SOME NUMERICAL INVARIANTS OF COMPLEX HYPERSURFACE SINGULARITIES

(KOMPLEKS HİPERYÜZEY TEKİLLİKLERİNİN BAZI SAYISAL DEĞİŞMEZLERİ)

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"An Fortschritt glauben heißt nicht glauben, daß ein Fortschritt schon geschehen ist. Das wäre kein Glauben."

Franz Kafka

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List of Symbols

$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{C}$	:	Complex number field.
$\mathbb{C}^*$	:	Set of all complex numbers except 0.
$\mathbb{C}[[\mathbf{z}]]$	:	Ring of formal power series in the variable $\mathbf{z} = (z_0, z_1, \cdots, z_n)$ over $\mathbb{C}$.
$\mathbb{C}\{\mathbf{z}\}$	:	Ring of convergent power series in the variable $\mathbf{z} = (z_0, z_1, \cdots, z_n)$ over $\mathbb{C}$.
$\mathcal{J}(f)$	:	Jacobian ideal of f.
$\nabla(f)$	:	Gradient of f.
$\mathbb{V}(f)$	:	Vanishing set of f.
$mult_{\mathbf{p}}(\mathbb{V}(f))$	:	Multiplicity of $\mathbb{V}(f)$ at $\mathbf{p}$.
$supp(f)$	:	Support of f.
$mt(f)$	:	Multiplicity of f.
$\mathcal{C}(f)$	:	Critical locus of f.
$\Sigma(f)$	:	Singular locus of $\mathbb{V}(f)$.
M_f	:	Milnor algebra of f.
T_f	:	Tjurina algebra of f.
$\mu(f)$	:	Milnor number of f.
$\tau(f)$	:	Tjurina number of f.
$conv(.)$	:	Convex hull of a set.
$\Gamma_+(f)$	:	Newton polyhedron of f.
$\Gamma(f)$	:	Newton boundary of f.
$\Gamma_-(f)$	:	Cone lying below $\Gamma(f)$ with apex at the origin.
f_0	:	Newtonian main part of f.
$det(.)$	:	Determinant of a square matrix.
$Vol_i(F)$	:	i-dimensional Euclidean volume of F.
$\nu(f)$	:	Newton number of f.

$\{\mathcal{S}_\alpha\}$	:	Analytic stratification.
$\Lambda_f^{(k)}$	:	kth Lê variety of f.
$\lambda_f^{(k)}(\mathbf{p})$	:	kth Lê number of f.
$\widetilde{\nu}_i(f)$	:	Modified Newton number of f with respect to the z_i-axis.
$\nu_i^s(f)$	:	Special modified Newton number of f with respect to the z_i-axis.

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ABSTRACT

In this thesis, we work on the explicit computation, algebraic and geometric points of view, of some numerical invariants of hypersurfaces with singularities.

In the case where the hypersurface has an isolated singularity, we reviewed the Milnor number and the Tjurina number of the defining analytic function f of the hypersurface. After, we present the Newton number attached to the Newton polyhedron of f. We compute it in the case when f is both convenient and not. Then, we examine the relation between these numerical invariants.

In the case where the hypersurface has non-isolated singularities, we are interested in the Lê varieties of an analytic function f and their intersection number with certain linear subspaces at the origin in the ambient space. This intersection number is called Lê number of f at the origin. On this subject, we mainly follow the book [21]. We examine the algebraic and geometric computation of the Lê number of a hypersurface. To compute Lê numbers from the geometric point of view we present the modified Newton number associated with the Newton polyhedron of the defining analytic function f.

We give good examples of calculating all these numerical invariants of a hypersurface, algebraically and geometrically. Our examples are mostly hypersurfaces whose singular locus of dimension 1, but we give also examples of a hypersurface with a 0-dimensional singular locus and with a singular locus of dimension 2.

Keywords: Milnor number, Tjurina number, Lê number, Newton number, modified Newton number, Newton polyhedron, isolated singularities, non-isolated singularities, hypersurfaces,

RÉSUMÉ

Dans ce ménoire, nous traitons du calcul explicite, d'un point de vue algébrique et géométrique, de quelques invariants numériques des singularités des hypersurfaces.

Dans le cas où l'hypersurface a une singularité isolée, nous avons revu le nombre de Milnor et le nombre de Tjurina de la fonction analytique définissante f de l'hypersurface. Ensuite, nous présentons le nombre de Newton qui dépend du polyèdre de Newton de f. Nous l'étudions par rapport à deux cas : lorsque f est commode et lorsque f n'est pas commode. D'ailleurs, nous étudions la relation du nombre de Newton avec le nombre de Milnor pour une hypersurface. De plus, nous avons brièvement énoncé la relation entre le nombre de Newton et le nombre de Tjurina pour une hypersurface.

Dans le cas où l'hypersurface possède des singularités non isolées, on s'intéresse aux variétés Lê d'une fonction analytique f et à leur nombre d'intersection avec certains sous-espaces linéaires à l'origine dans l'espace ambiant. Ce nombre d'intersection est appelé le nombre de Lê de f à l'origine. A ce sujet, nous suivons principalement le livre [21]. Nous examinons le calcul algébrique et géométrique du nombre Lê d'une hypersurface. Pour calculer les nombres de Lê du point de vue géométrique, nous présentons le nombre de Newton modifié associé au polyèdre de Newton de la fonction analytique définissante f.

Nous donnons de bons exemples de calcul de tous ces invariants numériques d'une hypersurface, algébriquement et géométriquement. Nos exemples sont principalement des hypersurfaces dont le singulier locus de dimension 1, mais nous donnons également des exemples d'une hypersurface avec le locus singulier de dimension 0 et avec le locus singulier de dimension 2.

Mots Clés: Nombre de Milnor, nombre de Tjurina, nombre de Lê, nombre de Newton, nombre de Newton modifié, polyèdre de Newton, singularités isolées, singularités non isolées, hypersurfaces.

ÖZET

Bu tezde, tekilliklere sahip hiperyüzeylerin bazı sayısal değişmezlerinin cebirsel ve geometrik bakış açısıyla açık hesaplaması üzerine çalışıyoruz.

Hiperyüzeylerin izole bir tekilliğe sahip olduğu durumda, hiperyüzeyin tanımlayıcısı f analitik fonksiyonunun Milnor sayısı ve Tjurina sayısını gözden geçiriyoruz. Sonrasında, f'in Newton çok yüzlüsüne bağlı olan Newton sayısını sunuyoruz. Bu sayıyı f'in uygun olduğu ve f'in uygun olmadığı iki farklı durumda inceliyoruz. Daha sonra, bir hiperyüzey için Newton sayısının Milnor sayısıyla arasındaki ilişkiyi inceliyoruz. Ayrıca bir hiperyüzey için Newton sayısı ile Tjurina sayısı arasındaki ilişkiyi kısaca ifade ediyoruz.

Hiperyüzeyin izole olmayan tekilliklere sahip olduğu durumda, f'nin Lê varyetesiyle ve bu varyetelerin belirli doğrusal alt uzaylarla orijindeki kesişim sayılarıyla ilgileniyoruz. Bu kesişim sayısına, f'nin orijindeki Lê sayısı denir. Bu konuda esas olarak [21] kitabını takip ediyoruz. Bir hiperyüzeyin Lê sayısının cebirsel ve geometrik olarak hesaplanmasını inceliyoruz. Lê sayılarını geometrik açıdan hesaplamak için, tanımlayıcı analitik fonksiyon f'nin Newton çok yüzlüsüne bağlı olan f'nin değiştirilmiş Newton sayısını sunuyoruz.

Bir hiperyüzeyin bütün bu sayısal değişmezlerini cebirsel ve geometrik açıdan hesaplanmasına iyi örnekler veriyoruz. Örneklerimiz çoğunlukla, tekil yerinin boyutu 1 olan hiperyüzeylerdir; ancak, tekil yeri 0 boyutlu ve tekil yeri 2 boyutlu olan hiperyüzeyler için de örnekler veriyoruz.

Anahtar Sözcükler: Milnor sayısı, Tjurina sayısı, Lê sayısı, Newton sayısı, değiştirilmiş Newton sayısı, Newton çok yüzlüsü, izole tekillikler, izole olmayan tekillikler, hiperyüzeyler.

1 Introduction

Consider a hypersurface $\mathbb{V}(f) := \{\mathbf{p} \in \mathbb{C}^{n+1} \mid f(\mathbf{p}) = 0\}$ where $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, \mathbf{0})$ is an analytic function. We say that two hypersurfaces $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are topologically the same if there exists a homeomorphism $\Phi : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}^{n+1}, \mathbf{0})$ such that $\Phi(\mathbb{V}(f)) = \mathbb{V}(g)$, In the case where $\mathbb{V}(f)$ has an isolated singularity at $\mathbf{0} \in \mathbb{V}(f)$ we obtain the topological equivalence by computing the *Milnor* number $\mu(f, \mathbf{0})$ introduced in [24]. By [16], if $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are topologically the same then $\mu(f) = \mu(g)$. The inverse implication is not true, meaning that f and g can be topologically different when $\mu(f) = \mu(g)$. Hence the Milnor number is a topological invariant of the singularity. In Section 3.1, we present some results on that number and compute it for a few hypersurfaces.

We associate another number with a hypersuface $\mathbb{V}(f)$, called the *Tjurina* number $\tau(f, \mathbf{0})$ of f. In [22], the authors show that if $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are analytically equivalent then $\tau(f) = \tau(g)$. In general, we have $\mu(f, \mathbf{0}) \geq \tau(f, \mathbf{0})$; the equality holds when f is weighted homogeneous (See [26]). In this work, we are interested in computing in different ways, algebraic and geometric, these invariants.

Another number associated with f is the *Newton number* $\nu(f)$ introduced in [15] and compute it using the *Newton polyhedron* of f in $\mathbb{R}^{n+1}$. But, $\nu(f)$ can be associated with an abstract bounded polyhedron in $\mathbb{R}^{n+1}$ (See Section 3.2.2). The computation is done by means of the subcones in the bounded polyhedron. Furthermore, we can explicitly calculate $\mu(f)$ and $\tau(f)$ using $\nu(f)$. We have $\mu(f) \geq \nu(f)$ and the equality holds for a Newton non-degenerate f; and, $\tau(f) \geq \dfrac{\nu(f)}{(n+1)}$ (see Section 3.2.2).

In the case where the hypersurface $\mathbb{V}(f)$ has non-isolated singularities with $d := dim(\Sigma f)$, we are interested in the Lê varieties introduced in [19].We show that the Lê varieties have the same information as Σf (see Corollary 4.1.12). We compute the Lê numbers which are the intersection number of Lê varieties with a certain linear subspace in an algebraic and geometric way and we give some results of Lê numbers with detailed examples.

We also give attention to *modified Newton number* and *special modified Newton number* introduced in [8] which are the certain numerical invariant attached to the Newton polyhedron of f in $\mathbb{R}^{n+1}$. When f is Newton non-degenerate, the computation of these invariants with respect to the Newton polyhedron of $f+z_0^m+\cdots+z_d^m$ with sufficiently large m gives an algorithm to obtain Lê numbers of f.

In Chapter 2, we recall the operations on the ideals in the polynomial ring $\mathbf{k}[x_1,x_2,\ldots,x_n]$ and the facts about the defining variety after such an operation.

In Chapter 3, we concentrate on the Newton number of an isolated hypersurface singularity. First, we review some basic definitions and results on the Milnor number and the Tjurina number of an isolated singularity. We explain roughly the relationship between these two invariants and show how these are calculated algebraically with the help of examples. Then, we present the Newton number of f which depends on its Newton polyhedron $\Gamma_+(f)$. We explain in detail how to calculate the Newton number whether f is convenient or not, and we examine the relationship between the Newton number and the Milnor number in both cases. We precise the conditions the Milnor can be computed directly by Newton number.

In Chapter 4, we concentrate on the Lê numbers of non-isolated singularities. First, we give the definition of the Lê varieties of f and we show some important results about them. We determine a good choice of ordering for the coordinates $(z_0,z_1,\ldots,z_n)$ in $\mathbb{C}^{n+1}$ to compute the Lê numbers. We show some important results in the computation of the Lê numbers. We give some examples for non-isolated singularities with 1-dimensional singular locus and for these singularities with 2-dimensional singular locus.

In Chapter 5, we give attention to some numerical invariants for non-isolated hypersurface singularities. We present the modified Newton number and the special modified Newton number of f obtained by computing the volumes of subcones of the Newton polyhedron of f. We give an algorithm to compute the Lê numbers of f from its Newton polyhedron calculating these numerical invariants for $f+z_0^m+\cdots+z_d^m$ with sufficiently large m where d is the dimension of the singular locus of f.

2 Preliminaries

In this section, we recall the operations on the ideals in the polynomial ring $\mathbf{k}[\mathbf{x}] := k[x_1,\ldots,x_n]$ including sums, intersections, product, radicals, and primary decomposition. We also recall the facts about the defining variety after such an operation.

2.1 Ideals in the polynomial ring

Let $\mathbf{k}$ be an algebraically closed field. Let I and J be two ideals in the polynomial ring $\mathbf{k}[\mathbf{x}]$. The sum of I and J is the set

$$I+J = \{f+g \mid f \in I \ and \ g \in J\}.$$

Example 2.1.1. Consider $I = \langle x^2+yz, z^2+y^3\rangle$ and $J = \langle xy^2\rangle$. Then,

$$\begin{aligned} I+J &= \{f+g|f = A(x^2+yz)+B(z^2+y^3) \ and \ g = Cxy^2 \ where \ A,B,C \in \mathbf{k}[x,y,z]\} \\ &= \{h \mid h = A(x^2+yz)+B(z^2+y^3)+Cxy^2 \ where \ A,B,C \in \mathbf{k}[x,y,z]\} \\ &= \langle x^2+yz, z^2+y^3, xy^2\rangle \end{aligned}$$

Remark 2.1.2. If $I = \langle f_1,\ldots,f_r\rangle$ and $J = \langle g_1,\ldots,g_s\rangle$, then $I+J = \langle f_1,\ldots,f_r,g_1,\ldots,g_s\rangle$ is the smallest ideal containing I and J.

The product IJ of two ideals in $\mathbf{k}[\mathbf{x}]$ is defined as follows: Let $I = \langle f_1,\ldots,f_r\rangle$ and $J = \langle g_1,\ldots,g_s\rangle$. The product is

$$IJ = \{f_i g_j \mid 1 \le i \le r, 1 \le j \le s\}.$$

Example 2.1.3. Consider $I = \langle xy, z\rangle$ and $J = \langle y, z\rangle$ in $\mathbf{k}[x,y,z]$.

$$\begin{aligned} IJ &= \{fg \mid f = Axy+Bz, g = Cy+Dz \ with \ A,B,C,D \in \mathbf{k}[x,y,z]\} \\ &= \{h \mid h = (Axy+Bz)(Cy+Dz) \ with \ A,B,C,D \in \mathbf{k}[x,y,z]\} \\ &= \{h \mid h = A'xy^2+B'yz+C'xyz+D'z^2 \ with \ A',B',C',D' \in \mathbf{k}[\mathbf{x}][x,y,z]\} \\ &= \langle xy^2, yz, xyz, z^2\rangle. \end{aligned}$$

The intersection $I \cap J$ of the ideals is the set

$$I \cap J = \{f \in \mathbf{k}[\mathbf{x}] \mid f \in I \ and \ f \in J\}$$

which is an ideal in $\mathbf{k}[\mathbf{x}]$.

Example 2.1.4. Consider the ideals $\langle x^2, y\rangle$ and $\langle x, y\rangle$ in $\mathbf{k}[x,y]$. We have:

$$\begin{aligned}\langle x^2,y\rangle \cap \langle x,y\rangle &= \{f \mid f = Ax^2+By \textit{ and } f = A'x+B'y \textit{ with } A,A',B,B' \in \mathbf{k}[x,y,z]\}\\ &= \{f \mid f = Cx^2+Dy \textit{ with } C,D \in k[x,y,z]\} = \langle x^2,y\rangle .\end{aligned}$$

Remark 2.1.5. We have $IJ \subseteq I \cap J$. In fact, the relation can be given by strict inclusion. For instance, the ideals $I = J = \langle x,y\rangle$ give

$$IJ = \langle x^2, xy, y^2\rangle \qquad \textit{and} \qquad I \cap J = \langle x,y\rangle \cap \langle x,y\rangle = \langle x,y\rangle .$$

Since $x \in I \cap J$ but $x \notin IJ$, we obtain $IJ \subset I \cap J$.

Algorithm to compute the intersection Let $I = \langle f_1,\ldots,f_r\rangle$ and $J = \langle g_1,\ldots,g_s\rangle$ two ideals in $\mathbf{k}[\mathbf{x}]$.

1. Consider $\langle tf_1,\ldots,tf_r,(1-t)g_1,\ldots,(1-t)g_s\rangle \subset \mathbf{k}[x_1,\ldots,x_n,t]$.

2. Compute a Gröbner basis of this ideal with respect to the order $t > x_n > \cdots > x_1$.

3. Eliminate the elements in this basis that contain the variable t

4. $I \cap J = tI \cap (1-t)J \cap \mathbf{k}[\mathbf{x}]$.

Example 2.1.6. Let $I = \langle x^2y\rangle$ and $J = \langle xy^2\rangle$ be ideals in $\mathbf{k}[x,y]$. Following the above algorithm we have

$$\langle tI,(1-t)J\rangle = \langle x^2yt, xy^2 - xy^2t\rangle \subset k[x,y,t]$$

whose Gröbner basis is $\{tx^2y, txy^2 - xy^2, x^2y^2\}$ with respect to the order $t > y > x$.Then, the elimination of t gives $I \cap J = \langle x^2y^2\rangle$.

The quotient of ideals in $\mathbf{k}[\mathbf{x}]$ is defined as

$$I : J := \{f \in \mathbf{k}[\mathbf{x}] \mid fg \in I \textit{ for all } g \in J\}.$$

Example 2.1.7. Consider $I = \langle xy^2, yz^3\rangle$ and $J = \langle y\rangle$. Then

$$\begin{aligned}I : J &= \{f \in \mathbf{k}[x,y,z] \mid fy \in I\} = \{f \in \mathbf{k}[x,y,z] \mid fy = Axy^2 + Byz^3\}\\ &= \{f \in \mathbf{k}[x,y,z] \mid f = Axy + Bz^3\} = \langle xy, z^3\rangle\end{aligned}$$

Remark 2.1.8. The quotient $I : J$ is an ideal and contains I.

Proposition 2.1.9. ([19], 0.1) Let I_1, I_2, I_3 and I_4 be ideals in $\mathbf{k}[\mathbf{x}]$. Then we have

(i) $(I_1+I_2):I_3=(I_1:I_3+I_2):I_3$,

(ii) if $I_3\subseteq I_1+I_2$, then

$-\ (I_4I_1+I_2):I_3=(I_4+I_2):I_3$
$-\ ((I_4\cap I_1)+I_2):I_3=(I_4+I_2):I_3$

Proposition 2.1.10. ([2], 1.12) Let $I,J,I_1,\ldots I_r$ be ideals in $\mathbf{k}[\mathbf{x}]$. Then,

(i) $\left(\bigcap\limits_{i=1}^{r} I_i\right):J=\bigcap\limits_{i=1}^{r}(I_i:J)$

(ii) $J:\left(\sum\limits_{i=1}^{r} I_i\right)=\bigcap\limits_{i=1}^{r}(J:I_i)$

In particular, for $J=\langle f_1,f_2,\ldots,f_r\rangle$ with $r\geq 1$ we have $I:J=\bigcap\limits_{i=1}^{r}(I:\langle f_i\rangle)$.

Example 2.1.11. Consider $I=\left\langle y^3,y^5-z^4\right\rangle$ and $J=\left\langle y,z^3\right\rangle$. By the previous proposition $I:J=I:\langle y\rangle\cap I:\left\langle z^3\right\rangle$ where

$$
\begin{aligned}
I:\langle y\rangle &= \{f\in\mathbf{k}[x,y,z]\mid fy\in I\}\\
&= \{f\in\mathbf{k}[x,y,z]\mid fy=Ay^3+B(y^5-z^4)\}\\
&= \{f\in\mathbf{k}[x,y,z]\mid f=Ay^2+B'(y^5-z^4)\}\\
&= \left\langle y^2,y^5-z^4\right\rangle\\
I:\left\langle z^3\right\rangle &= \{f\in\mathbf{k}[x,y,z]\mid fz^3\in I\}\\
&= \{f\in\mathbf{k}[x,y,z]\mid fz^3=Ay^3+B(y^5-z^4)\}\\
&= \{f\in\mathbf{k}[x,y,z]\mid f=A'y^3+B'y^5-Bz)\}\\
&= \left\langle y^3,z\right\rangle.
\end{aligned}
$$

That is, $I:J=\left\langle y^2,y^5-z^4\right\rangle\cap\left\langle y^3,z\right\rangle$. By using the algorithm given to compute the intersection of ideals, we obtain $I:J=\left\langle y^3,y^2z,z^4\right\rangle$.

Example 2.1.12. Let $I=\left\langle x^2y+y^3,z^3\right\rangle$ and $J=\langle y,z\rangle=\langle y\rangle+\langle z\rangle$ two ideals in $\mathbf{k}[x,y,z]$. We need to find the intersection of $I:\langle y\rangle$ and $I:\langle z\rangle$ because

$$I:J=(I:\langle y\rangle)\cap(I:\langle z\rangle).$$

We obtain

$$I=\left\langle y(x^2+y^2),z^3)\right\rangle=\left\langle y,z^3\right\rangle\cap\left\langle x^2+y^2,z^3\right\rangle.$$

So,

$$
\begin{aligned}
I:\langle y\rangle &= \left(\left\langle y,z^3\right\rangle:\langle y\rangle\right)\cap\left(\left\langle x^2+y^2,z^3\right\rangle:\langle y\rangle\right) \text{ and}\\
I:\langle z\rangle &= \left(\left\langle y,z^3\right\rangle:\langle z\rangle\right)\cap\left(\left\langle x^2+y^2,z^3\right\rangle:\langle z\rangle\right)
\end{aligned}
$$

Since

$$\begin{aligned}
\langle y,z^3\rangle : \langle y\rangle &= \{f\in \mathbf{k}[x,y,z] \mid fy\in \langle y,z^3\rangle\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid fy = Ay+Bz^3,\ \ A,B\in \mathbf{k}[x,y,z]\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid f = A+B'z^3\ \ A,B'\in k[x,y,z]\}\\
&= \mathbf{k}[x,y,z]\\
\langle x^2+y^2,z^3\rangle : \langle y\rangle &= \{f\in \mathbf{k}[x,y,z] \mid fy\in \langle x^2+y^2,z^3\rangle\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid fy = A(x^2+y^2)+Bz^3,\ \ A,B\in \mathbf{k}[x,y,z]\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid f = A'(x^2+y^2)+B'z^3,\ \ A',B'\in \mathbf{k}[x,y,z]\}\\
&= \langle x^2+y^2,z^3\rangle
\end{aligned}$$

we obtain

$$I:\langle y\rangle = \mathbf{k}[x,y,z]\cap \langle x^2+y^2,z^3\rangle = \langle x^2+y^2,z^3\rangle.$$

Now, we compute $I:\langle z\rangle$:

$$\begin{aligned}
\langle y,z^3\rangle : \langle z\rangle &= \{f\in \mathbf{k}[x,y,z] \mid fy\in \langle y,z^3\rangle\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid fz = Ay+Bz^3,\ \ A,B\in \mathbf{k}[x,y,z]\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid f = A'y+Bz^2,\ \ A',B\in \mathbf{k}[x,y,z]\}\\
&= \langle y,z^2\rangle\\
\langle x^2+y^2,z^3\rangle : \langle z\rangle &= \{f\in \mathbf{k}[x,y,z] \mid fy\in \langle x^2+y^2,z^3\rangle\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid fz = A(x^2+y^2)+Bz^3,\ \ A,B\in \mathbf{k}[x,y,z]\}\\
&= \{f\in \mathbf{k}[x,y,z] \mid f = A'(x^2+y^2)+Bz^2,\ \ A',B\in \mathbf{k}[x,y,z]\}\\
&= \langle x^2+y^2,z^2\rangle
\end{aligned}$$

Hence, we obtain

$$I:\langle z\rangle = \langle y,z^2\rangle\cap \langle x^2+y^2,z^2\rangle = \langle z^2,x^2y+y^3,x^2z^2+y^2z^2\rangle.$$

Therefore we have

$I:J = \langle x^2+y^2,z^3\rangle\cap\langle z^2,x^2y+y^3,x^2z^2+y^2z^2\rangle = \langle z^3,x^2y+y^3,x^2z^2+y^2z^2\rangle.$

2.2 Varieties in the affine space $\mathbf{k}^n$

Let I be an ideal in $\mathbf{k}[\mathbf{x}]$. The set

$$\mathbb{V}(I) = \{\mathbf{p} = (p_1,p_2,\ldots,p_n)\in \mathbf{k}^n \mid f(\mathbf{p}) = 0\ for\ all\ f\in I\}$$

is called the (affine) variety in $\mathbf{k}^n$ defined by the ideal I.

