SCHWARZ PROBLEM IN THE COMPLEX PLANE

by

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SCHWARZ PROBLEM IN THE COMPLEX PLANE

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To the memory of my father Ender Ulvi Gökgöz...

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ABSTRACT

SCHWARZ PROBLEM IN THE COMPLEX PLANE

This thesis consists of six chapters and the conclusion chapter. In the first chapter, an introduction to the history of boundary value problems, theorems and definitions to be used in the thesis are given. In the second chapter, for the sake of completeness, we review the Schwarz problem given in the unit disc. In the third chapter, we give the unique solution of the Schwarz problem for inhomogeneous Cauchy-Riemann equations and generalized Beltrami equations in a ring domain. In the fourth chapter, we extend the discussion to second-order equations and investigate the unique solution of the Schwarz problem for second-order linear equations in a ring domain. In the fifth chapter, we give the unique solution of the Schwarz problem for higher-order equations and higher-order linear equations in a ring domain. To analyze the unique solution of the Schwarz problem for higher-order linear equations in a ring domain, we also discuss the properties of higher-order operators. In the sixth chapter, we investigate the Schwarz problem for nonlinear higher-order equations. In the conclusion chapter, we summarize the research conducted and the key outcomes of this thesis.

ÖZET

KOMPLEKS DÜZLEMDE SCHWARZ PROBLEMİ

Bu tez altı bölüm ve sonuç bölümünden oluşmaktadır. Birinci bölümde sınır değer problemlerinin tarihçesine bir giriş, tezde kullanılacak teoremler ve tanımlar verilmiştir. İkinci bölümde, bütünlüğü sağlamak için, birim diskte verilen Schwarz problemini gözden geçiriyoruz. Üçüncü bölümde, bir halka bölgede homojen olmayan Cauchy-Riemann denklemleri ve genelleştirilmiş Beltrami denklemleri için Schwarz probleminin tek çözümünü veriyoruz. Dördüncü bölümde, tartışmayı ikinci mertebeden denklemlere genişletiyoruz ve bir halka bölgedeki ikinci mertebeden lineer denklemler için Schwarz probleminin tek çözümünü araştırıyoruz. Beşinci bölümde, bir halka bölgede yüksek mertebeden denklemler ve yüksek mertebeden lineer denklemler için Schwarz probleminin tek çözümünü veriyoruz. Bir halka bölgedeki yüksek mertebeden lineer denklemler için Schwarz probleminin tek çözümünü analiz etmek için, aynı zamanda yüksek mertebeden operatörlerin özelliklerini de tartışıyoruz. Altıncı bölümde, lineer olmayan yüksek mertebeden denklemler için Schwarz problemini inceliyoruz. Sonuç bölümünde, yürütülen araştırmayı ve bu tezin temel sonuçlarını özetliyoruz.

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LIST OF SYMBOLS/ABBREVIATIONS

$\mathbb{C}$	The space of complex numbers
$\mathbb{R}$	The space of real numbers
$\mathbb{R}^2$	The space of all points (x, y) where x and y are real numbers
$\overline{D}$	The closure of D
$C(\overline{D})$	The space of continuous functions in the closure of D
$C^1(D)$	All differentiable functions in D whose derivative is continuous
$w_{\overline{z}}$	The derivative of w with respect to the variable $\overline{z}$
w_z	The derivative of w with respect to the variable z
R	A ring domain
$B(X,Y)$	The space of bounded operators from X to Y
$K(X)$	The space of compact operators on X
$D(K)$	The domain of K
dim	Dimension
ker	Kernel
Im	Image
co	Complement
I	Identity operator
$F^n(y)$	The nth iterate of y under F
$\mathbb{D}$	The unit disc
∂R	The boundary of the ring domain
$\partial\mathbb{D}$	The boundary of the unit disc
Re	Real part
Im	Imaginary part
$L^p(X)$	All $p-th$ order integrable functions defined on X
$w_{\overline{z}\overline{z}}$	The second derivative of w with respect to the variable $\overline{z}$
$\partial_{\overline{z}}^k w$	The $k-th$ order derivative of w with respect to the variable $\overline{z}$
$\mathbb{C}^n$	Complex n-space
$W^{p,k}(R)$	The subset of functions f in $L^p(R)$ such that f and its weak derivatives up to order k have a finite L^p-norm

D^α	Multi-index

1. INTRODUCTION

Complex function theory is an important branches of mathematics, particularly in partial differential equations, complex analysis, geometry, mathematical physics, elasticity, fluid dynamics, electromagnetics, wave mechanics etc. Many problems in mathematics or mathematical physics are modeled by differential equations under various auxiliary conditions which are called as boundary value problems. Boundary value problems in $\mathbb{C}$ have initiated with Riemann. The relevant problem is stated as in Vekua [1]:

"*Find a function $w(z) = u + iv$ analytic in the domain G, which satisfies at every boundary point the relation*

$$F(u, v) = 0$$

on Γ."

Later, Hilbert converted the problem in such a way that it may be handled in the following alternative form [1].

"*Find in the domain G the solution $w(z) = u + iv$ of the equation*

$$\partial_{\bar{z}} w + A(z)w + B(z)\overline{w} = F(z)\, in\, G \tag{1.1}$$

satisfying the boundary condition

$$\alpha u + \beta v = \ Re\left[\overline{\lambda(z)}w\right] = \gamma(z)\ (on\, \Gamma)$$
$$\lambda(z) = \alpha(z) + i\beta(z),\ \gamma \in C_\nu(\Gamma),\ 0 < \nu \leq 1 \tag{1.2}$$

where $C_\nu(\Gamma)$ represents the Hölder continuous functions of order ν on Γ."

The above problem is called the generalized Riemann-Hilbert problem for a linear first-order equation. There are different boundary value problems derived from Riemann-Hilbert problem such as the Dirichlet problem, Neumann problem, Robin problem and Schwarz problem. Many researchers have been attracted by the boundary value problems in $\mathbb{C}$. Schwarz problems are treated in various types of domains. We can find the investigation in simply connected domains such as unit disc [2,3,4,5,6], plane [7], half plane [8], unbounded sectors [9], rectangle [10], triangle [11,12], lens-lunes [13,14,15] and multiply connected domains [16,17,18,19,20]. We are going to take a ring domain $R = \{z \in \mathbb{C} : r < |z| < 1\}$ which is a special case of multiply connected domains. Some boundary value problems are discussed in a ring domain, see for example [19,20,21,22,23,24,25,26,27]. We concentrate on the Schwarz problem in a ring domain. It is more difficult to investigate the Schwarz problem in a ring domain than in simply connected domains. Even if we consider $w_{\bar{z}} = 0$ which represents analytic functions, solvability conditions arise for the problem to have only one solution in a ring domain. Integral representation formulas are important to derive the solution of boundary value problems. We start with the complex form of Gauss Theorem in a regular domain D. From the Gauss Theorem, the Cauchy-Pompeiu integral representation formula in a regular domain D can be deduced.

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

The area integral in the above formula is called the Pompeiu operator. It is essential to know the properties of this operator. These properties were investigated by Vekua [1]. These are $\partial_{\bar{z}}Tf = f$ and $\partial_z Tf =: \Pi f$ where Π is a singular integral operator. But it is necessary to modify the Cauchy-Pompeiu integral representation formula appropriately and the operator Tf for a particular domain R. Kernel functions are also encountered. The kernel functions have been derived previously by Akhiezer [28] and Courant-Hilbert [29]. We define the operator T_R by the formula [19,20]

$$\begin{aligned} T_R f(z) := -\frac{1}{2\pi}\int\limits_{R} \Bigg\{ &\frac{f(\zeta)}{\zeta}\left[\frac{\zeta + z}{\zeta - z} + 1 + K_1(z,\zeta)\right] \\ &+ \frac{\overline{f(\zeta)}}{\bar{\zeta}}\left[\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} + 3 + K_2(z,\zeta)\right]\Bigg\} d\xi d\eta \end{aligned}$$

which has similar properties to the operator T.

(i) $\partial_{\bar{z}} T_R f(z) = f(z)$,

(ii) $Re T_R f(z) = 0, \quad z \in \partial R \quad ; \quad \frac{1}{2\pi i}\int\limits_{|z|=r} Im T_R f(z)\frac{dz}{z} = 0.$

In this thesis we consider the inhomogeneous first-order Cauchy-Riemann equations in a ring domain. We derive the solutions of relevant boundary value problem. We give the unique solution to the Schwarz problem for generalized Beltrami equations in a ring domain using the Fredholm alternative, [19].

To discuss the solutions of second-order equations in a ring domain we adjust the kernel functions for $w \in W^{p,2}(R;\mathbb{C})$ with the help of the iteration method. We investigate the unique solution of second-order linear equations in a ring domain by Fredholm theory.

We discuss the Schwarz problem for higher-order equations in which $w \in W^{p,n}(R;\mathbb{C})$, $n > 2$, [20]. While examining this case, we also find the properties of higher-order operators for a ring domain.

In the last part of this thesis, we transform the Schwarz problem for nonlinear higher-order equations into the system of integro-differential equations and we obtain the existence of unique solution using Banach fixed point theorem.

1.1. PRELIMINARIES

In order to find the solutions of boundary value problems in $\mathbb{C}$, it is very important to derive integral representations of the functions in $\mathbb{C}$. To find such an integral representation we start with the Gauss Theorem in a regular domain D, i.e., bounded domain with smooth boundary ∂D.

Theorem 1. *[2] (Gauss theorem, real form)*
Let (f,g) $\in C^1(D;\mathbb{R}^2) \cap C(\overline{D};\mathbb{R}^2)$ *be a differentiable real vector field in a regular domain* $D \subset \mathbb{R}^2$ *then*

$$\int_D \left(f_x(x,y) + g_y(x,y)\right) dxdy = -\int_{\partial D} \left(f(x,y)\, dy - g(x,y)\, dx\right).$$

The following form is the counterpart of Theorem 1 for complex valued functions.

Theorem 2. *[3] (Gauss theorem, complex form)*
Let $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ *in a regular domain of the complex plane* $\mathbb{C}$*, then*

$$\int_D w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz,$$

$$\int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)\overline{dz}$$

The following theorem gives the Cauchy-Pompeiu representations for functions which are not assumed to be holomorphic.

Theorem 3. *[3] Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{1.3}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\overline{d\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \tag{1.4}$$

hold in which $\zeta = \xi + i\eta$.

Observe that if $w_{\overline{\zeta}}$ is given in D and the values of w along the boundary is known, we can identify a unique function $w(z)$. This representation is an example of the solution of the Dirichlet problem. After some modification, Schwarz problem can be solved using the representation Equation (1.3) or Equation (1.4).

We will investigate the Schwarz problem in a ring domain. Throughout this thesis, $R = \{z : r < |z| < 1\}$ is given as the ring domain. We will also deal with the properties of the above integral operators. These integral operators are classified as follows.

Definition 1. *[30] Suppose D is a domain in $\mathbb{C}$. Also suppose that $F(z,\zeta)$ is a bounded function for z, $\zeta \in D$. Then*

$$K(z,\zeta) = \frac{F(z,\zeta)}{(\zeta - z)^{\alpha}}, \qquad 0 < \alpha \leq 2$$

is called a kernel with a weak or strong singularity depending on whether $\alpha < 2$ or $\alpha = 2$, *respectively. The operator*

$$Tu(z) = \int_D K(z,\zeta)u(\zeta)d\xi d\eta, \quad \zeta = \xi + i\eta$$

is called the corresponding singular integral operator.

In the sequel, we give a list of definitions, lemmas and theorems used in our investigations.

Lemma 1. *[30] (Schmidt's Inequality) If Ω is a domain in the complex plane having finite area, denoted by $m\Omega$, then*

$$\iint_\Omega \frac{1}{|\zeta - z|^\alpha} d\xi d\eta \leq \frac{2\pi}{2-\alpha}\left(\frac{m\Omega}{\pi}\right)^{1-\frac{\alpha}{2}}$$

holds for each z, provided that 0< α <2.

Lemma 2. *[30] If D is a domain of finite area, then for any two points $z_1, z_2 \in \mathbb{C}$ and $\alpha, \beta \in (1,2)$*

$$\iint_\Omega \frac{d\xi d\eta}{|\zeta - z_1|^\alpha |\zeta - z_2|^\beta} \leq \begin{cases} M_1(\alpha,\beta,D)\,|z_1 - z_2|^{2-\alpha-\beta}, \text{ if } \alpha+\beta > 2 \\ M_2(\alpha,\beta,D) + M_3(\alpha,\beta,D)\,|\log(z_1 - z_2)|, \text{ if } \alpha = \beta = 1 \end{cases}$$

Definition 2. *[31] If p and q are positive real numbers such that $p + q = pq$, or equivalently*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then we call p and q a pair of conjugate exponents.

Theorem 4. *[31] (Hölder's Inequality) Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable functions on X, with range in $[0,\infty]$. Then*

$$\int_X fg d\mu \leq \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} \left\{\int_X g^q d\mu\right\}^{\frac{1}{q}}.$$

Definition 3. *[32] The family* $\mathcal{F}$ *is said to be equicontinuous on compact set* K *if, given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $f \in \mathcal{F}$ *the condition*

$$z, w \in K, \ |z - w| < \delta \implies |f(z) - f(w)| < \epsilon$$

holds.

Theorem 5. *[31] (Arzela-Ascoli Theorem) Suppose that* F *is a pointwise bounded equicontinuous collection of complex functions on a metric space* X, *and that* X *contains a countable dense subset* E. *Every sequence* $\{f_n\}$ *in* F *has then a subsequence that converges uniformly on every compact subset of* X.

Theorem 6. *[32] (Montel Theorem) Let* F *be a family of holomorphic functions on an open set* U *of the plane. Then the following conditions are equivalent:*
a) Every sequence of functions of the family F *has a subsequence that is uniformly convergent on compact sets of* U.
b) The family F *is uniformly bounded on compact sets of* U.

Definition 4. *[33] Let* X, Y *be normed vector spaces. A linear operator* K *from* X *to* Y *is called compact (or completely continuous) if the domain* $D(K)$ *of* K *is* X *and for every sequence* $\{x_n\} \subset X$ *such that* $\|x_n\| \leq C$, *the sequence* $\{Kx_n\}$ *has a subsequence which converges in* Y.

Definition 5. *[33] Suppose* X, Y *are normed vector spaces and* $A \in B(X, Y)$, *the space of bounded linear operators from* X *to* Y. *For each* $y' \in Y'$, *the expression* $y'(Ax)$ *assigns a scalar to each* $x \in X$. *There is an* $x' \in X'$ *such that*

$$y'(Ax) = x'(x), \ x \in X.$$

We designate this assignment by $A' : Y' \to X'$ *such that*

$$y'(Ax) = A'y'(x).$$

The operator A' *is called the adjoint (or conjugate) of* A.

Now, we give a few theorems and definitions regarding Fredholm theory [34], [35], [33].

Definition 6. *Let X, Y be Banach spaces. An operator $A \in B(X,Y)$ is said to be a Fredholm operator from X to Y if the numbers $\alpha(A) = \dim\ker A$ and $\beta(A) = co\dim \operatorname{Im} A$ are finite. The set of Fredholm operators from X to Y is denoted by $\phi(X,Y)$. The index of a Fredholm operator is defined as $i(A) = \alpha(A) - \beta(A)$.*

Theorem 7. *For an index-zero Fredholm operator A, either of the following mutually exclusive events take place.*
(1) The homogeneous equation $Ax = 0$ has only the zero solution. The equation $Ax = y$ is solvable and has a unique solution given an arbitrary right hand side.
(2) The homogeneous equation $Ax = 0$ has a nonzero solution. The homogeneous equation $Ax = 0$ has finitely many linearly independent solutions $x_1, ..., x_n$. The conjugate equation $A'y' = 0$ has finitely many linearly independent solutions $y'_1, ...y'_n$. The equation $Ax = y$ is solvable if and only if $y'_1(y) = ... = y'_n(y) = 0$ and the general solution is the sum of a particular solution and the general solution of the homogeneous solution.

