Two Coalescing Brownian Flows: Correlated and Rank-Based

by

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A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Mathematics

KOÇ UNIVERSITY

August 4, 2017

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Koç University

Graduate School of Sciences and Engineering

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ABSTRACT

We consider a stochastic differential equation on the real line which is driven by two correlated Brownian motions B^+ and B^- respectively on the positive half line and the negative half line. We assume $|d\langle B^+, B^-\rangle_t| \leq \rho\, dt$ with $\rho \in [0, 1)$. We prove it has a unique flow solution. Then, we generalize this flow to a flow on the circle, which represents an oriented graph with two edges and two vertices. We prove that both flows are coalescing. Coalescence leads to the study of a correlated reflected Brownian motion on the quadrant. Moreover, we find the distribution of the hitting time to the origin of a reflected Brownian motion. This has implications for the effect of the correlation coefficient ρ on the coalescence time of our flows. Then, we study a stochastic differential equation with rank-based coefficients on the plane. The flow solution of this SDE follows from the first SDE. We also study coalescence of this SDE and its kernel solutions.

ÖZETÇE

Pozitif tarafta B^+ ve negatif tarafta B^- şeklinde ilintili iki Brown hareketi tarafndan belirlenen $\mathbb{R}$'de tanımlı bir stokastik diferansiyel denklemi inceledik. $\rho \in [0, 1)$ olacak şekilde $|d\langle B^+, B^-\rangle_t| \leq \rho\, dt$ kabul ettik. Bu diferansiyel denklemin akş çözümünün varlığını ve eşsizliğini ispatladık. Sonra, bu stokastik akışı çember üstünde stokastik akışa genelleştirdik. Her iki akşın kaynaşan olduğunu ispatladık. Kaynaşmanın incelenmesi sırasında $\mathbb{R}^2$'de pozitif bölgede tanımlı bir ilintili yansıtılmış Brown hareketini kullandık. Ayrıca, bu hareketin orijine çarpma anının dağılımını bulduk. Bunu kullanarak ρ sabitinin kaynaşma anı üstündeki etkisini inceledik. Sonra, düzlemde tanımlı ve sıraya bağlı katsayıları olan bir stokastik diferansiyel denklemi inceledik. Bu denklemin akış çözümünün varlığını ilk denklemi kullanarak ispatladık. Ayrıca bu denklemin kaynaşma özelliğini ve çekirdek akış çözümlerini inceledik.

ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor Prof. Mine Çağlar. She introduced me into this field of mathematics and supported me throughout my studies. I also would like to thank Prof. Atilla Yılmaz. He provided me help whenever I needed. I am thankful to Prof. Mehmet Öz for taking part in the thesis committee.

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INTRODUCTION

The theory of stochastic flows, as can be seen in [9], deals with homeomorphic flows that are solutions of SDEs. In this sense, a stochastic flow is a homeomorphism of R^n onto itself. However, the theory developed in [19] and [10] deals with stochastic flows that are weak or strong solutions to SDEs, but can also coalesce in finite time. They also consider stochastic flow of probability kernels which allows solution of an SDE to split.

Stochastic flows on graphs have attracted much interest recently [1, 2, 4, 5]. As a simple abstraction of change of randomness at a node connecting two edges, flows on $\mathbb{R}$ with the origin having a special role are considered [12]. In particular, a stochastic differential equation is used to represent different dynamics at the negative and positive axes for the motion of a single particle. In Chapter 2, we consider the equation

$$dX_t = 1_{\{X_t>0\}}dB_t^+ - 1_{\{X_t\leq 0\}}dB_t^-, \qquad X_0 = x$$

for the dynamics of the trajectory X in the flow, where B^+ and B^- are correlated standard Brownian motions adapted to the same filtration.

The same equation with B^+ and B^- are independent is considered in [12], and the existence of strong solution is proved by showing that this equation has a flow solution which is adapted to the filtration generated by the white noises driving the equation. A similar SDE on a star graph driven by independent Brownian motions on each ray is analyzed in [2]. The existence of strong solution when there are two rays is proved in [2] in a more direct manner compared to [12] and it is shown that the SDE admits only weak solutions, which are unique in law, when there are three or more rays. They also show that the solution of this SDE is Walsh's Brownian motion.

Since the usual conditions are not satisfied by the diffusion terms, we first investigate the existence of a strong solution. Then, we show that there exists a stochastic flow on $\mathbb{R}$ based on this differential equation and use it for generalizing to a stochastic flow on the circle.

More precisely, let B^+ and B^- be two Brownian motions, jointly defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ and adapted to a filtration $(\mathcal{F}_t)$ satisfying the usual conditions. By the assumption that B^+ and B^- are Brownian motions, (B^+, B^-) is a martingale with respect to $(\mathcal{F}_t)$ [15, pg.147], with cross variation process $H_t := \langle B^+, B^- \rangle_t$. Since H is of bounded variation [8], it is almost everywhere differentiable. Let h_t denote the derivative dH_t/dt when it exists. We assume

$$|h_t| \leq \rho \tag{1.1}$$

for almost every $t \geq 0$ with $\rho \in [0, 1)$ and it follows that

$$|H_t| \leq \rho t \ .$$

Examples of martingales such as (B^+, B^-) arise as solutions of stochastic differential equations. We seek for a flow of mapping φ which is coalescing, based on the SDE [10]. For the proof of coalescence, we make use of a reflected Brownian motion (RBM)on the quadrant. It is constructed from two particle motion by extracting the parts of the trajectories with opposite signs and assigning each partial trajectory to a coordinate of the RBM. Clearly, two particles have a chance to coalesce only if their signs are different. The coordinates of the RBM are correlated by construction. In the case when the correlation between B^+ and B^- is exactly ρt, we find the distribution of the hitting time T_0 of the RBM to the origin. In this way, we not only extend the results of [12] where the finiteness of T_0 is proved for $\rho = 0$, but also find the distribution of T_0, which is of independent interest as well. Its density is given by

$$f_\rho(t) = \frac{x}{2\sqrt{\pi t^3(1+\rho)}}\, e^{-\frac{x^2}{4t(1+\rho)}} \qquad t > 0\,.$$

We show that coalescence occurs faster as ρ gets closer to 1.

Our flows can be considered as interpolations between the flows associated to Tanaka's SDE [3, 11] and the basic Brownian flow considered in [12]. When $\langle B^+, B^-\rangle_t = \rho t$, $\rho = 1$ corresponds to Tanaka's SDE and the equation of Le Jan and Raimond [12] represents the independent case $\rho = 0$. We study the intermediate, but more general cross variation process H_t.

Second, we are interested in a flow on the unit circle $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. We embed an oriented graph with two edges and two vertices at 1 and e^{il} in $\mathcal{C}$ where $l \in (0, 2\pi]$ is fixed. In analogy with the original SDE, a particle is supposed to follow a different Brownian motion on each edge. This defines a special metric graph and the construction of a flow on this graph follows directly from the theory in [4]. Formally, we require the stochastic flow φ on the circle $\mathcal{C}$ to satisfy the equation

$$\begin{aligned} f(\varphi_{s,t}(z)) &= f(z) + \int_s^t f'(\varphi_{s,u}(z)) 1_{\{\arg(\varphi_{s,u}(z)) \in [0,l)\}} W^+(du) \\ &\quad - \int_s^t f'(\varphi_{s,u}(z)) 1_{\{\arg(\varphi_{s,u}(z)) \in [l,2\pi)\}} W^-(du) + \frac{1}{2}\int_s^t f''(\varphi_{s,u}(z))\, du \end{aligned} \tag{1.2}$$

for all $f \in C^2(\mathcal{C})$, as a generalization of the stochastic flow solution to the SDE on $\mathbb{R}$ given in Theorem 2.1.7. To that end, we refer to some flows of mappings φ^+ and φ^-, which are flow solutions to two forms of this SDE with consistent Brownian motions.

We prove that this flow is coalescing. We first take two particles moving on the plane such that when they are mapped on the circle they move with respect to the flow on the circle. Then, we construct an associated RBM on a bounded domain. We show that this process hits one of the corners in finite time, which in turn implies that coalescence occurs in finite time at one of the vertices. A further remarkable result is on coalescence time. We find its distribution explicitly and see that the closer ρ is to 1, the faster the coalescence occurs.

In Chapter 3, we find flow of mappings and flow of kernels solutions of a stochastic differential equation with rank-based characteristics on the plane. The existence of weak and strong solutions of this SDE are analyzed in detail in [18] where there is also an additional drift term. An example to this kind of SDE, as given in [18], is as follows:

$$\begin{aligned} dX_1(t) &= (g1_{\{X_1(t)\leq X_2(t)\}} - h1_{\{X_1(t)>X_2(t)\}})dt + (\rho 1_{\{X_1(t)>X_2(t)\}} + \sigma 1_{\{X_1(t)\leq X_2(t)\}})dB_1(t) \\ dX_1(t) &= (g1_{\{X_1(t)>X_2(t)\}} - h1_{\{X_1(t)\leq X_2(t)\}})dt + (\rho 1_{\{X_1(t)\leq X_2(t)\}} + \sigma 1_{\{X_1(t)>X_2(t)\}})dB_2(t) \end{aligned}$$

where B_1 and B_2 are one-dimensional independent Brownian Motions and ρ, σ, g, h are constants. In [21], the multidimensional case is analyzed further and under certain conditions the existence of strong solution is proved until three particles collide. Proving the existence of strong solution after this point is an open question. Here, we will only work with the two-dimensional equation. However studying the kernel solutions might help to analyze the open question for the multidimensional case.

In order to simplify our analysis, we first rotate the SDE clockwise by $\frac{\pi}{4}$ on the plane and the resulting SDE can be expressed in the following form:

$$\begin{aligned} d\tilde{X}_t^1 &= 1_{\{\tilde{X}_t^2<0\}} d\tilde{B}_t^1 + 1_{\{\tilde{X}_t^2\geq 0\}} d\tilde{B}_t^2 \\ d\tilde{X}_t^2 &= 1_{\{\tilde{X}_t^2<0\}} d\tilde{B}_t^3 + 1_{\{\tilde{X}_t^2\geq 0\}} d\tilde{B}_t^4 \end{aligned}$$

These Brownian motions are correlated as indicated by the following quadratic variations

$$\begin{aligned} d\langle \tilde{B}^1, \tilde{B}^2\rangle_t &= \frac{1}{2}\big(\rho\sigma(1+\varepsilon\delta)\cos(\varphi-\vartheta) + (\varepsilon-\delta)\sin(\varphi-\vartheta)\big) \\ d\langle \tilde{B}^3, \tilde{B}^4\rangle_t &= \frac{1}{2}\big(\rho\sigma(1+\varepsilon\delta)\cos(\varphi-\vartheta) - (\varepsilon-\delta)\sin(\varphi-\vartheta)\big) \\ d\langle \tilde{B}^1, \tilde{B}^4\rangle_t &= \frac{1}{2}\big(\rho\sigma(-1+\varepsilon\delta)\cos(\varphi-\vartheta) - (\varepsilon+\delta)\sin(\varphi-\vartheta)\big) = d\langle \tilde{B}^2, \tilde{B}^3\rangle_t \end{aligned}$$

where $\varepsilon, \sigma, \rho, \delta \in \{\pm 1\}$ and $0 \leq \vartheta\varphi \leq 2\pi$. Throughout the chapter we work with these rotated equations. It is worth noting that the second equation is the same stochastic differential equation we studied in Chapter 2. Next, using the results we found in Chapter 2, we give a characterization of the existence of weak and strong solutions of our rank-based SDE, recovering the same condition given in [18, Thm. 5.2].

The existence of the flow solutions follow from Chapter 2 for the rotated equations. Then, by a counterclockwise rotation, we obtain the flow solution for the rank-based SDE. We analyze and characterize the coalescence in a case by case manner. However, some cases are not completed in this thesis. Finally, the two-dimensional Tanaka equation is introduced and the kernel solutions of it are left as future work.

Chapter 2

CORRELATED COALESCING BROWNIAN FLOWS

2.1 The flow on $\mathbb{R}$

We first give the definition of stochastic flows with independent increments on $\mathbb{R}$ [10].

Definition 2.1.1. *A measurable stochastic flow of mappings (SFM) φ on $\mathbb{R}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a family $(\varphi_{s,t})_{s<t}$ such that*

1. *For all $s < t$, $\varphi_{s,t}$ is a measurable mapping from $(\Omega \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R} \otimes \mathcal{B}(\mathbb{R}))$.*

2. *For all $h \in \mathbb{R}$, $s < t$, $\varphi_{s+h,t+h}$ is distributed like $\varphi_{s,t}$.*

3. *For all $t_1 < t_2 < ... < t_n$, the family $\{\varphi_{t_i,t_{i+1}} : 1 \leq i \leq n-1\}$ is independent.*

4. *For all $s < t < u$, and all $x \in \mathbb{R}$, a.s. $\varphi_{s,u}(x) = \varphi_{t,u} \circ \varphi_{s,t}(x)$ and $\varphi_{s,s}$ equals the identity.*

5. *For all $f \in C_0(\mathbb{R})$, we have*

$$\lim_{(u,v)\to(s,t)} \sup_{x\in\mathbb{R}} \mathbb{E}[(f \circ \varphi_{u,v}(x) - f \circ \varphi_{s,t}(x))^2] = 0$$

6. *For all $f \in C_0(\mathbb{R})$, $x \in \mathbb{R}$, $s < t$, we have*

$$\lim_{y\to x} \mathbb{E}[(f \circ \varphi_{s,t}(y) - f \circ \varphi_{s,t}(x))^2] = 0$$

7. *For all $s < t$, $f \in C_0(\mathbb{R})$, we have*

$$\lim_{|x|\to\infty} \mathbb{E}[(f \circ \varphi_{s,t}(x))^2] = 0$$

Starting with the solution of SDE

$$dX_t = 1_{\{X_t>0\}} dB_t^+ - 1_{\{X_t\leq 0\}} dB_t^-, \qquad X_0 = x \tag{2.1}$$

we study its flow in this section.