Theoreme 2.2.1. ([14], 2.2.28) Let I and J be two ideals in $\mathbf{k}[\mathbf{x}]$. We have

$$\mathbb{V}(I+J) = \mathbb{V}(I)\cap\mathbb{V}(J).$$

Proof. Let $h\in I+J$. We have $h = f+g$ with $f\in I$ and $g\in J$. For $p\in\mathbb{V}(I)\cap\mathbb{V}(J)$ we have $h(p) = (f+g)(p) = f(p)+g(p)$. Since $p\in\mathbb{V}(I)$ and $p\in\mathbb{V}(J)$ we have $f(p) = 0$ and $g(p) = 0$. So, $h(p) = 0$. This says $p\in\mathbb{V}(I+J)$. Then, $\mathbb{V}(I)\cap\mathbb{V}(J)\subset\mathbb{V}(I+J)$.

For the opposite inclusion, recall that $I+J$ is the smallest ideal such that $I \subset I+J$ and $J \subset I+J$. So, we have $\mathbb{V}(I) \supset \mathbb{V}(I+J)$ and $\mathbb{V}(J) \supset \mathbb{V}(I+J)$. This implies that, for $q \in \mathbb{V}(I+J)$, we have $q \in \mathbb{V}(I)$ and $q \in \mathbb{V}(J)$. So, $q \in \mathbb{V}(I) \cup \mathbb{V}(J)$. Hence, $\mathbb{V}(I+J) \subset \mathbb{V}(I) \cap \mathbb{V}(J)$. Then $\mathbb{V}(I+J) = \mathbb{V}(I) \cap \mathbb{V}(J)$.

□

Example 2.2.2. Consider the ideals $I = \langle x^2+yz, z^2+y^3 \rangle$ and $J = \langle xy^2 \rangle$ in $\mathbf{k}[x,y,z]$. We get

$$\mathbb{V}(I+J) = \mathbb{V}(x^2+yz, z^2+y^3, xy^2) = \mathbb{V}(x^2+yz, z^2+y^3) \cap \mathbb{V}(xy^2).$$

Theoreme 2.2.3. ([14], 2.2.2) Let $I, J \in \mathbf{k}[\mathbf{x}]$. We have $\mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$.

Proof. Let $h \in IJ$. Let $p \in \mathbb{V}(I) \cup \mathbb{V}(J)$. Assume $h = f \cdot g$ with $f \in I$ and $g \in J$. If $p \in \mathbb{V}(I)$, then $f(p) = 0$. If $p \in \mathbb{V}(J)$, then $g(p) = 0$. In both cases, we have $p \in \mathbb{V}(IJ)$, so $\mathbb{V}(I) \cup \mathbb{V}(J) \subset \mathbb{V}(IJ)$.
To get the opposite inclusion, consider $q \in \mathbb{V}(IJ)$. For all $h \in IJ$, we have $h(q) = 0$. As $h = f \cdot g$ with $f \in I$ and $g \in J$ we have $f(q)g(q) = 0$. If $f(q) = 0$, then $q \in \mathbb{V}(I)$. If $f(q) \neq 0$, we must have $g(q) = 0$ such that $q \in \mathbb{V}(J)$. In both cases we have $q \in \mathbb{V}(I) \cup \mathbb{V}(J)$. Hence $\mathbb{V}(IJ) \subset \mathbb{V}(I) \cup \mathbb{V}(J)$. So, we get the equality given in the theorem.

□

Example 2.2.4. Consider the ideals $I = \langle xy, z \rangle$ and $J = \langle y, z \rangle$ in $\mathbf{k}[x,y,z]$. We have

$$\mathbb{V}(IJ) = \mathbb{V}(xy^2, yz, xyz, z^2) = \mathbb{V}(xy, z) \cup \mathbb{V}(y, z).$$

Theoreme 2.2.5. ([14], 2.2.2) Let $I, J \in \mathbf{k}[\mathbf{x}]$. We have

$$\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J).$$

Proof. Let $p \in \mathbb{V}(I) \cup \mathbb{V}(J)$. Since $I \cap J \subset I$ and $I \cap J \subset J$, we have $\mathbb{V}(I \cap J) \supset \mathbb{V}(I)$ and $\mathbb{V}(I \cap J) \supset \mathbb{V}(J)$. If $p \in \mathbb{V}(I)$ or, $p \in \mathbb{V}(J)$ then we have $p \in \mathbb{V}(I \cap J)$. Hence, $\mathbb{V}(I) \cup \mathbb{V}(J) \subset \mathbb{V}(I \cap J)$.
To get the opposite inclusion, consider $p \in \mathbb{V}(I \cap J)$. So there is $h \in I \cap J$ such that $h(p) = 0$. By definition we have $h = f \cdot g$ for $f \in I$, $g \in J$; hence $f \cdot g(p) = 0$. That is, $f(p) = 0$ or $g(p) = 0$. This says $p \in \mathbb{V}(I)$ or $p \in \mathbb{V}(J)$. So, $p \in \mathbb{V}(I) \cup \mathbb{V}(J)$. Hence, $\mathbb{V}(I \cap J) \subset \mathbb{V}(I) \cup \mathbb{V}(J)$. We get the equality in the theorem.

□

Example 2.2.6. For $\langle x^2, y \rangle = \langle x^2, y \rangle \cap \langle x, y \rangle$, we have $\mathbb{V}(x^2, y) = \mathbb{V}(x^2, y) \cup \mathbb{V}(x, y)$.

To define the variety of the quotient, we need to recall the radical of I which is

$$\sqrt{I} = \{f \in \mathbf{k}[\mathbf{x}] \mid f^m \in I \, for \, some \, m \in \mathbb{N}\}.$$

An ideal I is radical if and only if $\sqrt{I} = I$.

Theoreme 2.2.7. ([14], 2.2.38) Let I, J be two ideals in $\mathbf{k}[\mathbf{x}]$. Assume that J is a radical ideal. Then

$$\mathbb{V}(I : J) = \overline{\mathbb{V}(I) - \mathbb{V}(J)}$$

where $\overline{\mathbb{V}(I) - \mathbb{V}(J)}$ is the Zariski closure in $\mathbf{k}^n$.

Proof. Let us first show that $\mathbb{V}(I : J) \subset \mathbb{V}(\mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J)))$. To do this, we show

$$(*) \qquad \mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J)) \subset I : J.$$

Let $f \in \mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J))$. So $f(p) = 0$ for all $p \in \mathbb{V}(I) - \mathbb{V}(J)$. Let $g \in J$. We have $f \cdot g(q) = 0$ for all $q \in \mathbb{V}(I)$. That is, $f \cdot g \in \mathbb{I}(\mathbb{V}(I))$. Since I is radical, we have $fg \in I$ for all $g \in J$. Then $f \in I : J$, so we get $(*)$. By inclusion-reversing we get $\mathbb{V}(I : J) \subset \mathbb{V}(\mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J)))$. Now we want to show that $\mathbb{V}(\mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J))) \subset \mathbb{V}(I : J)$. Consider $f \in I : J$ and $p \in \mathbb{V}(I) - \mathbb{V}(J)$. By definition $f(p)g(p) = 0$ for all $g \in J$. Since $p \notin \mathbb{V}(J)$, we have $g(p) \neq 0$. So we obtain $f(p) = 0$. Since the argument is true for all $p \in \mathbb{V}(I) - \mathbb{V}(J)$, we get $f \in \mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J))$. Using inclusion-reversing, we get $\mathbb{V}(\mathbb{I}(\mathbb{V}(I) - \mathbb{V}(J))) \subset \mathbb{V}(I : J)$. Hence, we get the equality given in the theorem.

$\square$

Example 2.2.8. For $I = \langle xy^2, yz^3 \rangle$ and $J = \langle y \rangle$ above, we have $I : J = \langle xy, z^3 \rangle$ and

$$\mathbb{V}(I : J) = \mathbb{V}(xy, z^3) = \overline{\mathbb{V}(xy^2, yz^3) - \mathbb{V}(y)}.$$

We will mostly use the following remark about the variety defined by the quotient of ideals.

Remark 2.2.9. Let $f \in \mathbf{k}[\mathbf{x}]$. Let I be an ideal of $\mathbf{k}[\mathbf{x}]$. Consider a minimal primary decomposition

$$I = I_1 \cap I_2 \cap \cdots \cap I_r.$$

By Proposition 2.1 (i), we have

$$I : \langle f \rangle = \left(\bigcap_{i=1}^{r} I_i \right) : \langle f \rangle = \bigcap_{i=1}^{r} (I_i : \langle f \rangle).$$

So,

$$I:\langle f\rangle = \bigcap_i \{I_i \mid f \notin I_i\}.$$

Then we have

$$\mathbb{V}(I:f) := \bigcup_i \{\mathbb{V}(I_i) \mid f \notin I_i\}.$$

Decomposition of a variety: Let I be a radical ideal in $\mathbf{k}[\mathbf{x}]$. Consider $I = I_1 \cap \cdots \cap I_r$ is a minimal primary decomposition of I where I_i are primary ideal for all i. (See 6.0.6 in the Appendix.) Further, we have

$$\mathbb{V}(I) = \mathbb{V}(I_1 \cap \ldots I_r) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2) \cup \cdots \cup \mathbb{V}(I_r).$$

Thus, the primary decomposition gives us the decomposition of $\mathbb{V}(I)$ as the union of irreducible varieties $\mathbb{V}(I_i)$ for all $1 \leq i \leq r$. Moreover, we say that $\sqrt{I_i}$ is the associated prime of I and $Ass(I) := \{\sqrt{I_i} \mid 1 \leq i \leq r\}$.

Definition 2.2.10. *Let I be ideal in R. Let $p \in Ass(I)$.*
(i) $\mathbb{V}(p)$ is called the isolated (or non-embedded) component of $\mathbb{V}(I)$ if there is no element $q \in Ass(I)$ such that $q \subset p$.
(ii) $\mathbb{V}(p)$ is called embedded component of $\mathbb{V}(I)$ if there is at least one prime ideal $q \in Ass(I)$ such that $q \subset p$.

Notation. Let J be an ideal in $\mathbf{k}[\mathbf{x}]$. Assume that $I_{i_1}, I_{i_2}, \ldots, I_{i_s}$ are in the primary decomposition of I for some $s \leq r$. If $I_{i_1}, I_{i_2}, \ldots, I_{i_s} \subseteq J$, then we let $\mathbb{V}(I) \triangleleft \mathbb{V}(J)$ denote the variety defined by $I_{i_1} \cap Ii2 \cap \ldots \cap I_{i_s}$. That is,

$$\mathbb{V}(I) \triangleleft \mathbb{V}(J) := \mathbb{V}(I_{i_1} \cap Ii2 \cap \ldots \cap I_{i_s}).$$

Remark 2.2.11. The variety $\mathbb{V}(I:J)$ consists of the components of $\mathbb{V}(I)$ which are not contained in $\mathbb{V}(J)$. So, we get

$$\mathbb{V}(I) = \mathbb{V}(I:J) \cup (\mathbb{V}(I) \triangleleft \mathbb{V}(J)) \tag{2.2.1}$$

up to embedded components of $\mathbb{V}(I)$ and $\mathbb{V}(J)$. Moreover, if there is no isolated component of $\mathbb{V}(I)$ in $\mathbb{V}(J)$, then we have $\mathbb{V}(I) \triangleleft \mathbb{V}(J) = \emptyset$. Hence we get $\mathbb{V}(I) = \mathbb{V}(I:J)$ up to embedded components.

2.3 Singular points of a variety

Let $\mathbb{C}\{\mathbf{z}\} = \mathbb{C}\{z_0, z_1, \ldots, z_n\}$ be the ring of convergent power series. Let $f \in \mathbb{C}\{\mathbf{z}\}$. Let us first consider the variety

$$\mathbb{V}(f) := \{\mathbf{p} \in \mathbb{C}^{n+1} | f(\mathbf{p}) = 0\}$$

which is a *hypersurface* in $\mathbb{C}^{n+1}$. The set

$$C(f) := \{p \in \mathbb{C}^{n+1} \mid \frac{\partial f}{\partial z_0}(p) = \frac{\partial f}{\partial z_1}(p) = \cdots = \frac{\partial f}{\partial z_n}(p) = 0\}$$

is called the *critical locus* of f. The set

$$\Sigma(f) := \{p \in \mathbb{C}^{n+1} \mid f(p) = \frac{\partial f}{\partial z_0}(p) = \frac{\partial f}{\partial z_1}(p) = \cdots = \frac{\partial f}{\partial z_n}(p) = 0\} \tag{2.3.1}$$

is called the *singular locus* of $\mathbb{V}(f)$. A point $\mathbf{p}$ of $\mathbb{V}(f)$ is called *singular* if

$$\nabla f(\mathbf{p}) = \left(\frac{\partial f}{\partial z_0}(\mathbf{p}), \frac{\partial f}{\partial z_1}(\mathbf{p}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{p}) \right) = \mathbf{0}.$$

Otherwise, p is said to be *non-singular* (or *smooth*) point for $\mathbb{V}(f)$. In particular, a singular point p is said to be *isolated* if there exist a neighborhood U_p of p in $\mathbb{C}^{n+1}$ such that $U_{\mathbf{p}} - \{\mathbf{p}\}$ is smooth at each of its points. If $\mathbf{p}$ is not isolated then we say that $\mathbb{V}(f)$ has non-isolated singularities.

Example 2.3.1. **[E_7-type]** Consider $f(\mathbf{z}) = z_0^2 + z_1^3 + z_1 z_2^3 \in \mathbb{C}\{z_0, z_1, z_2\}$. We have

$$\nabla f(\mathbf{p}) = \left(2z_0, 3z_1^2 + z_2^3, 3z_1 z_2^2 \right)(\mathbf{p})$$

and the only point for which $\nabla f(\mathbf{p}) = 0$ is $\mathbf{p} = (0,0,0)$. So, p is an isolated singular point of $\mathbb{V}(f)$. That is, $dim(\Sigma f) = 0$.

Example 2.3.2. **[$E_{7,0}$-type]** Consider $f(\mathbf{z}) = z_2^3 + z_0^2 z_1 z_2 + z_1^4 \in \mathbb{C}\{z_0, z_1, z_2\}$. We have

$$\nabla f(\mathbf{p}) = \left(2z_0 z_1 z_2, z_0^2 z_2 + 4z_1^3, 3z_2^2 \right)(\mathbf{p})$$

For all points $\mathbf{p} = (z_0, 0, 0)$ with $z_0 \in \mathbb{C}$ we get $\nabla f(\mathbf{p}) = 0$. Hence $\mathbb{V}(f)$ has singularities along z_0-axis. So, $\mathbb{V}(f)$ has non-isolated singularities. In this particular case, we have $dim(\Sigma f) = 1$.

3 Milnor, Tjurina and Newton number of a hypersurface

3.1 Milnor number and Tjurina number

Let $f \in \mathbb{C}\{\mathbf{z}\}$ and $\mathbf{p} = (p_0, p_1, \ldots, p_n) \in \mathbb{V}(f)$. In this section, we will assume that $\mathbb{V}(f)$ has an isolated singularity at the origin in $\mathbb{C}^{n+1}$. Consider the Jacobian ideal of f:

$$\mathcal{J}(f) := \left\langle \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle$$

Definition 3.1.1. *The analytic algebras $M_{f,\boldsymbol{p}} := \mathbb{C}\{z_0 - p_0, z_1 - p_1, \cdots, z_n - p_n\}/\mathcal{J}(f)$ and $T_{f,\boldsymbol{p}} := \mathbb{C}\{z_0 - p_0, z_1 - p_1, \cdots, z_n - p_n\}/\langle f, \mathcal{J}(f)\rangle$ are respectively called Milnor algebra and Tjurina algebra of f at $\boldsymbol{p}$ and the respective dimensions*

$$\mu(f,\boldsymbol{p}) := dim_{\mathbb{C}}(M_{f,\boldsymbol{p}}) \quad and \quad \tau(f,\boldsymbol{p}) := dim_{\mathbb{C}}(T_{f,\boldsymbol{p}})$$

are called the Milnor number and Tjurina number of f at $\boldsymbol{p}$. Here $dim_{\mathbb{C}}$ is the dimension as a $\mathbb{C}$-vector space.

To simplify, we will use the following notation at the origin:

$$M_f := M_{f,\mathbf{0}}, \quad T_f := T_{f,\mathbf{0}}, \quad \mu(f) := \mu(f,\mathbf{0}) \quad and \quad \tau(f) := \tau(f,\mathbf{0}).$$

For a smooth point $\mathbf{p}$ of $\mathbb{V}(f)$, we have $\mu(f,\mathbf{p}) = \tau(f,\mathbf{p}) = 0$. Further, when the singularity is non-isolated we put $\mu(f,\mathbf{p}) = \tau(f,\mathbf{p}) = \infty$. In general, from the fact that $\mathcal{J}(f) \subset \langle f, \mathcal{J}(f)\rangle$ we have $\mu(f,\mathbf{p}) \geq \tau(f,\mathbf{p})$ (See [10, 18]). Moreover, we have the relation

$$\frac{\mu(f)}{\tau(f)} \leq n+1$$

if f has an isolated singularity at the origin. (See ([17],1.1))

Remark 3.1.2. Let $\mathbb{V}(f)$ and $\mathbb{V}(g)$ be two hypersurfaces in $\mathbb{C}^{n+1}$. Assume that they have an isolated singularity at the origin. Let $\Phi : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}^{n+1}, \mathbf{0})$ be a map such that $\Phi(\mathbb{V}(f)) = \mathbb{V}(g)$. We say that f and g have *the same topological type* if Φ is a homeomorphism. Furthermore, we have $\mu(f) = \mu(g)$ when if f and g have the same topological type (See [16]). We say that $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are analytically equivalent if Φ is biholomorphic. Moreover, $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are analytically equivalent if and only if their Tjurina algebras are isomorphic as $\mathbb{C}$–algebras. In particular, if $\mathbb{V}(f)$ and $\mathbb{V}(g)$ have the same analytic type then $\tau(f) = \tau(g)$ (See, [22]). In conclusion, the Milnor number is a topological invariant of the singularity while the Tjurina number is an analytical invariant of the singularity.

Example 3.1.3. [E_7**-type]** Consider $f = z_2^2 + z_1^3 + z_0^3 z_1$. The Milnor algebra of f is

$$M_f = \mathbb{C}\{z_0, z_1, z_2\} / \left\langle 3z_0^2 z_1, 3z_1^2 + z_0^3, 2z_2 \right\rangle = \left\langle 1, z_0, z_0^2, z_1, z_0 z_1, z_1^2, z_0 z_1^2 \right\rangle$$

and the Tjurina algebra of f is

$$\begin{aligned} T_f &= \mathbb{C}\{z_0, z_1, z_2\} / \langle z_2^2 + z_1^3 + z_0^3 z_1, 3z_0^2 z_1, 3z_1^2 + z_0^3, 2z_2 \rangle \\ &= \mathbb{C}\{z_0, z_1, z_2\} / \langle z_2, 3z_0^2 z_1, 3z_1^2 + z_0^3, z_1^5 \rangle \\ &= \langle 1, z_0, z_0^2, z_1, z_0 z_1, z_1^2, z_0 z_1^2 \rangle. \end{aligned}$$

Hence we obtain $\mu(f) = \tau(f) = 7$.

In the case of weighted homogenous f. Let $w_0, w_1, \ldots, w_n$ and D be positive integers. An element $f \in \mathbb{C}\{\mathbf{z}\}$ is said to be weighted homogeneous of weight $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ and of degree t if

$$f(\alpha^{w_0} z_0, \alpha^{w_1} z_1, \ldots, \alpha^{w_n} z_n) = \alpha^t f(z_0, z_1, \ldots, z_n)$$

for $\alpha \in \mathbb{C}$. In other words, for each monomial $z_0^{a_0} z_1^{a_1} \ldots z_n^{a_n}$ in f we have

$$a_0 w_0 + a_1 w_1 + \cdots + a_n w_n = t.$$

For instance, the following surface singularities known as ADE-type singularities are all weighted homogeneous:

A_n - type:	$z_2^2+z_1^2+z_0^{n+1}=0,$	$\mathbf{w}=(2,2k+1,2k+1),$	$t=4k+2,$	$n=2k \quad k\geq 1$
		$\mathbf{w}=(1,k+1,k+1),$	$t=2k+1,$	$n=2k+1 \quad k\geq 0$
D_n - type :	$z_2^2+z_0z_1^2+z_0^{n-1}=0,$	$\mathbf{w}=(1,n-2,n-1),$	$t=2n-2,$	for $\quad n\geq 4$
E_6 - type:	$z_2^2+z_1^3+z_0^4=0,$	$\mathbf{w}=(3,4,6),$	$t=12,$	
E_7 - type:	$z_2^2+z_1^3+z_0^3z_1=0,$	$\mathbf{w}=(4,6,9),$	$t=18,$	
E_8 - type:	$z_2^2+z_1^3+z_0^5=0,$	$\mathbf{w}=(6,10,15),$	$t=30.$	

Theoreme 3.1.4. ([25],Th.1) Assume that $\mathbb{V}(f)$ has an isolated singularity at the origin. If f is a weighted homogeneous of weight $\mathbf{w}$ and of degree t then

$$\mu(f)=\left(\frac{t}{w_0}-1\right)\cdots\left(\frac{t}{w_n}-1\right)$$

Example 3.1.5. Consider E_7-type singularity. The Milnor number of f is

$$\mu(f)=\left(\frac{18}{4}-1\right)\left(\frac{18}{6}-1\right)\left(\frac{18}{9}-1\right)=7.$$

Theoreme 3.1.6. *Let f be an analytic function. Assume that f has an isolated singularity at the origin. If f is weighted homogeneous polynomial then* $\dfrac{\mu(f)}{\tau(f)}=1.$

The statement follows from the fact that all weighted homogeneous polynomials of weight $(w_0,w_1,\dots,w_n)$ and of degree t satisfies the following relation:

$$tf(\mathbf{z})=w_0z_0\frac{\partial f}{\partial z_0}+w_1z_1\frac{\partial f}{\partial z_1}+\cdots+w_nz_n\frac{\partial f}{\partial z_n}$$

such that $f\in\mathcal{J}(f)$. This relation is known as *Euler Relation*. For instance, the theorem is true for all ADE-type singularities.

Remark 3.1.7. The inverse implication of the previous theorem is also true (See [26]).

3.2 Newton number

Let u, v be points in $\mathbb{R}^{n+1}$. A *line segment* in $\mathbb{R}^{n+1}$ is the set

$$[u\ , v] := \{\alpha u + (1-\alpha)v : 0 \leq \alpha \leq 1\}.$$

The set $S := \{v_0, v_1, \ldots, v_d\} \subseteq \mathbb{R}^{n+1}$ is said to be *convex* if $[v_i\ , v_j] \subseteq S$ for all $i, j \in \{0, 1, \ldots, d\}$. The *convex hull* of S is the set

$$conv(S) := \{\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_d v_d \mid d \geq 1, \alpha_i \geq 0, \sum_{i=0}^{d} \alpha_i = 1\}.$$

A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^{n+1}$. A *(positive) closed half space* is

$$H_+ := \{\mathbf{v} \in \mathbb{R}^{n+1} \mid \langle u, \mathbf{v} \rangle \geq a\}$$

where u is nonzero and $\langle -, - \rangle$ denotes the standard inner product. A *polyhedron* is an intersection of finitely many closed half-spaces in $\mathbb{R}^{n+1}$. A polyhedron is called *bounded* if it does not contain $\{u + tv \mid t \geq 0\}$ for any $v \neq 0$. In particular, a bounded polyhedron in $\mathbb{R}^{n+1}$ is a polytope in $\mathbb{R}^{n+1}$ which is known *Minkowski-Weyl Theorem*, see ([30],1.1). The *dimension* of a polytope is the dimension of the corresponding linear vector space. For instance, 0-dimensional polytopes are the *points* in $\mathbb{R}^{n+1}$, 1-dimensional polytopes are the *line segments* in $\mathbb{R}^{n+1}$ and 2-dimensional polytopes are the *polygons* in $\mathbb{R}^{n+1}$. A *d-simplex* is the convex hull of any $d+1$ affinely independent points in $\mathbb{R}^{d+1}$ with $d \leq n$. A d-simplex is a polytope of dimension d with $d+1$ vertices. We will use simplex to determine higher dimensional polytopes. A *face* of a polytope P is the intersection of P with a supporting hyperplane. Hence P itself is a face. The *dimension* of a face F of P is the minimum of dimensions of coordinate plane containing F. Indeed, if P is a polytope in $\mathbb{R}^{n+1}$, then

- 1-dimensional faces F_i of P are given by $P \cap \{z_{i_0} = 0, z_{i_1} = 0, \cdots, z_{i_{n-1}} = 0\}$ for all subset $\{i_0, i_1, \cdots, i_{n-1}\} \subset \{0, \cdots, n\}$,
- 2-dimensional faces A_i of P are given by $P \cap \{z_{i_0} = 0, z_{i_1} = 0, \ldots, z_{i_{n-2}} = 0\}$ for all subset $\{i_0, i_1, \ldots, i_{n-2}\} \subset \{0, \cdots, n\}$, and
- k-dimensional faces of P are given by $P \cap \{z_{i_0} = 0, z_{i_1} = 0, \cdots, z_{i_{n-k}} = 0\}$ for all subset $\{i_0, i_1, \ldots, i_{n-k}\} \subset \{0, \ldots, n\}$.

