DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

FINITE DIFFERENCE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

by
Fadime KARTAL

October, 2019
İZMİR

FINITE DIFFERENCE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

A Thesis Submitted to the
Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

by
Fadime KARTAL

October, 2019
İZMİR

M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled “**FINITE DIFFERENCE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS**” completed by **FADİME KARTAL** under supervision of **ASSIST. PROF. DR. MELDA DUMAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

....

Assist. Prof. Dr. Melda DUMAN

Supervisor

....

Assoc. Prof. Dr. Cetin Dişibüyük

Jury Member

....

Assist. Prof. Dr. Setenay AKDUMAN

Jury Member

Prof. Dr. Kadriye ERTEKİN
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

Foremost, I would like to thank my supervisor Assist. Prof. Dr. Melda Duman. The door to her office was always open whenever I had a question about my research and ran into problem.

Also, I would like to thank to Research Assist. Dr. Cem Çelik for sharing his information on Mathematica programming and LaTeX. I would like to thank to Assist. Prof. Dr. Celal Cem Sarıoğlu for suggestions on LaTeX.

Last but not the least, I am grateful to my dear parents and brother who encouraged me with their support. Thank you heartily for believing in me all the time.

Fadime KARTAL

FINITE DIFFERENCE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT

In this thesis, a difference approximation for the Caputo-Hadamard fractional derivative by using spline on a nonuniform mesh is studied. Accuracy of error for the Caputo-Hadamard derivative is investigated. Numerical experiment of the method is performed to find approximate solution of a fractional differential equation with initial and boundary conditions. The obtained numerical accuracy coincides with the theoretical result.

Keywords: Caputo-Hadamard fractional derivative, finite difference, spline curve

KESİRLİ DİFERANSİYEL DENKLEMLERİN SONLU FARKLAR ÇÖZÜMLERİ

ÖZ

Bu tezde, kesirli Caputo-Hadamard türevi için düzgün olmayan aralıktaki spline eğrisi kullanılarak fark yaklaşımı çalışılmıştır. Caputo-Hadamard türevi için hatanın doğruluğu araştırılmıştır. Yöntemin sayısal deneyi başlangıç ve sınır koşullu kesirli bir diferansiyel denklemin yaklaşık çözümünü bulmak için yapılmıştır. Elde edilen sayısal doğruluk teorik sonuç ile örtüşmektedir.

Anahtar kelimeler: Caputo-Hadamard kesirli türev, sonlu fark, spline eğrisi

CONTENTS

LIST OF FIGURES

Page

LIST OF TABLES

CHAPTER ONE
INTRODUCTION

1.1 Introduction

Several natural phenomena can better described by fractional differential equations, such as viscoelasticity, anomalous diffusion, signal processing and control theory. For comprehensive information on these areas, see Mainardi (2010), Metzler et al. (1994), Ortigueira & Machado (2003) Vinagre et al. (2000) and references therein.

In fractional calculus, there are several definitions for fractional derivatives. The Riemann-Liouville and the Caputo derivatives are the most well-known. In 1892, Hadamard introduced another derivative which is a generalization of n-th power of differential operator $x\frac{d}{dx}$, Samko et al. (1993). Detailed information about history of the Hadamard fractional integral and derivative can be found in the book Kilbas et al. (2006).

Recently, the Caputo-Hadamard fractional derivative was introduced by Jarad et al. (2012). The existence of solutions for the Cauchy problem was shown with upper and lower solution method by Bai & Kong (2017). Banach's fixed point theorem was used by Adjabi et al. (2016) to prove the existence and uniqueness of solutions. Numerical methods to approximate the Caputo-Hadamard fractional derivative developed in the literature. Almeida (2017) gave an approximation for the fractional derivative in an expansion formula. A discretization on nonuniform grid was considered by Almeida & Bastos (2016). The Euler and predictor-corrector methods were investigated by Gohar et al. (2019).

In chapter 2, some basic definitions, properties and relations are presented. In chapter 3, difference approximation by the help of non-polynomial spline curve for the Caputo-Hadamard fractional derivative is established. Second order accuracy is achieved in mesh and the efficiency of the derived method is shown.

CHAPTER TWO
THE HADAMARD FRACTIONAL DERIVATIVE

In this chapter, we review some basic definitions and properties of the left- sided Hadamard and the left-sided Caputo-Hadamard fractional derivatives on a finite interval. To find more knowledge see in Kilbas et al. (2006), Jarad et al. (2012) and Gambo et al. (2014).

2.1 Preliminaries

The Hadamard fractional integral that Hadamard introduced in 1892 has a kernel of logaritmic function of arbitrary exponents.

Definition 2.1.1 *(Hadamard fractional integral) Let* (a, b) *be a finite or infinite interval where* $0 \leqq a < b \leqq \infty$ *and* $\alpha > 0$*. The left-sided Hadamard fractional integral of order* α *is defined by*

$$(\mathcal{J}_{a^+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad (a < x < b), \tag{2.1}$$

where $\Gamma(\alpha)$ *is the Gamma function*

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Definition 2.1.2 *Let* $a < x < b$ *and* $\alpha \geq 0$*. The left-sided Hadamard fractional derivative of order* α *is defined by*

$$(\mathcal{D}_{a^+}^{\alpha} f)(x) := \delta^n (J_{a^+}^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx}\right)^n \int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha+1} f(t) \frac{dt}{t}, \tag{2.2}$$

where $\delta = x\dfrac{d}{dx}$*,* $n = [\alpha] + 1$*, and* $[\alpha]$ *is the integer part of* α*.*

Property 2.1.1 *Let* $\alpha > 0$, $\rho > 0$ *and* $0 < a < b < \infty$. *Then*

$$\left(\mathcal{J}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}\left(\log\frac{x}{a}\right)^{\rho+\alpha-1}, \tag{2.3}$$

$$\left(\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\left(\log\frac{x}{a}\right)^{\rho-\alpha-1}. \tag{2.4}$$

In general, the Hadamard fractional derivatives of a constant is not equal zero for the case of $\rho = 1$ *and* $\alpha \geqq 0$. *For instance,*

$$(\mathcal{D}_{a^+}^{\alpha}1) = \frac{1}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha}, \tag{2.5}$$

and when $0 < \alpha < 1$. *The functions* $\left(\log\frac{t}{a}\right)^{\alpha-j}$, $j = [\alpha]+1$, *are the zeros of* $\mathcal{D}_{a^+}^{\alpha}$ *(Kilbas et al. (2006), Property 2.24).*

Theorem 2.1.2 *Let* $\alpha > 0$, $n = [\alpha]$ *and* $0 < a < b < \infty$. *If* $f(x) \in L(a,b)$, *Lebesgue integrable on* $[a,b]$, *and*

$$(\mathcal{J}_{a^+}^{n-\alpha}f)(x) \in AC_{\delta}^{n}[a,b] = \left\{f:\ [a,b] \to \mathbb{R} \mid \delta^{n-1}f(x) \in AC[a,b],\ \delta = x\frac{d}{dx}\right\},$$

then

$$(\mathcal{J}_{a^+}^{\alpha}\mathcal{D}_{a^+}^{\alpha}f)(x) = f(x) - \sum_{k=1}^{n}\frac{(\delta^{n-k}(\mathcal{J}_{a^+}^{n-\alpha}f))(a)}{\Gamma(\alpha-k+1)}\left(\log\frac{x}{a}\right)^{\alpha-k}. \tag{2.6}$$

In particular, while $\alpha = n \in \mathbb{N}$ *and* $f(x) \in AC_{\delta}^{n}[a,b]$, *then*

$$(\mathcal{J}_{a^+}^{n}\mathcal{D}_{a^+}^{n}f)(x) = f(x) - \sum_{k=0}^{n-1}\frac{(\delta^{k}f)(a)}{k!}\left(\log\frac{x}{a}\right)^{k} \tag{2.7}$$

(Kilbas et al. (2006), Theorem 2.3).