2.1.1 *Strong solution for one-point motion*

In order to show that the strong solution exists, it is sufficient to show pathwise uniqueness and weak existence. The proof of this fact, as given in [8, pg.309-10] based on Yamada and Watanabe's theorem for SDE's driven by independent Brownian motions, is also valid for our SDE, where the Brownian motions B^+ and B^- are not independent. The crucial point is that (B^+, B^-) with cross variation process H still takes values in a Polish space, namely, $C(\mathbb{R}_+, \mathbb{R}^2)$. In view of this, there exist regular versions of the conditional probability distributions involved and the result follows; see also [15, pg.368].

For the existence of a weak solution of SDE (2.1), we demonstrate a probability space $(\Omega', \mathcal{H}', \mathbb{P}')$ with filtration $(\mathcal{F}_t')$ and a process $(X', B^{+'}, B^{-'})$ adapted to $(\mathcal{F}_t')$ that solves (2.1), as given in the following proposition.

Proposition 2.1.2. *There exists a weak solution of* (2.1).

Proof. Recall that (B^+, B^-) is a martingale on $(\Omega, \mathcal{H}, \mathbb{P}, (\mathcal{F}_t))$ with cross variation process H. When it exists, a solution X of SDE (2.1) is a Brownian motion by Lévy's characterization theorem, irrespective of the joint law of (B^+, B^-). Although the joint distribution of B^+ and B^- is not specified, it is fixed by H due to [15, Thm.V.3.9] as $dH_t = h_t\, dt$ for almost every $t > 0$. Accordingly, we can find a two-dimensional Brownian motion $B := (B^1, B^2)^T$ such that

$$(B_t^+, B_t^-)^T = \int_0^t \alpha_s\, dB_s \tag{2.2}$$

where α is a matrix-valued process given by

$$\alpha_s = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+h_s} & \sqrt{1-h_s} \\ \sqrt{1+h_s} & -\sqrt{1-h_s} \end{bmatrix}$$

as in [15, Thm.V.3.9]. Since α is invertible due to assumption $\rho < 1$ in (1.1), we have $(B^1_t, B^2_t)^T = \int_0^t \alpha_s^{-1}\, d(B^+_s, B^-_s)^T$. Then, B^1 and B^2 are independent Brownian motions adapted to $(\mathcal{F}_t)$ as $\langle B^1, B^2\rangle_t = 0$, for $t > 0$. Therefore, (2.2) characterizes the joint distribution of (B^+, B^-) uniquely in our original probability space. Now, let

$$
\begin{aligned}
X'_t &= B^+_t \\
B^{+'}_t &= \int_0^t 1_{\{X_s>0\}} dB^+_s - \int_0^t 1_{\{X_s\leq 0\}} dB^-_s \\
B^{-'}_t &= \int_0^t 1_{\{X_s>0\}} dB^+_s - \int_0^t 1_{\{X_s\leq 0\}} dB^-_s
\end{aligned}
$$

It follows that $(X', B^{+'}, B^{-'})$ is adapted to $(\mathcal{F}_t)$ and solves (2.1). It is easily checked that $\langle B^{+'}, B^{-'}\rangle_t = H_t$, and $(B^{+'}, B^{-'})$ has the same distribution as (B^+_t, B^-_t) by an analogous relation to (2.2) where B and (B^+, B^-) are replaced by B', and $(B^{+'}, B^{-'})$, respectively. Note that $\rho = 1$ is not problematic for demonstrating a weak solution. In that case, our arguments above can be modified by an enlargement of the probability space as in [15, Thm.V.3.9]. $\square$

Pathwise uniqueness will be shown next using [13, Thm.2], equivalently, its generalization, which is given in [18, Thm.6.1], based on the assumption $\rho < 1$. We also give an alternative proof of Proposition 2.1.3, which reveals the role of the magnitude of the correlation between B^+ and B^- by following the steps of [2, Prop.4.5], where $\rho < 1$ is again crucial.

Proposition 2.1.3. *Pathwise uniqueness holds for (2.1).*

Proof. Let us define $M_t = \frac{B^+_t + B^-_t}{2}$ and $N_t = \frac{B^+_t - B^-_t}{2}$. They are continuous local martingales which are strongly orthogonal, i.e., $\langle M, N\rangle_t = 0$. Then (2.1) reduces to

$$dX_t = \text{sgn}(X_t) dM_t + dN_t$$

We look for $c > 0$ such that $d\langle M\rangle_t \leq c d\langle N\rangle_t$. This is equivalent to $dH_t \leq \frac{c-1}{c+1}\, dt$. Since $h_t \leq \rho$ by (1.1), we have $dH_t \leq \rho\, dt$ with $\rho < 1$ and the domination relation is satisfied with $c = (1+\rho)/(1-\rho)$. By [13, Thm.2], we may conclude that pathwise uniqueness holds for (2.1). $\square$

Alternative Proof

Following the steps of [2, Prop.4.5], we give an alternative proof of Proposition 2.1.3, which reveals the role of the correlation between B^+ and B^-. Let (X, B^+, B^-) and (X', B^+, B^-) be two solutions with $X_0 = X'_0 = 0$. Set $\text{sgn}(x) = 1_{\{x>0\}} - 1_{\{x\leq 0\}}$. By the occupation times formula

$$\int_{(0,\infty]} L_t^a(X - X')\frac{da}{a} = \int_0^t 1_{\{X_s - X'_s > 0\}} \frac{d\langle X - X'\rangle_s}{X_s - X'_s}$$

Also observe that

$$d\langle X - X'\rangle_s = \left[1 - 1_{\{X_s>0,X'_s>0\}} - 1_{\{X_s\leq 0,X'_s\leq 0\}} + (1_{\{X_s>0,X'_s\leq 0\}} + 1_{\{X_s\leq 0,X'_s>0\}})h_s\right]2ds$$

So

$$|d\langle X - X'\rangle_s| \leq \left[|1 - 1_{\{X_s>0,X'_s>0\}} - 1_{\{X_s\leq 0,X'_s\leq 0\}}| + |1_{\{X_s>0,X'_s\leq 0\}} + 1_{\{X_s\leq 0,X'_s>0\}}|.|h_s|\right]2ds$$

$$\leq 2|\text{sgn}(X_s) - \text{sgn}(X'_s)|\, ds$$

since $|h_s| \leq \rho < 1$.

Let $\{f_n\} \subset C^1(\mathbb{R})$ such that $f_n \to \text{sgn}$ pointwise and $\{f_n\}$ is uniformly bounded in total variation. By Fatou's lemma we get

$$\begin{aligned}\int_{(0,\infty]} L_t^a(X - X')\frac{da}{a} &\leq 2\liminf_n \int_0^t 1_{\{X_s - X'_s>0\}} \frac{|f_n(X_s) - f_n(X'_s)|}{X_s - X'_s} ds \\ &\leq 2\liminf_n \int_0^t 1_{\{X_s - X'_s>0\}} \left| \int_0^1 f'_n(Z_s^u) du \right| ds\end{aligned}$$

where $Z_s^u = (1-u)X_s + uX'_s$. Now observe that

$$\frac{d\langle Z^u\rangle_s}{ds} = \begin{cases} 1 & \text{if} \quad X_s > 0, X'_s > 0 \quad \text{or} \quad X_s \leq 0, X'_s \leq 0 \\ 2(1+h_s)u^2 - 2(1+h_s)u + 1 & \text{if} \quad X_s > 0, X'_s \leq 0 \quad \text{or} \quad X_s \leq 0, X'_s > 0 \end{cases}$$

This shows that for all $0 \leq \rho < 1$ there exists a constant $A > 0$ such that for all $s \geq 0$, and $u \in [0,1]$ $d\langle Z^u\rangle_s \geq \frac{ds}{A}$, since this polynomial in u has its minimum for

$u = 1/2$ and it has roots $(1 \pm \sqrt{\frac{(h_s)^2-1}{h_s+1}})\frac{1}{2}$. For $|h_s| \leq \rho < 1$ it has no real roots. We have the following inequality:

$$\begin{aligned}\int_{(0,\infty]} L_t^a(X-X')\frac{da}{a} &\leq 2A\liminf_n \int_0^1\int_0^t |f_n'(Z_s^u)|d\langle Z^u\rangle_s du\\ &\leq 2A\liminf_n \int_0^1\int_{\mathbb{R}} |f_n'(a)|L_t^a(Z^u)dadu\end{aligned}$$

Now taking expectations and using Fatou's lemma we get

$$\mathbb{E}\left[\int_{(0,\infty]} L_t^a(X-X')\frac{da}{a}\right] \leq 2A\liminf_n \int_{\mathbb{R}} |f_n'(a)|da \sup_{a\in\mathbb{R},u\in[0,1]} \mathbb{E}[L_t^a(Z^u)]$$

By Tanaka's formula, we have

$$\begin{aligned}\mathbb{E}[L_t^a(Z^u)] &= \mathbb{E}[|Z_t^u-a|]-\mathbb{E}[|Z_0^u-a|]-\mathbb{E}\left[\int_0^t \operatorname{sgn}(Z_s^u-a)dZ_s^u\right]\\ &\leq \mathbb{E}[|Z_t^u-Z_0^u|]\end{aligned}$$

The right hand side is uniformly bounded with respect to (a,u) which gives us

$$\int_{(0,\infty]} L_t^a(X-X')\frac{da}{a} < \infty$$

Since $\lim_{a\downarrow 0} L^a(X-X') = L^0(X-X')$ this implies that $L_t^0(X-X') = 0$ and thus by Tanaka's formula, $|X-X'|$ is a local martingale which is also a nonnegative supermartingale with $|X_0-X_0'| = 0$ and finally X and X' are indistinguishable. $\square$

2.1.2 Coalescence of two particles

Let X and Y be two solutions of SDE (2.1) with $X_0 = 0, Y_0 = y > 0$. Define the coalescence time

$$T = \inf\{s \geq 0 : X_s = Y_s\}$$

and the quadrant

$$D = \{(x,y) \in \mathbb{R}^2 : x \leq 0, y \geq 0\}$$

which will be the domain of an RBM to be defined as follows. Let $A_t = \int_0^{t\wedge T} 1_{\{(X_s,Y_s)\in D\}} ds$. Let $\kappa_t = \inf\{s > 0 : A_s > t\}$ and $(X_t^r, Y_t^r) = (X_{\kappa_t}, Y_{\kappa_t})$ for $t \le A_T$. Let $L_t(X)$ and $L_t(Y)$ be the local times of X and Y at 0. Denote by L_t^1 and L_t^2, $\frac{1}{2}L_{\kappa_t}(X)$ and $\frac{1}{2}L_{\kappa_t}(Y)$, respectively.

Lemma 2.1.4. *The process (X^r, Y^r) is a correlated reflected Brownian motion in D, obliquely reflected at the boundary with angle $\pi/4$ and stopped when it hits $(0,0)$. That is, there exist Brownian motions B^1 and B^2 with $|d\langle B^1, B^2\rangle_t| \le \rho\, dt$ such that for all $t < A_T$*

$$X_t^r = -B_t^1 - L_t^1 + L_t^2\,, \qquad Y_t^r = y + B_t^2 - L_t^1 + L_t^2\,.$$

with $\langle B^1, B^2\rangle_t = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} h_s\, ds$.

