HOMOLOGY IN COMPLEXES

by

ÖZGÜR EKER

THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE ABANT IZZET BAYSAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF EDUCATION

FEBRUARY 2010

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Nihat ÇELEBİ
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Cenap ÖZEL
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Tahire ÖZEN
Supervisor

Examining Committee Members

Prof. Dr. Abdullah HARMANCI

Assist. Prof. Dr. Tahire ÖZEN

Assist. Prof. Dr. Cesim ÇELİK

ABSTRACT

HOMOLOGY IN COMPLEXES

Eker, Özgür

Master of Science, Department of Mathematics

Supervisor: Assist. Prof. Dr. Tahire Özen Öztürk

February 2010, 64 pages

In this thesis, we study on the generalizations of the some concepts in the category R-Mod (for ex. Projective module, injective module Ext, complete cotorsion pair, etc.) to the category of complexes on R-Mod. Moreover it is used Enochs' definition for Ext. This definition is different from Foxby and Avramov's.

This thesis consists of four chapters. In the first chapter, the first part contains some basic homological concepts in the category R-Mod and the second part contains some basic concepts in the category of complexes. . In the second chapter, it is studied the relationships between exact complexes and DG-projective and DG-injective complexes. Moreover we see that every complex has an exact cover with kernel DG-injective, an exact envelope with cokernel DG-projective and a DG-injective envelope with cokernel exact. In the third chapter, it is given a necessary condition to be that a quasi-isomorphism is an isomorphism. In the final chapter, if (C,D) is a complete cotorsion pair in the category of complexes such that C is suspension closed, we see that when we regard C and D as subcategories of the

homotopy category K(R-Mod), then the embedding functors $C \mapsto K(R\text{-Mod})$ and $D \mapsto K(R\text{-Mod})$ have left and rigt adjoints, respectively.

Keywords: projective - injective module, Ext, cover - envelope, DG-projective - DG-injective complex and injective - projective complex.

ÖZET

KOMPLEKSLERDE HOMOLOJİ

Eker, Özgür

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Tahire Özen Öztürk

Şubat 2010, 64 sayfa

Bu tezde sol (yada sağ) R-modüllerin kategorisindeki bazı kavramların (örneğin projective modül, injective modül, Ext, komplete kotorsion pair, vs.) sol (yada sağ) R-modüller üzerindeki komplekslerin kategorisine olan genellemeleri üzerinde çalıştık. Ayrıca Ext için Enochs' un tanımını kullandık. Bu tanım Foxby ve Avromov' un tanımından farklıdır.

Bu tez dört bölümden oluşmaktadır. Birinci bölümde ilk kısım sol (yada sağ)R-modüllerin kategorisindeki tezde kullandığımız bazı temel homolojiksel kavramları ve ikinci kısım ise komplekslerin kategorisindeki bazı temel kavramları kapsamaktadır. İkinci bölümde exact komplekslerin sınıfıyla DG-projective ve DG-injective komplekslerin sınıfı arasındaki ilişkiler incelenmiştir. Her kompleksin, kernelı DG-injective olan bir exact cover a, kokernelı DG-projective olan exact envelopa ve kokernelı exact olan DG-injective envelopa sahip olduğunu gördük. Üçüncü bölümde ise herhangi bir quasi-izomorfizmin izomorfizm olması için gerekli bir şart verilmiştir. Son bölümde ise (C,D) komplekslerin kategorisinde C suspension kapalı olacak şekilde bir komplete kotorsion çift ve ayrıca C ve D, homotopi kategorisi K(R-modül) ün alt kategorileri olarak alınırsa gömme functorları $C \mapsto K$(R-modül) ve $D \mapsto K$(R-modül) nın sırayla sol ve sağ adjointlere sahip olduğu incelenmiştir.

Anahtar Kelimeler: projective - injective modül, Ext, cover - envelope, DG-projective - DG-injective kompleks ve injective - projective kompleks.

To My Family,

ACKNOWLEDGEMENTS

I would like to express my gratitude to my thesis supervisor Assist. Prof. Dr. Tahire Özen who has guided and encouraged me throughout this thesis. I am grateful to her for a critical reading of my solutions, valuable suggestions in correcting my errors, her enlightening instructions and her patient. I also like to thank to Assoc. Prof. Dr. Cenap Özel for his endless encourges.

Finally, I would like to thank to my family, my friends and especially to my father for their sincere support and encouragement.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Some Basic Concepts in Homology

Definition 1.1 *(Category, Additive and Abelian Category) A category $\mathcal{C}$ consists of three ingredients: a class obj $\mathcal{C}$ of objects, a set of morphisms $Hom(A, B)$ for every ordered pair (A, B) of objects and composition $Hom(A, B) \times Hom(B, C) \longrightarrow Hom(A, C)$, denoted by $(f, g) \mapsto gf$, for every ordered triple A, B, C of objects. [We often write $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in Hom(A, B)$.] These ingredients are subject to the following axioms:*

- **(i)** *The Hom sets are pairwise disjoint; that is, each $f \in Hom(A, B)$ has a unique domain A and unique target B;*
- **(ii)** *For each object A, there is an identity morphism $f1_A = f$ and $1_B f = f$ for all $f : A \to B$;*
- **(iii)** *Composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h(gf) = (hg)f$.*

A category C is said to be additive if $Hom_C(A, B)$ is an abelian group such that if $f, f_1, f_2 \in Hom_C(A, B)$, $g, g_1, g_2 \in Hom_C(B, C)$, then $g(f_1 + f_2) = gf_1 + gf_2$ and $(g_1 + g_2)f = g_1 f + g_2 f$. We note that if C is an additive category, then $Hom_C(A, B) \neq \emptyset$ since the zero morphism is always in $Hom_C(A, B)$. This is denoted 0_{AB}. Easily, R-Mod(or Mod-R) is an additive category.

An additive category $Hom_C(A, B)$ is said to be an abelian category if it satisfies the following conditions:

(i) *$\mathcal{C}$ has products (and coproducts),*

(ii) *every morphism in $\mathcal{C}$ has a kernel and a cokernel where if $f : A \to B$ is a morphism, then a kernel of f, denoted by $ker(f)$ is a morphism $k : K \to A$ such that $fk = 0$ and for each morphism $g : C \to A$ with $fg = 0$, there exists a unique morphism $h : C \to K$ such that $g = kh$. K is denoted by $Ker(f)$. Dually, a cokernel of f, denoted by $coker(f)$ is a morphism $p : B \to C$ such that $pf = 0$ and for each morphism $g : B \to D$ with $gf = 0$, there exists a unique morphism $h : C \to D$ such that $hp = g$. C is denoted by $Coker(f)$,*

(iii) *for every morphism in $\mathcal{C}$ $f : A \to B$, the map $Coker(ker(f)) \to Ker(coker(f))$ is an isomorphism.*

For example, R- Mod and Mod-R are abelian categories.

Definition 1.2 *(Covariant Functor) If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a functor $T : \mathcal{C} \to \mathcal{D}$ is a function such that*

(i) *If $A \in obj\ \mathcal{C}$, then $T(A) \in obj\ \mathcal{D}$,*

(ii) *If $f : A \to A'$ in $\mathcal{C}$, then $T(f) : T(A) \to T(A')$ in $\mathcal{D}$,*

(iii) *If $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{C}$, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathcal{D}$, and $T(gf) = T(g)T(f)$,*

(iv) *$T(1_A) = 1_{T(A)}$ for every $A \in obj\ \mathcal{C}$.*

Definition 1.3 *(Contravariant Functor) A contravariant functor $T : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, is a function such that*

(i) *If $C \in obj\ \mathcal{C}$, then $T(C) \in obj\ \mathcal{D}$,*

(ii) *If $f : C \to C'$ in $\mathcal{C}$, then $T(f) : T(C') \to T(C)$ in $\mathcal{D}$ (note the reversal of arrows),*

(iii) *If $C \xrightarrow{f} C' \xrightarrow{g} C''$ in $\mathcal{C}$, then $T(C'') \xrightarrow{T(g)} T(C') \xrightarrow{T(f)} T(C)$ in $\mathcal{D}$ and $T(gf) = T(f)T(g)$,*

(iv) $T(1_A) = 1_{T(A)}$ *for every* $A \in obj\ C$.

Definition 1.4 *(Tensor Product) Given a ring R and modules A_R and $_RB$, then their tensor pruduct is an abelian group $A \otimes_R B$ and an R-biadditive function $h : A \times B \to A \otimes_R B$ such that, for every abelian group G and every R-biadditive $f : A \times B \to G$, there exists a unique $\mathbb{Z}$-homomorphism $\widetilde{f} : A \otimes_R B \to G$ making the following diagram commute. ($\widetilde{f}h = f$)*

$$\begin{tikzcd}
A \times B \arrow[r, "h"] \arrow[d, "f"'] & A \otimes_R B \arrow[dl, dotted, "\widetilde{f}"] \\
G &
\end{tikzcd}$$

Definition 1.5 *(Pushout) Given two morphisms $f : A \to B$ and $g : A \to C$ in a category C, a pushout (or fibered sum) is a triple $(D, \alpha, \beta)$with $\beta g = \alpha f$ that is a solution to the universal mapping problem: for every triple (Y, α', β') with $\beta' g = \alpha' f$, there exists a unique morphism $\theta : D \to Y$ making the diagram following commute.*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\beta"'] \arrow[ddr, "\beta'"] & \\
B \arrow[r, "\alpha"] \arrow[drr, "\alpha'"'] & D \arrow[dr, dotted, "\theta"] & \\
& & Y
\end{tikzcd}$$

Pushouts are unique up to isomorphism when they exist. In the above if we set $D = (B \oplus C)/K$ where $K = \{(f(a), -g(a)) : a \in A\}$ and let $\alpha(b) = (b, 0) + K$, $\beta(c) = (0, c) + K$, then D with morphisms α, β is a pushout. If g is one to one (onto), then so is α. And also if g is one to one, then $C/A \cong D/B$.

Definition 1.6 *(Pullback) Given two morphisms $f : B \to A$ and $g : C \to A$ in a category C, a pullback (or fibered product) is a triple (D, α, β) with $g\alpha = f\beta$ that is a solution to the universal mapping problem: for every triple (X, α', β') with $g\alpha' = f\beta'$, there exists a unique morphism $\theta : X \to D$ making the following diagram commute.*

$$\begin{tikzcd}
X \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] & & \\
& D \arrow[r, "\alpha"] \arrow[d, "\beta"] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

Pullbacks, when they exist, are unique up to isomorphism. If we set $D = \{(b, c) \in B \oplus C : f(b) = g(c)\}$ *and let* α *and* β *be projection morphisms, then* D *with morphisms* α, β *is a pullback. If* g *is one to one (onto), then so is* β*.*

Definition 1.7 *(Complex) Let* C *be an abelian category whose objects are modules. By a (chain) complex* C *of* R*-modules we mean a sequence* $C : \ldots \to C_2 \xrightarrow{\ell_2} C_1 \xrightarrow{\ell_1} C_0 \xrightarrow{\ell_0} C_{-1} \xrightarrow{\ell_{-1}} C_{-2} \to \ldots$ *of* R*-modules and* R*-homomorphisms such that,* $\ell_{n-1} \circ \ell_n = 0$*, for all* $n \in \mathbb{Z}$*. Notice that* $Im\ell_n \subseteq Ker\ell_{n-1}$*. Also* C *is denoted by* $((C_n), (\ell_n))$ *or* (C, ℓ) *where the* ℓ_n *are called the boundary operators of the complex* C *and* ℓ *is the differential of the complex* C *.*

Definition 1.8 *(Exact Sequence) A sequence of* R*-modules and* R*-homomorphisms* $\ldots \to M_2 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \to \ldots$ *is said to be exact if* $Imd_{i+1} = Kerd_i$ *for all* $i \in \mathbb{Z}$*. An exact sequence of the form* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is said to be a short exact sequence.*

Remark 1.9 *If* F *is a covariant additive functor in the some category of modules, then* $F(C) : \ldots \to F(C_2) \xrightarrow{F(\ell_2)} F(C_1) \xrightarrow{F(\ell_1)} F(C_0) \to \ldots$ *is also complex where the complex* $C : \ldots \to C_2 \xrightarrow{\ell_2} C_1 \xrightarrow{\ell_1} C_0 \to \ldots$*. Similarly, if* F *is a contravariant additive functor, then the sequence* $F(C) : \ldots \to F(C_{-1}) \xrightarrow{F(\ell_0)} F(C_0) \xrightarrow{F(\ell_1)} F(C_1) \to \ldots$ *is a complex. For example in the category of* R*-Modules if* A *is an* R*-Module, then* $Hom(A, -)$ *is a covariant functor and* $Hom(-, A)$ *is a contravariant functor. Then* $Hom(A, C) : \ldots \to Hom(A, C_2) \xrightarrow{\ell_2^*} Hom(A, C_1) \xrightarrow{\ell_1^*} Hom(A, C_0) \to \ldots$ *where* $\ell_n^* f = \ell_n f$ *for all* $f \in Hom(A, C_n)$ *and* $Hom(C, A) : \ldots \to Hom(C_0, A) \xrightarrow{\ell_1^*} Hom(C_1, A) \xrightarrow{\ell_2^*} Hom(C_2, A) \to \ldots$ *where* $\ell_n^* f = f\ell_n$ *for all* $f \in Hom(C_{n-1}, A)$*.*

Definition 1.10 *(Chain Map) If C and $C^{'}$ are complexes, then a chain map $f : C \longrightarrow C^{'}$ is a sequence of morphisms $f_n : C_n \longrightarrow C^{'}{}_n$ for all $n \in \mathbb{Z}$ making the following diagram commutative:*

$$\begin{tikzcd}
C : \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
C' : \ldots \arrow[r] & C^{'}_{n+1} \arrow[r, "d^{'}_{n+1}"'] & C^{'}_n \arrow[r, "d^{'}_n"'] & C^{'}_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

It is easy to check that the composite gf of two chain maps $f : C \to C'$ and $g : C' \to C''$ is itself a chain map, where $(gf)_n = g_n f_n$. The identity chain map 1_C on $((C_n), (d_n))$ is the sequence of identity morphisms $1_{C_n} : C_n \to C_n$.

Definition 1.11 *(nth Homology)If (A, d) is a complex, then its n-th homology is $H_n(A) = \frac{Kerd_n}{Imd_{n+1}}$. The elements of $Kerd_n$ are called n - cycles. The elements of Imd_{n+1} are called n - boundaries. And also we can show that $Kerd_n = Z_n(A) = Z_n$, $Imd_{n+1} = B_n(A) = B_n$ implies that $H_n(A) = \frac{Z_n(A)}{B_n(A)}$.*

Definition 1.12 *(Cochain Complex) A chain complex of the form $C : \ldots C^{-1} \xrightarrow{\partial^{-1}} C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \longrightarrow \ldots$ is called a cochain complex where $\partial^n \circ \partial^{n-1} = 0$ for all $n \in \mathbb{Z}$. Then nth cohomology $H^n(C) = \frac{Ker\partial^n}{Im\partial^{n-1}}$.*

Definition 1.13 *($\mathcal{F}$ − precover, $\mathcal{F}$ − cover) Let C be an abelian category and let $\mathcal{F}$ be a class of objects of C. Then for an object $M \in C$, a morphism $\varphi : F \longrightarrow M$ where $F \in \mathcal{F}$ is called an $\mathcal{F}$-precover of M, if $f : F' \longrightarrow M$ where $F' \in \mathcal{F}$ and the following diagram*

$$\begin{tikzcd}
F' \arrow[r, "f"] \arrow[d, dotted, "g"'] & M \\
F \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

can be completed to a commutative diagram. If when $F = F'$ and the only such g is automorphism of F, then $\varphi : F \longrightarrow M$ is called an $\mathcal{F}$-cover of M.

Definition 1.14 *($\mathcal{F}$-preenvelope , $\mathcal{F}$-envelope) Let C be an abelian category and let $\mathcal{F}$ be a class of objects of C. Then for an object $M \in C$ a morphism $\varphi : M \longrightarrow F$ where $F \in \mathcal{F}$ is called an $\mathcal{F}$-preenvelope of M, if $f : M \longrightarrow F'$ where $F' \in \mathcal{F}$ and the following diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"'] & F' \\ F \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

can be completed to a commutative diagram. If when $F = F'$ and the only such g is automorphism of F, then $\varphi : M \longrightarrow F$ is called an $\mathcal{F}$-envelope of M.