Theorem 8. *Let $K \in K(X)$ be the space of compact operators on X. Then $I - K$ is an index-zero Fredholm operator.*

Definition 7. *Let $A \in B(X,Y)$. An operator $L \in B(Y,X)$ is a left approximate inverse of A if $LA - I \in K(X)$. An operator $R \in B(Y,X)$ is a right approximate inverse of A if $AR - I \in K(Y)$. An operator $S \in B(Y,X)$ is an approximate inverse of A if S is a left and right approximate inverse of A. If an operator A has an approximate inverse S, then A is called approximately invertible.*

Theorem 9. *(Noether Criterion) An operator is a Fredholm operator if and only if it is approximately invertible.*

Theorem 10. *(Nikolskii Criterion) An operator is an index-zero Fredholm operator if and only if it is the sum of an invertible operator and a compact operator.*

Definition 8. *[36] A map $F : (X,d) \to (Y,\rho)$ of metric spaces that satisfies $\rho(F(x), F(z)) \leq Md(x,z)$ for some fixed constant M and all $x, z \in X$ is called Lipschitzian; the smallest such M is called the Lipschitz constant $L(F)$ of F. If $L(F) < 1$, the map F is called contractive* *with contraction constant $L(F)$.*

Definition 9. *[36] Let X be any space and f a map of X, or of a subset of X, into X. A point $x \in X$ is called a fixed point for f if $x = f(x)$.*

Theorem 11. *[36] (Banach Contraction Principle) Let (Y, d) be a complete metric space and $F : Y \to Y$ be contractive. Then F has a unique fixed point u, and $F^n(y) \to u$ for each $y \in Y$.*

Definition 10. *Let $f, g \in L^1(D)$. Then g is called the generalized (distributional) derivative of f with respect to $\overline{z}$ if for all $\varphi \in C_0^1(D)$*

$$\iint\limits_D \left(f \frac{\partial \varphi}{\partial \overline{z}} + g\varphi \right) dxdy = 0,$$

where $C_0^1(D)$ denotes the set of complex-valued functions in D continuosly differentiable and having compact support in D. The derivative is denoted by $g = f_{\overline{z}} = \partial_{\overline{z}} f$. Analogously the relation

$$\iint\limits_D \left(f \frac{\partial \varphi}{\partial z} + g\varphi \right) dxdy = 0$$

defines a generalized derivative with respect to z.

2. SCHWARZ PROBLEM IN THE UNIT DISC

In this chapter, we mainly present the Schwarz problem for first and higher-order equations in the unit disc. Firstly, we review the Schwarz problem for first-order equations.

Theorem 12. *[4] The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc $\mathbb{D}$ defined by*

$$w_{\bar{z}} = f \ in \ \mathbb{D}, Rew = \gamma \ on \ \partial\mathbb{D}, \ \ Imw(0) = c$$

is uniquely solvable in the distributional sense for $f \in L^1(\mathbb{D}), \gamma \in C(\partial\mathbb{D};\mathbb{R}), c \in \mathbb{R}$ by Cauchy-Schwarz-Poisson-Pompeiu formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + ic - \frac{1}{2\pi}\int\limits_{\mathbb{D}}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right] d\xi d\eta.$$

In the above formula, the second integral is called $S_1 f(z)$ in [37]. If homogeneous conditions are given, $S_1 f(z)$ is the only solution to the Schwarz problem for the inhomogeneous Cauchy-Riemann equation. Differentiability properties of the function $S_1 f(z)$ are given in the next lemma.

Lemma 3. *[37] For $f \in L^1(\mathbb{D})$, the function $S_1 f$ has generalized first order derivatives*

$$\frac{\partial}{\partial\overline{z}} S_1 f = f,$$

$$\frac{\partial}{\partial z} S_1 f = \Pi_1 f$$

where Π_1 is given by

$$\Pi_1 f = -\frac{1}{\pi}\int\limits_{\mathbb{D}}\left[\frac{f(\zeta)}{(\zeta-z)^2} + \frac{\overline{f(\zeta)}}{(1-z\overline{\zeta})^2}\right] d\xi d\eta, \ z \in \mathbb{D}.$$

The operator Π_1 is a strongly singular operator from $L^p(\mathbb{D})$ into itself, satisfying $\|\Pi_1\|_{L^2(\mathbb{D})} = 1$, which is proved in [1].

Secondly, we state the Schwarz problem for second-order equations.

Theorem 13. *[2], [5] The Schwarz problem for the inhomogeneous Bitsadze equation in the unit disc $\mathbb{D}$ defined by*

$$w_{\overline{z}\overline{z}} = f \ in \ \mathbb{D}, \ Rew = \gamma_0, \ Rew_{\overline{z}} = \gamma_1 \ on \ \partial\mathbb{D}, Imw(0) = c_0, \ Imw_{\overline{z}}(0) = c_1,$$

for $f \in L^1(\mathbb{D})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R})$, $c_0, c_1 \in \mathbb{R}$ is uniquely solvable in the distributional sense by

$$\begin{aligned} w(z) =& ic_0 + i(z+\overline{z})c_1 + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}(\zeta - z + \overline{\zeta - z})\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi}\int\limits_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right)(\zeta - z + \overline{\zeta - z})d\xi d\eta. \end{aligned}$$

Finally, we provide an overview of Schwarz problem for higher-order equations.

Theorem 14. *[5] The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc $\mathbb{D}$ defined by*

$$\partial_{\overline{z}}^k w = f \ in \ \mathbb{D}, \ Re\partial_{\overline{z}}^l w = \gamma_l \ \ on \ \partial\mathbb{D}, \ Im\partial_{\overline{z}}^l w(0) = c_l, \ 0 \le l \le k-1,$$

is uniquely solvable in the distributional sense for $f \in L^1(\mathbb{D})$, $\gamma_l \in C(\partial\mathbb{D};\mathbb{R})$, $c_l \in \mathbb{R}$, $0 \le l \le k-1$. The solution is

$$\begin{aligned} w(z) =& i\sum_{l=0}^{k-1}\frac{c_l}{l!}(z+\overline{z})^l + \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int\limits_{\partial\mathbb{D}} \gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\left(\zeta - z + \overline{\zeta - z}\right)^l\frac{d\zeta}{\zeta} \\ & + \frac{(-1)^k}{2\pi(k-1)!}\int\limits_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right)\left(\zeta - z + \overline{\zeta - z}\right)^{k-1} d\xi d\eta. \end{aligned}$$

3. SCHWARZ PROBLEM IN A RING DOMAIN FOR FIRST-ORDER EQUATIONS

In this chapter, we begin to investigate the Schwarz problem in a ring domain for first-order equations. Next, we deal with Schwarz problem in a ring domain for generalized Beltrami equations. These are studied in [19]. In the following the operators T_R and Π_R appears. These operators are defined over $L_p(R)$.

3.1. SCHWARZ PROBLEM IN A RING DOMAIN FOR INHOMOGENEOUS CAUCHY-RIEMANN EQUATIONS

We start investigating the Schwarz problem in a ring domain for first-order equations. Firstly it is defined by Vaitsiakhovich [16] but we think that there are inaccuracies in the content of the problem. So we modify the problem.

The Schwarz problem in a ring domain is:

Theorem 15. *[19] Find the solution of $w_{\overline{z}} = f(z)$ in R satisfying*

$$Rew(z) = \gamma(z) \text{ on } \partial R, \ \frac{1}{2\pi i}\int\limits_{|z|=r} Imw(z)\frac{dz}{z} = c$$

for $\gamma(z) \in C(\partial R;\mathbb{R})$, $c \in \mathbb{R}$ given. The Schwarz problem defined in R has the unique solution

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int\limits_{\partial R} \gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\right]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi}\int\limits_{|\zeta|=r} Imw(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right]d\xi d\eta \\ & - \frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} + 3 + K_2(z,\zeta)\right]d\xi d\eta \end{aligned} \tag{3.1}$$

subject to the solvability condition

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}=-\frac{3}{2\pi}\int\limits_{R}\left[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right]d\xi d\eta,$$

where

$$K_1(z,\zeta)=2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right),$$
$$K_2(z,\zeta)=2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right).$$

Before starting the proof, the operator T_R defined on a ring domain R similar to the discussion for the operator S_1 defined in a simply connected domain given in [37] which has a slight difference from his definition:

$$T_Rf(z):=-\frac{1}{2\pi}\int\limits_{R}\left\{\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\right.$$
$$\left.+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+3+K_2(z,\zeta)\right]\right\}d\xi d\eta$$

satisfies the properties

(i) $\partial_{\overline{z}}T_Rf(z)=f(z)$,

(ii) $ReT_Rf(z)=0, \quad z\in\partial R \quad ; \quad \frac{1}{2\pi i}\int\limits_{|z|=r}ImT_Rf(z)\frac{dz}{z}=0.$

For the first property, we can separate $T_Rf(z)$ into two integrals as

$$T_Rf(z):=-\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right)d\xi d\eta$$
$$-\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}\left(1+K_1(z,\zeta)\right)+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left(3+K_2(z,\zeta)\right)\right)d\xi d\eta.$$

In a regular domain, the derivative of the first integral in $T_Rf(z)$ with respect to $\overline{z}$ is f as in [37]. The second integral in $T_Rf(z)$ is analytic. Its derivative with respect to $\overline{z}$ is 0. So, the first property is satisfied. For the second property, the solution of the inhomogeneous first-order equation with homogeneous conditions is $w=T_Rf$. It comes naturally from the definition of the problem.

In the proof of Theorem 15, we use the Poisson kernels to derive the boundary condition. We make the following definition.

Definition 11. *The expression*

$$\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1$$

for $|z| < 1$, $|\zeta| = 1$ *or for* $|z| > r$, $|\zeta| = r$ *is the well known Poisson kernel for the unit disc* $|z| < 1$ *or for the domain* $|z| > r$. *Poisson kernels for* $|z| < 1$ *or* $|z| > r$ *possess the following important properties*

$$\lim_{|z|\to 1,\ |z|<1} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right] \frac{d\zeta}{\zeta} = \gamma(z)$$

$$\lim_{|z|\to r,\ |z|>r} \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right] \frac{d\zeta}{\zeta} = -\gamma(z)$$

for $\gamma \in C(\partial R; \mathbb{C})$.

Now, we can start the proof of Theorem 15.

Proof. The solution satisfies the inhomogeneous Cauchy-Riemann equation. Therefore, it is sufficient to check the boundary condition and the normalization condition. Firstly, we check the boundary condition.

$$\begin{aligned}
2Rew(z) =& \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \left[\frac{\zeta + z}{\zeta - z} + \frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}} + K_1(z,\zeta) + \overline{K_1(z,\zeta)} \right] \frac{d\zeta}{\zeta} \\
& - \frac{1}{2\pi} \int\limits_{R} \frac{f(\zeta)}{\zeta} \left[\frac{\zeta + z}{\zeta - z} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} + 4 + K_1(z,\zeta) + \overline{K_2(z,\zeta)} \right] d\xi d\eta \\
& - \frac{1}{2\pi} \int\limits_{R} \frac{\overline{f(\zeta)}}{\overline{\zeta}} \left[\frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} + 4 + \overline{K_1(z,\zeta)} + K_2(z,\zeta) \right] d\xi d\eta \\
& - \frac{2}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta}. \qquad (3.2)
\end{aligned}$$

When we restrict the real part of w to the boundary of the domain R, Equation (3.2) can be written as

$$
\begin{aligned}
2Rew(z)\Big|_{\partial R} &= \Bigg\{ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \Bigg[\frac{2\zeta}{\zeta - z} + \frac{2\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 2 + 2\sum_{n=1}^{\infty} \Bigg(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} \\
&+ \frac{r^{2n}z}{r^{2n}z - |z|^2\zeta} + \frac{r^{2n}|z|^2\zeta}{z - r^{2n}|z|^2\zeta} \Bigg) \Bigg] \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \Bigg[\frac{2\zeta}{\zeta - z} + \frac{2\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 2 \\
&+ 2\sum_{n=1}^{\infty} \Bigg(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} + \frac{r^{2(n+1)}z}{r^{2(n+1)}z - |z|^2\zeta} + \frac{r^{2(n-1)}|z|^2\zeta}{z - r^{2(n-1)}|z|^2\zeta} \Bigg) \Bigg] \frac{d\zeta}{\zeta} \\
&- \frac{1}{2\pi} \int\limits_{R} \frac{f(\zeta)}{\zeta} \Bigg[\frac{\zeta + z}{\zeta - z} + \frac{z + |z|^2\zeta}{z - |z|^2\zeta} + 4 + 2\sum_{n=1}^{\infty} \Bigg(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} \\
&+ \frac{r^{2n}z}{r^{2n}z - |z|^2\zeta} + \frac{r^{2n}|z|^2\zeta}{z - r^{2n}|z|^2\zeta} \Bigg) \Bigg] d\xi d\eta - \frac{1}{2\pi} \int\limits_{R} \frac{\overline{f(\zeta)}}{\overline{\zeta}} \Bigg[\frac{\overline{\zeta}z + |z|^2}{\overline{\zeta}z - |z|^2} + \frac{1 + \overline{\zeta}z}{1 - \overline{\zeta}z} \\
&+ 4 + 2\sum_{n=1}^{\infty} \Bigg(\frac{r^{2n}z\overline{\zeta}}{r^{2n}z\overline{\zeta} - |z|^2} + \frac{r^{2n}|z|^2}{z\overline{\zeta} - r^{2n}|z|^2} + \frac{r^{2n}}{r^{2n} - z\overline{\zeta}} + \frac{r^{2n}z\overline{\zeta}}{1 - r^{2n}z\overline{\zeta}} \Bigg) \Bigg] d\xi d\eta \Bigg\}\Bigg|_{\partial R} \\
&- \frac{2}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta}. \qquad (3.3)
\end{aligned}
$$

In the above formula, the first two integrals which are over $|\zeta| = 1$ and $|\zeta| = r$ tend to 2γ using Poisson kernels in Definition 11. So we obtain

$$
2Rew(z)\Big|_{\partial R} = 2\gamma - \frac{2}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} - \frac{3}{\pi} \int\limits_{R} \left(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}} \right) d\xi d\eta.
$$

Hence the boundary condition $Rew(z) = \gamma(z)$ on ∂R holds if

$$
\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} = -\frac{3}{2\pi} \int\limits_{R} \left(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}} \right) d\xi d\eta.
$$

For the normalization condition, we need to calculate the following integral.

$$
\frac{1}{2\pi i} \int\limits_{|z|=r} w(z) \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{1}{2\pi i} \int\limits_{|z|=r} \left[\frac{\zeta + z}{\zeta - z} + K_1(z, \zeta) \right] \frac{dz}{z} \frac{d\zeta}{\zeta}
$$

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{dz}{z}\frac{d\zeta}{\zeta}+\frac{1}{2\pi}\int\limits_{|\zeta|=r}Imw(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{dz}{z}\frac{d\zeta}{\zeta}\\
&-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\frac{dz}{z}d\xi d\eta\\
&-\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+3+K_2(z,\zeta)\right]\frac{dz}{z}d\xi d\eta
\end{aligned}
$$

where

$$
\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z}=-1,
$$

$$
\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+K_2(z,\zeta)\right]\frac{dz}{z}=1\ for\ \zeta\in R
$$

which are given in [16]. Substituting the above results, the integral becomes:

$$
\frac{1}{2\pi i}\int\limits_{|z|=r}w(z)\frac{dz}{z}=-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}+ic-\frac{1}{\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta.
$$

Taking its imaginary part, we obtain

$$
\frac{1}{2\pi i}\int\limits_{|z|=r}Imw(z)\frac{dz}{z}=c.
$$

Thus, the proof of Theorem 15 is completed.

□

3.2. SCHWARZ PROBLEM IN A RING DOMAIN FOR GENERALIZED BELTRAMI EQUATIONS

Now, we discuss the solution of the Schwarz problem

$$
w_{\overline{z}}+A_1(z)w_z+A_2(z)\overline{w_z}+B_1(z)w+B_2(z)\overline{w}=f(z) \tag{3.4}
$$

$$
Rew(z)=0 \text{ on } \partial R,\ \frac{1}{2\pi i}\int\limits_{|z|=r}Imw(z)\frac{dz}{z}=0 \tag{3.5}
$$

for generalized Beltrami equations in a ring domain [19].