The number of k-dimensional faces of a polytope with n vertices is at most $\binom{n}{k+1}$. Each faces F of P is a polytope (see [30], 2.3). Let $F := conv(\mathbf{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ be a k-dimensional face of P with $v_{i_j} \in \mathbb{R}^k$ and $\{i_1, \ldots, i_{n-k}\} \subset \{0, \cdots, n\}$. Then the k-dimensional Euclidean volume of F is

$$vol_k(F) := \frac{1}{k!}\left| det \begin{pmatrix} \mathbf{0} & v_{i_1} & v_{i_2} & \ldots & v_{i_k} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix} \right|$$

where the matrix has size $(k+1) \times (k+1)$.

3.2.1 Newton number of a polyhedron P

Let P be a bounded polyhedron in $\mathbb{R}^{n+1}$. We are interested in a convex polyhedron in $\mathbb{R}^{n+1}$ with vertices at integral points. Let V_i be the sum of i-dimensional Euclidean volumes of the intersection of P and i-dimensional coordinate plane sections of $\mathbb{R}^{n+1}$. That is,

- V_1 is the sum of lenghts of the 1-dimensional compact faces of P,
- V_2 is the sum of areas of the 2-dimensional compact faces of P
- V_k is the sum of the k-dimensional volume of the 2-dimensional compactt faces of P, and
- $V_n + 1$ is the volume of P.

Definition 3.2.1. [15] *The number*

$$\nu(P) := (n+1)!V_{n+1} - n!V_n + \cdots + (-1)^n 1!V_1 + (-1)^{n+1} \tag{3.2.1}$$

is called the Newton number of P.

Example 3.2.2. Consider the following polyhedron P:

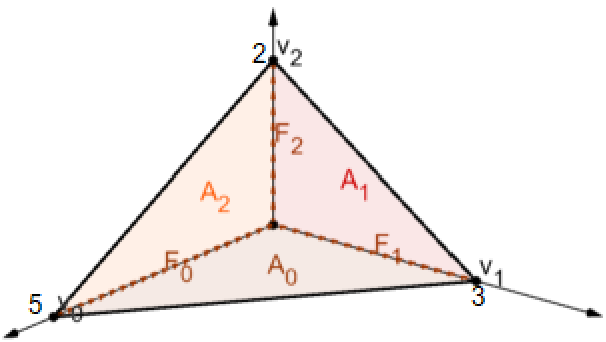

Figure 3.1: Polyhedron P

The notation in Figure 3.1 means the following:

$$A_0 = P \cap \{z_2 = 0\}, \qquad A_1 = P \cap \{z_0 = 0\}, \qquad A_2 = P \cap \{z_1 = 0\};$$
$$F_0 = P \cap \{z_1 = 0, z_2 = 0\}, \quad F_1 = P \cap \{z_0 = 0, z_2 = 0\}, \quad F_2 = P \cap \{z_0 = 0, z_1 = 0\}.$$

We need V_3 which is the volume of P, V_2 which is the sum of areas of A_i's and V_1 which is the sum of the lengths of F_i's for $i = 0, 1, 2$. We have $V_3 = Vol_3(P) = 5$, $V_2 = Vol_2(A_0) + Vol_2(A_1) + Vol_2(A_2) = \dfrac{31}{2}$ and finally $V_1 = Vol_1(F_0) + Vol_1(F_1) + Vol_1(F_2) = 10$. By the formula 3.2.1, we obtain $\nu(P) = 3!V_3 - 2!V_2 + V_1 - 1$. So, $\nu(P) = 8$.

3.2.2 Newton number of f

Let $f := \sum_{v \in \mathbb{N}^n} c_v \mathbf{z}^v$ with $c_v \in \mathbb{C}$. The support of f is the set

$$supp(f) := \{v \in \mathbb{N}^{n+1} \mid c_v \neq 0\}.$$

The *Newton polyhedron* of f in $\mathbb{R}^{n+1}$ is

$$\Gamma_+(f) := conv\{v + \mathbb{R}^{n+1} \mid v \in supp(f) - \{\mathbf{0}\}\}.$$

The *Newton boundary* is

$$\Gamma(f) := \bigcup_F \{F \text{ is a compact face of } \Gamma_+(f)\}.$$

The segments from the origin to the extremities in $\Gamma(f)$ defines an $(n+1)$-dimensional cone lying below $\Gamma(f)$ which will be denoted by $\Gamma_-(f)$. That is, $\Gamma(f)$ is contained in $\Gamma_-(f)$ as a union of n-dimensional compact faces. The restriction of f to the n-dimensional compact face

$$f_0 := \sum_{v \in \Gamma(f)} c_v \mathbf{z}^v$$

is called the *Newtonian main part* of f.

Example 3.2.3. [E_8-type] Consider $f = z_2^2 + z_1^3 + z_0^5$. We have

$$supp(f) = \{v_0 = (5,0,0), v_1 = (0,3,0), v_2 = (0,0,2)\}.$$

The Newton polyhedron $\Gamma_+(f)$ looks as below.

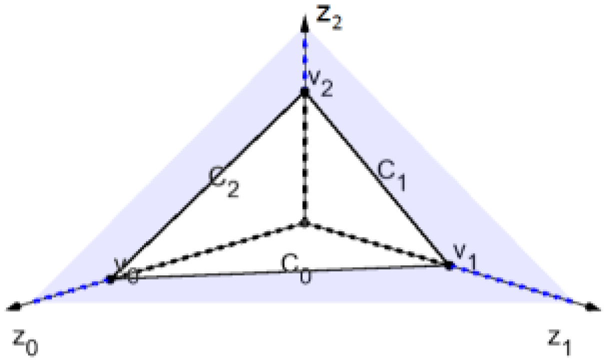

Figure 3.2: Newton Polyhedron of E_8-Type Singularity

The boundary $\Gamma(f)$ of $\Gamma_+(f)$ is

$$\Gamma(f) = v_0 \cup v_1 \cup v_2 \cup C_0 \cup C_1 \cup C_2 \cup C$$

where v_i's are 0-dimensional compact faces; $C_0 = conv(v_0 v_1)$, $C_1 = conv(v_1 v_2)$ and $C_2 = conv(v_2 v_0)$ are 1-dimensional compact faces; $C = conv(v_0 v_1 v_2)$ is the 2-dimensional compact face in $\Gamma_+(f)$. The cone $\Gamma_-(f)$ lying below $\Gamma(f)$ in $\Gamma_+(f)$ is the cone with vectors $\overrightarrow{0v_i}$.

Definition 3.2.4. *Let F be a compact face of $\Gamma_+(f)$. The restriction of f to F*

$$f\,|_F := \sum_{\nu \in F \cap \mathbb{N}^{n+1}} c_\nu z^\nu$$

is called the face function of f associated with F. The restriction of f to $\Gamma(f)$ is

$$f\,|_{\Gamma(f)} := \sum_{v \in \Gamma(f)} c_v z^v = f_0$$

where f_0 is the Newtonian main part of f.

Example 3.2.5. [E_8-type] Consider $\Gamma_+(f)$ as in Figure 3.2. The face functions of the 0-dimensional compact faces are

$$f\,|_{v_0} = \mathbf{z}^{v_0} = z_0^5 z_1^0 z_2^0 = z_0^5, \quad f\,|_{v_1} = \mathbf{z}^{v_1} = z_0^0 z_1^3 z_2^0 = z_1^3, \quad f\,|_{v_2} = \mathbf{z}^{v_2} = z_0^0 z_1^0 z_2^2 = z_2^2.$$

The face functions of the 1-dimensional compact faces are

$$f\,|_{C_0} = \mathbf{z}^{v_0} + \mathbf{z}^{v_1} = z_0^5 + z_1^3, \quad f\,|_{C_1} = \mathbf{z}^{v_1} + \mathbf{z}^{v_2} = z_1^3 + z_2^2, \quad f\,|_{C_2} = \mathbf{z}^{v_2} + \mathbf{z}^{v_0} = z_2^2 + z_0^5.$$

Finally, the face function of the unique 2-dimensional compact face $C = conv(v_0, v_1, v_2)$ of $\Gamma(f)$ is

$$f \mid_C = \mathbf{z}^{v_0} + \mathbf{z}^{v_1} + \mathbf{z}^{v_2} = z_0^5 + z_1^3 + z_2^2.$$

Note that $f \mid_C = f_0 = f$ which is Newtonian main part of f.

Remark 3.2.6. The Milnor number of $f|_K$ is

$$\mu(f|_F) = \mathbb{C}\{z_{i_0}, z_{i_1}, \cdots, z_{i_j}\}/\mathcal{J}(f|_F)$$

where $\{z_{i_j}\text{-}axis\} \cap F \neq \emptyset$.

Example 3.2.7. [E_8-type] The Milnor numbers $\mu(f|_F)$ for each compact face F we have $\mu(f_0) = \mu(f|_C) = \mathbb{C}[[z_0, z_1, z_2]]/\langle 5z_0^4 + 3z_1^2 + 2z_2\rangle = 8 = \mu(f)$ and

$$\begin{aligned}
\mu(f|_{C_0}) &= \mathbb{C}[z_0, z_1]/\langle 5z_0^4 + 3z_1^2\rangle = 8, & \mu(f|_{v_0}) &= \mathbb{C}[z_0]/\langle 5z_0^4\rangle = 4,\\
\mu(f|_{C_1}) &= \mathbb{C}[z_1, z_2]/\langle 3z_1^2 + 2z_2\rangle = 2, & \mu(f|_{v_1}) &= \mathbb{C}[z_1]/\langle 3z_1^2\rangle = 2,\\
\mu(f|_{C_2}) &= \mathbb{C}[z_0, z_2]/\langle 5z_0^4 + 2z_2\rangle = 4, & \mu(f|_{v_2}) &= \mathbb{C}[z_2]/\langle 2z_2\rangle = 1.
\end{aligned}$$

One can attribute the Milnor number to each face function and compute V_k by means of them. We will examine it in the next section.

Definition 3.2.8. *We say that f is convenient if $\Gamma_+(f)$ meets each of the coordinate axes in $\mathbb{R}^{n+1}$. Otherwise, f is called non-convenient.*

Let us first treat the case where f is convenient and extend the definition to the case where f is non-convenient.

When f is convenient The number $\nu(\Gamma_-(f))$ is called the Newton number of f, denoted by $\nu(f)$. That means,

$$\nu(f) = \nu(\Gamma_-(f))$$

In particular, the Newton number of an analytic function f is a lower bound for its Milnor number i.e. $\mu(f) \geq \nu(f)$ regardless of whether f is convenient or not (see [15], 1.10).

Example 3.2.9. In the case of E_8-type singularity, f is convenient as $\Gamma_+(f)$ meets each axis in $\mathbb{R}^3$ (see Figure 3.1). We have $\nu(f) = \nu(\Gamma_-(f)) = 8$ (see Example 3.2.2). Note that the definition is not applied in the case of E_7-type singularity since it is not convenient.

Remark 3.2.10. [12] Let Γ_+ and Γ'_+ be two convenient polyhedra in $\mathbb{R}^{n+1}$. If $\Gamma'_+ \subset \Gamma_+$, then $\nu(\Gamma'_+) \geq \nu(\Gamma_+)$.

Example 3.2.11. Consider $f = z_2^2 + z_1^3 + z_0^5$. We have

$$\Gamma_+(f) = conv(v_0 = (5,0,0) + \mathbb{R}^3, v_1 = (0,3,0) + \mathbb{R}^3, v_2 = (0,0,2) + \mathbb{R}^3)$$

and $\nu(\Gamma_-) = 8$ from Example 3.2.9. Let $g = z_2^2 + z_1^3 + z_0^6 \in \mathbb{C}\{\mathbf{z}\}$. We have

$$\Gamma_+(g) = conv(v_0 = (6,0,0) + \mathbb{R}^3, v_1 = (0,3,0) + \mathbb{R}^3, v_2 = (0,0,2) + \mathbb{R}^3)$$

such that $\Gamma_+(g) \subset \Gamma_+(f)$. We will compute the Newton number of $\Gamma_-(g)$. The cone $\Gamma_-(g)$ looks like in $\mathbb{R}^3$ as follows:

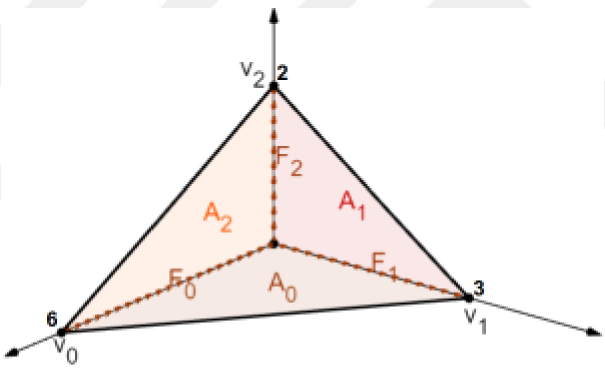

Figure 3.3: The cone $\Gamma_-(g)$

Here we have

$$V_1 = vol_1(conv(\mathbf{0}, v_0)) + vol_1(conv(\mathbf{0}, v_1)) + vol_1(conv(\mathbf{0}, v_2)) = 9$$

$$V_2 = vol_2(conv(\mathbf{0}, v_0, v_1)) + vol_2(conv(\mathbf{0}, v_1, v_2)) + vol_2(conv(\mathbf{0}, v_0, v_2)) = 18$$

$$V_3 = vol_3(\Gamma_-(g)) = 6.$$

Then we obtain $\nu(\Gamma_-(g)) = 3!6 - 2!18 + 9 - 1 = 10$. Hence, $\nu(g) > \nu(f)$.

When f is convenient, the face functions $f|_F$ are convenient. So, the Newton number of f is also given by means of the sum of the volumes of $\Gamma_-(f|_F)$ using the following formula:

$$V_k := \sum_{\substack{F \in \Gamma(f) \\ dim(F) = k-1}} Vol_k(\Gamma_-(f\,|_F)).$$

Here, $k \in \mathbb{N}^*$ and $V_{n+1} := Vol_{n+1}(\Gamma_-(f))$.

When f is non-convenient This means that there is at least one $i \in \{0,1,\ldots,n\}$ such that no monomial of the form z_i^k, $k \in \mathbb{N}$ (a pure power of z_i), appears in f. Renumbering the variables, we may assume that there exist $q \in \{0,\ldots,n-1\}$ such that pure powers of $z_0,\ldots,z_q$ do not appear in f, but pure powers of $z_{q+1},\ldots,z_n$ do. Put

$$\tilde{f}_m := f + z_0^m + z_1^m + \cdots + z_q^m$$

with $m \in \mathbb{N}$. Here, $\tilde{f}_m$ is convenient and the Newtonian main part of $\tilde{f}_m$ is

$$(\tilde{f}_m)_0 := f_0 + z_0^m + z_1^m + \cdots + z_q^m$$

where f_0 is the Newtonian main part of f. So the Newton number of f is defined by

$$\nu(f) := \sup_{m \in \mathbb{N}^*} \nu(\tilde{f}_m) \tag{3.2.2}$$

Let us clarify the choice of m: Assume that f does not contain the monomial $z_{i_0}^{v_{i_0}}$ but contains all other monomials $z_j^{v_j}$ for $j \neq i_0$. Put $\tilde{f}_m = f + z_{i_0}^m$ such that $\tilde{f}_m$ becomes convenient.

Example 3.2.12. **[E_7-type]** Consider $f = z_2^2 + z_1^3 + z_0^3 z_1$. The support set of f is

$$supp(f) = \{v_0 = (3,1,0), v_1 = (0,3,0), v_2 = (0,0,2)\}$$

and

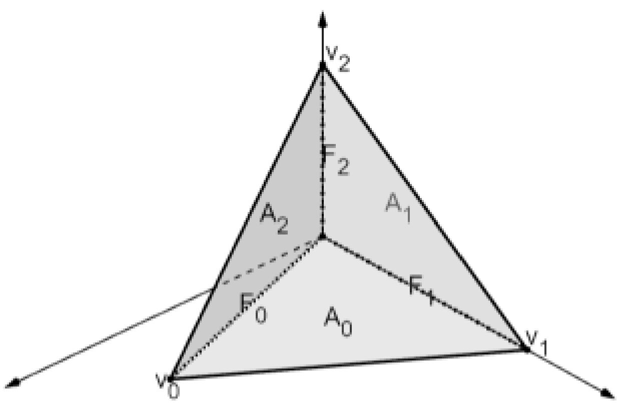

Figure 3.4: Newton boundary of E_7-type singularity

The Newton boundary $\Gamma(f)$ doesn't meet the z_0-axis, so f is non-convenient. Indeed, $F_0 \neq \Gamma_-(f) \cap \{z_1 = 0, z_2 = 0\}$ and $A_2 \neq \Gamma_-(f) \cap \{z_1 = 0\}$. Consider

$$\widetilde{f_m} = f + z_0^m.$$

We obtain

$$\nu(\widetilde{f_1}) = 0, \nu(\widetilde{f_2}) = 2, \nu(\widetilde{f_3}) = 4, \nu(\widetilde{f_4}) = 6, \; and \; \nu(\widetilde{f_m}) = 7 \; for \; m \geq 5$$

hence $sup\{0,2,4,6,7\} = \nu(f) = 7$.

Lemma 3.2.13. ([15],3.5) Let J be an ideal in $\mathbb{C}[[\mathbf{z}]]$. Assume that

$$dim\left(\mathbb{C}[[\mathbf{z}]]/(J + \mathfrak{m}^{l+1})\right) \leq l$$

with $l > 0$. Then $\mathfrak{m}^l \subseteq J$.

Proposition 3.2.14. *Assume that f has an isolated singularity at the origin. With preceding notation, $\mu(\tilde{f}) = \mu(f)$ for sufficiently large m.*

Proof. Assume that f does not contain the monomial of the form $z_{i_0}^{v_{i_0}}$ but contains all the monomials $z_j^{v_j}$ for $j \neq i_0$. Put $\widetilde{f_m} := f + z_{i_0}^m$ with $m \geq 2$ such that $\widetilde{f_m}$ is convenient. Let $m_0 := \mu(f)$. Consider $m \geq m_0 + 2$. If $\widetilde{f_m} - f = z_{i_0}^m$ then $\widetilde{f_m} - f \in \mathfrak{m}^{m_0+2}$ where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[\mathbf{z}]]$. Moreover we obtain $\dfrac{\partial(\widetilde{f_m} - f)}{\partial z_i} \in \mathfrak{m}^{m_0+1}$ That is,

$$\mathcal{J}(f) + \mathfrak{m}^{m_0+1} = \mathcal{J}(\widetilde{f}) + \mathfrak{m}^{m_0+1}.$$

As $\mu(f) < m_0 + 1$, we have $dim(\mathbb{C}[[\mathbf{z}]]/\left\langle \mathcal{J}(f) + \mathfrak{m}^{m_0+2} \right\rangle) < m_0 + 1$. Then, by the previous lemma, we obtain $\mathfrak{m}^{m_0+1} \subseteq \mathcal{J}(f)$ and so,

$$\mathcal{J}(f) = \mathcal{J}(\widetilde{f}) + \mathfrak{m}^{m_0+1}.$$

Furthermore,

$$m_0 = \mu(f) = dim(\mathbb{C}[[\mathbf{z}]]/\mathcal{J}(f)) = dim(\mathbb{C}[[\mathbf{z}]]/(\mathcal{J}(\widetilde{f}) + \mathfrak{m}^{m_0+1}) < m_0 + 1.$$

So $dim(\mathbb{C}[[\mathbf{z}]]/(\mathcal{J}(\widetilde{f}) + \mathfrak{m}^{m_0+2}) < m_0 + 1$ gives $\mathfrak{m}^{m_0+1} \subseteq \mathcal{J}(\widetilde{f})$ by the previous lemma.

Then,

$$\mathcal{J}(f) = \mathcal{J}(\widetilde{f_m}) + \mathfrak{m}^{m_0+1} = \mathcal{J}(\widetilde{f_m}).$$

Hence,

$$\mu(f) = dim(\mathbb{C}[[\mathbf{z}]]/(\mathcal{J}(\widetilde{f_m}) + \mathfrak{m}^{m_0+1}) = dim(\mathbb{C}[[\mathbf{z}]]/\mathcal{J}(\widetilde{f_m})) = \mu(\widetilde{f_m}).$$

$\square$

Proposition 3.2.15. ([5],3.6) Let f be a non-convenient function. Let $\nu(f) < \infty$. With preceding notation, then there exist $m \geq 2$ such that $\nu(\tilde{f}) = \nu(f)$.

Proof. Assume that f does not contain the monomial of the form $z_{i_0}^{v_{i_0}}$ but contains all the monomials $z_j^{v_j}$ for $j \neq i_0$. Put $\widetilde{f_m} := f + z_{i_0}^m$ with $m \geq 2$ such that $\widetilde{f_m}$ is convenient. For $m \leq m'$, we have $\Gamma_+(\tilde{f}_{m'}) \subset \Gamma_+(\tilde{f}_m)$. This gives $\nu(\Gamma_-(\tilde{f}_{m'})) \geq \nu(\Gamma_-(\tilde{f}_m))$ by Remark 3.2.10. Since $\Gamma_-(\widetilde{f_m})$'s are convenient we obtain $\nu(\tilde{f}_{m'}) \geq \nu(\tilde{f}_m)$. More generally, for an increasing sequence $2 \leq m < m_1 < m_2 < \dots$ of positive integers, we have

$$0 \leq \nu(\tilde{f}_m) \leq \nu(\tilde{f}_{m_1}) \leq \nu(\tilde{f}_{m_2}) \leq \dots.$$

From the fact that $\nu(f) = \sup_m(\nu(\widetilde{f_m})) < \infty$ the sequence $\{\nu(\widetilde{f_m})\}_m$ is bounded. Then it converge to its upper bound i.e. $\nu(\tilde{f}_m) = \nu(f)$ for $m \geq 2$. $\square$

Remark 3.2.16. ([15],1.25) Let f be a weighted homogeneous function of weight $(w_0, w_1, \dots, w_n)$ and of degree t. Consider f is non-convenient. By taking $m = (n+1)M$ with $M := \max_{0 \leq i \leq n} \left\lfloor \frac{t}{w_i} + 1 \right\rfloor$ we have $\nu(f) = \nu(f + z_0^m + \dots + z_n^m)$.

Example 3.2.17. [E_7-type] Consider $f = z_2^2 + z_1^3 + z_0^3 z_1$. f is weighted homogeneous of weight $\mathbf{w} = (4, 6, 9)$ and of degree $t = 18$ (see Table 3.1). So,

$$M = max\left\{ \left\lfloor \frac{18}{4} + 1 \right\rfloor, \left\lfloor \frac{18}{6} + 1 \right\rfloor, \left\lfloor \frac{18}{9} + 1 \right\rfloor \right\} = 5$$

which gives $m = 3.5 = 15$. Put $\tilde{f}_{15} := f + z_0^{15} + z_1^{15} + z_2^{15}$. The Newton boundary of $\tilde{f}_{15}$ is

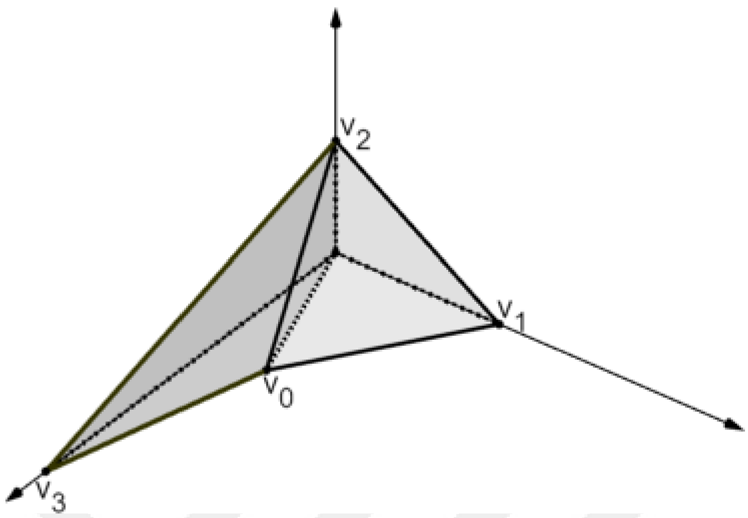

Figure 3.5: Newton boundary of $\tilde{f}_{15}$

where $v_0 = (3,1,0), v_1 = (0,3,0), v_2 = (0,0,2)$ and $v_3 = (15,0,0)$. Moreover, we have

$V_1 = vol_1(vol_1(conv(\mathbf{0},v_1)) + vol_1(conv(\mathbf{0},v_2)) + vol_1(conv(\mathbf{0},v_3)) = 20$

$V_2 = vol_2(conv(\mathbf{0},v_0,v_3)) + vol_2(conv(\mathbf{0},v_0,v_1) + vol_2(conv(\mathbf{0},v_1,v_2)) + vol_2(conv(\mathbf{0},v_2,v_3)) = 30$

$V_3 = vol_3(conv(\mathbf{0},v_0,v_1,v_2)) + vol_3(conv(\mathbf{0},v_0,v_2,v_3)) = 8.$

Then we obtain $\nu(\Gamma_-(\tilde{f}_{15})) = 3!8 - 2!30 + 20 - 1 = 7$. Moreover, since $\tilde{f}_{15}$ is convenient we have $\nu(\Gamma_-(\tilde{f}_{15})) = \nu(\tilde{f}_{15})$. Hence, $\nu(\tilde{f}_{15}) = 7 = \nu(f)$, see also Example 3.2.12 for $\nu(f)$.

