence, the shelter site allocation decisions are made in advance, in the preparation phase. Given the set of shelter sites, we use a mathematical model to decide which regions each one will serve so as to minimize the total weighted distance traversed by affected population groups. This problem is nontrivial as the number of people affected in each location is uncertain. Assuming that a region's demand has a symmetric probability distribution, our model ensures an equitable usage of shelters and mitigates the risk of having excessively low density in some shelters while others become relatively overcrowded.

In Table 4.5, we provide a list of the sets, parameters and decision variables used in the models discussed below.

Formulation **Shelter Allocation Nonlinear Model**

$$\min \quad \sum_{i\in I}\sum_{j\in J} n_i t_{ij} y_{ij} \tag{4.1a}$$

$$\text{s.t.} \sum_{j\in J} y_{ij} = 1 \qquad \forall i \in I \tag{4.1b}$$

$$\sum_{j\in J} a_{ij} y_{ij} = 1 \qquad \forall i \in M \tag{4.1c}$$

$$n_j^{max} = \frac{\sum_{i\in I} n_i y_{ij}}{cap_j} + \frac{\max_{H\subseteq I,|H|=\Gamma} \sum_{i\in H} \hat{n}_i y_{ij}}{cap_j} \qquad \forall j \in J \tag{4.1d}$$

$$n_j^{min} = \frac{\sum_{i\in I} n_i y_{ij}}{cap_j} - \frac{\max_{H\subseteq I,|H|=\Gamma} \sum_{i\in H} \hat{n}_i y_{ij}}{cap_j} \qquad \forall j \in J \tag{4.1e}$$

$$n_j^{max} \leq 1 \qquad \forall j \in J \tag{4.1f}$$

$$n_{j\omega} = v_{j\omega} n_j^{max} + (1 - v_{j\omega}) n_j^{min} \qquad \forall j \in J, \forall \omega \in \Omega \tag{4.1g}$$

$$|n_{k\omega} - n_{l\omega}| \leq d_{kl}^{\omega} \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \tag{4.1h}$$

$$\sum_{k\in J}\sum_{l>k\in J} d_{kl}^{\omega} \leq G|J| \sum_{j\in J} n_{j\omega} \qquad \forall \omega \in \Omega \tag{4.1i}$$

$$n_{j\omega} \geq 0 \qquad \forall j \in J, \forall \omega \in \Omega \tag{4.1j}$$

$$n_j^{max}, n_j^{min} \geq 0 \qquad \forall j \in J \tag{4.1k}$$

$$d_{kl}^{\omega} \geq 0 \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \tag{4.1l}$$

$$y_{ij} \in \{0, 1\} \qquad \forall i \in I, \forall j \in J \tag{4.1m}$$

Formulation **Shelter Allocation Linear Model**

$$\min \quad (4.1a)$$

$$\text{s.t.}(4.1b) - (4.1c), (4.1f) - (4.1g), (4.1i) - (4.1m)$$

$$n_j^{max} = \frac{\sum_{i \in I} n_i y_{ij}}{cap_j} + \sum_{i \in I} p_{ij} + \Gamma \lambda_j \qquad \forall j \in J \qquad (4.2a)$$

$$n_j^{min} = \frac{\sum_{i \in I} n_i y_{ij}}{cap_j} - \sum_{i \in I} p_{ij} - \Gamma \lambda_j \qquad \forall j \in J \qquad (4.2b)$$

$$\lambda_j + p_{ij} \geq \frac{\hat{n}_i y_{ij}}{cap_j} \qquad \forall i \in I, \forall j \in J \qquad (4.2c)$$

$$n_{k\omega} - n_{l\omega} \leq d_{kl}^{\omega} \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \qquad (4.2d)$$

$$n_{l\omega} - n_{k\omega} \leq d_{kl}^{\omega} \qquad \forall k \in J, \forall l > k \in J, \forall \omega \in \Omega \qquad (4.2e)$$

$$\lambda_j \geq 0 \qquad \forall j \in J \qquad (4.2f)$$

$$p_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J \qquad (4.2g)$$

In the objective function (4.1a), we minimize total weighted distance traversed by populations in each sub-district. (4.1b) ensures that each sub-district is assigned to exactly one shelter. In (4.1c), we guarantee that if a sub-district contains shelters, this sub-district is assigned to one of these shelters. In (4.1d)-(4.1e), we define the possible maximum and minimum shelter usage rates for each shelter j, respectively. (4.1f) prevents over-usage for each shelter. (4.1g) determines whether the utilization rate occurs at maximum or minimum level in a scenario. We define absolute pairwise differences of usage rates with (4.1h).(4.1i) keeps the Gini level of utilization rates under the threshold value G. (4.2a)-(4.2c) come from the linearization of (4.1d)-(4.1e) based on [27]. (4.2d)-(4.2e) come from the linearization of (4.1h). (4.1j)-(4.1l) and (4.2f)-(4.2g) are non-negativity constraints and (4.1m) defines binary decision variables.

In this case, we have focused on Kartal, which is a district of Istanbul. In our problem, we identified five candidate shelter locations based on the data shared by Kılcı et. al. [46]. We assigned capacities to the potential locations by dividing the 45 m^2 short of available area (given in [47]) by 3.5, as the Turkish Red Crescent assigns 3.5 m^2 of living space to each resident and 45 m^2 for group sanitary and dining facilities[46]. We use the data set of Kılcı et. al. [48] for the distance matrix between shelters and sub-districts.

In Kartal, there are 20 sub-districts. In our model, we assign these sub-districts to the candidate shelter locations. Ünal [47] and Kılcı et. al. [46] assumed that approximately 12.5% of the population would need to stay in shelter areas after an earthquake. Since the number of people affected by the earthquake is an uncertain parameter in our problem, we assume between 7.5 and 12.5% of the population would need to stay in shelter areas after an earthquake with a symmetric probability distribution whose mean value is 10%.

Table 4.6: Total Traversed Weighted Distance (people.kms)

Γ / G	0.05	0.1	0.15	0.20	0.25	0.30	1
0	287775	274765	274529	271051	271051	271051	271051
1	531309	287775	274765	274529	273478	271051	271051
2	Inf.	358729	284760	274765	274529	274529	274529
3	Inf.	Inf.	292510	274765	274529	274529	274529
4	Inf.	Inf.	298209	274765	274765	274529	274529
5	Inf.	Inf.	298209	274765	274765	274529	274529
6	Inf.	Inf.	298209	277192	274765	274529	274529
7	Inf.	Inf.	304389	277192	274765	274529	274529
8	Inf.	Inf.	304389	277192	274765	274529	274529

Since the Gini coefficient allows a structured parametric analysis between levels 0 and 1, we provide detailed results for the Gini based model in Table 4.6 and Figure 4.7. Table 4.7 and Table 4.8 list the solution times in central processing unit (CPU) seconds. Since for $\Gamma \in \{1, 2, ..., 19\}$, the minimum number of scenarios in S is 3200000, which results with the scenario pairs of $|S \times \Omega| = 3200000 \times 32$, we can only show the solution times for $\Gamma \in \{0, 20\}$ in FSBM. However, we only depend on 32 scenarios in Ω whatever Γ takes if we solve the Model 3.

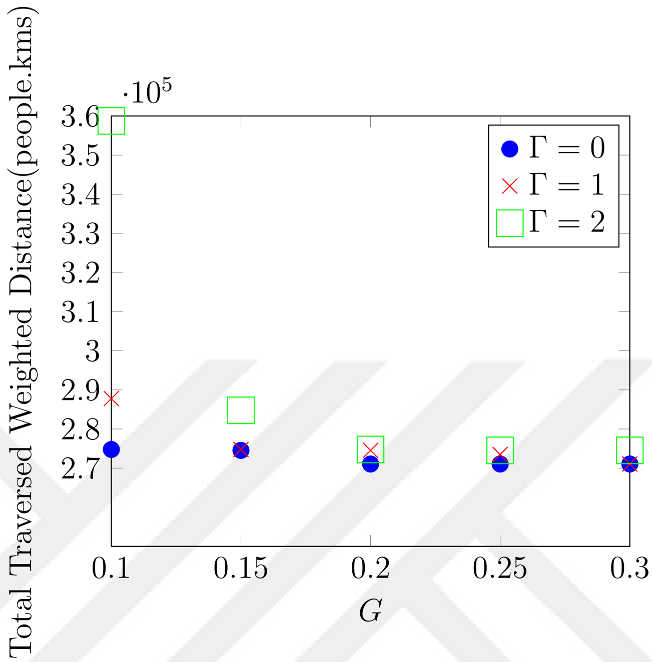


Figure 4.7: Total Traversed Weighted Distance

Table 4.7: Solution Times for Model 3

Γ / G	0.05	0.1	0.15	0.20	0.25	0.30	1
0	0.631	0.182	0.141	0.116	0.074	0.109	0.183
1	10.406	0.297	0.146	0.149	0.135	0.083	0.237
2	27.118	6.429	0.295	0.217	0.134	0.13	0.104
3	189.306	179.421	0.474	0.115	0.113	0.112	0.217
4	842.529	330.907	0.72	0.132	0.15	0.204	0.104
5	599.899	625.709	0.698	0.163	0.223	0.195	0.104
6	579.420	335.562	0.987	0.184	0.193	0.194	0.099
7	359.495	208.290	2.08	0.191	0.167	0.122	0.117
8	341.466	245.358	1.286	0.191	0.215	0.192	0.096