To solve the Schwarz problem for generalized Beltrami equation in a ring domain, we need to convert it into an integral equation. The solution of the inhomogeneous first-order Beltrami equation with homogeneous boundary condition is $w = T_R f$. But we are looking for the solution of the generalized Beltrami equation. We want to see if we can find a function $g \in L^p(R)$ for which $T_R g$ is a solution of the Equation (3.4). So we will substitute $T_R g$ for w to see whether we can determine g or not.

$$(T_R g)_{\bar{z}} + A_1(z)(T_R g)_z + A_2(z)\overline{(T_R g)_z} + B_1(z)T_R g + B_2(z)\overline{T_R g} = f(z).$$

We call $(T_R g)_z = \Pi_R g$ similar to the notation used in [38]. The equation assumes the form

$$g + A_1(z)\Pi_R g + A_2(z)\overline{\Pi_R g} + B_1(z)T_R g + B_2(z)\overline{T_R g} = f(z).$$

So, we need to investigate the boundedness of the operators T_R and Π_R. Firstly, we start with the properties of T_R. We separate the function $T_R g$ into two parts as $T_{1R}g(z)$ and $T_{2R}g(z)$:

$$T_{1R}g(z) = -\frac{1}{2\pi}\int_R \left(\frac{g(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{g(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right) d\xi d\eta,$$
$$T_{2R}g(z) = -\frac{1}{2\pi}\int_R \left(\frac{g(\zeta)}{\zeta}(1+K_1(z,\zeta)) + \frac{\overline{g(\zeta)}}{\bar{\zeta}}(3+K_2(z,\zeta))\right) d\xi d\eta.$$

We have to check the boundedness of two parts. Let us start with the boundedness of T_{1R}.

$$\begin{aligned}
|T_{1R}g(z)| &= \frac{1}{2\pi}\left|\int_R \left(\frac{g(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{g(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right) d\xi d\eta\right| \\
&\leq \frac{1}{2\pi}\int_R \left|\left(\frac{g(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{g(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)\right| d\xi d\eta \\
&\leq \frac{1}{2\pi}\int_R \left(\left|\frac{g(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}\right| + \left|\frac{\overline{g(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right|\right) d\xi d\eta
\end{aligned}$$

$$\leq \frac{1}{2\pi}\int_R \left(\left|\frac{g(\zeta)}{\zeta}\left(\frac{2\zeta}{\zeta-z}-1\right)\right|+\left|\frac{\overline{g(\zeta)}}{\overline{\zeta}}\left(\frac{2z\overline{\zeta}}{1-z\overline{\zeta}}+1\right)\right|\right)d\xi d\eta$$
$$\leq \frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta-z}\right|d\xi d\eta+\frac{1}{2\pi}\int_R \left|\frac{g(\zeta)}{\zeta}\right|d\xi d\eta$$
$$+\frac{1}{\pi}\int_R \left|\frac{z\overline{g(\zeta)}}{1-z\overline{\zeta}}\right|d\xi d\eta+\frac{1}{2\pi}\int_R \left|\frac{\overline{g(\zeta)}}{\overline{\zeta}}\right|d\xi d\eta$$
$$=\frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta-z}\right|d\xi d\eta+\frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta}\right|d\xi d\eta+\frac{1}{\pi}\int_R \left|\frac{z\overline{g(\zeta)}}{1-z\overline{\zeta}}\right|d\xi d\eta. \tag{3.6}$$

For the first integral in Equation (3.6), we use the Hölder inequality for its boundedness.

$$\frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta-z}\right|d\xi d\eta \leq \frac{1}{\pi}\left(\int_R |g|^p\, d\xi d\eta\right)^{\frac{1}{p}}\left(\int_R |\zeta-z|^{-q}\, d\xi d\eta\right)^{\frac{1}{q}}.$$

For the second integral on the right hand side of the above inequality, we can use Schmidt's inequality.

$$\frac{1}{\pi}\left(\int_R |\zeta-z|^{-q}\, d\xi d\eta\right)^{\frac{1}{q}} \leq \frac{1}{\pi}\left(\frac{2\pi}{2-q}\left(\frac{mR}{\pi}\right)^{1-\frac{q}{2}}\right)^{\frac{1}{q}} = M_1$$

for $q<2$. So,

$$\frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta-z}\right|d\xi d\eta \leq M_1\left\|g\right\|_p.$$

For the second integral in Equation (3.6), we obtain

$$\frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta}\right|d\xi d\eta \leq \frac{1}{\pi}\left(\int_R |g|^p\, d\xi d\eta\right)^{\frac{1}{p}}\left(\int_R \frac{1}{|\zeta|^q}d\xi d\eta\right)^{\frac{1}{q}}.$$

For the second integral on the right hand side of the above inequality, we have

$$\frac{1}{\pi}\left(\int_R \frac{1}{|\zeta|^q}d\xi d\eta\right)^{\frac{1}{q}} \leq \frac{1}{\pi}\left(\int_0^{2\pi}\int_r^1 \rho^{-q}\rho d\rho d\theta\right)^{\frac{1}{q}} \leq \frac{1}{\pi}\left(2\pi\left(\frac{1-r^{2-q}}{2-q}\right)\right)^{\frac{1}{q}} = M_2.$$

for $q<2$. Thus,

$$\frac{1}{\pi}\int_R \left|\frac{g(\zeta)}{\zeta}\right|d\xi d\eta \leq M_2\left\|g\right\|_p.$$

For the last integral in Equation (3.6), we obtain

$$\frac{1}{\pi}\int\limits_{R}\left|\frac{z\overline{g(\zeta)}}{1-z\overline{\zeta}}\right|d\xi d\eta \leq \frac{1}{\pi}\left(\int\limits_{R}\left|\overline{g(\zeta)}\right|^{p}d\xi d\eta\right)^{\frac{1}{p}}\left(\int\limits_{R}\left|\frac{z}{1-z\overline{\zeta}}\right|^{q}d\xi d\eta\right)^{\frac{1}{q}}.$$

Since it is trivial that

$$\left|\frac{z}{1-z\overline{\zeta}}\right| = \left|\frac{1}{\frac{1}{z}-\overline{\zeta}}\right| \geq \left|\frac{1}{|z|}-\left|\overline{\zeta}\right|\right| \geq \left|\frac{1}{r}-r\right| = \frac{1-r^{2}}{r},$$

we have

$$\begin{aligned}\frac{1}{\pi}\left(\int\limits_{R}\left|\frac{z}{1-z\overline{\zeta}}\right|^{q}d\xi d\eta\right)^{\frac{1}{q}} &\leq \frac{1}{\pi}\left(\int\limits_{R}\left(\frac{1-r^{2}}{r}\right)^{q}d\xi d\eta\right)^{\frac{1}{q}}\\ &= \frac{1}{\pi}\frac{\pi^{\frac{1}{q}}\left(1-r^{2}\right)^{\frac{1+q}{q}}}{r} = M_{3}\end{aligned}$$

for $q<2$ and

$$\frac{1}{\pi}\int\limits_{R}\left|\frac{z\overline{g(\zeta)}}{1-z\overline{\zeta}}\right|d\xi d\eta \leq M_{3}\left\|g\right\|_{p}.$$

Thus,

$$|T_{1R}g(z)| \leq (M_{1}+M_{2}+M_{3})\left\|g\right\|_{p},$$

$$M_{1}+M_{2}+M_{3} = \frac{1}{\pi}\left(\frac{2\pi}{2-q}\right)^{\frac{1}{q}}\left[\left(1-r^{2}\right)^{\frac{2-q}{2q}}+(1-r^{2-q})^{\frac{1}{q}}\right]+\frac{1}{\pi}\frac{\pi^{\frac{1}{q}}\left(1-r^{2}\right)^{\frac{1+q}{q}}}{r} = M,$$

$$|T_{1R}g(z)|^{p} \leq M^{p}\left\|g\right\|_{p}^{p},$$

$$\int\limits_{R}|T_{1R}g(z)|^{p}\,dxdy \leq M^{p}\left\|g\right\|_{p}^{p}\int\limits_{R}dxdy,$$

$$\left[\int\limits_{R}|T_{1R}g(z)|^{p}\,dxdy\right]^{\frac{1}{p}} \leq M\left\|g\right\|_{p}\pi^{\frac{1}{p}}\left(1-r^{2}\right)^{\frac{1}{p}},$$

$$\left\|T_{1R}g(z)\right\|_{p} \leq M\left[\pi\left(1-r^{2}\right)\right]^{\frac{1}{p}}\left\|g\right\|_{p},$$

$$\left\|T_{1R}\right\|_{p} \leq M\left[\pi\left(1-r^{2}\right)\right]^{\frac{1}{p}}.$$

Thus T_{1R} is bounded. Now, we check the boundedness of T_{2R}.

$$
\begin{aligned}
|T_{2R}g(z)| &= \frac{1}{2\pi}\left|\int\limits_R \left[\left(\frac{g(\zeta)}{\zeta}(1+K_1(z,\zeta))\right)+\left(\frac{\overline{g(\zeta)}}{\overline{\zeta}}(3+K_2(z,\zeta))\right)\right] d\xi d\eta\right| \\
&\leq \frac{1}{2\pi}\int\limits_R \left(\left|\frac{g(\zeta)}{\zeta}(1+K_1(z,\zeta))\right|+\left|\frac{\overline{g(\zeta)}}{\overline{\zeta}}(3+K_2(z,\zeta))\right|\right) d\xi d\eta \\
&\leq \frac{1}{2\pi}\left(\int\limits_R \left|\frac{g(\zeta)}{\zeta}\right|^p d\xi d\eta\right)^{\frac{1}{p}}\left(\int\limits_R |1+K_1(z,\zeta)|^q\, d\xi d\eta\right)^{\frac{1}{q}} \\
&+\frac{1}{2\pi}\left(\int\limits_R \left|\frac{g(\zeta)}{\zeta}\right|^p d\xi d\eta\right)^{\frac{1}{p}}\left(\int\limits_R |3+K_2(z,\zeta)|^q\, d\xi d\eta\right)^{\frac{1}{q}}.
\end{aligned}
\tag{3.7}
$$

For the first and third integral in (3.7), we obtain

$$
\left(\int\limits_R \left|\frac{g(\zeta)}{\zeta}\right|^p d\xi d\eta\right)^{\frac{1}{p}} \leq \|g\|_p\left(\int\limits_R \frac{1}{|\zeta|^p} d\xi d\eta\right)^{\frac{1}{p}} \leq \|g\|_p\, 2\pi\frac{1-r^{2-p}}{2-p}. \tag{3.8}
$$

For the second and fourth integral in (3.7), we should investigate the properties of the series

$$
\begin{aligned}
K_1(z,\zeta) &= 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z}+\frac{r^{2n}z}{\zeta - r^{2n}z}\right), \\
K_2(z,\zeta) &= 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r^{2n} - z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1 - r^{2n}z\overline{\zeta}}\right).
\end{aligned}
$$

We want to investigate convergence of these series. We apply ratio test to the first part of $K_1(z,\zeta)$ with

$$
a_n = \frac{r^{2n}\zeta}{r^{2n}\zeta - z},
$$

$$
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{r^{2n+2}\zeta}{r^{2n+2}\zeta - z}.\frac{r^{2n}\zeta - z}{r^{2n}\zeta}\right| = r^2 < 1.
$$

Again, we apply ratio test to the second part of $K_1(z,\zeta)$ with

$$
b_n = \frac{r^{2n}z}{\zeta - r^{2n}z},
$$

$$
\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{r^{2n+2}z}{\zeta - r^{2n+2}z}.\frac{\zeta - r^{2n}z}{r^{2n}z}\right| = r^2 < 1.
$$

So, $K_1(z,\zeta)$ is absolutely convergent.

Similarly, we apply ratio test to the first part of $K_2(z,\zeta)$ with

$$c_n = \frac{r^{2n}}{r^{2n} - z\overline{\zeta}},$$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{r^{2n+2}}{r^{2n+2} - z\overline{\zeta}}.\frac{r^{2n} - z\overline{\zeta}}{r^{2n}}\right| = \lim_{n\to\infty} r^2 \left|\frac{r^{2n} - z\overline{\zeta}}{r^{2n+2} - z\overline{\zeta}}\right| = r^2 < 1.$$

Again, we apply ratio test to the second part of $K_2(z,\zeta)$ with

$$d_n = \frac{r^{2n} z\overline{\zeta}}{1 - r^{2n} z\overline{\zeta}},$$

$$\lim_{n\to\infty}\left|\frac{d_{n+1}}{d_n}\right| = \lim_{n\to\infty}\left|\frac{r^{2n+2} z\overline{\zeta}}{1 - r^{2n+2} z\overline{\zeta}}.\frac{1 - r^{2n} z\overline{\zeta}}{r^{2n} z\overline{\zeta}}\right| = \lim_{n\to\infty} r^2 \left|\frac{1 - r^{2n} z\overline{\zeta}}{1 - r^{2n+2} z\overline{\zeta}}\right| = r^2 < 1.$$

So, $K_2(z,\zeta)$ is absolutely convergent. $K_1(z,\zeta)$ and $K_2(z,\zeta)$ are bounded holomorphic functions in R by Montel theorem [32]. For the upper bound of $K_1(z,\zeta)$ and $K_2(z,\zeta)$, they assume their maximum values on ∂R with respect to ζ. Let us start with

$$K_1(z,\zeta) = 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z}\right),$$

which gives

$$|K_1(z,\zeta)|\Big|_{|z|=1} \leq 2\left(\sum_{n=1}^{\infty}\frac{r^{2n}|\zeta|}{1 - r^{2n}|\zeta|} + \sum_{n=1}^{\infty}\frac{r^{2n}}{|\zeta| - r^{2n}}\right). \tag{3.9}$$

Observe that the general term $\frac{r^{2n}|\zeta|}{1-r^{2n}|\zeta|}$ of the first series in (3.9) is an increasing function of $|\zeta|$, and the general term $\frac{r^{2n}}{|\zeta|-r^{2n}}$ of the second series in (3.9) is a decreasing function of $|\zeta|$; hence

$$\begin{aligned}\max_{|z|=1,\, r\leq|\zeta|\leq 1} |K_1(z,\zeta)| &\leq 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{1 - r^{2n}} + \frac{r^{2n}}{r - r^{2n}}\right)\\ &\leq 4\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r - r^{2n}}\right)\end{aligned}$$

$$\le 4\sum_{n=1}^{\infty}\left(\frac{r^{2n-1}}{1-r^{2n-1}}\right) \le \frac{4}{1-r}\sum_{n=1}^{\infty} r^{2n-1}$$
$$\le \frac{4r}{1-r}\sum_{n=0}^{\infty} r^{2n} = \frac{4r}{1-r}\frac{1}{1-r^2} = \frac{4r}{(1-r)^2(1+r)}.$$

Now let us take

$$|K_1(z,\zeta)|\Big|_{|z|=r} \le 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}|\zeta|}{r-r^{2n}|\zeta|} + \frac{r^{2n+1}}{|\zeta|-r^{2n+1}}\right).$$

Using similar discussion given above, we have

$$\max_{|z|=r,\ r\le|\zeta|\le 1} |K_1(z,\zeta)| \le 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r-r^{2n}} + \frac{r^{2n+1}}{r-r^{2n+1}}\right) = 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r-r^{2n}} + \frac{r^{2n}}{1-r^{2n}}\right)$$
$$\le \frac{4r}{(1-r)^2(1+r)}.$$

Thus we have

$$\max_{z\in\partial R} |K_1(z,\zeta)| \le \frac{4r}{(1-r)^2(1+r)}.$$

Repeating a similar argument for

$$K_2(z,\zeta) = 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}} + \frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right),$$

we obtain

$$|K_2(z,\zeta)|\Big|_{|z|=1} \le 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{|\zeta|-r^{2n}} + \frac{r^{2n}|\zeta|}{1-r^{2n}|\zeta|}\right)$$
$$\le 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r-r^{2n}} + \frac{r^{2n+1}}{1-r^{2n+1}}\right)$$
$$\le 4\sum_{n=1}^{\infty}\frac{r^{2n}}{r-r^{2n}} \le \frac{4r}{(1-r^2)(1+r)}$$

for $|z| = 1$. For $|z| = r$ we find

$$|K_2(z,\zeta)|\Big|_{|z|=r} \le 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r|\zeta|-r^{2n}} + \frac{r^{2n+1}|\zeta|}{1-r^{2n+1}|\zeta|}\right)$$

$$\leq 2\sum_{n=1}^{\infty}\frac{r^{2n}}{r^2-r^{2n}}+2\sum_{n=1}^{\infty}\frac{r^{2n+1}}{1-r^{2n+1}}$$
$$=2\sum_{n=1}^{\infty}\frac{r^{2n-2}}{1-r^{2n-2}}+2\sum_{n=1}^{\infty}\frac{r^{2n+1}}{1-r^{2n+1}}$$
$$=2\sum_{n=1}^{\infty}\frac{r^{2n}}{1-r^{2n}}+2\sum_{n=1}^{\infty}\frac{r^{2n+1}}{1-r^{2n+1}}$$
$$\leq 4\sum_{n=1}^{\infty}\frac{r^{2n}}{1-r^{2n}}\leq\frac{4}{1-r^2}\sum_{n=1}^{\infty}r^{2n}.$$

So we find

$$\max_{z\in\partial R}|K_2(z,\zeta)|\leq\frac{4}{1-r^2}\frac{1}{1-r^2}=\frac{4}{(1-r^2)^2}$$

and

$$\max_{\overline{R}}\left(|K_1(z,\zeta)|,|K_2(z,\zeta)|\right)\leq\max\left(\frac{4r}{(1-r)^2(1+r)},\frac{4}{(1-r^2)^2}\right)=\frac{4}{(1-r^2)^2}.$$

For the upper bound of $\left(\int\limits_R |1+K_1(z,\zeta)|^q\,d\xi d\eta\right)^{\frac{1}{q}}$ in (3.7) we obtain

$$\begin{aligned}\left(\int\limits_R |1+K_1(z,\zeta)|^q\,d\xi d\eta\right)^{\frac{1}{q}} &\leq\left(\int\limits_R (1+|K_1(z,\zeta)|)^q\,d\xi d\eta\right)^{\frac{1}{q}}\\ &\leq\left(\int\limits_R\left(1+\frac{4r}{(1-r)^2(1+r)}\right)^q d\xi d\eta\right)^{\frac{1}{q}}\\ &\leq\left(\left(1+\frac{4r}{(1-r)^2(1+r)}\right)^q\int\limits_R d\xi d\eta\right)^{\frac{1}{q}}\\ &=\left(\left(1+\frac{4r}{(1-r)^2(1+r)}\right)^q\pi(1-r^2)\right)^{\frac{1}{q}}\\ &=\left(1+\frac{4r}{(1-r)^2(1+r)}\right)\pi^{\frac{1}{q}}(1-r^2)^{\frac{1}{q}}.\end{aligned}$$