Proof : According to SDE (2.1), the solutions X and Y satisfy

$$\begin{aligned} X_t &= \int_0^t 1_{\{X_s>0\}}\, dB_s^+ - \int_0^t 1_{\{X_s\le 0\}}\, dB_s^- \\ Y_t &= y + B_t^+ \end{aligned} \tag{2.3}$$

for Brownian motions B^+ and B^- with $\langle B^+, B^-\rangle_t = H_t$, until $\tau_1 := \inf\{t > 0 : Y_t = 0\}$. Note that (X_t^r, Y_t^r), $t \le \tau_1$, takes values in $\mathbb{R}^- \times \mathbb{R}$ and X^r is a reflected Brownian motion on $\mathbb{R}^-$. By Tanaka formula [8, Prop.3.6.8], there exists a Brownian motion B^1 satisfying

$$L_t^1 = -X_t^r - B_t^1$$

where L_t^1 is the local time of X_t^r and $B_t^1 = -\int_0^{\kappa_t} 1_{\{X_s\le 0\}}\, dX_s = \int_0^{\kappa_t} 1_{\{X_s\le 0\}}\, dB_s^-$. Putting $B_t^2 := \int_0^{\kappa_t} 1_{\{X_s\le 0\}}\, dB_s^+$, we observe that B^2 is also a Brownian motion and

$$Y_t^r = Y_{\kappa_t} = B_t^2 - L_t^1 + y$$

holds from (2.3) and the fact that $X_{\kappa_t} + B_t^1 = \int_0^{\kappa_t} 1_{\{X_s>0\}}\, dX_s$, which is $-L_t^1$. Now, we have

$$\langle B^1, B^2\rangle_t = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} d\langle B^+, B^-\rangle_s = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} dH_s = \int_0^{\kappa_t} 1_{\{X_s\le 0\}} h_s\, ds\,.$$

It follows that

$$|d\langle B^1, B^2\rangle_t| = |h_{\kappa_t}|\, dt \leq \rho\, dt$$

as $|h_t| \leq \rho$ for all t, and by definition of κ_t as right continuous inverse of $A_t = \int_0^t 1_{\{X_s \leq 0\}}\, ds$ in $(0, \tau_1]$. Thus, for $t \leq \tau_1$, (X^r_t, Y^r_t), is a correlated reflected Brownian motion on $\mathbb{R}^- \times \mathbb{R}$ with oblique reflection of angle $\pi/4$ (see e.g. [7] for Skorohod representation of multidimensional reflected diffusions, here until τ_1). More explicitly, we have

$$(X^r_t, Y^r_t) = (0, y) + (-B^1_t, B^2_t) - (L^1_t, L^1_t) \quad t \leq \tau_1\ .$$

Now, starting with $(x, 0) := (X_{\tau_1}, Y_{\tau_1})$, with $x < 0$, we replace X, Y, B^+, B^- with $X_{\tau_1+\cdot}, Y_{\tau_1+\cdot}, B^+_{\tau_1+\cdot} - B^+_{\tau_1}, B^-_{\tau_1+\cdot} - B^-_{\tau_1}$. In $(\tau_1, \tau_2]$ where $\tau_2 = \inf\{t > 0 : X_t = 0\}$, according to (2.1) we have

$$\begin{aligned} X_t &= x - B^-_t \\ Y_t &= \int_0^t 1_{\{Y_s > 0\}}\, dB^+_s - \int_0^t 1_{\{Y_s \leq 0\}}\, dB^-_s \end{aligned}$$

In $(\tau_1, \tau_2]$, similar to $[0, \tau_1]$ above, (X^r_t, Y^r_t) is a correlated reflected Brownian motion on $\mathbb{R} \times \mathbb{R}^+$ with angle of reflection $\pi/4$. In particular, Y^r is a reflected Brownian motion on $\mathbb{R}^+$ with local time

$$L^2_t = Y^r_t - B^2_t$$

where $B^2_t = \int_0^{\kappa_t} 1_{\{Y_s > 0\}}\, dY_s = \int_0^{\kappa_t} 1_{\{Y_s > 0\}}\, dB^+_s$ from Tanaka formula. This time, X is on the negative axis and satisfies

$$X^r_t = X_{\kappa_t} = x + L^2_t - B^1_t$$

where we put $B^1_t := -\int_0^{\kappa_t} 1_{\{Y_s \leq 0\}}\, dB^-_s$. It follows that

$$(X^r_t, Y^r_t) = (x, 0) + (-B^1_t, B^2_t) + (L^2_t, L^2_t)$$

and $|d\langle B^1, B^2\rangle_t| \leq \rho\, dt$ as before. Clearly, the process (X^r, Y^r) constructed this way is continuous by definition and satisfies

$$(X^r_t, Y^r_t) = (0, y) + (-B^1_t, B^2_t) - (L^1_t, L^1_t) + (L^2_t, L^2_t) \qquad \text{for} \quad t < A_T$$

since L^1 and L^2 are not positive at the same time. $\square$

Note that the proof of Lemma (2.1.4) considers a starting point of the form $(0, y)$ or $(x, 0)$. If the process (X, Y) starts from $(x, y) \in D$ with $x < 0 < y$, then with these starting points for (X^r, Y^r) the same proof applies after (X, Y) hits the boundary of D. Clearly, the case $y < 0 < x$ is symmetric with D replaced by $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$. If (x, y) is in the first or third quadrant, with the same sign, then we can define (X^r, Y^r) after the hitting time of one of X and Y to 0. The following lemma indicates coalescence.

Lemma 2.1.5. $\mathbb{P}\{T < \infty\} = 1$.

Proof : Let L^1, L^2 be as in Lemma 2.1.4 and define $L^r := L^1 + L^2$, which is the local time of (X^r, Y^r) at the boundary $\{x = 0\} \cup \{y = 0\}$. Let $T_0 = \inf\{t \geq 0 : X^r_t = Y^r_t = 0\}$, the hitting time of the RBM to the origin. Along the very same lines of the proof of [12, Lem.4.6], one can show that $\mathbb{P}\{L^r_{T_0} < \infty\} = 1$. The cross variation process of B^1 and B^2 and the upper bound ρ appear in an obvious way in the proof. Then, since $\frac{1}{2}(L_T(X) + L_T(Y)) = L^r_{T_0} < \infty$ and since X is a Brownian motion, it follows that $T < \infty$ a.s. $\square$

Since the solution of (2.1) is strong, when two particles meet they stay together thereafter.

2.1.3 *Feller property and the flow*

Since there exists a strong solution to (2.1), we may define $P^n_t(x, dy)$, $n \geq 1$, as the law of $(X^1, \ldots, X^n)$ which are n solutions of the same equation with initial conditions $X^i_0 = x_i$, $i = 1, \ldots, n$. We will prove $(P^n_t)_{n \geq 1}$ defines a compatible family of Feller semigroups next.

Lemma 2.1.6. *The family of Markovian semigroups corresponding to n-point motion corresponding to (2.1) is Feller, for each $n \geq 1$*

Proof : We will check Condition (C) of [10, Thm.4.1] as a sufficient condition for

Feller property. Let (X, Y) be the two-point motion. Condition (C) is verified if for every $t > 0$ and $\varepsilon > 0$

$$\lim_{|y-x|\to 0} \mathbb{P}^{(2)}_{(x,y)}\{\, |X_t - Y_t| > \varepsilon,\, t < T\} = 0 \tag{2.4}$$

Assume $0 < y - x < \varepsilon$, then

$$\begin{aligned}
\mathbb{P}^{(2)}_{(x,y)}\{Y_t - X_t > \varepsilon, t < T\} &\le \mathbb{P}^{(2)}_{(x,y)}\{\sup_{t\ge 0}(Y_t - X_t) \ge \varepsilon\} \\
&= \mathbb{P}^{(2)}_{(x,y)}\{Y_0 - X_0 \ge \varepsilon\} + \frac{1}{\varepsilon}\mathbb{E}^{(2)}_{(x,y)}[(y-x)1_{\{Y_0 - X_0 < \varepsilon\}}] \\
&= \frac{y-x}{\varepsilon}
\end{aligned}$$

where the first equality follows from [8, problem 1.3.28]. Then, (2.4) follows. □

Now, let W^+ and W^- be two given white noises with $|\langle W^+_{s,t}, W^-_{s,t}\rangle| \le \rho(t-s)$, for $s < t$. By the results of Section 2.1.1, the equation

$$\varphi_{s,t}(x) = x + \int_s^t 1_{\{\varphi_{s,u}(x)>0\}} W^+(du) - \int_s^t 1_{\{\varphi_{s,u}(x)\le 0\}} W^-(du) \tag{2.5}$$

has a strong solution φ for each $x \in \mathbb{R}$ and $s \le t$. As in the proof of [12, Thm.1.1], the mappings $\varphi_{s,t} : \mathbb{R} \to \mathbb{R}$, for s, t rational, can be defined on the same probability space. Then, the solutions can be extended to all $s, t \in \mathbb{R}_+$ and they will satisfy (2.5) by continuity in view of Lemma 2.1.6. Measurability and cocycle property follows by similar arguments as in [6, pg.81] and [9, pg.161]. Hence, we have the following

Theorem 2.1.7. *There exists a unique coalescing stochastic flow of mappings φ such that for all $x \in \mathbb{R}$ and $s \le t$*

$$\varphi_{s,t}(x) = x + \int_s^t 1_{\{\varphi_{s,u}(x)>0\}} W^+(du) - \int_s^t 1_{\{\varphi_{s,u}(x)\le 0\}} W^-(du)$$

where (W^+, W^-) is a white noise with $|\langle W^+_{s,t}, W^-_{u,v}\rangle| \le \rho|[s,t] \cap [u,v]|$.

2.1.4 Distribution of T_0

In this part, we find the distribution of the time it takes the reflected Brownian motion (X_r, Y_r) to hit the origin under the simplified condition that the cross variation process is exactly equal to ρt. Let (X, Y) be a two point motion of our SDE

$$dX_t = 1_{\{X_t>0\}}dB_t^+ - 1_{\{X_t\leq 0\}}dB_t^-$$

where $\langle B^+, B^-\rangle_t = \rho t$ with $X_0 = x > 0$ and $Y_0 = 0$. Remember the reflected Brownian motion constructed in subsection 2.1.2 as

$$X_t^r = x + B_t^1 + L_t^1 - L_t^2 \qquad Y_t^r = -B_t^2 + L_t^1 - L_t^2$$

which is on $\mathbb{R}^+ \times \mathbb{R}^-$. From the proof of Lemma 2.1.4, $\langle B^1, B^2\rangle_t = \rho t$ as a consequence of $\langle B^+, B^-\rangle_t = \rho t$. Let $V_t = \frac{1}{\sqrt{2}}(X_t^r - Y_t^r)$. Note that $V_t = 0$ if and only if $X_t^r = Y_t^r = 0$. Now define $M_t := V_t - V_0$ and $T(s) := \inf\{t \geq 0 : \langle M\rangle_t > s\}$. Then, $M_t \in \mathcal{M}_{loc}$, $\langle M\rangle_t = (1+\rho)t$, and $T(s) = \frac{s}{1+\rho}$. By [8, Thm. 3.4.6], we may conclude that the time changed process $B_s = M_{\frac{s}{1+\rho}}$ is a standard one dimensional Brownian motion. Now define the following stopping times:

$$T_0 = \inf\{t \geq 0 : V_t = 0\} \qquad S = \inf\{s \geq 0 : B_s = -\frac{x}{\sqrt{2}}\}$$

Note that T_0 is also the first hitting time of M_t to $-\frac{x}{\sqrt{2}}$. We know that $\mathbb{P}\{S \in ds\} = \frac{|x|}{2\sqrt{\pi s^3}}e^{-\frac{x^2}{4s}}ds$. Then, we may conclude that

$$\mathbb{P}\{T_0 \in dt\} = (1+\rho)^{-\frac{1}{2}}\frac{x}{2\sqrt{\pi t^3}}e^{-\frac{x^2}{4t(1+\rho)}}dt$$

Now let $\bar{F}_\rho(t) = \mathbb{P}(T_0 > t)$ and define $\bar{F}(t) = \bar{F}_0(t)$. Then, we have

$$\bar{F}_\rho(t) = \int_t^\infty (1+\rho)^{-1/2}\frac{x}{2\sqrt{\pi s^3}}e^{-\frac{x^2}{4s(1+\rho)}}ds$$

By a change of variable s to $s/(1+\rho)$, we get

$$\bar{F}_\rho(t) = \int_{t(1+\rho)}^\infty \frac{x}{2\sqrt{\pi s^3}}e^{-\frac{x^2}{4s}}ds$$

which gives the relation $\bar{F}_\rho(t) = \bar{F}(t(1+\rho))$. That is, the probability of the time it takes the reflected Brownian motion to hit the corner being greater than a fixed $t > 0$ is equal to $\bar{F}(t(1+\rho))$. This probability gets smaller, equivalently the probability of hitting the corner gets larger, as the correlation coefficient ρ increases.

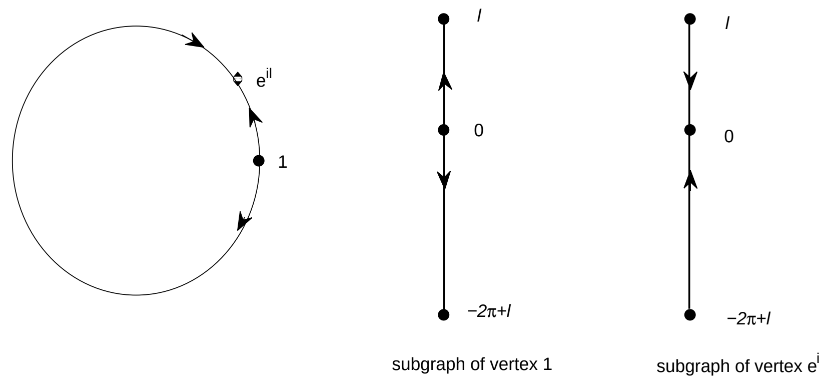

Figure 2.1: *Illustration of the circle and its subgraphs*

2.2 Flows on a Circle

In this section, $\mathbb{R}$ is replaced with a circle, a compact metric space, where we study our flow.