Proposition 3.2.18. ([17], 2.2) Assume that f is an analytic function at the origin with only isolated singularity. Then $\tau(f) \geq \dfrac{\nu(f)}{(n+1)}$.

Proof. First, note that if f has an isolated singularity then we have $\dfrac{\mu(f)}{\tau(f)} \leq n+1$ by the Equality 3.1. This gives $\dfrac{\mu(f)}{n+1} \leq \tau(f)$ but we know that $\mu(f) \geq \nu(f)$ for any analytic function f. So we obtain $\dfrac{\nu(f)}{n+1} \leq \dfrac{\mu(f)}{n+1} \leq \tau(f)$. That is, $\dfrac{\nu(f)}{n+1} \leq \tau(f)$. □

Example 3.2.19. Consider $f = z_2^3 + (z_0 + z_1^4)z_2^2 + 2z_0z_1^4z_2 + (z_0^2 + z_1^3)z_1^4 + z_0^8 \in \mathbb{C}\{\mathbf{z}\}$ with an isolated singularity at the origin. We have

$supp(f) = \{(0,0,3),(1,0,2),(0,4,2),(1,4,2),(0,8,0),(2,4,0),(0,7,0),(8,0,0)\}.$

So it is convenient. The Newton boundary of f is

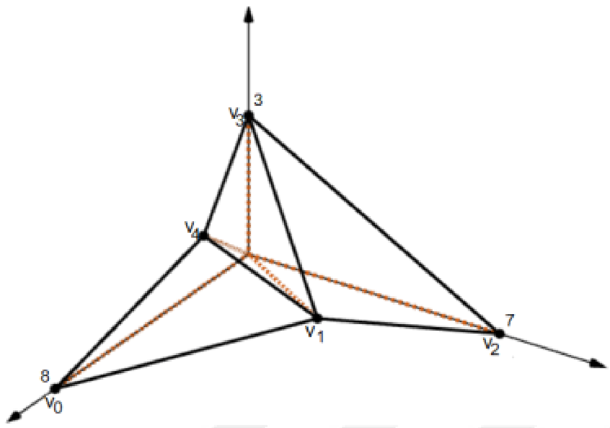

Figure 3.6: Newton boundary of f

where $v_0 = (8,0,0), v_1 = (2,4,0), v_2 = (0,7,0), v_3 = (0,0,3)$ and $v_4 = (1,0,2)$. In this case, $\Gamma_-(f)$ consists of 3 tetrahedra such that

$$\Gamma_-(f) = conv(\mathbf{0}, v_0, v_1, v_4) \cup conv(\mathbf{0}, v_1, v_3, v_4) \cup conv(\mathbf{0}, v_1, v_2, v_3).$$

So the volumes of compact faces in $\Gamma_-(f)$ are

$$\begin{array}{lll} vol_1(conv(\mathbf{0}, v_0)) = 8 & vol_1(conv(\mathbf{0}, v_1)) = 7 & vol_1(conv(\mathbf{0}, v_2)) = 3 \\ vol_2(conv(\mathbf{0}, v_0, v_1)) = 16 & vol_2(conv(\mathbf{0}, v_1, v_2)) = 7 & vol_2(conv(\mathbf{0}, v_2, v_3)) = \frac{3}{2} \\ vol_2(conv(\mathbf{0}, v_3, v_4)) = \frac{3}{2} & vol_2(conv(\mathbf{0}, v_0, v_4)) = 8 & \\ vol_3(conv(\mathbf{0}, v_0, v_1, v_4)) = \frac{32}{3} & vol_3(conv(\mathbf{0}, v_1, v_3, v_4)) = 2 & vol_3(conv(\mathbf{0}, v_1, v_2, v_3)) = 7 \end{array}$$

Thus $V_1 = 18, V_2 = 43$ and $V_3 = \frac{59}{3}$. Hence $\nu(\Gamma_-(f)) = 3!\frac{59}{3} - 2!43 + 18 - 1 = 39$. Since f is convenient we have $\nu(f) = \nu(\Gamma_-(f)) = 39$. Then, by the previous proposition, we have $\tau(f) \geq \dfrac{39}{3} = 13$.

3.2.3 Non-degeneracy of f

Let F be a compact face in $\Gamma(f)$ and f_F denote the face function associated with F.

Definition 3.2.20. *We say that f is non-degenerate on $F \in \Gamma(f)$ (in the sense of Kouchnirenko) if the system*

$$\frac{\partial f_F}{\partial z_0}(q) = \frac{\partial f_F}{\partial z_1}(q) = \cdots = \frac{\partial f_F}{\partial z_n}(q) = 0 \tag{3.2.3}$$

has no solution in $(\mathbb{C}^*)^{n+1}$*. We say that* f *is Newton non-degenerate (in the sense of Kouchnirenko) if it is non-degenerate with respect to each compact face in* $\Gamma(f)$*. We say that* f *is degenerate (in the sense of Kouchnirenko) if there exists at least one compact face* F *on which* f_F *is degenerate.*

Example 3.2.21. (i) The singularity of E_8-type is Newton non-degenerate. See Example 3.2.5 for the face functions.

(ii) The singularity E_7-type is Newton non-degenerate. The face functions (means, the restriction of f to the compact faces) are:

$$f|_{v_0} = z_0^3 z_1, \quad f|_{v_1} = z_1^3, \quad f|_{v_2} = z_2^2$$

$$f|_{C_0} = z_0^3 z_1 + z_1^3, \quad f|_{C_1} = z_1^3 + z_2^2, \quad f|_{C_2} = z_0^3 + z_2^2$$

Note that C_i's are the 1-dimensional compact faces of $\Gamma(f)$; and, we have $f|_C = f$ for the 2-dimensional compact face C of $\Gamma(f)$. Since any of the face functions has singular points except the origin, f is non-degenerate. See Figure 3.2.17 for $\Gamma(f)$.

Corollary 3.2.22. [15] Let f be convenient. Assume that f_0 is non-degenerate in Kouchnirenko sense and $\mu(f_0) = \mu(f)$. For a compact face F in $\Gamma_-(f)$, let us denote d_k^F the dimension of $\Gamma_-(f|_F)$ in $\mathbb{R}^{n+1}$. Then we have

$$k!V_k = \begin{cases} \displaystyle\sum_{\substack{F\in\Gamma_-(f)\\ d_k^F = k}} \left(\sum_{\tilde{F}\in\Gamma_-(f|_F)} \mu(f|_{\tilde{F}}) \right), & if\ k = 1,2,\cdots,n \\[2em] \displaystyle\sum_{F\in\Gamma_-(f)} \mu(f|_F), & if\ k = n+1. \end{cases}$$

Example 3.2.23. **[E_8-type]** For each compact face $F \in \Gamma(f)$ we have

$$3!V_3 = \mu(f|_{\mathbf{0}}) + \mu(f|_{v_0}) + \mu(f|_{v_1}) + \mu(f|_{v_2}) + \mu(f|_{C_0}) + \mu(f|_{C_1}) + \mu(f|_{C_2}) + \mu(f_0) = 30,$$
$$2!V_2 = 3\mu(f|_{\mathbf{0}}) + 2\mu(f|_{v_0}) + 2\mu(f|_{v_1}) + 2\mu(f_{v_2}) + \mu(f|_{C_0}) + \mu(f|_{C_1}) + \mu(f|_{C_2}) = 31,$$
$$1!V_1 = 3\mu(f|_{\mathbf{0}}) + \mu(f|_{v_0}) + \mu(f|_{v_1}) + \mu(f|_{v_2}) = 10.$$

By Formula 3.2.1 we obtain $\nu(f) = 30 - 31 + 10 - 1 = 8$.

Remark 3.2.24. The non-degeneracy of f does not depend on the fact that f is convenient or non-convenient.

Example 3.2.25. [1] Consider $f = z_0^2(z_0+z_1)^2 + z_0(z_0+z_1)^4 + z_0^5 + (z_0+z_1+z_2)^5$. The Newton boundary $\Gamma(f) \subset \mathbb{R}^3$ is

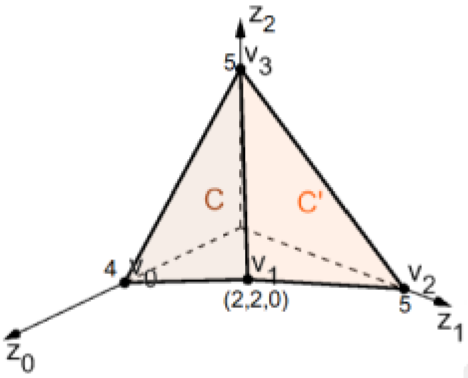

Figure 3.7: Newton boundary of f

We have $f|_C = z_0^4 + z_0^2 z_1^2 + z_2^5$ for the 2-dimensional compact face $C = conv(v_0, v_1, v_3)$, so

$$\begin{cases} \dfrac{\partial f|_C}{\partial z_0} = 4z_0^3 + 2z_0 z_1^2 = 0 \\ \\ \dfrac{\partial f|_C}{\partial z_1} = 2z_0^2 z_1 = 0 \\ \\ \dfrac{\partial f|_C}{\partial z_2} = 5z_2^4 = 0 \end{cases}$$

The system above admits a solution at each point $(0, z_1, 0)$ with $z_1 \in \mathbb{C}$. Hence f is degenerate on C. Then f is degenerate in the Kouchnirenko sense.

Proposition 3.2.26. ([5], 3.8) Consider f is Newton non-degenerate and non-convenient as in §3.2.2. Put $\tilde{f}_m := f + z_0^m + z_1^m + \cdots + z_q^m$. Then the Newtonian main part of $\tilde{f}_m$ is Newton non-degenerate for $m \geq 2$.

Example 3.2.27. [E_7-type] Consider $f = z_2^2 + z_1^3 + z_0^3 z_1$. Put $\widetilde{f} = f + z_0^5$. Note that f is non-convenient and Newton non-degenerate (see Example 3.2.21). The Newton boundary $\Gamma(\widetilde{f})$ gets the additional part and looks as below.

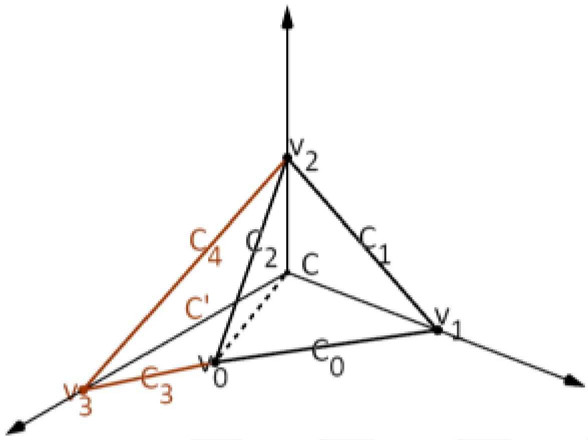

Figure 3.8: Newton boundary of $\widetilde{f}$

So $\widetilde{f}$ is clearly convenient. In the figure above, the face generated by v_0, v_1 and v_2 is $\Gamma(f)$. In Example 3.2.21, we saw that the compact faces C and C_i, v_i with $i = 0,1,2$ are Newton non-degenerate. So it remains to show that f is Newton non-degenerate on the compact faces v_3, C_3, C_4, and C'. We have $f\mid_{v_3} = \mathbf{z}^{v_0} = z_0^5$ and

$$f_{C_3} = z_0^5 + z_0^3 z_1, \quad f_{C_4} = z_0^5 + z_2^2, \quad f_{C'} = z_0^5 + z_0^3 z_1 + z_2^2.$$

As the only singular point of these face functions is the origin we conclude that $\widetilde{f}$ is Newton non-degenerate.

Theoreme 3.2.28. ([15],1.10) If the Newtonian main part f_0 of a polynomial f is non-degenerate then $\mu(f) = \nu(f)$.

Proof. First, assume that f is convenient. We know that the Newton number of f is given by the formula

$$\nu(f) = (n+1)!V_{n+1} - n!V_n + \cdots + (-1)^{(n+1-k)}V_k + \cdots + (-1)^n V_n + (-1)^{n+1}.$$

Using Remark 3.2.22 this formula becomes

$$\nu(f) = (n+1)! \sum_{F\in\Gamma_-(f)} \mu(f|_F) + \cdots + (-1)^{n+1-k} k! \sum_{\substack{F\in\Gamma(f)\\ d_k^F = k}} \sum_{\substack{\tilde{F}\in\Gamma_-(f)\\ \tilde{F}\subset F}} \mu(f|_{\tilde{F}}) + \cdots + (-1)^{n+1}$$

That is,

$$\nu(f) = \sum_{F\in\Gamma_-(f)} \sum_{\substack{\tilde{F}\in\Gamma_-(f)\\ \tilde{F}\subset F}} (-1)^{(n+1)-d_k^F} \mu(f|_{\tilde{F}}).$$

Let us rearrange it with respect to d_k^F. We obtain

$$\nu(f) = \sum_{F\in\Gamma_-(f)} \sum_{i=0}^{(n+1)-d_k^F} (-1)^{(n+1)-d_k^F-i} \binom{(n+1)-d_k^F}{i} \mu(f_F).$$

For $d_{n+1}^K = n+1$, the coefficient of $\mu(f_F)$ is equal to 1. Hence we get

$$\nu(f) = \mu(f_0) + \sum_{\substack{F\in\Gamma(f)\\ d_k^F\neq(n+1)}} \sum_{i=0}^{(n+1)-d_k^F} (-1)^{(n+1)-d_k^F-i} \binom{(n+1)-d_k^F}{i} \mu(f_F).$$

For $d_k^F = 1,2,\cdots,n$ and $N = (n+1)-d_k^F > 0$, the binomial theorem gives

$$\sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} = (1+(-1))^N = 0.$$

Then $\nu(f) = \mu(f_0)$. Since $\mu(f_0) = \mu(f)$ for a convenient function we have $\nu(f) = \mu(f)$.

Now, assume that f is non-convenient with $\nu(f) < \infty$. Without loss of generality, assume that f does not contain the monomial $z_{i_0}^{\nu_{i_0}}$ but contains all other monomials $z_j^{\nu_j}$ for $j \neq i_0$. Put $\tilde{f}_m = f + z_{i_0}^m$ with $m > 2$ such that $\tilde{f}_m$ is convenient. By Proposition 3.2.26 the Newtonian main part $(\tilde{f}_m)_0$ of $\tilde{f}_m$ is non-degenerate. Hence $\nu(f) = \nu(\tilde{f}_m) = \mu(\tilde{f}_m)$. As $\tilde{f}_m - f = z_{i_0}^m \in \mathfrak{m}^m$ where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{\mathbf{z}\}$, by Proposition 3.2.14 we have $\mu(\tilde{f}) = \mu(f)$ for sufficiently large m. Then,

$$\nu(f) = \nu(\tilde{f}) = \mu(\tilde{f}) = \mu(f).$$

It is clear that when $\nu(f) = \infty$, $\mu(f) = \infty$. $\square$

The following example shows that one can find f whose Newtonian main part f_0 is degenerate and $\nu(f) = \mu(f)$.

Example 3.2.29. (See [15], 1.21) Consider $f = (z_0 + z_1)^2 + z_0 z_2 + z_2^2$. We have $f_0 = f$ and

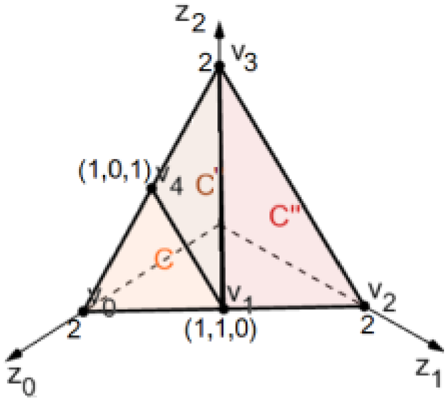

Figure 3.9: Newton Boundary of f

We have $f_C = z_0^2 + z_0z_1 + z_0z_2$. Since $\nabla f_C = \langle 2z_0 + z_1 + z_2, z_0, z_0\rangle$ vanish at the points $(0, z_1, -z_1)$, means we have a solution in $(\mathbb{C}^*)^3$ except the origin, f is Newton degenerate on C; so, f is Newton degenerate. Furthermore, we obtain $V_1 = 6, V_2 = 6$, and $V_3 = Vol_3(\Gamma_-(f)) = \dfrac{4}{3}$; hence $\nu(f) = 3!V_3 - 2!V_2 + V_1 - 1 = 1$. Let us compute the Milnor number of f.

$$\begin{aligned}\mu(f) &= \mathbb{C}\{z_0, z_1, z_2\}/\mathcal{J}(f)\\ &= \mathbb{C}\{z_0, z_1, z_2\}/\langle 2z_0 + 2z_1 + z_2, 2z_0 + 2z_1, z_0 + 2z_2\rangle\\ &= \mathbb{C}\{z_0, z_1, z_2\}/\langle z_0, z_1, z_2\rangle = 1.\end{aligned}$$

Then, we get $\nu(f) = \mu(f)$ when f is Newton degenerate.

For some particular f, the Newton number can be calculated regardless of whether it is convenient or not.

Proposition 3.2.30. *Let f be a weighted homogeneous with weight $(w_0, w_1, \dots, w_n)$ of degree t. Assume that f has an isolated singularity at the origin. If f is Newton non-degenerate then*

$$\nu(f) := \left(\frac{t}{w_0} - 1\right) \cdots \left(\frac{t}{w_n} - 1\right).$$

Proof. This fact follows from Theorem 3.2.28 and Theorem 3.1.4. $\square$

4 Lê numbers

4.1 Lê varieties

Let $f(\mathbf{z}) \in \mathbb{C}\{\mathbf{z}\}$. The singular locus of f is $\Sigma f := \mathbb{V}(\mathcal{J}(f))$ where $\mathcal{J}(f) := \left\langle \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right\rangle$ such that $dim(\Sigma f) \leq dim(\mathbb{V}(f))$. Consider the ideals

$$I_{k,\mathbf{z}} := \left\langle \frac{\partial f}{\partial z_k}, \cdots, \frac{\partial f}{\partial z_n} \right\rangle \text{ for each } k, \;\; 0 \leq k \leq n.$$

In this manner, $I_{k,\mathbf{z}}$ depends on the ordering of the coordinate system in $\mathbb{C}^{n+1}$. Here we consider the ordering as $\mathbf{z} = (z_0, z_1, \ldots, z_n)$ but we will see that the choice of the ordering of the coordinate system plays an important role in the computation of Lê numbers. Moreover, we have

$$\mathbf{0} \subseteq I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_k \subseteq \cdots \subseteq I_1 \subseteq I_0 = \mathcal{J}(f),$$

and, additionally,

$$\mathbf{0} \subseteq I_n : \mathcal{J}(f) \subseteq I_{n-1} : \mathcal{J}(f) \subseteq \cdots \subseteq I_k : \mathcal{J}(f) \subseteq \cdots \subseteq I_1 : \mathcal{J}(f) \subseteq I_0 : \mathcal{J}(f) = (1).$$

Furthermore,

$$\mathbb{V}(I_0 : \mathcal{J}(f)) \subseteq \mathbb{V}(I_1 : \mathcal{J}(f)) \subseteq \cdots \subseteq \mathbb{V}(I_k : \mathcal{J}(f)) \subseteq \cdots \subseteq \mathbb{V}(I_{n-1} : \mathcal{J}(f)) \subseteq \mathbb{V}(I_n : \mathcal{J}(f))$$

where $\mathbb{V}(I_0 : \mathcal{J}(f)) = \emptyset$. When $k = 1$ the variety $\mathbb{V}(I_1 : \mathcal{J}(f))$ is known as the *relative polar curve* of f with respect to the z_0-axis.
Notation. When the choice of the ordering of the coordinate system is clear we will use I_k instead of $I_{k,\mathbf{z}}$.

Definition 4.1.1. *The kth Lê variety of f at $\mathbf{0}$ is*

$$\Lambda_f^{(k)} := \left[\mathbb{V}\left(I_{k+1} : \mathcal{J}(f)\right) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_k}\right) \right] \triangleleft \Sigma f.$$

Remark 4.1.2. (i) For each k, the varieties $\mathbb{V}(I_k : \mathcal{J}(f))$ consists of the components of $\mathbb{V}(I_k)$ which are not contained in the singular locus of f.
(ii) The k-th Lê variety depends on the choice of the first k coordinates $z_0, z_1, \cdots, z_{k-1}$.

Example 4.1.3. **[$E_{6,0}$-type]** Consider $f = z_2^3 + z_1^3 z_2 + z_0^2 z_1^2$ such that $\mathbb{V}(f) \subset \mathbb{C}^3$. The Jacobian ideal of f is its primary decomposition are

$$\mathcal{J}(f) = \langle z_0 z_1^2, z_0^2 z_1 + z_1^2 z_2, z_1^3 + z_2^2 \rangle$$

with the primary decomposition

$$\mathcal{J}(f) = \langle z_1, z_2^2 \rangle \cap \langle z_0, z_1, z_2^2 \rangle \cap \langle z_0, z_1^3, z_2 \rangle \cap \langle z_0^2 + z_1 z_2, z_1^2, z_2^2 \rangle .$$

Let us compute the Lê varieties of f with respect to the different ordering of the coordinates.
(i) Fix the ordering (z_0, z_1, z_2). Note that the Lê varieties of f with respect to the ordering are

$$\Lambda_f^{(0)} = \left[\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_0})\right] \triangleleft \Sigma f$$
$$\Lambda_f^{(1)} = \left[\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_1})\right] \triangleleft \Sigma f$$

Here, we have

$$I_1 = \left\langle \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \right\rangle = \langle z_1(z_0^2 + z_1 z_2), z_1^3 + z_2^2 \rangle = \langle z_1, z_2^2 \rangle \cap \langle z_0^2 + z_1 z_2, z_1^3 + z_2^2 \rangle$$
$$I_2 = \langle \tfrac{\partial f}{\partial z_2} \rangle = \langle z_1^3 + z_2^2 \rangle .$$

together with

$$I_1 : \mathcal{J}(f) = \langle z_0^2 + z_1 z_2, z_1^3 + z_2^2 \rangle \quad \text{and} \quad I_2 : \mathcal{J}(f) = \langle z_1^3 + z_2^2 \rangle = I_2.$$

Let us compute firstly 0th Lê variety of f. Thus,

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_0}) = \mathbb{V}(z_0^2 + z_1 z_2, z_1^3 + z_2^2) \cap \mathbb{V}(z_0 z_1^2) = \mathbb{V}(z_0^2 + z_1 z_2, z_1^3 + z_2^2, z_0 z_1^2).$$

Here, the primary decomposition of the defining ideal is

$$\langle z_0^2 + z_1 z_2, z_1^3 + z_2^2, z_0 z_1^2 \rangle = \langle z_0, z_1, z_2^2 \rangle \cap \langle z_0, z_1^3, z_2 \rangle \cap \langle z_0^2 + z_1 z_2, z_1^2, z_2^2 \rangle .$$

Since each primary ideal in the decomposition is contained in $\mathcal{J}(f)$ i.e.

$$\langle z_0, z_1, z_2^2 \rangle , \langle z_0, z_1^3, z_2 \rangle , \langle z_0^2 + z_1 z_2, z_1^2, z_2^2 \rangle \subset \mathcal{J}(f)$$

we obtain 0th Lê variety of f

$$\Lambda_f^{(0)} = \mathbb{V}(z_0^2 + z_1 z_2, z_1^3 + z_2^2, z_0 z_1^2) \triangleleft \Sigma f = \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2) \cup \mathbb{V}(z_1^2, z_2^2, z_0^2 + z_1 z_2).$$

Now, we will compute 1st Lê variety of f. So,

$$\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right) = \mathbb{V}(z_1^3 + z_2^2) \cap \mathbb{V}(z_0^2 z_1 + z_1^2 z_2) = \mathbb{V}(z_1^3 + z_2^2, z_1(z_0^2 + z_1 z_2)).$$

Here, we have the primary decomposition

$$\langle z_1^3 + z_2^2, z_1(z_0^2 + z_1 z_2)\rangle = \langle z_1, z_2^2\rangle \cap \langle z_1^3 + z_2^2, z_0^2 + z_1 z_2\rangle$$

in which $\langle z_1^3 + z_2^2, z_0^2 + z_1 z_2\rangle \not\subseteq \mathcal{J}(f)$. Hence the 1th Lê variety of f is

$$\Lambda_f^{(1)} = \mathbb{V}(z_1^3 + z_2^2, z_1(z_0^2 + z_1 z_2)) \triangleleft \Sigma f = \mathbb{V}(z_1, z_2^2).$$

(ii) Now let us consider the ordering $(z_1, z_2, z_0) = (y_0, y_1, y_2)$. The Lê varieties of f are

$\Lambda_f^{(0)} = \left[\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_1})\right] \triangleleft \Sigma f$

$\Lambda_f^{(1)} = \left[\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_2})\right] \triangleleft \Sigma f$

Here, we have:

$I_1 = \left\langle \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2} \right\rangle = \left\langle \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_0} \right\rangle = \langle z_0, z_2^2 + z_1^3\rangle = \langle z_0, z_1^3 + z_2^2\rangle \cap \langle z_1^2, z_2^2\rangle$

$I_2 = \left\langle \frac{\partial f}{\partial y_2} \right\rangle = \left\langle \frac{\partial f}{\partial z_0} \right\rangle = \langle z_0 z_1^2\rangle$

along with $I_1 : \mathcal{J}(f) = \langle z_0 z_1, z_1^3 + z_2^2\rangle$ and $I_2 : \mathcal{J}(f) = \langle z_0 z_1^2\rangle = I_2$.