Definition 1.15 *(Enough Projective Objects) Let C be an abelian category. C is said to have enough projectives, if for every object A of C, there is a projective object $P \in C$ and an exact sequence $P \to A \to 0$ (epic).*

Definition 1.16 *(Enough Injective Objects) Let C be an abelian category. C said to have enough injectives, if for every object $A \in C$, there is an injective object $E \in C$ and exact sequence $0 \longrightarrow A \longrightarrow E$.*

Definition 1.17 *(Resolution) Let C be an abelian category and ε be a class of objects of C. By a left ε-resolution of an object M of C, we will mean a complex $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ (not necessarily exact) with each $F_i \in \varepsilon$ such that $Hom(\varepsilon, -)$ makes the complex exact. A right ε-resolution of an object M of C is $Hom(-, \varepsilon)$ exact complex $0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ (not necessarily exact) with each $E^i \in \varepsilon$.*

We can give an example for a right resolution:

Let an abelian category C have an ε-preenvelope. Let E^0 be an ε-preenvelope of $X \in C$, then there exists ϕ such that $0 \longrightarrow X \xrightarrow{\phi} E^0$ (not necessarily 1 - 1). Then $\frac{E^0}{\phi(X)} \in C$ implies that $\frac{E^0}{\phi(X)}$ has an ε-preenvelope E^1 such that,

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\phi"] & E^0 \arrow[r, "d^0"] \arrow[d, "\pi"'] & E^1 \\
& & \dfrac{E^0}{\phi(X)} \arrow[ur, dotted, "\phi_1"'] &
\end{tikzcd}$$

where $\phi_1\pi = d^0$. Then $d^0\phi = 0$. If we continue in this way, then we can find that a complex $0 \longrightarrow X \xrightarrow{\varepsilon} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow ...$ (not necessarily exact) such that all $E^i \in \varepsilon$. $Hom(-, F)$ makes the complex exact when $F \in \varepsilon$. That is, the sequence $... \rightarrow Hom(E^1, F) \rightarrow Hom(E^0, F) \rightarrow Hom(X, F) \rightarrow 0$ is exact. Such complex is a right ε - resolution of X.

Similarly we can give an example for a left resolution:

Let an abelian category C have an ε-precover. Let F_0 be an ε-precover of $X \in C$, then there exists ϕ such that $F_0 \xrightarrow{\phi} X \longrightarrow 0$ (not necessarily onto). Then $Ker(\phi) \in C$ and so it has an ε-precover F_1 such that,

$$\begin{tikzcd}
F_1 \arrow[r, "\lambda_1"] \arrow[dr, dotted, "\phi_1"'] & F_0 \arrow[r, "\phi"] & X \arrow[r] & 0 \\
& Ker\phi \arrow[u, "i"'] & &
\end{tikzcd}$$

where $i\phi_1 = \lambda_1$. Then $\phi\lambda_1 = 0$. If we continue in this way, then we can find that a complex $\cdots \longrightarrow F_2 \xrightarrow{\lambda_2} F_1 \xrightarrow{\lambda_1} F_0 \xrightarrow{\phi} X \longrightarrow 0$ (not necessarily exact) such that all $F_i \in \varepsilon$. $Hom(F, -)$ makes the complex exact when $F \in \varepsilon$. Such complex is a left ε - resolution of X.

Definition 1.18 *(Balanced) Let $\mathcal{F}$ and ε be two classes of objects of C such that every object has an $\mathcal{F}$ - precover and ε - preenvelope. We say that $Hom(-, -)$ is right balanced by $\mathcal{F} \times \varepsilon$, both if for every left $\mathcal{F}$ - resolution $... \rightarrow F_1 \rightarrow F_0 \rightarrow X \rightarrow 0$ of an object X of C, $Hom(-, E)$ makes the sequence exact for all $E \in \varepsilon$ and if for every right ε - resolution $0 \rightarrow Y \rightarrow E^0 \rightarrow E^1 \rightarrow ...$ of an object Y, $Hom(F, -)$ makes the* *sequence exact for all $F \in \mathcal{F}$. If $Hom(E, -)$ makes each such $... \to F_1 \to F_0 \to X \to 0$ exact when $E \in \varepsilon$ and $Hom(-, F)$ makes each such $0 \to Y \to E^0 \to ...$ exact when $F \in \mathcal{F}$, then we say $Hom(-, -)$ is left balanced by $\varepsilon \times \mathcal{F}$.*

Definition 1.19 *($\mathcal{F}$ - Dimension) If $\mathcal{F}$ is a precovering class of an abelian category C, then an object M of C is said to have left $\mathcal{F}$ - dimension $\leq n$, denoted left $\mathcal{F}$ - dim M $\leq n$, if there is a left $\mathcal{F}$ - resolution of the form $0 \to F_n \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$ of M. If n is the least, then we set left $\mathcal{F}$ - dim $M = n$ and there is no such n, we set left $\mathcal{F}$ - dim $M = \infty$. Similarly, we can define the right $\mathcal{F}$ - dimension of objects of C, denoted right $\mathcal{F}$ - dim, if $\mathcal{F}$ is a preenveloping class of C.*

Definition 1.20 *(Global $\mathcal{F}$ - Dimension) Let C be an abelian category and $\mathcal{F}$ be a precovering class of C. Then the global left $\mathcal{F}$ - dimension of C, denoted by gl left $\mathcal{F}$ - dimC, is defined by gl left $\mathcal{F}$ - dimC = sup $\{$left$\mathcal{F}$ - dimM: $M \in$ obj(C)$\}$ and is infinite otherwise. The global right $\mathcal{F}$ - dimension of C is defined similarly when $\mathcal{F}$ is a preenveloping class.*

Lemma 1.21 *(Wakamatsu's Lemma) If the class $\mathcal{F}$ is closed under extensions and $\phi : F \to M$ is an $\mathcal{F}$ - cover of an object M of C, then Ker(ϕ) $\in \mathcal{F}^{\perp}$ and if $\psi : M \to F$ is an $\mathcal{F}$ - envelope, then Coker(ψ) $\in {}^{\perp}\mathcal{F}$.*

Proof.

Let $A \in \mathcal{F}$ and $0 \to S \to P \to A \to 0$ be an exact sequence, with P projective. We know that; $Ext^1(A, Ker\phi) = 0$ if and only if Hom(P , Kerϕ) $\to$ Hom(S , Kerϕ) $\to O$ epic. Let $f : S \to$ Kerϕ. Then we have the following diagram:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[d, "0"] \\ Ker\phi \subseteq F \arrow[r, "\phi"'] & M \end{tikzcd}$$

where $\frac{P}{S} \in \mathcal{F}$. We can form the Pushout diagram as follow;

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[d, dotted, "\beta"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

where $G = \frac{F \oplus P}{\{(f(s), -i(s)) : s \in S\}}$. Then $\alpha : F \to G$ is an injection and $\mathrm{Coker}(F \to G) \cong \frac{P}{S} \in \mathcal{F}$. Since $\mathcal{F}$ is closed under extensions, $G \in F$. We can suppose $F \subset G$. By the properties of the pushout diagram, we have a linear $h : G \to M$ which agrees with ϕ on F. Since $\phi : F \to M$ is a cover, we have a linear $g : G \to F$ such that $\phi g = h$. Then, since $\phi \circ (g|_F) = \phi$ where $g|_F$ is the restriction of g on F, we have that $g|_F$ is an automorphism of F. If we replace g by $(g|_F)^{-1} \circ g$, we see that we can assume $g|_F = id_F$. So g makes the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \arrow[d] \arrow[dl, dotted] \\ F \arrow[r] & G \end{tikzcd}$$

commutative and hence makes commutative the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \arrow[d] \arrow[dl, dotted] \\ F \arrow[r] & M \end{tikzcd}$$

Thus $\phi g = 0$ implies $\phi g(p) = 0$ for all $p \in P$. Then $g(P) \subseteq Ker(\phi)$. Dually, we can prove that $coker(\psi) \in^{\perp} \mathcal{F}$, if $\psi : M \longrightarrow F$ is an $\mathcal{F}$-preenvelope.

□

Definition 1.22 *(Free module) Let R be a ring. A left R-module F is a free left R-module if F is isomorphic to a direct sum of copies of R: that is, there is a (possibly infinite) index set B with* $F = \bigoplus_{b \in B} R_b$, *where* $R_b = \langle b \rangle \cong R$ *for all* $b \in B$. *We call B a basis of F.*

Proposition 1.23 *(Proposition 2.33 in [1]) Let R be a ring. Given any set B, there* *exists a free left R-module F with basis B.*

Theorem 1.24 *(Theorem 2.35 in [1]) Every left R-module M is a quotient of a free left R-module F. Moreover, M is finitely generated if and only if F can be chosen to be finitely generated.*

Definition 1.25 *(Projective Module) Let R be a ring. A left R-module P is projective if, whenever p is surjective and h is any map, there exists a lifting g as the following; that is, there exists a map g making the following diagram commute:*

$$\begin{tikzcd}
 & P \arrow[d, "h"] \arrow[dl, dotted, "g"'] & \\
A \arrow[r, "p"'] & A'' \arrow[r] & 0
\end{tikzcd}$$

Lemma 1.26 *(Corollary 3.6 in [1])*

- **(i)** *Every direct summand of a projective module is itself projective.*
- **(ii)** *Every direct sum of projective modules is projective.*

Definition 1.27 *(Injective Module,Essential Submodule, Injective Envelope(Hull)) Let R be a ring. A left R-module E is injective if, whenever i is an injection, a dashed arrow exists making the following diagram commute:*

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, dotted, "g"']
\end{tikzcd}$$

Let M be a left R-module and N be a submodule such that the intersection of N with any nonzero submodule of M is nonzero. Then N is called the essential submodule of M, denoted by $N \subseteq^{ess} M$.

If a left R-module M is essential in an injective module E, then E is called the injective hull (or injective envelope) of M, denoted by $E(M)$. Then $E(M)$ is the smallest injective *module containing M and the greatest module containing M as essential. And also every R-module has a unique injective envelope up to isomorphism.*

Proposition 1.28 *(Proposition 3.25 in [1]) A left R-module E is injective if and only if $Hom_R(-, E)$ is an exact functor.*

Proposition 1.29 *(Corollary 3.27 in [1]) If an injective module E is a submodule of a module M, then E is a direct summand of M: there is a complement S with $M = E \oplus S$.*

Proposition 1.30 *(Proposition 3.28 in [1])*

(i) *If $(E_k)_{k\in K}$ is a family of injective left R-modules then $\prod_{k\in K} E_k$ is also an injective left R-module.*

(ii) *Every direct summand of an injective left R-module E is injective.*

Definition 1.31 *(Homotopy) Let (C, d) and (C', d') be chain complexes. Then the chain maps $f, g : (C, d) \to (C', d')$ are homotopic, denoted by $f \simeq g$, if for all n, there is a map $s = (s_n)$ where $s_n : C_n \to C'_{n+1}$ with*

$$f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n.$$

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"'] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

A chain map $f : (C, d) \to (C', d')$ is null-homotopic if $f \simeq 0$, where 0 is the zero chain map.

Theorem 1.32 *Homotopic chain maps induce the same morphism in homology: if $f, g : (C, d) \to (C', d')$ are chain maps and $f \simeq g$, then for all n,*

$$H_n(f) = H_n(g) : H_n(C) \to H_n(C').$$

Proof.

If z is an n-cycle, then $d_n z = 0$ and

$$f_n z - g_n z = d'_{n+1} s_n z + s_{n-1} d_n z = d'_{n+1} s_n z.$$

Therefore, $f_n z - g_n z \in B_n(C')$, and so $H_n(f) = H_n(g)$.

$\square$

Proposition 1.33 *Let $\mathcal{C}$ be an abelian category of complexes. If* $0 \to (C', d) \xrightarrow{i} (C, d) \xrightarrow{p} (C'', d'') \to 0$ *is an exact sequence in $\mathcal{C}$, then for each $n \in \mathbb{Z}$, then there is a morphism in $\mathcal{C}$ such that* $\partial_n : H_n(C'') \to H_{n-1}(C')$ *defined by* $\partial_n : z''_n + Imd''_{n+1} \mapsto i^{-1}_{n-1} d_n p^{-1}_n z''_n + Imd'_n$ *where $z''_n \in Kerd''_n$.*

Definition 1.34 *The morphisms $\partial_n : H_n(C'') \longrightarrow H_{n-1}(C')$ above are called connecting homomorphisms.*

Theorem 1.35 *(Theorem 6.10 in [1])(Long Exact Sequence)*

Let $\mathcal{C}$ an abelian category of complexes. If $0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 0$ is an exact sequence in $\mathcal{C}$, then there is an exact sequence in $\mathcal{C}$ such that $\cdots \to H_{n+1}(C'') \xrightarrow{\partial_{n+1}} H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(p)} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \to \cdots$.

Definition 1.36 *(Ext) Let an injective resolution of an R-module B is* $E : 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$, *then* $E^B : 0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$ *and* $0 \longrightarrow Hom(A, E^0) \xrightarrow{d^0_*} Hom(A, E^1) \xrightarrow{d^1_*} Hom(A, E^2) \longrightarrow \cdots$. *Then* $Ext^n_R(A, B) = H^n(Hom_R(A, E^B)) = \frac{kerd^n_*}{imd^{n-1}_*}$, *where* $d^n_* : Hom_R(A, E^n) \longrightarrow Hom_R(A, E^{n+1})$ *is defined, as usual, by* $d^n_* : f \mapsto d^n f$.

Let $\mathcal{C}$ be any abelian category and ε be a class of objects of $\mathcal{C}$. If $Ext^1(X, Y) = 0$ for all $X \in \varepsilon$, then $Y \in \varepsilon^{\perp}$. If $Ext^1(X, Y) = 0$ for all $Y \in \varepsilon$, then $X \in {}^{\perp}\varepsilon$.

Lemma 1.37 *The module $Ext_R^n(A, B)$ is independent of the choice of injective resolution of B.*

Proof.

We understand the proof by Proposition 6.40 in [1].

□

Corollary 1.38 *(Corollary 6.42 in [1]) Let $\mathcal{A}$ be an abelian category with enough injectives, let $B \in obj(\mathcal{A})$, and let $E : 0 \to B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \to \cdots$ be an injective resolution of B. Define $V_0 = im\eta$ and define $V_n = imd^{n-1}$ for all $n \geq 1$. For any left R-modules A and B,*

$$Ext_R^{n+1}(A, B) \cong Ext_R^n(A, V_0) \cong \cdots \cong Ext_R^1(A, V_{n-1}).$$

Remark 1.39 *By definition we understand $Hom_R(A, B) \cong Ext_R^0(A, B)$.*

Lemma 1.40 *(Corollary 6.46 in [1])*
If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is a short exact sequence of left R-modules, then for every left R-module A, there is a long exact sequence of abelian groups,

$0 \longrightarrow Hom_R(A, B') \longrightarrow Hom_R(A, B) \longrightarrow Hom_R(A, B'') \longrightarrow Ext_R^1(A, B') \longrightarrow Ext_R^1(A, B) \longrightarrow Ext_R^1(A, B'') \longrightarrow Ext_R^2(A, B') \longrightarrow Ext_R^2(A, B) \longrightarrow Ext_R^2(A, B'') \longrightarrow ....$
Thus, Ext repairs the loss of exactness after applying $Hom_R(A, -)$ to a short exact sequence.

Definition 1.41 *(ext) If the chosen projective resolution of an R-module A is P : $... \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ and $P_A : ... \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$, then*

$$ext_R^n(A, B) = H_n(Hom_R(P_A, B)) = \frac{ker\ d_n^*}{im\ d_{n-1}^*},$$

where $d_n^* : Hom_R(P_{n-1}, B) \longrightarrow Hom_R(P_n, B)$ is defined, as usual, by

$$d_n^* : f \mapsto fd_n .$$

Lemma 1.42 *(Corollary 6.57 in [1])*

The module $ext^n_R(A, B)$ *is independent of the choice of projective resolution of* A.

Theorem 1.43 *(Theorem 6.67 in [1])*

Let A *and* B *be left* R*-modules, let* $E : 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow ...$ *be an injective resolution of* B *and let* $P : ... \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ *be a projective resolution of* A. *Then* $H_n(Hom_R(P_A, B)) \cong H^n(Hom_R(A, E^B))$ *for all* $n \geq 0$ *and so*

$$Ext^n_R(A, B) \cong ext^n_R(A, B) .$$

Definition 1.44 *Given R-modules C and A, an extension of A by C is a short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

An extension is split if there exists an R-map $s : C \to B$ *with* $ps = 1_C$ *or equivalently there exists an R-map* $k : B \to A$ *with* $ki = 1_A$.

Proposition 1.45 *(Proposition 7.24 in [1]) If* $Ext^1_R(C, A) = \{0\}$, *then every extension of A by C splits.*

Corollary 1.46 *(Corollary 7.25 in [1])*

- **(i)** *A left R-module P is projective if and only if* $Ext^1_R(P, B) = \{0\}$ *for all every R-module B.*
- **(ii)** *A left R-module E is injective if and only if* $Ext^1_R(A, E) = \{0\}$ *for all every R-module A.*

1.2 Complexes

Definition 1.47 *If* $C : \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \longrightarrow \cdots$ *is a complex, then* $C' : \cdots \longrightarrow C'_2 \xrightarrow{d'_2} C'_1 \xrightarrow{d'_1} C'_0 \xrightarrow{d'_0} C'_{-1} \longrightarrow \cdots$ *is said to be a subcomplex of C if* $C'_m \subseteq C_m$ *for every* $m \in \mathbb{Z}$ *and if* d_m *agrees with* d'_m *on* C'_m. *If* $S \subseteq C$ *is a* *subcomplex, then for any* $m \in \mathbb{Z}$, $d_m : C_m \longrightarrow C_{m-1}$ *induces a map* $\frac{C_m}{S_m} \longrightarrow \frac{C_{m-1}}{S_{m-1}}$. *By these maps, we can obtain a complex* $\frac{C}{S}$. *Let* $f : C \longrightarrow D$ *be a morphism of complexes, then* $d_m(Ker(f_m)) \subseteq Ker(f_{m-1})$. *Then we can obtain a complex* $Ker f$ *with m-th term* $Ker(f_m)$. *Similarly, we can find a complex* $Im(f) \subseteq D$. *And also we have a complex* $Coker(f) = \frac{D}{Im(f)}$ *and* $Im(f) \cong C/Ker(f)$. *Thus we can see that the family of all complexes in R-Mod is an abelian category, denoted by* C*(R-mod).*

Remark 1.48 *Let* C *be a complex. Then we have a subcomplex* $Z(C) : \cdots \longrightarrow Ker(d_m) \longrightarrow Ker(d_{m-1}) \longrightarrow \cdots$ *where* $Z(C)_m = Ker(d_m)$ *for all* $m \in \mathbb{Z}$ *where* d *is the differential of the complex* C. *Notice that the differential of* $Z(C)$ *is* 0. *And a subcomplex* $B(C) \subseteq C$ *is defined as* $B(C)_m = Im(d_{m+1})$. *Since* $d_m \circ d_{m+1} = 0$, $B(C) \subseteq Z(C)$. *The quotient complex* $Z(C)/B(C)$ *is denoted by* $H(C)$ *where* $H(C)_m = H_m(C) = \frac{Z(C)_m}{B(C)_m}$. *Then the complex* C *becomes exact if* $H(C) = 0$, *or equivalently* $Ker(d_m) = Im(d_{m+1})$ *for all* $m \in \mathbb{Z}$.