Table 4.8: Solution Times for FSBM

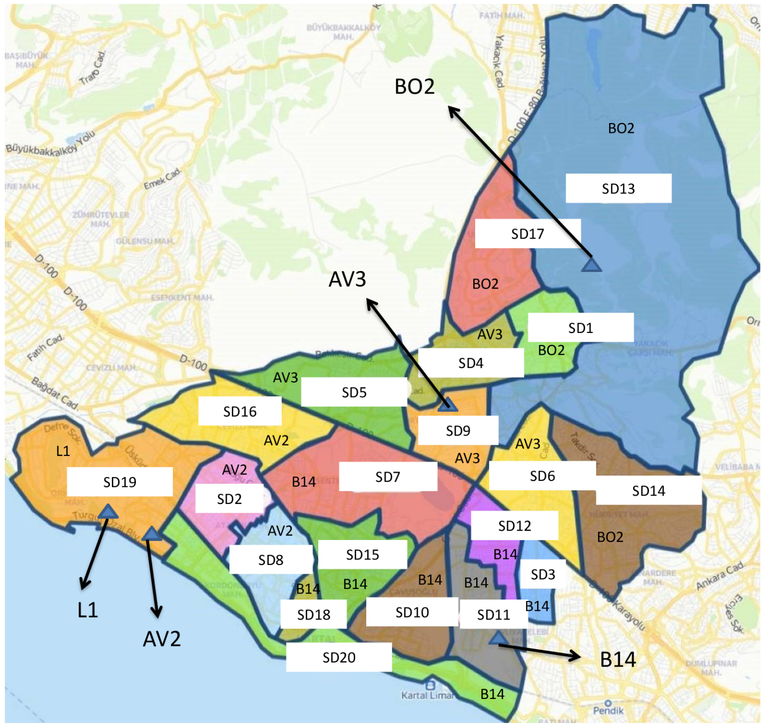
Γ / G	0.05	0.1	0.15	0.20	0.25	0.30	1
0	0.205	0.105	0.119	0.102	0.078	0.081	0.071
20	0.16	0.17	0.277	0.164	0.103	0.109	0.097

Table 4.9: Sub-district Assignments Based on Shelters ($\Gamma = 0, G = 0.1$ and $G = 1$)

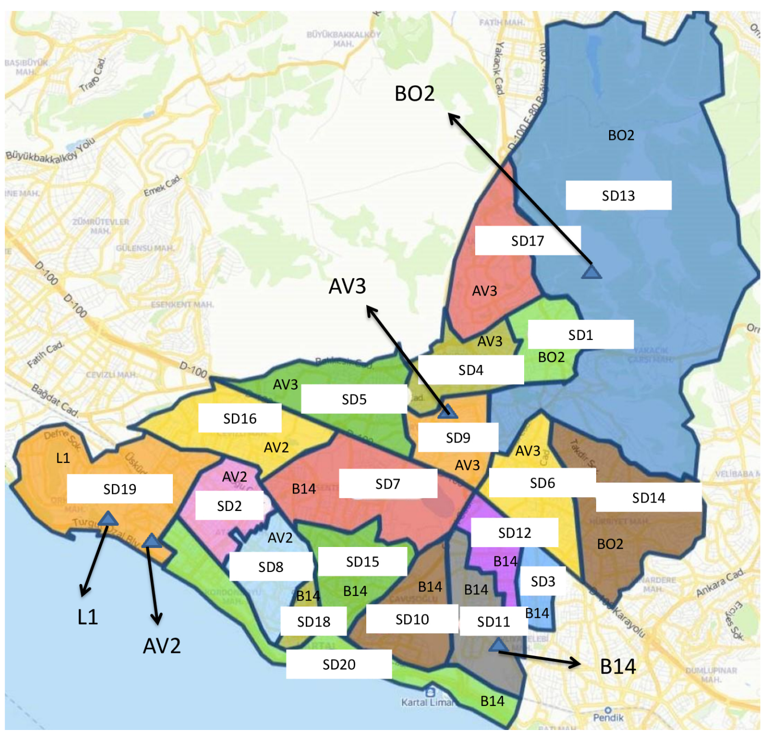
Shelters	$G = 0.1$	$G = 1$
L1	{SD19}	{SD19}
AV2	{SD2,SD8,SD16}	{SD2,SD8,SD16}
B14	{SD3,SD7,SD10,SD11,SD12,SD15,SD18,SD20}	{SD3,SD7,SD10,SD11,SD12,SD15,SD18,SD20}
AV3	{SD4,SD5,SD6,SD9}	{SD4,SD5,SD6,SD9,**SD17**}
BO2	{SD1,SD13,SD14,**SD17**}	{SD1,SD13,SD14}

We provide details on the assignment decisions for ($\Gamma = 0$) in Tables 4.9 and 4.10 and Figure 4.8 for two levels of G. We can observe that changing the inequity threshold G affects the objective value and assignment decisions slightly. When there is no concern for equity in shelter utilization, i.e. when $G = 1$, all sub-districts are assigned to their closest shelters. When the threshold is tightened to $G = 0.1$, the only difference in the assignment decisions is in the assignment of SD17, which leads to a difference of only 3714 people.kms in the total weighted traversed distance.

Figure 4.9 shows that a distinctive change occurs in the utilizations of AV3 and BO2 with the displacement of SD17 since SD17 has the second largest population. When the threshold is 0.1, the utilization vector is [0.44, 0.61, 0.43, 0.60, 0.51] for shelters L1, AV2, B14, AV3, BO2 respectively, which leads to a Gini coefficient value of 0.08. If the equity constraint is relaxed ($G = 1$), the new utilization vector becomes [0.44, 0.61, 0.43, 0.89, 0.33], leading to a Gini level of 0.19. The results imply that when there is no uncertainty, as long as the capacity allows, assigning districts to their closest shelter already leads to a good level of utilization equality among shelters in this example.



(a) $G = 0.1$



(b) $G = 1$

Figure 4.8: $\Gamma = 0$

Table 4.10: Sub-district Assignments Based on Sub-districts ($\Gamma = 0, G = 0.1$ and $G = 1$)

Subdistrict /Shelter	L1	AV2	B14	AV3	BO2	L1	AV2	B14	AV3	BO2
SD1					✓					✓
SD2		✓					✓			
SD3			✓					✓		
SD4				✓					✓	
SD5				✓					✓	
SD6				✓					✓	
SD7			✓					✓		
SD8		✓					✓			
SD9				✓					✓	
SD10			✓					✓		
SD11			✓					✓		
SD12			✓					✓		
SD13					✓					✓
SD14					✓					✓
SD15			✓					✓		
SD16		✓					✓			
SD17					✓				✓	
SD18			✓					✓		
SD19	✓					✓				
SD20			✓					✓		

