# CLONING-PROOF SOCIAL CHOICE CORRESPONDENCES

A Master's Thesis

by

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Ankara

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Graduate School of Economics and Social Sciences
of
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ANKARA


August 2011

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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# ABSTRACT

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In this thesis study, we provide axiomatic characterizations of the well-known Condorcet and Plurality rules via consistency axioms when the alternative set is endogeneous, namely hereditariness and cloning-proofness. Cloning-proofness is the requirement that the social choice rule be insensitive to the replication of alternatives, whereas hereditariness requires insensitivity to withdrawal of alternatives.

*Keywords:* Social Choice Theory, Cloning-Proofness, Hereditariness, Plurality Rule, Condorcet Rule, Axiomatic Characterization.

# ÖZET

# KLONLAMAYA KARŞI DİRENÇLİ SOSYAL SEÇME KURALLARI

ÖZTÜRK, Zeliha Emel

Yüksek Lisans, Ekonomi Bölümü

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Bu tez çalşmamızda alternatif ve seçmen kümesinin değişken olduğu durumlara dair tutarlılık ilkeleri olan aktarımsallık ve klonlamaya karşı direnç ilkelerini kullanarak yaygın olarak bilinen Çoğunluk ve Condorcet sosyal seçme kurallarının aksiyomatik karakterizasyonlarını yapıyoruz. Klonlamaya karşı direnç ilkesi alternatif kümesinin eleman sayısının arttığı, aktarımsallık ise azaldığı durumlarda sosyal seçme kuralının bu değişimlerden belirli koşullar altında etkilenmemesini talep eden ilkelerdir.

*Anahtar Kelimeler:* Sosyal Seçim, Klonlamaya Karşı Direnç, Aktarımsallık, Çoğunluk Yöntemi, Condorcet Kuralı, Karakterizasyon.

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# CHAPTER 1

# INTRODUCTION

A considerable amount of social choice literature exists on the manipulability of social choice rules by voters misrepresenting their preferences. However this is not the only kind of strategy to manipulate the outcomes in an election.

If the set of alternatives is endogenous (not fixed by nature), then the outcome of a social choice procedure can be manipulated via the addition to or removal of some alternatives from the set of alternatives. For example consider a scenario where three national parties A, B and C can contest an election in which the winner is decided by Plurality rule. Although candidate A may have the highest number of first preference votes, it may still fail to win the election if, for instance, candidate B drops out.

Consistency axioms requiring choice rules or social choice rules to be non-manipulable via withdrawal of alternatives are widely studied in the literature commonly yielding impossibility results (see Arrow, 1959; Plott, 1973; Kelly, 1978 for examples). Manipulation via addition of alternatives, on the other hand, was first studied by Tideman (1987) who introduced the concept of independence of clones which requires the addition/deletion of clones not to affect the winning status of non-clones or whether the clones are among winners, where a group of alternatives is called *clones* if no voter ranks any alternative outside the group between any members of the group, i.e. they stand next to each other in every individual ranking. And by *cloning* of an alternative, we mean the addition of clones of that alternative to the alternative set from which the choice is to be done. Withdrawal of an alternative could also be interpreted as a special case of cloning, where the withdrawn alternative is cloned zero times.

Given a finite alternative set A, social choice correspondences are defined to be mappings from linear order profiles on A to subsets of it. When some alternatives are cloned, the procedure we use is as follows: Let $R$ be a profile on $A = \{a_1, ..., a_m\}$. We refer to a vector $(A_1^{'}, ..., A_m^{'})$ as a *cloning structure* for $A_m$ if $A_1^{'}, ..., A_m^{'}$ are sets such that, for each $i \in \{1, ..., m\}$, $a_i \in A_i^{'}$ whenever $A_i^{'}$ is non-empty. Given a cloning structure $(A_1^{'}, ..., A_m^{'})$ for $A_m$ and $R \in \mathcal{L}(A_m)^{I_n}$, a linear order profile $R^{'}$ on $A^{'} = A_1^{'} \cup ... \cup A_m^{'}$ obtained by replacing each occurrence of $a_i$ in $R$ by some linear order on $A_i^{'}$ whenever $A_i^{'} \neq \emptyset$ and deleting $a_i$ whenever $A_i^{'} = \emptyset$, is called a clone-extended profile obtained from $R$ for the cloning structure $(A_1^{'}, ..., A_m^{'})$. We emphasize the following two features of any clone-extended profile: in each linear order of $R^{'}$ all clones of the same alternative are standing together but the order on these clones may be different from one linear order of $R^{'}$ to another. We call a social choice correspondence F as cloning-proof if cloning of some alternatives by a positive number does not change the outcome of the election whereas we call it hereditary if the winning status of an alternative a does not change when some alternatives (but not a) are withdrawn from the alternative set.

As mentioned before, hereditariness is a commonly used concept in the literature. Cloning-proofness is also studied in some papers. One of those is by Koray and Slinko (2007). The authors name this property as resistance to cloning of essential alternatives and study the self-selectivity properties of the selections of SCCs which are resistant to cloning.

In this paper, the two social choice correspondences we focus on are the Condorcet rule and scoring rules. All scoring rules except one turn out to be very sensitive to any type of cloning (including withdrawals) since such an action changes the ranks of alternatives in a preference profile. Condorcet rule, on the other hand, is immune to any such manipulation when an appropriate $R'$ is used because with such an $R'$ any Condorcet winner continues to be a Condorcet winner whenever it is cloned or some alternatives are withdrawn from the election. One of the most popular scoring rules, the Plurality rule, is also sensitive to withdrawal of alternatives. Yet, it turns out to be immune to cloning. We show that it is actually the unique scoring rule with this property. Thus, hereditariness and cloning-proofness are two consistency properties that could be used to compare the Condorcet rule and scoring rules.

To the best of our knowledge, Laslier (2000) is the first to use a form of cloning-proofness (what he calls cloning-consistency) in the characterization of a social choice correspondence, namely the one which chooses the set of essential alternatives in every preference profile.

The organization of the paper is as follows: In Chapter 2, we study hereditary social choice correspondences and provide three axiomatic characterizations of the Condorcet rule. In Chapter 3, we provide three axiomatic characterizations of the Plurality rule using cloning-proofness in Theorems 6 and 7 and tops-onliness in Theorem 8. Chapter 4 is devoted to concluding remarks and we show that the axioms used in our theorems are logically independent in the Appendix.

# CHAPTER 2

# HEREDITARINESS

## 2.1 Preliminaries

Let N be a finite non-empty society with cardinality n that will be kept fixed throughout this chapter. Let $\mathbb{N}$ denote the set of natural numbers, set $I_m$=$\{1, \ldots, m\}$ and denote the set of all linear orders[1] on $A_m$ by $\mathcal{L}(A_m)$ for each $m \in \mathbb{N}$. We call a mapping

$$F : \bigcup_{m \in \mathbb{N}} \mathcal{L}(I_m)^N \to 2^{I_m}$$

a social choice correspondence (SCC) if and only if, for each $m \in \mathbb{N}$ and $R \in \mathcal{L}(I_m)^N$, one has $F(R) \subseteq I_m$.