Furthermore , we have

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right) = \mathbb{V}(z_0 z_1, z_1^3 + z_2^2) \cap \mathbb{V}(z_1(z_0^2 + z_1 z_2)) = \mathbb{V}(z_0 z_1, z_1^3 + z_2^2, z_1(z_0^2 + z_1 z_2)).$$

Since $\langle z_0 z_1, z_1^3 + z_2^2, z_1(z_0^2 + z_1 z_2)\rangle = \langle z_1, z_2^2\rangle \cap \langle z_0, z_1, z_2^2\rangle \cap \langle z_0, z_1^3, z_2\rangle \cap \langle z_0^2 + z_1 z_2, z_1, z_2^2\rangle$, we get

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right) = \mathbb{V}(\langle z_1, z_2^2\rangle \cap \langle z_0, z_1, z_2^2\rangle \cap \langle z_0, z_1^3, z_2\rangle \cap \langle z_0^2 + z_1 z_2, z_1, z_2^2\rangle)$$

moreover

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right) = \mathbb{V}(z_1, z_2^2) \cup \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2).$$

Since $\langle z_1, z_2^2\rangle, \langle z_0, z_1, z_2^2\rangle, \langle z_0, z_1^3, z_2\rangle, \langle z_0^2 + z_1z_2, z_1, z_2^2\rangle \subset \mathcal{J}(f)$, we have

$$\left[\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right)\right] \triangleleft \Sigma f = \mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right).$$

Hence, the 0th Lê variety of f is

$$\Lambda_f^{(0)} = \mathbb{V}(z_1, z_2^2) \cup \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2).$$

Now, let us compute 1st Lê variety of f. Thus,

$$\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_2}) = \mathbb{V}(z_0z_1^2) \cap \mathbb{V}(z_1^3 + z_2^2) = \mathbb{V}(z_0z_1^2, z_1^3 + z_2^2).$$

Note that

$$\langle z_0z_1^2, z_1^3 + z_2^2\rangle = \langle z_0, z_1^3 + z_2\rangle \cap \langle z_1^2, z_2^2\rangle$$

in which $\langle z_0, z_1^3 + z_2\rangle, \langle z_1^2, z_2^2\rangle \subset \mathcal{J}(f)$. Then, the 1st Lê number of f is

$$\Lambda_f^{(1)} = \mathbb{V}(z_0, z_1^3 + z_2) \cup \mathbb{V}(z_1^2, z_2^2).$$

Notation. We will use the usual ordering $(z_0, z_1, \cdots, z_n)$ if we don't specify.

Proposition 4.1.4. ([21], 1.14) An irreducible component of $\Lambda_f^{(k)}$ has dimension $\geq k$.

Proof. Fix a k and put $V := \mathbb{V}(I_{k+1} : \mathcal{J}(f))$. Let $V = \bigcup V_i$ where V_i is the irreducible component of V. So, the ideal $\mathbb{I}(V)$ is generated by $(n+1) - (k+1)$ polynomials and the ideal $\mathbb{I}(V_i)$ is a minimal prime ideal in $\mathbb{I}(V)$. Hence we have $ht(\mathbb{I}(\mathbb{V}_i)) \leq (n+1) - (k+1)$; so, $dim(V_i) = (n+1) - ht(\mathbb{I}(V_i))$. This says that each V_i has dimension at least $k+1$ and the intersection $V \cap \mathbb{V}(\frac{\partial f}{\partial z_k})$ has dimension at least k.

$\square$

Example 4.1.5. **[$E_{6,0}$-type]** Consider $f = z_2^3 + z_1^3z_2 + z_0^2z_1^2$ in the example above. The components of 0th Lê variety of f are $\mathbb{V}(z_0, z_1, z_2^2)$, $\mathbb{V}(z_0, z_1^3, z_2)$ and $\mathbb{V}(z_1^2, z_2^2, z_0^3 + z_1z_2)$. Each of which is of dimension 0. The 1st Lê variety of f has a unique component for which $dim(\mathbb{V}(z_1, z_2^2)) = 1$.

Definition 4.1.6. *Let I and J be two ideals in $\mathbb{C}\{z\}$. We say that $\mathbb{V}(I)$ and $\mathbb{V}(J)$ equal up to embedded components if their isolated components are equal.*

Lemma 4.1.7. ([19],1.8) With preceding notation, we have

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)=\mathbb{V}(I_k:\mathcal{J}(f))\cup\Lambda_f^{(k)}$$

for all k, up to embedded components.

Proof. Let $0\leq k\leq n$. We have

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)=\mathbb{V}\left(I_{k+1}:\mathcal{J}(f)+\left\langle\frac{\partial f}{\partial z_k}\right\rangle\right).$$

Put $\mathbb{V}(I):=\mathbb{V}\left(I_{k+1}:\mathcal{J}(f)+\left\langle\frac{\partial f}{\partial z_k}\right\rangle\right)$. By Proposition 2.2.1 we have

$$\mathbb{V}(I)=\mathbb{V}(I:\mathcal{J}(f))\cup\mathbb{V}(I)\triangleleft\Sigma f$$

up to embedded components. We have $\mathbb{V}(I)\triangleleft\Sigma f=\Lambda_f^{(k)}$ and, by Proposition 2.1.9,

$$\mathbb{V}(I:\mathcal{J}(f))=\mathbb{V}\left(\left(I_{k+1}+\left\langle\frac{\partial f}{\partial z_k}\right\rangle\right):\mathcal{J}(f)\right)=\mathbb{V}\left(I_k:\mathcal{J}(f)\right).$$

Hence, $\mathbb{V}(I)=\mathbb{V}\left(I_k:\mathcal{J}(f)\right)\cup\Lambda_f^{(k)}$. That's what we want to show. $\square$

Proposition 4.1.8. ([21], 1.14) With preceding notation, $\Lambda_f^{(k)}=\emptyset$ for all $d<k<n+1$.

The statement follows from Lemma 4.1.7, Definition 4.1.1 and from the fact that each component of $\mathbb{V}(I_{k+1}:\mathcal{J}(f))$ has dimension at least $k+1$ (See the proof of Prop.4.1.4).

Example 4.1.9. [E_7-Type] Consider $f=z_2^2+z_1^3+z_0^3z_1$. The singular locus of f is

$$\Sigma f=\mathbb{V}(\mathcal{J}(f))=\mathbb{V}\left(3z_0^2z_1,z_0^3+3z_1^2,2z_2\right).$$

We have

$$I_0=\mathcal{J}(f),\quad I_1=\left\langle z_0^3+3z_1^2,2z_2\right\rangle\quad \textit{and}\quad I_2=\langle 2z_2\rangle$$

and $\mathbf{0}\subseteq I_2\subseteq I_1\subseteq I_0=\mathcal{J}(f)$. Since

$$height(I_0)=3,\ \ height(I_1)=2\ and\ height(I_2)=1$$

$$codim(\mathbb{V}(I_0))=3,\ \ codim(\mathbb{V}(I_1))=2\ and\ codim(\mathbb{V}(I_2))=1.$$

So the variety $\mathbb{V}(I_i)$ for each i is complete intersection, hence Cohen-Macaulay and $\mathbb{V}(I_i) = \mathbb{V}(I_i : \mathcal{J}(f))$. The later equality can be found by applying the quotient, for instance:

$$\begin{aligned} I_1 : \mathcal{J}(f) &= \{f \in \mathbb{C}[z_0,z_1,z_2] \mid fg \in I_1 \ \ for\ all\ \ g \in \mathcal{J}(f)\} \\ &= \{f \in \mathbb{C}[z_0,z_1,z_2] \mid fg \in I_1\ with\ g = A(3z_0^2z_1) + B(z_0^3+3z_1^2) + C(2z_2)\ \ with \\ &\qquad A,B,C \in \mathbb{C}[z_0,z_1,z_2]\} \\ &= \{f \in \mathbb{C}[z_0,z_1,z_2] \mid fg \in I_1 \ \ with\ \ g = A(3z_0^2z_1) + I_1,\ \ A \in \mathbb{C}[z_0,z_1,z_2]\} \\ &= \{f \in \mathbb{C}[z_0,z_1,z_2] \mid f \in I_1\} = I_1. \end{aligned}$$

We get $\Lambda_f^{(0)} = \left[\mathbb{V}(I_1) \cap \mathbb{V}(3z_0^2z_1)\right] \triangleleft \Sigma f$. Since

$$\mathbb{V}(I_1) \cap \mathbb{V}(3z_0^2z_1) = \mathbb{V}(I_1 + \langle 3z_0^2z_1 \rangle) = \mathbb{V}(z_0^3+3z_1^2, 2z_2, 3z_0^2z_1) = \mathbb{V}(I_0) = \Sigma f.$$

Hence we obtain, $\Lambda_f^{(0)} = \Sigma f \triangleleft \Sigma f = \Sigma f$.
Now, let us describe explicitly the 1st Lê variety

$$\Lambda_f^{(1)} = \left[\mathbb{V}\,(I_2 : J(f)) \cap \mathbb{V}(z_0^3+3z_1^2)\right] \triangleleft \Sigma f.$$

We have $\mathbb{V}\,(I_2 : \mathcal{J}(f)) = \mathbb{V}(I_2)$ and $\Lambda_f^{(1)} = \left[\mathbb{V}(I_2) \cap \mathbb{V}(z_0^3+3z_1^2)\right] \triangleleft \Sigma f$. By using Theorem 2.2.1, we obtain $\Lambda_f^{(1)} = \mathbb{V}(2z_2, z_0^3+3z_1^2) \triangleleft \Sigma f$ which gives

$$\Lambda_f^{(1)} = \mathbb{V}(I_1) \triangleleft \Sigma f.$$

Since $\mathbb{V}(I_1)$ is Cohen-Macaulay, $\mathbb{V}(I_1)$ has no embedded subvarieties. Then, $\Lambda_f^{(1)} = \emptyset$.

Proposition 4.1.10. *Let $\mathbb{V}(f) \subset \mathbb{C}^{n+1}$ be a hypersurface with an isolated singularity at the origin. Then, $\Lambda_f^{(k)} = \emptyset$ for $k > 0$.*

Proof. First, note that the singular locus of f is $\Sigma f = \mathbb{V}\left(\left\langle \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right\rangle\right)$ and $0 \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_1 \subseteq I_0 = \mathcal{J}(f)$. So, for all i, we have $height(I_i) = n+1-i$. Thus, $\mathbb{V}(I_i)$ is a complete intersection for all $i = 0, \cdots, n$. Then, $\mathbb{V}(I_i)$ is Cohen-Macaulay i.e. it has no embedded subvarieties. In addition, we have $\mathbb{V}(I_i) \triangleleft \Sigma f = \emptyset$. Furthermore, $\mathbb{V}(I_i) = \mathbb{V}(I_i : \mathcal{J}(f)) \cup (\mathbb{V}(I_i) \triangleleft \Sigma f) = VV(I_i : \mathcal{J}(f))$ for all i. The 0-th Lê variety is

$$\Lambda_f^{(0)} = \left[\mathbb{V}(I_1) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_0})\right] \triangleleft \Sigma f = \left[\mathbb{V}\left(I_1 + \left\langle \frac{\partial f}{\partial z_0} \right\rangle\right)\right] \triangleleft \Sigma f = \mathbb{V}(I_0) \triangleleft \Sigma f = \Sigma f$$

knowing that $I_0 = \mathcal{J}(f)$. For $k \neq 0$, we have

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)=\mathbb{V}(I_{k+1})\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)=\mathbb{V}(I_k).$$

Moreover,

$$\Lambda_f^{(k)}=\left[\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\right]\triangleleft\Sigma f=\mathbb{V}(I_k)\triangleleft\Sigma f.$$

As $\mathbb{V}(I_k)\triangleleft\Sigma f=\emptyset$, we obtain $\Lambda_f^{(k)}=\emptyset$ for $0<k\leq n$.

□

Proposition 4.1.11. ([21], 1.15) For each k we have

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\Sigma f=\Lambda_f^{(0)}\cup\Lambda_f^{(1)}\cup\cdots\cup\Lambda_f^{(k)} \tag{4.1.1}$$

as sets.

Proof. By induction on k. When $k=0$,

$$\begin{aligned}\mathbb{V}(I_1:\mathcal{J}(f))\cap\Sigma f&=\mathbb{V}(I_1:\mathcal{J}(f))\cap\mathbb{V}(\tfrac{\partial f}{\partial z_0})\cap\Sigma f\\&=\left(\mathbb{V}(I_0:\mathcal{J}(f))\cup\Lambda_f^{(0)}\right)\cap\Sigma f \text{ by Lemma 4.1.7}\\&=\Lambda_f^{(0)}\cap\Sigma f \quad \text{because } \mathbb{V}(I_0):\mathcal{J}(f)=\emptyset.\\&=\Lambda_f^{(0)} \quad \text{by definition.}\end{aligned}$$

When $k=1$,

$$\begin{aligned}\mathbb{V}(I_2:\mathcal{J}(f))\cap\Sigma f&=\mathbb{V}(I_2:\mathcal{J}(f))\cap\mathbb{V}(\tfrac{\partial f}{\partial z_1})\cap\Sigma f\\&=\left(\mathbb{V}(I_1:\mathcal{J}(f))\cup\Lambda_f^{(1)}\right)\cap\Sigma f \text{ by Lemma 4.1.7}\\&=(\mathbb{V}(I_1:\mathcal{J}(f))\cap\Sigma f)\cup\left(\Lambda_f^{(1)}\cap\Sigma f\right)\\&=\Lambda_f^{(0)}\cup\left(\Lambda_f^{(1)}\cap\Sigma f\right)=\Lambda_f^{(0)}\cup\Lambda_f^{(1)}.\end{aligned}$$

Now assume that the Proposition is true for k. Let us show it for $k+1$:

$$\begin{aligned}\mathbb{V}(I_{k+2}:\mathcal{J}(f))\cap\Sigma f&=\mathbb{V}(I_{k+2}:\mathcal{J}(f))\cap\mathbb{V}(\tfrac{\partial f}{\partial z_{k+1}})\cap\Sigma f\\&=\left(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cup\Lambda_f^{(k+1)}\right)\cap\Sigma f \text{ by Lemma 4.1.7}\\&=(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\Sigma f)\cup\left(\Lambda_f^{(k+1)}\cap\Sigma f\right)\\&=(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\Sigma f)\cup\Lambda_f^{(k+1)}.\\&=\Lambda_f^{(0)}\cup\Lambda_f^{(1)}\cup\cdots\cup\Lambda_f^{(k)}\cup\Lambda_f^{(k+1)}.\end{aligned}$$

Then we have the affirmation of the proposition is true.

□

Corollary 4.1.12. ([21], 1.15) We have

$$\Sigma f = \Lambda_f^{(0)} \cup \Lambda_f^{(1)} \cup \cdots \cup \Lambda_f^{(d)}$$

as germ of sets at $\mathbf{p} \in \mathbb{V}(f)$.

Before proving this corollary let us give an example.

Example 4.1.13. [$E_{6,0}$-type] Consider $f = z_2^3 + z_1^3 z_2 + z_0^2 z_1^2$. Fix the coordinates (z_0, z_1, z_2) for f. Note that the Jacobian ideal of f is

$$\mathcal{J}(f) = \left\langle z_0 z_1^2, z_0^2 z_1 + z_1^2 z_2, z_1^3 + z_2^2 \right\rangle.$$

So the singular locus of f is the variety defined by $z_0 = 0$ i.e. $dim(\Sigma f) = 1$. The primary decomposition of $\mathcal{J}(f)$ gives

$$\mathcal{J}(f) = \left\langle z_1, z_2^2 \right\rangle \cap \left\langle z_0, z_1, z_2^2 \right\rangle \cap \left\langle z_0, z_1^3, z_2 \right\rangle \cap \left\langle z_0^2 + z_1 z_2, z_1^2, z_2^2 \right\rangle.$$

Furthermore from Example 4.1.3 we have

$$\Lambda_f^{(0)} = \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2) \cup \mathbb{V}(z_1^2, z_2^2, z_0^2 + z_1 z_2) \quad \textit{and} \quad \Lambda_f^{(1)} = \mathbb{V}(z_1, z_2^2).$$

Hence we get

$$\Sigma f = \Lambda_f^{(0)} \cup \Lambda_f^{(1)}.$$

Proof of Corollary 4.1.12. By Proposition 4.1.11 we have

$$\mathbb{V}(I_{d+1} : \mathcal{J}(f)) \cap \Sigma f = \Lambda_f^{(0)} \cup \Lambda_f^{(1)} \cup \cdots \cup \Lambda_f^{(d)}.$$

So, it suffices to show that $\mathbb{V}(I_{d+1} : \mathcal{J}(f)) \cap \Sigma f = \Sigma f$. Note that $\Sigma f = \mathbb{V}(\mathcal{J}(f))$ and

$$\mathbb{V}(I_{d+1}) = \mathbb{V}(I_{d+1} : \mathcal{J}(f)) \cup (\mathbb{V}(I_{d+1}) \triangleleft \Sigma f)$$

up to embedded components. Indeed, $\mathbb{V}(I_{d+1})$, consists of the components lying in Σf and those which are not in Σf. By the discussion in the proof of Proposition 4.1.4 each component of $\mathbb{V}(I_{d+1})$ has dimension at least $d+1$. Since $dim(\Sigma f) = d$, the variety $\mathbb{V}(I_{d+1})$ has no components in Σf, that is, $\mathbb{V}(I_{d+1}) \triangleleft \Sigma f = \emptyset$. Thus, we get $\mathbb{V}(I_{d+1}) = \mathbb{V}(I_{d+1} : \mathcal{J}(f))$ up to embedded components. Furthermore,

$$\mathbb{V}(I_{d+1} : \mathcal{J}(f)) \cap \Sigma f = \mathbb{V}(I_{d+1}) \cap \Sigma f = \Sigma f$$

since $I_{d+1} \subset \mathcal{J}(f)$. Then, the singular locus of f is the union of the first $d+1$ Lê varieties of f.

□

4.2 Lê tuples

In this section, we will determine a good choice of the ordering for the coordinates $(z_0, z_1, \cdots, z_n)$ in $\mathbb{C}^{n+1}$ regarding Remark 4.1.2.

Definition 4.2.1. *Fix the ordering as $(z_0, z_1, \cdots, z_n)$ in $\mathbb{C}^{n+1}$. Let $1 \leq k \leq n$. We say that $(z_0, z_1, \cdots, z_k)$ is a Lê tuple if*

$$\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_k}\right) \cap \mathbb{V}(z_0, z_1, \cdots, z_{k-1})$$

is 0-dimensional or empty at the origin. In particular, when $k = 0$ we say that z_0 is a Lê axis if $\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0})$ is 0-dimensional or empty.

In other words, $(z_0, z_1, \cdots, z_k)$ is a Lê tuple if the intersection $\Lambda_f^{(k)} \cap \mathbb{V}(z_0, z_1, \cdots, z_{k-1})$ is of dimension 0 or empty at the origin.

Example 4.2.2. [$E_{6,0}$-type] Consider $f = z_2^3 + z_1^3 z_2 + z_0^2 z_1^2$. By (i) of Example 4.1.3, we have

$$\begin{aligned}\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_0}) &= \mathbb{V}(z_0^2 + z_1 z_2, z_1^3 + z_2^2, z_0 z_1^2) \\ &= \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2) \cup \mathbb{V}(z_0^2 + z_1 z_2, z_1^2, z_2^2).\end{aligned}$$

As each component on the right hand side is 0-dimensional, $\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0})$ is 0-dimensional. So, z_0 is a Lê axis for f at the origin. By (ii) of Example 4.1.3, we have

$$\begin{aligned}\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_1}) \cap \mathbb{V}(z_0) &= \mathbb{V}(z_0, z_1^3 + z_2^2, z_1(z_0^2 + z_1 z_2)) \\ &= \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2)\end{aligned}$$

which is of dimension 0. So, (z_0, z_1) is a Lê tuple for f at the origin. In this case, we will say that the ordering (z_0, z_1, z_2) is a good choice for f at the origin. However, (z_1, z_2, z_0) is not a good choice: Indeed, (ii) of Example 4.1.3 says that

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}\right) = \mathbb{V}(z_1, z_2^2) \cup \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2) \cup \mathbb{V}(z_0^2 + z_1 z_2, z_1, z_2^2)$$

where $dim_0(\mathbb{V}(z_0,z_1,z_2^2)) = dim_0(\mathbb{V}(z_0,z_1^3,z_2)) = dim_0(\mathbb{V}(z_0^2+z_1z_2,z_1,z_2^2)) = 0$ and $dim_0(\mathbb{V}(z_1,z_2^2)) = 1$. So $dim_0(\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_1})) = 1$ and z_1 is not a Lê axis for f; hence (z_1,z_2) is not a Lê tuple for f at the origin.

Remark 4.2.3. When z_0 is a Lê axis for f at the origin, $\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0})$ has no embedded components at the origin. Furthermore, we have

$$\Lambda_f^{(0)} = \mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0}).$$

Proposition 4.2.4. ([19], 2.4) Consider a Lê tuple $(z_0,\cdots,z_k)$ for f. Then $\Lambda_f^{(k)}$ is purely k-dimensional at the origin.

Proof. By definition, the variety $\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_k}) \cap \mathbb{V}(z_0,\cdots,z_{k-1})$ has dimension 0 or empty at the origin while $\mathbb{V}(I_{k+1} : \mathcal{J}(f))$ has dimension at least $k+1$. From Lemma 4.1.7 we have

$$\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_k}\right) = \mathbb{V}(I_k : \mathcal{J}(f)) \cup \Lambda_f^{(k)}.$$

The variety $\mathbb{V}(I_{k+1} : \mathcal{J}(f))$ should be purely $k+1$ dimensional at the origin. The fact that $(z_0,\ldots,z_k)$ is a Lê tuple for f implies that $(z_0,\ldots,z_{k-1})$ is a Lê tuple for f. So the variety $\mathbb{V}(I_k : \mathcal{J}(f))$ is purely k-dimensional at the origin. As the left hand side of the equality above is purely k dimensional, $\Lambda_f^{(k)}$ is purely k-dimensional at the origin.

□

Corollary 4.2.5. ([19], 2.4) Consider a Lê tuple $(z_0,\cdots,z_k)$ for f. Then, $\Lambda_f^{(k)}$ contains all k-dimensional components of Σf through $\mathbf{p}$.

Proof. Let C be a component of the singular locus of dimension k. One can find a component C' of $\mathbb{V}(I_{k+1})$ such that $C \subset C'$. As the dimension of C' is at least $k+1$, C' can not be a component of Σf. Then C' is a component of $\mathbb{V}(I_{k+1} : \mathcal{J}(f))$, so does C. This gives,

$$C \cap \mathbb{V}(\frac{\partial f}{\partial z_k}) \subset \mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_k}).$$

By Lemma 4.1.7 we obtain

$$C \subset \mathbb{V}(I_k : \mathcal{J}(f)) \cup \Lambda_f^{(k)}$$

where $C = C \cap \mathbb{V}(\frac{\partial f}{\partial z_k})$. As $\mathbb{V}(I_k : \mathcal{J}(f))$ does not contain any component of Σf, C can not be in $\mathbb{V}(I_k : \mathcal{J}(f))$. Then we have $C \subset \Lambda_f^{(k)}$ since $\Lambda_f^{(k)}$ is purely k-dimensional at the origin. That is what we want to show.

□

Proposition 4.2.6. ([19], 2.8) Let $dim(\Sigma f) = d \geq 1$. A tuple $(z_0, \cdots, z_d)$ is a Lê tuple for f if and only if $\Sigma\left(f|_{\mathbb{V}(z_0,\cdots,z_{d-1})}\right)$ is of dimension 0.