Property 2.1.3 *Let* $\alpha > \rho > 0$, $0 < a < b < \infty$ *and* $1 \leqq p \leqq \infty$. *Then, for* $f \in L_p(a,b)$,

$$\mathcal{D}_{a^+}^{\rho}\mathcal{J}_{a^+}^{\alpha}f = \mathcal{J}_{a^+}^{\alpha-\rho}f, \tag{2.8}$$

$$\mathcal{J}_{a^+}^{\alpha}\mathcal{J}_{a^+}^{\rho}f = \mathcal{J}_{a^+}^{\alpha+\rho}f \tag{2.9}$$

(Kilbas et al. (2006), Property 2.27).

Lemma 2.1.4 *Let $n = [\alpha] + 1$, $\alpha \geqq 0$ and let $f(x) \in AC_{\delta}^{n}[a, b]$, $0 < a < b < \infty$. Then the Hadamard fractional derivative $\mathcal{D}_{a^+}^{\alpha}f$ exists and the representation formula*

$$(\mathcal{D}_{a^+}^{\alpha}f)(x) = \sum_{k=0}^{n-1}\frac{(\delta^k f)(a)}{\Gamma(1+k-\alpha)}\left(\log\frac{x}{a}\right)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{n-\alpha-1}(\delta^n f)(t)dt \tag{2.10}$$

holds almost everywhere on $[a, b]$ (Kilbas et al. (2006), Lemma 2.34).

For instance, if $1 < \alpha < 2$, then

$$\begin{aligned}(\mathcal{D}_{a^+}^{\alpha}f)(x) \quad &= \quad \frac{f(a)}{\Gamma(1-\alpha)}\left(\log\frac{x}{a}\right)^{-\alpha} + \frac{\delta f(a)}{\Gamma(2-\alpha)}\left(\log\frac{x}{a}\right)^{1-\alpha} \\ &\quad + \frac{1}{\Gamma(2-\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{1-\alpha}\delta^2 f(t)\frac{dt}{t},\end{aligned} \tag{2.11}$$

where $f(x) \in AC_{\delta}^{2}[a, b]$.

Here the last term given by integral in (2.10) known as the left-sided Caputo-Hadamard fractional derivative of order α, notated by ${}^{C}\mathcal{D}_{a^+}^{\alpha}$. In other words,

$$({}^{C}\mathcal{D}_{a^+}^{\alpha}f)(x) = \mathcal{D}_{a^+}^{\alpha}\left[(f)(t) - \sum_{k=0}^{n-1}\frac{\delta^k f(a)}{k!}\left(\log\frac{t}{a}\right)^k\right](x), \tag{2.12}$$

where $\alpha \geq 0$, $n = [\alpha] + 1$ and $f(x) \in AC_{\delta}^{n}[a, b]$, $0 < a < b < \infty$.

Theorem 2.1.5 *Let $\alpha \geq 0$, $n = [\alpha] + 1$ and $f \in AC_{\delta}^{n}[a, b]$, $0 < a < b < \infty$. Then ${}^{C}\mathcal{D}_{a^+}^{\alpha}f$ exists almost everywhere on $[a, b]$ and* *(i) if* $\alpha \notin \mathbb{N}_0$,

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{a}^{x} \left(\log\frac{x}{t}\right)^{n-\alpha-1} \delta^n f(t)\frac{dt}{t} = \mathcal{J}^{n-\alpha}_{a^+}\delta^n f(x), \quad (2.13)$$

(ii) if $\alpha = n \in \mathbb{N}_0$*,*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}f)(x) = \delta^n f(x). \quad (2.14)$$

In particular, $({}^{C}\mathcal{D}^{0}_{a^+}f)(x) = f(x)$ *(Jarad et al. (2012), Theorem 2.1).*

Let $C^n_\delta[a,b] = \left\{ f : \|f\|_{C^n_\delta} = \sum\limits_{k=0}^{n-1} \|\delta^k f\|_C + \|\delta^n f\|_{C_\delta} \right\}$, $C^0_\delta[a,b] = C[a,b]$ (Kilbas et al. (2006), p.5).

Theorem 2.1.6 *Let* $\alpha \geq 0$ *and* $n = [\alpha]+1$*. If* $f \in C^n_\delta[a,b]$*, where* $0 < a < b < \infty$*. Then* ${}^{C}\mathcal{D}^{\alpha}_{a^+}f(x)$ *is continuous on* $[a,b]$ *and the followings are valid.*

(i) $({}^{C}\mathcal{D}^{\alpha}_{a^+}f)(x)$ *can be represented by* (2.13) *except for* $\alpha \notin \mathbb{N}_0$*.*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}f)(a) = 0. \quad (2.15)$$

(ii) If $\alpha = n \in \mathbb{N}_0$*, then the formula in* (2.14) *hold (Jarad et al. (2012), Theorem 2.2).*

Lemma 2.1.7 *Let* $\alpha \geqq 0$ *and* $n = [\alpha]+1$*.*

(i) If $\alpha \notin \mathbb{N}_0$*, then* ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ *is bounded from the space* $C^n_\delta[a,b]$ *to the space* $C_a[a,b] = \{f \in C[a,b] \ s.t. \ f(a) = 0\}$ *and*

$$\|{}^{C}\mathcal{D}^{\alpha}_{a^+}f\|_{C_a} \leq \frac{\left(\log\frac{b}{a}\right)^{n-\alpha}}{\mid \Gamma(n-\alpha) \mid (n-\alpha)} \|f\|_{C^n_\delta}. \quad (2.16)$$

(ii) If $\alpha = n \in \mathbb{N}_0$, then ${}^C\mathcal{D}^{\alpha}_{a+}$ bounded from $C^n_\delta[a,b]$ to $C[a,b]$ and

$$\|{}^C\mathcal{D}^{n}_{a+} f\|_C \leq \|f\|_{C^n_\delta} \tag{2.17}$$

(Jarad et al. (2012), Corollary 2.3).

Lemma 2.1.8 *Let $\alpha \geqq 0$, $n = [\alpha] + 1$ and $f \in C[a,b]$.*

(i) If $\alpha \neq 0$, or $\alpha \in \mathbb{N}$, then

$$ {}^C\mathcal{D}^{\alpha}_{a+}(\mathcal{J}^{\alpha}_{a+} f)(x) = f(x). \tag{2.18}$$

(ii) If $\alpha \in \mathbb{N}$, and $\alpha \neq 0$, then

$$ {}^C\mathcal{D}^{\alpha}_{a+}(\mathcal{J}^{\alpha}_{a+} f)(x) = f(x) - \frac{\mathcal{J}^{\alpha-n+1}_{a+} f(a)}{\Gamma(n-\alpha)} \left(\log \frac{x}{a}\right)^{n-\alpha} \tag{2.19}$$

(Jarad et al. (2012), Lemma 2.4).