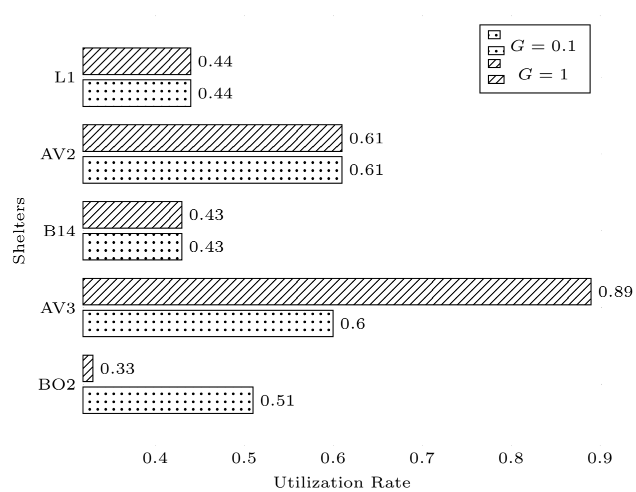
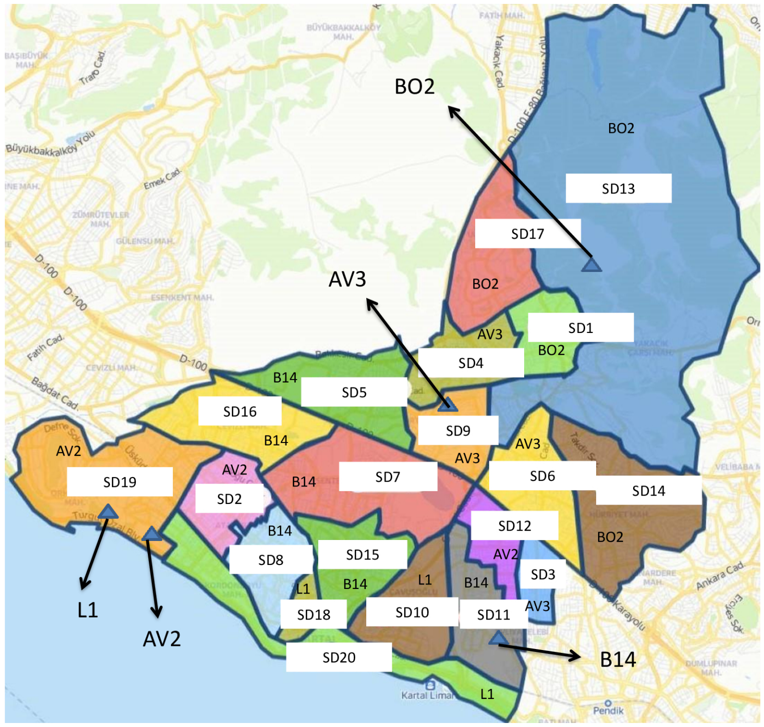
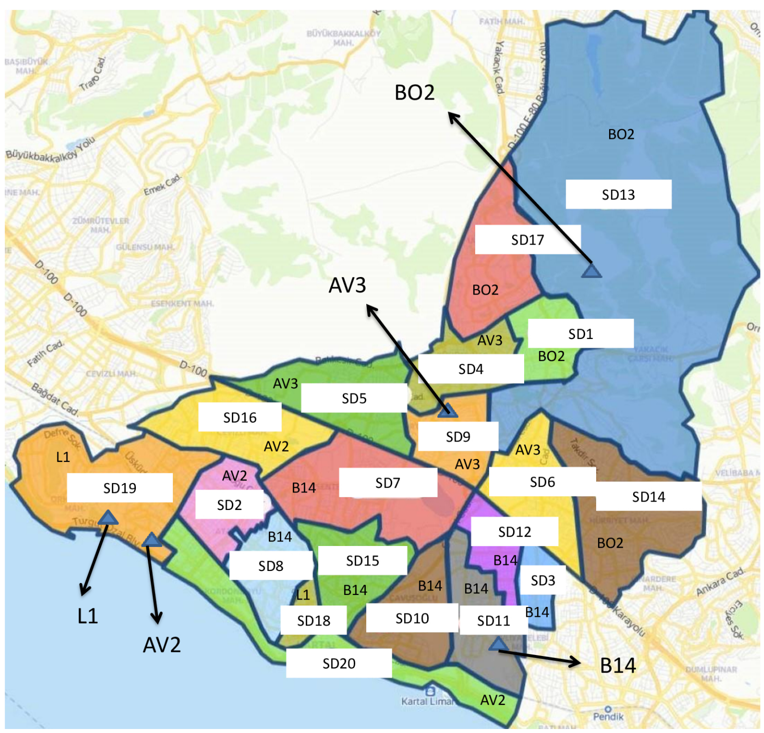


Figure 4.9: Utilizations of Shelters ($\Gamma = 0$)



(a) $G = 0.1$



(b) $G = 0.15$

Figure 4.10: $\Gamma = 2$

Table 4.11: Sub-district Assignments Based on Shelters ($\Gamma = 2, G = 0.1$ and $G = 0.15$)

Shelters	$G = 0.1$	$G = 0.15$
L1	{**SD10**,SD18,**SD20**}	{SD18,**SD19**}
AV2	{SD2,**SD12**,**SD19**}	{SD2,**SD16**,**SD20**}
B14	{**SD5**,SD7,SD8,SD11,SD15,**SD16**}	{**SD3**,SD7,SD8,**SD10**,SD11,**SD12**,SD15}
AV3	{**SD3**,SD4,SD6,SD9}	{SD4,**SD5**,SD6,SD9}
BO2	{SD1,SD13,SD14,SD17}	{SD1,SD13,SD14,SD17}

We compare the results of ($G = 0.1$) and ($G = 0.15$) for $\Gamma = 2$ to see the effect of inequity threshold at the same level of conservatism. The detailed results are in Tables 4.11 and 4.12 and Figure 4.10 . When $G = 0.1$, the solution deviates from the one that makes assignments to the closest shelters. SD10 and SD18 are assigned to shelter L1. SD12 is also assigned to shelter AV2. These assignments lead to slightly larger travel distances, an additional 30 kms, and larger weighted traversed distances, an additional 36697 people.kms compared to the assignments to the closest shelters. When $G = 0.1$, ensuring the inequity constraint under the worst scenario increases the total traversed weighted distance to 358729 people.kms and makes 10 out of 20 assignments to the closest shelter. Relaxing the inequity threshold and using $G = 0.15$ reduces this distance to 284760 people.kms and makes more assignments (16 assignments) to closest shelters. We can see the utilization ranges of shelters in Figure 4.11. When $G = 0.15$, the minimum utilization vector is [0.42, 0.39, 0.41, 0.51, 0.41] for shelters L1, AV2, B14, AV3 and BO2 respectively. The maximum utilization vector is also [0.70, 0.59, 0.51, 0.69, 0.60]. In a specific scenario where AV2 has the minimum utilization rate while other shelters have the maximum utilization rates ($v = [1, 0, 1, 1, 1]$), the realized Gini level is 0.11, which makes the solution obtained when $G = 0.15$ infeasible for $G = 0.1$. When $G = 0.1$, the minimum utilization vector is [0.40, 0.40, 0.47, 0.43, 0.41] and the maximum utilization vector is [0.58, 0.61, 0.57, 0.58, 0.60]. Although the differences between maximum and minimum utilization rates in $G = 0.1$ and $G = 0.15$ are close to each other for other shelters, the assignment for L1 changes in a way that the difference between maximum and minimum utilization rate increases sharply when $G = 0.15$ with respect to $G = 0.1$. When

Table 4.12: Sub-districts Assignments Based on Sub-districts ($\Gamma = 2, G = 0.1$ and $G = 0.15$)

Subdistrict /Shelter	L1	AV2	B14	AV3	BO2	L1	AV2	B14	AV3	BO2
SD1					✓					✓
SD2		✓					✓			
SD3				✓				✓		
SD4				✓					✓	
SD5			✓						✓	
SD6				✓					✓	
SD7			✓					✓		
SD8			✓					✓		
SD9				✓					✓	
SD10	✓							✓		
SD11			✓					✓		
SD12		✓						✓		
SD13					✓					✓
SD14					✓					✓
SD15			✓					✓		
SD16			✓				✓			
SD17					✓					✓
SD18	✓					✓				
SD19		✓				✓				
SD20	✓						✓			

$G = 0.1$, SD10, SD18 and SD20 are assigned to L1. When $G = 0.15$, SD18 is still assigned to L1, but SD19 replaces SD10 and SD20. Since the total deviation of SD10 and SD20 is 636 and the deviation of SD19 is 754, we can observe a larger utilization range for L1 when $G = 0.15$.

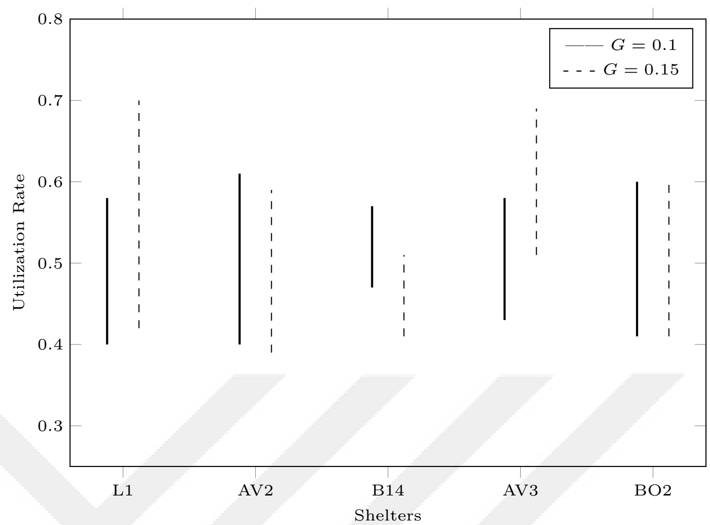


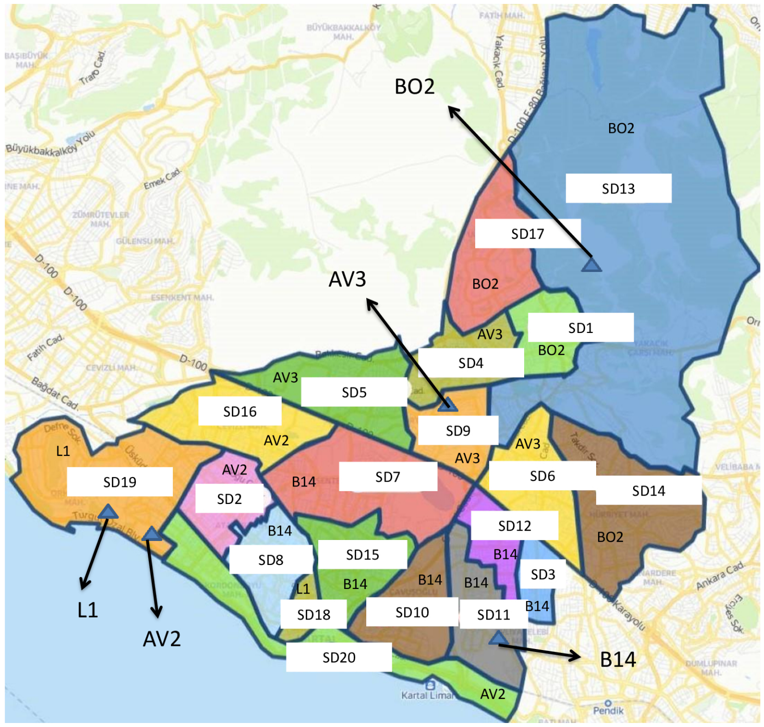
Figure 4.11: Utilization Ranges of Shelters ($\Gamma = 2$)

To see the effect of different levels of conservatism (different Γ parameters) for the same level of fairness (G level), we compare the results of ($\Gamma = 2$) and ($\Gamma = 7$) for $G = 0.15$. As expected, decreasing the robustness parameter Γ leads to an improvement in the total traversed weighted distance. (It reduces from 304389 to 284760 people.kms). The resulting assignments are also considerably different in Tables 4.13 and 4.14 and Figure 4.12.