2.2.1 *Construction of the flow*

We aim to construct a flow solution of the equation on the unit circle $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$, which will be denoted by φ . A function f is said to be differentiable at $z \in \mathcal{C}$ if

$$f'(z) = \lim_{h \to 0} \frac{f(ze^{ih}) - f(z)}{h}$$

exists. Assume that an oriented graph with two edges and two vertices at 1 and e^{il} is embedded in $\mathcal{C}$, where $l \in (0, \pi]$ is fixed. We assume that vertex 1 is the tail and e^{il} is the head for both edges. Let $\mathcal{C}^+ = \{z \in \mathcal{C} : \arg(z) \in (0, l)\}$ and $\mathcal{C}^- = \{z \in \mathcal{C} : \arg(z) \in (l, 2\pi)\}$. See Figure for an illustration.

For all $f \in C^2(\mathcal{C})$, we require φ to satisfy

$$\begin{aligned} f(\varphi_{s,t}(z)) &= f(z) + \int_s^t f'(\varphi_{s,u}(z)) 1_{\{\arg(\varphi_{s,u}(z)) \in [0,l)\}} W^+(du) \\ &\quad - \int_s^t f'(\varphi_{s,u}(z)) 1_{\{\arg(\varphi_{s,u}(z)) \in [l,2\pi)\}} W^-(du) + \frac{1}{2} \int_s^t f''(\varphi_{s,u}(z))\, du \end{aligned} \tag{2.6}$$

by Ito formula, where W^+, W^- are correlated white noises with $\langle W^+_{s,t}, W^-_{u,v} \rangle \leq \rho\left(|\,[s,t] \cap [u,v]\,|\right)$, $\rho \in [0,1)$. Note that $\rho = 1$ gives Tanaka's flow on the circle as studied in [5], and $\rho = 0$ yields a flow on the circle as a special case of [4].

We construct φ on $\mathcal{C}$ using φ^+ and φ^- on $\mathbb{R}$, which are the respective flow solutions of

$$d\varphi^+_{s,t}(x) = 1_{\{\varphi^+_{s,t}(x) > 0\}} W^+(dt) - 1_{\{\varphi^+_{s,t}(x) \leq 0\}} W^-(dt)$$

and

$$d\varphi^-_{s,t}(x) = 1_{\{\varphi^-_{s,t}(x) \leq 0\}} W^-(dt) - 1_{\{\varphi^-_{s,t}(x) > 0\}} W^+(dt)$$

with $x \in \mathbb{R}$. For $z \in \mathcal{C}$, let

$$\tau^z_s = \inf\{r \geq s : 1_{\{z \in \mathcal{C}^+\}} z e^{iW^+_{s,r}} + 1_{\{z \in \mathcal{C}^-\}} z e^{-iW^-_{s,r}} = 1 \text{ or } e^{il}\}$$

which is the hitting time of the particle to one of the vertices $\{1, e^{il}\}$. Let

$$A_{s,t} = \{\sup_{s<u<v<t} \max(|W^+_{u,v}|, |W^-_{u,v}|) < l\}\ .$$

On $A_{s,t}$, let

$$\varphi^0_{s,t}(z) = \begin{cases} 1_{\{z \in \mathcal{C}^+\}} z e^{iW^+_{s,t}} + 1_{\{z \in \mathcal{C}^-\}} z e^{-iW^-_{s,t}} & \text{if } \ t \leq \tau^z_s \\ e^{i\varphi^+_{s,t}(x)} 1_{\{\varphi^0_{s,\tau^z_s}(z) = 1\}} + e^{i(l - \varphi^-_{s,t}(l-x))} 1_{\{\varphi^0_{s,\tau^z_s}(z) = e^{il}\}} & \text{if } \ t > \tau^z_s \end{cases}$$

with $x = \arg(z)$, and set $\varphi^0_{s,t}(z) = z$ on $A^c_{s,t}$.

For $n \in \mathbb{N}$, let $\mathbb{D}_n = \{k2^{-n} : k \in \mathbb{Z}\}$. For $s > 0$, let $s_n = \sup\{u \in \mathbb{D}_n : u \leq s\}$ and $s^+_n = s_n + 2^{-n}$. For every $n \geq 1$ and $s \leq t$, we define

$$\varphi^n_{s,t} = \varphi^0_{t_n,t} \circ \varphi^0_{t_n - 2^{-n}, t_n} \circ \ldots \circ \varphi^0_{s^+_n, s^+_n + 2^{-n}} \circ \varphi^0_{s, s^+_n} \tag{2.7}$$

Let $\Omega^n_{s,t} = \{\sup_{\{s<u<v<t : |v-u| \leq 2^{-n}\}} \max(|W^+_{u,v}|, |W^-_{u,v}|) < l\}$, and let $\Omega_{s,t} = \cup_n \Omega^n_{s,t}$. Due to continuity of W, we have $\mathbb{P}(\Omega_{s,t}) = 1$. Then, φ is constructed as

$$\varphi_{s,t}(\omega) := \varphi^n_{s,t}(\omega) \tag{2.8}$$

for $\omega \in \Omega_{s,t}$, where $n = n_{s,t} = \inf\{k : \omega \in \Omega^k_{s,t}\}$, and $\varphi(\omega, z) := z$ for $\omega \in \Omega^c_{s,t}$.

2.2.2 Flow and Feller property

We first prove the flow property.

Lemma 2.2.1. *φ satisfies the flow property: for $s < t < u$, $\varphi_{s,u} = \varphi_{t,u} \circ \varphi_{s,t}$.*

Proof :

We first prove the flow property for φ^0. We will show that

$$\varphi_{u,v} = \varphi_{t,v} \circ \varphi_{u,t} \tag{2.9}$$

for all $u < t < v$, almost surely on $A_{u,v}$. Note that $A_{u,v} \subset A_{u,t} \cap A_{t,v}$. Suppose $z \in \mathcal{C}^+$, and assume $\varphi^0_{u,\tau^z_u}(z) = 1$ if $\tau^z_u < v$ for brevity of notation. When $v \leq \tau^z_u$, we have $Z := \phi^0_{u,t}(z) = ze^{iW^+_{u,t}}$. Then, $\tau^Z_t = \tau^z_u$ and (2.9) holds by additivity of the white noise. If $t \leq \tau^z_u < v$, then $\varphi^0_{u,t}(z) =: Z$, $\varphi^0_{u,v}(z) = e^{\varphi^+_{u,v}(x)}$ with $x = \arg(z)$, and $\tau^Z_t = \tau^z_u < v$. Therefore, we have

$$\varphi^0_{t,v} \circ \varphi^0_{u,t}(z) = \varphi^0_{t,v}(Z) = e^{\varphi^+_{t,v}(X)} = e^{\varphi^+_{t,v} \circ \varphi^+_{u,t}(x)} = e^{\varphi^+_{u,v}(x)} = \varphi^0_{u,v}(z)$$

where $X = \arg(Z)$, and we used the fact that φ^+ is a flow. If $t > \tau^z_u$, we have

$$\varphi^0_{u,v}(z) = e^{\varphi^+_{u,v}(x)} = e^{\varphi^+_{t,v} \circ \varphi^+_{u,t}(x)}$$

by the flow property of φ^+. On the event $A_{u,v} \cap \{t > \tau^z_u\}$, $\phi^+_{t,v}(Y)$ takes values in $(0, 2\pi)$ for $Y \in (0, 2\pi)$. Since $X := \varphi^+_{u,t}(x) \in (0, 2\pi)$ also in this case, this implies that $e^{\varphi^+_{t,v}(\varphi^+_{u,t}(x))} = e^{\varphi^+_{t,v}(X)} \in (0, 2\pi)$. Therefore, we can write

$$e^{\varphi^+_{t,v} \circ \varphi^+_{u,t}(x)} = e^{\varphi^+_{t,v}(X)} = \varphi^0_{t,v}(e^{iX}) = \varphi^0_{t,v}(Z) = \varphi^0_{t,v}(\varphi^0_{u,t}(z))$$

on $A_{u,v} \cap \{t > \tau^z_u\}$. Then, (2.9) follows. The same arguments are valid if $\varphi^0_{u,\tau^z_u}(z) = e^{il}$, but with slightly more involved notation.

Now by definition of $\varphi_{s,t}$ as given in (2.8) almost surely, we have $\varphi_{s,t} = \varphi^n_{s,t}$. Also, for all $s \leq u < v \leq t$ such that $|v - u| \leq 2^{-n}$, we have $\Omega^n_{s,t} \subset A_{u,v}$. Therefore, we can apply the flow property (2.9) on each interval $(u, v) \in \{(s, s^+_n), (s^+_n, s^+_n +$

$2^{-n}), \ldots, (t_n, t)\}$. In particular, we consider the finer mesh $s_m^+, s_m^+ + 2^{-m}, \ldots, t_m$ since $s_n^+, s_n^+ + 2^{-n}, \ldots, t_n$ are also in $\mathbb{D}_m$, for $m \geq n_{s,t}$. We get

$$\varphi_{s,t} = \varphi_{s,t}^n = \varphi_{t_m,t}^0 \circ \varphi_{t_m-2^{-m},t_m}^0 \circ \ldots \circ \varphi_{s_m^+,s_m^++2^{-m}}^0 \circ \varphi_{s,s_m^+}^0 = \varphi_{s,t}^m$$

for all $m \geq n_{s,t}$ almost surely.

For $m \geq \max(n_{s,u}, n_{s,t}, n_{t,u})$ and $s < t < u$, we have

$$\begin{aligned}
\varphi_{s,u} = \varphi_{s,u}^m &= \varphi_{u_m,u}^m \circ \ldots \circ \varphi_{t_m^+,t_m^++2^{-m}}^m \circ \varphi_{t_m,t_m^+}^m \circ \varphi_{s,t_m}^m \\
&= \varphi_{u_m,u}^m \circ \ldots \circ \varphi_{t_m^+,t_m^++2^{-m}}^m \circ \varphi_{t,t_m^+}^m \circ \varphi_{t_m,t}^m \circ \varphi_{s,t_m}^m \\
&= \varphi_{t,u}^m \circ \varphi_{s,t}^m \;\;=\;\; \varphi_{t,u} \circ \varphi_{s,u}
\end{aligned}$$

by the definition of φ^m in (2.7), and the flow property (2.9) for φ^0 on (t_m, t_m^+), equivalently for φ^m on this interval. $\square$

In the following lemma, we prove a sufficient condition for Feller property of φ. By [10, Lem. 1.11], an additional condition is

$$\lim_{t \to 0} \mathbb{E}[f(\varphi_{0,t}(z)] = f(z)$$

for all $z \in \mathcal{C}$, but this is satisfied trivially by bounded convergence theorem.

Lemma 2.2.2. *For all $f \in C(\mathcal{C})$ and $s \leq t$*

$$\lim_{d(x,y) \to 0} \mathbb{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] = 0$$

for every $x, y \in \mathcal{C}$.

Proof : First we will show that $\varphi_{s,t}^0$ is Fellerian. For this it is enough to show that $d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y))$ converges to zero in probability for all $t > 0$, and $x \in \mathcal{C}$ as $y \to x$. The proof relies on two facts: φ^+, φ^- are Fellerian and $W_{\tau_0^y}^+$ will converge to $W_{\tau_0^x}^+$ in probability as $y \to x$ (similarly for W^-), so $\mathbb{P}(\varphi_{0,\tau_0^y}^0(y) \neq \varphi_{0,\tau_0^x}^0(x))$ will converge to 0 as $y \to x$. Note that

$$d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y)) = d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y))1_{A_{0,t}} + d(x,y)1_{A_{0,t}^c}$$

Now assume that $x \in \mathcal{C}^+$. Fix $t > 0$. On the event $\{t < \tau_0^x\} \cap \{t < \tau_0^y\}$ we have $d(\varphi_{0,t}^0(x), \varphi_{0,t}^0(y)) = |\arg(y) - \arg(x)|$. The probability of the event $\{(t < \tau_0^x) \cap (t \geq \tau_0^y)\}$ will converge to 0 as $y \to x$, and on the event $\{t \geq \tau_0^x\} \cap \{t \geq \tau_0^y\}$, the Feller property of φ^+ and φ^- will give us the desired result. The other cases are similar.

Note that the flow $\varphi_{s,t}^n$ constructed in (6) is Fellerian since it is a composition of Fellerian mappings which are independent from each other. Now let $\epsilon > 0$ and $k \in \mathbb{N}$ such that $\mathbb{P}\{n_{s,t} > k\} < \epsilon$. Then for all $x, y \in \mathcal{C}$, since $\mathbb{P}(\Omega_{s,t}) = 1$, we have

$$\mathbb{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] \leq \sum_{n \leq k} \mathbb{E}[(f \circ \varphi_{s,t}^n(x) - f \circ \varphi_{s,t}^n(y))^2] + 4\epsilon ||f||_\infty^2$$

since $\varphi_{s,t}^n$ is Fellerian for all n, we get

$$\lim_{d(x,y) \to 0} \mathbb{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] \leq 4\epsilon ||f||_\infty^2 \ .$$

As ϵ is arbitrary, the result follows. $\square$

The following is an explicit formula for the flow.