Lemma 2.1.9 *Let $f \in AC^n_\delta[a,b]$ or $C^n_\delta[a,b]$, and $\alpha \in \mathbb{R}$, then*

$$\mathcal{J}^{\alpha}_{a+}({}^C\mathcal{D}^{\alpha}_{a+} f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} \left(\log \frac{x}{a}\right)^{k} \tag{2.20}$$

(Jarad et al. (2012), Lemma 2.5).

Specially, when $f(x) \in C^n_\delta[a,b]$ and $\alpha = n$, the relation (2.20) yields the following representation

$$f(x) = \frac{1}{(n-1)!} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} (\delta^n f)(t) \frac{dt}{t} + \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} \left(\log \frac{x}{a}\right)^{k}. \tag{2.21}$$

Property 2.1.10 *If $\alpha \geq 0$, $n = [\alpha] + 1$ and $\rho > 0$. Then*

$$ {}^C\mathcal{D}^{\alpha}_{a+}\Big(\log \frac{x}{a}\Big)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\Big(\log \frac{x}{a}\Big)^{\rho-\alpha-1}, \quad \rho > n, \tag{2.22}$$

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+}\Big(\log\frac{x}{a}\Big)^{k} = 0, \ \ k = 0, 1, ..., n-1. \tag{2.23} $$

Specially, we have

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+}1 = 0. \tag{2.24} $$

Lemma 2.1.11 *Let $\alpha \geq 0$ and $n = [\alpha] + 1$. If f is a function such that ${}^{C}\mathcal{D}^{\alpha}_{a^+}f$ and $\mathcal{D}^{\alpha}_{a^+}f$ exist, then*

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+}f(x) = \mathcal{D}^{\alpha}_{a^+}f(x) - \sum_{k=0}^{n-1}\frac{\delta^k f(a)}{\Gamma(k-\alpha+1)}\Big(\log\frac{x}{a}\Big)^{k-\alpha} \tag{2.25} $$

(Gambo et al. (2014), Lemma 6).

We have under the condition of Lemma (2.1.11)

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+}f(x) = \mathcal{D}^{\alpha}_{a^+}f(x), \tag{2.26} $$

if and only if f has an $n-$fold zero at a.

CHAPTER THREE
FINITE DIFFERENCE APPROXIMATION TO THE CAPUTO-HADAMARD FRACTIONAL DERIVATIVE

In this chapter, we obtain a difference expression for the fractional derivative via a piecewise approximating function and the Taylor expansions. The theoretical and computational study of accuracy are given.

3.1 A Non-polynomial B-Spline Curve of Order One

We constitute a logarithmic B-spline curve of order one based on the description from the article Dişibüyük & Goldman (2016). Let define the space $\pi_1(\gamma_1, \gamma_2) = span\{\gamma_1, \gamma_2\}$ where $\gamma_1 = 1$ and $\gamma_2 = \log x$ are differentiable and linearly independent on the interval $\Omega = [a, b]$. The point $(1, \log x)$ has non-negative barycentric coordinates

$$\omega_1(x) = \frac{\log\theta_2 - \log x}{\log\theta_2 - \log\theta_1}, \quad \omega_2(x) = \frac{\log x - \log\theta_1}{\log\theta_2 - \log\theta_1}$$

relative the end points of the arc $\Gamma(1, \log t)$ for $\theta_1 \leq t \leq \theta_2$ and $a \leq \theta_1 < \theta_2 \leq b$. The B-spline basis functions of order one are defined recursively by

$$N_{k,0}(x) = \begin{cases} 1, & \text{if } x \in [x_k, x_{k+1}) \\ 0, & \text{otherwise,} \end{cases}$$

$$N_{k,1}(x) = \frac{\log x - \log x_k}{\log x_{k+1} - \log x_k} N_{k,0}(x) + \frac{\log x_{k+2} - \log x}{\log x_{k+2} - \log x_{k+1}} N_{k+1,0}(x).$$

We wish to interpolate for given data $y_k = f(x_k)$, $k = 0, 1, ..., m$ at the given knots $x_0 < x_1 < \cdots < x_m$, by spline function $S_m(x) \in \sigma_1(\gamma_1, \gamma_2)$. Denote by $\Psi_k(x)$ B-spline curve segments in $\pi_1(\gamma_1, \gamma_2)$ on each interval $[x_k, x_{k+1}]$ satisfying conditions

$\Psi_k(x_{k+1}) = \Psi_{k+1}(x_{k+1}),\ k = 0, 1, ..., m-2$. The continuous spline has the form

$$S_m(x) = \begin{cases} \Psi_0(x) & , \quad x_0 \le x \le x_1 \\ \Psi_1(x) & , \quad x_1 \le x \le x_2 \\ \vdots & \qquad \vdots \\ \Psi_{m-1}(x), & x_{m-1} \le x \le x_m. \end{cases} \tag{3.1}$$

On each interval $[x_i, x_{i+1}]$ there are two interpolation conditions $S(x_i) = y_i$ and $S(x_{i+1}) = y_{i+1}$. Since $N_{k,1}(x) \neq 0$ over $[x_i, x_{i+1}]$ for $k = i-1$ and $k = i$, we have

$$\Psi_i(x) = \sum_{k=i-1,i} N_{k,1}(x) P_k.$$

The solution of $\Psi_i(x_j) = y_j$, $j = i, i+1$, that is,

$$\Psi_i(x_i) = N_{i-1,1}(x_i)P_{i-1} + N_{i,1}(x_i)P_i = y_i,$$
$$\Psi_i(x_{i+1}) = N_{i-1,1}(x_{i+1})P_{i-1} + N_{i,1}(x_{i+1})P_i = y_{i+1}$$

is $P_{i-1} = y_i$ and $P_i = y_{i+1}$ determined by uniquely. Hence,

$$S_m(x) = \sum_{k=-1}^{m-1} f(x_{k+1}) N_{k,1}(x) \tag{3.2}$$

is interpolating spline to $f(x)$. On each interval $[x_i, x_{i+1}]$, we have

$$S_m(x) = \frac{\log x_{i+1} - \log x}{\log x_{i+1} - \log x_i} f(x_i) + \frac{\log x - \log x_i}{\log x_{i+1} - \log x_i} f(x_{i+1}). \tag{3.3}$$