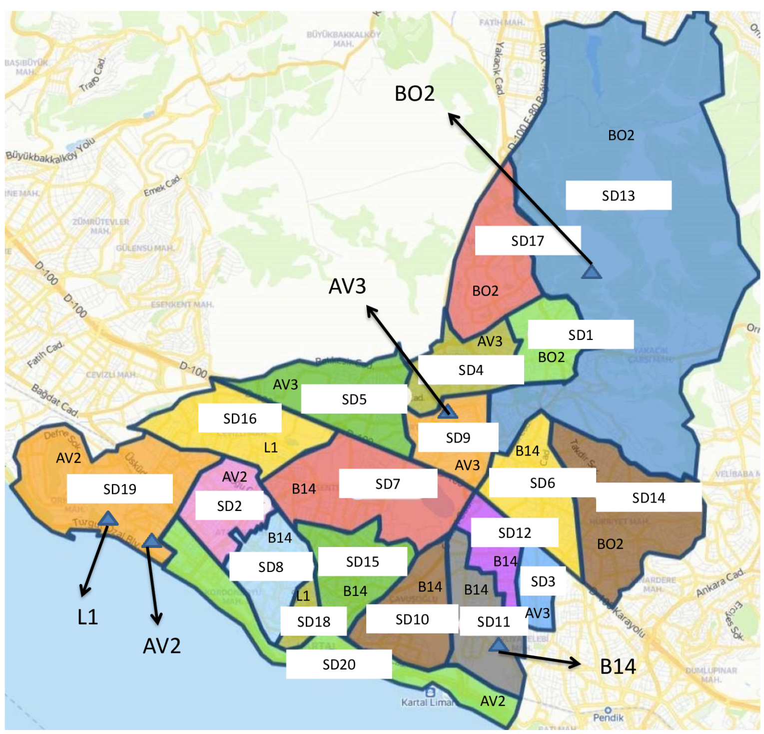
Table 4.13: Sub-district Assignments Based on Shelters ($G = 0.15$, $\Gamma = 7$ and $\Gamma = 2$)

Shelters	$\Gamma = 7$	$\Gamma = 2$
L1	{**SD16**,SD18}	{SD18,**SD19**}
AV2	{SD2,**SD19**,SD20}	{SD2,**SD16**,SD20}
B14	{**SD6**,SD7,SD8,SD10,SD11,SD12,SD15}	{**SD3**,SD7,SD8,SD10,SD11,SD12,SD15}
AV3	{**SD3**,SD4,SD5,SD9}	{SD4,SD5,**SD6**,SD9}
BO2	{SD1,SD13,SD14,SD17}	{SD1,SD13,SD14,SD17}

When $\Gamma = 7$, sub-districts SD16 and SD18 are assigned to shelter L1. When $\Gamma = 2$, SD16 is assigned to AV2, but SD18 is still assigned to L1. Additionally, $\Gamma = 2$ makes the assignments for SD3 and SD6 to the closest shelters.



(a) $\Gamma = 2$



(b) $\Gamma = 7$

Figure 4.12: $G = 0.15$

As we can see from Figure 4.13, when $\Gamma = 7$, the minimum utilization vector is [0.40, 0.38, 0.37, 0.41, 0.38] and the maximum utilization vector is [0.67, 0.63, 0.61, 0.68, 0.63]. If we keep the assignments of $\Gamma = 2$ the same and apply the uncertainty based on $\Gamma = 7$, the minimum utilization vector is [0.42, 0.37, 0.35, 0.45, 0.38] and the maximum utilization vector is [0.70, 0.62, 0.58, 0.75, 0.63]. In a specific scenario where AV2 and B14 have the minimum utilization rates whereas other shelters have the maximum utilization rates (v=[1, 0, 0, 1, 1]), the realized Gini level is 0.162, which makes the solution obtained when $\Gamma = 2$ infeasible for $\Gamma = 7$ and $G = 0.15$.

Table 4.14: Sub-districts Assignments Based on Sub-districts ($G = 0.15, \Gamma = 2$ and $\Gamma = 7$)

Subdistrict /Shelter	L1	AV2	B14	AV3	BO2	L1	AV2	B14	AV3	BO2
SD1					✓					✓
SD2		✓					✓			
SD3			✓						✓	
SD4				✓					✓	
SD5				✓					✓	
SD6				✓				✓		
SD7			✓					✓		
SD8			✓					✓		
SD9				✓					✓	
SD10			✓					✓		
SD11			✓					✓		
SD12			✓					✓		
SD13					✓					✓
SD14					✓					✓
SD15			✓					✓		
SD16		✓				✓				
SD17					✓					✓
SD18	✓					✓				
SD19	✓						✓			
SD20		✓					✓			

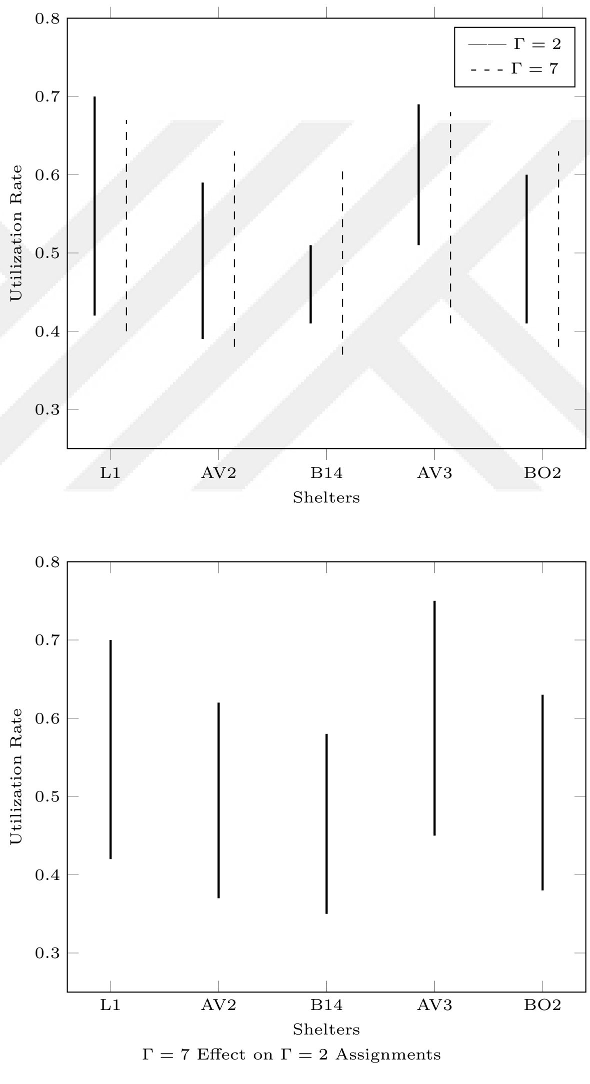


Figure 4.13: Utilization Ranges of Shelters ($G = 0.15$)

Chapter 5

Conclusion

In this work, we focus on resource allocation problems through multiple beneficiaries with fairness concern. In the problem settings, we consider a stochastic environment with uncertain system parameters such as benefit levels in project selection or the number of affected people after an earthquake, and a decision maker aiming to distribute the resource specified in the problem to the beneficiaries in an equitable manner. Due to the uncertain parameters, we propose a robust programming approach maximizing system efficiency such as the total expected benefit while guaranteeing an equitable resource allocation even under the worst scenario. Therefore, we adopt a parametric approach controlling the degree of conservatism through a parameter (Γ).

Firstly, we take this problem into consideration as a binary knapsack problem in a stochastic environment where items have uncertain benefit levels and certain costs. Since the non-linearity of the fairness related functions used in the problem precludes a direct application of the controllable robust approach, we show that the maximum inequity levels occurs at extreme allocation levels. Then, we use this result to find robustly fair solutions to knapsack problems building mixed integer linear models. We also provide the fully scenario based model for SOPD metric to show how we eliminate the complexity and excessive number of scenarios with our controllable robust approach. Then we provide the results of a parametric analysis over difference levels of uncertainty and inequity threshold for three inequity metrics. For the same threshold (resp. Γ), the total expected output is non-increasing with Γ (resp. threshold).