Definition 1.49 *(The Tensor Product Of The Complexes A and B) If* $A = (A, \Delta')$ *and* $B = (B, \Delta'')$, *then* $A \otimes B$ *is the complex defined by* $(A \otimes B)_n = \coprod_{p+q=n} A_p \otimes B_q$ *and* $d_n : (A \otimes B)_n \longrightarrow (A \otimes B)_{n-1}$ *with* $a_p \otimes b_q \longrightarrow \Delta' a_p \otimes b_q + (-1)^p a_p \otimes \Delta'' b_q$ *where* $a_p \in A_p$ *and* $b_q \in B_q$.

Definition 1.50 *(*$\mathcal{H}om(A, C)$*) If* (A, Δ') *and* (C, Δ'') *are complexes, then* $\mathcal{H}om(A, C)$ *is the complex defined by* $\mathcal{H}om(A, C)_n = \prod_{p+q=n} Hom(A_{-p}, C_q)$ *and* $D_n : \mathcal{H}om(A, C)_n \longrightarrow \mathcal{H}om(A, C)_{n-1}$ *such that* $D_n : f \longrightarrow D_n f = \{g_{p,q}\}$ *where* $g_{p,q} : A_{-p} \longrightarrow C_q$ *with* $q + p = n - 1$ *is defined by* $g_{p,q} = \Delta''_{q+1} f_{p,q+1} + (-1)^{p+q} f_{p+1,q} \Delta'_{-p}$.

Lemma 1.51 *Let* $A : \ldots \rightarrow A_2 \xrightarrow{\Delta'_2} A_1 \xrightarrow{\Delta'_1} A_0 \xrightarrow{\Delta'_0} A_{-1} \rightarrow \ldots$ *and* $C : \ldots \rightarrow C_2 \xrightarrow{\Delta''_2} C_1 \xrightarrow{\Delta''_1} C_0 \xrightarrow{\Delta''_0} C_{-1} \rightarrow \ldots$ *be chain complexes. Then,* $\mathcal{H}om(A, C) : \ldots \rightarrow \mathcal{H}om(A, C)_2 \xrightarrow{D_2} \mathcal{H}om(A, C)_1 \xrightarrow{D_1} \mathcal{H}om(A, C)_0 \xrightarrow{D_0} \ldots$ *is a chain complex.*

Proof.

Our aim is to find that $D_{n-1}D_n = 0$. So we will show that $D_{n-1}D_n(f) = 0$ for all $f \in \mathcal{H}om(A, C)_n$. Let $D_{n-1}D_n(f) = \{h_{p,q}\}$ with $p + q = n - 2$ and $D_n(f) = \{g_{x,y}\}$

with $x+y = n-1$ and $g_{x,y} = \Delta''_{y+1} f_{x,y+1} + (-1)^{x+y} f_{x+1,y}\Delta'_{-x}$. Then $h_{p,q} = A_{-p} \to C_q$ with $h_{p,q} = \Delta''_{q+1} g_{p,q+1} + (-1)^{p+q} g_{p+1,q}\Delta'_{-p}$ implies that $h_{p,q} = \Delta''_{q+1}(\Delta''_{q+2} f_{p,q+2} + (-1)^{p+q+1} f_{p+1,q+1}\Delta'_{-p}) + (-1)^{p+q}(\Delta''_{q+1} f_{p+1,q+1} + (-1)^{p+1+q} f_{p+2,q}\Delta'_{-p-1})\Delta'_{-p} = 0$.

□

Lemma 1.52 *$Ker(D_0)$ = The abelian group $Hom(A, C)$ of chain maps from A to C.*

Proof. Let $A : ... \to A_2 \xrightarrow{\Delta'_2} A_1 \xrightarrow{\Delta'_1} A_0 \xrightarrow{\Delta'_0} A_{-1} \xrightarrow{\Delta'_{-1}} ...$ and $C : ... \to C_2 \xrightarrow{\Delta''_2} C_1 \xrightarrow{\Delta''_1} C_0 \xrightarrow{\Delta''_0} C_{-1} \xrightarrow{\Delta''_{-1}} ....$ $Ker(D_0)$ =?

$$\begin{array}{llll} D_0 : & \mathcal{H}om(A, C)_0 & \longrightarrow & \mathcal{H}om(A, C)_{-1} \\ & f & \longrightarrow & D_0(f) \end{array}$$

Since $\mathcal{H}om(A, C)_0 = \prod_{p\in\mathbb{Z}} Hom(A_p, C_p)$ and $\mathcal{H}om(A, C)_{-1} = \prod_{p\in\mathbb{Z}} Hom(A_p, C_{p-1})$, for all $f \in KerD_0$ $D_0(f) = \{g_{p,p-1}\} = \{0\}$ and then $g_{p,p-1} = \Delta''_p f_{p,p} + (-1)^{-1} f_{p-1,p-1}\Delta'_p = 0$. Therefore we have that $\Delta''_p f_{p,p} = f_{p-1,p-1}\Delta'_p$. This gives that f is a chain map from A to C.

□

Definition 1.53 *A chain map $f : A \longrightarrow A'$ is an isomorphism if and only if each f_n is an isomorphism as follow,*

$$\begin{array}{lccccccc} A : & \cdots & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\ & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ A' : & \cdots & \longrightarrow & A'_n & \longrightarrow & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

Definition 1.54 *If $\{A^k : k \in K\}$ is a family of complexes, where $A^k : ... \longrightarrow A^k_n \xrightarrow{d^k_n} A^k_{n-1} \longrightarrow ...$, then the coproduct $\coprod A^k$ is defined as the complex $\coprod A^k : ... \longrightarrow \coprod_k A^k_n \xrightarrow{\coprod d^k_n} \coprod_k A^k_{n-1} \longrightarrow ...$. Then $H_n(\coprod_k A^k) \cong \coprod_k H_n(A^k)$ for every $n \in \mathbb{Z}$. If K is a quasi - ordered set, then $limA^k : ... \longrightarrow limA^k_n \longrightarrow limA^k_{n-1}$ (see for direct limit [4]). If K is directed, then $H_n(\varinjlim A^k) \cong \varinjlim H_n(A^k)$ for all $n \in \mathbb{Z}$.*

Definition 1.55 *(Suspension)The suspension of a complex C is the complex denoted by $S(C)$ or $C[1]$ where $S(C)_m = C[1]_m = C_{m-1}$ (or $S(C)^m = C[1]^m = C^{m+1}$) and whose differential is $-d$ where d is the differential of C. Then we can define $S^k(C)$ for any $k \in \mathbb{Z}$ as $S^k(C)_m = C[k]_m = C_{m-k}$ (or $S^k(C)^m = C[m]^k = C^{m+k}$) whose differential is $(-1)^k d$ where d is the differential of C.*

Definition 1.56 *(Mapping Cone)Let $f : (X, \lambda) \longrightarrow (Y, d)$ be a chain map. For each n, define*

$$\begin{array}{lccccccccccc}
X: & \cdots & \longrightarrow & X^{-1} & \xrightarrow{\lambda^{-1}} & X^0 & \xrightarrow{\lambda^0} & X^1 & \xrightarrow{\lambda^1} & X^2 & \longrightarrow & \cdots \\
 & & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\
Y: & \cdots & \longrightarrow & Y^{-1} & \xrightarrow[d^{-1}]{} & Y^0 & \xrightarrow[d^0]{} & Y^1 & \xrightarrow[d^1]{} & Y^2 & \longrightarrow & \cdots
\end{array}$$

and $M(f)^n = X^{n+1} \oplus Y^n$. Then the complex $M(f) : \cdots M(f)^{n-1} \xrightarrow{\partial^{n-1}} M(f)^n \xrightarrow{\partial^n}\longrightarrow \cdots$ whose boundary operators are $\partial^n(x, y) = (-\lambda^{n+1}x, f^{n+1}(x) + d^n y)$ for $(x, y) \in X^{n+1} \oplus Y^n$ is called a mapping cone. So there exists an exact sequence, $0 \longrightarrow Y \xrightarrow{i} M(f) \xrightarrow{p} X[1] \longrightarrow 0$.

Definition 1.57 *(Quasi - Isomorphism (or Homology Isomorphism))A morphism $f : (X, \lambda) \longrightarrow (Y, d)$ of complexes such that $H(f) : H(X) \longrightarrow H(Y)$ is an isomorphism will be called a quasi - isomorphism where $H^n(f) : H^n(X) \longrightarrow H^n(Y)$ with $x + Im\lambda^{n-1} \longrightarrow f^n(x) + Imd^{n-1}$ for all $x \in Ker\lambda^n$.*

Definition 1.58 *If M is a module, then $\overline{M} : \cdots 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$ where $\overline{M}_1 = M$ and $\overline{M}_0 = M$ and $\underline{M} : \cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$ where $\underline{M}_0 = M$. Then $\underline{M} \subseteq \overline{M}$.*

Definition 1.59 *(Graded Set) A graded set A is a family of sets $(A_m)_{m\in\mathbb{Z}}$. If we assume that $A_m \cap A_n = \emptyset$ if $m \neq n$, then we write $x \in A$ to mean, $x \in \bigcup_{m\in\mathbb{Z}} A_m$ and we write $deg(x) = m$ if $x \in A_m$.*

If A and B are graded sets by a morphism $f : A \longrightarrow B$ of degree $p \in \mathbb{Z}$ we mean a family $(f_m)_{m\in\mathbb{Z}}$ of functions f_m where $f_m : A_m \longrightarrow B_{m+p}$ for all $m \in \mathbb{Z}$. So we have a category with $Hom(A, B)$, denoting the set of morphisms $A \longrightarrow B$ (of any degree). $Hom(A, B)_p$ means the set of all morphisms $f : A \longrightarrow B$ of degree p. By the cardinality of graded set A we mean $\sum_{m\in\mathbb{Z}} | A_m |$ (or $\sum_{m\in\mathbb{Z}} card(A_m)$). Let C be a complex and A be a graded set, then $A \subseteq C$ means that for all $m \in \mathbb{Z}$ $A_m \subseteq C_m$.

Definition 1.60 *If A is a graded subset of the complex C, we say $S \subseteq C$ is the subcomplex generated by A if S is the intersection of all subcomplexes of C containing A. C is said to be a finitely generated complex if there is a finite set $A \subseteq C$ generating C. C is said to be finitely presented if there is an exact sequence $E \longrightarrow D \longrightarrow C \longrightarrow 0$ of complexes where E and D are finitely generated complexes.*

Definition 1.61 *(Free Complex) A complex F said to be a free complex with base B if $B \subseteq F$ is a graded subset of F such that for any complex C and any morphism $B \longrightarrow C$ of graded sets of degree 0, there is a unique morphism $F \longrightarrow C$ of complexes that agrees with the morphism $B \longrightarrow C$. We say F is free if it has a base. If F is free, then all F_m are free.*
For example, the complex $\overline{R} : \cdots 0 \longrightarrow R \xrightarrow{id} R \longrightarrow 0 \longrightarrow \cdots$ where $\overline{R}_1 = R$ and $\overline{R}_0 = R$ is free with the base consisting of the $1 \in R$ of degree -1. Now we note that if F is free with base $B \subseteq F$, then for any $k \in \mathbb{Z}$, $S^k(F)$ is free with base $S^k(B)$. Also, if $(F^i)_{i\in I}$ is any family of free complexes, then $\bigoplus_{i\in I} F^i$ is also free.

Proposition 1.62 *Given any complex C. Then there is a free complex F and an epimorphism $F \longrightarrow C$.*

Proof. It suffices to find a free F with a base B such that $| B_m | \geq | C_m |$. Then there is a degree 0 epimorphism $B \longrightarrow C$ (of graded sets). So the corresponding $F \longrightarrow C$ is necessarily an epimorphism.

□

Definition 1.63 *A complex P is said to be a projective complex if for any morphism $P \longrightarrow D$ and any epimorphism $C \longrightarrow D$, the diagram*

$$\begin{tikzcd}
& P \arrow[d] \arrow[dl, dotted] & \\
C \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

can be completed to a commutative diagram by a morphism $P \longrightarrow C$.

Proposition 1.64 *Any free complex F is projective.*

Proposition 1.65 *If P is a projective module, then $\overline{P}$ is a projective complex.*

Proof. Consider the diagram in $C(R-Mod)$:

$$\begin{tikzcd}
& P \arrow[dl, dotted, "h_k"'] \arrow[d, "f_k"] \arrow[dr, "1_P"] & \\
C_k \arrow[r, "g_k"'] \arrow[dr, "d_k"'] & C'_k \arrow[dr, "d'_k"] & P \arrow[dl, dotted, "h_{k-1}"'] \arrow[d, "f_{k-1}"] \\
& C_{k-1} \arrow[r, "g_{k-1}"'] & C'_{k-1}
\end{tikzcd}$$

where $g : C \longrightarrow C' \longrightarrow 0$ is an epic morphism and $f : \overline{P} \longrightarrow C$ is a morphism and d , d' are differentials of C , C', respectively. Since P is a projective module, there is a morphism $h_k : P \longrightarrow C_k$ with $g_k h_k = f_k$. Define $h_{k-1} : P \longrightarrow C_{k-1}$ with $h_{k-1} = d_k h_k$. Then $g_{k-1}h_{k-1} = g_{k-1}d_k h_k = d'_k g_k h_k = d'_k f_k = f_{k-1}$. Thus we have a morphism $h : \overline{P} \longrightarrow C$ which is a chain map. □

Notice that for all $m \in \mathbb{Z}$, $S^m(\overline{P})$ is a projective complex whenever P is a projective module. And also any direct sum of projective complexes is a projective complex.

Corollary 1.66 *A complex P is projective if and only if it is a direct summand of a free complex.*

The following remark shows that if P is a projective complex, then all P_m are projective, but the converse may not be true.

Remark 1.67 *Since a projective complex P is isomorphic to a direct summand of F, we get P_m is isomorphic to a direct summand of F_m. And also if F is free, then each F_m is a free module. Hence P_m is a projective module. But the converse may not be true. that is, each P_m can be projective module, but P may not be a projective complex. For example, $\overline{R}$ has $\underline{R}$ as a subcomplex and $\overline{R}/\underline{R} = S(\underline{R})$. Since $Hom_{C(R)}(S(\underline{R}), \overline{R}) = 0$, the diagram*

$$\begin{tikzcd} & S(\underline{R}) \arrow[dl, dotted] \arrow[d, "id"] & \\ \overline{R} \arrow[r] & S(\underline{R}) \arrow[r] & 0 \end{tikzcd}$$

cannot be completed to a commutative diagram. So $S(\underline{R})$ is not a projective complex even though each term of $S(\underline{R})$ is a projective module.

Theorem 1.68 *Let P be a complex. Then the following are equivalent:*

(i) *P is projective.*

(ii) *P is a direct summand of a free complex.*

(iii) *There is a family $(P^m)_{m\in\mathbb{Z}}$ of projective modules such that $P \cong \bigoplus_{m\in\mathbb{Z}} S^m(\overline{P^m})$.*

(iv) *P is exact and $Z(P)$ has all terms projective.*

Furthermore, for a projective P the family $(P^m)_{m\in\mathbb{Z}}$ in the part (iii) is unique up to isomorphism.

Definition 1.69 *(Injective Complex) I is said to be an injective complex if for any morphism $C \longrightarrow I$ and any morphism $C \longrightarrow D$, the diagram*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d] & D \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

can be completed to a commutative diagram by a morphism $D \longrightarrow I$.