Theorem 2.2.3. *There exists a stochastic flow φ on $\mathcal{C}$ satisfying (1.2) and such that*

$$\varphi_{s,t}(z) = \begin{cases} e^{i\varphi_{s,t}^+(arg(z))} & \text{if } t < \gamma_s^{z,+} \\ e^{il - i\varphi_{s,t}^-(l - arg(z))} & \text{if } t < \gamma_s^{z,-} \end{cases} \tag{2.10}$$

where $arg(z) \in [-2\pi + l, l]$, $z \in \mathcal{C}$, almost surely for all $s < t$ where $\gamma_s^{z,+} = \inf\{r \geq s : \varphi_{s,r}^+(arg(z)) \notin (-2\pi + l, l)\}$ and $\gamma_s^{z,-} = \inf\{r \geq s : \varphi_{s,r}^-(l - arg(z)) \notin (-2\pi + l, l)\}$.

Proof. SDE (2.1) is satisfied on each edge by construction of φ as a stochastic flow and (2.6) follows. We have $\varphi_{s,t} = \varphi_{s,t}^0$ on $A_{s,t}$. It follows that (2.10) holds on $A_{s,t}$ as both pieces of the definition of φ^0 coincide with φ^+ (and φ^-) after a transformation. We will show this on $\{t < \gamma_s^{z,+}\}$ ($t < \gamma_s^{z,-}$ is similar). For $m \geq n_{s,t}$, we have

$$\varphi_{s,t} = \varphi_{t_m,t}^0 \circ \ldots \circ \varphi_{s_m^+, s_m^+ + 2^{-m}}^0 \circ \varphi_{s,s_m^+}^0 \tag{2.11}$$

If $t < \gamma_s^{z,+}$, then each point in the mesh $s_m^+, s_m^+ + 2^{-m}, \ldots, t_m$ is also less than $\gamma_s^{z,+}$. Therefore, each $\varphi_{u,v}^0$, where $(u, v) \in \{(s, s_m^+), (s_m^+, s_m^+ + 2^{-m}), \ldots, (t_m, t)\}$, will map into $\mathcal{C}^- \cup \{1\} \cup \mathcal{C}^+$, and will coincide with $\varphi^+_{u,v}$ after the isometric transformation from $\mathcal{C}$ to $\mathbb{R}$. That is, we have

$$\begin{aligned}\varphi_{s_m^+, s_m^+ + 2^{-m}} \circ \varphi_{s, s_m^+}(z) &= \exp[i\varphi^+_{s_m^+, s_m^+ + 2^{-m}}(\text{arg}(e^{i\varphi^+_{s,s_m^+}(x)}))] \\ &= \exp[i\,\varphi^+_{s_m^+, s_m^+ + 2^{-m}} \circ \varphi^+_{s,s_m^+}(x)] \;=\; e^{i\varphi^+_{s,s_m^+ + 2^{-m}}(arg(z))}\end{aligned}$$

with $x = \text{arg}(z)$, and in view of (2.11) we get

$$\varphi_{s,t}(z) = e^{i\varphi^+_{s,t}(arg(z))}$$

almost surely, when $t < \gamma_s^{z,+}$. □

2.2.3 Coalescence

In this section, we will prove that two particles on the unit circle $\mathcal{C}$ with respect to the flow given in Theorem 2.2.3 meet in finite time almost surely. Consider the two particle motion as represented by (e^{iX_t}, e^{iY_t}) where X and Y are solutions of SDE (2.1), with $(e^{iX_0}, e^{iY_0}) = (e^{ix}, 1)$, $0 < x < l$. This is an embedding of $\mathcal{C}$ into the plane which allows us to focus on the processes X and Y, where each one has the same increments as B^+ in $\bigcup_{n\in\mathbb{Z}} (2n\pi, 2n\pi + l]$ and as $-B^-$ elsewhere. Furthermore, we identify each interval of the form $[2n\pi, 2n\pi + l]$, $n \neq 0$, with the interval $[0, l]$, and the intervals of the form $[-2n\pi + l, -2(n-1)\pi]$, $n \neq 1$, with $[-2\pi + l, 0]$ since the argument of a point on the circle is mapped to the latter intervals. Then, the coalescence time is given by

$$T = \inf\{s \geq 0 : (X_s, Y_s) = (0, 0) \text{ or } (X_s, Y_s) = (l, -2\pi + l)\}$$

and the region

$$D = [0, l] \times [-2\pi + l, 0]$$

is considered. Define, as before, $A_t = \int_0^{t\wedge T} 1_{\{(X_s,Y_s)\in D\}} ds$, its right continuous inverse $\kappa_t = \inf\{s > 0 : A_s > t\}$, and the process $(X_t^r, Y_t^r) = (X_{\kappa_t}, Y_{\kappa_t})$ together with the local times

$$L_t^1 = \frac{1}{2} L_{\kappa_t}(X, 0) \quad L_t^2 = \frac{1}{2} L_{\kappa_t}(X, l) \quad L_t^3 = \frac{1}{2} L_{\kappa_t}(Y, 0) \quad L_t^4 = \frac{1}{2} L_{\kappa_t}(Y, -2\pi + l)$$

where $L_t(X,a)$ and $L_t(Y,b)$ denote the local times of X and Y at a and b, respectively, for $a,b \in \mathbb{R}$.

Lemma 2.2.4. *Suppose $(X_0,Y_0) = (x,0)$ with $0 < x < l$. The process (X^r,Y^r) is a correlated reflected Brownian motion in the bounded domain D which is stopped when it hits $(0,0)$ or $(l,-2\pi+l)$. That is, there exist two Brownian motions B^1 and B^2 with $|\langle B^1,B^2\rangle_t| \leq \rho\, dt$ such that for all $t < A_T$*

$$(X_t^r,Y_t^r) = (x,0) + (B_t^1,-B_t^2) + (L_t^1,L_t^1) - (L_t^2,L_t^2) - (L_t^3,L_t^3) + (L_t^4,L_t^4)$$

Proof : Let B^+ and B^- be two Brownian motions with $H_t := \langle B^+,B^-\rangle_t$. Then, we consider

$$X_t = x + B_t^+ \qquad Y_t = \int_0^t 1_{\{Y_s>0\}} dB_s^+ - \int_0^t 1_{\{Y_s\leq 0\}} dB_s^-$$

according to SDE (2.1) until the stopping time $\bar\tau_1 = \tau_1 \wedge \tau_2 \wedge \tau_3$ where

$$\tau_1 = \inf\{s\geq 0 : X_s = 0\} \quad \tau_2 = \inf\{s \geq 0 : X_s = l\} \quad \tau_3 = \inf\{s\geq 0 : Y_s = -2\pi + l\}\ .$$

Note that for $t \leq \bar\tau_1$, (X_t^r,Y_t^r) takes values in $D = [0,l]\times[-2\pi+l,0]$ and Y_t^r is a reflected Brownian motion on $[0,2\pi-l]$. By Tanaka formula, there is a Brownian motion B^2 satisfying

$$L_t^3 = -Y_t^r - B_t^2$$

where L^3 is the local time of Y_t^r at 0, and $B_t^2 = -\int_0^{\kappa_t} 1_{\{Y_s\leq 0\}} dY_s = \int_0^{\kappa_t} 1_{\{Y_s\leq 0\}} dB_s^-$. Putting $B_t^1 = \int_0^{\kappa_t} 1_{\{Y_s\leq 0\}} dB_s^+$, we observe that it is also a Brownian motion and

$$X_t^r = x + B_t^1 - L_t^3$$

holds by the construction of X and the fact that $\int_0^{\kappa_t} 1_{\{Y_s>0\}} dY_s = -L_t^3$. Note that we have $|d\langle B^1,B^2\rangle_t| \leq \rho\, dt$ just as in the proof of Lemma 2.1.4. Thus, (X_t^r,Y_t^r) is a correlated reflected Brownian motion on $[0,l]\times[-2\pi+l,0]$. More explicitly, we have

$$(X_t^r,Y_t^r) = (x,0) + (B_t^1,-B_t^2) - (L_t^3,L_t^3) \qquad t \leq \bar\tau_1\ .$$

We will consider the three possibilities at $\bar{\tau}_1$ in i)-iii) below depending on the value of $\bar{\tau}_1$ and continue the construction by considering a new origin and a new subgraph for X or Y to start with.

i) If $\bar{\tau}_1 = \tau_1$, we have $(X_{\bar{\tau}_1}, Y_{\bar{\tau}_1}) = (0, y)$, for some $y \in [-2\pi + l, 0]$. Now, replace X, Y, B^+, B^- with $X_{\bar{\tau}_1+.}, Y_{\bar{\tau}_1+.}, B^+_{\bar{\tau}_1+.}, B^-_{\bar{\tau}_1+.}$. Define the following stopping times

$$\tau_4 = \inf\{s \geq 0 : X_s = l\} \quad \tau_5 = \inf\{s \geq 0 : Y_s = 0\} \quad \tau_6 = \inf\{s \geq 0 : Y_s = -2\pi + l\}$$

and let $\bar{\tau}_2 = \tau_4 \wedge \tau_5 \wedge \tau_6$ to confine the process in region D. Construct X and Y in the interval $(\bar{\tau}_1, \bar{\tau}_2]$ as

$$Y_t = y - B^-_t \qquad X_t = \int_0^t 1_{\{X_s > 0\}} dB^+_s - \int_0^t 1_{\{X_s \leq 0\}} dB^-_s$$

Then, X^r_t is a reflected Brownian motion on $[0, l]$ with local time L^1_t at 0:

$$L^1_t = X^r_t - B^1_t$$

by Tanaka formula and again by construction Y satisfies

$$Y^r_t = -B^2_t + L^1_t$$

where $B^1_t = \int_0^{\kappa_t} 1_{\{X_s>0\}} dB^+_s$ and $B^2_t = \int_0^{\kappa_t} 1_{\{X_s>0\}} dB^-_s$ as before. More explicitly, we have

$$(X^r_t, Y^r_t) = (0, y) + (B^1_t, -B^2_t) + (L^1_t, L^1_t) \qquad t \in (\bar{\tau}_1, \bar{\tau}_2]$$

ii) If $\bar{\tau}_1 = \tau_2$, then $(X_{\bar{\tau}_1}, Y_{\bar{\tau}_1}) = (l, y)$, $y \in [-2\pi + l, 0]$. We replace X, Y, B^+, B^- with $X_{\bar{\tau}_1+.}, Y_{\bar{\tau}_1+.}, B^+_{\bar{\tau}_1+.}, B^-_{\bar{\tau}_1+.}$. Define the following stopping times

$$\tau_4 = \inf\{s \geq 0 : X_s = 0\} \quad \tau_5 = \inf\{s \geq 0 : Y_s = 0\} \quad \tau_6 = \inf\{s \geq 0 : Y_s = -2\pi + l\}$$

and let $\bar{\tau}_2 = \tau_4 \wedge \tau_5 \wedge \tau_6$. Construct X and Y in the interval $(\bar{\tau}_1, \bar{\tau}_2]$ as

$$Y_t = y - B^-_t \qquad X_t = l - \int_0^t 1_{\{X_s > l\}} dB^-_s + \int_0^t 1_{\{X_s \leq l\}} dB^+_s$$

Then, X^r_t is a reflected Brownian motion on $[0, l]$ with local time L^2_t at l:

$$L^2_t = l + B^1_t - X^r_t$$

by Tanaka formula where $B_t^1 = \int_0^{\kappa_t} 1_{\{X_s \le l\}} dB_s^+$ is a Brownian motion. Again, by construction Y satisfies

$$Y_t^r = y - B_t^2 - L_t^2$$

where $B_t^2 = \int_0^{\kappa_t} 1_{\{X_s \le l\}} dB_s^-$ is a Brownian motion. We get

$$(X_t^r, Y_t^r) = (l, y) + (B_t^1, -B_t^2) - (L_t^2, L_t^2) \qquad t \in (\bar{\tau}_1, \bar{\tau}_2]$$

iii) If $\bar{\tau}_1 = \tau_3$, then $(X_{\bar{\tau}_1}, Y_{\bar{\tau}_1}) = (x', -2\pi + l)$, for some $x' \in (0, l)$. Then, we replace X, Y, B^+, B^- with $X_{\bar{\tau}_1+.}, Y_{\bar{\tau}_1+.}, B^+_{\bar{\tau}_1+.}, B^-_{\bar{\tau}_1+}$ and define

$$\tau_4 = \inf\{s \ge 0 : X_s = 0\} \quad \tau_5 = \inf\{s \ge 0 : X_s = l\} \quad \tau_6 = \inf\{s \ge 0 : Y_s = 0\}$$

and let $\bar{\tau}_2 = \tau_4 \wedge \tau_5 \wedge \tau_6$. In $(\bar{\tau}_1, \bar{\tau}_2]$, X and Y satisfy

$$X_t = x' + B_t^+ \qquad Y_t = -2\pi + l + \int_0^t 1_{\{Y_s \le -2\pi + l\}} dB_s^+ - \int_0^t 1_{\{Y_s > -2\pi + l\}} dB_s^-$$