Proof. We know that each component of $\mathbb{V}(I_{d+1})$ has dimension at least $d+1$. As sets, we have $\mathbb{V}(I_{d+1} : \mathcal{J}(f)) = \mathbb{V}(I_{d+1})$. The singular locus of the restriction of f is

$$\begin{aligned}\Sigma\left(f|_{\mathbb{V}(z_0,\cdots,z_{d-1})}\right) &= \mathbb{V}(z_0,\cdots,z_{d-1},\tfrac{\partial f}{\partial z_d},\cdots,\tfrac{\partial f}{\partial z_n}) \\ &= \mathbb{V}(z_0,\cdots,z_{d-1})\cap\mathbb{V}(\tfrac{\partial f}{\partial z_d})\cap\mathbb{V}(I_{d+1}) \\ &= \mathbb{V}(z_0,\cdots,z_{d-1})\cap\mathbb{V}(\tfrac{\partial f}{\partial z_d})\cap\mathbb{V}(I_{d+1} : \mathcal{J}(f)).\end{aligned}$$

If $(z_0,\cdots,z_d)$ is a Lê tuple for f then $\mathbb{V}(I_{d+1} : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_d}) \cap \mathbb{V}(z_0,\cdots,z_{d-1})$ is of dimension 0. By the equality above, the singular locus of the restriction of f is of dimension 0. To show inverse implication, suppose that $dim\left(\Sigma\left(f|_{\mathbb{V}(z_0,\cdots,z_{d-1})}\right)\right) = 0$. Thus, $\mathbb{V}(I_{d+1} : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_d}) \cap \mathbb{V}(z_0,\cdots,z_{d-1})$ is of dimension 0. Then, by definition, $(z_0,\cdots,z_d)$ is a Lê tuple for f.

□

Remark 4.2.7. When $d = 0$ in the proposition, the variety $\mathbb{V}(I_{k+1})$ is $k+1$-dimensional, it has no components in Σf and no embedded components. So $\mathbb{V}(I_{k+1} : \mathcal{J}(f)) = \mathbb{V}(I_{k+1})$ (as sets) for $k \geq 0$. Furthermore, we have

$$\mathbb{V}\left(I_{k+1} : \mathcal{J}(f)\right)\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\cap\mathbb{V}(z_0,\cdots,z_{k-1}) = \mathbb{V}(z_0,z_1,\cdots,z_{k-1},\frac{\partial f}{\partial z_k},\frac{\partial f}{\partial z_{k+1}},\cdots,\frac{\partial f}{\partial z_n})$$

which equals the singular locus of the restriction $f|_{\mathbb{V}(z_0,\cdots,z_{k-1})}$. Hence $(z_0,z_1,\cdots,z_k)$ is a Lê tuple for f at the origin if and only if $\mathbb{V}(z_0,z_1,\cdots,z_{k-1})$ intersects transversely the smooth part of $\mathbb{V}(f)$.

Generic Lê tuples. To define the genericity of a Lê tuple we need first to have a good stratification of $\mathbb{V}(f)$. Recall that a stratification of $\mathbb{V}(f)$ is a locally finite partition of $\mathbb{V}(f)$ into analytic submanifolds called *strata*, such that the closure of each stratum is analytic and $\mathbb{V}(f)$ is a union of strata. We say that stratification of $\mathbb{V}(f)$ is good if the smooth part of $\mathbb{V}(f)$ is a stratum and the stratification satisfies *Thom's a_f condition* with respect to $\mathcal{U} - \mathbb{V}(f)$ where $\mathcal{U}$ is an open subset of $\mathbb{C}^{n+1}$. (See [23], p.28). When $d = 1$ the stratification

$$\{\mathbb{V}(f) - \Sigma f, \Sigma f - \mathbf{p}, \mathbf{p}\}$$

is good for $\mathbb{V}(f)$. (See [21],1.29).
A Lê axis z_i is called *generic* for f at $\mathbf{p} \in \mathbb{V}(f)$ with respect to the good stratification of $\mathbb{V}(f)$ if $\mathbb{V}(z_i - p_i)$ intersects transversely each stratum, except perhaps the stratum $\mathbf{p}$, in a neighborhood of $\mathbf{p}$. For $0 \leq k \leq n$, a Lê tuple $(z_0, z_1, \ldots, z_k)$ is called *generic* for f at $\mathbf{p}$ with respect to the good stratification if $\mathbb{V}(z_0 - p_0)$ transversely intersects each stratum except perhaps the stratum $\{\mathbf{p}\}$ itself, in a neighborhood of $\mathbf{p}$.

4.3 Lê Numbers

Let $\mathbb{V}(f) \subseteq \mathbb{C}^{n+1}$, $\mathbf{p} = (p_0, \ldots, p_n) \in \mathbb{V}(f)$ and $d := dim(\Sigma f)$.

Definition 4.3.1. *Let $(z_0, z_1, \cdots, z_k)$ be a Lê tuple for f. The intersection number at $\boldsymbol{p}$*

$$\lambda_f^{(k)}(\boldsymbol{p}) := (\Lambda_f^{(k)} \cdot \mathbb{V}(z_0 - p_0, z_1 - p_1, \cdots, z_{k-1} - p_{k-1}))_{\boldsymbol{p}}$$

is called the k-th Lê number of f at $\boldsymbol{p}$. In particular, $\lambda_f^{(0)}(\boldsymbol{p}) := mult_{\boldsymbol{p}}(\Lambda_f^{(0)})$.

Here, we denote by $mult_{\mathbf{p}}(\Lambda_f^{(0)})$ the multiplicity of $\Lambda_f^{(0)}$ at the point $\mathbf{p}$. Note that $\lambda_f^{(k)}(\mathbf{p})$ is not *defined* when $\Lambda_f^{(k)} \cdot \mathbb{V}(z_0 - p_0, z_1 - p_1, \cdots, z_{k-1} - p_{k-1})$ is not purely 0-dimensional at $\mathbf{p}$.

Proposition 4.3.2. ([21], 1.14) Then $\lambda_f^{(k)} = 0$ for all $d < k < n+1$.

Proof. This follows from Proposition 4.1.4 and Proposition 4.1.8.

□

Remark 4.3.3. Assume that $\mathbb{V}(f)$ has an isolated singularity at the origin. By Corollary 4.1.10, we have

$$\Lambda_f^{(0)} = \Sigma f \quad and \quad \Lambda_f^{(k)} = \emptyset \; for \; all \; k > 0.$$

Moreover, we know that $\lambda_f^{(0)}$ equals the multiplicity of $\Lambda_f^{(0)}$ at the origin. That is, $\lambda_f^{(0)}$ is the multiplicity of the singular locus of f at the origin which is exactly the Milnor number of f. Hence, for an isolated singularity, we always have $\lambda_f^{(0)} = \mu(f)$.

Example 4.3.4. **[E_7-Type]** Consider $f = z_2^2 + z_1^3 + z_0^3 z_1$ having an isolated singularity at the origin. From Example 4.1.9 we have:

$$\Lambda_f^{(0)} = \mathbb{V}(3z_1^2, 2z_2, 3z_0^2) \cup \mathbb{V}(z_0^3, 2z_2, z_1) \;\text{ and }\; \Lambda_f^{(1)} = \emptyset.$$

The 0th Lê number at $\mathbf{0}$ is

$$\lambda_f^{(0)}(0) = (\Lambda_f^{(0)} \cdot \mathbb{C}^3)_0 = \big(\mathbb{V}(3z_1^2, 2z_2, 3z_0^2)\big)_0 + \big(\mathbb{V}(z_0^3, 2z_2, z_1)\big)_0 = 2.2 + 3 = 7 = \mu(f).$$

(See Example 3.1.3).

Example 4.3.5. **[$E_{6,0}$-Type]** Consider $f = z_2^3 + z_1^3 z_2 + z_0^2 z_1^2$. We have $dim_{\mathbf{0}}\Sigma f = 1$ and the singularity is along z_0*-axis*. The 0th and 1th Lê varieties are

$$\Lambda_f^{(0)} = \mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2) \cup \mathbb{V}(z_1^2, z_2^2, z_0^2 + z_1 z_2) \text{ and } \Lambda_f^{(1)} = \mathbb{V}(z_1, z_2^2).$$

and the Lê numbers are

$$\begin{aligned}
\lambda_f^{(0)} &= \big(\mathbb{V}(z_0, z_1, z_2^2) \cup \mathbb{V}(z_0, z_1^3, z_2) \cup \mathbb{V}(z_1^2, z_2^2, z_0^2 + z_1 z_2)\big)_{\mathbf{0}} \\
&= \big(\mathbb{V}(z_0, z_1, z_2^2)\big)_{\mathbf{0}} + \big(\mathbb{V}(z_0, z_1^3, z_2)\big)_{\mathbf{0}} + \big(\mathbb{V}(z_1^2, z_2^2, z_0^2 + z_1 z_2)\big)_{\mathbf{0}} \\
&= 2 + 3 + 8 = 13, \\
\lambda_f^{(1)} &= \big(\mathbb{V}(z_1, z_2^2).\mathbb{V}(z_0)\big)_{\mathbf{0}} = \big(\mathbb{V}(z_1, z_2^2,)) \cap \mathbb{V}(z_0)\big)_{\mathbf{0}} = 2.
\end{aligned}$$

Generalized Lê-Iomdine Formula. Let $\Sigma f = d \geq 1$. Fix the ordering of coordinates $\mathbf{z} = (z_0, z_1, \ldots, z_n)$ for $\mathbb{C}^{n+1}$ such that it is an extension of Lê tuples of f. So, $\lambda_f^{(k)}$ is defined for all $k \leq d$. Put $\widetilde{f_m} = f + z_{i_0}^m$ such that $dim(\Sigma\widetilde{f_m}) = d - 1$ for $m > 2 + \lambda_f^{(0)}$.

Remark 4.3.6. We take $m > max\{1, deg(f)\}$ whenever f is weighted homogeneous.

We will use the ordering of the coordinates as $\tilde{\mathbf{z}} = (z_0, z_1, \ldots, \widehat{z_{i_0}}, \ldots, z_n, z_{i_0})$ for $\widetilde{f_m}$.

Lemma 4.3.7. ([19], 3.1) With the preceding notation, for all $m > 2 + \lambda_f^{(0)}$ we have

(i) $\mathbb{V}(I_1 : \mathcal{J}(f)) \cdot \mathbb{V}(\frac{\partial f}{\partial z_{i_0}} + m z_{i_0}^{m-1}) = \mathbb{V}(I_1 : \mathcal{J}(f)) \cdot \mathbb{V}(\frac{\partial f}{\partial z_{i_0}})$, and

(ii) $\Sigma\widetilde{f_m} = \Sigma f \cap \mathbb{V}(z_{i_0})$ as sets.

Proof. Without loss of generality put $\widetilde{f_m} = f + z_0^m$. Use the ordering of coordinates $(z_1, z_2, \ldots, z_n, z_0)$ for $\widetilde{f_m}$.
(i) We will show that $\mathbb{V}(I_1 : \mathcal{J}(f)) \cdot \mathbb{V}(\frac{\partial f}{\partial z_0} + m z_0^{m-1}) = \mathbb{V}(I_1 : \mathcal{J}(f)) \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})$. First note that $\mathbb{V}(I_1)$ is of dimension 1 and $\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0})$ is of dimension 0 since $\lambda_f^{(0)}$ is defined. Let $\mathcal{C}$ be an irreducible component of $\mathbb{V}(I_1 : \mathcal{J}(f))$. It suffices to show the equality

$$(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0} + m z_0^{m-1}))_{\mathbf{0}} = (\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0}))_{\mathbf{0}}.$$

If $\mathcal{C} \subseteq \mathbb{V}(z_0)$ the equality is clear. Assume $\mathcal{C} \nsubseteq \mathbb{V}(z_0)$. Note that

$$\lambda_f^{(0)} = mult_{\mathbf{0}}(\Lambda_f^{(0)}) = mult_{\mathbf{0}}(\mathbb{V}(I_1 : \mathcal{J}(f))) \cap \mathbb{V}(\frac{\partial f}{\partial z_0}) = (\mathbb{V}(I_1 : \mathcal{J}(f)) \cdot \mathbb{V}(\frac{\partial f}{\partial z_0}))_{\mathbf{0}}.$$

Furthermore, when $\mathcal{C} \nsubseteq \mathbb{V}(z_0)$ we have $(\mathcal{C} \cdot \mathbb{V}(z_0))_{\mathbf{0}} \neq 0$ and so

$$\lambda_f^{(0)} = \left(\mathbb{V}(I_1 : \mathcal{J}(f)) \cdot \mathbb{V}(\tfrac{\partial f}{\partial z_0}\right)_{\mathbf{0}} > \frac{\left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})\right)_{\mathbf{0}}}{(\mathcal{C} \cdot \mathbb{V}(z_0))_{\mathbf{0}}}.$$

Since $m > 2 + \lambda_f^{(0)}$ we have $m - 1 > 1 + \lambda_f^{(0)} > \lambda_f^{(0)}$ and so $\frac{1}{\lambda_f^{(0)}} > \frac{1}{m-1}$. This gives

$$1 > \frac{\left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})\right)_{\mathbf{0}}}{\lambda_f^{(0)}\,(\mathcal{C} \cdot \mathbb{V}(z_0))_{\mathbf{0}}} > \frac{\left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})\right)_{\mathbf{0}}}{(m-1)\,(\mathcal{C} \cdot \mathbb{V}(z_0))_{\mathbf{0}}} = \frac{\left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})\right)_{\mathbf{0}}}{\left(\mathcal{C} \cdot \mathbb{V}(z_0^{m-1})\right)_{\mathbf{0}}}.$$

Hence, we get $\left(\mathcal{C} \cdot \mathbb{V}(z_0^{m-1})\right)_{\mathbf{0}} > \left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})\right)_{\mathbf{0}}$. On the other hand, by [9], we know that

$$(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0} + mz_0^{m-1}))_{\mathbf{0}} = min\{(\mathcal{C} \cdot \mathbb{V}\left(\frac{\partial f}{\partial z_0}\right))_{\mathbf{0}}, (\mathcal{C} \cdot \mathbb{V}(z_0^{m-1}))_{\mathbf{0}}\}.$$

So we get $\left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0} + mz_0^{m-1})\right)_{\mathbf{0}} = \left(\mathcal{C} \cdot \mathbb{V}(\frac{\partial f}{\partial z_0})\right)_{\mathbf{0}}$.

(ii) We will show that $\Sigma\widetilde{f_m} = \Sigma f \cap \mathbb{V}(z_0)$ as sets. We have

$$\Sigma\widetilde{f_m} = \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) = \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}\right) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right).$$

That is, $\Sigma\widetilde{f_m} = \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}\right) \cap \mathbb{V}(I_1)$. So from the fact that

$$\mathbb{V}(I_1) = \mathbb{V}(I_1 : \mathcal{J}(f)) \cup \mathbb{V}(I_1) \triangleleft \Sigma f = \mathbb{V}(I_1 : \mathcal{J}(f)) \cup \Sigma f$$

we obtain $\Sigma\widetilde{f_m} = \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}\right) \cap (\mathbb{V}(I_1 : \mathcal{J}(f)) \cup \mathbb{V}(I_1) \triangleleft \Sigma f)$. Thus we have

$$\Sigma\widetilde{f_m} = \left(\Sigma f \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}\right)\right) \cup \left(\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0} + mz_0^{m-1})\right).$$

By (i) we obtain

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0} + mz_0^{m-1}) = \mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0}\right).$$

This gives

$$\Sigma\widetilde{f_m} = \left(\Sigma f \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}\right)\right) \cup \mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0}\right).$$

As $\lambda_f^{(0)}$ is defined, $\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0}\right)$ is 0-dimensional or empty. So we obtain

$$\Sigma\widetilde{f_m} = \left(\Sigma f \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + mz_0^{m-1}\right)\right) = \Sigma f \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0}\right) \cap \mathbb{V}(mz_0^{m-1}).$$

As $\mathbb{V}\left(\frac{\partial f}{\partial z_0}\right) \subset \Sigma f$ then we have

$$\Sigma\widetilde{f_m} = \Sigma f \cap \mathbb{V}(z_0^{m-1}) = \Sigma f \cap \mathbb{V}(z_0)$$

as sets.

$\square$

Remark 4.3.8. Let $(z_0, z_1, \ldots, z_k)$ be Lê tuple for f with $k \leq d$. Put

$$\widetilde{f_m} := f + z_0^m + z_1^m + \cdots + z_k^m.$$

Lemma 4.3.7 can be generalized in this case by induction on k. That is,

$$\Sigma\widetilde{f} = \Sigma f \cap \mathbb{V}(z_0, z_1, \ldots, z_k).$$

Moreover,

$$\Sigma\widetilde{f} = \Sigma f \cap \mathbb{V}(z_0, z_1, \ldots, z_k) = \mathbb{V}(z_0, z_1, \ldots, z_k, \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) = \Sigma\left(f|_{\mathbb{V}(z_0,\ldots,z_k)}\right)$$

as sets.

Proposition 4.3.9. *With the preceding notation, fix the ordering of the coordinates* $(z_0, z_1, \ldots, z_n)$ *such that it is an extension of the Lê tuples of f. Assume* $d \geq 1$. *Then* $dim(\Sigma\widetilde{f_m}) = d - 1$.

Proof. Without loss of generality put $\widetilde{f_m} = f + z_0^m$. So, we have $\Sigma\widetilde{f_m} = \Sigma f \cap \mathbb{V}(z_0)$ by Lemma 4.3.7. Furthermore, we get from Corollary 4.1.12,

$$\Sigma\widetilde{f_m} = \Sigma f \cap \mathbb{V}(z_0) = \bigcup_{k=0}^{d} \left(\Lambda_f^{(k)} \cap \mathbb{V}(z_0)\right)$$

as sets. Thus, if $dim(\Sigma\widetilde{f_m}) \geq d$ then the dimension of the union of $\Lambda_f^{(k)} \cap \mathbb{V}(z_0)$ is at least d, i.e., $\mathbb{V}(z_0)$ contains some of d-dimensional component of some $\Lambda_f^{(k)}$ but this is not the case. Indeed, as $\lambda_f^{(k)}(\mathbf{0})$ are defined for all $k \leq d$, we have $\Lambda_f^{(k)} \cap \mathbb{V}(z_0, z_1, \ldots, z_{k-1})$ is 0-dimensional or empty. Thus, $\mathbb{V}(z_0)$ does not contain any d-dimensional component of $\Lambda_f^{(k)}$ for all k. Hence we obtain $dim(\Sigma\widetilde{f_m}) \leq d-1$. As we have $\Sigma f \cap \mathbb{V}(z_0) = \Sigma\left(f|_{\mathbb{V}(z_0)}\right)$ by Remark 4.3.8 then

$$dim(\Sigma\widetilde{f_m}) = dim(\Sigma f \cap \mathbb{V}(z_0)) = d-1.$$

□

Lemma 4.3.10. ([20], 5.16) Fix the ordering of the coordinates $(z_0, z_1, \ldots, z_n)$ for $\mathbb{C}^{n+1}$ such that it is an extension of the Lê tuples of f. Then

$$dim\left(\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \Sigma f \cap \mathbb{V}(z_0)\right) \leq k-1.$$

Proof. By Lemma 4.3.7 we have

$$\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \Sigma f \cap \mathbb{V}(z_0) = \left(\Lambda_f^{(0)} \cup \ldots \Lambda_f^{(k)}\right) \cap \mathbb{V}(z_0) = \bigcup_{i=0}^{k}(\Lambda_f^{(i)} \cap \mathbb{V}(z_0)).$$

So, if $dim\left(\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \Sigma f \cap \mathbb{V}(z_0)\right) \geq k$ then the finite union of $\Lambda_f^{(i)} \cap \mathbb{V}(z_0)$ has dimension at least k i.e. $\mathbb{V}(z_0)$ contains some k-dimensional component of $\Lambda_f^{(i)}$ for some i. However, as all Lê numbers of f are defined by assumption, $\Lambda_f^{(i)} \cap \mathbb{V}(z_0, \ldots z_{i-1})$ is 0-dimensional or empty. Thus $\mathbb{V}(z_0)$ can not contain any k-dimensional component of $\Lambda_f^{(i)}$ for all i. Hence, we get

$$dim\left(\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \Sigma f \cap \mathbb{V}(z_0)\right) \leq k-1.$$

□

Lemma 4.3.11. ([19], 3.2) Fix the ordering of the coordinates $(z_0, z_1, \ldots, z_n)$ for $\mathbb{C}^{n+1}$ such that it is an extension of the Lê tuples of f. With preceding notations, we have

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}} : \mathcal{J}(\tilde{f}_m)) := \mathbb{V}(I_{k+1,\mathbf{z}} : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_{i_0}} + m z_{i-0}^{m-1}).$$

Proof. Without loss of generality consider $\tilde{f}_m = f + z_0^m$ with $m > 2 + \lambda_f^{(0)}$. So, by

Theorem 2.2.7, we have

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}} : \mathcal{J}(\tilde{f}_m)) = \overline{\mathbb{V}(I_{k,\tilde{\mathbf{z}}}) - \Sigma\widetilde{f}_m}.$$

Furthermore,

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}} : \mathcal{J}(\tilde{f}_m)) = \overline{\mathbb{V}(I_{k,\tilde{\mathbf{z}}}) - (\Sigma f \cap \mathbb{V}(z_0))}$$

by Lemma 4.3.7. Note that

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}}) = \mathbb{V}\left(\frac{\partial f}{\partial z_{k+1}}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial z_0} + m z_0^{m-1}\right) = \mathbb{V}(I_{k+1}) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right).$$

From the fact that

$$\mathbb{V}(I_{k+1}) = \mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cup (\mathbb{V}(I_{k+1}) \triangleleft \Sigma f)$$

we get

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}}) = (\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cup (\mathbb{V}(I_{k+1}) \triangleleft \Sigma f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right).$$

Then we obtain

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}}) = \left(\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right)\right) \cup (\mathbb{V}(I_{k+1}) \triangleleft \Sigma f) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right).$$

Here as we have

$$(\mathbb{V}(I_{k+1}) \triangleleft \Sigma f) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right) \subseteq \Sigma f \cap \mathbb{V}(z_0)$$

then we obtain

$$\overline{(\mathbb{V}(I_{k+1}) \triangleleft \Sigma f) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right) - \Sigma f \cap \mathbb{V}(z_0)} = \emptyset.$$

Hence we get

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}} : \mathcal{J}(\tilde{f}_m)) = \overline{\left(\mathbb{V}(I_{k+1} : \mathcal{J}(f)) \cap \mathbb{V}\left(\frac{\partial f}{\partial z_0} + m z_0^{m-1}\right)\right) - \Sigma f \cap \mathbb{V}(z_0)}.$$

Since the dimension of

$$\left(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_0}+mz_0^{m-1}\right)\right)$$

is at least k but from Lemma 4.3.10

$$\left(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_0}+mz_0^{m-1}\right)\cap\Sigma f\cap\mathbb{V}(z_0)\right)=(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\Sigma f\cap\mathbb{V}(z_0))$$

is of dimension $\leq k-1$. That is, $\left(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_0}+mz_0^{m-1}\right)\right)$ does not contain any component of $\Sigma f\cap\mathbb{V}(z_0))$. Hence,

$$\mathbb{V}(I_{k,\tilde{\mathbf{z}}}:\mathcal{J}(\tilde{f}_m))=\left(\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_0}+mz_0^{m-1}\right)\right)$$

□

Proposition 4.3.12. ([19], 3.3) Suppose that $(z_0,z_1,\dots,z_n)$ is an extension of Lê tuples for f. Then $(z_0,z_1,\dots,\widehat{z_{i_0}},\dots,z_n,z_{i_0})$ is an extension of the Lê tuples for $\widetilde{f_m}$ with $m>2+\lambda_f^{(0)}$.