Example 1 *Let $f(x) = x^3$ where $x \in [1, e]$. Choose the knots $x_0 = 1, x_1 = \frac{e+1}{2}$ and $x_2 = e$. Then the following B-spline basis functions active on $[1, e]$. We get the restrictions*

$$\begin{aligned} N_{-1,1}(x) &= \begin{cases} \log \dfrac{x_1}{x} / \log \dfrac{x_1}{x_0}, & x_0 \le x < x_1 \\ 0 & , \ \textit{otherwise} \end{cases} \\ &= \begin{cases} 1 - (\log x / \log \frac{e+1}{2}), & 1 \le x < \frac{e+1}{2} \\ 0 & , \ \textit{otherwise,} \end{cases} \end{aligned}$$

$$
\begin{aligned}
N_{0,1}(x) &= \frac{\log x - \log x_0}{\log x_1 - \log x_0} N_{0,0}(x) + \frac{\log x_2 - \log x}{\log x_2 - \log x_1} N_{1,0}(x) \\
&= \begin{cases} \log x / \log \frac{e+1}{2} & , \quad 1 \leq x < \frac{e+1}{2} \\ (1 - \log x)/(1 - \log \frac{e+1}{2}), & \frac{e+1}{2} \leq x < e \\ 0 & , \quad \textit{otherwise} \end{cases}
\end{aligned}
$$

and

$$
N_{1,1}(x) = \begin{cases} \log \frac{2x}{e+1} / \log \frac{2e}{e+1}, & \frac{e+1}{2} \leq x < e \\ 0 & , \quad \textit{otherwise.} \end{cases}
$$

Moreover, the computed convergence order in L_2-*norm is* 2.00.

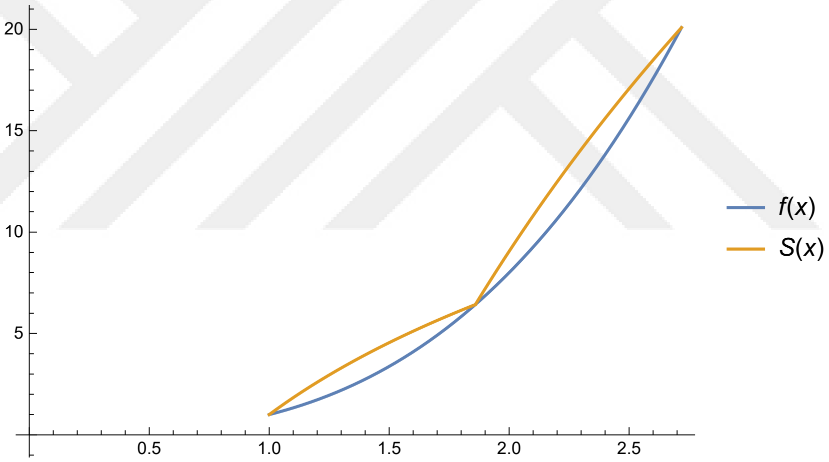

Figure 3.1 $S(x)$ is the spline that consists of two joined curve segments

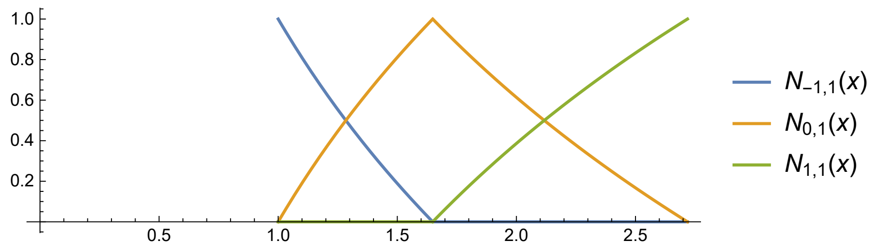

Figure 3.2 $N_{-1,1}(x)$, $N_{0,1}(x)$ and $N_{1,1}(x)$ when the knots are $\{x_j\}_{j=0}^{2} = \{1, \frac{e+1}{2}, e\}$

3.2 Error in Interpolation

Let $g(x) \in \pi_1(1, \log x)$ be approximate function on $[x_j, x_{j+1}]$ that interpolates $f(x)$ at the knots x_j, x_{j+1} as

$$g(x) = N_{j-1,1}(x) f(x_j) + N_{j,1}(x) f(x_{j+1}) \tag{3.4}$$

or equivalently

$$g(x) = \frac{\log x_{j+1} - \log x}{\log x_{j+1} - \log xj} f(x_j) + \frac{\log x - \log x_j}{\log x_{j+1} - \log x_j} f(x_{j+1}), \tag{3.5}$$

where $g(x_j) = f(x_j),\ g(x_{j+1}) = f(x_{j+1})$.

Lemma 3.2.1 *Let $u(x) \in C_\delta^5[1, e]$. Then*

$$\delta^2 u(x) - S(x) = \frac{1}{2} \delta^4 u(x_k) \log \frac{x}{x_k} \log \frac{x}{x_{k+1}} + O(\Delta\eta^3) \tag{3.6}$$

where $x_k < x < x_{k+1},\ \ k = 0, 1, ..., N-1$ and

$$S(x) = \frac{\log(x_{k+1}/x)}{\log(x_{k+1}/x_k)} \delta^2 u(x_k) + \frac{\log(x/x_k)}{\log(x_{k+1}/x_k)} \delta^2 u(x_{k+1}) \tag{3.7}$$

is the spline curve segment.

Proof. Consider the representation of $g(x) \in C_\delta^3[1, e]$

$$g(x) = g(x_k) + \delta g(x_k) \log \frac{x}{x_k} + \frac{1}{2!} \delta^2 g(x_k) \Big(\log \frac{x}{x_k} \Big)^2 + \frac{1}{2!} \int\limits_{x_k}^{x} \Big(\log \frac{x}{t} \Big)^2 \delta^3 g(t) \frac{dt}{t}. \tag{3.8}$$

Differentiating $S(x)$, we get

$$\delta^2 S(x) = x S^{'}(x) + x^2 S^{''}(x) = 0.$$

Utilizing (3.8), we can write the error for $x_k < x < x_{k+1}$,

$$\begin{aligned} E(x) &= \delta^2 u(x) - S(x) \\ &= \left(\delta^2 u(x_k) + \log\frac{x}{x_k}\delta^3 u(x_k) + \frac{1}{2!}\left(\log\frac{x}{x_k}\right)^2 \delta^4 u(x_k)\right. \\ &\quad \left. + \frac{1}{2!}\int_{x_k}^{x}\left(\log\frac{x}{t}\right)^2 \delta^5 u(t)\frac{dt}{t}\right) - \left(S(x_k) + \left(\log\frac{x}{x_k}\right)\delta S(x_k) + 0\right) \end{aligned} \quad (3.9)$$