To show the usability of our approach, we consider a real-life based R&D project selection problem described in [1]. Our parametric analysis demonstrates that the expected total output decreases as inequity aversion increases for a fixed level of conservatism. If the decision maker becomes more conservative for a fixed level of inequity aversion, the expected benefit tends to decrease as well.

We also work on a shelter allocation problem to show that the applicability of our methodology is not limited to knapsack problems. Our model ensures an equitable usage of shelters and mitigates the risk of having excessively low density in some shelters while others become relatively overcrowded so that people can have access to a service quality that is as close to each other as in each shelter.

In the future work, one can work on problems where both costs and benefits are uncertain. Additionally, one can consider a setting in which entities have different production utility functions that convert resources to utilities.

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Chapter 6

Appendices

6.1 Proof of Remarks 1 and 2

Proof. Proof of Remarks 1 and 2

Let $b \in \mathbb{R}^n$ be an entity benefit distribution vector. Assume that b is ordered as follows: $b_1 \leq b_2 \leq ... \leq b_n$. This is without loss of generality because for any permutation $b_{\pi(1)}, ..., b_{\pi(n)}$ of b the following holds:

$$\sum_{\pi(i) \in N} \sum_{\pi(j) > \pi(i) \in N} \mid b_{\pi(i)} - b_{\pi(j)} \mid = \sum_{i \in N} \sum_{j > i \in N} \mid b_i - b_j \mid$$

, where $N = \{1, 2, ..., n\}$.

We will first show that the sum of distances between all n points is a weighted sum of the differences between specific pairs in the ordered distribution b as follows:

$$\sum_{i \in N} \sum_{j > i \in N} \mid b_i - b_j \mid = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (n - 2k + 1) \mid b_{n-k+1} - b_k \mid \tag{6.1}$$

We will prove this by induction:

Equation 6.1 trivially holds for $n = 1$.

Assume it holds for an arbitrary value $n = m$.

$$\sum_{i\in N}\sum_{j>i\in N} \mid b_i - b_j \mid = \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} (m - 2k + 1) \mid b_{m-k+1} - b_k \mid$$

, where $N = \{1, 2, ..., m\}$.

We will show that it also holds for $n = m + 1$. Let $N = \{1, 2, ..., m\}$. By equation , we can re-write the sum of pairwise differences as follows:

$$\sum_{i\in N}\sum_{j>i\in N} \mid b_i - b_j \mid + \sum_{k=1}^{m} \mid b_{m+1} - b_k \mid = \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} (m-2k+1) \mid b_{m-k+1} - b_k \mid + \sum_{k=1}^{m} \mid b_{m+1} - b_k \mid$$

We will show the following:

$$\sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} (m-2k+1) \mid b_{m-k+1} - b_k \mid + \sum_{k=1}^{m} \mid b_{m+1} - b_k \mid = \sum_{k=1}^{\lfloor \frac{m+2}{2} \rfloor} (m-2k+2) \mid b_{m-k+2} - b_k \mid \tag{6.2}$$

Let A be difference between the left hand side and the right hand side of Equation 6.2.

Case 1: m is even. Then A becomes:

$$\sum_{k=1}^{\frac{m}{2}} (m - 2k + 1)(b_{m-k+1} - b_k) + \sum_{k=1}^{m} (b_{m+1} - b_k) - \sum_{k=1}^{\frac{m}{2}+1} (m - 2k + 2)(b_{m-k+2} - b_k)$$

Let $l = k - 1$. Note that :

$$\sum_{k=1}^{\frac{m}{2}+1} (m - 2k + 2)(b_{m-k+2} - b_k) = \sum_{l=0}^{\frac{m}{2}} (m - 2l) b_{m-l+1} - \sum_{k=1}^{\frac{m}{2}+1} (m - 2k + 2) b_k$$

$$= \sum_{l=1}^{\frac{m}{2}} (m - 2l) b_{m-l+1} + m b_{m+1} - \sum_{k=1}^{\frac{m}{2}} (m - 2k + 2) b_k$$

Then

$$A = \sum_{k=1}^{\frac{m}{2}}(m-2k)b_{m-k+1} + \sum_{k=1}^{\frac{m}{2}} b_{m-k+1} - \sum_{k=1}^{\frac{m}{2}}(m-2k+1)b_k + mb_{m+1}$$
$$-\sum_{k=1}^{m} b_k - \sum_{l=1}^{\frac{m}{2}}(m-2l)b_{m-l+1} - mb_{m+1} + \sum_{k=1}^{\frac{m}{2}}(m-2k+2)b_k$$

Since $\sum_{k=1}^{\frac{m}{2}} b_{m-k+1} = \sum_{k=\frac{m}{2}+1}^{m} b_k$, all terms cancel out, hence the difference, A, is 0.

Case 2: m is odd. Then A becomes:

$$\sum_{k=1}^{\frac{m+1}{2}}(m-2k+1)(b_{m-k+1}-b_k) + \sum_{k=1}^{m}(b_{m+1}-b_k) - \sum_{k=1}^{\frac{m+1}{2}}(m-2k+2)(b_{m-k+2}-b_k)$$

Let $l = k-1$. Note that :

$$\sum_{k=1}^{\frac{m+1}{2}}(m-2k+2)(b_{m-k+2}-b_k) = \sum_{l=0}^{\frac{m-1}{2}}(m-2l)b_{m-l+1} - \sum_{k=1}^{\frac{m+1}{2}}(m-2k+2)b_k$$
$$= \sum_{l=1}^{\frac{m-1}{2}}(m-2l)b_{m-l+1} + mb_{m+1} - \sum_{k=1}^{\frac{m-1}{2}}(m-2k+2)b_k - b_{\frac{m+1}{2}}$$

Then

$$A = \sum_{k=1}^{\frac{m-1}{2}}(m-2k)b_{m-k+1} + \sum_{k=1}^{\frac{m-1}{2}} b_{m-k+1} - \sum_{k=1}^{\frac{m-1}{2}}(m-2k+1)b_k + mb_{m+1}$$
$$-\sum_{k=1}^{m} b_k - \sum_{l=1}^{\frac{m-1}{2}}(m-2l)b_{m-l+1} - mb_{m+1} + \sum_{k=1}^{\frac{m-1}{2}}(m-2k+2)b_k + b_{\frac{m+1}{2}}$$

Since $\sum_{k=1}^{\frac{m-1}{2}} b_{m-k+1} = \sum_{k=\frac{m+3}{2}}^{m} b_k$, all terms cancel out, hence the difference, A, is 0.

Hence we showed that

$$\sum_{i\in N}\sum_{j>i\in N} \mid b_i - b_j \mid = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (n-2k+1) \mid b_{n-k+1} - b_k \mid$$

Thus, we can write the sum of absolute pairwise differences among entity benefits as the linear combination of line segments between the first $\frac{n-1}{2}$ points and the last $\frac{n-1}{2}$ points in the following form if n is odd:

$$(n-2+1)|b_n - b_1| + (n-4+1)|b_{n-1} - b_2| + ... + (n-2(\frac{n+1}{2}-1)+1 = 2)|b_{\frac{n+3}{2}} - b_{\frac{n-1}{2}}|$$

If n is even, $\lfloor \frac{n+1}{2} \rfloor = \frac{n}{2}$. The summation above can be written as the linear combination of line segments between the first $\frac{n}{2}$ points and the last $\frac{n}{2}$ points in the following form:

$$(n-2+1)|b_n - b_1| + (n-4+1)|b_{n-1} - b_2| + ... + (n-2\frac{n}{2}+1 = 1)|b_{\frac{n}{2}+1} - b_{\frac{n}{2}}|$$

The first $\frac{n-1}{2}$ elements should be at their minimum possible value and the last $\frac{n-1}{2}$ elements should be at their maximum possible value to maximize this summation if n is odd. Similarly, first $\frac{n}{2}$ elements should be at their minimum possible value and the last $\frac{n}{2}$ elements should be at their minimum possible value to maximize this summation if n is even.

□

6.2 Proof of Proposition 1

Proof. Proof of Proposition 1. Define a protection function for each entity j, given a solution $x^* \in X$ to Model 2 (resp. nonlinear Gini model) and parameter Γ_j as follows: $\beta(x^*, \Gamma_j) = \max_{H\subseteq I, |H|=\Gamma_j} \sum_{i\in H} \hat{o}_i^j x_i^*$. Note that, we take $\Gamma_j = \Gamma$, $\forall j \in J$ in this work without loss of generality. The value of the protection function is equal to the optimal value of the following linear programming problem (see [27], Proposition 1):

$$\text{Formulation } \textbf{Primal } (j) \tag{6.3}$$

$$\beta(x^*, \Gamma) = \max \quad \sum_{i \in I} \hat{o}_i^j x_i^* z_{ij} \tag{6.3a}$$

$$\text{s.t.} \sum_{i \in I} z_{ij} \leq \Gamma \qquad (\lambda_j) \tag{6.3b}$$

$$0 \leq z_{ij} \leq 1 \qquad \forall i \in I \quad (p_{ij}) \tag{6.3c}$$

The optimal solution value of the Primal problem consists of Γ variables at 1. This is equivalent to the selection of subset $H \subseteq I, \mid H \mid = \Gamma$ with corresponding profit function $\sum_{i \in H} \hat{o}_i^j x_i^*$.