For each $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^N$ and every permutation $\sigma_m$ on $I_m$, we define the permuted linear order profile $R^{\sigma_m}$ on $I_m$ as follows:

$$\text{For all } i \in N,\ a, b \in I_m\text{: } aR_i^{\sigma_m}b \Longleftrightarrow \sigma_m(a)R_i^{\sigma_m}\sigma_m(b).$$

An SCC $F$ is called *neutral* if and only if, for each $m \in \mathbb{N}$ and every permutation $\sigma_m$ on $I_m$, one has

$$\sigma_m(F(R^{\sigma_m})) = F(R).$$

[1]A linear order R is a complete, transitive and antisymmetric binary relation. A binary relation R on some set A is complete if and only if, $\forall a, b \in A$: either aRb or bRa holds; transitive if and only if, $\forall a, b, c \in A$: aRb and bRc implies aRc; antisymmetric if and only if, $\forall a, b \in A$: aRb and bRa implies a=b.

We will restrict ourselves to neutral SCCs throughout the paper. Given the neutrality of F, we can extend the domain of F to linear order profiles on any finite nonempty set in a natural fashion. Take any finite set $A_m$ with cardinality m. Let $\mu : I_m \rightarrow A_m$ be a bijection (i.e. a one-to-one and onto function). Any linear order profile L on $A_m$ induces a linear order profile $L^{\mu}$, on $I_m$ like above, where, for all $i \in N$ and $k, l \in I_m$ one has $kL_i^{\mu}l$ if and only if $\mu(k)L_i\mu(l)$. Now it is clear that $\mu(F(L^{\mu})) = \upsilon(F(L^{\upsilon}))$ for any two bijections $\mu, \upsilon : I_m \rightarrow A_m$ if F is neutral. We will simply define $F(L) = \mu(F(L^{\mu}))$, where $\mu$ is a bijection from $I_m$ onto $A_m$.

## 2.2 Characterization of Hereditary Social Choice Correspondences

In this section, we provide a characterization of hereditary social choice rules.

**Definition 1.** Given $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$, an SCC F is said to be *hereditary* at $R$ if and only if, for any $B \subseteq A_m$, one has $F(R) \cap B \subseteq F(R\mid_B)$[2]. F is hereditary if and only if it is hereditary at each $R \in \mathcal{L}(A_m)^N$ for every $m \in \mathbb{N}$.

Hereditariness requires that if an alternative $a \in A$ is chosen from A then it should also be chosen from every subset B of A with $a \in B$. This property is widely known as Sen's $\alpha$ (Sen, 1971) or Chernoff (Chernoff, 1954) condition in the literature on rationalizability of choice functions (see Moulin, 1985 for a general survey on rational choice).

We provide three more definitions before we state the main result of this section. To do so, take any $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$.

[2] $R\mid_B$ stands for the restriction of R into B.

**Definition 2.** An SCC F satisfies $\alpha^*$ at $R$ if and only if, for all subsets B of $A_m$ there exists $a \in F(R\mid_B)$ such that for all $C \subseteq B$, $a \in C$ implies $a \in F(R\mid_C)$. F satisfies $\alpha^*$ if and only if it satisfies $\alpha^*$ at each $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$.

This property was first introduced by Deb (1983) in choice theoretic terms. Deb (1983) has shown that it is equivalent to the weak rationalization of a choice rule T, where, given a finite alternative set A, T is weakly rationalizable if and only if, there exists an ordering R on A such that $\emptyset \neq M(B,R) \subseteq T(B)$ for all non-empty subsets B of A[3].

**Definition 3.** An SCR F satisfies $\beta^*$ at $R$ if and only if, for any $a, b \in F(R)$, $a \in F(R\mid_B)$ for some subset B of $A_m$ and $b \in B$ implies $b \in F(R\mid_B)$.[4] F satisfies $\beta^*$ if and only if it satisfies $\beta^*$ at each $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$.

**Definition 4.** An SCR F satisfies $\gamma^*$ at $R$ if and only if, for any two alternatives a,b in $A_m$: $a \in F(R)$, $b \in A_m \setminus F(R)$ implies $\nexists B \subseteq A_m$ with $a, b \in B$ and $b \in F(R\mid_B)$ where $a \notin F(R\mid_B)$. F satisfies $\gamma^*$ if and only if it satisfies $\gamma^*$ at each $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$.

Now we are ready to state our result.

**Theorem 1.** *Let F be a non-empty valued social choice correspondence. Now F is hereditary if and only if it satisfies $\alpha^*$, $\beta^*$, and $\gamma^*$.*

*Proof.* It is obvious that a hereditary social choice rule F satisfies $\alpha^*$, $\beta^*$, and $\gamma^*$.

Now let F satisfy $\alpha^*$, $\beta^*$, and $\gamma^*$. We will show that F is hereditary. Take any $m \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^N$ and $a \in F(R)$. We must show that $a \in F(R\mid_B)$ for any subset B of $A_m$ with $a \in B$. By $\alpha^*$, there exists some $b' \in F(R)$ such that $b' \in F(R\mid_B)$ for every $B \subseteq A_m$ with $b' \in B$. Thus by $\beta^*$, $a \in F(R\mid_B)$ for every B such that $a, b' \in B$.

Take any $C \subseteq A_m$ such that $a \in C$ and $b' \notin C$. Since F is non-empty valued,

[3] M(B,R) denotes the maximal elements of B with respect to the ordering R.
[4] $\beta^*$ can be thought of as inverse of Sen's $\beta$.

there exists some $c \in F(R\mid_C)$. Now if $c \in F(R)$, then $a \in F(R\mid_C)$ by $\beta^*$. If, on the other hand, $c \notin F(R)$, then $a \in F(R\mid_C)$ by $\gamma^*$ and non-emptiness of F. So, $a \in F(R\mid_B)$ for every B, $\emptyset \neq B \subseteq A_m$.

Thus, F is hereditary. $\square$

We are not aware of any equivalent formulation of hereditariness in choice theory. Yet, it is clear that the above result can be directly transmitted to choice theory with direct formulations of the axioms. Thus, the above theorem also characterizes the well-known property Sen's alpha.

## 2.3 Characterizations of the Condorcet Rule

Given any $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$, set $n(a, b, R) = \mid \{i \in N : aR_ib\} \mid$[5] for any $a, b \in A_m$ with $a \neq b$. We say that $a \in A_m$ is a *Condorcet winner* for $R$ if and only if, $n(a, b, R) \geq \frac{N}{2}$ for each $b \in A_m \setminus \{a\}$. We denote the set of all Condorcet winners at $R$ by CW(R).

Condorcet rule is the social choice rule C such that C$(R) = CW(R)$ at every $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$.

We say that an SCC F is *top-majoritarian* if and only if, for any $m \in \mathbb{N}$, $a \in A_m$, $R \in \mathcal{L}(A_m)^N$,

$$[n(a, b, R) > \tfrac{N}{2} \text{ for every } b \in A_m \setminus \{a\}] \text{ implies } F(R) = \{a\}.$$

First we provide the following proposition without proof since it is straightforward to see that Condorcet rule is both top-majoritarian and hereditary.

**Proposition 1.** *Condorcet rule is top-majoritarian and hereditary.*

Given the above proposition, we now state our first characterization result for the Condorcet rule.

[5] $\mid A \mid$ stands for the cardinality of any set A.

**Theorem 2.** *Condorcet rule is the maximal top-majoritarian, hereditary social choice correspondence, i.e.* $F(R) \subseteq \mathrm{C}(R)$ *at every* $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$*, where* $F$ *is any top-majoritarian, hereditary social choice correspondence..*

*Proof.* It is straightforward to verify that the Condorcet rule is both hereditary and top-majoritarian.

To see that it is the maximal such correspondence, take any $m \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^N$ and let F be a hereditary, top-majoritarian SCC. Assume, on the contrary, that $F(R) \nsubseteq \mathrm{C}(R)$.