Using the interpolation condition $\delta^2 u(x_k) = S(x_k)$, replacing

$$\delta S(x_k) = -\frac{1}{\log(x_{k+1}/x_k)}\delta^2 u(x_k) + \frac{1}{\log(x_{k+1}/x_k)}\delta^2 u(x_{k+1}), \quad (3.10)$$

and expanding $\delta^2 u(x_{k+1})$ about $x = x_k$, we get

$$\begin{aligned} E(x) &= \left(\left(\log\frac{x}{x_k}\right)\delta^3 u(x_k) + \frac{1}{2!}\left(\log\frac{x}{x_k}\right)^2 \delta^4 u(x_k) + \frac{1}{2!}\int_{x_k}^{x}\left(\log\frac{x}{t}\right)^2 \delta^5 u(t)\frac{dt}{t}\right) \\ &\quad - \left(\left(\log\frac{x}{x_k}\right)\frac{1}{\log(x_{k+1}/x_k)}\left[-\delta^2 u(x_k) + \delta^2 u(x_k) + \left(\log\frac{x_{k+1}}{x_k}\right)\delta^3 u(x_k)\right.\right. \\ &\quad \left.\left. + \frac{1}{2!}\left(\log\frac{x_{k+1}}{x_k}\right)^2 \delta^4 u(x_k) + \frac{1}{2!}\int_{x_k}^{x_{k+1}}\left(\log\frac{x_{k+1}}{t}\right)^2 \delta^5 u(t)\frac{dt}{t}\right]\right). \end{aligned}$$

$$\begin{aligned} \left|E(x) - \frac{1}{2!}\left(\log\frac{x}{x_k}\right)\left(\log\frac{x}{x_{k+1}}\right)\delta^4 u(x_k)\right| &< \left|\int_{x_k}^{x}\left(\log\frac{x}{t}\right)^2 \delta^5 u(t)\frac{dt}{t}\right| \\ &\quad + \left|\log\frac{x}{x_k}\frac{1}{\log\frac{x_{k+1}}{x_k}}\int_{x_k}^{x_{k+1}}\left(\log\frac{x_{k+1}}{t}\right)^2 \delta^5 u(t)\frac{dt}{t}\right|. \end{aligned}$$

The left side of above inequality are accurate to third order in $\Delta\eta$ if $\delta^5 u \in C[1, e]$. Thus, we have

$$E(x) = \frac{1}{2!}\left(\log\frac{x}{x_k}\right)\left(\log\frac{x}{x_{k+1}}\right)\delta^4 u(x_k) + O(\Delta\eta^3). \quad (3.11)$$

As a result, Lemma (3.2.1) is proved.

Now, we can find a bound for the error in (x_k, x_{k+1}). From the Mean-Value Theorem, there exists a number $\rho \in (x_k, x)$ such that

$$\log \frac{x}{x_k} = \frac{1}{\rho}(x - x_k).$$

Hence,

$$\max_k \left| \log \frac{x}{x_k} \log \frac{x}{x_{k+1}} \right| < |(x - x_k)(x - x_{k+1})| \,. \tag{3.12}$$

3.3 Approximation to the Caputo-Hadamard Fractional Derivative

Consider the left-sided Caputo-Hadamard derivative of order α

$$({}^C D^{\alpha}_{1+} u)(x) = \frac{1}{\Gamma(2-\alpha)} \int\limits_a^x \left(\log \frac{x}{\xi} \right)^{1-\alpha} \delta^2 u(\xi) \frac{d\xi}{\xi}, \;\; 1 < \alpha < 2, \tag{3.13}$$

where $u \in C^2_{\delta}[a, b]$. Let the subintervals $[x_{j-1}, x_j]$ be the division of interval $[a, b]$. Define grid points $x_0, x_1, \ldots, x_N$ with size $\Delta x_j = x_j - x_{j-1}$ and

$$\frac{\Delta x_j}{\Delta x_{j-1}} = r, \;\; 1 \leq j \leq N,$$

where r is a constant near 1. Approximate to $\delta^2 u$ by spline curve $S_j(x)$ with knots $x_k, k = 0, 1, 2, \ldots, j$, given in (3.2),

$$S_j(x) = \sum_{k=0}^{j} \delta^2 u(x_k) N_{k-1,1}(x). \tag{3.14}$$

For $1 \leq k \leq j - 1$,

$$N_{k-1,1}(x) = \begin{cases} \log \dfrac{x}{x_{k-1}} \Big/ \log \dfrac{x_k}{x_{k-1}}, & x_{k-1} \leq x < x_k \\ \log \dfrac{x_{k+1}}{x} \Big/ \log \dfrac{x_{k+1}}{x_k}, & x_k \leq x < x_{k+1} \\ 0 \hspace{3cm}, & \text{otherwise.} \end{cases}$$

For $k = 0$,

$$N_{-1,1}(x) = \begin{cases} \log \dfrac{x_1}{x} / \log \dfrac{x_1}{x_0}, & x_0 \le x < x_1 \\ 0 \quad , & \text{otherwise.} \end{cases}$$

For $k = j$,

$$N_{j-1,1}(x) = \begin{cases} \log \dfrac{x}{x_{j-1}} / \log \dfrac{x_j}{x_{j-1}}, & x_{j-1} \le x < x_j \\ 0 \quad , & \text{otherwise.} \end{cases}$$

$$\begin{aligned} ({}^C D^{\alpha}_{1^+} u)(x) &= \frac{1}{\Gamma(2-\alpha)} \int\limits_{1}^{x_j} \left(\log \frac{x_j}{\xi} \right)^{1-\alpha} \sum_{k=0}^{j} \delta^2 u(x_k) N_{k-1,1}(\xi) \frac{d\xi}{\xi} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \delta^2 u(x_k) \int\limits_{x_0}^{x_j} \left(\log \frac{x_j}{\xi} \right)^{1-\alpha} N_{k-1,1}(\xi) \frac{d\xi}{\xi}. \end{aligned} \tag{3.15}$$

We evaluate the following integrals. For $k = 0$,

$$\begin{aligned} I_{j,0} &= \frac{1}{\Gamma(2-\alpha)} \int\limits_{x_0}^{x_j} \left(\log \frac{x_j}{\xi} \right)^{1-\alpha} N_{-1,1}(\xi) \frac{d\xi}{\xi} \\ &= \frac{1}{\Gamma(2-\alpha)} \left(\log \frac{x_1}{x_0} \right)^{-1} \int\limits_{x_0}^{x_1} \left(\log \frac{x_j}{\xi} \right)^{1-\alpha} \left(\log \frac{x_1}{\xi} \right) \frac{d\xi}{\xi} \\ &= \frac{1}{(2-\alpha)(3-\alpha)\Gamma(2-\alpha)} \left(\log \frac{x_1}{x_0} \right)^{-1} \\ &\quad \left[\left(\log \frac{x_j}{x_1} \right)^{3-\alpha} - \left(\log \frac{x_j}{x_0} \right)^{2-\alpha} \left((\alpha - 3) \log \frac{x_1}{x_0} + \log \frac{x_j}{x_0} \right) \right] \\ &= \frac{1}{\Gamma(4-\alpha)} \left(\log \frac{x_1}{x_0} \right)^{-1} \left[\left(\log \frac{x_j}{x_1} \right)^{3-\alpha} - \left(\log \frac{x_j}{x_0} \right)^{2-\alpha} \left((\alpha - 3) \log \frac{x_1}{x_0} + \log \frac{x_j}{x_0} \right) \right]. \end{aligned}$$