Then, we consider the dual of this problem:

$$\text{Formulation } \textbf{Dual } (j) \tag{6.4}$$

$$\min \quad \sum_{i \in I} p_{ij} + \Gamma \lambda_j \tag{6.4a}$$

$$\text{s.t.} \lambda_j + p_{ij} \geq \hat{o}_i^j x_i^* \qquad \forall i \in I \tag{6.4b}$$

$$p_{ij} \geq 0 \qquad \forall i \in I \tag{6.4c}$$

$$\lambda_j \geq 0 \tag{6.4d}$$

For any j, we have

$$\max_{H \subseteq I, |H| = \Gamma} \sum_{i \in H} \hat{o}_i^j x_i^* = \sum_{i \in I} \hat{o}_i^j x_i^* z_{ij} \leq \sum_{i \in I} p'_{ij} + \Gamma \lambda'_j \tag{6.5}$$

, for any $x^* \in X$ and any feasible solution to the dual problem (p', λ') by weak duality.

Take (3.3c) and (3.3d)

$$o_j^{max} = \sum_{i \in I} o_i^j x_i + \max_{H \subseteq I, |H| = \Gamma} \sum_{i \in H} \hat{o}_i^j x_i, \forall j \in J(3.3c)$$

$$o_j^{min} = \sum_{i \in I} o_i^j x_i - \max_{H \subseteq I, |H| = \Gamma} \sum_{i \in H} \hat{o}_i^j x_i, \forall j \in J (3.3d)$$

and (3.4c) and (3.4d)

$$\tilde{o}_j^{max} = \sum_{i \in I} o_i^j x_i + \sum_{i \in I} p_{ij} + \Gamma \lambda_j, \forall j \in J (3.4c)$$

$$\tilde{o}_j^{min} = \sum_{i \in I} o_i^j x_i - \sum_{i \in I} p_{ij} - \Gamma \lambda_j, \forall j \in J (3.4d)$$

with new notations as above. Now, we can say that

$$\tilde{o}_j^{max} \geq o_j^{max}, \tilde{o}_j^{min} \leq o_j^{min}, \forall j \in J$$

Since $[o_j^{min}, o_j^{max}] \subseteq [\tilde{o}_j^{min}, \tilde{o}_j^{max}], \forall j \in J$, by Remark 1

$$\sum_{k \in J} \sum_{l > k \in J} d_{kl}^{\omega'} \leq \sum_{k \in J} \sum_{l > k \in J} \tilde{d}_{kl}^{\omega''} \leq G$$

and by Remark 3

$$\frac{\sum_{k \in J} \sum_{l > k \in J} d_{kl}^{\omega'}}{|J| \sum_{j \in J} o_{j\omega}} \leq \frac{\sum_{k \in J} \sum_{l > k \in J} \tilde{d}_{kl}^{\omega''}}{|J| \sum_{j \in J} \tilde{o}_{j\omega}} \leq G$$

, where ω' and ω'' are the scenarios leading to the maximum sum of absolute pairwise differences (resp. Gini level) in Models 2 (resp. nonlinear Gini model) and 3 (resp. linear Gini model), respectively.

This shows that an optimal solution (x^*, p^*, λ^*) to Model 3 (resp. linear Gini model), which is also naturally a feasible solution to Model 3 (resp. linear Gini model), corresponds to a feasible solution (x^*) to Model 2 (resp. nonlinear Gini model) (The other variables are auxiliary variables, hence we only mention the main variables). To see why x^* is optimal for Model 2 (resp. nonlinear Gini model), assume there exists x' such that x' is feasible for Model 2 (resp. nonlinear Gini model) and $o^T x' > o^T x^*$. From the protection model and the dual of it, which are defined for x', one can obtain optimal dual variables p' and λ' for which constraint (3.4i) (resp. constraint (3.5a)) is guaranteed to hold since if primal is feasible and bounded then dual is feasible and bounded (by strong duality), but such a solution (x', p', λ') would also be feasible for Model 3(resp. linear Gini model), contradicting (x^*, p^*, λ^*)'s optimality to Model 3 (resp. linear Gini model).

□

6.3 Proof of Remark 3

Proof. Proof of Remark 3. Let $b \in \mathbb{R}^n_+$ be an ordered permutation of a realized entity benefit distribution vector (This is, again, without loss of generality). We will first consider the cases with two entities ($n = 2$) and then the ones where $n > 2$.

6.3.1 For $n = 2$:

First of all, we consider a benefit distribution (b_1,b_2) where $b_1 \leq b_2$, and $n = 2$. Then, $G(b) = \frac{b_2 - b_1}{2(b_1 + b_2)}$. We assume that b_1 corresponds to a benefit level which is neither the maximum nor the minimum possible benefit for an entity in cases i,ii,and iii.

Case i: Let b' be the distribution obtained by decreasing b_1 in b: $b'_1 = b_1 - \epsilon$ and $b'_2 = b_2$ for some $0 \leq \epsilon \leq b_1$. Then, $G(b') = \frac{b_2 - b_1 + \epsilon}{2(b_1 + b_2 - \epsilon)}$. We can clearly observe that $G(b') \geq G(b)$.

Case ii: Let b' be the distribution obtained by increasing b_1 in b: $b'_1 = b_1 + \epsilon$ and $b'_2 = b_2$ for some $0 \le \epsilon \le b_2 - b_1$. Then, $G(b') = \frac{b_2 - b_1 - \epsilon}{2(b_1+b_2+\epsilon)}$. We can clearly observe that $G(b) \ge G(b')$.

According to case i and ii, we can say that decreasing the benefit level b_1 without changing its order increases the Gini coefficient.

Case iii: If $b_1^2 - b_2^2 + \epsilon b_1 \ge 0$, then increasing the benefit level b_1 increases the Gini coefficient by changing the benefit order as shown below: Let b' be the distribution obtained by increasing b_1 in b: $b'_2 = b_1 + \epsilon$ and $b'_1 = b_2$ for some $\epsilon \ge b_2 - b_1$. Then, $G(b') = \frac{b_1 + \epsilon - b_2}{2(b_1+b_2+\epsilon)}$.

$$
\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{b_1 + \epsilon - b_2}{2(b_1 + b_2 + \epsilon)} - \frac{b_2 - b_1}{2(b_1 + b_2)} \\
&= \frac{b_1^2 - b_2^2 + \epsilon b_1}{(b_1 + b_2)(b_1 + b_2 + \epsilon)} \ge 0, \quad (\text{By assumption } b_1^2 - b_2^2 + \epsilon b_1 \ge 0) \\
H'(\epsilon) &= \frac{b_2}{(b_1 + b_2 + \epsilon)^2} \ge 0
\end{aligned}
\tag{6.6}
$$

Since $H(\epsilon)$ is a non-decreasing and non-negative function, $G(b') \nless G(b)$. If $b_1^2 - b_2^2 + \epsilon b_1 < 0$, then increasing the benefit level b_1 by ϵ unit decreases the Gini coefficient with a change of benefit order. However, Case i still leads to increasing the Gini coefficient if we decrease the benefit level b_1 by ϵ unit. Showing that it, maximum Gini coefficient cannot occur in intermediate levels of b_1.

Now, we assume that b_2 corresponds to a benefit level which is neither the maximum nor the minimum possible benefit for an entity in cases iv,v,and vi.

Case iv: Let b' be the distribution obtained by increasing b_2 in b: $b'_2 = b_2 + \epsilon$ and $b'_1 = b_1$ for some $\epsilon \ge 0$. Then, $G(b') = \frac{b_2 - b_1 + \epsilon}{2(b_1+b_2+\epsilon)}$.

$$
\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{b_2 + \epsilon - b_1}{2(b_1 + b_2 + \epsilon)} - \frac{b_2 - b_1}{2(b_1 + b_2)} \\
&= \frac{\epsilon b_1}{(b_1 + b_2)(b_1 + b_2 + \epsilon)} \geq 0 \\
H'(\epsilon) &= \frac{(b_1 b_2 + b_1^2)}{(b_1 + b_2)(b_1 + b_2 + \epsilon)^2} \geq 0
\end{aligned}
\tag{6.7}
$$

Since $H(\epsilon)$ is a non-decreasing and non-negative function, $G(b') \not< G(b)$.

Case v: Let b' be the distribution obtained by decreasing b_2 in b: $b'_2 = b_2 - \epsilon$ and $b'_1 = b_1$ for some $0 \leq \epsilon \leq b_2 - b_1$. Then, $G(b') = \frac{b_2 - b_1 - \epsilon}{2(b_1 + b_2 - \epsilon)}$.

$$
\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{b_2 - \epsilon - b_1}{2(b_1 + b_2 - \epsilon)} - \frac{b_2 - b_1}{2(b_1 + b_2)} \\
&= \frac{-\epsilon b_1}{(b_1 + b_2)(b_1 + b_2 - \epsilon)} \leq 0
\end{aligned}
\tag{6.8}
$$

According to case iv and v, we can say that increasing the benefit level b_2 without changing its order increases the Gini coefficient.