Proposition 1.70 *If M is injective module, then $\overline{M}$ is an injective complex.*

Proof. Let $0 \longrightarrow C \longrightarrow C'$ be a monic morphism. Then we have the following diagram

$$\begin{tikzcd}
 & M \arrow[dr, "1_M"] & \\
C_k \arrow[ur, "h_k"] \arrow[r, "g_k"'] \arrow[dr, "d_k"'] & C'_k \arrow[u, dotted, "f'_k"'] \arrow[dr, "d'_k"] & M \\
 & C_{k-1} \arrow[ur, "h_{k-1}"'] \arrow[r, "g_{k-1}"'] & C'_{k-1} \arrow[u, dotted, "f'_{k-1}"']
\end{tikzcd}$$

Here, g_k and g_{k-1} are parts of a monic chain map $C \longrightarrow C'$. Since M is injective, there is $f'_{k-1} : C'_{k-1} \longrightarrow M$ with $f'_{k-1}g_{k-1} = h_{k-1}$. Let $f'_k : C'_k \longrightarrow M$ such that $f'_k = f'_{k-1}d'_k$. Then $f'_k g_k = f'_{k-1}d'_k g_k = f'_{k-1}g_{k-1}d_k = h_{k-1}d_k = h_k$. So we have a chain morphism $f' : C' \longrightarrow M$. Therefore $\overline{M}$ is an injective complex. □

Notice that for all $m \in \mathbb{Z}$, $S^m(\overline{M})$ is an injective complex whenever M is an injective module. And also any direct product of injective complexes is an injective complex. Let C be any complex. Then we have an injective linear monomorphism $C^m \longrightarrow I^m$ and then form the corresponding morphism $C \longrightarrow \overline{I^m}$. Then we have $C \longrightarrow \prod_{m\in\mathbb{Z}} \overline{I^m}$ is a monomorphism as a product of injective complexes where $\prod_{m\in\mathbb{Z}} \overline{I^m}$ is an injective complex. Then $\prod_{m\in\mathbb{Z}} \overline{I^m}$ is an injective preenvelope of C. Thus there are enough injective complexes. If C itself is injective, then C is isomorphic to a direct summand of $\prod_{m\in\mathbb{Z}} \overline{I^m}$ and also it is exact. If A is a non zero subcomplexes of a complex C such that $A \cap B \neq 0$ for all nonzero subcomplexes B , then A is called an essential subcomplex of C . There exists an injective complex containing essentially the complex C. It is called the injective envelope of the complex C.

Theorem 1.71 *A complex E is injective if and only if it is injective for each $S^m(\overline{R})$.*

Note that subcomplexes of $\overline{R}$ are of the form; $\cdots \longrightarrow 0 \longrightarrow I \hookrightarrow J \longrightarrow 0 \longrightarrow \cdots$ where I and J are left ideals of R. Then we understand that E is injective for $\overline{R}$ only if every morphism $\cdots \longrightarrow 0 \longrightarrow I \hookrightarrow R \longrightarrow 0 \longrightarrow \cdots$ (where $J = R$) into E can be extended to a morphism $\overline{R} \longrightarrow E$. Then we have the corresponding claim for $S^m(\overline{R})$ and E (for any $m \in \mathbb{Z}$).

Definition 1.72 *(Ext in Complexes) Let A and B complexes in the category of complexes of left R-modules $\mathcal{C}$. The definition of Ext in complexes is the same as the definition of Ext in R-Mod. We can obtain an injective resolution of the complex B since every complex has an enough injective complex. If the chosen injective resolution of B is $E = 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$, then $Ext^n(A, B) = H^n(Hom(A, E^B)) = \frac{ker d_*^n}{im d_*^{n-1}}$, where $d_*^n : Hom(A, E^n) \longrightarrow Hom(A, E^{n+1})$ is defined, as usual, by $d_*^n : f \mapsto d^n f$.*

From definition we understand that Lemma 1.37, Corollary 1.38, Remark 1.39, Lemma 1.40 , Lemma 1.42 and Theorem 1.43 in R-Mod are satisfied also in complexes.

CHAPTER 2

ORTHOGONALITY IN THE CATEGORY OF COMPLEXES

Let ε be the class of exact complexes and $C(R-Mod)$ (or shortly C) be the category of cochain complex of left R-modules throughout this section , $\mathcal{H}om(X,Y)$ denote the usual complex formed from the two complexes X and Y (See 1.50) and $Hom(X,Y)$ denote the group of cochain maps from X to Y. We call a complex P $DG-projective$ if each P^n is projective and if $\mathcal{H}om(P,E)$ is an exact complex for all $E \in \varepsilon$. Thus every projective complex is DG -projective. A complex I is called $DG-injective$ if each I^n is injective and $\mathcal{H}om(E,I)$ is exact for all $E \in \varepsilon$. Thus every injective complex is DG -injective. The classes of DG-projective and DG-injective complexes are closed under taking finite sums and taking summands. For examples, if I is such that all I^n are injective and such that for some n_0, $I^n = 0$ for $n \leq n_0$ (i.e. I is bounded below), then I is DG-injective. If $l.gl.dimR < \infty$, then any complex I with all I^n injective is DG-injective. A bounded above complex P with all P^n projective is DG-projective and if $l.gl.dimR < \infty$, any P with all P^n projective is DG-projective (see [2] and [3]). For any n, X[n] denotes the complex such that $X[n]^m = X^{n+m}$ and whose boundary operators are $(-1)^n\partial^{n+m}$ where ∂ is a boundary operator of the complex X. If X is DG-projective (injective), then so is $X[n]$ for any n. And also if M is any injective left R-module, then the complex $\cdots \to 0 \to M \xrightarrow{id} M \to 0 \to \cdots\cdots$ is injective (with the first R-module M in the n-th place). In fact, any injective complex is uniquely up to isomorphism the direct sum of such complexes. Moreover we know that $\varepsilon^{\perp} = \{X \in C(R-Mod) : Ext^1(E,X) = 0 \text{ for all } E \in \varepsilon\}$ and ${}^{\perp}\varepsilon = \{X \in C(R-Mod) : Ext^1(X,E) = 0 \text{ for all } E \in \varepsilon\}$.

Lemma 2.1 *If $I \in \varepsilon^{\perp}$, then each I^n is injective.*

Proof. Let $S \subseteq M$ be a submodule of a left $R-$ module and $f : S \longrightarrow I^n$ be linear form the pushout,

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "i"] \arrow[d, "\alpha"'] & M \arrow[d, dotted, "i_1"] \\
& I^n \arrow[r, dotted, "i_2"'] & \dfrac{I^n \oplus M}{A}
\end{tikzcd}$$

where $A = \langle(\alpha(s), -s) : s \in S\rangle$. Thus i_2 is $1-1$ the same as i. Then, $\overline{I} : ... \longrightarrow I^{n-1} \longrightarrow \frac{I^n \oplus M}{A} \longrightarrow I^{n+1} \oplus \frac{M}{S} \longrightarrow I^{n+2} \longrightarrow ...$ is a complex. If we denote $\frac{I^n \oplus M}{A}$ by $I^n \oplus_S M$, then we can obtain the following diagram where all rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & I^{n-1} \arrow[r] \arrow[d] & I^{n-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^n \arrow[r] \arrow[d] & I^n \oplus_S M \arrow[r] \arrow[d] & \dfrac{M}{S} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & I^{n+1} \arrow[r] \arrow[d] & I^{n+1} \oplus \dfrac{M}{S} \arrow[r] \arrow[d] & \dfrac{M}{S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+2} \arrow[r] & I^{n+2} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Therefore we have an exact sequence $0 \longrightarrow I \xrightarrow{f} \overline{I} \longrightarrow E \longrightarrow 0$ where E : $... \longrightarrow 0 \longrightarrow \frac{M}{S} \longrightarrow \frac{M}{S} \longrightarrow 0 \longrightarrow ...$ and thus we have an exact complex $0 \rightarrow Hom(E, I) \longrightarrow Hom(\overline{I}, I) \longrightarrow Hom(I, I) \longrightarrow Ext^1(E, I) = 0$ since $I \in \varepsilon^{\perp}$. So $Hom(\overline{I}, I) \longrightarrow Hom(I, I) \rightarrow 0$ is epic. This implies that we can find that $\overline{f} : \overline{I} \longrightarrow I$ such that $\overline{f} f = 1$. Therefore there exists a function $\overline{f}^n : I^n \oplus_S M \longrightarrow I^n$ such that $\overline{f}^n f^n = 1$. Then $S \longrightarrow I^n$ can be extended to $\overline{f}^n i_1 : M \longrightarrow I^n$. So I^n is injective.

□

Lemma 2.2 *Let $f : X \longrightarrow Y$ be a morphism of complexes. Then the exact sequence*

$0 \longrightarrow Y \longrightarrow M(f) \longrightarrow X[1] \longrightarrow 0$ *associated with the mapping cone* $M(f)$ *splits in* C *if and only if* f *is homotopic to* 0.

Proof.

$(\Longrightarrow)$

Since the sequence splits, there exists a morphism $g : X[1] \longrightarrow M(f)$ with $g(x) = (x, -s(x))$ where s is a morphism from $X[1]$ to Y. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y^{n-1} \arrow[r, "i^{n-1}"] \arrow[d, "\partial_1^{n-1}"'] & X^n \oplus Y^{n-1} \arrow[r, dotted, shift left, "p^{n-1}"] \arrow[d, "\theta^{n-1}"'] & X^n \arrow[l, dotted, shift left, "g^n"] \arrow[r] \arrow[d, "-\partial^n"] & 0 \\
0 \arrow[r] & Y^n \arrow[r, "i^n"'] & X^{n+1} \oplus Y^n \arrow[r, dotted, shift left, "p^n"] & X^{n+1} \arrow[l, dotted, shift left, "g^{n+1}"] \arrow[r] & 0
\end{tikzcd}$$

where $\theta^{n-1}(x, y) = (-\partial^n x, f^n(x) + \partial_1^{n-1} y)$ for all $(x, y) \in X^n \oplus Y^{n-1}$. By the commutative diagram, we can find that for all $x \in X^n$

$$\theta^{n-1}(g^n(x)) = -g^{n+1}\partial^n(x)$$

$$\theta^{n-1}(x, -s^n(x)) = (-\partial^n(x), s^{n+1}\partial^n(x))$$

$$(-\partial^n x, f^n(x) + \partial_1^{n-1}(-s^n(x))) = (-\partial^n(x), s^{n+1}\partial^n(x)).$$

This implies that $f^n(x) = \partial_1^{n-1} s^n(x) + s^{n+1}\partial^n(x)$, for all $x \in X^n$. Therefore, $f^n = \partial^{n-1} s^n + s^{n+1}\partial^n$ which implies that f is homotopic to 0.

$(\Longleftarrow)$

Suppose that f is homotopic to 0. If s is such a homotopy, then we can construct a function,

$$g : X[1] \longrightarrow M(f)$$

$$x \longmapsto (x, -s(x))$$

where $s^n : X^n \longrightarrow Y^{n-1}$ such that $f^n = \partial^{n-1} s^n + s^{n+1}\partial^n$. Then $p^{n-1} g^n(x) = p^{n-1}(x, -s^n(x)) = I(x)$ for all $x \in X^n$ and so $p^{n-1} g^n = I$. So the exact sequence $0 \longrightarrow Y \longrightarrow M(f) \longrightarrow$

$X[1] \longrightarrow 0$ splits.

□

Lemma 2.3 *If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is an split exact sequence, then* $B \cong A \oplus C$.

Proof. Since $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact, there exist both a morphism $h : B \longrightarrow A$ such that $hf = 1_A$ and a morphism $\theta : C \longrightarrow B$ such that $g\theta = 1_C$. So for all $b \in B$ we can write that $b = b - fh(b) + fh(b)$ where $b - fh(b) \in Kerh$. Therefore $B = Kerh \oplus Imf$. Since f is $1-1$, $Imf \cong A$. Since the sequence $0 \longrightarrow C \xrightarrow{\theta} B \xrightarrow{h} A \longrightarrow 0$ is exact, $Kerh = Im\theta \cong C$. Thus $B \cong C \oplus A$.

□

Corollary 2.4 *If* X *is a complex, then* $\mathcal{H}om(X, I)$ *is exact for a complex* I *such that* I^m *is injective for all* $m \in \mathbb{Z}$ *if and only if* $Ext^1(X, I[n]) = 0$ *for all* $n \in \mathbb{Z}$.

Proof.

($\Longrightarrow$)

Suppose that $\mathcal{H}om(X, I)$ is exact for a complex I such that I^m is injective for all $m \in \mathbb{Z}$. Since $\mathcal{H}om(X, I)$ being exact is equivalent to the claim that for each $n \in \mathbb{Z}$ each morphism $f : X \longrightarrow I[n]$ is homotopic to 0, by Lemma 2.2 the exact sequence $0 \longrightarrow I[n] \longrightarrow M(f) \xrightarrow{\beta} X[1] \longrightarrow 0$ splits, so we have an exact sequence $0 \longrightarrow Hom(X, I[n]) \longrightarrow Hom(X, M(f)) \xrightarrow{\beta^*} Hom(X, X[1]) \longrightarrow Ext^1(X, I[n]) \longrightarrow Ext^1(X, M(f)) \longrightarrow ...$ and $\beta' : X[1] \to M(f)$ with $\beta\beta' = 1$. Thus for all $g \in Hom(X, X[1])$ there exists at least one $\beta' g \in Hom(X, M(f))$ such that $\beta^*(\beta' g) = \beta(\beta' g) = g$, that is, β^* is onto. So $Ext^1(X, I[n]) = 0$ for all $n \in \mathbb{N}$.

($\Longleftarrow$)

Since for all $n \in \mathbb{Z}$, $Ext^1(X, I[n]) = 0$, this implies that any exact sequence $0 \longrightarrow I[n] \longrightarrow Y \longrightarrow X \longrightarrow 0$ of complexes splits. For, we have the exact sequence $0 \longrightarrow Hom(X, I[n]) \longrightarrow Hom(Y, I[n]) \longrightarrow Hom(I[n], I[n]) \longrightarrow \underbrace{Ext^1(X, I[n])}_{=0}$.

Since $0 \longrightarrow I[n] \longrightarrow Y \longrightarrow X \longrightarrow 0$ splits, $Y \cong I[n] \oplus X$ by Lemma 2.3. So this sequence is isomorphic to a sequence

$$0 \longrightarrow I[n] \longrightarrow M(f) \longrightarrow X \longrightarrow 0$$

where $f : X[-1] \longrightarrow I[n]$. Therefore $0 \longrightarrow I[n] \longrightarrow M(f) \longrightarrow X \longrightarrow 0$ splits. By Lemma 2.2 $f : X[-1] \longrightarrow I[n]$ is homotopic to 0 for all $n \in \mathbb{Z}$. So $f' : X \longrightarrow I[n+1]$ is homotopic to 0 for all $n \in \mathbb{Z}$. Thus we understand from the explanation above that $\mathcal{H}om(X, I)$ is exact.

□

Proposition 2.5 *$\varepsilon^{\perp}$ is the class of DG - injective complexes.*

Proof. If $I \in \varepsilon^{\perp}$, then by Lemma 2.1 all I^n are injective. Since $I \in \varepsilon^{\perp}$, for all $E \in \varepsilon$ $Ext^1(E, I) = 0$. By Corollary 2.4, $\mathcal{H}om(E, I)$ is exact for all $E \in \varepsilon$. So I is DG - injective.

□

Dual arguments gives the following proposition;

Proposition 2.6 *${}^{\perp}\varepsilon$ is the class of DG - projective complexes.*

Proposition 2.7 ${}^{\perp}(\varepsilon^{\perp}) = \varepsilon$

Proof. Clearly $\varepsilon \subseteq {}^{\perp}(\varepsilon^{\perp})$. Let $E \in {}^{\perp}(\varepsilon^{\perp})$. We will show that E is exact. Let $I : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow I^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$ where I^0 is an injective cogenerator (see [4] for more details). Then I is (so also any $I[n], n \in \mathbb{Z}$) DG-injective complex. By Proposition 2.5, $I \in \varepsilon^{\perp}$. Since $E \in {}^{\perp}(\varepsilon^{\perp})$, $Ext^1(E, I[n]) = 0$ for all $n \in \mathbb{Z}$. Let $E : \cdots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$. Then we have the sequence $\cdots \to Hom(E^2, I^0) \to Hom(E^1, I^0) \to Hom(E^0, I^0) \to \cdots$. This sequence is equal to $\mathcal{H}om(E, I)$. By Corollary 2.4, $\mathcal{H}om(E, I)$ is exact. Since I^0 is an injective cogenerator, the complex E is exact.

$\square$

Remark 2.8 *If $I \in \varepsilon^{\perp}$, then for all $E \in \varepsilon$ and all $i \geq 1$ $Ext^i(E, I) = 0$.*

Because, we can find that an exact sequence of complexes with P projective complex

$$0 \to S \xrightarrow{f} P \xrightarrow{g} E \to 0.$$

Then we have the following diagram;

$$\begin{tikzcd}
0 \arrow[r] & S^{m-1} \arrow[r] \arrow[d, "\beta^{m-1}"] & P^{m-1} \arrow[r] \arrow[d, "\phi^{m-1}"] & E^{m-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^{m} \arrow[r] \arrow[d, "\beta^{m}"] & P^{m} \arrow[r] \arrow[d, "\phi^{m}"] & E^{m} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^{m+1} \arrow[r] & P^{m+1} \arrow[r] & E^{m+1} \arrow[r] & 0
\end{tikzcd}$$

where $\phi^{m-1} |_{S^{m-1}} = \beta^{m-1}$ and $\phi^m |_{S^m} = \beta^m$. Therefore S is also an exact sequence. Since $Ext^1(S, I) = Ext^2(E, I)$ and $I \in \varepsilon^{\perp}$ and $S \in \varepsilon$, $Ext^2(E, I) = 0$. Using the induction we can write that for all $i \geq 1$ $Ext^i(E, I) = 0$.