For the upper bound of $\left(\int\limits_R |3+K_2(z,\zeta)|^q \, d\xi d\eta\right)^{\frac{1}{q}}$ in (3.7) we find

$$\begin{aligned}
\left(\int\limits_R |3+K_2(z,\zeta)|^q \, d\xi d\eta\right)^{\frac{1}{q}} &\leq \left(\int\limits_R (3+|K_2(z,\zeta)|)^q \, d\xi d\eta\right)^{\frac{1}{q}} \\
&\leq \left(\int\limits_R \left(3+\frac{4}{(1-r^2)^2}\right)^q \, d\xi d\eta\right)^{\frac{1}{q}} \\
&\leq \left(\left(3+\frac{4}{(1-r^2)^2}\right)^q \int\limits_R d\xi d\eta\right)^{\frac{1}{q}} \\
&= \left(3+\frac{4}{(1-r^2)^2}\right)\pi^{\frac{1}{q}}(1-r^2)^{\frac{1}{q}}.
\end{aligned}$$

So

$$\begin{aligned}
|T_{2R}g(z)| &\leq \frac{1}{2\pi}2\pi\frac{1-r^{2-p}}{2-p}\pi^{\frac{1}{q}}(1-r^2)^{\frac{1}{q}}\,\|g\|_p\left(1+\frac{4r}{(1-r)^2(1+r)}+3+\frac{4}{(1-r^2)^2}\right) \\
&= \frac{1-r^{2-p}}{2-p}\pi^{\frac{1}{q}}(1-r^2)^{\frac{1}{q}}4\left(1+(1-r)(r^2+r+1\right)\|g\|_p
\end{aligned}$$

$$|T_{2R}g(z)|^p \leq A^p\,\|g\|_p^p\,,$$

$$\int\limits_R |T_{2R}g(z)|^p \, dxdy \leq A^p\,\|g\|_p^p\int\limits_R dxdy,$$

$$\left[\int\limits_R |T_{2R}g(z)|^p \, dxdy\right]^{\frac{1}{p}} \leq A\,\|g\|_p\,\pi^{\frac{1}{p}}\left(1-r^2\right)^{\frac{1}{p}},$$

$$\|T_{2R}g(z)\|_p \leq A\left[\pi\left(1-r^2\right)\right]^{\frac{1}{p}}\|g\|_p\,,$$

$$\|T_{2R}\|_p \leq A\left[\pi\left(1-r^2\right)\right]^{\frac{1}{p}}.$$

T_{2R} is bounded. Since the sum of the bounded operators is bounded, $T_{1R}+T_{2R}$ is bounded.

Now, we show the equicontinuity of the operator T_R. For this, we examine the equicontinuity of $T_{1R}g(z)$ and $T_{2R}g(z)$. We start with $T_{1R}g(z)$.

$$|T_{1R}g(z_1) - T_{1R}g(z_2)| \leq \left|\frac{1}{2\pi}\int\limits_R \left(-\frac{g(\zeta)}{\zeta}\frac{\zeta+z_1}{\zeta-z_1} + \frac{g(\zeta)}{\zeta}\frac{\zeta+z_2}{\zeta-z_2}\right) d\xi d\eta\right|$$

$$+\left|\frac{1}{2\pi}\int\limits_R \left(-\frac{\overline{g(\zeta)}}{\overline{\zeta}}\frac{1+z_1\overline{\zeta}}{1-z_1\overline{\zeta}} + \frac{\overline{g(\zeta)}}{\overline{\zeta}}\frac{1+z_2\overline{\zeta}}{1-z_2\overline{\zeta}}\right) d\xi d\eta\right|$$

$$= \frac{1}{2\pi}\left|-\int\limits_R \frac{g(\zeta)}{\zeta}\left(\frac{\zeta+z_1}{\zeta-z_1} - \frac{\zeta+z_2}{\zeta-z_2}\right) d\xi d\eta\right| + \frac{1}{2\pi}\left|-\int\limits_R \frac{\overline{g(\zeta)}}{\overline{\zeta}}\left(\frac{1+z_1\overline{\zeta}}{1-z_1\overline{\zeta}} - \frac{1+z_2\overline{\zeta}}{1-z_2\overline{\zeta}}\right) d\xi d\eta\right|$$

$$= \frac{1}{\pi}\left|\int\limits_R \frac{g(\zeta)(z_1-z_2)}{(\zeta-z_1)(\zeta-z_2)} d\xi d\eta\right| + \frac{1}{\pi}\left|\int\limits_R \frac{\overline{g(\zeta)}(z_1-z_2)}{(1-z_1\overline{\zeta})(1-z_2\overline{\zeta})} d\xi d\eta\right|$$

$$\leq |z_1-z_2|\left(\int\limits_R |g(\zeta)|^p \, d\xi d\eta\right)^{\frac{1}{p}} \frac{1}{\pi}\left(\int\limits_R \left(\frac{1}{|\zeta-z_1|\,|\zeta-z_2|}\right)^q d\xi d\eta\right)^{\frac{1}{q}}$$

$$+\,|z_1-z_2|\left(\int\limits_R |g(\zeta)|^p \, d\xi d\eta\right)^{\frac{1}{p}} \frac{1}{\pi}\left(\int\limits_R \left(\frac{1}{\left|1-z_1\overline{\zeta}\right|\left|1-z_2\overline{\zeta}\right|}\right)^q d\xi d\eta\right)^{\frac{1}{q}}$$

$$\leq |z_1-z_2|\,\|g\|_p \,\frac{1}{\pi}\left(\int\limits_R \left(\frac{1}{|\zeta-z_1|\,|\zeta-z_2|}\right)^q d\xi d\eta\right)^{\frac{1}{q}}$$

$$+\,|z_1-z_2|\,\|g\|_p \,\frac{1}{\pi}\left(\int\limits_R \left(\frac{1}{\left|1-z_1\overline{\zeta}\right|\left|1-z_2\overline{\zeta}\right|}\right)^q d\xi d\eta\right)^{\frac{1}{q}}. \tag{3.10}$$

By Lemma 2 the first integral in (3.10) is bounded. In [1], the second integral in (3.10) is bounded in a regular domain. Thus,

$$|T_{1R}g(z_1) - T_{1R}g(z_2)| \leq (C_1 + C_2)\,\|g\|_p\,|z_1-z_2| = C_3\,\|g\|_p\,|z_1-z_2|\,.$$

Hence, $T_{1R}g$ is equicontinuous.

For the equicontinuity of $T_{2R}g(z)$, we find

$$|T_{2R}g(z_3)-T_{2R}g(z_4)| \leq \left|\frac{1}{2\pi}\int_R \left(-\frac{g(\zeta)}{\zeta}(-1+K_1(z_3,\zeta))+\frac{g(\zeta)}{\zeta}(-1+K_1(z_4,\zeta))\right)d\xi d\eta\right|$$

$$+\left|\frac{1}{2\pi}\int_R \left(-\frac{\overline{g(\zeta)}}{\overline{\zeta}}(-1+K_1(z_4,\zeta))+\frac{\overline{g(\zeta)}}{\overline{\zeta}}(-1+K_2(z_4,\zeta))\right)d\xi d\eta\right|$$

$$=\frac{1}{2\pi}\left|-\int_R \frac{g(\zeta)}{\zeta}(K_1(z_3,\zeta)-K_1(z_4,\zeta))d\xi d\eta\right|$$
$$+\frac{1}{2\pi}\left|-\int_R \frac{\overline{g(\zeta)}}{\overline{\zeta}}(K_2(z_3,\zeta)-K_2(z_4,\zeta))d\xi d\eta\right|$$

$$=\frac{1}{2\pi}\left|\int_R \frac{g(\zeta)}{\zeta}\left(2\sum_{n=1}^{\infty}\frac{(z_3-z_4)r^{2n}\zeta}{(r^{2n}\zeta-z_3)(r^{2n}\zeta-z_4)}+2\sum_{n=1}^{\infty}\frac{(z_3-z_4)r^{2n}\zeta}{(\zeta-r^{2n}z_4)(\zeta-r^{2n}z_3)}\right)d\xi d\eta\right|$$

$$+\frac{1}{2\pi}\left|\int_R \frac{\overline{g(\zeta)}}{\overline{\zeta}}\left(2\sum_{n=1}^{\infty}\frac{r^{2n}(z_3-z_4)\overline{\zeta}}{(r^{2n}-z_3\overline{\zeta})(r^{2n}-z_4\overline{\zeta})}+2\sum_{n=1}^{\infty}\frac{r^{2n}(z_3-z_4)\overline{\zeta}}{\left(1-r^{2n}z_3\overline{\zeta}\right)\left(1-r^{2n}z_4\overline{\zeta}\right)}\right)d\xi d\eta\right|$$

$$\leq \frac{1}{\pi}|z_3-z_4|\left(\int_R |g(\zeta)|^p\,d\xi d\eta\right)^{\frac{1}{p}}\left(\int_R\left|\sum_{n=1}^{\infty}\frac{r^{2n}}{(r^{2n}\zeta-z_3)(r^{2n}\zeta-z_4)}\right.\right.$$
$$\left.\left.+\sum_{n=1}^{\infty}\frac{r^{2n}}{(\zeta-r^{2n}z_4)\left(\zeta-r^{2n}z_3\right)}\right|^q d\xi d\eta\right)^{\frac{1}{q}}$$
$$+\frac{1}{\pi}|z_3-z_4|\left(\int_R |g(\zeta)|^p\,d\xi d\eta\right)^{\frac{1}{p}}\left(\int_R\left|\sum_{n=1}^{\infty}\frac{r^{2n}}{\left(r^{2n}-z_3\overline{\zeta}\right)\left(r^{2n}-z_4\overline{\zeta}\right)}\right.\right.$$
$$\left.\left.+\sum_{n=1}^{\infty}\frac{r^{2n}}{\left(1-r^{2n}z_3\overline{\zeta}\right)\left(1-r^{2n}z_4\overline{\zeta}\right)}\right|^q d\xi d\eta\right)^{\frac{1}{q}}.$$

Let us now consider $\sum_{n=1}^{\infty} \frac{r^{2n}}{(r^{2n}\zeta - z_3)(r^{2n}\zeta - z_4)}$. We have

$$\lim_{n\to\infty} \left| \frac{r^{2n+2}}{(r^{2n+2}\zeta - z_3)(r^{2n+2}\zeta - z_4)} . \frac{(r^{2n}\zeta - z_3)(r^{2n}\zeta - z_4)}{r^{2n}} \right|$$

$$= r^2 \lim_{n\to\infty} \left| \frac{(r^{2n}\zeta - z_3)(r^{2n}\zeta - z_4)}{(r^{2n+2}\zeta - z_3)(r^{2n+2}\zeta - z_4)} \right| = r^2 < 1.$$

For $\sum_{n=1}^{\infty} \frac{r^{2n}}{(\zeta - r^{2n}z_4)(\zeta - r^{2n}z_3)}$, we have

$$\lim_{n\to\infty} \left| \frac{r^{2n+2}}{\left(\zeta - r^{2n+2}z_4\right)\left(\zeta - r^{2n+2}z_3\right)} . \frac{\left(\zeta - r^{2n}z_4\right)\left(\zeta - r^{2n}z_3\right)}{r^{2n}} \right|$$

$$= r^2 \lim_{n\to\infty} \left| \frac{\left(\zeta - r^{2n}z_4\right)\left(\zeta - r^{2n}z_3\right)}{\left(\zeta - r^{2n+2}z_4\right)\left(\zeta - r^{2n+2}z_3\right)} \right| = r^2 < 1.$$

The above two series are absolutely convergent. The sum of two absolutely convergent series also converges absolutely. The sum of these two series converges to a holomorphic function of ζ in R.

For $\sum_{n=1}^{\infty} \frac{r^{2n}}{(r^{2n} - z_3\overline{\zeta})(r^{2n} - z_4\overline{\zeta})}$, we have

$$\lim_{n\to\infty} \left| \frac{r^{2n+2}}{\left(r^{2n+2} - z_3\overline{\zeta}\right)\left(r^{2n+2} - z_4\overline{\zeta}\right)} . \frac{\left(r^{2n} - z_3\overline{\zeta}\right)\left(r^{2n} - z_4\overline{\zeta}\right)}{r^{2n}} \right|$$

$$= r^2 \lim_{n\to\infty} \left| \frac{\left(r^{2n} - z_3\overline{\zeta}\right)\left(r^{2n} - z_4\overline{\zeta}\right)}{\left(r^{2n+2} - z_3\overline{\zeta}\right)\left(r^{2n+2} - z_4\overline{\zeta}\right)} \right| = r^2 < 1.$$

For $\sum_{n=1}^{\infty} \frac{r^{2n}}{(1 - r^{2n}z_3\overline{\zeta})(1 - r^{2n}z_4\overline{\zeta})}$, we have

$$\lim_{n\to\infty} \left| \frac{r^{2n+2}}{\left(1 - r^{2n+2}z_3\overline{\zeta}\right)\left(1 - r^{2n+2}z_4\overline{\zeta}\right)} . \frac{\left(1 - r^{2n}z_3\overline{\zeta}\right)\left(1 - r^{2n}z_4\overline{\zeta}\right)}{r^{2n}} \right|$$

$$= r^2 \lim_{n\to\infty} \left| \frac{\left(1 - r^{2n}z_3\overline{\zeta}\right)\left(1 - r^{2n}z_4\overline{\zeta}\right)}{\left(1 - r^{2n+2}z_3\overline{\zeta}\right)\left(1 - r^{2n+2}z_4\overline{\zeta}\right)} \right| = r^2 < 1.$$

Similarly, the above two series are absolutely convergent. The sum of two absolutely convergent series also converges absolutely. The sum of these two series converges to a holomorphic function of ζ in R. So

$$\begin{aligned}|T_{2R}g(z_3) - T_{2R}g(z_4)| \leq C_4 |z_3 - z_4| \|g\|_p + C_5 |z_3 - z_4| \|g\|_p &= (C_4 + C_5) |z_3 - z_4| \|g\|_p \\ &= C_6 \|g\|_p .\end{aligned}$$

Hence, $T_{2R}g$ is equicontinuous. By Arzela-Ascoli Theorem [31], T_R is compact. We separate the function $\Pi_R g$ into two parts as $\Pi_{1R}g(z)$ and $\Pi_{2R}g(z)$:

$$\Pi_{1R}g(z) = -\frac{1}{\pi}\int\limits_R \left(\frac{g(\zeta)}{(\zeta - z)^2} + \frac{\overline{g(\zeta)}}{\left(1 - z\overline{\zeta}\right)^2}\right) d\xi d\eta,$$

$$\Pi_{2R}g(z) = -\frac{1}{2\pi}\int\limits_R \left[\frac{g(\zeta)}{\zeta}\partial_z K_1(z,\zeta) + \frac{\overline{g(\zeta)}}{\overline{\zeta}}\partial_z K_2(z,\zeta)\right] d\xi d\eta.$$

We start with the boundedness of Π_{1R}. We separate it into two parts as:

$$\Pi_{11R}g(z) = -\frac{1}{\pi}\int\limits_R \frac{g(\zeta)}{(\zeta - z)^2} d\xi d\eta,$$

$$\Pi_{12R}g(z) = -\frac{1}{\pi}\int\limits_R \frac{\overline{g(\zeta)}}{(1 - z\overline{\zeta})^2} d\xi d\eta$$

In [1], Π_{11R} is shown to be bounded in a regular domain G. We can choose the domain as the ring domain (particular case of G). So Π_{11R} is bounded. For the boundedness of Π_{12R}, we use a simple extension. Let

$$\tilde{f} = \begin{cases} 0, & f \in \mathbb{D}\backslash R \\ f(z), & f \in R \end{cases}$$

We obtain

$$\left\|\Pi_{12R}\tilde{f}\right\|_{L^p(\mathbb{D})} \leq C \left\|\tilde{f}\right\|_{L^p(\mathbb{D})} = C \|f\|_{L^p(R)} ,$$

or,

$$\|\Pi_{12R}\|_{L^p(R)} \leq \|\Pi_{12R}\|_{L^p(\mathbb{D})} \leq C.$$

So, Π_{1R} is bounded. For the boundedness of the operator Π_{2R}, we have

$$|\Pi_{2R}g(z)| \leq \left| -\frac{1}{2\pi} \int_R \left(\frac{g(\zeta)}{\zeta} \partial_z K_1(z,\zeta) + \frac{\overline{g(\zeta)}}{\overline{\zeta}} \partial_z K_2(z,\zeta) \right) d\xi d\eta \right|$$

$$\begin{aligned} &\leq \frac{1}{2\pi} \left(\int_R \left| \frac{g(\zeta)}{\zeta} \right|^p d\xi d\eta \right)^{\frac{1}{p}} \left(\int_R |\partial_z K_1(z,\zeta)|^q \, d\xi d\eta \right)^{\frac{1}{q}} \\ &+ \frac{1}{2\pi} \left(\int_R \left| \frac{g(\zeta)}{\zeta} \right|^p d\xi d\eta \right)^{\frac{1}{p}} \left(\int_R |\partial_z K_2(z,\zeta)|^q \, d\xi d\eta \right)^{\frac{1}{q}}. \end{aligned} \tag{3.11}$$

We have shown in (3.8) that the first and third integrals in (3.11) are bounded. Other integrals include derivatives of holomorphic functions in R. Thus, Π_{2R} is bounded. Since the sum of two bounded operators is bounded, Π_R is bounded, too.
Thus, we have converted the generalized Beltrami equation into an integral equation. The resulting equation is

$$(I + \widehat{\Pi}_R + \widehat{K}_R)g(z) = f(z)$$

where

$$\begin{aligned} \widehat{\Pi}_R g &= A_1(z)\Pi_R g + A_2(z)\overline{\Pi_R g} \\ \widehat{K}_R g &= B_1(z)T_R g + B_2(z)\overline{T_R g}. \end{aligned}$$

We also investigate the existence of a unique solution. $\widehat{K}_R$ is a compact operator by Arzela-Ascoli theorem [31]. $\widehat{\Pi}_R$ is a bounded strongly singular operator of Calderon-Zygmund type. $I + \widehat{\Pi}_R$ is invertible if there exists q_0 such that

$$q_0 \|\widehat{\Pi}_R\|_{L^p(R)} < 1 \text{ for } p > 2.$$

By Nikolskii criterion, an operator is an index-zero Fredholm operator if and only if it is the sum of an invertible operator and a compact operator. So $I + \widehat{\Pi}_R + \widehat{K}_R$ is a Fredholm operator with index-zero. Hence Equation (3.4) subject to the condition (3.5) has a unique solution of the form $w = T_R g$ where $g \in L^p(R)$, $p > 2$.