Then there exists an alternative $a \in F(R)$ with $a \notin CW(R)$, i.e. there exists another alternative $b \in A_m$ such that $n(b, a, R) > \frac{N}{2}$. Then, by top-majoritarianity, $F(R\mid_{\{a,b\}}) = \{b\}$, a contradiction with $F$ being hereditary. Thus, $F(R) \subseteq \mathrm{C}(R)$. □

Koray (1998) also deals with consistency in the sense of self-selectivity and rediscovers the Condorcet rule as the maximal self-selective social choice corresponclence among neutral and top-majoritarian social choice rules.

Before we proceed to the next theorem, we need one more definition.

**Definition 5.** Given $m \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^N$, an SCC $F$ is said to satisfy *binary dominance consistency* (BDC) at $R$ if and only if, for any $a \in A_m$,

$$[a \in F(R\mid_{\{a,b\}}) \text{ for all } b \in A_m \setminus \{a\}] \Longrightarrow a \in F(R).$$

F satisfies BDC if and only if, it satisfies BDC at each $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$.

Binary dominance consistency requires that an alternative that is among the chosen alternatives in all two-element subsets of a set A should be among the chosen alternatives from A as a whole. It is the social choice rule formulation of what is known as Sen's $\gamma$ (Sen 1971) or Generalized Condorcet Property (Plott, (1973); Blair et al. (1976)). Plott shows that BDC conjoined with Path Independence is equivalent to the quasi-transitive rationalization of a choice function. Blair et al. (1976) shows that BDC conjoined with

Chernoff Condition is equivalent to the acyclic rationalization of a choice function.

**Theorem 3.** *Given an odd number of individuals, Condorcet rule is the unique SCC which is top-majoritarian, hereditary and binary dominance consistent.*

*Proof.* It is, again, obvious that the Condorcet rule satisfies all three axioms stated in the theorem.
Moreover, given $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$, we already established that if an SCC $F$ is hereditary and top-majoritarian, then $F(R) \subseteq \mathrm{C}(R)$. So, we only need to show that $\mathrm{C}(R) \subseteq F(R)$. Note that, since the number of individuals is odd, we have either $CW(R) = \emptyset$ or $\mid CW(R) \mid= 1$. To complete the proof we need only to show that $F(R) \neq \emptyset$ if $\mid CW(R) \mid= 1$. To do so, assume $CW(R) = \{a\}$. Then, $n(a, b, R) > \frac{N}{2}$ for all $b \in A_m \setminus \{b\}$. Then, by top-majoritarianity, $F(R \mid_{\{a,b\}}) = \{a\}$ for all $b \in A_m \setminus \{a\}$ which implies $a \in F(R)$ by binary dominance consistency.
Hence, $F(R) = CW(R)$, i.e. F is the Condorcet rule. $\square$

Finally, in this section, we provide a characterization of the Condorcet rule without any restriction on the number of individuals. To do so, we need one more definition.

**Definition 6.** Given $m \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^N$, an SCC F is said to satisfy *cancellation* (CA) at R if and only if,

$$[\mid \{i \in N : aR_ib\} \mid=\mid \{i \in N : bR_ia\} \mid \text{ for all } a, b \in A_m] \Longrightarrow F(R) = A_m.$$

F satisfies cancellation if and only if it satisfies cancellation at each $R \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$.

Cancellation simply requires that if for any pair of alternatives a, b the number of individuals preferring a to b is equal to the number of individuals preferring b to a, then the whole set of alternatives should be chosen. It is introduced by Young (1974) and used in the characterization of the Borda rule.

Now we are ready to state the last theorem of this chapter.

**Theorem 4.** *Condorcet rule is the unique SCC which is top-majoritarian, hereditary and binary dominance consistent satisfying cancellation.*

*Proof.* It is obvious that Condorcet satisfies all four axioms. Moreover, given $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$, we already established that if an SCC $F$ is hereditary and top-majoritarian, then $F(R) \subseteq \mathrm{C}(R)$. So, again it remains to show that $a \in F(R)$ whenever $a \in CW(R)$. To see that, take any $a \in CW(R)$ and any $b \in A \setminus \{a\}$. Now by definition of a Condorcet winner, we have one of the following:

1. $| \{i \in N : aR_ib\} | > N/2$
2. $| \{i \in N : aR_ib\} | = | \{i \in N : bR_ia\} |$

If 1 holds, then, by top-majoritarianity, $\{a\} = F(R \mid_{\{a,b\}})$. If, on the other hand, 2 holds, then by cancellation, $\{a, b\} = F(R \mid_{\{a,b\}})$. Either way, $a \in F(R \mid_{\{a,b\}})$. Thus, by BDC, we have $a \in F(R)$. So, $C(R) \subseteq F(R)$.
Thus $F(R) = C(R)$, completing the proof.

□

# CHAPTER 3

# CLONING-PROOFNESS

## 3.1 Preliminaries

We kept the number of individuals fixed whereas we allowed the number of alternatives vary throughout the previous chapter. In this chapter, we allow both the number of individuals and the number of alternatives vary. Let $A_m$=$\{1, \ldots, m\}$ be the set of alternatives and $I_n = \{1, \ldots, n\}$ be the set of individuals for each $m, n \in \mathbb{N}$. We call a correspondence

$$F : \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^{I_n} \rightarrow \bigcup_{m\in\mathbb{N}} A_m$$

a social choice rule (SCR) if and only if, $\forall m \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall R \in \mathcal{L}(A_m)^{I_n}$, $\emptyset \neq F(R) \subseteq A_m$. Throughout the chapter, $a_i, a_j, ...$ will denote the generic elements of the alternative set and $i, j, ...$ will denote those of the individual set.

Note that the domain of F can be extended to any set $A_m$ with cardinality m as in Chapter 2 since we deal only with neutral social choice rules.

Starting with May's (1952) characterization of the majority rule, an extensive literature emerged on axiomatic characterizations of scoring rules. In this chapter we provide three different characterizations of the Plurality rule. The two well-known characterizations of the same rule is due to Richelson (1978) and Ching (1996). Richelson (1978) is the first to characterize the

Plurality rule and shows that it is the only rule satisfying anonymity, neutrality, independence of dominated candidates, reinforcement, and continuity. Ching (1996) shows that continuity is redundant in the characterization.

Our Theorem 5 produces Ching's (1996) (and hence Richelson's) characterization of the same rule as a corollary.

In order to state the definitions of this chapter, take any $m, n \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^{I_n}$.

**Definition 7.** We say that a non-empty subset $C$ of $A_m$ is *a set of clone alternatives* at R if and only if, for each $c, c^{'} \in C$, for each $x \in A_m \setminus C$ and for each $i \in I_n$:

$$xR_ic \Longleftrightarrow xR_ic^{'}$$
$$cR_ix \Longleftrightarrow c^{'}R_ix.$$

Two alternatives are clones of each other if they stand next to each other in every individual ranking, i.e. they perform identically in pairwise comparisons with other alternatives.

**Definition 8.** We refer to a vector $(A_1^{'}, ..., A_m^{'})$ as a *cloning structure* for $A_m$ if $A_1^{'}, ..., A_m^{'}$ are sets such that, $A_i^{'} \cap A_j^{'} = \emptyset$ for any distinct $i, j \in \{1, ..., m\}$ and for each $i \in \{1, ..., m\}$, $a_i \in A_i^{'}$ whenever $A_i^{'}$ is non-empty. Given a cloning structure $(A_1^{'}, ..., A_m^{'})$ for $A_m$ and $R \in \mathcal{L}(A_m)^{I_n}$, a linear order profile $R^{'}$ on $A^{'} = A_1^{'} \cup ... \cup A_m^{'}$ obtained by replacing each occurrence of $a_i$ in $R$ by some linear order on $A_i^{'}$ whenever $A_i^{'} \neq \emptyset$ and deleting $a_i$ whenever $A_i^{'} = \emptyset$, is called a clone-extended profile obtained from $R$ for the cloning structure $(A_1^{'}, ..., A_m^{'})$. We denote the set of all clone-extended profiles obtained from $R$ for the cloning structure $(A_1^{'}, ..., A_m^{'})$ by $\mathcal{L}^{I_n}[R; (A_1^{'}, ..., A_m^{'})]$.