Similarly, if $P \in {}^{\perp}\varepsilon$, then for all $i \geq 1$, $Ext^i(P, E) = 0$.

Thus if $0 \to I' \to I \to I'' \to 0$ is an exact sequence of complexes and I' and I'' are DG-injective, then by Proposition 2.5, I' and $I'' \in \varepsilon^{\perp}$ and for all $E \in \varepsilon$, we can find the exact sequence;

$0 \to Hom(E, I') \to Hom(E, I) \to Hom(E, I'') \to Ext^1(E, I') \to Ext^1(E, I) \to Ext^1(E, I'') \to Ext^2(E, I') \to \cdots$.

Since $Ext^1(E, I') = 0 = Ext^1(E, I'')$, $Ext^1(E, I) = 0$, so $I \in \varepsilon^{\perp}$ and hence I is a DG-injective complex.

And also if I' and I are DG-injective, then so is I''.

Similarly, if $0 \to P' \to P \to P'' \to 0$ is exact sequence of complexes and P and P'' are DG-projective, then for all $E \in \varepsilon$, we can find the exact sequence;
$0 \to Hom(P'', E) \to Hom(P, E) \to Hom(P', E) \to Ext^1(P'', E) \to Ext^1(P, E) \to Ext^1(P', E) \to Ext^2(P'', E) \to \cdots.$
Since $Ext^1(P, E) = Ext^2(P'', E) = 0$, $Ext^1(P', E) = 0$, so $P' \in {}^{\perp}\varepsilon$. By Proposition 2.6, P' is also DG-projective.

And also P' and P'' are DG-projective, then so is P.

Lemma 2.9 *Let $0 \to E \to I \to \frac{I}{E} \to 0$ be an exact sequence of complexes and let E and I be exact complexes. Then $\frac{I}{E}$ is an exact complex , too. Similarly if I and $\frac{I}{E}$ are exact, then E is exact. And also if E and $\frac{I}{E}$ are exact, then I is exact.*

Proof. We will give only the proof of the first statement. We can write the following commutative diagram;

$$\begin{tikzcd}
0 \arrow[r] & E^n \arrow[r, "\alpha^n"] \arrow[d, "f^n"'] & I^n \arrow[r, "\beta^n"] \arrow[d, "g^n"'] & \frac{I^n}{E^n} \arrow[r] \arrow[d, "h^n"'] & 0 \\
0 \arrow[r] & E^{n+1} \arrow[r, "\alpha^{n+1}"] \arrow[d, "f^{n+1}"'] & I^{n+1} \arrow[r, "\beta^{n+1}"] \arrow[d, "g^{n+1}"'] & \frac{I^{n+1}}{E^{n+1}} \arrow[r] \arrow[d, "h^{n+1}"'] & 0 \\
0 \arrow[r] & E^{n+2} \arrow[r, "\alpha^{n+2}"] & I^{n+2} \arrow[r, "\beta^{n+2}"] & \frac{I^{n+2}}{E^{n+2}} \arrow[r] & 0
\end{tikzcd}$$

where $Imf^n = Kerf^{n+1}$ and $Img^n = Kerg^{n+1}$. We will prove that $Imh^n = Kerh^{n+1}$. Let $x+E^{n+1} \in Imh^n$. Then there exists at least one $y+E^n \in \frac{I^n}{E^n}$ such that $h^n(y+E^n) = x+E^{n+1}$. So $g^n(y) + E^{n+1} = x + E^{n+1}$. Then

$$h^{n+1}(x + E^{n+1}) = h^{n+1}(g^n(y) + E^{n+1}) = g^{n+1}(g^n(y)) + E^{n+2} = E^{n+2}.$$

Therefore $x + E^{n+1} \in Ker(h^{n+1})$. Notice that this implies that $\frac{I}{E}$ is a complex.

Now let $x+E^{n+1} \in Ker(h^{n+1})$. Then $h^{n+1}(x+E^{n+1}) = g^{n+1}(x)+E^{n+2} = E^{n+2}$. So $g^{n+1}(x) \in E^{n+2}$. Since $\beta^{n+2}(g^{n+1}(x)) = h^{n+1}(\beta^{n+1}(x)) = 0$, $\beta^{n+1}(x) \in Kerh^{n+1}$, so $\beta^{n+1}(x) \in Imh^n$. So there exists at least one $a + E^n \in \frac{I^n}{E^n}$ such that, $h^n(a + E^n) = \beta^{n+1}(x) = x + E^{n+1}$, so $x + E^{n+1} \in Imh^n$. As a result, we can write that $Imh^n = Kerh^{n+1}$.

□

Proposition 2.10 *$\varepsilon \cap \varepsilon^{\perp}$ is the class of injective complexes and $\varepsilon \cap {}^{\perp}\varepsilon$ is the class of projective complexes.*

Proof. Let $I \in \varepsilon \cap \varepsilon^{\perp}$. We can find an injective (exact) complex J such that $I \subseteq J$. Then we have an exact sequence of complexes, $0 \to I \to J \to \frac{J}{I} \to 0$. Since I and J are exact, $\frac{J}{I}$ is an exact complex by Lemma 2.9. So $Ext^1(\frac{J}{I}, I) = 0$. Since we have the exact sequence

$$0 \to Hom(\frac{J}{I}, I) \to Hom(J, I) \to Hom(I, I) \to Ext^1(\frac{J}{I}, I) = 0 \to \cdots ,$$

the exact sequence $0 \to I \to J \to \frac{J}{I} \to 0$ splits. Thus I is a direct summand of J by Lemma 2.3, so it is injective. Similarly, we can prove that $\varepsilon \cap {}^{\perp}\varepsilon$ is the class of projective complexes.

□

Proposition 2.11 $({}^{\perp}\varepsilon)^{\perp} = \varepsilon$.

Proof. It is proved dual to proof of Proposition 2.7.

□

Theorem 2.12 *([5]) For every complex X there are quasi-isomorphisms $P \to X$ and $X \to I$ where P is DG-projective and I is DG-injective.*

Lemma 2.13 *Let X and I complexes in C and $f : X \to I$ be a $1-1$ morphism. Then $\frac{I}{X}$ is exact if and only if $X \to I$ is a homology isomorphism. And also if $g : I \to X$ is onto morphism, then $Kerg$ is exact if and only if $I \to X$ is a homology isomorphism.*

Proof. We understand the proof by the long exact sequence in Theorem 6.10 in [1].

□

Corollary 2.14 *Every complex X has a DG-projective precover in which kernel is exact and a DG-injective preenvelope.*

Proof. Let $P \to X$ be as in the Theorem 2.12. Since there exists an onto morphism $Q \to X$ with Q projective (so it is exact) complex, we can find an onto morphism $P \oplus Q \to X$. And $H(Q) = 0$ implies that $P \oplus Q \to X$ is a quasi-isomorphism. Hence we have an exact sequence of complexes, $0 \to E \to P \oplus Q \to X \to O$ where E is exact by Lemma 2.13. Let P' be any DG-projective complex. Then by Proposition 2.6 $Ext^1(P', E) = 0$, so $Hom(P', P \oplus Q) \to Hom(P', X) \to 0$ is onto. Therefore we have the commutative diagram

$$\begin{tikzcd}
P' \arrow[d, dotted] \arrow[dr] & & \\
P \oplus Q \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Therefore $P \oplus Q \to X$ is a DG-projective precover. Similarly, we get that every X has a DG-injective preenvelope $X \to I$.

□

Remark 2.15 *We see that such precovers $P \to X$ and preenvelopes $X \to I$ are necessarily surjective and injective, respectively, since projective complexes are DG-projective and injective complexes are DG-injective.*

Lemma 2.16 *If $E \in \varepsilon$ and $E \to I$ is an injective envelope, then $E \to I$ is a DG-injective envelope.*

Proof. Let $0 \to E \xrightarrow{\theta} I \xrightarrow{\beta} E' \to 0$ be exact. Since E and I are exact, by Lemma 2.9 E' is exact, too. If I' is DG-injective, then $Ext^1(E', I') = 0$ by Proposition 2.5 and so $Hom(I, I') \to Hom(E, I') \to 0$ is exact. Since I is DG-injective, $E \to I$ is DG-injective preenvelope. Since it is an injective envelope, we see that it is a DG-injective envelope.

□

Theorem 2.17 *Every complex X has a DG-injective envelope $X \to I$. Such an envelope is injective and is a quasi-isomorphism. A morphism $X \longrightarrow I$ of complexes is a DG-injective envelope of X if and only if it is injective, if I is DG-injective, if $X \longrightarrow I$ is a quasi-isomorphism and if there are no exact subcomplexes $E \subset I$ such that $X \cap E = 0$ and $E \neq 0$.*

Proof. The proof that X has a DG-injective envelope is a straight forward modification of the proof of Theorem 6.1 in [6]. In that proof we replace the class of modules of finite projective dimension with the class of exact complexes and note that this class of complexes is closed under inductive limits. Since every DG-injective preenvelope $X \longrightarrow I$ is injective, so is such an envelope. Let U be any DG-injective complex. Then the exact sequence $0 \to X \to I \to \frac{I}{X} \to 0$ implies that the exact sequence;
$0 \to Hom(\frac{I}{X}, U) \to Hom(I, U) \to Hom(X, U) \to Ext^1(\frac{I}{X}, U) \to \cdots$.
Since $X \longrightarrow I$ is a DG-injective preenvelope, $Ext^1(\frac{I}{X}, U) = 0$. So $\frac{I}{X} \in^{\perp} (\varepsilon^{\perp})$ by Proposition 2.5. Since ${}^{\perp}(\varepsilon^{\perp}) = \varepsilon$ by Proposition 2.7, $\frac{I}{X}$ is exact and therefore by Lemma 2.13 $X \longrightarrow I$ is a homology-isomorphism.
Now let $X \longrightarrow I$ be a DG-injective envelope (so $X \longrightarrow I$ is an injection) and $E \subset I$ be an exact complex with $X \cap E = 0$. Since the injective envelope J of E is the DG-injective envelope by Lemma 2.16, $E \xrightarrow{1-1} I$ can be extended to a map $f : J \longrightarrow I$. Since E is essential in J, f is injective. Since $f(E) = E$ is essential in $f(J)$, $X \cap f(J) = 0$. Since J is the also exact, we can assume the original E is injective. Then $\frac{I}{E}$ is also DG-injective by Remark 2.8. Since $X \longrightarrow I$ is a homology isomorphism and $H(E) = 0$, by Theorem 1.35 we have $X \longrightarrow \frac{I}{E}$ is a homology isomorphism. By

Lemma 2.13 we have an exact sequence $0 \longrightarrow X \longrightarrow \frac{I}{E} \longrightarrow F \longrightarrow 0$ with F exact. Therefore $X \longrightarrow \frac{I}{E}$ is a DG-injective preenvelope. Because if U is any DG-injective complex, then $0 \longrightarrow Hom(F,U) \longrightarrow Hom(\frac{I}{E},U) \longrightarrow Hom(X,U) \longrightarrow Ext^1(F,U) = 0$ by Proposition 2.5. Hence there is a commutative diagram

$$\begin{tikzcd} & I \arrow[d, "\theta"] \\ X \arrow[ur] \arrow[r] \arrow[dr] & \frac{I}{E} \arrow[d, "\beta"] \\ & I \end{tikzcd}$$

with $I \longrightarrow \frac{I}{E}$ the canonical surjection. Since $X \longrightarrow I$ is an envelope, $I \xrightarrow{\theta} \frac{I}{E} \xrightarrow{\beta} I$ is an automorphism of I. Let $e \in E$. Then $\beta\theta(e) = \beta(\theta(e)) = \beta(e + E) = \beta(0) = 0$ and since $\beta\theta$ is an automorphism, $e = 0$. So $E = 0$.

Conversely, we know that there exists a DG-injective envelope of X such that $0 \longrightarrow X \longrightarrow I$. Let I' be DG-injective, $0 \longrightarrow X \longrightarrow I'$ be a quasi-isomorphism and there are no exact subcomplexes $E \subset I'$ such that $X \cap E = 0$ and $E \neq 0$. Since $0 \longrightarrow X \longrightarrow I'$ is a quasi-isomorphism and so $\frac{I'}{X}$ is exact, $0 \longrightarrow X \longrightarrow I'$ is a DG-injective preenvelope. So we have the following commutative diagram

$$\begin{tikzcd} & I \arrow[d] \\ X \arrow[ur] \arrow[r] \arrow[dr] & I' \arrow[d] \\ & I \end{tikzcd}$$

Since $I \longrightarrow I' \longrightarrow I$ is an automorphism of I, $I' \cong I \oplus J$ where $J = Ker(I' \longrightarrow I)$, so J is also DG-injective. Since $X \longrightarrow I$ and $X \longrightarrow I'$ are homology isomorphism, for all $n \in \mathbb{Z}$; $H_n(I) \cong H_n(I')$. Since $0 \longrightarrow I \longrightarrow I'$ is a quasi-isomorphism, $\frac{I'}{I}$ is exact by Lemma 2.13 and $I' \cong I \oplus J$ implies that J is exact, so $J \in \varepsilon \cap \varepsilon^{\perp}$. By Proposition 2.10,

J is injective. From diagram we understand that $X \cap J = 0$. This implies that $J = 0$ and so $I' \longrightarrow I$ is an isomorphism. Hence $X \longrightarrow I'$ is an envelope.

□

Corollary 2.18 *The DG-injective envelope $X \longrightarrow I$ of a complex X is such that I is injective if and only if X exact.*

Proof. If X is exact, then since $0 \longrightarrow X \longrightarrow I$ is DG-injective preenvelope, $H_n(X) \cong H_n(I)$ and so $\frac{I}{X}$ is exact, therefore I is exact. So $I \in \varepsilon^{\perp} \cap \varepsilon$ and hence by Proposition 2.10 I is injective.

Conversely, let $X \longrightarrow I$ be a DG-injective envelope and assume that I is injective. If $0 \longrightarrow X \longrightarrow I \longrightarrow E \longrightarrow 0$ is exact, then $\frac{I}{X} \cong E$ is exact by Lemma 2.13, since $H_n(X) \cong H_n(I)$ by Theorem 2.17. Since I and E are exact, X is also exact by Lemma 2.9.

□

Corollary 2.19 *If $0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$ is a minimal DG-injective resolution of X, then each I_{-i} for $i \geq 1$ is injective, I_0 is injective if and only if X is exact.*

Proof. By Theorem 2.17, $X \longrightarrow I_0$ is injective and a quasi-isomorphism, so $\frac{I_0}{X}$ is exact. By Corollary 2.18 I_{-1} is injective. Repeating this argument we get I_{-2} , I_{-3} , $\cdots$ injective.

The converse follows from Corollary 2.18.

□

Proposition 2.20 *For a DG-injective complex I, the following are equivalent:*

(a) *If $E \subset I$ is an exact complex, then $E = 0$;*

(b) *If $J \subset I$ is an injective subcomplex, then $J = 0$;*

(c) *For each n,* $Z^n(I)$ *is essential in* I^n*.*

Proof.

$(a) \Rightarrow (b)$ is satisfied since injective complexes are exact.

To argue $(b) \Rightarrow (c)$, assume $Z^n(I)$ is not essential in I^n. Then there is an injective submodule $M \subset I^n$ with $Z^n(I) \cap M = 0$ and $M \neq 0$. Then $\partial^n \mid_M$ is an injection . But the complex $\cdots 0 \to M \to \partial^n(M) \to 0 \cdots$ is a non-zero injective subcomplex of I. This is a contradiction.

To argue $(c) \Rightarrow (b)$, an injective complex $J \subset I$ is a direct sum of complexes of the form $\cdots 0 \to M \xrightarrow{id} M \to 0 \to \cdots$ where M is an injective module. But by (c), every such M must be 0. Hence $J = 0$.

To argue $(b) \Rightarrow (a)$, E has an injective envelope J such that E is essential in J. By Lemma 2.16 $E \longrightarrow J$ is a DG -injective envelope. If $E \subset J$ is exact, then we have the following commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & \\
0 \arrow[r] & E \arrow[r, "i_1"] \arrow[d, "i_2"'] & I \\
& J \arrow[ur, "f"] &
\end{tikzcd}$$

where $fi_2 = i_1$. Then $Ker(f) \cap E = 0$. Since E is essential in J, $Ker(f) = 0$. So we can take $E \subset J \subset I$ with J an injective envelope of E. But then $J = 0$ implies $E = 0$.