Case vi: If $b_1^2 - b_2^2 + \epsilon b_2 \geq 0$, then decreasing the benefit level b_2 increases the Gini coefficient by changing the benefit order as shown below: Let b' be the distribution obtained by decreasing b_2 in b: $b'_1 = b_2 - \epsilon$ and $b'_2 = b_1$ for some $b_2 - b_1 \leq \epsilon \leq b_2$. Then, $G(b') = \frac{b_1 + \epsilon - b_2}{2(b_1 + b_2 - \epsilon)}$.

$$
\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{b_1 + \epsilon - b_2}{2(b_1 + b_2 - \epsilon)} - \frac{b_2 - b_1}{2(b_1 + b_2)} \\
&= \frac{b_1^2 - b_2^2 + \epsilon b_2}{(b_1 + b_2)(b_1 + b_2 - \epsilon)} \geq 0, \quad \text{(By assumption } b_1^2 - b_2^2 + \epsilon b_2 \geq 0\text{)} \\
H'(\epsilon) &= \frac{b_1}{(b_1 + b_2 - \epsilon)^2} \geq 0
\end{aligned}
\tag{6.9}
$$

Since $H(\epsilon)$ is a non-decreasing and non-negative function, $G(b') \nless G(b)$. If $b_1^2 - b_2^2 + \epsilon b_2 < 0$, then decreasing the benefit level b_2 by ϵ unit decreases the Gini coefficient with a change of benefit order. However, Case iv still leads to increasing the Gini coefficient if we increase the benefit level b_2 by ϵ unit. Showing that it, maximum Gini coefficient cannot occur in intermediate levels of b_2.

6.3.2 For $n \geq 3$:

From now on, we focus on $n \geq 3$ for all cases below. Let the sum of pairwise differences for b be $D = \sum_{i \in N} \sum_{j > i \in N} \mid b_i - b_j \mid$ and let $T = \sum_{i \in N} b_i, G(b) = \frac{D}{nT}$. To the contrary assume that for an arbitrary k, b_k corresponds to a benefit level which is neither the maximum nor the minimum possible benefit. Note that the rank of b_k may or may not change with any increase or decrease. We will first provide proofs for the cases where the rank does not change and then the ones for the cases where the rank changes.

Rank order stays the same with change

Two cases are possible here:

Case 1: If $k \leq \lfloor (n+1)/2 \rfloor$, then decreasing the benefit level increases the Gini coefficient as shown below: Let b' be the distribution obtained by decreasing kth element in b: $b'_i = b_i \forall i \neq k$ and $b'_k = b_k - \epsilon$ for some $0 \leq \epsilon \leq b_k - b_{k-1}$. Then $G(b') = \frac{D+(n-2k+1)\epsilon}{n(T-\epsilon)}$.

$$\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{D + (n-2k+1)\epsilon}{n(T-\epsilon)} - \frac{D}{nT} \\
&= \frac{DT + T\epsilon(n-2k+1) - DT + D\epsilon}{nT(T-\epsilon)} \\
&= \frac{D\epsilon + T\epsilon(n-2k+1)}{nT(T-\epsilon)} \geq 0 \\
H'(\epsilon) = \frac{D + T(n-2k+1)}{n(T-\epsilon)^2} &\geq 0, \quad (n-2k+1) \geq 0 \text{ for } k \leq \lfloor (n+1)/2 \rfloor)
\end{aligned} \tag{6.10}$$

Hence *ceteris paribus*, decreasing the benefit level to its minimum maximizes the Gini coefficient.

Case 2: If $k > \lfloor (n+1)/2 \rfloor$, then either decreasing the benefit level to its minimum or increasing the benefit increases the Gini coefficient as shown below:

Case 2.1: If $D \leq (2k-n-1)T$ then increasing the benefit level to its maximum maximizes the Gini coefficient.

Let b' be the distribution obtained by increasing kth element in b: $b'_i = b_i \forall i \neq k$ and $b'_k = b_k + \epsilon$ for some $0 \leq \epsilon \leq b_{k+1} - b_k$. Then $G(b') = \frac{D+(2k-n-1)\epsilon}{n(T+\epsilon)}$.

$$\begin{aligned} H(\epsilon) = G(b') - G(b) &= \frac{D + (2k-n-1)\epsilon}{n(T+\epsilon)} - \frac{D}{nT} \\ &= \frac{DT + T\epsilon(2k-n-1) - DT - D\epsilon}{nT(T+\epsilon)} \\ &= \frac{-D\epsilon + T\epsilon(2k-n-1)}{nT(T+\epsilon)} \geq 0 \end{aligned} \tag{6.11}$$

$$H'(\epsilon) = \frac{T(2k-n-1) - D}{n(T+\epsilon)^2} \geq 0, \quad \text{(By assumption } D \leq (2k-n-1)T\text{)}$$

If $D = (2k-n-1)T$, $H'(\epsilon) = 0$. Then, $H(\epsilon)$ is a non-decreasing and non-negative function and $G(b') \not< G(b)$.

Case 2.2: If $D \geq (2k-n-1)T$ then decreasing the benefit level increases the Gini coefficient.

Let b' be the distribution obtained by decreasing kth element in b: $b'_i = b_i \forall i \neq k$ and $b'_k = b_k - \epsilon$ for some $0 \leq \epsilon \leq b_k - b_{k-1}$. Then $G(b') = \frac{D-(2k-n-1)\epsilon}{n(T-\epsilon)}$.

$$
\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{D - (2k-n-1)\epsilon}{n(T-\epsilon)} - \frac{D}{nT} \\
&= \frac{DT - T\epsilon(2k-n-1) - DT + D\epsilon}{nT(T-\epsilon)} \\
&= \frac{D\epsilon - T\epsilon(2k-n-1)}{nT(T-\epsilon)} \geq 0
\end{aligned}
\tag{6.12}
$$

$$
H'(\epsilon) = \frac{D - T(2k-n-1)}{n(T-\epsilon)^2} \geq 0, \quad \text{(By assumption } D \geq (2k-n-1)T)
$$

If $D = (2k-n-1)T, \quad H'(\epsilon) = 0$. Then, H($\epsilon$) is a non-decreasing and non-negative function and G$(b') \not< G(b)$.

Rank order changes:

Now, let us change the order of b_k by increasing or decreasing ϵ unit. As a result of this change, we allow only one unit shift. We again have two main possible cases:

Case 3: If $k < \lfloor (n+1)/2 \rfloor$, then decreasing the benefit level increases the Gini coefficient as shown below: Let b' be the distribution obtained by decreasing kth element in b: $b'_i = b_i \forall i \neq k, k-1$ and $b'_k = b_{k-1}, b'_{k-1} = b_k - \epsilon$ for some $b_k - b_{k-1} \leq \epsilon \leq b_k - b_{k-2}$. Then $G(b') = \frac{D+(n-2k+2)\epsilon+\epsilon-2d}{n(T-\epsilon)}$ where $d = b_k - b_{k-1}$.

$$
\begin{aligned}
H(\epsilon) = G(b') - G(b) &= \frac{D + (n-2k+2)\epsilon + \epsilon - 2d}{n(T-\epsilon)} - \frac{D}{nT} \\
&= \frac{D\epsilon + T\epsilon(n-2k+3) - 2dT}{nT(T-\epsilon)} \geq 0
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
H'(\epsilon) = \frac{D + T(n-2k+3) - 2d}{n(T-\epsilon)^2} &\geq 0, \quad (n-2k+3) \geq 4 \\
&\text{for } k < \lfloor (n+1)/2 \rfloor \text{ and } n \geq 3, d \leq \epsilon
\end{aligned}
$$

Hence *ceteris paribus*, decreasing the benefit level to its minimum maximizes the Gini coefficient.

Case 4: If $k \geq \lfloor (n+1)/2 \rfloor$, then either decreasing the benefit level to its minimum or increasing the benefit increases the Gini coefficient as shown below:

Case 4.1: If $D \leq (2k-n+1)T - \frac{2d_1 T}{\epsilon_1}$ then increasing the benefit level to its maximum maximizes the Gini coefficient where $d_1 = b_{k+1} - b_k \leq \epsilon_1$.

Let b' be the distribution obtained by increasing kth element in b: $b'_i = b_i \forall i \neq k$ and $b'_k = b_{k+1}, b'_{k+1} = b_k + \epsilon_1$ for some $b_{k+1} - b_k \leq \epsilon_1 \leq b_{k+2} - b_k$. Then $G(b') = \frac{D+(2k-n)\epsilon_1+\epsilon_1-2d_1}{n(T+\epsilon_1)}$.

$$
\begin{aligned}
H(\epsilon_1) = G(b') - G(b) &= \frac{D+(2k-n)\epsilon_1+\epsilon_1-2d_1}{n(T+\epsilon_1)} - \frac{D}{nT} \\
&= \frac{-2d_1T - D\epsilon_1 + T\epsilon_1(2k-n+1)}{nT(T+\epsilon_1)} \geq 0
\end{aligned}
$$

$$
H'(\epsilon_1) = \frac{2d_1 + T(2k-n+1) - D}{n(T+\epsilon_1)^2} \geq 0, \quad \text{(By assumption } D \leq (2k-n+1)T - \frac{2d_1T}{\epsilon_1}\text{)} \tag{6.14}
$$

Since $2k-n+1 \geq 2$ for $n \geq 3$ and $k \geq \lfloor (n+1)/2 \rfloor$.