4. SCHWARZ PROBLEM IN A RING DOMAIN FOR SECOND-ORDER EQUATIONS

The second-order equations and its higher-order generalizations are investigated in [20].

4.1. SCHWARZ PROBLEM IN A RING DOMAIN FOR INHOMOGENEOUS CAUCHY-RIEMANN EQUATIONS

We start with a remark on the integral representation for Schwarz problem in a ring domain. The following theorem and its proof are given previously in Section 3.1.

Theorem 16. *[19] The Schwarz problem defined in R has the unique solution*

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta} \\ & +ic-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]d\xi d\eta \\ & -\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+3+K_2(z,\zeta)\right]d\xi d\eta \end{aligned} \tag{4.1}$$

subject to the solvability condition

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}=-\frac{3}{2\pi}\int\limits_{R}\left[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right]d\xi d\eta, \tag{4.2}$$

where

$$\begin{aligned} K_1(z,\zeta) &= 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right), \\ K_2(z,\zeta) &= 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right). \end{aligned}$$

An observation: We want to see what happens if we write "a" instead of 1, "b" instead of 3 with $a, b \in \mathbb{R}$. In such a case we can write the corresponding formula to Equation (4.1) as

$$\begin{aligned} w_{(a,b)}(z) = & \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \left[\frac{\zeta + z}{\zeta - z} + K_1(z,\zeta) \right] \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} \\ & + ic - \frac{1}{2\pi} \int\limits_R \left[\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}} \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} \right] d\xi d\eta \\ & - \frac{1}{2\pi} \int\limits_R \left[\frac{f(\zeta)}{\zeta} (a + K_1(z,\zeta)) + \frac{\overline{f(\zeta)}}{\overline{\zeta}} (b + K_2(z,\zeta)) \right] d\xi d\eta, \end{aligned}$$

with the exception of the last term in the above representation is a holomorphic function. It is easy to observe that

$$\partial_{\overline{z}} w_{(a,b)}(z) = f(z),$$

so $w_{(a,b)}(z)$ is also a solution of inhomogeneous Cauchy-Riemann equation. Now we need to check the auxiliary conditions. In the following theorem we show that only solvability conditions are effected by the choice of a, b instead of $1, 3$.

Theorem 17. *The Schwarz problem defined in R has the unique solution*

$$\begin{aligned} w_{(a,b)}(z) = & \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \left[\frac{\zeta + z}{\zeta - z} + K_1(z,\zeta) \right] \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} \\ & + ic - \frac{1}{2\pi} \int\limits_R \frac{f(\zeta)}{\zeta} \left[\frac{\zeta + z}{\zeta - z} + a + K_1(z,\zeta) \right] d\xi d\eta \\ & - \frac{1}{2\pi} \int\limits_R \frac{\overline{f(\zeta)}}{\overline{\zeta}} \left[\frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} + b + K_2(z,\zeta) \right] d\xi d\eta \end{aligned} \tag{4.3}$$

subject to the solvability condition

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} + \frac{(a+b+2)}{4\pi} \int\limits_R \left[\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}} \right] d\xi d\eta = 0, \tag{4.4}$$

if boundary integral over $|\zeta| = r$ is computed in the positive sense where

$$K_1(z,\zeta) = 2 \sum_{n=1}^{\infty} \left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} \right),$$

$$K_2(z,\zeta) = 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}} + \frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right).$$

Proof. For the proof of this theorem, it is easy to see that the solution satisfies the Cauchy-Riemann equation. To determine the solvability condition and the normalization condition we must compute $2Rew_{(a,b)}(z)$ and $\frac{1}{2\pi i}\int\limits_{|z|=r} Imw_{(a,b)}(z)\frac{dz}{z}$. Firstly we compute $2Rew_{(a,b)}(z)$. We rearrange $w_{(a,b)}(z)$ using Equation (3.1).

$$\begin{aligned}
w_{(a,b)}(z) = {} & \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta} \\
& + \frac{1}{2\pi}\int\limits_{|\zeta|=r} Imw_{(a,b)}(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1-1+a+K_1(z,\zeta)\right]d\xi d\eta \\
& - \frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+3-3+b+K_2(z,\zeta)\right]d\xi d\eta \\
= {} & \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta} \\
& + \frac{1}{2\pi}\int\limits_{|\zeta|=r} Imw_{(a,b)}(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]d\xi d\eta \\
& - \frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+3+K_2(z,\zeta)\right]d\xi d\eta \\
& - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}(a-1)d\xi d\eta - \frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(b-3)d\xi d\eta.
\end{aligned}$$

$$\begin{aligned}
2Rew_{(a,b)}(z) = {} & \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z}+\frac{\overline{\zeta}+\overline{z}}{\overline{\zeta}-\overline{z}}+K_1(z,\zeta)+\overline{K_1(z,\zeta)}\right]\frac{d\zeta}{\zeta} \\
& -\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+4+K_1(z,\zeta)+\overline{K_2(z,\zeta)}\right]d\xi d\eta \\
& -\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{\overline{\zeta}+\overline{z}}{\overline{\zeta}-\overline{z}}+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+4+\overline{K_1(z,\zeta)}+K_2(z,\zeta)\right]d\xi d\eta \\
& -\frac{2}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}(a-1)d\xi d\eta
\end{aligned}$$

$$-\frac{1}{2\pi}\int_R \frac{\overline{f(\zeta)}}{\overline{\zeta}}(b-3)d\xi d\eta-\frac{1}{2\pi}\int_R \frac{\overline{f(\zeta)}}{\overline{\zeta}}(a-1)d\xi d\eta$$
$$-\frac{1}{2\pi}\int_R \frac{f(\zeta)}{\zeta}(b-3)d\xi d\eta$$
$$=\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z}+\frac{\overline{\zeta}+\overline{z}}{\overline{\zeta}-\overline{z}}+K_1(z,\zeta)+\overline{K_1(z,\zeta)}\right]\frac{d\zeta}{\zeta}$$
$$-\frac{1}{2\pi}\int_R \frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+4+K_1(z,\zeta)+\overline{K_2(z,\zeta)}\right]d\xi d\eta$$
$$-\frac{1}{2\pi}\int_R \frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{\overline{\zeta}+\overline{z}}{\overline{\zeta}-\overline{z}}+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+4+\overline{K_1(z,\zeta)}+K_2(z,\zeta)\right]d\xi d\eta$$
$$-\frac{2}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi}\int_R \frac{f(\zeta)}{\zeta}(a+b-4)d\xi d\eta$$
$$-\frac{1}{2\pi}\int_R \frac{\overline{f(\zeta)}}{\overline{\zeta}}(a+b-4)d\xi d\eta.$$

When we restrict the real part of w to the boundary of the domain R, we obtain

$$2Rew_{(a,b)}(z)\Big|_{\partial R}=2\gamma-\frac{2}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}-\frac{3}{\pi}\int_R\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta$$
$$-\frac{a+b-4}{4\pi}\int_R\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta,$$
$$Rew_{(a,b)}(z)\Big|_{\partial R}=\gamma-\left[\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}+\frac{a+b+2}{4\pi}\int_R\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta\right].$$

We have obtained solvability condition.

Secondly, we compute the normalization condition.

$$\frac{1}{2\pi i}\int_{|z|=r}w_{(a,b)}(z)\frac{dz}{z}=\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{1}{2\pi i}\int_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{1}{2\pi i}\int_{|z|=r}\frac{dz}{z}\frac{d\zeta}{\zeta}+\frac{1}{2\pi}\int_{|\zeta|=r}Imw_{(a,b)}(\zeta)\frac{1}{2\pi i}\int_{|z|=r}\frac{dz}{z}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi}\int_{R}\frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+a+K_1(z,\zeta)\right]\frac{dz}{z}d\xi d\eta$$

$$-\frac{1}{2\pi}\int_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1}{2\pi i}\int_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+b+K_2(z,\zeta)\right]\frac{dz}{z}d\xi d\eta$$

where

$$\frac{1}{2\pi i}\int_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z}=-1,$$

$$\frac{1}{2\pi i}\int_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+K_2(z,\zeta)\right]\frac{dz}{z}=1\ for\ \zeta\in R$$

and

$$\frac{1}{2\pi i}\int_{|z|=r}a\frac{dz}{z}=-a$$
$$\frac{1}{2\pi i}\int_{|z|=r}b\frac{dz}{z}=-b$$

for $a,b\in\mathbb{R}$. Now, we check the normalization condition.

Since

$$\frac{1}{2\pi i}\int_{|z|=r}w_{(a,b)}(z)\frac{dz}{z}=-\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}+ic$$
$$-\frac{1}{2\pi}\int_{R}\frac{f(\zeta)}{\zeta}(-1-a)d\xi d\eta-\frac{1}{2\pi}\int_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(1-b)d\xi d\eta,$$

we have

$$\frac{1}{2\pi i}\int_{|z|=r}w_{(a,b)}(z)\frac{dz}{z}-\frac{1}{2\pi i}\int_{|z|=r}\overline{w_{(a,b)}(z)}\frac{dz}{z}=-\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}+ic$$

$$-\frac{1}{2\pi}\int_{R}\frac{f(\zeta)}{\zeta}(-1-a)d\xi d\eta-\frac{1}{2\pi}\int_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(1-b)$$

$$-\left\{\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}-ic\right.$$

$$-\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(-1-a)d\xi d\eta-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}(1-b)d\xi d\eta\Bigg\}$$

$$=-\frac{1}{\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}+2ic$$

$$-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}(a+b)d\xi d\eta-\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(a+b)d\xi d\eta$$

$$=-\frac{2}{\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi}(a+b)\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta$$

$$+2ic.$$

So,

$$2iIm\frac{1}{2\pi i}\int\limits_{|z|=r}w_{(a,b)}(z)\frac{dz}{z}=2ic.$$

or

$$\frac{1}{2\pi i}\int\limits_{|z|=r}Imw_{(a,b)}(z)\frac{dz}{z}=c.$$

□

Thus, we may take any pair for (a,b) in Theorem 17. In the sequel we take $a=1, b=-1$.

The statement of the Schwarz problem for second-order equations is as follows.

"*Find the solution of* $w_{\overline{z}\overline{z}}=f(z)$ *in* R *satisfying*

$$Rew(z)=\gamma_0(z)\ on\ \partial R,\ \frac{1}{2\pi i}\int\limits_{|z|=r}Imw(z)\frac{dz}{z}=c_0$$

$$Rew_{\overline{z}}(z)=\gamma_1(z)\ on\ \partial R,\ \frac{1}{2\pi i}\int\limits_{|z|=r}Imw_{\overline{z}}(z)\frac{dz}{z}=c_1$$

for $\gamma_0,\gamma_1\in C(\partial R;\mathbb{R})$, $c_0,c_1\in\mathbb{R}$ *given.*"

Theorem 18. *The Schwarz problem defined in R has the unique solution*

$$w(z)=ic_0+i(z+\overline{z})c_1+\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_0(\zeta)-\gamma_1(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)\right]\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}$$

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_0\left(\zeta\right)-\gamma_1\left(\zeta\right)\left(\zeta-z+\overline{\zeta-z}\right)\right]\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\left(\zeta-z+\overline{\zeta-z}\right)d\xi d\eta\\
&+\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]\left(\zeta-z+\overline{\zeta-z}\right)d\xi d\eta
\end{aligned}
\tag{4.5}
$$

subject to the solvability conditions

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})\right]\frac{d\zeta}{\zeta}\\
&\quad+\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)(\zeta-z+\overline{\zeta-z})d\xi d\eta=0,
\end{aligned}
\tag{4.6}
$$

$$
\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1\left(\zeta\right)\frac{d\zeta}{\zeta}-\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta=0
\tag{4.7}
$$

where

$$
\begin{aligned}
K_1(z,\zeta)&=2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right),\\
K_2(z,\zeta)&=2\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right).
\end{aligned}
$$

Proof. Similar to the discussion of the properties of the operator T_R, we see that the above representation is the solution of the Bitsadze equation. To see that this problem is uniquely solvable, we first need to check if it satisfies the boundary condition. We use the Poisson kernels given in Definition 11.

$$
\begin{aligned}
Rew(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}&\left[\gamma_0(\zeta)-\gamma_1(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)\right]\left[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_0(\zeta)-\gamma_1(\zeta)\left(\zeta-z+\overline{\zeta-z}\right)\right]\left[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2(n-1)}|z|^2\zeta}{r^{2(n-1)}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2(n+1)}z}{|z|^2\zeta-r^{2(n+1)}z}\right)\right]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})\right]\frac{d\zeta}{\zeta}+\frac{1}{4\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}\right.\\
&+\frac{z+|z|^2\zeta}{z-|z|^2\zeta}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}+\frac{r^{2n}z}{r^{2n}z-|z|^2\zeta}\right.\\
&\left.\left.+\frac{r^{2n}|z|^2\zeta}{z-r^{2n}|z|^2\zeta}\right)(\zeta-z+\overline{\zeta-z})d\xi d\eta+\frac{1}{4\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{z\overline{\zeta}+|z|^2}{z\overline{\zeta}-|z|^2}\right.\right.\\
&+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}z\overline{\zeta}}{r^{2n}z\overline{\zeta}-|z|^2}+\frac{r^{2n}|z|^2}{z\overline{\zeta}-r^{2n}|z|^2}+\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}\right.\\
&\left.\left.+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right)\right](\zeta-z+\overline{\zeta-z})d\xi d\eta.
\end{aligned}
$$

If we restrict $Rew(z)$ to the boundary ∂R, we obtain Equation (4.6).

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})\right]\frac{d\zeta}{\zeta}\\
&\qquad+\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)(\zeta-z+\overline{\zeta-z})d\xi d\eta=0.
\end{aligned}
$$

To satisfy the second boundary condition, we take the derivative of w with respect to $\overline{z}$.

$$
w_{\overline{z}}(z)=ic_1+\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}
$$

$$
-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]d\xi d\eta
$$

$$
-\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]d\xi d\eta.
$$

Now, we take the real part of $w_{\overline{z}}(z)$.

$$
\begin{aligned}
Rew_{\overline{z}}(z) = & \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\left[\frac{\zeta}{\zeta-z}+\overline{\frac{\zeta}{\zeta-z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\left[\frac{\zeta}{\zeta-z}+\overline{\frac{\zeta}{\zeta-z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2(n-1)}|z|^2\zeta}{r^{2(n-1)}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2(n+1)}z}{|z|^2\zeta-r^{2(n+1)}z}\right)\right]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{4\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+\frac{z+|z|^2\zeta}{z-|z|^2\zeta}\right.\\
&\left.+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}+\frac{r^{2n}z}{r^{2n}z-|z|^2\zeta}+\frac{r^{2n}|z|^2\zeta}{z-r^{2n}|z|^2\zeta}\right)\right]d\xi d\eta\\
&+\frac{1}{4\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{z\overline{\zeta}+|z|^2}{z\overline{\zeta}-|z|^2}+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}z\overline{\zeta}}{r^{2n}z\overline{\zeta}-|z|^2}+\frac{r^{2n}|z|^2}{z\overline{\zeta}-r^{2n}|z|^2}\right.\right.\\
&\left.\left.+\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\right)\right]d\xi d\eta.
\end{aligned}
$$

If we restrict $Rew_{\overline{z}}(z)$ to the boundary ∂R, we obtain

$$
\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)d\xi d\eta=0.
$$

That is, we obtain Equation (4.7).