Proof. Without loss of generality consider $\tilde{f}_m=f+z_0^m$ with $m>2+\lambda_f^{(0)}$. Assume that $(z_0,\dots,z_k)$ is a Lê tuples of f with $k\leq d$. To show the statement is true it suffices to show that $(z_1,\dots,z_k)$ is a Lê tuple for $\tilde{f}_m$. That is, we will show that

$$\mathbb{V}(I_k:\mathcal{J}(\tilde{f}_m))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\cap\mathbb{V}(z_1,\dots,z_{k-1})$$

is of dimension 0 or empty. From Lemma 4.3.11 we have

$$\mathbb{V}(I_{k,\tilde{z}}:\mathcal{J}(\tilde{f}_m))=\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1}).$$

It suffices to show that

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1})\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\cap\mathbb{V}(z_1,\dots,z_{k-1})$$

is 0 dimension or empty. Put

$$S:=\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\cap\mathbb{V}(z_1,\dots,z_{k-1}).$$

As $(z_0,\dots,z_k)$ is a Lê tuple for f, each component of $\mathbb{V}(I_{k+1}:\mathcal{J}(f))$ has dimension $k+1$ and then the dimension of S is equal to 1. Furthermore, if $S\cap\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1})$ is of dimension 1 then $\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1})$ contains a component of S. Consider $\mathcal{C}$ is a component of S such that

$$\mathcal{C}\subseteq\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1}).$$

Indeed, $\mathcal{C}$ is contained by $\mathbb{V}(\frac{\partial f}{\partial z_0}+m_1z_0^{m_1-1})$ and $\mathbb{V}(\frac{\partial f}{\partial z_0}+m_2z_0^{m_2-1})$ for any sufficiently large m_1 and m_2 with $m_1\neq m_2$. This show that z_0 vanishes along $\mathcal{C}$ but this is not the case since from the fact that $(z_0,z_1,\dots,z_k)$ is a Lê tuple for f we have

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\cap\mathbb{V}(z_0,\dots,z_{k-1})=S\cap\mathbb{V}(z_0)$$

is of dimension 0 or empty. Then $\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1})$ does not contain any component of S. Thus $S\cap\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1})$ can not be 1-dimensional. Then,

$$\mathbb{V}(I_{k+1}:\mathcal{J}(f))\cap\mathbb{V}(\frac{\partial f}{\partial z_0}+mz_0^{m-1})\cap\mathbb{V}\left(\frac{\partial f}{\partial z_k}\right)\cap\mathbb{V}(z_1,\dots,z_{k-1})$$

is of dimension 0. Hence, $(z_1,\dots,z_k)$ is a Lê tuple for $\tilde{f}_m$ for $m>2+\lambda_f^{(0)}$. $\square$

Theoreme 4.3.13. ([20], 1.14) With the preceding notation, the Lê numbers of $\widetilde{f_m}$ are

(i) $\lambda^{(0)}_{\widetilde{f_m}}(\mathbf{0})=\lambda_f^{(0)}(\mathbf{0})+(m-1)\lambda_f^{(1)}(\mathbf{0})$,

(ii) $\lambda^{(k)}_{\widetilde{f_m}}(\mathbf{0})=(m-1)\lambda_f^{(k+1)}(\mathbf{0})\quad\text{for}\quad 1\leq k\leq d-1.$

Example 4.3.14. Consider $f=z_2^3+z_1^5+z_0z_1^2+z_3z_1^3$. We have $\Sigma f=\mathbb{V}(z_1,z_2)$ with $\mathcal{J}(f)=\langle z_1^2,z_1^4+z_0z_1+z_3z_1^2,z_2^2,z_1^3\rangle$ and $dim(\Sigma f)=2$. We get $\Lambda_f^{(k)}=\emptyset$ and $\lambda_f^{(k)}(\mathbf{0})=0$ for all $k>2$. Let us first show that (z_3), (z_3,z_0) and (z_3,z_0,z_1) are Lê tuples for f: In this case, the ideals I_k with respect to the ordering $(z_3,z_0,z_1,z_2)=:(y_0,y_1,y_2,y_3)$ are

$$I_1=\left\langle\frac{\partial f}{\partial z_0},\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2}\right\rangle=\langle z_1^2,z_1(z_0+z_1^3+z_1z_3),z_2^2\rangle=\langle z_1,z_2^2\rangle\cap\langle z_1^2,z_2^2,z_2,z_0+z_1z_3\rangle,$$

$$I_2=\left\langle\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2}\right\rangle=\langle z_1^4+z_0z_1+z_1^2z_3,z_2^2\rangle,\text{ and}$$

$$I_3=\left\langle\frac{\partial f}{\partial z_2}\right\rangle=\langle z_2^2\rangle.$$

Since $\mathcal{J}(f) = \langle z_1, z_2^2 \rangle \cap \langle z_0, z_1^2, z_2^2 \rangle$, we have $I_1 : \mathcal{J}(f) = 1$, so $\mathbb{V}(I_1 : \mathcal{J}(f)) = \emptyset$. Thus

$$\mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_3}) = \emptyset$$

implies that (z_3) is a Lê tuple for f. Since $I_2 : \mathcal{J}(f) = \langle z_2^2, z_0 + z_1 z_3 + z_1^3 \rangle$ and

$$\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_0}) \cap \mathbb{V}(z_3) = \mathbb{V}(z_2^2, z_0 + z_1 z_3 + z_1^3, z_1^2, z_3) = \mathbb{V}(z_0, z_1^2, z_2^2, z_3)$$

is of dimension 0, (z_3, z_0) is a Lê tuple for f. Finally we have $I_3 : \mathcal{J}(f) = < z_2^2 >$ and

$$\mathbb{V}(I_3 : \mathcal{J}(f)) \cap \mathbb{V}(\frac{\partial f}{\partial z_1}) \cap \mathbb{V}(z_3, z_0) = \mathbb{V}(z_2^2, z_1^4 + z_0 z_1 + z_3 z_1^2, z_3, z_0) = \mathbb{V}(z_0, z_1^4, z_2^2, z_3)$$

This last variety is 0-dimensional. Hence (z_3, z_0, z_1) is a Lê tuple for f. In consequence (z_3, z_0, z_1, z_2) is a good choice for f. With respect to this ordering the Lê varieties of f are

$$\Lambda_f^{(0)} = \emptyset \text{ since } \mathbb{V}(I_1 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_3}) = \emptyset$$

$$\Lambda_f^{(1)} = \left[\mathbb{V}(I_2 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_0}) \right] \triangleleft \Sigma f = \mathbb{V}(z_1^2, z_2^2, z_0 + z_1 z_3) \triangleleft \Sigma f = \mathbb{V}(z_1^2, z_2^2, z_0 + z_1 z_3)$$

$$\begin{aligned}\Lambda_f^{(2)} &= \left[\mathbb{V}(I_3 : \mathcal{J}(f)) \cap \mathbb{V}(\tfrac{\partial f}{\partial z_1}) \right] \triangleleft \Sigma f = \mathbb{V}(z_2^2, z_1^4 + z_0 z_1 + z_3 z_1^2) \triangleleft \Sigma f \\ &= \mathbb{V}(z_2^2, z_1^4 + z_0 z_1 + z_3 z_1^2).\end{aligned}$$

Then the Lê numbers of f are as follows.

$$\lambda_f^{(0)}(\mathbf{0}) = 0 \text{ since } \Lambda_f^{(0)} = \emptyset,$$

$$\begin{aligned}\lambda_f^{(1)}(\mathbf{0}) &= \left(\mathbb{V}(z_2^2, z_0 + z_1 z_3 + z_1^3, z_1^2) \cdot \mathbb{V}(z_3)\right)_{\mathbf{0}} = \left(\mathbb{V}(z_2^2, z_0 + z_1 z_3 + z_1^3, z_1^2) \cap \mathbb{V}(z_3)\right)_{\mathbf{0}} \\ &= mult_{\mathbf{0}}(\mathbb{V}(z_0, z_1^2, z_2^2, z_3)) = 4, \text{ and}\end{aligned}$$

$$\begin{aligned}\lambda_f^{(2)}(\mathbf{0}) &= \left(\mathbb{V}(z_2^2, z_1^4 + z_0 z_1 + z_3 z_1^2) \cdot \mathbb{V}(z_3, z_0)\right)_{\mathbf{0}} = \left(\mathbb{V}(z_2^2, z_1^4 + z_0 z_1 + z_3 z_1^2) \cap \mathbb{V}(z_3, z_0)\right)_{\mathbf{0}} \\ &= mult_{\mathbf{0}}(\mathbb{V}(z_0, z_1^4, z_2^2, z_3)) = 8.\end{aligned}$$

Note that f is weighted homogeneous polynomial with $deg(f) = 5$. Put $\widetilde{f} := f + z_3^6$. We get $dim(\Sigma \widetilde{f}) = 1$ and $\lambda_{\widetilde{f}}^{(i)} = 0$ for $i > 1$. Fix the ordering of the coordinates as (z_0, z_1, z_2, z_3) for $\widetilde{f}$ such that it is an extension of the Lê tuples of $\widetilde{(}f)$. By previous theorem the Lê numbers of $\widetilde{f}$ at the origin are:

$\lambda^{(0)}_{\widetilde{f}}(\mathbf{0}) = \lambda^{(0)}_{f}(\mathbf{0}) + 5\lambda^{(1)}_{f}(\mathbf{0}) = 20$, and $\lambda^{(1)}_{\widetilde{f}}(\mathbf{0}) = 5\lambda^{(2)}_{f}(\mathbf{0}) = 40$.

Corollary 4.3.15. ([21], 4.6) Consider $(z_0, z_1, \dots, z_n)$ for $\mathbb{C}^{n+1}$ such that $(z_0, z_1, \dots, z_k)$ is a Lê tuple of f. Let $d := dim(\Sigma f)$ such that $d \geq 1$. Put

$$\widetilde{f} := f + z_0^{m_0} + \cdots + z_{d-1}^{m_{d-1}}$$

where $m_0 \ll m_1 \ll \cdots \ll m_{d_1 - 1}$ and $m_i > 2 + \lambda^{(0)}_{f}(\mathbf{0})$ for all i. Then $\widetilde{f}$ has an isolated singularity at the origin and

$$\mu(\widetilde{f}) = \sum_{i=0}^{d} \left(\lambda^{(i)}_{f}(\mathbf{0}) \prod_{k=0}^{i-1} (m_k - 1) \right).$$

Remark 4.3.16. Let f be an analytic function. Assume that $dim(\Sigma f) = 1$. Fix the ordering of coordinates as $(z_0, z_1, \dots, z_n)$ for $\mathbb{C}^{n+1}$ such that it is an extension of the Lê tuples of f. So, $\lambda^{(k)}_{f}(\mathbf{0})$ is defined for each k and $\lambda^{(k)}_{f}(\mathbf{0}) = 0$ for $k \geq 2$. Put $\widetilde{f} = f + z_0^m$ with $m \geq 2$. Let us use the ordering $(z_1, z_2, \dots, z_n, z_0)$ for $\widetilde{f}$. As $\Sigma\widetilde{f} = \Sigma f \cap \mathbb{V}(z_0)$ by Lemma 4.3.7, we have $dim(\Sigma(\widetilde{f}) = dim(\Sigma(f)) - 1 = 0$. This implies that 0-th Lê number of $\widetilde{f}$ is defined and non-zero, furthermore, we have $\lambda^{(0)}_{\widetilde{f}}(\mathbf{0}) = \mu(\widetilde{f})$ by Remark 4.3.3. Hence we obtain

$$\mu(\widetilde{f}) = \lambda^{(0)}_{f}(\mathbf{0}) + (m-1)\lambda^{(1)}_{f}(\mathbf{0})$$

by applying Theorem 4.3.13.

Corollary 4.3.17. ([21], 4.7) Let f be a homogeneous polynomial. Let $d := dim(\Sigma f)$ with $d \geq 1$. Consider the order of coordinates $(z_0, z_1, \cdots, z_n)$ is an extension of the Lê tuples of f. Then we have

$$\sum_{k=0}^{d} (deg(f) - 1)^k \lambda^{(k)}_{f}(\mathbf{0}) = (deg(f) - 1)^{n+1}.$$

Example 4.3.18. Consider $f = z_2^3 + z_1^3 + z_0 z_1^2$. It is homogeneous of degree 3. We have

$$\mathcal{J}(f) = \langle z_1^2 3z_1^2 + 2z_0 z_1, 3z_2^2, \rangle = \langle z_1, z_2^2 \rangle \cap \langle z_0, z_1^2, z_2^2 \rangle$$

and f has singularities along z_0-axis and $dim(\Sigma f) = 1$. Thus, $\Lambda^{(k)}_{f} = \emptyset$ and $\lambda^{(k)}_{f}(\mathbf{0}) = 0$ for all $k > 1$. To compute $\lambda^{(0)}_{f}$ and $\lambda^{(1)}_{f}$ we need first to see that (z_0) and (z_0, z_1) are Lê tuples for f. We have

$$I_1 = \left\langle \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \right\rangle = \left\langle 3z_1^2 + 2z_0z_1, 3z_2^2 \right\rangle, \text{ and } I_1 : \mathcal{J}(f) = \langle z_0 + z_1, z_2^2 \rangle,$$

$$\mathbb{V}(z_0 + z_1, z_2^2) \cap \mathbb{V}(z_1^2) = \mathbb{V}(z_0 + z_1, z_2^2, z_1^2)$$

is of dimension 0, hence z_0 is a Lê axis. The second tople (z_0, z_1) is also a Lê tuple for f since $I_2 = \left\langle \frac{\partial f}{\partial z_2} \right\rangle = \langle 3z_2^2 \rangle$ and $I_2 : \mathcal{J}(f) = \langle 3z_2^2 \rangle$ with

$$\mathbb{V}(3z_2^2) \cap \mathbb{V}(3z_1^2 + 2z_0z_1) \cap \mathbb{V}(z_0) = \mathbb{V}(3z_2^2, 3z_1^2 + 2z_0z_1, z_0)$$

which is 0-dimensional. So the Lê varieties of f are

$$\Lambda_f^{(0)} = \mathbb{V}(z_0 + z_1, z_2^2, z_1^2) \triangleleft \Sigma f = \mathbb{V}(z_0 + z_1, z_2^2, z_1^2), \text{ and}$$

$$\Lambda_f^{(1)} = \mathbb{V}(3z_2^2, 3z_1^2 + 2z_0z_1) \triangleleft \Sigma f = \mathbb{V}(3z_2^2, 3z_1^2 + 2z_0z_1).$$

and, the Lê numbers of f are

$$\lambda_f^{(0)}(\mathbf{0}) = mult_{\mathbf{0}} \left(\mathbb{V}(z_0 + z_1, z_2^2, z_1^2) \right) = 4, \text{ and}$$

$$\begin{aligned} \lambda_f^{(1)}(\mathbf{0}) &= \left(\mathbb{V}(3z_2^2, 3z_1^2 + 2z_0z_1) \cdot \mathbb{V}(z_0) \right)_{\mathbf{0}} = \left(\mathbb{V}(3z_2^2, 3z_1^2 + 2z_0z_1) \cap \mathbb{V}(z_0) \right)_{\mathbf{0}} \\ &= mult_{\mathbf{0}}(\mathbb{V}(3z_2^2, 3z_1^2 + 2z_0z_1, z_0)) = 2 \end{aligned}$$

Note that we have

$$(3-1)^0 \lambda_f^{(0)}(\mathbf{0}) + (3-1)\lambda_f(1)(\mathbf{0}) = 4 + 2.2 = 8 = (3-1)^3.$$

5 Modified Newton Numbers

5.1 Modified Newton Number

Let $f \in \mathbb{C}\{\mathbf{z}\}$ and $\Gamma_-(f) \subset \mathbb{R}^{n+1}$. Assume that $\Gamma_-(f)$ intersect at least one of the coordinate axes of $\mathbb{R}^{n+1}$. Let $\mathbf{K}$ be a set of k-dimensional compact faces in $\Gamma_-(f)$ intersecting with k-dimensional coordinate planes sections of $\mathbb{R}^{n+1}$ for all $0 < k \leq n+1$. Let us fix one of the axes, say z_1-axis.

$$\mathbf{K}_1 := \{F \in \mathbf{K} \mid F \cap \{z_1\text{-}axis\}\; is\; a\; 1\text{-}dimensional\; face\; of\; \Gamma_-(f)\}.$$

According to the choice of z_1, let $\widetilde{F}$ be a compact face obtained from $F \in \mathbf{K}_1$ replacing v_1 by unit vector e_1 in $\mathbb{R}^{n+1}$. We denote by $\widetilde{\mathbf{K}_1}$ the set of such compact faces $\widetilde{F}$. Let us see this sets on a concrete example.

Example 5.1.1. **[E_7-Type]** Let us recall the Newton polyhedron of f:

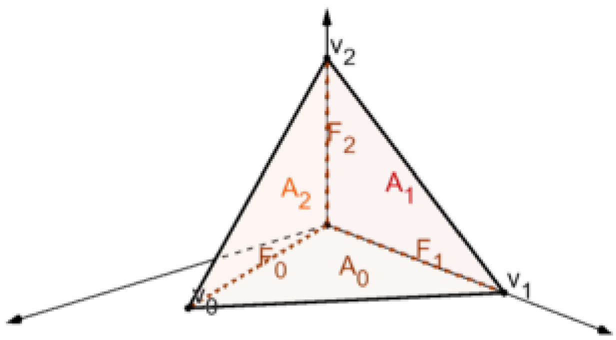

Figure 5.1: Newton polyhedron of f

The 1-dimensional compact faces of $\Gamma_-(f)$ are $F_0 = [\mathbf{0}\; v_0]$, $F_1 = [\mathbf{0}\; v_1]$, and $F_2 = [\mathbf{0}\; v_2]$; and the 2-dimensional compact faces of $\Gamma_-(f)$ are $A_0 = [\mathbf{0}\; v_0\; v_1]$, $A_1 = [\mathbf{0}\; v_1\; v_2]$ and $A_2 = [\mathbf{0}\; v_0\; v_2]$.
We have $\mathbf{K} = \{F_1, F_2, A_0, A_1, \Gamma_-(f)\}$. For $i = 0,1,2$ the subsets $\mathbf{K}_i$ of $\mathbf{K}$ with respect to z_i-axis are as follows:

$$\mathbf{K}_0 = \emptyset, \quad \mathbf{K}_1 = \{F_1, A_0, A_1, \Gamma_-(f)\}, \quad \mathbf{K}_2 = \{F_2, A_1, \Gamma_-(f)\}.$$

So we have $\widetilde{\mathbf{K}}_0 = \emptyset$.
In the case of choosing z_1-axis, we replace v_1 of all compact faces in $\mathbf{K}_1$ with the unit vector e_1 and we get

$$\widetilde{\mathbf{K}}_1 = \{\widetilde{F}_1,\, \widetilde{A}_0,\, \widetilde{A}_1,\, \widetilde{\Gamma_-}(f)\}.$$

Note that $\widetilde{F}_1 = conv(\mathbf{0}, e_1)$, $\widetilde{A}_0 = conv(\mathbf{0}, v_0, e_1)$, $\widetilde{A}_1 = conv(\mathbf{0}, e_1, v_2)$ and $\widetilde{\Gamma_-}(f) = conv(\mathbf{0}, v_0, e_1, v_2)$.
In a similar way, replacing v_2 in $\mathbf{K}_2$ by the unit vector e_2 we obtain

$$\widetilde{\mathbf{K}}_2 = \{\widetilde{F}_2,\, \widetilde{A}_1,\, \widetilde{\Gamma_-}(f)\}$$

with $\widetilde{F}_2 = conv(\mathbf{0}, e_2)$,$\widetilde{A}_1 = conv(\mathbf{0}, v_1, e_2)$, and $\widetilde{\Gamma_-}(f) = conv(\mathbf{0}, v_0, v_1, e_2)$.

Example 5.1.2. [$E_{6,0}$-Type] Consider

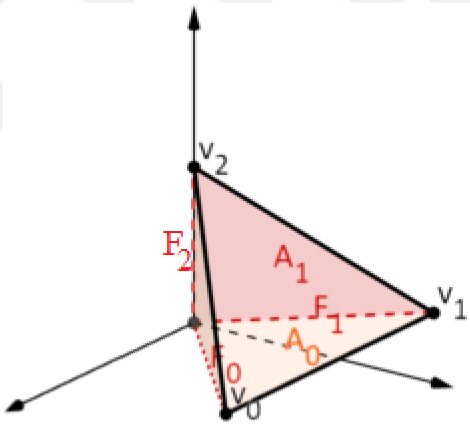

Figure 5.2: Newton boundary of $E_{6,0}$-type singularity

The 1-dimensional compact faces of $\Gamma_-(f)$ are $F_0 = conv(\mathbf{0}, v_0)$, $F_1 = conv(\mathbf{0}, v_1)$, and $F_2 = conv(\mathbf{0}, v_2)$; the 2-dimensional compact faces of $\Gamma_-(f)$ are $A_0 = conv(\mathbf{0}, v_0, v_))$, $A_1 = conv(\mathbf{0}, v_1, v_2)$ and $A_2 = conv(\mathbf{0}, v_0, v_2)$. Put $\mathbf{K} = \{F_2, A_1, \Gamma_-(f)\}$. For $i = 0, 1, 2$ the subsets $\mathbf{K}_i$ of $\mathbf{K}$ with respect to z_i-axis are as follows:

$$\mathbf{K}_0 = \mathbf{K}_1 = \emptyset \quad \textit{and}\ \mathbf{K}_2 = \{F_2, A_1, \Gamma_-(f)\}.$$

So we have $\widetilde{\mathbf{K}}_0 = \widetilde{\mathbf{K}}_1 = \emptyset$. Replacing v_2 in $\mathbf{K}_2$ with the unit vector e_2 we obtain

$$\widetilde{\mathbf{K}}_2 = \{\widetilde{F}_2,\, \widetilde{A}_1,\, \widetilde{\Gamma_-}(f)\}$$

with $\widetilde{F}_2 = conv(\mathbf{0}, e_2)$, $\widetilde{A}_1 = conv(\mathbf{0}, v_1, e_2)$, and $\widetilde{\Gamma_-}(f) = conv(\mathbf{0}, v_0, v_1, e_2)$.

Definition 5.1.3. *With the preceding notation, let V_k be the sum of volumes of k-dimensional faces in $\widetilde{\boldsymbol{K}}_i$. The number*

$$\widetilde{\nu}_i(f) := (n+1)!V_{n+1} - n!V_n + \cdots + (-1)^{(n+1)-k}k!V_k + \cdots + (-1)^n V_1$$

is called modified Newton number of f with respect to z_i-axis (corresponding to $\widetilde{\boldsymbol{K}}_i$).

Example 5.1.4. [E_7-type] By Using Example 5.1.1, we have

$$\widetilde{\mathbf{K}}_0 = \emptyset, \quad \widetilde{\mathbf{K}}_1 = \{\widetilde{F}_1, \widetilde{A}_0, \widetilde{A}_1, \widetilde{\Gamma_-}(f)\} \quad \textit{and} \quad \widetilde{\mathbf{K}}_2 = \{\widetilde{F}_2, \widetilde{A}_1, \widetilde{\Gamma_-}(f)\}.$$

So, f has modified Newton numbers with respect to the z_1 and z_2-axes. First we compute the one $\widetilde{\nu}_1(f)$ of f with respect to the z_1-axis: The volumes V_k with respect to $\widetilde{\mathbf{K}}_1$ are:

$$V_3 = Vol_3(\widetilde{\Gamma_-}(f)) = 1,\ V_2 = Vol_2(\widetilde{A}_0) + Vol_2(\widetilde{A}_1) = \frac{3}{2} + 1 = \frac{5}{2}\ \textit{and}\ V_1 = Vol_1(\widetilde{F}_1) = 1.$$

Hence, we have $\widetilde{\nu}_1(f) = 2$.
Now, let us compute the modified Newton number $\widetilde{\nu}_2(f)$ of f with respect to the z_2-axis. The volumes V_k with respect to $\widetilde{\mathbf{K}}_2$ are

$$V_3 = Vol_3(\widetilde{\Gamma_-}(f)) = \frac{3}{2},\ V_2 = Vol_2(\widetilde{A}_1) = \frac{3}{2}\ \textit{and}\ V_1 = Vol_1(\widetilde{F}_2) = 1.$$

Hence, the modified Newton number is $\widetilde{\nu}_2(f) = 7$.

Example 5.1.5. [$E_{6,0}$-type] By Example 5.1.2 we have

$$\widetilde{\mathbf{K}}_0 = \widetilde{\mathbf{K}}_1 = \emptyset \quad \textit{and} \quad \widetilde{\mathbf{K}}_2 = \{\widetilde{F}_2, \widetilde{A}_1, \widetilde{\Gamma_-}(f)\}.$$

This means that f may get the modified Newton numbers with respect to the z_2-axis.
To compute $\widetilde{\nu}_2(f)$ the volumes V_k in $\widetilde{\mathbf{K}}_2$: are

$$V_3 = Vol_3(\widetilde{\Gamma_-}(f)) = \frac{2}{3},\ V_2 = Vol_2(\widetilde{A}_1) = 1,\ \textit{and}\ V_1 = Vol_1(\widetilde{F}_2) = 1.$$

Hence the modified Newton number is $\widetilde{\nu}_2(f) = 3$.

5.1.1 Special modified Newton number

Let us fix one of z_i-axes as before, put

$$\mathbf{K}^s_{i_0} := \{F \in \mathbf{K} | F \cap \{z_{i_0}\text{-}axis\}\ \textit{is not a}\ 1\text{-}\textit{dimensional face on}\ \{z_{i_0}\text{-}axis\}\}.$$

Definition 5.1.6. *Let V_k be the sum of volumes of k-dimensional compact faces in $\boldsymbol{K}^{si_0}$. The number*

$$\nu^s_{i_0}(f) := (n+1)!V_{n+1} - n!V_n + \cdots + (-1)^{(n+1)-k}k!V_k + \cdots + (-1)^n V_1$$

is called the special modified Newton number of f.