If $D = (2k-n+1)T - \frac{2d_1T}{\epsilon_1}$, $H'(\epsilon_1) \geq 0$. Then, $H(\epsilon_1)$ is a nondecreasing function and $G(b') \not< G(b)$.

Case 4.2: If $D \geq \frac{2d_2T}{\epsilon_2} + (2k-n-3)T$ then decreasing the benefit level increases the Gini coefficient.

Let b' be the distribution obtained by decreasing kth element in b: $b'_i = b_i \forall i \neq k-1, k$ and $b'_k = b_{k-1}, b'_{k-1} = b_k - \epsilon_2$ for some $b_k - b_{k-1} \leq \epsilon_2 \leq b_k - b_{k-2}$. Then $G(b') = \frac{D+(n-2k+2)\epsilon_2+\epsilon_2-2d_2}{n(T-\epsilon_2)}$ where $d_2 = b_k - b_{k-1} \leq \epsilon_2$.

$$
\begin{aligned}
H(\epsilon_2) = G(b') - G(b) &= \frac{D+(n-2k+2)\epsilon_2+\epsilon_2-2d_2}{n(T-\epsilon_2)} - \frac{D}{nT} \\
&= \frac{D\epsilon_2 + T\epsilon_2(n-2k+3) - 2d_2T}{nT(T-\epsilon_2)} \geq 0
\end{aligned}
$$

$$
H'(\epsilon_2) = \frac{D - T(2k-n-3) - 2d_2}{n(T-\epsilon_2)^2} \geq 0, \quad \text{(By assumption } D \geq \frac{2d_2T}{\epsilon_2} + (2k-n-3)T)\text{)} \tag{6.15}
$$

Note that $\epsilon_2 \leq T$ because $b_i \geq 0, \forall i \in N$. If $D = \frac{2d_2T}{\epsilon_2} + (2k-n-3)T, H'(\epsilon_2) \geq 0$. Then, H($\epsilon_2$) is a non-decreasing function and G(b') $\not< G(b)$.

Since $d_1 \leq \epsilon_1$ and $d_2 \leq \epsilon_2$, $(2k-n+1)T - \frac{2d_1T}{\epsilon_1} \geq \frac{2d_2T}{\epsilon_2} + (2k-n-3)$.

Case 4.3: If $(2k-n+1)T - \frac{2d_1T}{\epsilon_1} \geq D \geq \frac{2d_2T}{\epsilon_2} + (2k-n-3)$ then both decreasing or increasing the benefit level increases the Gini coefficient because both Case 2.1 and Case 2.2 realize simultaneously.

$\square$

6.4 Details of R & D Project Selection Data

Table 6.1: Problem Data from [1]

Project	Type	Cost	Mean Benefit	Project	Type	Cost	Mean Benefit
1	1	0.19	1.39	21	2	0.88	1.71
2	1	0.16	1.13	22	2	0.86	1.34
3	1	0.30	1.67	23	3	0.05	2.15
4	1	0.29	1.48	24	3	0.18	2.47
5	1	0.55	2.13	25	3	0.16	1.96
6	1	0.57	1.43	26	3	0.31	3.42
7	1	0.96	1.50	27	3	0.43	3.92
8	1	0.99	1.44	28	3	0.42	3.42
9	1	0.74	0.99	29	3	0.42	2.97
10	1	0.67	0.85	30	3	0.33	2.29
11	2	0.21	3.13	31	3	0.37	1.67
12	2	0.28	2.52	32	3	0.59	2.60
13	2	0.28	2.11	33	3	0.42	1.79
14	2	0.40	2.43	34	3	0.96	4.08
15	2	0.24	1.49	35	3	0.54	2.11
16	2	0.58	2.91	36	3	0.54	2.08
17	2	0.95	3.15	37	3	0.90	3.25
18	2	0.89	2.82	38	3	0.75	2.20
19	2	0.91	2.47	39	3	0.81	2.06
20	2	0.61	1.57				

6.5 Results of R & D Project Selection

Table 6.2: Total Expected Benefits for Different Parameters (Gini)

Γ / G	1	0.4	0.3	0.2	0.1	0
0	59.32	59.32	58.72	55.75	49.97	39.06
4	59.32	59.32	58.18	55.05	48.24	0
8	59.32	59.32	58.04	53.89	47.11	0
12	59.32	59.32	58.04	53.89	46.45	0

Table 6.3: Total Expected Benefits for Different Parameters (SOPD & Rawlsian)

Metric	SOPD						Rawlsian					
Γ / G	∞	8	6	4	2	0	0	8	10	12	14	16
0	59.32	47.11	45.53	44.60	43.92	39.06	59.32	58.67	56.92	53.75	48.23	Inf.
4	59.32	44.72	43.93	39.96	14.81	0	59.32	58.67	56.39	50.83	Inf.	Inf.
8	59.32	44.10	42.45	29.97	14.81	0	59.32	58.67	54.52	48.23	Inf.	Inf.
12	59.32	43.93	42.24	29.97	14.81	0	59.32	58.67	54.52	48.23	Inf.	Inf.

Table 6.4: Solution Times for Model 3 (Gini)

Γ / G	1	0.4	0.3	0.2	0.1	0
0	0.077	0.086	0.117	0.195	0.106	0.644
4	0.127	0.094	0.102	0.074	0.082	0.049
8	0.082	0.127	0.126	0.102	0.095	0.049
12	0.086	0.084	0.125	0.087	0.159	0.049

Table 6.5: Solution Times for Model 3 (SOPD & Rawlsian)

Metric	SOPD						Rawlsian					
Γ / G	∞	8	6	4	2	0	0	8	10	12	14	16
0	0.076	0.11	0.132	0.085	0.14	0.647	0.074	0.067	0.077	0.08	0.053	0.33
4	0.077	0.143	0.13	2.201	1.303	0.094	0.067	0.072	0.07	0.079	0.372	0.254
8	0.109	0.142	0.211	2.385	0.761	0.05	0.077	0.073	0.06	0.065	0.207	0.179
12	0.14	0.127	0.304	1.156	0.289	0.075	0.076	0.077	0.059	0.07	0.391	0.155

Table 6.6: Solution Times for FSBM (Gini)

Γ / G	1	0.4	0.3	0.2	0.1	0
0	0.254	0.131	0.215	0.196	0.115	0.757
39	0.083	0.077	0.123	0.079	0.154	0.046

6.6 Details of Shelter Allocation Case Study Data

Table 6.7: Sub-district Data

Subdistrict	Interval	Mean	Deviation	Subdistrict	Interval	Mean	Deviation
Yakacık Yeni (SD1)	[1068 1780]	1424	356	Yunus (SD11)	[1112 1854]	1483	371
Atalar (SD2)	[2250 3750]	3000	750	Topselvi (SD12)	[879 1465]	1172	293
Yalı (SD3)	[772 1288]	1030	258	Çarşı (SD13)	[1028 1714]	1371	343
Gümüşpınar (SD4)	[1722 2872]	2297	575	Hürriyet (SD14)	[3257 5429]	4343	1086
Soğanlık Yeni (SD5)	[1678 2798]	2238	560	Karlıktepe (SD15)	[2068 3446]	2757	689
Cumhuriyet (SD6)	[1304 2174]	1739	435	Cevizli (SD16)	[2144 3574]	2859	715
Esentepe (SD7)	[1894 3158]	2526	632	Uğurmumcu (SD17)	[2785 4643]	3714	929
Petrol-İş (SD8)	[2184 3640]	2912	728	Yukarı (SD18)	[606 1012]	809	203
Orta (SD9)	[1077 1795]	1436	359	Orhantepe (SD19)	[2261 3769]	3015	754
Çavuşoğlu (SD10)	[1030 1718]	1374	344	Kordonboyu (SD20)	[872 1456]	1164	292

Table 6.8: Distance Matrix (in km's)

Subdistrict / Shelter	L1	AV2	B14	AV3	BO2	Subdistrict / Shelter	L1	AV2	AV3	B14	BO2
Yakacık Yeni	28	27	15	8	5	Yunus	19	18	1	11	14
Atalar	8	7	13	17	22	Topselvi	20	19	5	9	13
Yalı	21	21	4	10	11	Çarşı	30	29	17	10	6
Gümüşpınar	23	23	14	4	8	Hürriyet	28	27	10	8	5
Soğanlık Yeni	18	18	15	7	12	Karlıktepe	13	12	7	12	16
Cumhuriyet	25	24	11	5	7	Cevizli	11	11	15	16	22
Esentepe	14	13	8	12	17	Uğur Mumcu	29	28	17	10	11
Petroliş	10	9	10	15	20	Yukarı	11	10	8	14	18
Orta	23	23	12	2	9	Orhantepe	4	5	19	21	27
Çavuşoğlu	16	15	3	10	14	Kordonboyu	12	11	7	14	18

Table 6.9: Candidate Location Data

Candidate Shelter Location	Area (m^2)	Capacity
L1	24000	6844
AV2	50000	14272
B14	100000	28558
AV3	45000	12844
BO2	75000	21415