Similar to previous cases, Y_t^r is a reflected Brownian motion on $[-2\pi + l, 0]$ with local time L_t^4 at $-2\pi + l$:

$$L_t^4 = Y_t^r + B_t^2 + 2\pi - l$$

where $B_t^2 = \int_0^{\kappa_t} 1_{\{Y_s > -2\pi + l\}} dB_s^-$, and $X_t^r = x' + B_t^1 + L_t^4$ where $B^1 = \int_0^{\kappa_t} 1_{\{Y_s > -2\pi + l\}} dB_s^+$. More explicitly, we have

$$(X_t^r, Y_t^r) = (x, -2\pi + l) + (B_t^1, -B_t^2) + (L_t^4, L_t^4) \quad t \in (\bar{\tau}_1, \bar{\tau}_2]$$

As a result, we get the desired representation. $\square$

Note that other starting points in D can be handled similarly. Now, consider the process $V_t := \frac{X_t^r - Y_t^r}{\sqrt{2}}$. It follows that $V_t = \frac{x}{\sqrt{2}} + \frac{B_t^1 + B_t^2}{\sqrt{2}}$, which is a martingale with $\lim_{t \to \infty} \langle V \rangle_t = \infty$. That is, V_t is a time changed Brownian motion. Define $T_0 = \inf\{t \ge 0 : V_t = 0 \text{ or } V_t = 2\pi\}$. Then, $T_0 < \infty$ almost surely. To see this, note that $0 \le V_t \le 2\pi$ for all t, and $V_t = 0$ if and only if $X_t = Y_t = 0$ and $V_t = 2\pi$ if and only if $X_t = l$ and $Y_t = -2\pi + l$. Therefore, the correlated reflected Brownian motion will hit one of the corners $(0, 0)$ or $(l, -2\pi + l)$ in D in finite time, almost surely.

Lemma 2.2.5. $\mathbb{P}\{T < \infty\} = 1.$

Proof : Let $U_t = \frac{X_t^r + Y_t^r}{\sqrt{2}}$, and note that $\frac{-2\pi + l}{\sqrt{2}} \leq U_t \leq \frac{l}{\sqrt{2}}$. Putting $L^r = \sqrt{2}(L^1 - L^2 - L^3 + L^4)$, we get

$$U_t = \frac{x}{\sqrt{2}} + \frac{B_t^1 - B_t^2}{\sqrt{2}} + L_t^r$$

and

$$\mathbb{E}[U_{t \wedge T_0}] = \frac{x}{\sqrt{2}} + \mathbb{E}[L_{t \wedge T_0}^r]$$

It follows that $L_{t \wedge T_0}^r$ is finite for each $t > 0$ since U_t is bounded. This yields $L_{T_0}^r < \infty$. In particular, $L_{T_0}^1$, which is equal to $\frac{1}{2} L_T(X, 0)$, is finite almost surely. Since X is a Brownian motion, this implies that $T < \infty$ almost surely. □

2.2.4 *Distribution of* T_0

In this section, we will use the notation of Lemma 2.2.4 and the subsequent definitions, and assume that $H_t = \rho t$ with $\rho \in [0, 1)$. Note that $\langle V \rangle_t = (1 + \rho)t$. Define $M_t = V_t - V_0$. Then M is a martingale with $M_0 = 0$. Since $\langle M \rangle_t = (1 + \rho)t$ it is a time changed Brownian motion. Then, by the same arguments as given in Section 2.1.4, $B_s := M_{s/(1+\rho)}$ is a one dimensional Brownian motion. Recall that $T_0 = \inf\{t \geq 0 : V_t = 0 \text{ or } V_t = 2\pi\}$ and let

$$S = \inf\{s \geq 0 : B_s = -\frac{x}{\sqrt{2}} \ \text{ or } \ B_s = 2\pi - \frac{x}{\sqrt{2}}\} \ .$$

We have $\mathbb{P}\{T_0 \in dt\} = P\{S \in (1 + \rho)dt\}$. Since the distribution of S is known [17] as

$$\mathbb{P}\{S \in dt\} = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi t^3}} e^{-\frac{(2\pi(k+1) - \frac{x}{\sqrt{2}})^2}{2t}} dt \ ,$$

the distribution of T_0 is given by

$$\mathbb{P}\{T_0 \in dt\} = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi(1+\rho)t)^3}} e^{-\frac{(2\pi(k+1) - \frac{x}{\sqrt{2}})^2}{2(1+\rho)t}} (1 + \rho)dt$$

This leads to the complementary distribution function

$$\bar{F}_\rho(t) = P\{T_0 \geq t\} = \sum_{k=-\infty}^{+\infty} \int_t^\infty (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi(1+\rho)s)^3}} e^{-\frac{(2\pi(k+1)-\frac{x}{\sqrt{2}})^2}{2(1+\rho)s}} (1+\rho)ds$$

where the interchange of integration and summation is justified by uniform convergence of the series. By a change of variable from $(1+\rho)s$ to s, we get

$$\begin{aligned}
\mathbb{P}\{T_0 \geq t\} &= \sum_{k=-\infty}^{+\infty} \int_{(1+\rho)t}^\infty (-1)^k \frac{2\pi(k+1) - \frac{x}{\sqrt{2}}}{\sqrt{2\pi s^3}} e^{-\frac{(2\pi(k+1)-\frac{x}{\sqrt{2}})^2}{2s}} ds \\
&= \int_{(1+\rho)t}^\infty P\{S \in ds\} = \mathbb{P}\{S \geq (1+\rho)t\}
\end{aligned}$$

As a result, it becomes more likely for the reflected process to hit one of the corners before a fixed time t when the correlation coefficient ρ increases.

Chapter 3

BROWNIAN FLOWS WITH RANK-BASED CHARACTERISTICS AND TWO-DIMENSIONAL TANAKA EQUATION

3.1 The Equation

Using the same notation as in [18], we define

$$\Sigma_+ = \begin{pmatrix} \rho\cos\varphi & -\rho\sin\varphi \\ \sigma\varepsilon\sin\varphi & \sigma\varepsilon\cos\varphi \end{pmatrix} \quad \Sigma_- = \begin{pmatrix} \sigma\cos\vartheta & -\sigma\sin\vartheta \\ \rho\delta\sin\vartheta & \rho\delta\cos\vartheta \end{pmatrix}$$

where $\rho, \sigma, \varepsilon, \delta \in \{-1, +1\}$, $0 \le \varphi, \vartheta \le 2\pi$ and we set $\Sigma(x_1, x_2) = \Sigma_+ \mathbf{1}_{\{x_1 > x_2\}} + \Sigma_- \mathbf{1}_{\{x_1 \le x_2\}}$. We study the following stochastic differential equation

$$d\mathbf{X}(t) = \Sigma(\mathbf{X}(t))d\mathbf{W}(t) \tag{3.1}$$

where $\mathbf{W} = (W^1, W^2)'$ is a two-dimensional standard Brownian motion, and apostrophe denotes transposition. In [18], $\rho^2 + \sigma^2 = 1$ is assumed. However, we choose them to be ± 1 so that one-point motion of (3.1) is a two-dimensional Brownian motion.

We perform a clockwise rotation on $\mathbb{R}^2$ by $\pi/4$ with the matrix $R = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and set $\tilde{\mathbf{X}} = R\mathbf{X}$, to get

$$d\tilde{X}_t^1 = 1_{\{\tilde{X}_t^2 < 0\}} d\tilde{B}_t^1 + 1_{\{\tilde{X}_t^2 \ge 0\}} d\tilde{B}_t^2 \tag{3.2}$$

$$d\tilde{X}_t^2 = 1_{\{\tilde{X}_t^2 < 0\}} d\tilde{B}_t^3 + 1_{\{\tilde{X}_t^2 \ge 0\}} d\tilde{B}_t^4 \tag{3.3}$$

where $R\Sigma_+\mathbf{W} = (\tilde{B}^1, \tilde{B}^3)'$ and $R\Sigma_-\mathbf{W} = (\tilde{B}^2, \tilde{B}^4)'$ and they are two-dimensional Brownian motions. These Brownian motions also satisfy

$$\begin{aligned}
d\langle \tilde{B}^1, \tilde{B}^2\rangle_t &= \frac{1}{2}\big(\rho\sigma(1+\varepsilon\delta)\cos(\varphi-\vartheta) + (\varepsilon-\delta)\sin(\varphi-\vartheta)\big)\\
d\langle \tilde{B}^3, \tilde{B}^4\rangle_t &= \frac{1}{2}\big(\rho\sigma(1+\varepsilon\delta)\cos(\varphi-\vartheta) - (\varepsilon-\delta)\sin(\varphi-\vartheta)\big)\\
d\langle \tilde{B}^1, \tilde{B}^4\rangle_t &= \frac{1}{2}\big(\rho\sigma(-1+\varepsilon\delta)\cos(\varphi-\vartheta) - (\varepsilon+\delta)\sin(\varphi-\vartheta)\big) = d\langle \tilde{B}^2, \tilde{B}^3\rangle_t
\end{aligned}$$

This rotation will simplify the analysis of our problem and while proving the existence of weak and strong solutions and the flow solution we will work with (3.2) and (3.3).

3.2 Strong and Weak Solutions

We know from [18, Thm. 5.2] that Equation (3.1) fails to have a strong solution if and only if $(\mathbf{e}_1 - \mathbf{e}_2)'\Sigma_- = -(\mathbf{e}_1 - \mathbf{e}_2)'\Sigma_+$, where $\mathbf{e}_1 = (1,0)'$ and $\mathbf{e}_2 = (0,1)'$. On the other hand, we emphasize the equivalence of (3.1) and (3.3) in terms of the existence of weak and strong solutions in the following

Proposition 3.2.1. *Equation (3.1) fails to have a strong solution if and only if Equation (3.3) is Tanaka equation.*

Proof. First, we note that if $\mathbf{X} = (X^1, X^2)'$ is a solution of the Equation (3.1) then, $\mathcal{F}^{(\tilde{X}^1,\tilde{X}^2)} = \mathcal{F}^{(X^1,X^2)}$, and $\mathcal{F}^{\tilde{B}^j} \subset \mathcal{F}^{(W^1,W^2)}$ holds for $j = 1,2,3,4$.

If (3.3) is not Tanaka equation then, either $d\langle \tilde{B}^3, \tilde{B}^4\rangle_t = 1$ or $|d\langle \tilde{B}^3, \tilde{B}^4\rangle_t| < 1$. In the first case, it trivially has a strong solution. In the second case, it has a strong solution by Theorem 2.1.7. Therefore the relation $\mathcal{F}^{\tilde{X}^2} \subset \mathcal{F}^{(\tilde{B}^3,\tilde{B}^4)}$ holds. It is also clear that the relation $\mathcal{F}^{\tilde{X}^1} \subset \mathcal{F}^{(\tilde{X}^2,\tilde{B}^1,\tilde{B}^2)}$ holds, too. It follows that $\mathcal{F}^{(X^1,X^2)} \subset \mathcal{F}^{(W^1,W^2)}$.

We first note that $\mathcal{F}^{(W^1,W^2)} = \mathcal{F}^{(\tilde{B}^1,\tilde{B}^3)}$. Now, assume that (3.3) is Tanaka equation, that is $d\langle \tilde{B}^3, \tilde{B}^4\rangle_t = -1$, so that $\mathcal{F}^{\tilde{B}^3} \subsetneq \mathcal{F}^{\tilde{X}^2}$ and (3.1) has a strong solution. Then we have the following relations

$$\mathcal{F}^{(\tilde{X}^2,\tilde{B}^1)} \subset \mathcal{F}^{(W^1,W^2)} = \mathcal{F}^{(\tilde{B}^3,\tilde{B}^1)} \subsetneq \mathcal{F}^{(\tilde{X}^2,\tilde{B}^1)}$$

which is a contradiction.

We also note that (3.3) is Tanaka equation if and only if $(\mathbf{e}_1 - \mathbf{e}_2)'\Sigma_- = -(\mathbf{e}_1 - \mathbf{e}_2)'\Sigma_+$ which is exactly the condition in [18, Thm. 5.2].