□

Definition 2.21 *A DG-injective complex I is said to be minimal if I satisfies one of the equivalent conditions* $(a), (b)$ *or* (c) *in Proposition 2.20.*

Proposition 2.22 *A DG-injective complex is the direct sum of an injective complex and a minimal DG-injective complex. This direct sum decomposition is unique up to isomorphism.*

Proof. Let I be a DG-injective complex. Using the Zorn's Lemma, for each n we can find a maximal submodule $J^n \subseteq I^n$ such that, $J^n \cap Ker(I^n \to I^{n+1}) = 0$. Since I^n is injective, we can find an injective module K^n such that, $J^n \subseteq^{ess} K^n \subseteq I^n$. Then it must be $K^n = J^n$ and so J^n is injective. Then I has the injective subcomplex; $\cdots \to 0 \to J^n \longrightarrow \partial^n(J^n) \to 0 \to \cdots$. Since $\partial^{n-1}(J^{n-1}) \subseteq Ker(I^n \to I^{n+1})$, $\partial^{n-1}(J^{n-1}) \cap J^n = 0$. So the sum of these complexes is direct and is isomorphic to their external direct product, because the sum is

$$I_1 : \cdots \to J^{n-1} \oplus \partial^{n-1}(J^{n-1}) \to I^{n+1} \oplus \partial^n(J^n) \to \cdots .$$

Hence the sum is injective. So I_1 is a direct summand of I and we have $I = I_1 \oplus I_2$. By construction, we understand that I_2 is minimal. Now suppose we have another decomposition $I = I_1' \oplus I_2'$ with I_1' injective and I_2' minimal. Put $S = I_1 \cap I_2'$. Let the injective envelope of S be $E(S) \subseteq I_1$. Then the projection $\phi : E(S) \subseteq I_1' \oplus I_2' \to I_2'$ is injective, otherwise $Ker\phi \neq 0$ and $S \cap Ker\phi \neq 0$ since $S \subseteq^{ess} E(S)$, but $Ker\phi \subseteq I_1'$ and $S \cap I_1' = 0$, but this is a contradiction. So ϕ is injective. Thus $\phi(E(S))$ is injective. But since I_2' is minimal, $\phi(E(S)) = 0$ and so $E(S) = 0 = S$. Hence I_1 projects injectively into I_1' with image $J \subseteq I_1'$. So J is injective and hence J is a direct summand of I_1', so we have a decomposition $I = J \oplus K \oplus I_2'$. Since I_1 projects isomorphically onto J, we have $I \cong I_1 \oplus K \oplus I_2'$. But then $I_2 \cong \frac{I}{I_1} \cong K \oplus I_2'$. Since K is injective and I_2 is minimal, $K = 0$ and so I_1 projects isomorphically I_1'. And also $I_2 \cong I_2'$.

□

Corollary 2.23 *Given any complex X, there exists a quasi-isomorphism $X \to I$ with I a minimal DG-injective complex.*

Proof. By Theorem 2.12, there is a quasi-isomorphism $X \to I$ with I a DG-injective complex. If $I = I_1 \oplus I_2$ with I_1 injective and I_2 minimal from the proposition above, then since $H(I_1) = 0$, $X \to I_2$ is the desired homology isomorphism.

□

Theorem 2.24 *Every complex X has an exact cover $E \longrightarrow X$. Every exact cover $E \longrightarrow X$ is surjective and $Ker(E \longrightarrow X)$ is DG-injective. A morphism $E \longrightarrow X$ of complexes is an exact cover of X if and only if E is exact, $E \longrightarrow X$ is surjective and $Ker(E \longrightarrow X)$ is a minimal DG-injective complex. If $E \longrightarrow X$ is an exact cover, E is injective if and only if X is DG-injective.*

Proof. Given X, by Theorem 2.12 there is a homology isomorphism $f : X \longrightarrow I$ with I DG-injective. So we have an exact sequence; $0 \to I \to M(f) \to X[1] \to 0$. Since $H_n(X) \cong H_n(I)$ and $H_n(X) = H_{n-1}(X[1])$, by Theorem 1.35; $M(f)$ is exact. Let $E = M(f)[-1]$. Then we have a surjective morphism $E \longrightarrow X$ with the kernel $I[-1]$ being DG-injective by Corollary 2.4.

We claim that $E \longrightarrow X$ is an exact precover. For, if E' is an exact complex, by Proposition 2.5; $Ext^1(E', I[-1]) = 0$ and so $Hom(E', E) \to Hom(E', X) \to 0$ is exact. So $E \longrightarrow X$ is an exact precover. Since ε is closed under inductive limits, if we modify Corollary 5.2.7 in [4] to complexes, then we can find that X has an exact cover. If $\phi : E \longrightarrow X$ is an exact cover, since there is a projective complex P such that $P \longrightarrow X$ is surjective, so ϕ must be surjective. By Lemma 1.21; $I = Ker(E \longrightarrow X) \in \varepsilon^{\perp}$ and so it is DG-injective. If I were not minimal, then by Proposition 2.22 it would have an injective subcomplex $I_1 \neq O$. Then if $E = I_1 \oplus E_2$, then the projection map $f : E \longrightarrow E_2$ with for all $a \in I_1 \subseteq I$ and for all $b \in E_2$, $f(a+b) = b$ gives the following commutative diagram

$$\begin{tikzcd}
E \arrow[d, "f"'] \arrow[dr, "\phi"] & \\
E_2 \subseteq E \arrow[r, "\phi"'] & X
\end{tikzcd}$$

where $\phi f = \phi$ and f is not automorphism. Since $\phi : E \longrightarrow X$ is an exact cover, this implies a contradiction. Hence I must be minimal.

Conversely, suppose $\phi : E \longrightarrow X$ is a surjective morphism with E exact and $I = Ker(\phi)$ a minimal DG-injective complex. So if $E' \in \varepsilon$, then we have $Hom(E', E) \longrightarrow$

$Hom(E', X) \longrightarrow Ext'(E', Ker\phi) = 0$. This implies that $\phi : E \longrightarrow X$ is an exact precover. Let $\overline{E} \longrightarrow X$ be an exact cover. Then we have the following commutative diagram:

$$\begin{tikzcd}
\overline{E} \arrow[d, dotted] \arrow[dr] & \\
E \arrow[r] \arrow[d, dotted] & X \\
\overline{E} \arrow[ur] &
\end{tikzcd}$$

But $\overline{E} \longrightarrow E \longrightarrow \overline{E}$ is an automorphism of $\overline{E}$. So $Ker(E \longrightarrow \overline{E})$ is a direct summand of E (so is exact) contained in $I = Ker(\phi)$. Since I is minimal, $Ker(E \longrightarrow \overline{E}) = 0$ and so $E \longrightarrow \overline{E}$ is an isomorphism. Hence $E \longrightarrow X$ is a cover.

Now suppose that X is DG-injective and $E \longrightarrow X$ an exact cover with $I = Ker(E \longrightarrow X)$. Then since I is DG-injective and $0 \to I \to E \to X \to 0$ is exact, E is DG-injective. But since E is also exact, by Proposition 2.10, E is injective.

If on the other hand; $E \longrightarrow X$ is an exact cover and E is injective, then with $I = Ker(E \longrightarrow X)$, the exactness of $0 \to I \to E \to X \to 0$ where I and E are DG-injective, gives that X is DG-injective by Remark 2.8.

□

Now using the theorem above we can give an example that how we can determine the exact cover of a complex.

Example 2.25 *Let $X : \cdots 0 \longrightarrow X^0 \longrightarrow 0 \longrightarrow \cdots$ where $X^0 \in R$-Mod be a complex. If $E : \cdots \longrightarrow 0 \longrightarrow 0 \to X^0 \to X^1 \to X^2 \to X^3 \to \cdots$ is a minimal injective resolution of the module X^0, then the map $E \longrightarrow X \longrightarrow 0$ such that,*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & X^0 \arrow[r] \arrow[d, "1"'] & X^1 \arrow[r] \arrow[d, "0"'] & X^2 \arrow[r] \arrow[d, "0"'] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & X^0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is an exact cover of X, since we see that $I = Ker(E \to X) : \cdots \to 0 \to 0 \to X^1 \to$

$X^2 \to \cdots$ and for each n, $Z^n(I)$ is essential in X^n where $n \geq 1$ which we see from construction of injective resolution E and so $Ker(E \to X)$ is a minimal DG-injective complex.

Corollary 2.26 *If $\cdots \to E_2 \to E_1 \to E_0 \to X \to 0$ is a minimal exact left resolution of a complex X, then each E_i ($i \geq 1$) is injective and E_0 is injective if and only if X is DG-injective.*

Proof. By Theorem 2.24 the complex X has an exact cover $E_0 \to X$ such that; $Ker(E_0 \to X)$ is DG-injective. By Theorem 2.24, the exact cover of $Ker(E_0 \to X)$ exists, say E_1 such that $E_1 \to Ker(E_0 \to X) \to 0$ and E_1 is injective. If we continue in this way, then for all $i \geq 1$ each E_i is injective. Again by the same theorem, the other part is proved, too.

□

Theorem 2.27 *In C, $Hom(-,-)$ is left balanced by ${}^{\perp}\varepsilon \times \varepsilon^{\perp}$.*

Proof. If $0 \to Y \to I_0 \to I_{-1} \to I_{-2} \cdots$ is any right $\varepsilon^{\perp}$-resolution of Y and $P \in {}^{\perp}\varepsilon$, we must show that $Hom(P,-)$ leaves the sequence exact. Using Theorem 2.17, we can construct the sequence $0 \to Y \to I_0 \to I_{-1} \to I_{-2} \cdots$, we see that we only need show that if $Y \to I$ is a DG-injective preenvelope and if $C = Coker(Y \to I)$, then $Hom(P, I) \to Hom(P, C) \to 0$ is exact.
Given another such DG-injective preenvelope $Y \to \overline{I}$ with cokernel $\overline{C}$, by the definition of preenvelopes, we see that if the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\parallel"'] & \overline{I} \arrow[r] \arrow[d] & \overline{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & I \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative, then the vertical maps give a homotopy equivalence between the two complexes. Hence if $Hom(P,-)$ leaves one of these exact, then it leaves the other exact.

By Theorem 2.17 we have an exact sequence; $0 \to Y \to I \to E \to 0$ where I is DG-injective and since $0 \to Y \to I$ is a quasi-isomorphism, E is exact by Lemma 2.13.

Since E is an epimorphic image of a free (projective) complex, we have a surjection $Q \to E \to 0$. Since E and Q are exact, $Ker(Q \to E)$ is exact and so $Ext^1(P, Ker(Q \to E)) = 0$. This implies that $Hom(P \to Q) \to Hom(P, E) \to 0$. Since Q is a projective complex, any map $Q \to E$ has a lifting $Q \to I$. By the following diagram

$$\begin{tikzcd}
P \arrow[r, dotted] \arrow[dr] & Q \arrow[d] \arrow[dl, dotted] & \\
I \arrow[r] & E \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

we understand that $Hom(P, I) \to Hom(P, E) \to 0$ is exact.

A dual argument shows that if $0 \to D \to P \to X \to 0$ is exact with $P \to X$ a DG-projective precover, then whenever I is DG-injective, $Hom(P, I) \to Hom(D, I) \to 0$ is exact. □

Lemma 2.28 *Let I be a DG-injective complex and let $id : I \longrightarrow I$ give the exact sequence* $0 \longrightarrow I \longrightarrow M(id) \longrightarrow I[1] \longrightarrow 0$. *Then $M(id)$ is injective and $M(id) \longrightarrow I[1]$ is an exact precover. If I is minimal, then $I \longrightarrow M(id)$ is an injective envelope and $M(id) \longrightarrow I[1]$ is an exact cover.*

Proof. Since id is a homology isomorphism, $M(id)$ is exact by the long exact sequence. Since I and so $I[1]$ are DG-injective, so is $M(id)$. Then by Proposition 2.10 $M(id)$ is injective. Since $Ext^1(E, I) = 0$ for any exact E, $Hom(E, M(id)) \longrightarrow Hom(E, I[1]) \longrightarrow 0$ exact. So $M(id) \longrightarrow I[1]$ is an exact precover.

Now suppose I is minimal. If $M(id)$ is not an injective envelope of I, then I is not essential in $M(id)$. So there is a nonzero complex $J' \subseteq M(id)$ such that $J' \cap I = 0$. And also we have the injective envelope of J', say J such that J' essential in J. Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & J \arrow[d] \\
0 \arrow[r] & J' \arrow[ur] \arrow[r] \arrow[dr] & M(id) \arrow[d] \\
0 \arrow[ur] & & J
\end{tikzcd}$$

such that $J \longrightarrow M(id) \longrightarrow J$ is an automorphism, so $J' \subset^{ess} J \subset M(id)$ where J is exact and $J \cap I = 0$. But J maps isomorphically to a subcomplex of $I[1]$. Since J is exact and $I[1]$ is minimal, we get a contradiction. So $M(id)$ is an injective envelope of I. When I is minimal we get that $M(id) \longrightarrow I[1]$ is an exact cover by Theorem 2.24.

□

Corollary 2.29 *A DG-injective complex I has injective dimension 0 or ∞.*

Proof. If I itself is injective, then the injective dimension is 0. Let I be not injective. Then we have the following diagram by Lemma 2.28

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & M_1(id) \arrow[r] \arrow[d] & M_2(id) \arrow[r] \arrow[d] & M_3(id) \arrow[r] & \cdots \\
& & I[1] \arrow[ur] & I[2] \arrow[ur] & & \\
& 0 \arrow[ur] & 0 \arrow[ur] & & &
\end{tikzcd}$$

where $M_j(id) = I[j-1] \oplus I[j]$ for $j \geq 1$ .Since I is not injective, for all $j \geq 1$ $I[j]$ cannot be injective. So the sequence above goes to infinity. So injective dimension is ∞.

□

Proposition 2.30 *If P is a DG-projective complex and $id : P \longrightarrow P$ has the associated exact sequence $0 \to P \to M(id) \to P[1] \to 0$, then $M(id)$ is projective and $P \longrightarrow M(id)$ is an exact preenvelope of P.*

Proof. A dual argument of Lemma 2.28

□

Corollary 2.31 *A DG-projective complex has projective dimension 0 or ∞.*

Proof. Clear from Proposition 2.30

□

Lemma 2.32 *Let X, Y be complexes, $f : X \longrightarrow Y$ a morphism and let the exact sequence $0 \longrightarrow Y[-1] \longrightarrow M(id)[-1] \longrightarrow Y \longrightarrow 0$ be gotten from $id : Y \longrightarrow Y$. Then $f : X \longrightarrow Y$ has a lifting $X \longrightarrow M(id)[-1]$ if and only if f is homotopic to 0.*

Proof.

$(\Longrightarrow)$

Let $f : X \longrightarrow Y$ have a lifting $X \longrightarrow M(id)[-1]$. Then we have the following diagram

$$\begin{tikzcd} Mid[-1] \arrow[r, "\pi"] & Y \arrow[r] & 0 \\ X \arrow[u, "g"] \arrow[ur, "f"'] & & \end{tikzcd}$$

where $\pi g = f$. Let $\pi' : M(id)[-1] \longrightarrow Y[-1]$ be a projection and $s = \pi' g$. Since $g : X \longrightarrow M(id)[-1]$ is a chain morphism, by the following diagram

$$\begin{tikzcd}
X^{n-1} \arrow[r, "\lambda^{n-1}"] \arrow[d, "g^{n-1}"'] & X^n \arrow[r, "\lambda^n"] \arrow[d, "g^n"'] & X^{n+1} \arrow[d, "g^{n+1}"] \\
Y^{n-1} \oplus Y^{n-2} \arrow[r, "u^{n-1}"'] & Y^n \oplus Y^{n-1} \arrow[r, "u^n"'] & Y^{n+1} \oplus Y^n
\end{tikzcd}$$

where

$$u^n : M(id)^n[-1] = \qquad Y^n \oplus Y^{n-1} \longrightarrow Y^{n+1} \oplus Y^n$$

$$(a, b) \longrightarrow (-\gamma^n(a), a - \gamma^{n-1}(b))$$

such that $g^{n+1}\lambda^n = u^n g^n$, we have $s^{n+1}\lambda^n + \gamma^{n-1}s^n = \pi'^{n+1}g^{n+1}\lambda^n + \gamma^{n-1}\pi'^n g^n = \pi'^{n+1}u^n g^n + \gamma^{n-1}\pi'^n g^n = (\pi'^{n+1}u^n + \gamma^{n-1}\pi'^n)g^n$. For all $(a, b) \in Y^{n+1} \oplus Y^n$ $(\pi'^{n+1}u^n + \gamma^{n-1}\pi'^n)(a, b) = \pi'^{n+1}(-\gamma^n(a), a - \gamma^{n-1}(b)) + \gamma^{n-1}(b) = a - \gamma^{n-1}(b) + \gamma^{n-1}(b) = \pi(a, b)$. So $s^{n+1}\lambda^n + \gamma^{n-1}s^n = \pi^n g^n = f^n$ and therefore f is homotopic to 0 .

($\Longleftarrow$)

Let f be homotopic to 0 . Then there exists a homotopy $s : X \longrightarrow Y[-1]$ such that we have the following diagram:

$$\begin{tikzcd}
X^{n-1} \arrow[r, "\lambda^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "\lambda^n"] \arrow[d, "f^n"] \arrow[dl, "s^n"] & X^{n+1} \arrow[d, "f^{n+1}"] \arrow[dl, "s^{n+1}"] \\
Y^{n-1} \arrow[r, "\gamma^{n-1}"'] & Y^n \arrow[r, "\gamma^n"'] & Y^{n+1}
\end{tikzcd}$$

where $s^{n+1}\lambda^n + \gamma^{n-1}s^n = f^n$. Then the required lifting is as the following:

$$g : \qquad X \longrightarrow M(id)[-1]$$

$$x \longrightarrow (f(x), s(x))$$

□

Lemma 2.33 *Every complex C has an ε-envelope with cokernel DG-projective complex.*

Proof. Let $C_0 = C$. We will construct an increasing sequence of complexes $C_0 \subset C_1 \subset C_2 \subset \cdots$ with the properties that $\frac{C_n}{C_{n-1}}$ has all its terms projective and $\delta = 0$ for all $n \geq 1$, and such that $Z(C_n) \subset B(C_{n+1})$ for all $n \geq 0$. For then if $E = \cup C_n$, E is exact and by the Lemma 3.1 in [7], $\frac{E}{C}$ will be DG-projective. Clearly we only need show how to construct C_1. Then the same procedure allows us to construct $C_2 \subset C_2 \subset \cdots$.