For $k = j$,

$$\begin{aligned}
I_{j,j} &= \frac{1}{\Gamma(2-\alpha)} \int\limits_{x_0}^{x_j} \Big(\log \frac{x_j}{\xi}\Big)^{1-\alpha} N_{j-1,1}(\xi) \frac{d\xi}{\xi} \\
&= \frac{1}{\Gamma(2-\alpha)} \int\limits_{x_{j-1}}^{x_j} \Big(\log \frac{x_j}{\xi}\Big)^{1-\alpha} \Big(\log \frac{\xi}{x_{j-1}} / \log \frac{x_j}{x_{j-1}}\Big) \frac{d\xi}{\xi} \\
&= \frac{1}{\Gamma(2-\alpha)} \Big(\log \frac{x_j}{x_{j-1}}\Big)^{-1} \int\limits_{x_{j-1}}^{x_j} \Big(\log \frac{x_j}{\xi}\Big)^{1-\alpha} \Big(\log \frac{\xi}{x_{j-1}}\Big) \frac{d\xi}{\xi} \\
&= \frac{1}{(2-\alpha)(3-\alpha)\Gamma(2-\alpha)} \Big(\log \frac{x_j}{x_{j-1}}\Big)^{2-\alpha} \\
&= \frac{1}{\Gamma(4-\alpha)} \Big(\log \frac{x_j}{x_{j-1}}\Big)^{2-\alpha}.
\end{aligned}$$

For $1 \le k \le j-1$,

$$\begin{aligned}
I_{j,k} &= \frac{1}{\Gamma(2-\alpha)} \int\limits_{x_0}^{x_j} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} N_{k-1,1}(\xi) \frac{d\xi}{\xi} \\
&= \frac{1}{\Gamma(2-\alpha)} \int\limits_{x_{k-1}}^{x_k} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} \frac{\log \dfrac{\xi}{x_{k-1}}}{\log \dfrac{x_k}{x_{k-1}}} \frac{d\xi}{\xi} \\
&\quad + \int\limits_{x_k}^{x_{k+1}} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} \frac{\log \dfrac{x_{k+1}}{\xi}}{\log \dfrac{x_{k+1}}{x_k}} \frac{d\xi}{\xi} \\
&= \frac{1}{\Gamma(2-\alpha)} \left(\log \frac{x_k}{x_{k-1}}\right)^{-1} \int\limits_{x_{k-1}}^{x_k} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} \left(\log \frac{\xi}{x_{k-1}}\right) \frac{d\xi}{\xi} \\
&\quad + \frac{1}{\Gamma(2-\alpha)} \left(\log \frac{x_{k+1}}{x_k}\right)^{-1} \int\limits_{x_k}^{x_{k+1}} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} \left(\log \frac{x_{k+1}}{\xi}\right) \frac{d\xi}{\xi} \\
&= \frac{1}{\Gamma(4-\alpha)} \Bigg[\left(\log \frac{x_k}{x_{k-1}}\right)^{-1} \left(\left(\log \frac{x_j}{x_{k-1}}\right)^{3-\alpha} - \left(\log \frac{x_j}{x_k}\right)^{3-\alpha} \right) \\
&\quad + \left(\log \frac{x_{k+1}}{x_k}\right)^{-1} \left(\left(\log \frac{x_j}{x_{k+1}}\right)^{3-\alpha} - \left(\log \frac{x_j}{x_k}\right)^{3-\alpha} \right) \Bigg].
\end{aligned}$$

Introduce the grid mapping $x(\eta) = ae^{\eta \log \frac{b}{a}}$ from the interval $[0,1]$ to $[a,b]$. The N-equal partition in $[0,1]$ such that $\eta_k = \dfrac{k}{N}$ converts to the nonuniform grid points $x_k = x(\eta_k)$ in the interval $[a,b]$.

For simplicity, we replace $[a,b] = [1,e]$. Then,

$$I_{j,0} = \frac{(\Delta\eta)^{2-\alpha}}{\Gamma(4-\alpha)} c_{j,0}, \quad I_{j,j} = \frac{(\Delta\eta)^{2-\alpha}}{\Gamma(4-\alpha)} c_{j,j}, \quad I_{j,k} = \frac{(\Delta\eta)^{2-\alpha}}{\Gamma(4-\alpha)} c_{j,k},$$

where $\Delta\eta = \dfrac{1}{N}$ and

$$c_{j,k} = \begin{cases} (j-1)^{3-\alpha} - (j)^{2-\alpha}(\alpha - 3 + j) & , \quad k = 0 \\ (j-k+1)^{3-\alpha} - 2(j-k)^{3-\alpha} + (j-k-1)^{3-\alpha}, & \quad 1 \le k \le j-1 \\ 1 & , \quad k = j. \end{cases}$$

It follows that

$$({}^C D^{\alpha}_{1+} u)(x_j) \approx \frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=0}^{j} \delta^2 u_k c_{j,k}. \tag{3.16}$$

Here the numbers $c_{j,k}$ are the same as (Sousa, 2011). Besides this, the numbers $c_{j,k}$ are used by Maidi & Corriou (2019).

Writing Taylor expansions of $u(x_{k+1})$ and $u(x_{k-1})$ at $x = x_k$, Veldman & Rinzema (1992), one can obtain

$$\begin{aligned} u'(x_k) \quad = \quad & \frac{u(x_{k+1}) - u(x_{k-1})}{x_{k+1} - x_{k-1}} - \frac{\Delta x_{k+1}^2 - \Delta x_k^2}{2!(x_{k+1} - x_{k-1})} u''(x_k) \\ & - \frac{\Delta x_{k+1}^3 + \Delta x_k^3}{3!(x_{k+1} - x_{k-1})} u'''(x_k) - \cdots \end{aligned}$$

and

$$\begin{aligned} u''(x_k) \quad = \quad & \frac{2}{\Delta x_k \Delta x_{k+1}} \Bigg(\frac{\Delta x_k}{(x_{k+1} - x_{k-1})} u(x_{k+1}) + \frac{\Delta x_{k+1}}{(x_{k+1} - x_{k-1})} u(x_{k-1}) \\ & - \frac{\Delta x_{k+1} + \Delta x_k}{(x_{k+1} - x_{k-1})} u(x_k) - \frac{\Delta x_k \Delta x_{k+1}^3 - \Delta x_{k+1} \Delta x_k^3}{3!(x_{k+1} - x_{k-1})} u'''(x_k) + \cdots \Bigg). \end{aligned}$$

Therefore,

$$\begin{aligned}\delta^2 u_k &= x_k u_k^{'} + x_k^2 u_k^{''} \\ &= \frac{x_k}{\Delta x_k \Delta x_{k+1}(x_{k+1}-x_{k-1})}\Big((x_{k-1}+x_k)\Delta x_{k+1} u_{k-1} \\ &\quad -2x_k(x_{k+1}-x_{k-1})u_k + (x_k + x_{k-1})\Delta x_k u_{k+1}\Big) + R(x_k).\end{aligned}$$

Writing $\Delta x_{k+1} = r\Delta x_k$ and $r = e^{\Delta\eta}$, we abbreviate the remainder

$$R(x_k) = \frac{1}{6}x_k \Delta x_k \left[(1-r)(3u_k^{(2)} + 2x_k u_k^{(3)}) - \Delta x_k (1 - r + r^2)u_k^{(3)} + \cdots\right], \tag{3.17}$$

which is second order accurate where $\underset{k}{max}\,(\Delta x_k) = \Delta x$ and $\Delta x = e(1-e^{-\Delta\eta})$ if $u_k^{(2)}$ and all of its higher orders are bounded. Thus,

$$R(x_k) = \frac{1}{6}x_k \Delta x_k \left[(1-r)(3u_k^{(2)} + 2x_k u_k^{(3)})\right] + O(\Delta x^2). \tag{3.18}$$