*Remark* 1. Note that, given any $m, n \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^{I_n}$, we derive any clone-extended profile $R^{'}$ for any cloning-structure in such a way that all alternatives at any $A_i^{'}$ are clone alternatives.

**Definition 9.** An SCR $F$ is said to be

*cloning-proof* (CP) at $R$ if and only if, for every cloning structure

$(A_1^{'}, ..., A_m^{'})$ with $A_i^{'} \neq \emptyset$ for all $i \in \{1, ..., m\}$, one has $F(R) = F(R^{'})$ where $R^{'}$ is any clone extended profile obtained from $R$ with the following property: for each $i \in I_n$, $a_j \in A_m$: $a_j R_i^{'} a_j^{'}$ for all $a_j^{'} \in A_j^{'} \setminus \{a_j\}$. F is CP if and only if it is CP at each $R \in \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \mathcal{L}(A_m)^{I_n}$.

**Definition 10.** Let $\pi : I_n \rightarrow I_n$ be a permutation on $I_n$ and $\Pi$ be the collection of all such permutations. Let $R_{\pi(I_n)} = (R_{\pi(i)})_{i \in I_n}$. Now, an SCC F satisfies *anonymity* if and only if, for all $\pi \in \Pi$, $F(R_{\pi(I_n)}) = F(R)$.

**Definition 11.** Let $N$ and $N^{'}$ be two disjoint sets of individuals. F satisfies *reinforcement* if and only if, for all $R_N \in \mathcal{L}(A)^N$ and all $R_{N'} \in \mathcal{L}(A)^{N^{'}}$, if $F(R_N) \cap F(R_{N'}) \neq \emptyset$, then $F(R_N + R_{N'}) = F(R_N) \cap F(R_{N'})$, where $R_N + R_{N'}$ is the profile obtained by concatenating the two preference profiles.

Given $i \in I_n$ and i's preference ordering $R_i$, let $\tau(R_i)$ be the top-ranked alternative by agent i and $\tau(R)$ be the set of top-ranked alternatives at $R$. Moreover, for any $a_k \in A_m$ let $N_k(R) = \{i \in I_n : \tau(R_i) = a_k\}$ and let $n_k(R) =\mid N_k(R) \mid$.

**Definition 12.** An SCC F is said to be *unanimous* if and only if, for any $a_k \in A_m$, $\tau(R) = \{a_k\}$ implies $F(R) = \{a_k\}$ at each $R \in \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \mathcal{L}(A_m)^{I_n}$.

Finally, we are ready to state the definition of the Plurality rule and the first result of this chapter.

**Plurality Rule P** :

$$P(R) = \{a_k \in A_m : \forall a_j \in A_m, n_k(R) \geq n_j(R)\}.$$

## 3.2 Characterizations of the Plurality Rule

**Theorem 5.** *Plurality rule is the unique neutral social choice correspondence that satisfies reinforcement, cloning-proofness, and anonymity.*

*Proof.* It is easy to see that Plurality rule satisfies all axioms stated in the theorem.

Now, let F be an SCC satisfying all of the axioms. Take any $m, n \in \mathbb{N}$. We first claim that if at any $R \in \mathcal{L}(A_m)^{I_n}$, the top-ranked alternatives of all individuals are distinct (thus $n \leq m$), then $F(R) = \tau(R)$. To show that, we proceed by induction on the cardinality of $I_n$.

1. Suppose $\mid I_n \mid$=1. Without loss of generality, assume $\tau(R) = \{a\}$. Then, all other alternatives could be considered to be clones of a, and hence by cloning-proofness and non-emptiness of F, one has $F(R) = \tau(R)$.

2. Suppose $\mid I_n \mid$=n. Assume, without loss of generality, $R$ is as follows:

$$R = \begin{bmatrix} \underline{R_1} & . & . & \underline{R_{n-1}} & \underline{R_n} \\ a_1 & . & . & a_{n-1} & a_{s_1} \\ . & . & . & . & a_{s_2} \\ . & . & . & . & . \\ . & . & . & . & a_{s_{m-1}} \\ . & . & . & . & a_{s_m} \end{bmatrix}$$

   where $a_{s_1} = a_n$. By the induction hypothesis, $F(R_1, ..., R_{n-1}) = \{a_1, ..., a_{n-1}\}$. Now let $I_n^{'} = \{n+1, ..., n+m-1\}$ and $R^{'} \in \mathcal{L}(A_m)^{I_n^{'}}$ be as follows[1]:

$$R^{'} = \begin{bmatrix} \underline{R_{n+1}} & . & . & \underline{R_{n+m-1}} \\ a_{s_2} & . & . & a_{s_m} \\ a_{s_3} & . & . & a_{s_1} \\ . & . & . & . \\ . & . & . & . \\ a_{s_m} & . & . & a_{s_{m-2}} \\ a_{s_1} & . & . & a_{s_{m-1}} \end{bmatrix}$$

[1]Ching (1995) has the same construction of profiles, which is originally due to Richelson (1978).

Neutrality and anonymity imply $F(R_n + R^{'}) = A_m$. Moreover, reinforcement implies that,

(a) $F(R+R^{'}) = F(R_1, ..., R_{n-1}) \cap F(R_n, R_{n+1}, ..., R_{n+m-1}) = \{a_1, ..., a_{n-1}\}$.

Now let $A^{'} = \{a_{s_2}, ..., a_{s_m}\}$ and $R^{''} \in \mathcal{L}(A^{'})^{I^{'}_n}$ be as follows:

$$R^{''} = \begin{bmatrix} \underline{R_{n+1}} & . & . & \underline{R_{n+m-1}} \\ a_{s_2} & . & . & a_{s_m} \\ a_{s_3} & . & . & a_{s_2} \\ . & . & . & . \\ . & . & . & . \\ a_{s_m} & . & . & a_{s_{m-1}} \end{bmatrix}$$

Again, by neutrality and anonymity, $F(R^{''}) = A^{'}$. Notice that, at $R^{'}$, $a_{s_1}, a_{s_m}$ are clone alternatives. Thus, $R^{'}$ is a clone-extended profile obtained from $R^{''}$ for the cloning structure $(A^{'}_{s_2}, ..., A^{'}_{s_m})$ where $A^{'}_{s_i} = \{a_{s_i}\}$ for all $i \in \{2, ..., m-1\}$ and $A^{'}_{s_m} = \{a_{s_1}, a_{s_m}\}$. Thus, by CP,

(b) $F(R^{'}) = F(R^{''}) = \{a_{s_2}, ..., a_{s_m}\} = \{a_1, ..., a_{n-1}\}$.

Then, combining (a), (b) and reinforcement, we have,

(c)

$$F(R) = \begin{cases} \{a_n\} & \text{or,} \\ \{a_1, ..., a_{n-1}\} & \text{or} \\ \{a_1, ..., a_n\}. & \end{cases}$$

Similarly, let $\bar{R}$ be a profile such that $(R_{n-1}, \bar{R})$ are symmetric as in $(R_n, R^{'})$. Following the exact similar arguments, one obtains:

(d)

$$F(R) = \begin{cases} \{a_{n-1}\} & \text{or,} \\ \{a_1, ..., a_{n-2}, a_n\} & \text{or} \\ \{a_1, ..., a_n\}. \end{cases}$$

Now, (c) and (d) imply that $F(R_N) = \{a_1, ..., a_n\}$, proving our claim.