Finally, we need to satisfy the normalization conditions to see if this problem is uniquely solvable. For this, we calculate the following integrals.

$$
\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|z|=r}w(z)\frac{dz}{z} = & ic_0\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{dz}{z}+i\frac{1}{2\pi i}\int\limits_{|z|=r}(z+\overline{z})c_1\frac{dz}{z}\\
&+\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z}\frac{d\zeta}{\zeta}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\,\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\left(\zeta-z+\overline{\zeta-z}\right)\frac{dz}{z}\frac{d\zeta}{\zeta}\\
&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_0(\zeta)\,\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{dz}{z}\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\,\frac{1}{2\pi i}\int\limits_{|z|=r}\left(\zeta-z+\overline{\zeta-z}\right)\frac{dz}{z}\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\left(\zeta-z+\overline{\zeta-z}\right)\frac{dz}{z}d\xi d\eta\\
&+\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]\left(\zeta-z+\overline{\zeta-z}\right)\frac{dz}{z}d\xi d\eta.
\end{aligned}
$$

When calculating the above integrals, we use the following integrals computed by Vaitsiakhovich [16].

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]dz=-\frac{2r^2\zeta}{(1-r^2)},$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z}=-1,$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z^2}=\frac{2}{\zeta}+\frac{2r^2}{\left(1-r^2\right)\zeta},$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+K_2(z,\zeta)\right]dz=-\frac{2r^2}{\left(1-r^2\right)\zeta},$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+K_2(z,\zeta)\right]\frac{dz}{z}=1,$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+K_2(z,\zeta)\right]\frac{dz}{z^2}=2\overline{\zeta}+\frac{2r^2\overline{\zeta}}{1-r^2}.$$

When we substitute the values of the above integrals, we obtain

$$\frac{1}{2\pi i}\int\limits_{|z|=r}w(z)\frac{dz}{z}=ic_0-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\left[\zeta+\overline{\zeta}-\frac{2r^2(1-\zeta^2)}{(1-r^2)\zeta}\right]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_0(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[2\zeta+2\overline{\zeta}-\frac{2r^2(1-\zeta^2)}{(1-r^2)\zeta}\right]d\xi d\eta$$

$$+\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[2\zeta+2\overline{\zeta}-\frac{2r^2(1-\overline{\zeta}^2)}{(1-r^2)\overline{\zeta}}\right]d\xi d\eta,$$

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|z|=r}w(z)\frac{dz}{z}=&ic_0-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\left[\zeta+\overline{\zeta}-\frac{2r^2(1-\zeta^2)}{(1-r^2)\zeta}\right]\frac{d\zeta}{\zeta}\\&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_0(\zeta)\frac{d\zeta}{\zeta}\\&+\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right)\left(2\zeta+2\overline{\zeta}\right)d\xi d\eta\\&-\frac{1}{2\pi}\int\limits_{R}\left(\frac{f(\zeta)}{\zeta}\frac{2r^2(1-\zeta^2)}{(1-r^2)\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{2r^2(1-\overline{\zeta}^2)}{(1-r^2)\overline{\zeta}}\right)d\xi d\eta.\end{aligned}$$

We take the imaginary part, so we see that the first normalization condition

$$\frac{1}{2\pi i}\int\limits_{|z|=r}Imw(z)\frac{dz}{z}=c_0$$

is satisfied. To satisfy the second normalization condition, we compute the integral over $|z|=r$.

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|z|=r}w_{\overline{z}}(z)\frac{dz}{z}=&ic_1\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{dz}{z}+\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{dz}{z}\frac{d\zeta}{\zeta}\\&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{dz}{z}\frac{d\zeta}{\zeta}\\&-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\frac{dz}{z}d\xi d\eta\\&-\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1}{2\pi i}\int\limits_{|z|=r}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]\frac{dz}{z}d\xi d\eta.\end{aligned}$$

If we write the results computed by Vaitsiakhovich [16], we obtain

$$\frac{1}{2\pi i}\int\limits_{|z|=r} w_{\overline{z}}(z)\frac{dz}{z} = ic_1 - \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{d\zeta}{\zeta}.$$

Taking the imaginary part, we obtain the second normalization condition:

$$\int\limits_{|z|=r} Imw_{\overline{z}}(z)\frac{dz}{z} = c_1$$

The proof of the theorem is completed. □

4.2. SCHWARZ PROBLEM IN A RING DOMAIN FOR SECOND-ORDER LINEAR EQUATIONS

In this section, we investigate Schwarz problem for second-order linear equations in a ring domain. The solution of second-order Schwarz problem with homogeneous conditions is given in [20]:

$$\begin{aligned} w(z) &= -\frac{1}{2\pi}\int\limits_{R}\left\{\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\right. \\ &\left.+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]\right\}\left(\zeta-z+\overline{\zeta-z}\right)d\xi d\eta \\ &= T_R^2 f(z). \end{aligned}$$

We take the linear equation:

$$w_{\overline{z}\overline{z}} + A_1(z)w_{\overline{z}z} + A_2(z)w_{zz} + B_1(z)w_{\overline{z}} + B_2(z)w_z + C(z)w = f. \tag{4.8}$$

Assume there exists a function g such that $T_R^2 g = w$ for the linear equation. Then:

$$Ig + A_1(z)\Pi_R^1 g + A_2(z)\Pi_R^2 g + B_1(z)T_R^1 g + B_2(z)\Pi_R^1 g + C(z)T_R^2 g = f$$

or

$$\left(I + A_1(z)\Pi_R^1 + A_2(z)\Pi_R^2 + B_1(z)T_R^1 + B_2(z)\Pi_R^1 + C(z)T_R^2\right) g := f \tag{4.9}$$

where

$$T_R^1 g = -\frac{1}{2\pi}\int_R \left\{\frac{g(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right] + \frac{\overline{g(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} - 1 + K_2(z,\zeta)\right]\right\} d\xi d\eta,$$

$$\begin{aligned} T_R^2 g = &-\frac{1}{2\pi}\int_R \left\{\frac{g(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right]\right. \\ &\left. + \frac{\overline{g(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} - 1 + K_2(z,\zeta)\right]\right\}\left(\zeta - z + \overline{\zeta - z}\right) d\xi d\eta \end{aligned}$$

and

$$\begin{aligned} \Pi_R^1 g = \partial_z T_R^1 g(z) = &-\frac{1}{\pi}\int_R \left(\frac{g(\zeta)}{(\zeta-z)^2} + \frac{\overline{g(\zeta)}}{\left(1-z\overline{\zeta}\right)^2}\right) d\xi d\eta \\ &-\frac{1}{2\pi}\int_R \left[\frac{g(\zeta)}{\zeta}\partial_z K_1(z,\zeta) + \frac{\overline{g(\zeta)}}{\overline{\zeta}}\partial_z K_2(z,\zeta)\right] d\xi d\eta, \end{aligned}$$

$$\begin{aligned} \Pi_R^2 g = \partial_z T_R^2 g(z) = &-\frac{1}{2\pi}\int_R \frac{g(\zeta)}{\zeta}\left[\frac{2\zeta}{(\zeta-z)^2} + \partial_z K_1(z,\zeta)\right]\left(\zeta - z + \overline{\zeta} - \overline{z}\right) d\zeta d\eta \\ &-\frac{1}{2\pi}\int_R \frac{\overline{g(\zeta)}}{\overline{\zeta}}\left[\frac{2\overline{\zeta}}{(1-z\overline{\zeta})^2} + \partial_z K_2(z,\zeta)\right]\left(\zeta - z + \overline{\zeta} - \overline{z}\right) d\zeta d\eta \\ &+\frac{1}{2\pi}\int_R \frac{g(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right] d\zeta d\eta \\ &+\frac{1}{2\pi}\int_R \frac{\overline{g(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} - 1 + K_2(z,\zeta)\right] d\zeta d\eta \\ = &-\frac{1}{\pi}\int_R \left(\frac{g(\zeta)}{(\zeta-z)^2} + \frac{\overline{g(\zeta)}}{\left(1-z\overline{\zeta}\right)^2}\right)\left(\zeta - z + \overline{\zeta} - \overline{z}\right) d\xi d\eta \\ &+\frac{1}{2\pi}\int_R \left[\frac{g(\zeta)}{\zeta}\partial_z K_1(z,\zeta) + \frac{\overline{g(\zeta)}}{\overline{\zeta}}\partial_z K_2(z,\zeta)\right]\left(\zeta - z + \overline{\zeta} - \overline{z}\right) d\xi d\eta \end{aligned}$$

$$
-\frac{1}{2\pi}\int\limits_{R}\left\{\frac{g(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]\right.
$$
$$
\left.+\frac{\overline{g(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]\right\}d\xi d\eta.
$$

We need upper bounds for $\left\|\Pi_R^1\right\|_p$, $\left\|T_R^1\right\|_p$, $\left\|\Pi_R^2\right\|_p$ and $\left\|T_R^2\right\|_p$. In Section 3.2, we found upper bounds for $\|T_{1R}\|_p$, $\|T_{2R}\|_p$. Hence,

$$
\begin{aligned}
\left\|T_R^1 g(z)\right\|_p &\le \|T_{1R}g(z)\|_p + \|T_{2R}g(z)\|_p \\
&\le M_1\left[\pi\left(1-r^2\right)\right]^{\frac{1}{p}}\|g\|_p + A_1\left[\pi\left(1-r^2\right)\right]^{\frac{1}{p}}\|g\|_p \\
\left\|T_R^1 g(z)\right\|_p &\le \left[\pi(1-r^2)\right]^{\frac{1}{p}}.(M_1+A_1).\|g\|_p. \\
\left\|T_R^2 g(z)\right\|_p &\le 4\left\|T_R^1 g(z)\right\|_p = 4\left[\pi(1-r^2)\right]^{\frac{1}{p}}.(M_1+A_1).\|g\|_p.
\end{aligned}
$$

So, T_R^1 and T_R^2 are bounded.

Also, $T_R^1 g(z)$ is the sum of $T_{1R}g(z)$ and $T_{2R}g(z)$. In Section 3.2, we showed that $T_{1R}g(z)$ and $T_{2R}g(z)$ are equicontinuous. The sum of two equicontinuous functions are also equicontinuous. Thus, $T_R^1 g(z)$ is equicontinuous. We know that $T_R^2 g(z) = \left(\zeta - z + \overline{\zeta - z}\right)T_R^1 g(z)$. Similar to the operations in Page 24, it can be shown to be equicontinuous.
Besides, in Section 3.2 we know that $\|\Pi_{1R}\|_p$ and $\|\Pi_{2R}\|_p$ are bounded. Therefore,

$$
\begin{aligned}
\left\|\Pi_R^1 g(z)\right\|_p &\le \|\Pi_{1R}g(z)\|_p + \|\Pi_{2R}g(z)\|_p \\
\left\|\Pi_R^2 g(z)\right\|_p &\le 4\|\Pi_{1R}g(z)\|_p + 4\|\Pi_{2R}g(z)\|_p + \left\|T_R^1 g(z)\right\|_p.
\end{aligned}
$$

Thus, Π_R^1 and Π_R^2 are bounded. Now we prove the existence and uniqueness theorem for the solutions of Schwarz problem in a ring domain for second-order linear equations.

Theorem 19. *There exists a unique solution $w \in W^{p,2}(R)$ of the Schwarz problem for Equation* (4.8) *subject to the homogeneous boundary conditions*

$$Rew(z) = 0 \ on \ \partial R, \ \frac{1}{2\pi i}\int\limits_{|z|=r} Imw(z)\frac{dz}{z} = 0$$

$$Rew_{\overline{z}}(z) = 0 \ on \ \partial R, \ \frac{1}{2\pi i}\int\limits_{|z|=r} Imw_{\overline{z}}(z)\frac{dz}{z} = 0$$

where $A_1(z)$, $A_2(z)$, $B_1(z)$ and $B_2(z)$ are bounded measurable functions with

$$|A_1(z)| + |A_2(z)| \leq q_0 < 1$$

satisfying

$$q_0\|\widehat{\Pi}_R\|_{L^p(R)} < 1$$

and $C, f \in L^p(R)$, $p > 2$.

Proof. Assume that $w = T_R^2 g$, $g \in L^p(R)$ is a solution of Equation (4.8). Then we observe that g should be the solution of

$$(I + \widehat{\Pi}_R + \widehat{K}_R)g(z) = f(z) \tag{4.10}$$

where

$$\widehat{\Pi}_R g = A_1(z)\Pi_R^1 g + A_2(z)\Pi_R^2 g$$
$$\widehat{K}_R g = B_1(z)T_R^1 g + B_2(z)\Pi_R^1 g + C(z)T_R^2 g.$$

Thus we have converted Equation (4.8) into an integral equation. $\widehat{K}_R$ is a compact operator by Arzela-Ascoli theorem, and $\widehat{\Pi}_R$ is a bounded strongly singular operator of Calderon-Zygmund type. $I + \widehat{\Pi}_R$ is invertible if there exists q_0 such that

$$q_0\|\widehat{\Pi}_R\|_{L^p(R)} < 1 \text{ for } p > 2.$$

So $I + \widehat{\Pi}_R + \widehat{K}_R$ is a Fredholm operator with index-zero. Hence Equation (4.8) has a unique solution of the form $w = T_R^2 g$ where $g \in L^p(R)$, $p > 2$, is a solution of Equation (4.10). $\square$

5. SCHWARZ PROBLEM IN A RING DOMAIN FOR HIGHER-ORDER EQUATIONS

Now, we investigate higher-order equations. To examine higher-order equations, we need to know the properties of higher-order operators. Thus, in this chapter, we also give the properties of higher-order operators.

5.1. SCHWARZ PROBLEM IN A RING DOMAIN FOR INHOMOGENEOUS CAUCHY-RIEMANN EQUATIONS

In Section 4.1 an integral representation for $w \in W^{p,1}(R;\mathbb{C})$ depending on $w_{\overline{z}}$ is given. This helps us to find an integral representation for $w \in W^{p,n}(R;\mathbb{C})$ in terms of the derivatives of w with respect to $\overline{z}$. Using an induction method we obtain the integral representation

$$\begin{aligned}
w(z) =& i\sum_{j=0}^{n-1}\frac{1}{j!}c_j(z+\overline{z})^j - \frac{1}{2\pi i}\int_{|\zeta|=r}\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}Re\partial_{\overline{\zeta}}^{j}w(\zeta)(\zeta - z+\overline{\zeta}-\overline{z})^j\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi i}\int_{\partial R}\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}Re\partial_{\overline{\zeta}}^{j}w(\zeta)(\zeta - z+\overline{\zeta}-\overline{z})^j\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2\pi}\int_R\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial_{\overline{\zeta}}^{n}w(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right](\zeta - z+\overline{\zeta}-\overline{z})^{n-1}d\xi d\eta\\
&-\frac{1}{2\pi}\int_R\frac{(-1)^{n-1}}{(n-1)!}\frac{\overline{\partial_{\overline{\zeta}}^{n}w(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right](\zeta - z+\overline{\zeta}-\overline{z})^{n-1}d\xi d\eta.
\end{aligned} \tag{5.1}$$

It is easy to observe that

$$\begin{aligned}
\partial_{\overline{z}}w(z) =& i\sum_{j=1}^{n-1}\frac{1}{(j-1)!}c_j(z+\overline{z})^{j-1} - \frac{1}{2\pi i}\int_{|\zeta|=r}\sum_{j=1}^{n-1}\frac{(-1)^{j-1}}{(j-1)!}Re\partial_{\overline{\zeta}}^{j-1}w(\zeta)(\zeta - z+\overline{\zeta}-\overline{z})^{j-1}\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi i}\int_{\partial R}\sum_{j=1}^{n-1}\frac{(-1)^{j-1}}{(j-1)!}Re\partial_{\overline{\zeta}}^{j-1}w(\zeta)(\zeta - z+\overline{\zeta}-\overline{z})^{j-1}\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}
\end{aligned}$$

$$-\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-2}}{(n-2)!}\frac{\partial_{\zeta}^{n}w(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right](\zeta-z+\overline{\zeta}-\overline{z})^{n-2}d\xi d\eta$$

$$-\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-2}}{(n-2)!}\frac{\overline{\partial_{\zeta}^{n}w(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right](\zeta-z+\overline{\zeta}-\overline{z})^{n-2}d\xi d\eta.$$

We compute iteratively $\partial_{\overline{z}}^{n-1}w(z)$ as follows.