□

3.3 Flow Solutions

A flow solution of (3.1) is a stochastic flow of mappings $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)$[10, Definition 1.6]. That is, $\{\boldsymbol{\varphi}_{s,t}, s \leq t\}$ is a family of random variables, and for all $s \leq t$, $\boldsymbol{\varphi}_{s,t}$ is a measurable mapping from $\left(\mathbb{R}^2 \times \Omega, \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{A}\right)$ to $\left(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)\right)$ satisfying

$$d\boldsymbol{\varphi}_{s,t} = \Sigma(\boldsymbol{\varphi}_{s,t})d\boldsymbol{W}_{s,t}$$

where $\mathbf{W}_{s,t} = (W^1_{s,t}, W^2_{s,t})'$ is a white noise, or equivalently

$$\varphi^1_{s,t}(x) = x_1 + \int_s^t 1_{\{\varphi^1_{s,u} > \varphi^2_{s,u}\}} dB^1_{s,u} + \int_s^t 1_{\{\varphi^1_{s,u} \leq \varphi^2_{s,u}\}} dB^2_{s,u} \tag{3.4}$$

$$\varphi^1_{s,t}(x) = x_2 + \int_s^t 1_{\{\varphi^1_{s,u} > \varphi^2_{s,u}\}} dB^3_{s,u} + \int_s^t 1_{\{\varphi^1_{s,u} \leq \varphi^2_{s,u}\}} dB^4_{s,u} \tag{3.5}$$

where $(B^1, B^3)' = \Sigma_+ \boldsymbol{W}$ and $(B^2, B^4)' = \Sigma_- \boldsymbol{W}$. We also have $R(B^1, B^3)' = (\tilde{B}^1, \tilde{B}^3)'$ and $R(B^2, B^4)' = (\tilde{B}^2, \tilde{B}^4)'$.

There are three cases for $\tilde{B}^3$ and $\tilde{B}^4$ in (3.3). $d\langle \tilde{B}^3, \tilde{B}^4 \rangle_t$ is equal to 1, -1, or a value between the two. In the first case, it trivially has a flow solution since it is a one-dimensional Brownian motion. In the second and third cases, we know from Theorem 2.1.7, [20, Thm. 1], [19, Prop. 7.1], and [10, Example 4.4.3] that Equation (3.3) has a flow solution. On the other hand, the right hand side of Equation (3.2) does not depend on $\tilde{X}^1$ so we can always integrate it, after $\tilde{X}^2$ is solved for. Therefore, we have the following flow solution $\tilde{\boldsymbol{\varphi}}$ for Equations (3.2) and (3.3)

$$\tilde{\varphi}^1_{s,t}(\tilde{x}_1,\tilde{x}_2) = \tilde{x}_1 + \int_s^t 1_{\{\tilde{\varphi}^2_{s,u}(\tilde{x}_2)<0\}} d\tilde{B}^1_{s,u} + \int_s^t 1_{\{\tilde{\varphi}^2_{s,u}(\tilde{x}_2)\geq 0\}} d\tilde{B}^2_{s,u} \tag{3.6}$$

$$\tilde{\varphi}^2_{s,t}(\tilde{x}_2) = \tilde{x}_2 + \int_s^t 1_{\{\tilde{\varphi}^2_{s,u}(\tilde{x}_2)<0\}} d\tilde{B}^3_{s,u} + \int_s^t 1_{\{\tilde{\varphi}^2_{s,u}(\tilde{x}_2)\geq 0\}} d\tilde{B}^4_{s,u} \tag{3.7}$$

Setting $\boldsymbol{\varphi}_{s,t} = R^{-1}\tilde{\boldsymbol{\varphi}}_{s,t}$ we see that Equations (3.4) and (3.5) are satisfied, so we have the following

Theorem 3.3.1. *There is a flow solution of Equation (3.1) which is defined as $\boldsymbol{\varphi} = R^{-1}\tilde{\boldsymbol{\varphi}}$.*

Proof. Since $\tilde{\boldsymbol{\varphi}}$ is a flow and R^{-1} is a deterministic rotation the result follows.

□

3.4 Coalescence

Given two solutions $\boldsymbol{X}$ and $\boldsymbol{Y}$ of (3.1) starting at different positions $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{y} \in \mathbb{R}^2$ respectively, if we can find a stopping time T which is finite with positive probability and satisfies $\boldsymbol{X}_T = \boldsymbol{Y}_T$ then we say the flow is coalescing. We study coalescence because as it is proved in [10], all solutions of the SDE can be given by filtering the coalescing solution.

We have $\boldsymbol{X}_T = \boldsymbol{Y}_T$ if and only if $\tilde{\boldsymbol{X}}_T = \tilde{\boldsymbol{Y}}_T$ where $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Y}}$ start at $R\boldsymbol{x} = \tilde{\boldsymbol{x}}$ and $R\boldsymbol{y} = \tilde{\boldsymbol{y}}$ respectively. Therefore we will study coalescence for the system of stochastic differential equations in (3.2) and (3.3), instead of (3.1).

The Brownian motions in (3.2) and (3.3) have the following cross variations

$$d\langle \tilde{B}^1, \tilde{B}^2\rangle_t = \frac{1}{2}\big(\rho\sigma(1+\varepsilon\delta)\cos(\varphi-\vartheta) + (\varepsilon-\delta)\sin(\varphi-\vartheta)\big) \tag{3.8}$$

$$d\langle \tilde{B}^3, \tilde{B}^4\rangle_t = \frac{1}{2}\big(\rho\sigma(1+\varepsilon\delta)\cos(\varphi-\vartheta) - (\varepsilon-\delta)\sin(\varphi-\vartheta)\big) \tag{3.9}$$

We study coalescence in two broad categories, namely $\varepsilon = -\delta$ and $\varepsilon = \delta$.

3.4.1 $\varepsilon = -\delta$

In this case, the cross variation processes in (3.8) and (3.9) simplify to

$$d\langle \tilde{B}^1, \tilde{B}^2 \rangle_t = \varepsilon \sin(\varphi - \vartheta) \quad d\langle \tilde{B}^3, \tilde{B}^4 \rangle_t = -\varepsilon \sin(\varphi - \vartheta)$$

Now, we have three possibilities: $\varepsilon \sin(\varphi - \vartheta) = 1$, $\varepsilon \sin(\varphi - \vartheta) = -1$, or $|\sin(\varphi - \vartheta)| < 1$.

If $\epsilon \sin(\varphi - \vartheta) = 1$, then (3.2) and (3.3) turn into

$$\begin{aligned} d\tilde{X}_t^1 &= d\tilde{B}^1 \\ d\tilde{X}_t^2 &= \operatorname{sgn}(\tilde{X}_t^2) d\tilde{B}_t^3 \end{aligned}$$

Consider the flow solutions $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ of these equations as given in (3.6) and (3.7).

$$\begin{aligned} \tilde{\varphi}_{s,t}^1(\tilde{x}_1) &= \tilde{x}_1 + \int_s^t d\tilde{B}_{s,u}^1 \\ \tilde{\varphi}_{s,t}^2(\tilde{x}_2) &= \tilde{x}_2 + \int_s^t \operatorname{sgn}(\tilde{\varphi}_{s,u}^2) d\tilde{B}_{s,u}^3 \end{aligned}$$

Starting with two different initial positions $(\tilde{x}_1, \tilde{x}_2)$ and $(\tilde{y}_1, \tilde{y}_2)$, the two particles evolve according to this flow equation. Note that the difference $\tilde{\varphi}_{s,t}^1(\tilde{x}_1) - \tilde{\varphi}_{s,t}^1(\tilde{y}_1) = \tilde{x}_1 - \tilde{y}_1$ is fixed. Also, it is known that given $\tilde{x}_2$, $\tilde{y}_2$ there exists an almost surely finite stopping time T such that $\tilde{\varphi}_{s,T}^2(\tilde{x}_2) = \tilde{\varphi}_{s,T}^2(\tilde{y}_2)$[20, Thm. 1], [19, Prop. 7.1], [10, Example 4.4.3]. If $\tilde{x}_1 = \tilde{y}_1$ then, we also have $\tilde{\varphi}_{s,T}^1(\tilde{x}_1) = \tilde{\varphi}_{s,T}^1(\tilde{y}_1)$. Therefore, we have coalescence if and only if $\tilde{x}_1 = \tilde{y}_1$,that is, $x_1 + x_2 = y_1 + y_2$.

If $\varepsilon \sin(\varphi - \vartheta) = -1$, then (3.2) and (3.3) turn into

$$\begin{aligned} d\tilde{X}_t^1 &= \operatorname{sgn}(\tilde{X}_t^2) d\tilde{B}_t^1 \\ d\tilde{X}_t^2 &= d\tilde{B}_t^3 \end{aligned}$$

Consider the flow solutions $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ of these equations as given in (3.6) and (3.7).

$$\begin{aligned} \tilde{\varphi}_{s,t}^1(\tilde{x}_1) &= \tilde{x}_1 + \int_s^t \operatorname{sgn}(\tilde{\varphi}_{s,u}^2) d\tilde{B}_{s,u}^1 \\ \tilde{\varphi}_{s,t}^2(\tilde{x}_2) &= \tilde{x}_2 + \int_s^t d\tilde{B}_{s,u}^3 \end{aligned}$$

Note that given $\tilde{x}_2$ and $\tilde{y}_2$ the difference $\tilde{\varphi}^2_{s,t}(\tilde{x}_2) - \tilde{\varphi}^2_{s,t}(\tilde{y}_2) = \tilde{x}_2 - \tilde{y}_2$ is fixed. Then, for any possibility of coalescence we need $\tilde{x}_2 = \tilde{y}_2$. However, this implies that $\tilde{\varphi}^1_{s,t}(\tilde{x}_1) - \tilde{\varphi}^1_{s,t}(\tilde{y}_1) = \tilde{x}_1 - \tilde{y}_1$ for any $\tilde{x}_1$ and $\tilde{y}_1$. But if we also assume that $\tilde{x}_1 = \tilde{y}_1$, then the two particles will start at the same position. Hence, if two particles start at different positions we know that they will not meet. Therefore, there is no coalescence.

If $|\sin(\varphi - \vartheta)| < 1$, then this case needs to be analyzed in detail.

3.4.2 $\varepsilon = \delta$

In this case, the cross variation processes in (3.8) and (3.9) simplify to

$$d\langle \tilde{B}^1, \tilde{B}^2 \rangle_t = d\langle \tilde{B}^3, \tilde{B}^4 \rangle_t = \rho\sigma \cos(\varphi - \vartheta)$$

Again, we have three possibilities: $\rho\sigma\cos(\varphi - \vartheta) = 1$, $\rho\sigma\cos(\varphi - \vartheta) = -1$, or $|\cos(\varphi - \vartheta)| < 1$.

If $\rho\sigma\cos(\varphi - \vartheta) = 1$, then this suggests that $\tilde{B}^1 = \tilde{B}^2$ and $\tilde{B}^3 = \tilde{B}^4$, so given two solutions of (3.2) and (3.3) the distance between them will be fixed at all times. Therefore, there is no coalescence.

If $\rho\sigma\cos(\varphi - \vartheta) = -1$, then this case needs to be analyzed in detail.

If $|\cos(\varphi - \vartheta)| < 1$, then this case needs to be analyzed in detail.

3.4.3 *Reflected Process*

Our purpose in this chapter is to construct a reflected process which we will use in our future work when analyzing the coalescence. We also conjecture that this process is a reflected Brownian motion.

Let $\mathcal{H}^+ := \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$, $\mathcal{H}^- := \{(x,y) \in \mathbb{R}^2 : y \leq 0\}$ and $D := \mathcal{H}^- \times \mathcal{H}^+$. Given two solutions $\boldsymbol{X} = (X^1, X^2)'$ and $\boldsymbol{Y} = (Y^1, Y^2)'$ of the system of differential equations in (3.2) and (3.3), starting at $(\tilde{x}_1, \tilde{x}_2)$ and $(\tilde{y}_1, \tilde{y}_2)$ with $\tilde{x}_2 = 0$ and $\tilde{y}_2 > 0$ respectively, we define the stopping time $T = \inf\{t \geq 0 : X_t = Y_t\}$. Also, let $A_t = \int_0^{t \wedge T} 1_{\{(X_s, Y_s) \in D\}} ds$ and $\kappa_t = \inf\{s \geq 0 : A_s > t\}$, and set $(X^r, Y^r)_t = (X, Y)(\kappa_t)$,

$L_t^r(X^2) = L_{\kappa_t}(X^2)$ and $L_t^r(Y^2) = L_{\kappa_t}(Y^2)$ where $L_t(X^2)$ and $L_t(Y^2)$ are local times of X^2 and Y^2 at the origin respectively.