Given any $x \in Z^n(C) \subset C^n$, we can construct a complex $\cdots \longrightarrow C^{n-2} \longrightarrow C^{n-1} \oplus R \longrightarrow C^n \longrightarrow C^{n+1} \cdots$ having C as a subcomplex and such that $C^{n-1} \oplus R \longrightarrow C^n$ maps $(0, 1)$ to x. It is clear that in fact this is a complex say D with $x \in B(D)$ and with $\frac{D}{C}$ the complex $\cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$ with R in the $(n-1)$-th place. With this procedure we see that by attaching the direct sum of as many copies of R as necessary to C (at different C^k's) we can construct the complex C_1 having the desired properties. This completes the proof.

□

Proposition 2.34 *In C, $Hom(-,-)$ is left balanced by $\varepsilon \times \varepsilon$.*

Proof. By the Proposition and by Theorem 2.24 we get the required resolutions. The other requirement of balance is immediate.

□

CHAPTER 3

MINIMAL INJECTIVE RESOLUTIONS

The term DG-injective resolution is used in [2] and [8]. In this paper, if X is bounded below, then X has so-called ”*minimal injective resolution*”. This means that there is a quasi-isomorphism $X \to I$ where I is a minimal DG-injective complex. Such an I will also be bounded below. If I_1 and I_2 are bounded below minimal DG-injective complexes, any quasi-isomorphism $I_1 \to I_2$ is an isomorphism. This fact can be used to prove that a ”minimal injective resolution” $X \to I$ as above is unique up to isomorphism. In this section we show that as a result of Spaltenstein’s work we no longer need to restrict complexes to those which are bounded below to get these results. We use quotation marks with the term ”minimal injective resolution” since the definition above is not in agreement with our usage in this section. In the rest of the section we will refer to these resolutions by giving their properties.

Lemma 3.1 *Every complex K has an injective envelope $K \longrightarrow E(K)$ where $E(K)$ is the injective envelope of K. If $0 \longrightarrow K \longrightarrow E$ is an injective preenvelope and there are no injective subcomplexes $J \subseteq E$ such that $K \cap J = 0$ and $J \neq 0$, then $K \longrightarrow E$ is an injective envelope.*

Proof. The first part is clear.

We have the following diagram:

$$\begin{tikzcd}
& & E(K) \arrow[d, dotted] \\
0 \arrow[r] & K \arrow[ur] \arrow[r] \arrow[dr] & E \arrow[d, dotted] \\
0 \arrow[ur] & & E(K)
\end{tikzcd}$$

Then $E(K) \longrightarrow E \longrightarrow E(K)$ is an automorphism, say $J = Ker(E \longrightarrow E(K))$. Then $K \cap J = 0$ since $E(K) \longrightarrow E \longrightarrow E(K)$ is an automorphism. Since $E = J \oplus Im(E(K) \longrightarrow E)$, J is injective. By hypothesis of lemma, $J = 0$. So $E \cong E(K)$. Since $K \longrightarrow E(K)$ is envelope, so is $K \longrightarrow E$.

□

Lemma 3.2 *If I is a minimal DG-injective complex and $E \longrightarrow I$ is an exact cover with kernel K and $E = M(id)[-1]$ for $id : I \longrightarrow I$, then K is minimal DG-injective complex and $K \longrightarrow E$ is an injective envelope of K. And conversely if K is a minimal DG-injective complex and $K \longrightarrow E$ is an injective envelope with cokernel I, then I is a minimal DG-injective complex and $E \longrightarrow I$ is an exact cover.*

Proof. Given a minimal DG-injective complex I, let $E = M(id)[-1]$ for $id : I \longrightarrow I$. Then we have the exact sequence

$$0 \longrightarrow I[-1] \longrightarrow M(id)[-1] \longrightarrow I \longrightarrow 0$$

Clearly $I[-1]$ is a minimal DG-injective complex. Also since E is DG-injective and exact by the long exact sequence, E is injective by Proposition 2.10. By Theorem 2.24 $E \longrightarrow I$ is an exact cover. So letting $K = I[-1]$ we get the first part of the claim.

If $0 \longrightarrow K \longrightarrow E$ were not an injective envelope (but notice that $0 \longrightarrow K \longrightarrow E$ is an injective preenvelope), then by Lemma 3.1 there exists an injective subcomplex $J \subseteq E$ such that $K \cap J = 0$ and $J \neq 0$. But then $E \longrightarrow I$ would map J isomorphically onto an injective (so exact) subcomplex of I. Since I is minimal, we get a contradiction. So $K \longrightarrow E$ is an injective envelope.

For the second part of Lemma , let $I = K[1]$ and construct $0 \longrightarrow I[-1] \longrightarrow M(id)[-1] \longrightarrow I \longrightarrow 0$ as above. Then $I[-1] = K$ and by the proceeding $K(= I[-1]) \longrightarrow E(= M(id)[-1])$ is an injective envelope and we also have $I = K[1]$ (that is, the cokernel of $K \longrightarrow E$) is a minimal DG-injective complex.

□

Lemma 3.3 *Let I be a minimal DG-injective complex and let $f : X \longrightarrow I$ be a morphism of complexes. Then the following are equivalent:*

(a) *f is homotopic to* 0.

(b) *f can be factored through an exact cover $E \longrightarrow I$.*

(c) *f can be factored through some exact complex F.*

(d) *f can be factored through some injective complex J.*

(e) $H(f) = 0$

Proof. For $id : I \longrightarrow I$ we have the exact sequence $0 \longrightarrow I[-1] \longrightarrow M(id)[-1] \longrightarrow I \longrightarrow 0$. By Lemma 2.28, $M(id)[-1] \longrightarrow I$ is an exact cover of I.
By Lemma 2.32, f has a lifting to $M(id)[-1]$ (i.e., f can be factored through $M(id)[-1] \rightarrow I$) if and only if f is homotopic to 0. Hence (a) and (b) are equivalent.
Since by Lemma 2.28, the exact cover $E = M(id)[-1] \longrightarrow I$ of I is injective (and of course exact), (b) implies (c) and (d).
But each of (c) and (d) implies (b) since $E \longrightarrow I$ is a cover (and since injective complexes I are exact).
Each of (a) , (b) , (c) and (d) implies (e). And (e) implies (d) by Lemma 2.28 and 2.32.
□

Corollary 3.4 *If I is a minimal DG-injective complex and $S = End(I)$ (the endomorphism ring of I), then any $g \in S$ which is homotopic to* 0 *is in the Jacobson radical of S.*

Proof. By the preceding lemma, any such $\theta : I \longrightarrow I$ can be factored through $E \longrightarrow I$ where $E \longrightarrow I$ is an exact cover of I. Hence using the notation of Lemma 3.2 we get a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, "id+g"] & 0 \\
0 \arrow[r] & K \arrow[r] & E \arrow[r] & I \arrow[r] & 0
\end{tikzcd}$$

Since $E \longrightarrow I$ is an exact cover, $E \longrightarrow E$ is an isomorphism. By the five lemma (see Proposition 2.72 in [1]) $id + g$ is an isomorphism. Since the set of all $g \in S$ homotopic to 0 is a two-sided ideal, the claim is established. □

Theorem 3.5 *If* I_1 *and* I_2 *are minimal DG-injective complexes, then any quasi-isomorphism* $f : I_1 \longrightarrow I_2$ *is an isomorphism.*

Proof. Let $s : I_1 \longrightarrow J$ be an injective where J is an injective complex. Then $\theta' : I_1 \longrightarrow I_2 \oplus J$ is an injective homology isomorphism since $H_n(I_2 \oplus J) \cong H_n(I_2)$. So $E = \frac{I_2 \oplus I}{I_1}$ is exact. So we have an exact sequence

$$0 \longrightarrow I_1 \xrightarrow{\theta'} I_2 \oplus J \xrightarrow{\beta'} E \longrightarrow 0$$

with E exact. By Proposition 2.5, $\varepsilon^{\perp}$ is the class of DG-injective complexes. So

$$0 \to Hom(E, I_1) \to Hom(E, I_2 \oplus J) \to Hom(E, F) \to Ext^1(E, I) = 0$$

and this implies

$$0 \to I_1 \xrightarrow{\theta'} I_2 \oplus J \xrightarrow{\beta'} E \to 0$$

splits. Therefore there exist $\theta : I_2 \oplus J \longrightarrow I_2$ and $\beta : E \longrightarrow I_2 \oplus J$ such that $\theta\theta' = id$ and $\beta\beta' = 1$. Hence we have a homology isomorphism $I_2 \xrightarrow{i} I_2 \oplus J \xrightarrow{\theta} I_1$.
Take $g : I_2 \longrightarrow I_1$ such that $g = \theta i$. Then if $id : I_1 \longrightarrow I_1$, then we have the following diagram;

$$
\begin{tikzcd}
J \arrow[dr, "i'"] & \\
I_2 \oplus J \arrow[u, "\pi"] & I_2 \oplus J \arrow[d, "\theta"] \\
I_1 \arrow[u, "if-\theta'"] \arrow[r] & I_1
\end{tikzcd}
$$

Then using the morphism $\theta if - \theta\theta' = \theta(if - \theta') = gf - id$, we have that $g \circ f - id$ is homotopic to 0 by Lemma 3.3. By Corollary 3.4 $(g \circ f - id) + id = g \circ f$ is an automorphism of I_1. So g is onto.

Similarly we can find a morphism $h : I_1 \longrightarrow I_2$ such that $h \circ g$ is an automorphism. Thus g is $1-1$. So g has an inverse and since $hg \circ f$ is an automorphism, f is an automorphism.

$\square$

Theorem 3.6 *Given any complex X, there is a homology isomorphism $X \longrightarrow I$ where I is a minimal DG-injective complex. If $\phi_1 : X \longrightarrow I_1$ and $\phi_2 : X \longrightarrow I_2$ are two such morphisms, then there is a morphism $f : I_1 \longrightarrow I_2$ such that $f \circ \phi_1$ is homotopic to ϕ_2. Any such f is an isomorphism. Furthermore f is unique up to homotopy.*

Proof. In Corollary 2.23 it was established that given any complex X there is a homology isomorphism $X \longrightarrow I$ with I a minimal DG-injective complex.

Let $\phi_1 : X \longrightarrow I_1$ and $\phi_2 : X \longrightarrow I_2$ be as in the theorem. Let $0 \longrightarrow X \longrightarrow J$ be an injective envelope of X. Then $X \longrightarrow I_1 \oplus J$ and $X \longrightarrow I_2 \oplus J$ are DG-injective preenvelopes. Because we have the following diagram;

$$0 \longrightarrow X \longrightarrow I_1 \oplus J \longrightarrow \frac{I_1 \oplus J}{X} \longrightarrow 0$$

where $\frac{I_1 \oplus J}{X}$ is exact since $X \longrightarrow I_1 \oplus J$ is a homology isomorphism. Then for all DG-injective complexes A, we have the diagram;

$$0 \rightarrow Hom(\frac{I_1 \oplus J}{X}, A) \rightarrow Hom(I_2 \oplus J, A) \rightarrow Hom(A, X) \rightarrow Ext^1(\frac{I_1 \oplus J}{X}, A) = 0 \ .$$

So $X \longrightarrow I_1 \oplus J$ and similarly $X \longrightarrow I_2 \oplus J$ are DG- injective preenvelopes. Thus there is a commutative diagram;

$$\begin{tikzcd} & I_1 \oplus J \arrow[d] \\ X \arrow[ur] \arrow[r] & I_2 \oplus J \end{tikzcd}$$

Let f be the composition $I_1 \longrightarrow I_1 \oplus J \xrightarrow{f} I_2 \oplus J \longrightarrow I_2$. Then f is a quasi isomorphism. And $f\phi_1 - \phi_2$ can be factored through J, so by Lemma 3.3 $f\phi_1$ is homotopic to ϕ_2 . By Theorem 3.5 $f : I_1 \longrightarrow I_2$ is an isomorphism and uniqueness of f follows from Lemma 3.3.

□

CHAPTER 4

COTORSION PAIRS IN C($\mathcal{R}$-MOD)

Let $S, T : \mathcal{A} \longrightarrow \mathcal{B}$ be covariant functors. A natural transformation $\tau : S \longrightarrow T$ is a one-parameter family of morphisms in $\mathcal{B}$,

$$\tau = (T_A : SA \longrightarrow TA)_{A \in obj\,(\mathcal{A})},$$

making the following diagram commute for all $f : A \longrightarrow A'$ in $\mathcal{A}$:

$$\begin{tikzcd} SA \arrow[r, "\tau_A"] \arrow[d, "S_f"'] & TA \arrow[d, "T_f"] \\ SA' \arrow[r, "\tau_{A'}"'] & TA' \end{tikzcd}$$

Natural transformations between contravariant functors are defined similarly. A natural isomorphism is a natural transformation τ for which each τ_A is an isomorphism.

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be covariant functors. The ordered pair (F, G) is an *adjoint pair* if, for each $C \in obj(\mathcal{C})$ and $D \in obj(\mathcal{D})$, there are bijections

$$\tau_{C,D} : Hom_{\mathcal{D}}(FC, D) \longrightarrow Hom_{\mathcal{C}}(C, GD)$$

that are natural transformations in $\mathcal{C}$ and $\mathcal{D}$.

In more detail, naturality says that the following two diagrams commute for all $f : C' \longrightarrow C$ in $\mathcal{C}$ and $g : D' \longrightarrow D$ in $\mathcal{D}$:

$$\begin{tikzcd} Hom_{\mathcal{D}}(FC, D) \arrow[r, "(Ff)^*"] \arrow[d, "\tau_{C,D}"'] & Hom_{\mathcal{D}}(FC', D) \arrow[d, "\tau_{C',D}"] \\ Hom_{\mathcal{C}}(C, GD) \arrow[r, "(Gf)^*"'] & Hom_{\mathcal{C}}(C', GD) \end{tikzcd}$$

$$\begin{tikzcd}
Hom_{\mathcal{D}}(FC,D) \arrow[r, "(F_g)_*"] \arrow[d, "\tau_{C,D}"'] & Hom_{\mathcal{D}}(FC,D') \arrow[d, "\tau_{C,D'}"] \\
Hom_{C}(C,GD) \arrow[r, "(G_g)_*"'] & Hom_{C}(C,GD')
\end{tikzcd}$$

Let $C \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} C$ be functors. If (F,G) is an adjoint pair, then we say that F has a right adjoint and that G has left adjoint. If $C \in obj(C)$, then setting $D = FC$ gives a bijection

$$\tau_{C,FC} : Hom_{\mathcal{D}}(FC,FC) \longrightarrow Hom_{C}(C,GFC),$$

so that η_C , defined by $\eta_C = \tau(1_{FC})$ is a morphism $C \longrightarrow GFC$. Thus $\eta : 1_C \longrightarrow GF$ is a natural transformation; it is called the unit of the adjoint pair. If $D \in obj(\mathcal{D})$, then setting $C = GD$ gives a bijection

$$\tau^{-1}_{GD,D} : Hom(GD,GD) \longrightarrow Hom(FGD,D)$$

with $\tau^{-1}(1_{GD}) = \varepsilon_D : FGD \longrightarrow D$. Thus $\varepsilon : FG \longrightarrow 1_D$ is a natural transformation. It is called the counit of the adjoint pair.

In 1979 L Salce introduced the notion of cotorsion pair $(\mathcal{A},\mathcal{B})$ in the category of abelian groups. But this definitions and basic results carry over to more general abelian categories and have proved useful in a variety of settings. In this chapter we will consider complete cotorsion pairs $(C,\mathcal{D})$ in the category $C(R - Mod)$ of complexes of left R-modules over some ring R. If $(C,\mathcal{D})$ is such a pair and if C is called under taking suspensions, we will show when we regard C and $\mathcal{D}$ as subcategories of the homotopy category $K(R - Mod)$, then the embedding functors $C \longrightarrow K(R - Mod)$ and $\mathcal{D} \longrightarrow K(R - Mod)$ have left and right adjoints respectively. In finding examples of such pairs we will describe a procedure for using Hovey's results in [9] to find more model structures on $C(R - Mod)$.

Given f , $g \in Hom(C,D)$ we will let $f \cong g$ mean that f and g are homotopic. If we want to indicate the homotopy s we write $f \overset{s}{\cong} g$. The symbol $K(R - Mod)$ will denote the homotopy category of $C(R-Mod)$.

A pair $(\mathcal{C},\mathcal{D})$ of classes of objects of $C(R-Mod)$ will said to be a cotorsion pair in $C(R-Mod)$ if $\mathcal{D}=\mathcal{C}^{\perp}=\{D|Ext^1(C,D)=0$, for all $C\in\mathcal{C}\}$ and if $\mathcal{C}={}^{\perp}\mathcal{D}=\{C|Ext^1(C,D)=0$, for all $D\in\mathcal{D}\}$.