$$\begin{aligned}\partial_{\overline{z}}^{n-1}w(z) = {} & \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_{n-1}(\zeta)\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_{n-1}(\zeta)\frac{d\zeta}{\zeta}\\ & +ic_{n-1}-\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right]d\xi d\eta\\ & -\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}-1+K_2(z,\zeta)\right]d\xi d\eta,\end{aligned}$$

and differentiation of the above formula with respect to $\overline{z}$ gives

$$\partial_{\overline{z}}^{n}w = f(z).$$

Hence Equation (5.1) is the required integral representation for $W^{p,n}(R;\mathbb{C})$. Thus we may employ this formula for the solutions of the boundary value problems, similar to our previous discussion for the nonhomogeneous Cauchy-Riemann equations. Now recall the statement of the Schwarz problem for higher-order inhomogeneous Cauchy-Riemann equation in a ring domain given in [20]:

"*The Schwarz problem for inhomogeneous higher-order Cauchy-Riemann equation in a ring domain is to find the solution of*

$$\partial_{\overline{z}}^{n}w = f(z)$$

subject to

$$Re\partial_{\overline{z}}^{j}w(z) = \gamma_j(z) \text{ on } \partial R\ ;\ \frac{1}{2\pi i}\int\limits_{|z|=r} Im\partial_{\overline{z}}^{j}w(z)\frac{dz}{z} = c_j$$

for $\gamma_j(z)\in C(\partial R;\mathbb{R})$ *and* $c_j\in\mathbb{R},\ j=0,1,...,n-1.$"

It is trivial that the proposed solution is

$$
\begin{aligned}
w(z) =& i\sum_{j=0}^{n-1}\frac{1}{j!}c_j(z+\overline{z})^j - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}\gamma_j(\zeta)(\zeta - z + \overline{\zeta} - \overline{z})^j\frac{d\zeta}{\zeta}\\
&+ \frac{1}{2\pi i}\int\limits_{\partial R}\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}\gamma_j(\zeta)(\zeta - z + \overline{\zeta} - \overline{z})^j\left[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}\\
&- \frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-1}}{(n-1)!}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right](\zeta - z + \overline{\zeta} - \overline{z})^{n-1}d\xi d\eta\\
&- \frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-1}}{(n-1)!}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} - 1 + K_2(z,\zeta)\right](\zeta - z + \overline{\zeta} - \overline{z})^{n-1}d\xi d\eta.
\end{aligned}
\tag{5.2}
$$

since $\partial_{\overline{z}}^n w = f(z)$. Thus we have to observe if the auxiliary conditions are satisfied. Hence we start with the computation of the derivative of Equation (5.2) with respect to $\overline{z}$ of order s where $0 \leq s \leq n-1$.

$$
\begin{aligned}
\partial_{\overline{z}}^s w(z) =& i\sum_{j=s}^{n-1}\frac{1}{(j-s)!}c_j(z+\overline{z})^{j-s} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta - z + \overline{\zeta} - \overline{z})^{j-s}\frac{d\zeta}{\zeta}\\
&+ \frac{1}{2\pi i}\int\limits_{\partial R}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta - z + \overline{\zeta} - \overline{z})^{j-s}\left[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\right]\frac{d\zeta}{\zeta}\\
&- \frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right](\zeta - z + \overline{\zeta} - \overline{z})^{n-s-1}d\xi d\eta\\
&- \frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} - 1 + K_2(z,\zeta)\right](\zeta - z + \overline{\zeta} - \overline{z})^{n-s-1}d\xi d\eta.
\end{aligned}
$$

Restricting the real part of $\partial_{\overline{z}}^s w(z)$ to the boundary ∂R we obtain

$$
2Re\partial_{\overline{z}}^s w(z)\bigg|_{\partial R} = \Bigg\{ - \frac{1}{\pi i}\int\limits_{|\zeta|=r}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta - z + \overline{\zeta} - \overline{z})^{j-s}\frac{d\zeta}{\zeta}
$$

$$
+\frac{1}{2\pi i}\int\limits_{\partial R}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}2\left(\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}} - 1\right)\gamma_j(\zeta)(\zeta - z + \overline{\zeta} - \overline{z})^{j-s}\frac{d\zeta}{\zeta}
$$

$$-\frac{1}{\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\left[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right](\zeta-z+\overline{\zeta}-\overline{z})^{n-s-1}d\xi d\eta\Bigg\}\Bigg|_{\partial R}$$

$$=2\gamma_s(z)-\frac{1}{\pi i}\int\limits_{|\zeta|=r}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta-z+\overline{\zeta}-\overline{z})^{j-s}\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\left[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right](\zeta-z+\overline{\zeta}-\overline{z})^{n-s-1}d\xi d\eta.$$

Hence the solvability conditions for the inhomogeneous polyanalytic equation related with $Re\partial_{\overline{z}}^{s}w(z)\big|_{\partial R}=\gamma_s(z)$ are

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta-z+\overline{\zeta}-\overline{z})^{j-s}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\left[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\right](\zeta-z+\overline{\zeta}-\overline{z})^{n-s-1}d\xi d\eta=0,$$

for $j=0,1,2,...,n-1$. We use a similar approach to the one Vaitekhovich [17] has given to check the normalization conditions

$$\frac{1}{2\pi i}\int\limits_{|z|=r}Im\partial_{\overline{z}}^{j}w(z)\frac{dz}{z}=c_j,\qquad j=0,1,2,...,n-1.$$

To give an idea about this computation, we assume $n=2$. Then we have to check if the following holds:

$$\frac{1}{2\pi i}\int\limits_{|z|=r}Imw(z)\frac{dz}{z}=c_0,\quad \frac{1}{2\pi i}\int\limits_{|z|=r}Imw_{\overline{z}}(z)\frac{dz}{z}=c_1.$$

These validities of normalization condions are available in Section 4.1. So combining the computations presented above with the normalization conditions we obtain

Theorem 20. *[20] The inhomogeneous Schwarz problem for higher-order equation*

$$\partial_{\overline{z}}^{n}w=f(z)$$

in a ring domain, subject to

$$Re\partial_{\bar{z}}^{j}w(z)=\gamma_j(z)\ on\ \partial R\ ;\ \frac{1}{2\pi i}\int\limits_{|z|=r} Im\partial_{\bar{z}}^{j}w(z)\frac{dz}{z}=c_j$$

for $\gamma_j(z)\in C(\partial R;\mathbb{R})$ *and* $c_j\in\mathbb{R},\ \ j=0,1,...,n-1$ *has a unique solution in* $W^{p,n}(R;\mathbb{C})$ *if the set of solvability conditions*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta-z+\bar{\zeta}-\bar{z})^{j-s}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\left[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\right](\zeta-z+\bar{\zeta}-\bar{z})^{n-s-1}d\xi d\eta=0$$

hold for $j=0,1,...,n-1$.

5.2. PROPERTIES OF HIGHER-ORDER OPERATORS

The operator $T_1{:}L^p(\mathbb{D})\to L^p(\mathbb{D})$

$$T_1f(z)=-\frac{1}{2\pi}\int\limits_{\mathbb{D}}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right]d\xi d\eta$$

has been defined by Vekua [1]. The iterations of T_1 have been denoted with the symbol S_n or T_k. Its properties are investigated by several researchers [38,37]. We will prefer the notation T_k in our computations for iterations of T_1. For $k\in\mathbb{N}$

$$T_kf(z)=\frac{(-1)^k}{2\pi(k-1)!}\iint\limits_{\mathbb{D}}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right]\left(\zeta-z+\bar{\zeta}-\bar{z}\right)^{k-1}d\xi d\eta$$

and $T_0f(z)=f(z)$. These operators satisfy

(i) $\partial_{\bar{z}}^{l}T_kf=T_{k-l}f,\quad 1\le l\le k,$

(ii) $Re\partial_{\bar{z}}^{l}T_kf=0\ on\ \partial\mathbb{D},\quad 0\le l\le k-1,$

(iii) $Im\partial_{\bar{z}}^{l}T_kf(0)=0,\quad 0\le l\le k-1.$

Besides

$$\partial_z^l T_k f(z) = \sum_{\lambda=0}^{l} \binom{l}{\lambda} \frac{(-1)^{k-\lambda}(l-\lambda)!}{(k-\lambda-1)!} \frac{1}{\pi} \iint_{\mathbb{D}} (\zeta - z + \overline{\zeta - z})^{k-\lambda-1} \left[\frac{f(\zeta)}{(\zeta - z)^{l-\lambda+1}} + \frac{\overline{\zeta}^{l-\lambda-1}}{(1-\overline{z}\zeta)^{l-\lambda+1}} \right] d\xi d\eta + \frac{(-1)^{k-l-1}}{(k-l-1)!} \frac{1}{2\pi} \iint_{\mathbb{D}} (\zeta - z + \overline{\zeta - z})^{k-l-1} \left[\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}} \right] d\xi d\eta$$

is a weakly singular integral operator for $0 \leq l \leq k-1$ since $k-\lambda-1-(l-\lambda+1) = k-l-2 \geq -1$, however

$$\begin{aligned} \Pi_k f(z) &:= \partial_z^k T_k f(z) \\ &= \frac{(-1)^k k!}{\pi} \iint_{\mathbb{D}} \left[\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} \right. \\ &\quad \left. + \left(\frac{\zeta - z + \overline{\zeta} - \overline{z}}{1 - z\overline{\zeta}} \overline{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\overline{\zeta})^2} \right] d\xi d\eta \end{aligned}$$

is a Calderon-Zygmund type strongly singular integral operator. It is proved [38,37] that T_k are compact and Π_k are bounded singular operators over $\mathbb{D}$.

Now we need to generalize the integral operator T_R for the particular solutions of $\partial_{\overline{z}}^n w = f(z)$ in R. Let us start with a simple case. Take the equation

$$\partial_{\overline{z}}^2 w = f(z).$$

We may convert it into a system of equations:

$$\partial_{\overline{z}} w = w_1,$$

$$\partial_{\overline{z}} w_1 = f(z).$$

Their solutions are

$$w = T_R w_1,$$

$$w_1 = T_R f.$$

Thus

$$w = T_R\left(T_R f(z)\right) = T_R^2 f(z).$$

Iterating the operator T_R with itself using the rule $T_R^k f(z) = T_R\left(T_R^{k-1} f(z)\right)$ gives us

$$\begin{aligned} T_R^k f(z) = &\frac{(-1)^k}{2\pi(k-1)!}\int_R \left(\zeta - z + \overline{\zeta} - \overline{z}\right)^{k-1}\left[\frac{f(\zeta)}{\zeta}\left(\frac{\zeta+z}{\zeta-z} + 1 + K_1(z,\zeta)\right)\right. \\ &\left. + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\left(\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} - 1 + K_2(z,\zeta)\right)\right] d\xi d\eta \\ := &T_{1R}^k f(z) + T_{2R}^k f(z) \end{aligned} \tag{5.3}$$

where

$$T_{1R}^k f(z) = \frac{(-1)^k}{2\pi(k-1)!}\int_R \left(\zeta - z + \overline{\zeta} - \overline{z}\right)^{k-1}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right] d\xi d\eta,$$

$$\begin{aligned} T_{2R}^k f(z) = &\frac{(-1)^k}{2\pi(k-1)!}\int_R \left(\zeta - z + \overline{\zeta} - \overline{z}\right)^{k-1}\left[\frac{f(\zeta)}{\zeta}\left(1 + K_1(z,\zeta)\right)\right. \\ &\left. + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\left(-1 + K_2(z,\zeta)\right)\right] d\xi d\eta. \end{aligned}$$

Similar to the case in which the problem is discussed in $\mathbb{D}$, we may derive the following properties for $f \in L^p(R)$, $p > 2$ and $k \in \mathbb{N}$:

(i) $\partial_{\overline{z}}^l T_R^k f(z) = \partial_{\overline{z}}^l T_{1R}^k f(z) + \partial_{\overline{z}}^l T_{2R}^k f(z) = T_{1R}^{k-l} f(z) + T_{2R}^{k-l} f(z) = T_R^{k-l} f(z)$,

(ii) $Re\,\partial_{\overline{z}}^l T_R^k f(z) = Re\,T_R^{k-l} f(z) = 0$ on ∂R for $0 \le l < k$,

(iii) $\frac{1}{2\pi i}\int\limits_{|z|=r} Im\,\partial_{\overline{z}}^l T_R^k f(z)\frac{dz}{z} = 0,\ 0 \le l < k$

under the solvability condition

$$\int_R \left[\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\right](\zeta - z + \overline{\zeta} - \overline{z})^{n-s-1} d\xi d\eta\bigg|_{\partial R} = 0$$

for $s = 0, 1, 2, ..., n-1$. Let us note that these three properties may also be derived starting with the definition of T_R^k. Previously it is proved in [6,37] that $\partial_{\bar{z}}^l T_{\mathbb{D}}^k$ are compact operators for $0 \le l < k$. They are also compact in subsets of $\mathbb{D}$. Hence $\partial_{\bar{z}}^l T_{1R}^k$, $0 \le l < k$, are also compact in R. To discuss the properties of $\partial_{\bar{z}}^l T_{2R}^k$, we must know the properties of $K_1(z,\zeta)$ and $K_2(z,\zeta)$. In Section 3.2, we have proved that these series are convergent and they define holomorphic functions of ζ in R. So $\|T_{2R}^k\|$ is bounded, hence $\partial_{\bar{z}}^l T_R^k$ are compact for $0 \le l < k$. Now let us compute

$$\begin{aligned}
\Pi_R^k f(z) :=& \partial_z^k T_R^k f(z) = \partial_z^k T_{1R}^k f(z) + \partial_z^k T_{2R}^k f(z) \\
=& \frac{(-1)^k k}{\pi} \int_R \left[\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} \right. \\
&+ \left. \left(\frac{\zeta - z + \overline{\zeta} - \overline{z}}{1 - z\overline{\zeta}} \overline{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\overline{\zeta})^2} \right] d\xi d\eta \\
&+ \partial_z^k \left[\frac{(-1)^k}{2\pi (k-1)!} \int_R \left(\zeta - z + \overline{\zeta} - \overline{z} \right)^{k-1} \left[\frac{f(\zeta)}{\zeta} (1 + K_1(z,\zeta)) \right. \right. \\
&+ \left. \left. \frac{\overline{f(\zeta)}}{\overline{\zeta}} (-1 + K_2(z,\zeta)) \right] d\xi d\eta. \right.
\end{aligned}$$

Then $\partial_z^k T_{1R}^k$: $L^p(R) \to L^p(R)$, $p > 2$ is a Calderon-Zygmund type strongly singular operator and $\partial_z^k T_{2R}^k$ is a bounded operator. Thus Π_R^k: $L^p(R) \to L^p(R)$, $p > 2$, is a bounded integral operator.

5.3. SCHWARZ PROBLEM IN A RING DOMAIN FOR HIGHER-ORDER LINEAR EQUATIONS

In this section we study the solvability of the Schwarz problem of higher-order linear equation stated as:

"*Find $w \in W^{p,k}(R)$ as a solution to the $k-th$ order complex differential equation*

$$\partial_{\bar{z}}^k w + \sum_{j=1}^{k} \left[q_{1j}(z) \partial_z^j \partial_{\bar{z}}^{k-j} w + q_{2j}(z) \partial_{\bar{z}}^j \partial_z^{k-j} \overline{w} \right]$$

$$+\sum_{l=0}^{k}\sum_{m=0}^{l}\left[a_{ml}(z)\partial_z^m\partial_{\overline{z}}^{l-m}w+b_{ml}(z)\partial_z^m\partial_{\overline{z}}^{l-m}\overline{w}\right]=f(z)\ in\ R \tag{5.4}$$

subject to the homogeneous Schwarz boundary conditions

$$Re\partial_{\overline{z}}^l w=0\ on\ \partial R,\ \frac{1}{2\pi i}\int\limits_{|z|=r} Im\partial_{\overline{z}}^l w(z)\frac{dz}{z}=0,\ 0\le l\le k-1 \tag{5.5}$$

where

$$a_{ml},b_{ml}\in L^p(R),\ f\in L^p(R) \tag{5.6}$$

and q_{1j} and q_{2j} for $j=1,2,...,k$ are measurable bounded functions satisfying

$$\sum_{j=1}^{k}\left(\left|q_{1j}(z)\right|+\left|q_{2j}(z)\right|\right)\le q_0<1 \tag{5.7}$$

in R."