Now, we define the stopping time $T_1 = \inf\{t \geq 0 : Y_t^2 = 0\}$. For $t \leq T_1$, we have

$$\begin{aligned} Y_t^1 &= \tilde{y}_1 + \tilde{B}_t^2 \quad Y_t^2 = \tilde{y}_2 + \tilde{B}_t^4 \\ X_t^1 &= \tilde{x}_1 + \int_0^t 1_{\{X_s^2<0\}} d\tilde{B}_s^1 + \int_0^t 1_{\{X_s^2\geq 0\}} d\tilde{B}_s^2 \\ X_t^2 &= \int_0^t 1_{\{X_s^2<0\}} d\tilde{B}_s^3 + \int_0^t 1_{\{X_s^2\geq 0\}} d\tilde{B}_s^4 \end{aligned}$$

Then, by Tanaka formula we have

$$X_t^{2,r} = B_t^3 - L_t^r(X^2)$$

where $B_t^3 = \int_0^{\kappa_t} 1_{\{X_s^2<0\}} d\tilde{B}_s^3$ is a Brownian motion. Also, upon setting $B_t^4 = \int_0^{\kappa_t} 1_{\{X_s^2<0\}} d\tilde{B}_s^4$ which is also a Brownian motion, we get

$$Y_t^{2,r} = \tilde{y}_2 + B_t^4 - L_t^r(X^2)$$

For $X_t^{1,r}$ and $Y_t^{1,r}$ we have the following representations

$$X_t^{1,r} = \tilde{x}_1 + B_t^1 + \mathcal{L}_t^r(X^1) \quad Y_t^{1,r} = \tilde{y}_1 + B_t^2 + \mathcal{L}_t^r(X^1)$$

where $\mathcal{L}_t^r(X^1) = \int_0^{\kappa_t} 1_{\{X_s^2\geq 0\}} d\tilde{B}_s^2$, $B_t^1 = \int_0^{\kappa_t} 1_{\{X_s^2<0\}} d\tilde{B}_s^1$, and $B_t^2 = \int_0^{\kappa_t} 1_{\{X_s^2<0\}} d\tilde{B}_s^2$

Now, we define the stopping time $T_2 = \inf\{t \geq T_1 : X_t^2 = 0\}$. For $T_1 \leq t \leq T_2$, we have

$$\begin{aligned} X_t^1 &= X_{T_1}^1 + \tilde{B}_t^1 \quad X_t^2 = X_{T_1}^2 + \tilde{B}_t^3 \\ Y_t^1 &= Y_{T_1}^1 + \int_0^t 1_{\{Y_s^2<0\}} d\tilde{B}_s^1 + \int_0^t 1_{\{Y_s^2\geq 0\}} d\tilde{B}_s^2 \\ Y_t^2 &= \int_0^t 1_{\{Y_s^2<0\}} d\tilde{B}_s^3 + \int_0^t 1_{\{Y_s^2\geq 0\}} d\tilde{B}_s^4 \end{aligned}$$

Then, as it is done above, we again use Tanaka formula and define $B_t^4 = \int_0^{\kappa_t} 1_{\{Y_s^2\geq 0\}} d\tilde{B}_s^4$, $B_t^3 = \int_0^{\kappa_t} 1_{\{Y_s^2\geq 0\}} d\tilde{B}_s^3$, and $\mathcal{L}_t^r(Y^1) = \int_0^{\kappa_t} 1_{\{Y_s^2<0\}} d\tilde{B}_s^1$ to arrive at the following representations for $T_1 \leq t \leq T_2$

$$\begin{aligned} Y_t^{2,r} &= B_t^4 + L_t^r(Y^2) \qquad & X_t^{2,r} &= X_{T_1}^2 + B_t^3 + L_t^r(Y^2) \\ Y_t^{1,r} &= Y_{T_1}^1 + B_t^2 + \mathcal{L}_t^r(Y^1) \qquad & X_t^{1,r} &= X_{T_1}^1 + B_t^1 + \mathcal{L}_t^r(Y^1) \end{aligned}$$

Therefore, for all $t \geq 0$ we have

$$\begin{aligned} X_t^{1,r} &= \tilde{x}_1 + B_t^1 + \mathcal{L}_t^r(X^1) + \mathcal{L}_t^r(Y^1) \\ X_t^{2,r} &= \tilde{x}_2 + B_t^3 - L_t^r(X^2) + L_t^r(Y^2) \\ Y_t^{1,r} &= \tilde{y}_1 + B_t^2 + \mathcal{L}_t^r(X^1) + \mathcal{L}_t^r(Y^1) \\ Y_t^{2,r} &= \tilde{y}_2 + B_t^4 - L_t^r(X^2) + L_t^r(Y^2) \end{aligned}$$

In this representation, we can take $(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$ and $(\tilde{y}_1, \tilde{y}_2) \in \mathbb{R}^2$ with the condition $\tilde{y}_2 > \tilde{x}_2$. Also, whatever the correlation between the Brownian motions $\{\tilde{B}^j\}_1^4$ it passes on to the Brownian motions $\{B^j\}_1^4$. The process X_t^r takes its values on $\mathbb{R} \times \mathbb{R}^-$ whereas Y_t^r takes its values on $\mathbb{R} \times \mathbb{R}^+$. This representation holds for $t \leq \lim_{n\to\infty} T_n = \bar{T}$ which is finite almost surely but not necessarily equal to T.

3.5 Kernel Solutions and Two-Dimensional Tanaka Equation

First, we give the definition of flow of kernels [10].

Definition 3.5.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Then a family of (E, ϵ)-valued random variables $(K_{s,t})_{s,t}$ is called a measurable stochastic flow of kernels if, for all $s \leq t$,*

$$(x, \omega) \to K_{s,t}(x, \omega)$$

is a measurable mapping from $(\Omega \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}))$ onto $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$ and it satisfies the following properties:

1. *For all $s < u < t$ and $x \in \mathbb{R}$, $\mathbb{P}$-a.s., for every $f \in C_0(\mathbb{R})$, $K_{s,t}f(x) = K_{s,u}(K_{u,t}f)(x)$*
2. *For all $s \leq t$, the law of $K_{s,t}$ only depends on t-s*
3. *For all $t_1 < t_2 < ... < t_n$, the family $\{K_{t_i,t_{i+1}} : 1 \leq i \leq n-1\}$ is independent.*
4. *For every $f \in C_0(\mathbb{R})$*

$$\lim_{(u,v)\to(s,t)} \sup_{x\in\mathbb{R}} \mathbb{E}[(K_{u,v}f(x) - K_{s,t}f(x))^2] = 0$$

5. *For all* $f \in C_0(\mathbb{R})$ *and* $s < t$*, we have*

$$\lim_{y \to x} \mathbb{E}[(K_{s,t}f(y) - K_{s,t}f(x))^2] = 0$$

6. *For all* $s < t$*,* $f \in C_0(\mathbb{R})$*, we have*

$$\lim_{|x| \to \infty} \mathbb{E}[(K_{s,t}f(x))^2] = 0$$

The two-dimensional Brownian motions in (3.1) defined as $(B^1, B^3)' = \Sigma_+ \boldsymbol{W}$ and $(B^2, B^4)' = \Sigma_- \boldsymbol{W}$, which are also viewed as white noises by an abuse of notation in (3.4) and (3.5), give rise to the following cross variation processes

$$d\langle B^1, B^2\rangle_t = \rho\sigma\cos(\varphi - \vartheta) \qquad d\langle B^3, B^4\rangle_t = \rho\sigma\delta\varepsilon\cos(\varphi - \vartheta) \tag{3.10}$$

$$d\langle B^1, B^4\rangle_t = -\delta\sin(\varphi - \vartheta) \qquad d\langle B^2, B^3\rangle_t = \varepsilon\sin(\varphi - \vartheta) \tag{3.11}$$

The kernel solution of (3.1) is the trivial solution $K = \delta_\varphi$ when it has a strong solution. It is called the Wiener solution. Therefore we will look for kernel solutions when there is only weak solution, that is, when $(\mathbf{e}_1 - \mathbf{e}_2)'\Sigma_- = -(\mathbf{e}_1 - \mathbf{e}_2)'\Sigma_+$. Explicitly, we have

$$-\rho\cos(\varphi) + \sigma\varepsilon\sin(\varphi) = \sigma\cos(\vartheta) - \rho\delta\sin(\vartheta) \tag{3.12}$$

$$\rho\sin(\varphi) + \sigma\varepsilon\cos(\varphi) = -\sigma\sin(\vartheta) - \rho\delta\cos(\vartheta) \tag{3.13}$$

This, in turn, implies

$$(\delta - \varepsilon)\cos(\varphi - \vartheta) - \rho\sigma(1 + \varepsilon\delta)\sin(\varphi - \vartheta) = 0 \tag{3.14}$$

Now, we have two cases: $\varepsilon = \delta$ and $\varepsilon = -\delta$.

3.5.1 $\varepsilon = \delta$

In this case, (3.14) simplifies to $\sin(\varphi - \vartheta) = 0$. Hence, either $\varphi = \vartheta$ or $\varphi = \vartheta \pm \pi$. Putting these back into (3.12) and (3.13), we get two solutions: $\{\varepsilon = \delta, \rho = \sigma, \varphi =$

$\vartheta \pm \pi, \vartheta = 0, \pm\pi\}$ and $\{\varepsilon = \delta, \rho = -\sigma, \varphi = \vartheta, 0 \leq \vartheta \leq 2\pi\}$. Under these conditions, (3.10) and (3.11) simplify to

$$d\langle B^1, B^2\rangle_t = d\langle B^3, B^4\rangle_t = -1 \quad d\langle B^1, B^4\rangle_t = d\langle B^2, B^3\rangle_t = 0$$

This turns (3.1) into

$$dX_t^1 = 1_{\{X_t^1 > X_t^2\}} dB_t^1 - 1_{\{X_t^1 \leq X_t^2\}} dB_t^1 \tag{3.15}$$

$$dX_t^2 = 1_{\{X_t^1 > X_t^2\}} dB_t^3 - 1_{\{X_t^1 \leq X_t^2\}} dB_t^3 \tag{3.16}$$

The solution of this equation follows $(-B^1, -B^3)$ when it is above the diagonal $x_1 = x_2$ and it follows (B^1, B^3) when it is below the diagonal. We will call this equation two-dimensional Tanaka equation because of its resemblance to one-dimensional Tanaka equation. In a similar manner, this equation has a weak solution but not a strong solution. The kernel solution K of this equation is a family of random variables $\{K_{s,t} : s \leq t\}$ which is called a measurable stochastic flow of kernels [10, Definition 2.3] and it satisfies

$$\begin{aligned} K_{s,t}f(x) = f(x_0) \quad &+ \quad \int_s^t K_{s,u}\Big(\frac{\partial f}{\partial x_1}(1_{\{x_1 > x_2\}} - 1_{\{x_1 \leq x_2\}})\Big)(x_1, x_2) dB_{s,u}^1 \\ &+ \quad \int_s^t K_{s,u}\Big(\frac{\partial f}{\partial x_2}(1_{\{x_1 > x_2\}} - 1_{\{x_1 \leq x_2\}})\Big)(x_1, x_2) dB_{s,u}^3 \\ &+ \quad \frac{1}{2}\int_s^t K_{s,u}\Big(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\Big) du \end{aligned}$$

for all twice continuously differentiable functions with compact support. Coalescence for (3.15) and (3.16) is yet to be studied as indicated in section 3.4.2 under the case $\rho\sigma\cos(\varphi - \vartheta) = -1$.

3.5.2 $\varepsilon = -\delta$

In this case, (3.14) simplifies to $\cos(\varphi - \vartheta) = 0$. Hence, either $\varphi = \vartheta \pm \frac{\pi}{2}$ or $\varphi = \vartheta \pm \frac{3\pi}{2}$. Putting these back into (3.12) and (3.13), we get two solutions: $\{\varepsilon = -\delta, \varepsilon = 1, \varphi = \vartheta + \frac{\pi}{2} \quad \text{or} \quad \vartheta - \frac{3\pi}{2}, 0 \leq \vartheta \leq 2\pi\}$ and $\{\varepsilon = -\delta, \varepsilon = -1, \varphi = \vartheta - \frac{\pi}{2} \quad \text{or} \quad \vartheta + \frac{3\pi}{2}, 0 \leq \vartheta \leq 2\pi\}$. Under these conditions, (3.10) and (3.11) simplify to

$$d\langle B^1, B^2\rangle_t = d\langle B^3, B^4\rangle_t = 0 \quad d\langle B^1, B^4\rangle_t = d\langle B^2, B^3\rangle_t = 1$$

This turns (3.1) into

$$dX_t^1 = 1_{\{X_t^1 > X_t^2\}} dB_t^1 + 1_{\{X_t^1 \le X_t^2\}} dB_t^2 \tag{3.17}$$

$$dX_t^2 = 1_{\{X_t^1 > X_t^2\}} dB_t^2 + 1_{\{X_t^1 \le X_t^2\}} dB_t^1 \tag{3.18}$$

The solution of this equation follows (B^2, B^1) when it is above the diagonal $x_1 = x_2$ and it follows (B^1, B^2) when it is below the diagonal. It is referred to as two-dimensional Tanaka equation in [18, Equations 2.14,2.15].

The kernel solution K of (3.17) and (3.18) satisfies

$$\begin{aligned} K_{s,t} f(x) = f(x_0) \quad &+ \quad \int_s^t K_{s,u} \Big(\frac{\partial f}{\partial x_1} 1_{\{x_1 > x_2\}} + \frac{\partial f}{\partial x_2} 1_{\{x_1 \le x_2\}} \Big)(x_1, x_2) dB_{s,u}^1 \\ &+ \quad \int_s^t K_{s,u} \Big(\frac{\partial f}{\partial x_2} 1_{\{x_1 > x_2\}} + \frac{\partial f}{\partial x_1} 1_{\{x_1 \le x_2\}} \Big)(x_1, x_2) dB_{s,u}^2 \\ &+ \quad \frac{1}{2} \int_s^t K_{s,u} \Big(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \Big) du \end{aligned}$$

for all twice continuously differentiable functions with compact support.

Note that coalescence for (3.17) and (3.18) is studied in section 3.4.1 under the case $\varepsilon \sin(\varphi - \vartheta) = 1$. We know under which conditions it occurs.

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