Having proven our claim, now we show that F is the Plurality rule. Take any $m, n \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^{I_n}$.

Suppose, without loss of generality, $n_1 \leq n_2 \leq ... \leq n_m$. Let $(R_{N_1}, ..., R_{N_m})$ be such that, for each $h \in \{1, ..., m\}$, $n_h(R_{N_h}) = ... = n_m(R_{N_h}) = n_h(R) - n_{h-1}(R)$, where $n_0(R) = 0$. By the above claim and applying reinforcement repeatedly, for all $h \in \{1, ..., m\}$, with $n_h(R_{N_h}) > 0$, we have $F(R_{N_h}) = \{a_h, ..., a_m\}$. Now, let $\bar{h} = min\{h : n_h(R) = n_m(R)\}$. By applying reinforcement repeatedly, $F(R) = \{a_{\bar{h}}, ..., a_m\}$.

Thus F is the Plurality rule. □

As already mentioned before, a characterization result of the Plurality rule in the literature is due to Ching (1995), where he shows Plurality is the unique rule that satisfies neutrality, anonymity, independence of dominated candidates and reinforcement.An SCC F is said to be independent of dominated candidates if and only if, for all $R \in \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^{I_n}$ and all $a_h \in A_m$, if there exists $a_k \in A_m$ such that $a_k R_i a_h$ for all $i \in I_n$, then $F(R) = F(R\ |_{A_m \setminus \{a_h\}})$. An SCR that satisfies independence of dominated candidates, however, is CP, thus his result follows as a corollary to our theorem.

**Corrolary 1.** *Plurality is the unique rule that satisfies neutrality, anonymity, independence of dominated candidates and reinforcement.*

Now, we proceed to the next characterization result where we use a weakened form of CP and the famous axiom of independence of irrelevant alternatives (IIA).

**Definition 13.** An SCR $F$ is said to be
*weakly cloning-proof* (wCP) at $R$ if and only if, for every cloning structure $(A_1^{'}, ..., A_m^{'})$ with $A_i^{'} \neq \emptyset$ for all $i \in \{1, ..., m\}$, there exists $R^{'} \in \mathcal{L}^{I_n}[R; (A_1^{'}, ..., A_m^{'})$ such that $F(R) = F(R^{'})$. F is wCP if and only if it is wCP at each $R \in \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \mathcal{L}(A_m)^{I_n}$.

Next, we define independence of totally irrelevant alternatives. Independence of irrelevant alternatives requires that the addition of a new candidate x to an election does not change the winner set unless x itself is the winner. A weakened form of IIA is the independence from totally irrelevant alternatives (ITA). To define ITA, take any $m \in \mathbb{N}$, $n \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^{I_n}$ and $x \notin A_m$. Let $R^{'} \in \mathcal{L}(A_m \cup \{x\})^{I_n}$ be a profile such that $R^{'} \mid_{A_m} = R \mid_{A_m}$ and for each $i \in I_n$, for all $a \in A_m$: $aR_ix$. Now we say that an SCR F satisfies ITA at R if and only if,

$$F(R^{'}) = F(R).$$

Next, we define scoring rules.
For any $m \in \mathbb{N}$, a score vector is an m-tuple $s = (s_1, ..., s_m) \in \mathbb{R}^m$ with $s_i \geq s_{i+1}$ for all $i \in \{1, ..., m\}$. Given $m \in \mathbb{N}$, $n \in \mathbb{N}$, some alternative $a \in A_m$ and $R \in \mathcal{L}(A_m)^{I_n}$, let $\sigma(a, R_i)$ denote the ranking of a in individual i's preference, i.e. $\sigma(a, R_i) = \mid \{b \in A : bR_ia\} \mid$ and s(a,R) the score of a at R with respect to s, i.e.
$s(a, R) = \sum_{i \in N} s_{\sigma(a,R_i)}$. We say that F is a simple scoring rule (SSR) if and only if, there exists a score vector $s \in \mathbb{R}^m$ such that for any $R \in \mathcal{L}(A_m)^{I_n}$,

$$F(R) = \{a \in A_m : s(a, R) \geq s(b, R) \text{ for any } b \in A_m\}.$$

*Remark* 2. For any two scoring rules $F^t, F^w$ with associated score vectors $t, w \in \mathbb{R}^m$, we have $F^t = F^w$ whenever w is a positive linear transformation of t, i.e. $w = at + be$, where $a, b \in \mathbb{R}$, $a > 0$ and $e = (1, ...1) \in \mathbb{R}^m$. Thus, for any $s = (s_1, ..., s_m)$ with $s_1 > s_m$, we could equivalently write the score vector s as $s^{'} = (1, ..., 0)$ where the linear transformation is given by,

$$s_i^{'} = \frac{s_i - s_m}{s_1 - s_m} = \frac{1}{s_1 - s_m} s_i + \frac{-s_m}{s_1 - s_m}.$$

Plurality rule (P) in this context is the simple scoring rule with the associated score vector $s = (1, 0, ..., 0)$.

Now let $s, t$ be two score vectors in $\mathbb{R}^m$ and $F^s, F^t$ be the associated SSRs. The composition of $F^s, F^t$, denoted by $F^s \circ F^t$ is defined by,

1. $a \in (F^s \circ F^t)(R) \iff a \in F^t(R)$ and $\sum_{i \in N} s_{\sigma(a,R_i)} \geq \sum_{i \in N} s_{\sigma(b,R_i)}$ for any $b \in F^t(R)$.

Applying 1 recursively, we define the composite scoring rule (CSR) of order m to be any SCC of the form $F^{s_\alpha} \circ F^{s_{\alpha-1}} \circ \ldots \circ F^{s_1}$, $\alpha \geq 1$, where $s_1, ..., s_\alpha \in \mathbb{R}^m$.

Before we proceed to the statement and proof of the characterization theorem, we state Young's famous characterization of scoring rules.

**Theorem 6.** (**Young(75)**) *Given $m \in \mathbb{N}$, $n \in \mathbb{N}$, an SCC F satisfies neutrality, anonymity and reinforcement if and only if it is a scoring rule, i.e. it is representable by a scoring rule $F(R) = F^{s_\alpha} \circ F^{s_{\alpha-1}} \circ \ldots \circ F^{s_1}$ for all $R \in \mathcal{L}(A_m)^{I_n}$, where $s_\alpha, ..., s_1$ denote the associated score vectors.*

Now we can state the main theorem of this section.

**Theorem 7.** *Let F be a neutral, anonymous, wCP, independent of totally irrelevant alternatives social choice rule that satisfies reinforcement. Then F is the Plurality rule.*

*Proof.* It is obvious that Plurality rule satisfies all of the axioms.

Now take any $m \in \mathbb{N}$, $n \in \mathbb{N}$ and let F be an SCC satisfying all five axioms. Then by Theorem 2, F is representable by a composite scoring rule,i.e. $F(R) = F^{s_\alpha} \circ F^{s_{\alpha-1}} \circ \ldots \circ F^{s_1}$ for all $R \in \mathcal{L}(A_m)^{I_n}$. Now take an alternative set $A = \{a_1, ..., a_m\}$, a set of individuals $I_m = \{1, ..., m\}$ and fix any $\beta$,

$1 \leq \beta \leq \alpha$.