To find the solution we need to convert Equation (5.4) into a singular integral equation. We recall the definition of $w=T_R^k g$ is the unique solution of

$$\partial_{\overline{z}}^k w=g$$

satisfying the homogeneous boundary conditions.Then we can find

$$\partial_z^j\partial_{\overline{z}}^{k-j}w=\partial_z^j\partial_{\overline{z}}^{k-j}\left(T_R^k g\right)=\partial_z^j\left(T_R^j g\right)=\Pi_R^j g,$$

and

$$\partial_{\overline{z}}^j\partial_z^{k-j}\overline{w}=\partial_{\overline{z}}^j\left(\overline{\partial_{\overline{z}}^{k-j}\left(T_R^k g\right)}\right)=\partial_{\overline{z}}^j\overline{\left(T_R^j g\right)}=\overline{\Pi_R^j g}$$

for $1\le j\le k$. Similarly

$$\partial_z^m\partial_{\overline{z}}^{l-m}w=\partial_z^m\partial_{\overline{z}}^{l-m}\left(T_R^k g\right)=\partial_z^m T_R^{k-l+m}g,$$

and

$$\partial_z^{l-m}\partial_{\overline{z}}^m\overline{w}=\partial_{\overline{z}}^m\left(\overline{\partial_{\overline{z}}^{l-m}T_R^k g}\right)=\partial_{\overline{z}}^m\overline{T_R^{k-l+m}g}$$

for $1 \leq m \leq l \leq k$. Thus we obtain

$$\left(I + \widehat{\Pi}_R + \widehat{K}_R\right) g = f. \tag{5.8}$$

So we may combine the above discussion as

Lemma 4. *The Schwarz problem* (5.4)-(5.5) *stated in the annular domain R is equivalent to the singular integral equation*

$$\left(I + \widehat{\Pi}_R + \widehat{K}_R\right) g = f \tag{5.9}$$

where $w = T_R^k g$,

$$\widehat{\Pi}_R g = \sum_{j=1}^{k} \left(q_{1j} \Pi_R^j g + q_{2j} \overline{\Pi_R^j g} \right), \tag{5.10}$$

and

$$\widehat{K}_R g = \sum_{l=0}^{k} \sum_{m=0}^{l} \left[a_{ml} \partial_z^m T_R^{k-l+m} g + b_{ml} \partial_{\overline{z}}^m \overline{T_R^{k-l+m} g} \right]. \tag{5.11}$$

So Equation (5.4) is transformed into the singular integral equation (5.9). Now we have to recall the Fredholm alternative [33] to discuss the invertibility of $I + \widehat{\Pi}_R + \widehat{K}_R$.

Lemma 5. *If*

$$q_0 \max_{1 \leq j \leq k} \|\widehat{\Pi}_R^j\|_{L^p(R)} < 1 \tag{5.12}$$

then $I + \widehat{\Pi}_R$ *is invertible for* $p > 2$.

Proof. We need the norm of the operator $\widehat{\Pi}_R$:

$$\begin{aligned} \|\widehat{\Pi}_R\|_{L^p(R)} &\leq \sum_{j=1}^{k} \left(|q_{1j}| + |q_{2j}| \right) \|\widehat{\Pi}_R^j\|_{L^p(R)} \\ &\leq q_0 \max_{1 \leq j \leq k} \|\widehat{\Pi}_R^j\|_{L^p(R)}. \end{aligned}$$

Under the restriction (5.12),

$$\|\widehat{\Pi}_R\|_{L^p(R)} < 1.$$

So $I + \widehat{\Pi}_R$ is invertible. □

Now, for the invertibility of $I + \widehat{\Pi}_R + \widehat{K}_R$, we will see if $\widehat{K}_R$ is compact. We have decomposed the operator T_R^k into two parts as in (5.3). The part

$$T_{1R}^k f(z) = \frac{(-1)^k}{2\pi(k-1)!}\int\limits_R \left(\zeta - z + \overline{\zeta} - \overline{z}\right)^{k-1}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right]d\xi d\eta$$

has been studied previously [6,37] in a simply connected domain. Following the line of that proof we observe that $\partial_z^l T_{1R}^k$ is a compact operator on $L^p(R)$.
The second part has the form

$$\begin{aligned}T_{2R}^k f(z) =& \frac{(-1)^k}{2\pi(k-1)!}\int\limits_R \left(\zeta - z + \overline{\zeta} - \overline{z}\right)^{k-1}\left[\frac{f(\zeta)}{\zeta}\left(1+K_1(z,\zeta)\right)\right.\\ &\left.+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left(-1+K_2(z,\zeta)\right)\right]d\xi d\eta.\end{aligned}$$

$K_1(z,\zeta)$ and $K_2(z,\zeta)$ are analytic functions in R by Montel theorem [32]. So

$$T_R^k f(z) = T_{1R}^k f(z) + T_{2R}^k f(z)$$

which defines the compact operators T_{1R}^k and T_{2R}^k. Eventually

Theorem 21. *Under the condition* (5.6) $\widehat{K}_R$ *is a compact operator in* $L^p(R)$ *for* $p > 2$.

Proof. As in [6,37] we know that $\partial_z^l T_R^k g$ is equicontinuous on R. Besides Arzela-Ascoli theorem holds. Thus $\partial_z^l T_R^k$ for $0 \le l \le k-1$ are compact operators on $L^p(R)$. Now let us compute $\|\widehat{K}_R\|$:

$$|\widehat{K}_R g|^p \le \sum_{l=0}^{k}\sum_{m=0}^{l}\left(|a_{ml}(z)|\left|\partial_z^m T_R^{k-l+m} g\right| + |b_{ml}(z)|\left|\partial_z^m T_R^{k-l+m}\right|\right)^p$$

or by [6,37] we obtain

$$\|\widehat{K}_R g\|_{L^p(R)}^p \le C_1\|g\|_{L^p(R)}^p \int\limits_R \left[\left(\sum_{l=0}^{k-1}\sum_{m=0}^{l}|a_{ml}(z)|\right)^p + \left(\sum_{l=0}^{k-1}\sum_{m=0}^{l}|b_{ml}(z)|\right)^p\right]dxdy.$$

Using the Minkowski inequality we obtain

$$\|\widehat{K}_R g\|_{L^p(R)} \le M\|g\|_{L^p(R)}.$$

Let $\{g_n\}$ be a sequence of bounded functions in $L^p(R)$. Then $\{\partial_z^l T_R^k g_n\}$ is bounded in $L^p(R)$. By the Arzela-Ascoli theorem it has a convergent subsequence on R. Hence $\widehat{K}_R$ is a compact operator from $L^p(R)$ into itself for $p > 2$ and is a solution of (5.9). □

Now, we can state the unique solution of Equation (5.8).

Theorem 22. *If* (5.12) *holds, the equation* (5.4) *subject to the homogeneous conditions has a solution of the form* $w = T_R^k g$ *where* $g \in L^p(R)$, $p > 2$ *and the solvability conditions*

$$\frac{1}{2\pi}\int_R \frac{(-1)^{n-s-1}}{(n-s-1)!}\left[\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\right](\zeta - z + \overline{\zeta} - \overline{z})^{n-s-1} d\xi d\eta = 0$$

for $s = 0, 1, 2, ..., n-1$ *are satisfied.*

Proof. Since $\widehat{K}_R$ is compact, the operator $I + \widehat{\Pi}_R + \widehat{K}_R$ is the sum of the invertible operator $I + \widehat{\Pi}_R$ and compact operator $\widehat{K}_R$ which implies that $I + \widehat{\Pi}_R + \widehat{K}_R$ is a Fredholm operator with index-zero. The Fredholm alternative applies to singular integral equation (5.9). Thus $w = T_R^k g$ is the solution of (5.4) subject to the homogeneous conditions. □

6. SCHWARZ PROBLEM IN A RING DOMAIN FOR NONLINEAR HIGHER-ORDER EQUATIONS

The boundary value problems for nonlinear elliptic equations have been studied by several researchers, for example, Mshimba [39,40,41], Akal and Begehr [7], Aksoy, Begehr, Çelebi [8], in the books by Wen [42] and others. In this chapter we consider the Schwarz boundary value problem for a nonlinear differential equation in a ring domain.
Thus we start with elliptic equation

$$\partial_{\bar{z}}^{n} w = F(z, w, D^{\alpha_1} w, ..., D^{\alpha_n} w) \tag{6.1}$$

of order n in a ring domain R. As in [8] we employ the notations

$$D = (\partial_z, \partial_{\bar{z}}), \ \alpha_j = (k, l), \ \left|\alpha_j\right| = k + l = j, \ j = 0, 1, 2, ...n$$

with restriction $(k, l) \neq (0, n)$. To simplify the notation, we take

$$\{D^{\alpha_1} w, D^{\alpha_2} w, ..., D^{\alpha_n} w\} = \left\{w_{k,l}\right\}.$$

Hence Equation (6.1) may be written as

$$\partial_{\bar{z}}^{n} w = F(z, \left\{w_{k,l}\right\}). \tag{6.2}$$

To discuss the boundary value problem, we need some assumptions on $F(z, \left\{w_{k,l}\right\})$ defined in $R = \{z : r < |z| < 1\}$:
Assumptions:

(i) The function $F(z, \left\{w_{k,l}\right\})$ is continuous with respect to all of its variables,

(ii) For multiples $(z, \left\{w_{k,l}\right\})$ if $\left\{w_{k,l}\right\} \in L^p(R)$, $p > 2$, then $F(z, \left\{w_{k,l}\right\}) \in L^p(R)$,

(iii) The function $F(z, \left\{w_{k,l}\right\})$ satisfies the Lipschitz condition

$$\left|F(z,\{w_{k,l}\})-F(z,\{w^*_{k,l}\})\right| \leq \sum_{k+l=0}^{n} L_{kl}\left|w_{k,l}-w^*_{k,l}\right| \leq L\sum_{k+l=0}^{n}\left|w_{k,l}-w^*_{k,l}\right|$$

where $L=\max_{k+l\leq n} L_{kl}$, $(k,l)\neq(0,n)$ in which L_{kl} are constants.

Now we define the relevant Schwarz problem:

"*Find a solution* $w\in W^{n,p}(R)$ *with* $2<p<\infty$ *of Equation* (6.2) *subject to*

$$Re\partial^{\mu}_{\bar{z}}w=\gamma_{\mu} \ on \ \partial R, \ \frac{1}{2\pi i}\int\limits_{|z|=r} Im\partial^{\mu}_{\bar{z}}w(z)\frac{dz}{z}=c_{\mu} \tag{6.3}$$

for $\gamma_{\mu}(z)\in C(\partial R;\mathbb{R})$ *and* $c_{\mu}\in\mathbb{R}$, $\mu=0,1,...,n-1$."

To solve this problem we employ the operators

$$\begin{aligned} T^n_R w(z) = &-\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^n_{\bar{\zeta}}w(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z}+1+K_1(z,\zeta)\right](\zeta-z+\bar{\zeta}-\bar{z})^{n-1}d\xi d\eta \\ &-\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-1}}{(n-1)!}\frac{\overline{\partial^n_{\bar{\zeta}}w(\zeta)}}{\bar{\zeta}}\left[\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}-1+K_2(z,\zeta)\right](\zeta-z+\bar{\zeta}-\bar{z})^{n-1}d\xi d\eta. \end{aligned}$$

As in [8] we transform the Schwarz problem for nonlinear equation into the following system of integro-differential equations:

$$\begin{aligned} w(z) &= \phi_w(z)+T^n_R F(z,\{w_{k,l}\})(z) \\ w_{k,l}(z) &= \partial^k_z\partial^l_{\bar{z}}\phi_w(z)+\partial^k_z T^{n-l}_R F(z,\{w_{k,l}\})(z) \end{aligned}$$

for $0\leq k+l\leq n$, with the restriction $(k,l)\neq(0,n)$ and

$$\begin{aligned} \phi_w(z) = i&\sum_{j=0}^{n-1}\frac{1}{j!}c_j(z+\bar{z})^j-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}Re\partial^j_{\bar{\zeta}}w(\zeta)(\zeta-z+\bar{\zeta}-\bar{z})^j\frac{d\zeta}{\zeta} \\ &+\frac{1}{2\pi i}\int\limits_{\partial R}\sum_{j=0}^{n-1}\frac{(-1)^j}{j!}Re\partial^j_{\bar{\zeta}}w(\zeta)(\zeta-z+\bar{\zeta}-\bar{z})^j\left[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\right]\frac{d\zeta}{\zeta} \end{aligned} \tag{6.4}$$

It is trivial that ϕ_w is a polyanalytic function. The relevant solvability condition is

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\sum_{j=s}^{n-1}\frac{(-1)^{j-s}}{(j-s)!}\gamma_j(\zeta)(\zeta-z+\overline{\zeta}-\overline{z})^{j-s}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{2\pi}\int\limits_{R}\frac{(-1)^{n-s-1}}{(n-s-1)!}\left[\frac{F(\zeta,\{w_{k,l}(\zeta)\})}{\zeta}\right.$$
$$\left.+\frac{\overline{F(\zeta,\{w_{k,l}(\zeta)\})}}{\overline{\zeta}}\right](\zeta-z+\overline{\zeta}-\overline{z})^{n-s-1}d\xi d\eta=0, \tag{6.5}$$

for $j=0,1,2,...,n-1$.

Now we may discuss the solvability of the Schwarz problem. We use the notation given in [8]. Firstly we introduce the function space $\mathfrak{L}^{\mathfrak{p}}(R)$, $p>2$:

$$\mathfrak{L}^{\mathfrak{p}}(R)=\left(\{w_{k,l}\}:w_{k,l}\in L^p(R),\ p>2\right)$$

where $0\leq k+l\leq n$, $(k,l)\neq(0,n)$. $\mathfrak{L}^{\mathfrak{p}}(R)$ is a Banach space with norm

$$\left\|\{w_{k,l}\}\right\|_{\mathfrak{L}^{\mathfrak{p}}(R)}=\max_{0\leq k+l\leq n}\left\|w_{k,l}\right\|_{L^p(R)}$$

We introduce an operator Q by

$$Q\left(\{w_{k,l}\}\right)=\left(\{W_{k,l}\}\right)$$

if

$$W(z)=\phi_w(z)+T_R^nF(z,\{w_{k,l}\})(z)$$
$$W_{k,l}(z)=\partial_z^k\partial_{\overline{z}}^l\phi_w+\partial_z^kT_R^{n-l}F(z,\{w_{k,l}\})(z)$$

where $\phi_w(z)$ is given by Equation (6.4). Thus we obtain a system of integro-differential equations. Then the fixed point of Q is the solution of Equation (6.2), satisfying the Schwarz boundary conditions Equation (6.3) subject to the solvability requirements.

Now we try to show that Q has a fixed point. We take the images of $\{w_{k,l}\}$ and $\{\overline{w}_{k,l}\}$ in

$\mathfrak{L}^{\mathfrak{p}}(R)$ which are denoted by $\{W_{k,l}\}$ and $\{\overline{W}_{k,l}\}$ respectively. Then

$$\begin{aligned}\left\|W - \overline{W}\right\|_{L^p(R)} &\leq \|\phi_w - \phi_{\overline{w}}\|_{L^p(R)} + \left\|T_R^n F(., \{w_{k,l}(.)\}) - T_R^n F(., \{\overline{w}_{k,l}(.)\})\right\|_{L^p(R)} \\ &\leq L \left\|T_R^n\right\|_{L^p(R)} \left\|\{w_{k,l}\} - \{\overline{w}_{k,l}\}\right\|_{\mathfrak{L}^{\mathfrak{p}}(R)}\end{aligned}$$

and

$$\left\|W_{k,l} - \overline{W}_{k,l}\right\|_{L^p(R)} \leq L \left\|\partial_z^k T_R^{n-l}\right\|_{L^p(R)} \left\|\{w_{k,l}\} - \{\overline{w}_{k,l}\}\right\|_{\mathfrak{L}^{\mathfrak{p}}(R)} \tag{6.6}$$

for $0 \leq k + l < n$ and

$$\left\|W_{k,l} - \overline{W}_{k,l}\right\|_{L^p(R)} \leq L \left\|\Pi_R^n\right\|_{L^p(R)} \left\|\{w_{k,l}\} - \{\overline{w}_{k,l}\}\right\|_{\mathfrak{L}^{\mathfrak{p}}(R)} \tag{6.7}$$

for $k + l = n$. Combining (6.6) and (6.7) we obtain

$$\left\|\{W_{k,l}\} - \{\overline{W}_{k,l}\}\right\|_{\mathfrak{L}^{\mathfrak{p}}(R)} \leq M \left\|\{w_{k,l}\} - \{\overline{w}_{k,l}\}\right\|_{\mathfrak{L}^{\mathfrak{p}}(R)}$$

where

$$M = L \max \left\{\max \left\|\partial_z^k T_R^{n-l}\right\|_{L^p(R)}, \left\|\Pi_R^n\right\|_{L^p(R)}\right\}.$$

Thus the operator Q is contractive if $M < 1$. From Banach fixed point theorem [36] we obtain the existence of the unique solution $w_{k,l}$.

7. CONCLUSION

In this thesis, firstly we have examined the Schwarz problem for first-order equations in a ring domain which was studied by Vaitsiakhovich [16] in which the problem has more restrictions than the usual cases. We obtained the unique solution to Schwarz problem for first-order equations in a ring domain [19]. Then we employed a different perspective to second-order equations as well. We derived the unique solution of the Schwarz problem for second-order equations in a ring domain subject to the solvability conditions. Moreover, we extended the investigations to higher-order equations. We achieved the unique solution of the Schwarz problem for higher-order equations in a ring domain and developed the properties of higher-order operators for a ring domain. This part has been published as an article [20]. On the last chapter, we investigated the existence of the unique solution of the Schwarz problem for nonlinear higher-order equations using the idea given in [8].

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