Replace the following estimation for $\delta^2 u_k$.

$$\Lambda u_k = \frac{x_k(\Delta x_{k+1}(x_{k-1}+x_k)u_{k-1} - 2x_k(x_{k+1}-x_{k-1})u_k + \Delta x_k(x_k + x_{k+1})u_{k+1})}{\Delta x_{k+1}\Delta x_k (x_{k+1}-x_{k-1})},$$

or explicitly

$$\Lambda u_k = \frac{e^{\Delta\eta}}{(e^{\Delta\eta}-1)^2}(u_{k-1} - 2u_k + u_{k+1}). \tag{3.19}$$

Therefore we obtain the discretization at $x = x_j$, $1 \le j \le N-1$,

$$({}^C D_{1^+}^{\alpha} u)(x_j) \approx \frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=0}^{j} \Lambda u_k c_{j,k}, \tag{3.20}$$

which can be written as the following a difference formula

$$({}^C D_{1^+}^{\alpha} u)(x_j) \approx \sum_{p=-1}^{j} d_{j,j-p} u_{j-p}. \tag{3.21}$$

Theorem 3.3.1 *If the function $u(x)$ is in $C^5[1,e]$, then*

$$({}^C D^{\alpha}_{1^+} u)(x_j) = \sum_{p=-1}^{j} d_{j,j-p} u_{j-p} + e(x_j), \tag{3.22}$$

where

$$d_{j,k-p} = \frac{\Delta\eta^{2-\alpha} e^{\Delta\eta}}{\Gamma(4-\alpha)(e^{\Delta\eta}-1)^2} \begin{cases} c_{j,j} & , \quad p=-1 \\ c_{j,j-1} - 2c_{j,j} & , \quad p=0 \\ c_{j,j-p-1} - 2c_{j,j-p} + c_{j,j-p+1}, & 1 \le p \le j \end{cases} \tag{3.23}$$

and the error $e(x_j) = O(\Delta x^2)$ for $\max\limits_{k} (\Delta x_k) = \Delta x = e(1- e^{-\Delta\eta})$.

Proof. Due to the Taylor expansion of $\delta^2 u_k$

$$\frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=0}^{j} \delta^2 u_k c_{j,k} = \frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=0}^{j} \Lambda u_k c_{j,k} + E_1(x_j), \tag{3.24}$$

it becomes the following error

$$E_1(x_j) = \frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=0}^{j} c_{j,k} R(x_k). \tag{3.25}$$

There is a positive constant C_1 such that

$$\begin{aligned} |E_1(x_j)| &\le C_1 \left| \frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=0}^{j} c_{j,k}((1-e^{\Delta\eta})\Delta x_k + \Delta x^2) \right| \\ &< C_1 \left| \frac{\Delta\eta^{2-\alpha}}{\Gamma(4-\alpha)} \right| \left((e^{\Delta\eta}-1)\Delta x \left| \sum_{k=0}^{j} c_{j,k} \right| + \left| \Delta x^2 \sum_{k=0}^{j} c_{j,k} \right| \right) \\ &\le C_1 \left| \frac{(j\Delta\eta)^{2-\alpha}\Delta x^2}{\Gamma(3-\alpha)} \right| \\ &= C_1 \frac{(\log(x_j/x_0))^{2-\alpha}}{\Gamma(3-\alpha)} \Delta x^2. \end{aligned} \tag{3.26}$$

Note that

$$\sum_{k=0}^{j} c_{j,k} = (3-\alpha) j^{2-\alpha},$$

$$e^{\Delta\eta} - 1 = e^{\Delta\eta}(1 - e^{-\Delta\eta}) = e^{\Delta\eta}e^{-1}\Delta x < \Delta x,$$

and

$$\log \frac{x_j}{x_0} = \log \frac{x_j}{x_{j-1}} \frac{x_{j-1}}{x_{j-2}} \cdots \frac{x_1}{x_0} = j\Delta\eta.$$

Hence, $E_1(x_j) = O(\Delta x^2)$.

From now on, we need to show that the error $E_2(x_j)$ of a spline function that approximates $\delta^2 u(x)$. That is,

$$E_2(x_j) = \frac{1}{\Gamma(2-\alpha)} \int_{x_0}^{x_j} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} (\delta^2 u(\xi) - S_j(\xi)) \frac{d\xi}{\xi}. \tag{3.27}$$

We can write

$$\begin{aligned} |E_2(x_j)| &\leq C_2 \max_{x_0 \leq \xi \leq x_j} \left|\delta^2 u(\xi) - S_j(\xi)\right| \left| \frac{1}{\Gamma(2-\alpha)} \int_{x_0}^{x_j} \left(\log \frac{x_j}{\xi}\right)^{1-\alpha} \frac{1}{\xi} d\xi \right| \\ &= C_2 \max_{x_0 \leq \xi \leq x_j} \left|\delta^2 u(x) - S_j(x)\right| \frac{(\log(x_j/x_0))^{2-\alpha}}{|\Gamma(3-\alpha)|} \end{aligned} \tag{3.28}$$

for some $C_2 > 0$. From Lemma (3.2.1), we have

$$|E_2(x_j)| \leq C_3 K \Delta x^2,$$

where $K = \max\limits_{1 \leq \xi \leq e} |\delta^4 u(\xi)|$ and $C_3 > 0$. Combining (3.25) and (3.27), we conclude that

$$\max_{1 \leq j \leq N} \{|E_1(x_j)| + |E_2(x_j)|\} = O(\Delta x^2). \tag{3.29}$$

We have prepared a test example to illustrate accuracy.

Example 2 *Consider the following fractional differential equation*

$$({}^C D^{\alpha}_{1^+} u)(x) + 3u(x) = \frac{\Gamma(3)}{\Gamma(3-\alpha)} (\log x)^{2-\alpha} + 3(\log x)^2, \;\; 1 < x < e,$$

with the initial and boundary conditions $u(1) = 0$, $u'(1) = 0$ *and* $u(e) = 1$.

The exact solution is $u(x) = (\log x)^2$. In the Table 3.1 and Table 3.2, we give the observed order of convergence using average of

$$\log_2 \frac{\max\limits_i |u_i - U_i|}{\max\limits_i |u_i - U_{2i}|}, \tag{3.30}$$

where U_i denotes the approximate solution of u_i at grid points $x = x_i$ for $N = 100$ and $N = 500$.