First let $m \geq 3$. Consider the below profile:

$$R = \begin{bmatrix} \underline{R_1} & \underline{R_2} & . & . & \underline{R_m} \\ a_1 & a_2 & . & . & a_m \\ a_2 & a_3 & . & . & a_1 \\ . & . & . & . & . \\ a_{m-1} & a_m & . & . & a_{m-2} \\ a_m & a_1 & . & . & a_{m-1} \end{bmatrix}$$

By anonymity and neutrality, $F(R) = A$. Then, by definition, $F^{s_\beta}(R) = A$. Now consider the cloning structure $(A_1^{'}, ..., A_m^{'})$, where $A_1^{'} = \{a_1, a_1^{'}\}$ and $A_i^{'} = \{a_i\}$ for all $i \in \{2, ..., m\}$. Let $A^{'} = A_1^{'} \cup ... \cup A_m^{'}$. Let $s_\beta = (s_\beta^1, ..., s_\beta^m) \in \mathbb{R}^m$ and $\tilde{s}_\beta = (\tilde{s}_\beta^1, \tilde{s}_\beta^2, ..., \tilde{s}_\beta^{m+1}) \in \mathbb{R}^{m+1}$ denote the associated score vectors used by $F^{s_\beta}$ respectively on A and $A^{'}$. At any $R^{'} \in \mathcal{L}^{I_m}[R; (A_1^{'}, ..., A_m^{'})]$, we have the following:

(1) $\tilde{s}_\beta(a_2, R^{'}) = \tilde{s}_\beta^3 + \tilde{s}_\beta^1 + \tilde{s}_\beta^{m+1} + ... + \tilde{s}_\beta^4$

(2) $\tilde{s}_\beta(a_3, R^{'}) = \tilde{s}_\beta^4 + \tilde{s}_\beta^2 + \tilde{s}_\beta^1 + ... + \tilde{s}_\beta^5$

(3) .

(4) .

(m-1) $\tilde{s}_\beta(a_m, R^{'}) = \tilde{s}_\beta^{m+1} + \tilde{s}_\beta^{m-1} + ... + \tilde{s}_\beta^1$

Since F is wCP, there exists some $\bar{R} \in \mathcal{L}^{I_m}[R; (A_1^{'}, ..., A_m^{'})]$ such that $F(R) = F(\bar{R}) = A$, i.e.

(*) $A \in F^{\tilde{s_\beta}}(\bar{R})$, i.e. $\tilde{s}_\beta(a_2, R^{'}) = ... = \tilde{s}_\beta(a_m, R^{'})$.

Note that equations (1) to (m-1) hold for $\bar{R}$. Combining (*) and equations (1) to (m-1) yields $\tilde{s}_\beta^2 = \tilde{s}_\beta^3 = ... = \tilde{s}_\beta^m$. Moreover, F being wCP implies that $\tilde{s}_\beta^1 > \tilde{s}_\beta^{m+1}$. Thus we have shown that, for $m \geq 4$, the scoring vector used by

$F^{s_\beta}$ has the form $s_\beta = (1, s_\beta^m, s_\beta^m, ..., s_\beta^m, 0)$.[2]

Now let $s_\beta^3 = (1, s_3^\beta, 0)$ be the score vector used by $F^{s_\beta}$ when m=3. We will show that $s_3^\beta < 1$. To do so, let $B = \{a_1, a_2\}$ and consider,

$$Q = \begin{bmatrix} \underline{Q_1} & \underline{Q_2} \\ a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}$$

By neutrality and anonymity, $F(Q) = B$. Let $B' = B_1' \cup B_2'$ where $B_1' = \{a_1, a_1'\}$ and $B_2' = \{a_2\}$. The four possible cloned profiles on $B'$ are as follows:

$$Q_1' = \begin{bmatrix} \underline{Q_1} & \underline{Q_2} \\ a_1 & a_2 \\ a_1' & a_1 \\ a_2 & a_1' \end{bmatrix}$$

$$Q_2' = \begin{bmatrix} \underline{Q_1} & \underline{Q_2} \\ a_1' & a_2 \\ a_1 & a_1 \\ a_2 & a_1' \end{bmatrix}$$

$$Q_3' = \begin{bmatrix} \underline{Q_1} & \underline{Q_2} \\ a_1 & a_2 \\ a_1' & a_1' \\ a_2 & a_1 \end{bmatrix}$$

[2]see Remark 1.

$$Q_4^{'} = \begin{bmatrix} \underline{Q_1} & \underline{Q_2} \\ a_1^{'} & a_2 \\ a_1 & a_1^{'} \\ a_2 & a_1 \end{bmatrix}$$

Let $\bar{Q}$ be the cloned profile where $F(\bar{Q}) = B$ (and hence $a_1, a_2 \in F^{s_\beta}(\bar{Q})$.) We have,

1. $s_\beta^3(a_1, Q_1^{'}) = 1 + s_3^\beta$

   $s_\beta^3(a_2, Q_1^{'}) = 1$

   Thus, either $s_3^\beta = 0$ or $Q_1^{'} \neq \bar{Q}$.

2. $s_\beta^3(a_1, Q_2^{'}) = 2s_3^\beta$

   $s_\beta^3(a_2, Q_2^{'}) = 1$

   $s_\beta^3(a_1^{'}, Q_2^{'}) = 1$

   If $Q_2^{'} = \bar{Q}$, then we should have $2s_3^\beta = 1$,i.e. $s_3^\beta = \frac{1}{2}$.

3. $s_\beta^3(a_1, Q_3^{'}) = 1$

   $s_\beta^3(a_2, Q_3^{'}) = 1$

   $s_\beta^3(a_1^{'}, Q_3^{'}) = 2s_3^\beta$

   If $Q_3^{'} = \bar{Q}$, then we should have $2s_3^\beta \leq 1$,i.e. $s_3^\beta \leq \frac{1}{2}$.

4. $s_\beta^3(a_1, Q_4^{'}) = s_3^\beta$

   $s_\beta^3(a_2, Q_4^{'}) = 1$

   $s_\beta^3(a_1^{'}, Q_4^{'}) = 1 + s_3^\beta$

   If $Q_4^{'} = \bar{Q}$, then we should have $s_3^\beta = 1$, but then, $F^{s_\beta}(\bar{Q}) = \{a_1^{'}\}$, a contradiction. Thus $Q_4^{'} \neq \bar{Q}$.

   Combining equations 1 to 4 yields that $s_3^\beta \leq \frac{1}{2}$, i.e. $s_3^\beta < 1$.

Next, we show that $s_\beta^m < 1$ for $m \geq 4$. Assume not, i.e. $s_\beta = (1, 1, ...1, 0)$. Consider, again,

$$R = \begin{bmatrix} \underline{R_1} & \underline{R_2} & . & . & \underline{R_m} \\ a_1 & a_2 & . & . & a_m \\ a_2 & a_3 & . & . & a_1 \\ . & . & . & . & . \\ a_{m-1} & a_m & . & . & a_{m-2} \\ a_m & a_1 & . & . & a_{m-1} \end{bmatrix}$$

By anonymity and neutrality, $F(R) = A$, i.e. $F^{s_\beta}(R) = A$. Now again consider the cloning structure $(A_1^{'}, ..., A_m^{'})$, where $A_1^{'} = \{a_1, a_1^{'}\}$ and $A_i^{'} = \{a_i\}$ for all $i \in \{2, ..., m\}$. Let $A^{'} = A_1^{'} \cup ... \cup A_m^{'}$. Let $R^{'} \in \mathcal{L}^{I_m}[R; (A_1^{'}, ..., A_m^{'})]$ be as follows:

$$R^{'} = \begin{bmatrix} \underline{R_1^{'}} & \underline{R_2^{'}} & . & . & \underline{R_m^{'}} \\ a_1 & a_2 & . & . & a_m \\ a_1^{'} & a_3 & . & . & a_1 \\ . & . & . & . & . \\ a_{m-1} & a_1 & . & . & a_{m-2} \\ a_m & a_1^{'} & . & . & a_{m-1} \end{bmatrix}$$

$s_\beta(a_2, R^{'}) = s_\beta(a_3, R^{'}) = ... = s_\beta(a_m, R^{'}) = m - 1$.