Example 5.1.7. [E_7-type] We have

$$\mathbf{K} = \{F_1, F_2, A_0, A_1, \Gamma_-(f)\}.$$

Let us fix z_0-axis. So $\mathbf{K}^s_0 = \mathbf{K}$. Then, the special modified Newton number of f with respect to the z_0- axis is

$$\nu^s_0(f) = 3!3 - 2!\frac{15}{2} + 5 = 8 = \nu(\Gamma_-(f)) + 1 = 7 + 1.$$

Now le us fix z_1-axis. So, $\mathbf{K}^s_1 = \{F_2\}$. The special modified Newton number of f with respect to the z_1-*axis* is $\nu^s_1 = 2$. Finally, by fixing z_2-axis we obtain $\mathbf{K}^s_2 = \{A_0, F_1\}$. In this case, the special modified Newton number is

$$\nu^s_2(f) = -2!\frac{9}{2} + 3 = -6.$$

Example 5.1.8. [$E_{6,0}$-type] Recall that $\mathbf{K} = \{F_2, A_1, \Gamma_-(f)\}$ and

$$Vol_1(F_2) = 3, Vol_2(A_1) = 3 \textit{ and } Vol_3(\Gamma_-(f)) = 2$$

If we fix z_0-axis, we get $\mathbf{K}^s_0 = \mathbf{K}^s_1 = \mathbf{K}$. Then, the special modified Newton numbers of f with respect to the z_0 and z_1- axes are

$$\nu^s_0(f) = \nu^s_1(f) = 9.$$

Since there is no compact faces in $\mathbf{K}$ intersecting the z_2-axis we have $\mathbf{K}^s_2 = \emptyset$. Hence, the special modified Newton number with respect to the z_2-axis does not exist.

5.2 Modified Newton numbers and Lê numbers

Let f be a non-degenerate analytic function in Kouchnirenko sense. Let $d := dim(\Sigma f)$ such that $d \geq 1$. Assume that $(z_0, z_1, \ldots, z_n)$ is an extension of the Lê tuples of f. Put

$$\widetilde{f} := f + z_0^{m_0} + \cdots + z_{d-1}^{m_{d-1}}$$

where $m_i > 2 + \lambda_f^{(0)}(\mathbf{0})$ for $i = 0, 1, \ldots, d-1$ such that $m_0 \ll m_1 \ll \cdots \ll m_{d-1}$. Here, we take $m_i > max\{2, deg(f)\}$ whenever f is weighted homogeneous polynomial. Let us use the rotated coordinates $(z_d, z_{d+1}, \ldots, , z_n, z_0, z_1, \ldots, z_{d-1})$ for $\widetilde{f}$.

Theoreme 5.2.1. ([8], 4.1) If f is non-degenerate then the Lê numbers are:

$$\lambda_f^{(0)}(0) = (-1)^{n+1} + \nu^s_{\{0,1,\cdots,d-1\}}(\widetilde{f}) + \tilde{\nu}_0(\widetilde{f});$$

$$\lambda_f^{(k)}(0) = (-1)^{k-1}(\tilde{\nu}_{k-1}(\widetilde{f}) - \tilde{\nu}_k(\widetilde{f}) \text{ for } 1 \leq k \leq d-1 \text{ (if } d \geq 2);$$

$$\lambda_f^{(d)}(0) = (-1)^{d-1}(\tilde{\nu}_{d-1}\widetilde{f}));$$

Remark 5.2.2. When $d = 1$, let us put $\widetilde{f}(\mathbf{z}) := f(\mathbf{z}) + z_0^{m_0}$ where $m_0 > 2 + \lambda_f^{(0)}$. By the previous theorem, we have

$$\lambda_f^{(0)}(0) = (-1)^{n+1} + \nu_0^s(\widetilde{f}) + \tilde{\nu}_0(\widetilde{f}) \text{ and } \lambda_f^{(1)}(0) = \tilde{\nu}_0(\widetilde{f}).$$

Moreover, all other Lê number of f is defined and equal to zero at the origin, that is, $\lambda_f^{(k)} = 0$ for all $k > 1$.

Example 5.2.3. **[$E_{6,0}$-Type]** Consider $f = z_2^3 + z_1^3 z_2 + z_0^2 z_1^2$ with $deg(f) = 4$. So, f has a line singularity along z_0-*axis* such that $dim_{\mathbf{0}}\Sigma f = 1$. Let us fix the order (z_0, z_1, z_2) for $\mathbb{C}^3$ such that z_0 a is Lê-axis for f. Since f is weighted homogeneous of weight $(5,4,6)$ and of degree 18, one can take $m > 4$. Put $\widetilde{f} = f + z_0^5$. To compute Lê numbers of f we will first calculate the modified Newton number and the special modified Newton number for $\widetilde{f}$ with respect to z_0-axis. Note that the Newton boundary of $\widetilde{f}$ is

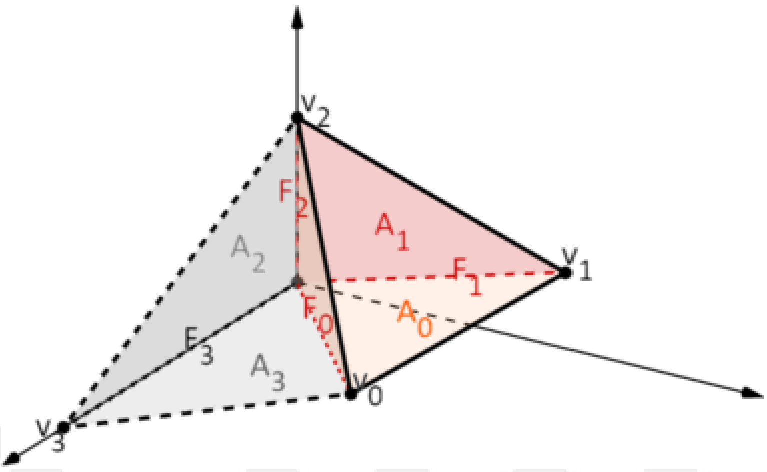

Figure 5.3: Newtonboundary of $\widetilde{f}$

Here, $\Gamma_-(\widetilde{f})$ contains two tetrahedra as 3-dimensional compact faces

$$\{(0,0,0), v_0 = (2,2,0), v_1 = (0,3,1), v_2 = (0,0,3)\} \text{ and}$$
$$\{(0,0,0), v_0 = (2,2,0), v_2 = (0,0,3), v_3 = (5,0,0)\}$$

such that first one is the same as $\Gamma_-(f)$. In the figure $F_i = conv(\mathbf{0}, v_i)$ are 1-dimensional compact faces of $\Gamma_-(f)$ and $A_i = conv(\mathbf{0}, v_i, v_{i+1})$ are its 2-dimensional compact faces for $i = 0,1,2$, furthermore, $A_3 = conv(\mathbf{0}, v_3, v_0)$ is also 2-dimensional compact faces of $\Gamma_-(f)$. Note that to compute modified Newton numbers of $\widetilde{f}$ we will use the compact faces of $\Gamma_-(f)$ in the set

$$\mathbf{K} = \{F_1, F_2, F_3, A_1, A_2, A_3, conv(\mathbf{0}, v_0, v_1, v_2), conv(\mathbf{0}, v_0, v_2, v_3)\}.$$

In particular, we are interested in two subsets of $\mathbf{K}$:

$$\mathbf{K}_0 = \{F_3, A_2, A_3, conv(\mathbf{0}, v_0, v_2, v_3)\} \; and \; \mathbf{K}_0^s = \{F_2, A_1, conv(\mathbf{0}, v_0, v_1, v_2)\}.$$

Let us put the reduced form of the compact faces in $\mathbf{K}_0$ into the set

$$\widetilde{\mathbf{K}_0} = \{\widetilde{F_3} = conv(\mathbf{0}, \widetilde{v_3}), \widetilde{A_2} = conv(\mathbf{0}, v_2, \widetilde{v_3}), \widetilde{A_3} = conv(\mathbf{0}, \widetilde{v_3}, v_0), conv(\mathbf{0}, v_0, v_2, \widetilde{v_3})\}$$

where $\widetilde{v_3} = (1,0,0)$. To compute the Modified Newton number of f with respect to the z_0-axis, $\nu_0(f)$, note that the volumes:

$$vol_1(\widetilde{F_3}) = 1,\ vol_2(\widetilde{A_2}) = 1,\ vol_2(\widetilde{A_3}) = \frac{3}{2},\ \text{and}\ vol_3(conv(\mathbf{0}\,v_0\,v_2\,\widetilde{v_3})) = 1.$$

Thus,

$$\widetilde{\nu}_0(f) = 3!V_3 - 2!V_2 + V_1 = 6 - 2(1+\frac{3}{2}) + 1 = 2$$

from where $V_3 = vol_3(conv(\mathbf{0}\,v_0\,v_2\,\widetilde{v_3})])$, $V_2 = vol_2(\widetilde{A_2}) + vol_2(\widetilde{A_3})$, and $V_1 = vol_1(\widetilde{F_3})$. Now, note that the volumes of the faces in $\mathbf{K}_0^s$ are

$$vol_1(F_2) = 3,\ vol_2(A_1) = \frac{9}{2},\ and\ vol_3(conv(\mathbf{0}\,v_0\,v_1\,v_2)) = 3.$$

we obtain $\nu_0^s(f) = 3!V - 3 - 2!V_2 + V_1 = 3!3 - 2!\frac{9}{2} + 3 = 12$ where $V_i = vol_i$ for each i. Then the Lê numbers of f are

$$\lambda_f^{(0)}(\mathbf{0}) = (-1)^3 + \nu_0^s(\widetilde{f}) + \widetilde{\nu}_0(\widetilde{f}) = -1 + 12 + 2 = 13,\ \text{and}\ \lambda_f^{(1)}(\mathbf{0}) = \nu_0(\widetilde{f}) = 2.$$

Remark 5.2.4. Let f be a non-degenerate function in Kouchnirenko sense. Consider $\Gamma_-(f) \cap \{z_{i_0}\text{-}axis\} = \emptyset$. When $d = 0$, from Remark 4.3.3 we have $\lambda_f^{(0)}(\mathbf{0}) = \mu(f)$ and $\lambda_f^{(k)}(\mathbf{0}) = 0$ for all $k > 0$. If we take $\widetilde{f} = f$ then Theorem 5.2.1 gives

$$\lambda_f^{(0)}(f) = (-1)^{n+1} + \nu_{i_0}^s(f) + \widetilde{\nu}_{i_0}(f).$$

Since $\Gamma_-(f) \cap \{z_{i_0}\text{-}axis\} = \emptyset$ we have $K_{i_0} = \emptyset$ and so $\widetilde{\nu}_{i_0}(f) = 0$. Furthermore if f is non-degenerate then $\mu(f) = \nu(f)$ by Theorem 3.2.28. Hence,

$$\nu(f) = (-1)^{n+1} + \nu_{i_0}^s(f).$$

Example 5.2.5. [E_7-Type] Note that $\Gamma_-(f) \cap \{z_0\} = \emptyset$, see Figure 5.1. From Example 5.1.7 we have $\mathbf{K} = \{F_1, F_2, A_0, A_1, \Gamma_-(f)\} = \mathbf{K}_0^s$ and $\nu_0^s(f) = 8$. Then, $\nu(f) = (-1)^3 + \nu_0^s(f) = -1 + 8 = 7$. See Example 3.2.12, for $\nu(f)$.

Corollary 5.2.6. ([8], 5.1) Let f, g be two non-degenerate functions. Consider the coordinates system $(z_0, z_1, \ldots, z_n)$ for $\mathbb{C}^{(n+1)}$ is an extension of the Lê tuples of f and of g. Assume that $\Gamma(f) = \Gamma(g)$. Then we have

$$dim(\Sigma(f)) = dim(\Sigma(g)) \quad and \quad \lambda_f^{(k)}(0) = \lambda_g^{(k)}(0)$$

for all k.

Proof. Let $d_1 := dim(\Sigma f)$ and $d_2 := dim(\Sigma g)$. Suppose for instance that $d_1 < d_2$ and $d_1 \geq 1$. Put

$$\widetilde{f} := f + z_0^{m_0} + \cdots + z_{d_1-1}^{m_{d_1-1}} \quad \textit{and} \quad \widetilde{g} := g + z_0^{m_0} + \cdots + z_{d_1-1}^{m_{d_1-1}}$$

where $m_0 \ll m_1 \ll \cdots \ll m_{d_1-1}$ and $m_i > 2 + \lambda_f^{(0)}(\mathbf{0})$ for all i. Note that $\widetilde{f}$ and $\widetilde{g}$ are Newton nondegenerate by Proposition 3.2.26 and they have an isolated singularity at the origin by Corollary 4.3.15. Thus $\mu(\widetilde{f}) = \nu(\widetilde{f})$ and $\mu(\widetilde{g}) = \nu(\widetilde{g})$ by Theorem 3.2.28. From the fact that $\Gamma(f) = \Gamma(g)$ we have $\Gamma(\widetilde{f}) = \Gamma(\widetilde{g})$. Hence, $\nu(\Gamma_{\widetilde{f}}) = \nu(\Gamma_{\widetilde{g}})$ and than $\nu(\widetilde{f}) = \nu(\widetilde{g})$. On the other hand, since $(z_0, z_1, \ldots, z_n)$ is an extension of the Lê tuples of f and of g we have

$$\Sigma\widetilde{f} = \Sigma f \cap \mathbb{V}(z_0, z_1, \ldots, z_{d_1-1}) \quad \textit{and} \quad \Sigma\widetilde{g} = \Sigma g \cap \mathbb{V}(z_0, z_1, \ldots, z_{d_1-1})$$

and $dim(\Sigma\widetilde{f}) = dim(\Sigma f) - d_1 = 0$ and $dim(\Sigma\widetilde{g}) = dim(\Sigma g) - d_1 = d_2 - d_1 > 0$. This is in contradiction with the fact that $\widetilde{g}$ has an isolated singularity at the origin. Then $d_1 = d_2$ when $d_1 \geq 1$. Moreover, as $\widetilde{f}$ and $\widetilde{g}$ are nondegenerate and $d_1 \geq 1$ one can apply Theorem 5.2.1 to calculate the Lê numbers. Indeed, $\Gamma(\widetilde{f}) = \Gamma(\widetilde{g})$ implies $\nu_i(\widetilde{f}) = \nu_i(\widetilde{g})$ and $\nu_i^s(\widetilde{f}) = \nu_i^s(\widetilde{g})$ for all i. This gives $\lambda_f^{(k)}(\mathbf{0}) = \lambda_g^{(k)}(\mathbf{0})$ for all k when $d_1 \geq 1$. Now suppose that $d_1 = 0$. So we have $\lambda_f^{(k)}(\mathbf{0}) = 0$ for all $k > 0$ and $\lambda_f^{(0)}(\mathbf{0}) = \mu(f)$. If f and g is Newton non degenerate then we have $\mu(f) = \nu(f)$ and $\mu(g) = \nu(g)$. As $\Gamma(f) = \Gamma(g)$ we have $\nu(f) = \nu(g)$. So, $\mu(f) = \nu(f) = \nu(g) = \mu(g)$ and $d_2 = 0$. This concludes the proof. $\square$

Example 5.2.7. Consider f is $E_{6,0}$-type singularity. Let us fix the coordinates system (z_0, z_1, z_2) for $\mathbb{C}^3$. Let

$$g := z_2^6 + z_0^2 z_1^3 z_2^4 + (z_1^4 + 1)z_2^3 + (z_0^3 z_1^3 + z_1^3)z_2 + z_0 z_1^5 + z_0^2 z_1^4 + z_0^2 z_1^2.$$

Note that

$$supp(g) = \{(0,0,6),(2,3,4),(0,4,3),(0,0,3),(3,3,1),(0,3,1),(1,5,0),(2,4,0),(2,2,0)\}.$$

So $\Gamma(g) = conv((0,0,3),(2,2,0),(0,3,1)) = \Gamma(f)$, see Example 5.2.3 for $\Gamma(f)$. Moreover, we have the system

$$\begin{cases} \dfrac{\partial g}{\partial z_0} = z_0 z_1^3 z_2^4 + z_0^2 z_1^3 z_2 + z_1^5 + z_0 z_1^4 + z_0 z_1^2 = 0 \\ \\ \dfrac{\partial g}{\partial z_1} = z_0^2 z_1^2 z_2^4 + z_1^3 z_2^3 + z_0^3 z_1^2 z_2 + z_1^2 z_2 + z_0 z_1^4 + z_0^2 z_1 = 0 \\ \\ \dfrac{\partial g}{\partial z_2} = z_2^5 + z_0^2 z_1^3 z_2^3 + z_1^4 z_2^2 + z_2^2 + z_0^3 z_1^3 + z_1^3 = 0 \end{cases} .$$

As all points $(z_0, 0, 0)$ satisfy the system above, g has singularities along the z_0-axis. Thus,

$$dim(\Sigma g) = 1 = dim(\Sigma f).$$

Note that the coordinates system (z_0, z_1, z_2) is also an extension of the Lê tuples of g. Then we obtain $\lambda_g^{(k)}(\mathbf{0}) = 0$ for all $k > 1$. Since g is a weighted homogenous polynomial, one can take $m > deg(g) = 9$. Put $\widetilde{g} = g + z_0^{10}$ with $dim(\Sigma \tilde{g}) = 0$. So we get $\Gamma_-(\tilde{g})$ as in the Figure 5.3 by replacing the vertex v_3 with the vertex $v_4 := (10, 0, 0)$. To compute Lê numbers of g we need to compute the modified Newton numbers of $\tilde{g}$ with respect to the z_0 axis: $\tilde{\nu}_0(\tilde{g})$ and $\nu_0^s(g)$. Since the sets

$$\widetilde{\mathbf{K}_0} = \{conv(\mathbf{0}, \tilde{v_4}), conv(\mathbf{0}, v_2, \tilde{v_4}), conv(\mathbf{0}, \tilde{v_4}, v_0), conv(\mathbf{0}, v_0, v_2, \tilde{v_4})\}$$

with $\tilde{v_4} = (1, 0, 0)$ and $\mathbf{K}_0^s = \{conv(\mathbf{0}, v_2), conv(\mathbf{0}, v_1, v_2), conv(\mathbf{0}, v_0, v_1, v_2)\}$ are the same as in Example 5.2.3, the corresponding modified Newton numbers are the same as in the Example i.e. $\tilde{\nu}_0(\tilde{g}) = 2$ and $\nu_0^s(\tilde{g}) = 12$. This gives $\lambda_f^{(0)}(\mathbf{0}) = \lambda_g^{(0)}(\mathbf{0})$ and $\lambda_f^{(1)}(\mathbf{0}) = \lambda_g^{(1)}(\mathbf{0})$ by Theorem 5.2.1. Hence, $\lambda_f^{(k)}(\mathbf{0}) = \lambda_g^{(k)}(\mathbf{0})$ for all k.

6 Conclusion

In this thesis, we aim to understand the relation between numerical invariants of f using the Newton polyhedron. When $d = 0$, we present roughly the Milnor number and Tjurina number of f. The Milnor number is a topological invariant of f while the Tjurina number is an analytic invariant of f. We give some results about the relation between them as $\mu(f) \geq \nu(f)$ and $\tau(f) \geq \frac{\nu(f)}{n+1}$. We have $\mu(f) = \nu(f)$ when f is Newton non-degenerate. When $d \geq 1$, we present Lê varieties and we explain in detail how to compute it in an algebraic way. We show that the singular locus of f is the union of the first d Lê varieties of f. The Le number can be seen as a generalization of the Milnor number defined for isolated singularities to non-isolated singularities. In particular, the Milnor number of an isolated singularity equals its 0th Lê number. We present another numerical invariant of f associated with the Newton polyhedron of f, called the modified Newton number. We explain in detail an algorithm which give Lê numbers of f using the modified Newton number of $f + z_o^m + z_1^m + \cdots + z_d^m$ for sufficiently large m. Almost all examples in this thesis are new, and don't appear in the literature; we aim that these examples lead us to relate the Tjurina number, Lê number, and Lojasiewicz number.

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APPENDIX

On Primary Decomposition of Ideals

Let R be a Noetherian ring. An ideal $I \subset R$ in R is said to be a *primary* if $xy \in I$ and $x \notin I$ implies $y^s \in I$ for some s.

Proposition 6.0.1. ([14], 1.4.8) Let $I \subset R$ be an ideal. If the radical $\sqrt{I}$ is a maximal ideal, then I is a primary ideal.

Proof. Suppose that $\sqrt{I}$ is a maximal ideal. Put $\mathfrak{m} := \sqrt{I}$. We want to show that I is a primary ideal. Let $xy \in I$ with $x \notin I$. Assume that $y^s \notin I$ for all $s \in \mathbb{N}$. So, $y \notin \mathfrak{m}$. As $\mathfrak{m}$ is a maximal ideal, we have $\langle y \rangle + \mathfrak{m} = R$. That is, $1 = ry + m$ with $r \in R$ and $m \in \mathfrak{m}$. Moreover, we have $x = 1x = (ry+m)x = ryx + mx$ and its iteration gives

$$x = r(1+m+m^2+\cdots+m^k)yx + m^{k+1}x$$

for all $k > 0$. Hence, $m^{k+1} \in I$. Then we get $x \in I$. This contradicts the assumption. Hence, $y^s \in I$ for some s . Therefore, I is primary. $\square$

Proposition 6.0.2. ([14], 1.4.8) If $I \subset R$ is a primary ideal, then $\sqrt{I}$ is prime.

Proof. Let $xy \in \sqrt{I}$. We have $(xy)^s \in I$ for some s. Assume that $x \notin \sqrt{I}$. So, $x^s \notin I$. This gives $(y^s)^r \in I$ for some r. Hence, $y \in \sqrt{I}$. Then, $\sqrt{I}$ is a prime ideal. $\square$

Remark 6.0.3. The inverse implication of the statement in the previous proposition is not always true. That is, an ideal whose radical is prime is not necessarily primary.

Example 6.0.4. Let $R = \mathbf{k}[x,y,z]/\left\langle xy - z^2 \right\rangle$. Consider $I = \langle x, z \rangle \subset R$. We have

$$R/I \cong \mathbf{k}[x,y,z]/\langle x,z \rangle \cong \mathbf{k}[y]$$

which is an integral domain. So, I is a prime ideal and we have $I^2 = \left\langle x^2, xz, z^2 \right\rangle$. Since $z^2 \in I^2$ and $z^2 = xy$ in R, we have $xy \in I^2$. And, we have $x \notin I^2$. As $y^s \notin I^2$ for any $s \in \mathbb{N}$, I^2 is not primary.

Theoreme 6.0.5. ([14], 1.4.11) *Let $I \subset R$ be an ideal. Then there exist finitely many primary ideals $I_1, I_2, \cdots, I_r$ in R such that*

$$I = I_1 \cap I_2 \cap \cdots \cap I_r.$$

Definition 6.0.6. *Let I be ideal in R. A primary decomposition $I = I_1 \cap I_2 \cap \ldots \cap I_r$ of I is said to be minimal if $I_i \neq I_j$, $I_i \not\subset I_j$ for all $i \neq j$ and $P_i := \sqrt{I_i}$ are pairwise different for all i.*

Example 6.0.7. Consider $I = \langle x^2y, y^2x \rangle$. A decomposition into ideals of I is:

$$I = \langle x^2y - y^2x, y^2x \rangle = \langle xy(x-y), y^2x \rangle = \langle y \rangle \cap \langle x(x-y), y^2x \rangle = \langle x \rangle \cap \langle y \rangle \cap \langle x-y, y^2 \rangle$$

To say that it is a primary decomposition for I each ideal in the decomposition should be a primary ideal. Obviously, $\langle x \rangle$ and $\langle y \rangle$ are primary. For the last ideal on the right-hand side, we have

$$\sqrt{\langle x-y, y^2 \rangle} = \langle x-y, y \rangle = \langle x, y \rangle$$

which is a maximal ideal in $\mathbf{k}[x,y]$. By Proposition 6.0.1 $\langle x-y, y^2 \rangle$ is primary. Hence, the right-hand side of equality above is a primary decomposition for I. Furthermore, as the primary ideals in the decomposition are pairwise different, the primary decomposition is the minimal primary decomposition of I.

Consider $I = I_1 \cap \cdots \cap I_r$ is a minimal primary decomposition of I. Then, $\sqrt{I_i}$ is called the associated prime of I and $Ass(I) := \{\sqrt{I_i} \mid 1 \leq i \leq r\}$.

Definition 6.0.8. *Let I be ideal in R. Let $p \in Ass(I)$.*
(i) $\mathbb{V}(p)$ is called the isolated (or non-embedded) component of $\mathbb{V}(I)$ if there is no element $q \in Ass(I)$ such that $q \subset p$.
(ii) $\mathbb{V}(p)$ is called embedded component of $\mathbb{V}(I)$ if there is at least one prime ideal $q \in Ass(I)$ such that $q \subset p$.

Example 6.0.9. Consider $I = \langle x^2y, y^2x \rangle$ as in Example 6.0.7. We have

$$I = \langle x \rangle \cap \langle y \rangle \cap \langle x-y, y^2 \rangle$$

where $\sqrt{\langle x-y, y^2 \rangle} = \langle x, y \rangle$. So, all associated primes are pairwise different. Hence, the primary decomposition is minimal and we have

$$Ass(I) = \{\langle x \rangle, \langle y \rangle, \langle x, y \rangle\}.$$

Since $\langle x\rangle, \langle y\rangle \subset \langle x,y\rangle$, the ideals $\mathbb{V}(x)$ and $\mathbb{V}(y)$ are isolated components of $\mathbb{V}(I)$ and $\mathbb{V}(x,y)$ is an embedded component of $\mathbb{V}(I)$.

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