Table 3.1 Maximum errors and observed orders for $\Delta\eta = \frac{1}{100}$, and different values of α

α	$\max\limits_i \lvert u_i - U_i\rvert$	observed order
1.1	1.7394×10^{-4}	2.06
1.3	4.11527×10^{-5}	2.12
1.5	1.27858×10^{-5}	2.06
1.7	6.31782×10^{-6}	2.02
1.9	3.77859×10^{-6}	2.00

Table 3.2 Maximum errors and observed orders for $\Delta\eta = \frac{1}{500}$, and different values of α

α	$\max\limits_i \lvert u_i - U_i\rvert$	observed order
1.1	6.05783×10^{-6}	2.03
1.3	1.27483×10^{-6}	2.07
1.5	4.71214×10^{-7}	2.02
1.7	2.46503×10^{-7}	2.00
1.9	1.50409×10^{-7}	2.00

CHAPTER FOUR

CONCLUSION

A difference approximation to the Caputo-Hadamard fractional derivative is given. The discretization of the Caputo-Hadamard fractional derivative of a sufficiently smooth function is accomplished by the method based on spline function and finite difference. Second order accuracy of the derived difference formula on a nonuniform grid is obtained. The presented method is implemented to a fractional differential equation with initial and boundary conditions. Maximum errors and observed orders in the test problem for some values of meshes and order of fractional derivatives are illustrated. Numerical results confirm the theoretical study.

REFERENCES

Adjabi, Y., Jarad, F., Baleanu, D., & Abdeljawad, T. (2016). On Caputo problems with Caputo Hadamard fractional derivatives. *Journal Computational Analysis and Applications*, *21*, 661-681.

Almeida, R. (2017). Caputo-Hadamard fractional derivatives of variable order. *Numerical Functional Analysis and Optimization*, *38*(1), 1-19.

Almeida, R., & Bastos, N. R. (2016). A discretization of the Hadamard fractional derivative. *Mathematical Sciences and Applications E-Notes*, *4*(1), 31-39.

Bai, Y., & Kong, H. (2017). Existence of solutions for nonlinear Caputo-Hadamard fractional differential equations via the method of upper and lower solutions. *Journal of Nonlinear Sciences and Applications*, *10*(11), 5744-5752.

Dişibüyük, Ç., & Goldman, R. (2016). A unified approach to non-polynomial B-spline curves based on a novel variant of the polar form. *Calcolo. A Quarterly on Numerical Analysis and Theory of Computation*, *53*(4), 751–781.

Gambo, Y. Y., Jarad, F., Baleanu, D., & Abdeljawad, T. (2014). On Caputo modification of the Hadamard fractional derivatives. *Advances in Difference Equations*, 2014(10), 12.

Gohar, M., Li, C., & Yin, C. (2019). On Caputo–Hadamard fractional differential equations. *International Journal of Computer Mathematics*, 1-25.

Jarad, F., Abdeljawad, T., & Baleanu, D. (2012). Caputo-type modification of the Hadamard fractional derivatives. *Advances in Difference Equations*, *2012*(142).

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*. Amsterdam: Elsevier Science B.V.

Maidi, A., & Corriou, J.-P. (2019). Distributed feedback control of a fractional diffusion process. *International Journal of Dynamics and Control*, *7*(3), 1091-1100.

Mainardi, F. (2010). *Fractional calculus and waves in linear viscoelasticity*. London: Imperial College Press.

Metzler, R., Glöckle, W. G., & Nonnenmacher, T. F. (1994). Fractional model equation for anomalous diffusion. *Physica A: Statistical Mechanics and its Applications*, *211*(1), 13 - 24.

Ortigueira, M. D., & Machado, J. A. T. (2003). Fractional signal processing and applications. *Signal Processing*, *83*(11), 2285–2286.

Samko, S., Kilbas, A., & Marichev, O. (1993). *Theory and applications of fractional differential equations*. Switzerland: Gordon and Breach Science.

Sousa, E. (2011). Numerical approximation for fractional diffusion equations via splines. *Computers and Mathematics with Applications*, *62*(3), 938-944.

Veldman, A., & Rinzema, K. (1992). Playing with nonuniform grids. *Journal of Engineering Mathematics*, *26*(1), 119-130.

Vinagre, B., Podlubny, I., Hernandez, A., & Feliu, V. (2000). Some approximations of fractional order operators used in control theory and applications. *Journal of Fractional Calculus and Applied Analysis*, *3*, 231–248.

APPENDICES

Appendix 1: Non-polynomial B-spline curves of order n

Definition A.1.1 *A spline is a piecewice function consists of some curve segments joined together with certain smoothness condition.*

The B-spline curve described by Dişibüyük & Goldman (2016) is a follows.

Consider two differentiable locally linearly independent functions $\gamma_1(x)$ and $\gamma_2(x)$. Define the space $\pi_n(\gamma_1, \gamma_2)$ whose elements are homogeneous polynomials of degree n spanned by the functions $\gamma_1^{n-k}\gamma_2^k, k = 0, 1, 2, ..., n$.

Definition A.1.2 *The barycentric coordinates $\omega_1(x)$ and $\omega_2(x)$ of a given point on planar curve $(\gamma_1(x), \gamma_2(x))$ with respect to the end points of the arc*

$$\Gamma(t) = (\gamma_1(t), \gamma_2(t)),\ \theta_1 \leq t \leq \theta_2$$

are of the form

$$\omega_1(x) = \frac{d(x, \theta_2)}{d(\theta_1, \theta_2)},\ \ \omega_2(x) = \frac{d(\theta_1, x)}{d(\theta_1, \theta_2)}, \tag{A.1}$$

where $d(u, v) = \gamma_1(u)\gamma_2(v) - \gamma_2(u)\gamma_1(v)$.

Definition A.1.3 *A sequence $\{x\}_{i=1}^{2n}$ of $2n$ real numbers satisfying the constraints $d(x_j, x_{i+n}) \neq 0$ for all $i \leq j \leq n$ is called progressive sequence.*

For a fixed progressive knot sequence, let $\sigma_n\ (\gamma_1, \gamma_2)$ denotes the space of splines whose curve segments lie in $\pi_n(\gamma_1, \gamma_2)$. A B-spline curve in $\sigma_n\ (\gamma_1, \gamma_2)$ is a spline generated by the de Boor algorithm (Dişibüyük & Goldman, 2016).

Definition A.1.4 *The B-spline basis functions in* $\sigma_n\ (\gamma_1, \gamma_2)$ *of order* n *are defined recursively by*

$$N_{k,0}(x) = \begin{cases} 1, & \textit{if } x \in [x_k, x_{k+1}) \\ 0, & \textit{otherwise} \end{cases},$$

and if $n > 0$

$$N_{k,n}(x) = \frac{d(x_k, x)}{d(x_k, x_{k+n})} N_{k,n-1}(x) + \frac{d(x, x_{k+n+1})}{d(x_{k+1}, x_{k+n+1})} N_{k+1,n-1}(x). \tag{A.2}$$

$N_{k,n}(x)$ *has compact support* $[x_k, x_{k+n+1}]$ *which contains* $n+1$ *knots.*

B-spline curve segments $\Psi_k(x)$ in $\pi_n(\gamma_1, \gamma_2)$ over the knot intervals $[x_k, x_{k+1}]$ generated by fixed progressive knot sequence satisfy matching condition

$$\Psi_k^{(r)}(x_{k+1}) = \Psi_{k+1}^{(r)}(x_{k+1}), r = 0, 1, ..., n-1.$$

Any spline function $S(x) \in \sigma_n(\gamma_1, \gamma_2)$ with control points P_k and knots x_k represented as

$$S(x) = \sum_k N_{k,n}(x) P_k$$

(Dişibüyük & Goldman, 2016).