$s_\beta(a_1, R^{'}) = m$, i.e. $F^{s_\beta}(R^{'}) = \{a_1\}$. Thus, we need another clone extended profile $\hat{R}$ where the score of $a_1$ is less than $s_\beta(a_1, R^{'})$. The only change in the score of $a_1$ could be obtained by moving it below its clone at $R_2^{'}$. Let $\hat{R}$ be the profile obtained from $R^{'}$ by doing so. But, then,

$s_\beta(a_1, \hat{R}) =_\beta (a_2, \hat{R}) = ... = s_\beta(a_m, \hat{R}) = m - 1$ whereas $s_\beta(a_1^{'}, \hat{R}) = m$, i.e. $F^{s_\beta}(\hat{R}) = \{a_1^{'}\}$, a contradiction. Hence, $s_\beta^m < 1$ for $m \geq 4$, too.

Finally, we will show that $s_\beta = (1, 0, 0..., 0)$ for each m. To do so, take any $m \in \mathbb{N}$ and let $s_\beta = (1, s_\beta^m, s_\beta^m, ..., s_\beta^m, 0)$. Consider the following profile:

$$L = \begin{bmatrix} \underline{L_1} & \underline{L_2} \\ a_1 & a_2 \\ a_2 & a_3 \\ . & . \\ . & . \\ a_m & a_1 \end{bmatrix}$$

$s_\beta(a_1, L) = 1$

$s_\beta(a_2, L) = 1 + s_\beta^m$

$s_\beta^m(a_3, L) = s_\beta^m(a_4, L) = ... = s_\beta^m(a_{m-1}, L) = 2s_\beta^m$

$s_\beta^m(a_m, L) = s_\beta^m$

Next, assume a new alternative x is added to the alternative set as follows:

$$L^{'} = \begin{bmatrix} \underline{L_1} & \underline{L_2} \\ a_1 & a_2 \\ a_2 & a_3 \\ . & . \\ . & . \\ a_m & a_1 \\ x & x \end{bmatrix}$$

Now,

$s_\beta(a_1, L^{'}) = 1 + s_\beta^{m+1}$

$s_\beta(a_2, L^{'}) = 1 + s_\beta^{m+1}$

$s_\beta^m(a_3, L^{'}) = s_\beta^m(a_4, L) = ... = s_\beta^m(a_m, L^{'}) = 2s_\beta^{m+1}$

Thus, $F^{s_\beta}(L^{'}) = \{a_1, a_2\}$, i.e. $F(L^{'}) = \{a_1, a_2\}$. Thus, by ITA, we should have $F^{s_\beta}(L) = \{a_1, a_2\}$, i.e. $s_\beta^m = 0$

Thus, we have shown that $s_\beta = (1, 0, 0..., 0)$ for any number of alternatives. Hence, F is the Plurality rule.

$\square$

Finally, we provide one more characterization of the Plurality rule where we replace cloning-proofness with another consistency axiom.

**Definition 14.** An SCC F is said to be *tops-only* if and only if for each $n \in \mathbb{N}$, for each $R, R^{'} \in \bigcup_{m\in\mathbb{N}} \mathcal{L}(A_m)^N$,

$$[\tau(R) = \tau(R^{'})] \Longrightarrow F(R) = F(R^{'}).$$

**Theorem 8.** *Plurality rule is the unique neutral, anonymous, independent of totally irrelevant alternatives and tops-only social choice correspondence satisfying reinforcement.*

*Proof.* It is clear that Plurality rule satisfies all of the axioms stated in the theorem.

Now take any $m, n \in \mathbb{N}$ and let F be an SCC satisfying all of the axioms. Again, we first claim that if at any $R \in \mathcal{L}(A_m)^{I_n}$, the top-ranked alternatives of all individuals are distinct (thus $n \leq m$), then $F(R) = \tau(R)$. To do so, let $A_m = \{a_1, \ldots, a_m\}$.

1. Suppose $| I_n |$=1. Then, by ITA and non-emptiness of F, $F(R) = \tau(R)$.

2. Suppose $| I_n | \geq 2$. Let $R, R^{'} \in \mathcal{L}(A_m)^{I_n}$ be as follows:

$$R = \begin{bmatrix} \underline{R_1} & . & . & \underline{R_n} \\ a_1 & . & . & a_n \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$$

$$R' = \begin{bmatrix} \underline{R'_1} & \underline{R'_2} & \ldots & \underline{R'_n} \\ a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & a_1 \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_{n-1} & a_n & \ldots & a_{n-2} \\ a_n & a_1 & \ldots & a_{n-1} \\ a_{n+1} & a_{n+1} & \ldots & a_{n+1} \\ a_{n+2} & a_{n+2} & \ldots & a_{n+2} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_m & a_m & \ldots & a_m \end{bmatrix}$$

Moreover, let $\bar{R} = R' \mid_{\{a_1,...,a_n\}}$. By neutrality and anonymity, $F(\bar{R}) = \{a_1, a_2, ..., a_n\}$. Moreover, applying ITA repeatedly, we have $F(R') = F(\bar{R}) = \{a_1, a_2, ..., a_n\}$. Finally, $F(R) = F(R')$ since F is tops-only. Thus, we have proven our claim.

The rest of the proof follows the same steps with that of Theorem 5. Thus we have proven that F is the Plurality rule. $\square$

# CHAPTER 4

# CONCLUSION

In this thesis we have provided axiomatic characterizations of the Condorcet and Plurality rules as well as the class of social choice correspondences that are hereditary.

The two main concepts we focused on are hereditariness and cloning-proofness, both of which are consistency axioms defined when the alternative set is not fixed by nature. However, the definitions we used are not the only ways one could define these consistency axioms. Further research could be conducted on various definitions of these axioms, and we conjecture that those variations would also lead to characterizations of several other social choice rules studied in the literature.

# BIBLIOGRAPHY


Arrow, K.J. (1959): Rational Choice Functions and Orderings. Economica, 26: 121-127.

Blair and et al. (1976): Impossibility Theorems without Collective Rationality. Journal of Economic Theory, 13: 361-379.

Chernoff, H. (1954): Rational Selections of Decision Functions. Econometrica, 22: 423-43.

Ching, S. (1996): A Simple Characterization of Plurality Rule. Journal of Economic Theory, 71: 298302.

Deb, R. (1983): Binariness and Rational Choice. Mathematical Social Sciences, 5: 97-106.

Kelly, Jerry S. (1978): Arrow's Impossibility Theorems. New York: Academic Press.

Koray, S. (1998): Consistency in Electoral System Design. Mimeo, Bilkent University, Ankara.

Koray, S., and Slinko, A. (2008): Self-Selective Social Choice Functions. Social Choice and Welfare, 31: 129-149.

Laslier, J.F. (2000): Aggregation of Preferences with a Variable Set of Alternatives. Social Choice and Welfare, 17: 269-282.

May, K.O. (1952): A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions. Econometrica, 20: 680-684.

Moulin, H. (1985): Choice Functions Over a Finite Set: A Summary. Social Choice and Welfare, 3: 147-160.

Moulin, H. (1986): Choosing from a Tournament. Social Choice and Welfare, 3: 271-291.

Plott, C.R. (1973): Path Independence, Rationality and Social Choice. Econometrica, 41: 1075-1091.

Richelson, J. (1978): A Characterization Result for the Plurality Rule. Journal of Economic Theory, 19: 548-550.

Sen, A.K. (1971): Choice Functions and Revealed Preference. The Review of Economic Studies, 38(3): 307-317.

Tideman, T.N. (1987): Independence of Clones as a Criterion for Voting Rules. Social Choice and Welfare, 4: 185-206.

Yeh, Chun-Hsien. (2008): An Efficiency Characterization of Plurality Rule in Collective Choice Problems. Economic Theory, 34: 575-583.

Young, H.P. (1974): An Axiomatization of Borda' s Rule. Journal of Economic Theory, 9: 43-52.

Young, H.P. (1975): Social Choice Scoring Functions. SIAM Journal of Applied Mathematics, 28: 824-838.

# APPENDIX

We now show that the axioms listed in our theorems are logically independent, starting with those of Theorem 4.

To do so, let N be the set of individuals with cardinality n as in Chapter 1. First rule we introduce is the Pareto rule **PR**, which chooses the set of all Pareto optimal alternatives. That is, for each $m \in \mathbb{N}$, for each $R \in \mathcal{L}(A_m)^N$, $\mathbf{PR}(R) = PO(R) = \{a \in A_m$: $\nexists b \in A_m$ such that $bR_i a$ for all $i \in N\}$.

Second is the Top-cycle correspondence, **TC** defined as follows: for each a, b in $A_m$, we write $aDb$ if a strict majority of agents prefer a to b. **TC** chooses the smallest subset of $A_m$ with the property that nothing outside the set is preferred by a strict majority to anything in the set. Formally, for each $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$, $\mathbf{TC}(R) = \cap\{B \subseteq A_m : a \in B,\, b \notin B \text{ implies } aDb\}$.

Next is the rule **F***. To define **F***, take any $m \in \mathbb{N}$. Set $\mathfrak{R}^m = \{R \in \mathcal{L}(A_m)^N :\mid \{i \in N : aR_i b\} \mid = \mid \{i \in N : bR_i a\} \mid \text{ for all } a, b \in A_m\}$. Moreover, take some fixed $\bar{a} \in A_m$. Now,

$$\mathbf{F^*}(R) = \begin{cases} A_m & \text{if } R \in \mathfrak{R}^m,\ m \geq 3, \\ \{\bar{a}\} & \text{if } R \notin \mathfrak{R}^m;\ \bar{a} \in CW(R);\ m \geq 3\ , \\ CW(R) & \text{otherwise} \end{cases}$$

Finally, we define **F**** as follows: for each $m \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^N$, set $SM(R) = \{a \in A_m : n(a,b,R) > \frac{N}{2}$ for every $b \in A_m \setminus \{a\}\}$. Now, $\mathbf{F^{**}}(R) = SM(R)$ if $SM(R) \neq \emptyset$ and $\mathbf{F^{**}}(R) = \emptyset$ if $SM(R) = \emptyset$.

Table 1 below shows that axioms listed in Theorem 4 are logically independent[1]. In the table, TM stands for top-majoritarianity, H for hereditariness, BDC for binary dominance consistency and CA for cancellation.

| **Table 1** | TM | H | BDC | CA |
|---|---|---|---|---|
| **PR** | No | Yes | Yes | Yes |
| **TC** | Yes | No | Yes | Yes |
| **F*** | Yes | Yes | No | Yes |
| **F**** | Yes | Yes | Yes | No |

Next, we show that axioms in Theorem 5 are logically independent. Again, we need additional social choice correspondences.

First rule we use is **CC** defined as follows: there exists some $\bar{a}$ such that for every $m, n \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^{I_n}$,

$$\mathbf{CC}(R) = \begin{cases} \{\bar{a}\} & \text{if } \bar{a} \in \tau(R), \\ \{a_k \in A_m : \forall a_j \in A_m, n_k(R) \geq n_j(R)\} & \text{otherwise} \end{cases}$$

Second is the class of priority rules. Let $\Theta$ denote the set of all linear orders on $\mathbb{N}$, with generic element $\prec$. The notation $i \prec j$ means that agent i has priority over agent j. Given $n \in \mathbb{N}$, the T-priority rule relative to $\prec$, **PT**, most preferred alternative of the agent with highest priority in $I_n$. Formally, let $D^{\prec}_{I_n} = \{i \in I_n : \text{for each } j \in I_n \setminus \{i\}, i \prec j\}$. For each $m, n \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^{I_n}$, $\mathbf{PT}(R) = \{a \in A_m : \forall b \in A_m, aR_{D^{\prec}_{I_n}} b\}$.

Third is the Borda rule, **BR**, which is the simple scoring rule with the associated score vector $(m, m-1, ..., 1) \in \mathbb{R}^m$ for each $m \in \mathbb{N}$.

Next, let **T** stand for the top-rule, that chooses the set of all top-ranked alternatives, i.e. for every $m, n \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^{I_n}$, $\mathbf{T}(R) = \tau(R)$.

Table 2 below shows the axioms of Theorem 5 are logically independent.

[1]"Yes" ("No") means that a certain rule satisfies (violates) a certain property.

| **Table 2** | neutral | anonymous | CP1 | reinforcement |
|---|---|---|---|---|
| **CC** | No | Yes | Yes | Yes |
| **PT** | Yes | No | Yes | Yes |
| **BR** | Yes | Yes | No | Yes |
| **T** | Yes | Yes | Yes | No |

Next, we show that axioms in Theorem 7 are logically independent.

Next is the rule **IP**. Take any $m, n \in \mathbb{N}$. For each $k \in \{1, ..., m\}$, set $b_k = \mid \{i \in I_n : \forall a_j \in A_m \setminus \{a_k\} : a_j R_i a_k\} \mid$. Then, for any $m, n \in \mathbb{N}$, $R \in \mathcal{L}(A_m)^{I_n}$,

$$\mathbf{IP}(R) = \{a_k \in A_m : \forall a_j \in A_m, b_k(R) \geq b_j(R)\}.$$

Table 3 below shows that axioms in Theorem 7 are logically independent.

| **Table 3** | neutral | anonymous | wCP1 | ITA | Reinforcement |
|---|---|---|---|---|---|
| **CC** | No | Yes | Yes | Yes | Yes |
| **PT** | Yes | No | Yes | Yes | Yes |
| **BR** | Yes | Yes | No | Yes | Yes |
| **IP** | Yes | Yes | Yes | No | Yes |
| **T** | Yes | Yes | Yes | Yes | No |

Finally, in Table 4 below, we show that axioms in Theorem 8 are independent.

To do so, we need one last social choice rule. **U** is the rule that chooses the whole set of alternatives, i.e $\mathbf{U}(R) = A_m$ for any $m, n \in \mathbb{N}$ and $R \in \mathcal{L}(A_m)^{I_n}$.

| **Table 4** | neutral | anonymous | ITA | Tops-only | Reinforcement |
|---|---|---|---|---|---|
| **CC** | No | Yes | Yes | Yes | Yes |
| **PT** | Yes | No | Yes | Yes | Yes |
| **U** | Yes | Yes | No | Yes | Yes |
| **BR** | Yes | Yes | Yes | No | Yes |
| **T** | Yes | Yes | Yes | Yes | No |