A cotorsion theory $(\mathcal{C},\mathcal{D})$ is said to have enough injectives if for every $X\in C(R-Mod)$ there is an exact sequence $0\longrightarrow X\longrightarrow D'\longrightarrow C'\longrightarrow 0$ with $D'\in\mathcal{D}$ and $C'\in\mathcal{C}$. We say that $(\mathcal{C},\mathcal{D})$ has enough projectives if for every $X\in C(R-Mod)$ there is an exact sequence $0\longrightarrow D\longrightarrow C\longrightarrow X\longrightarrow 0$ with $C\in\mathcal{C}$ and $D\in\mathcal{D}$. The cotorsion pair $(\mathcal{C},\mathcal{D})$ will be said to be complete if $(\mathcal{C},\mathcal{D})$ have both enough injectives and enough projectives.

Notice that if $(\mathcal{C},\mathcal{D})$ is complete cotorsion, then $\mathcal{C}$ and $\mathcal{D}$ are closed under extensions. If we have such sequences and if $\overline{C}\in\mathcal{C}$, then we have the exact sequence

$$Hom(\overline{C},C)\longrightarrow Hom(\overline{C},X)\longrightarrow Ext^1(\overline{C},D)=0\ .$$

This gives that $C\longrightarrow X$ is a $\mathcal{C}$ - precover. Similarly we get that $X\longrightarrow D'$ is a $\mathcal{D}$ - preenvelope.

If C is a complex, then $x\in C$ will mean that $x\in C_n$ for some unique $n\in\mathbb{Z}$. A cotorsion pair $(\mathcal{C},\mathcal{D})$ is said to be hereditary if $Ext^n(C,D)=0$ for all $n\geq 1$ and all $C\in\mathcal{C}$ and $D\in\mathcal{D}$. We will also consider cotorsion pairs $(\mathcal{A},\mathcal{B})$ in the category R -Mod. The definitions and terminology are essentially the same as those above.

Lemma 4.1 *Given $f\in Hom(C,D)$ and the associated exact sequence*

$$0\longrightarrow D\xrightarrow{i} M(f)\xrightarrow{p} C[1]\longrightarrow 0$$

.

If there exists a chain morphism $r: M(f)\longrightarrow D$ being homotopic to the projection morphism $p_1: M(f)\longrightarrow D$, then there exists a (chain) morphism $\theta: M(f)\longrightarrow D$ such that $\theta i=1$.

Proof. Let t be the corresponding homotopy. Then we have the following diagram:

$$\begin{tikzcd}
M(f)^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[d, "p_1^{n-1},r^{n-1}"'] & M(f)^n \arrow[r, "\partial^n"] \arrow[d, "p_1^n,r^n"'] \arrow[dl, dotted, "t^n"'] & M(f)^{n+1} \arrow[d, "p_1^{n+1},r^{n+1}"] \arrow[dl, dotted, "t^{n+1}"'] \\
D^{n-1} \arrow[r, "d^{n-1}"'] & D^n \arrow[r, "d^n"'] & D^{n+1}
\end{tikzcd}$$

where $d^{n-1}t^n + t^{n+1}\partial^n = r^n - p_1^n$ and $\partial^n(x,y) = (-d_1^{n+1}(x), f^{n+1}(x) + d^n(y))$ with $d_1^n : C^n \longrightarrow C^{n+1}$.

Let

$$\theta^n(x,y) = y + t^{n+1}i^{n+1}f^{n+1}(x) + r^n(x,0)$$

for all $(x,y) \in M(f)^n = C^{n+1} \oplus D^n$. Then $\theta^n i^n(y) = \theta(0,y) = y + t^{n+1}i^{n+1}f^{n+1}(0) + r^n(0,0) = y$, that is $\theta i = 1_D$.

Now let's find that θ is a chain map from $M(f)$ to D . For all $(x,y) \in M(f)^{n-1}$,

$\theta^n\partial^{n-1}(x,y) = \theta^n(-d_1^n(x), f^n(x) + d^{n-1}(y)) = f^n(x) + d^{n-1}(y) + t^{n+1}i^{n+1}f^{n+1}(-d_1^n(x)) + r^n(-d_1^n(x),0) = f^n(x) + d^{n-1}(y) - t^{n+1}(0, f^{n+1}d_1^n(x)) - r^n(d_1^n(x),0)$ and $d^{n-1}\theta^{n-1}(x,y) = d^{n-1}(y + t^n i^n f^n(x) + r^{n-1}(x,0)) = d^{n-1}(y) + d^{n-1}t^n(0, f^n(x)) + d^{n-1}r^{n-1}(x,0)$.

Since $d^{n-1}r^{n-1} = r^n\partial^{n-1}$, $d^{n-1}\theta^{n-1}(x,y) = d^{n-1}(y) + d^{n-1}t^n(0, f^n(x)) + r^n\partial^{n-1}(x,0) = d^{n-1}(y) + d^{n-1}t^n(0, f^n(x)) + r^n(-d_1^n(x), f^n(x))$. And also since $d^{n-1}t^n(0, f^n(x)) = -t^{n+1}\partial^n(0, f^n(x))r^n(0, f^n(x)) - p_1^n(0, f^n(x))$, so we have the following equality:

$d^{n-1}\theta^{n-1}(x,y) = d^{n-1}(y) + r^n(-d_1^x(x), f^n(x)) - t^{n+1}\partial^n(0, f^n(x)) + r^n(0, f^n(x)) - p_1^n(0, f^n(x)) = d^{n-1}(y) - r^n(d_1^n(x),0) - t^{n+1}(0, d^n f^n(x))$.

Since $d^n f^n = f^{n+1}d_1^n$, $\theta^n\partial^{n-1} = d^{n-1}\theta^{n-1}$, that is, θ is a chain map such that $\theta i = 1_D$. Thus $0 \longrightarrow D \longrightarrow M(f) \longrightarrow C[1] \longrightarrow 0$ is an split exact sequence.

□

Lemma 4.2 *Given* $f \in Hom(C, D)$ *and the associated exact sequence*

$$0 \longrightarrow D \overset{i}{\longrightarrow} M(f) \overset{p}{\longrightarrow} C[1] \longrightarrow 0 \ .$$

If there exists a chain map $s : C[1] \longrightarrow M(f)$ *being homotopic to the injection* $i_1 : C[1] \longrightarrow M(f)$ *, then there exists a chain map* $\theta : C[1] \longrightarrow M(f)$ *such that* $p\theta = 1$.

Proof. Since $s : C[1] \longrightarrow M(f)$ is chain map for all $x \in C[1]$, $s(x) = (u(x), v(x))$ where $u^n : C^n \longrightarrow C^n$ and $v^n : C^n \longrightarrow D^{n-1}$ and $\partial^{n-1}s^n(x) = -s^{n+1}d_1^n(x)$ for all C^n . So

$$(-d_1^{n+1}(u^n(x)) \ , \ f^n(u^n(x)) \ + \ d^{n-1}(v^n(x))) = (-u^{n+1}(d_1^n(x)) \ , \ -v^{n+1}(d_1^n(x))) \ ... \ (*)$$

Let ω be the associated homotopy such that

$$\omega_1^n : C^n \longrightarrow C^{n-1}$$

and

$$\omega_2^n : C^n \longrightarrow D^{n-2} \ .$$

Thus we have the following equality

$$(-\omega^{n+1}d_1^n + \partial^{n-2}\omega^n)(x) = (x, 0) - (u^n(x), v^n(x)) \ .$$

So we can have

$$-\omega_1^{n+1}d_1^n(x) - d_1^{n-1}(\omega_1^n(x)) = x - u^n(x) \ ,$$

and hence

$$-f^n\omega_1^{n+1}d_1^n(x) - f^n d_1^{n-1}(\omega_1^n(x)) = f^n(x) - f^n(u^n(x))$$

$$-f^n\omega_1^{n+1}d_1^n(x) - d^{n-1}f^{n-1}(\omega_1^n(x)) = f^n(x) - f^n(u^n(x)) \;...\; (**)$$

Let $\theta : C[1] \longrightarrow M(f)$ with for all $x \in C^n$ $\theta^n(x) : (x, f^{n-1}p^{n-2}(\omega^n(x) + v^n(x))$. Then $p\,\theta = 1$. Now we will investigate that θ is a chain map such that $-\theta^{n+1}d_1^n = \partial^{n-1}\theta^n$

$$-\theta^{n+1}d_1^n(x) = (-d_1^n(x)\ ,\ -f^n p^{n-1}\omega^{n+1}(d_1^n(x)) - v^{n+1}(d_1^n(x)))$$

$$= (-d_1^n(x)\ ,\ -f^n\omega_1^{n+1}(d_1^n(x))\ +\ f^n(u^n(x))\ +\ d^{n-1}(v^n(x)))$$

by $(*)$.

$$\partial^{n-1}\theta^n(x) = \partial^{n-1}(x\ ,\ f^{n-1}p^{n-2}\omega^n(x)\ +\ v^n(x))$$

$$= (-d_1^n(x)\ ,\ f^n(x)\ +\ d^{n-1}f^{n-1}\omega_1^n(x)\ +\ d^{n-1}v^n(x))$$

By $(**)$

$$\partial^{n-1}\theta^n(x) = (-d_1^n(x), -f^n\omega_1^{n+1}d_1^n(x) + f^n(u^n(x)) + d^{n-1}v^n(x)) = -\theta^{n+1}d_1^n(x)$$

for all $x \in C^n$.

So θ is a chain map such that $p\,\theta = 1$.

(That is, $p : M(f) \longrightarrow C[1]$ has a section in $C(R-Mod)$.)

$\square$

Lemma 4.3 *If* $C, D \in C(R{-}Mod)$ *and if* $Ext^1(C[1], D) = 0$ *, then every* $f : C \longrightarrow D$ *is homotopic to* 0 .

Proof. By Lemma 2.2 , we know that $0 \longrightarrow D \longrightarrow M(f) \longrightarrow C[1] \longrightarrow 0$ is split exact if and only if $f \cong 0$. Since $Ext^1(C[1], D) = 0$, the sequence

$$0 \longrightarrow Hom(C[1], D) \longrightarrow Hom(M(f), D) \longrightarrow Hom(D, D) \longrightarrow 0$$

is exact. So $0 \longrightarrow D \longrightarrow M(f) \longrightarrow C[1] \longrightarrow 0$ is split exact. By the explanation above $f \cong 0$.

□

Proposition 4.4 *Suppose that* $(\mathcal{C}, \mathcal{D})$ *is a complete cotorsion pair in* $\mathcal{C}(R - Mod)$ *where* $\mathcal{C}$ *is closed under taking suspensions. For* $X \in \mathcal{C}(R - Mod)$ *let* $0 \longrightarrow D \longrightarrow C \xrightarrow{u} X \longrightarrow 0$ *be exact where* $C \in \mathcal{C}$ *and* $D \in \mathcal{D}$ *. Then if* $C' \in \mathcal{C}$ *and* $f_i \in Hom(C', X)$ *for* $i = 1, 2$ *let* $g_i \in Hom(C', C)$ *for* $i = 1, 2$ *be such that*

$$\begin{tikzcd} C' \arrow[d, "g_i"'] \arrow[dr, "f_i"] & \\ C \arrow[r] & X \end{tikzcd}$$

is commutative for $i = 1, 2$ *. Then if* $f_1 \cong f_2$ *, then we have* $g_1 \cong g_2$ *.*

Proof. Letting $f = f_1 - f_2$ and $g = g_1 - g_2$ we see that we only need to show that when $f \cong 0$ we have $g \cong 0$. Using the commutative diagram we get with such an f and g we get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "u"] & M(g) \arrow[r] \arrow[d, "t"] & C'[1] \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & X \arrow[r, shift left, "i'"] & M(f) \arrow[l, shift left, "k'"] \arrow[r] & C'[1] \arrow[r] & 0 \end{tikzcd}$$

Since $f \cong 0$, we get that the lower short exact sequence splits such that $k'i' = 1$.

Then we have the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "u"'] & M(g) \arrow[dl, "k't"] \\ X & \end{tikzcd}$$

such that $k'ti = u$. Since $\mathcal{C}$ is closed under extensions and suspensions we have $M(g) \in \mathcal{C}$. Since $C \longrightarrow X$ is a $\mathcal{C}$-precover we get a lifting $k : M(g) \longrightarrow C$ such that $uk = k't$ and $uki = u$.

Now we want to prove that $C \longrightarrow M(g) \longrightarrow C$ is homotopic to id_C (the identity map on C). We have the commutative diagram

$$\begin{tikzcd} C \arrow[d, "i"'] \arrow[dr, "u"] & \\ M(g) \arrow[r, "k't"] \arrow[d, "k"'] & X \\ C \arrow[ur, "u"'] & \end{tikzcd}$$

Let h denote the composition $C \xrightarrow{i} M(g) \xrightarrow{k} C$. Then we have $u \circ id_C = u$ and $u \circ h = u$. So we get $u \circ (id_C - h) = 0$. This gives that $Im(id_C - h) \subseteq Ker(u) = D$. So $id_C - h : C \longrightarrow D$. Since $Ext'(C[1], D) = 0$, by Lemma 4.3 $id_C - h$ is homotopic to 0. By Lemma 4.2 we can find a morphism $\phi : M(g) \longrightarrow C$ such that $\phi i = 1$. So $0 \longrightarrow C \longrightarrow M(g) \longrightarrow C'[1] \longrightarrow 0$ is split exact. By Lemma 2.2 $g \cong 0$.

□

Theorem 4.5 *If* $(\mathcal{C}, \mathcal{D})$ *is a complete cotorsion pair in* $\mathcal{C}(R-Mod)$ *and if* $\mathcal{C}$ *is closed under taking suspensions, then the embeddings* $\mathcal{C} \longrightarrow \mathcal{K}(R-Mod)$ *and* $\mathcal{D} \longrightarrow \mathcal{K}(R-Mod)$ *have right and left adjoints, respectively.*

Proof. A right adjoint T of $\mathcal{C} \longrightarrow \mathcal{K}(R-Mod)$ can be constructed as follow. For $X \in \mathcal{C}(R-Mod)$, let $0 \longrightarrow D \longrightarrow C \longrightarrow X \longrightarrow 0$ be exact in $\mathcal{C}(R-Mod)$ with

$C \in \mathcal{C}$ and with $D \in \mathcal{D}$. Letting $T(X) = C$ and using Proposition 4.4 we see that we get the desired functor. The argument for a left adjoint of $\mathcal{D} \longrightarrow \mathcal{K}(R - Mod)$ is dual to this argument.

□

Remark 4.6 *If D above is exact, then $T(X) = C \longrightarrow X$ is a homology isomorphism. So if all such D are exact the counit of the adjunction consists of homology isomorphisms. If all $C \in \mathcal{C}$ are exact, we get a similar claim about the unit of the left adjoint of $\mathcal{D} \longrightarrow \mathcal{K}(R - mod)$.*

4.1 Examples and Applications

In this section we point out how we can find complete cotorsion pairs in $\mathcal{C}(R - Mod)$. One of the principal tool in constructing complete cotorsion pairs in $R - Mod$ is the Eklof - Trlifaj Theorem (see [10] or [4] , Theorem 7.4.1) They proved that S is any set of left R- modules, then if $\mathcal{B} = S^{\perp}$, there is a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$. In this situation we say that the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is generated by a set (some says ”cogenerated by a set”). We note that the Eklof - Trlifaj proof works as well in the category $\mathcal{C}(R - Mod)$, and so is a source of complete cotorsion pairs in this category. If such a pair $(\mathcal{C}, \mathcal{D})$ in $\mathcal{C}(R - Mod)$ is cogenerated by the set S of complexes and if S is closed under taking suspensions, then so is $\mathcal{C}$. Hence we can then apply Theorem 4.5 and get the associated adjoint functors. We now recall a method of Gillespie for creating cotorsion pairs in $\mathcal{C}(R - Mod)$ from pairs in $R - Mod$. Here ε is the class of exact complexes.

Proposition 4.7 *(Gillespie, Proposition 4.4 of [11]) If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $R - Mod$ which is generated by a set, then $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}) \cap \varepsilon$ are the second components of cotorsion pairs in $\mathcal{C}(R - Mod)$ which are generated by a set.*

It is also true $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{A}) \cap \varepsilon$ are the first components of cotorsion pairs cogenerated by sets. We will not give the proof but point out that this can be shown using

Stovicek and Trilifaj's version of the Hill Lemma (see [12]). And also we will use the symbol $R-$Proj to denote the category of projective R-modules. Then we will use other variations of this notation.

Example 4.8 *We have the complete cotorsion pair (R-Proj, R-Mod) which generated by the set R. Then* $(C(R-Proj), C(R-Proj)^{\perp})$ *is a cotorsion pair which is generated by a set by the explanation above. So it is a complete cotorsion pair by Theorem 7.4.1 in [4]. Consequently by Theorem 4.5* $\mathcal{K}(R-Proj) \longrightarrow \mathcal{K}(R-Mod)$ *has a right adjoint.*

Example 4.9 $(R-Mod, R-Inj)$ *is generated by the set* $\frac{R}{I}$ *where* I *is a left ideal of* R *by Theorem 3.1.9 in [4]. Then using Proposition 4.1 and proceeding as in Example 4.8 we get that* $\mathcal{K}(R-Inj) \longrightarrow \mathcal{K}(R-Mod)$ *has a right adjoint.*

Example 4.10 *We know that an R-module* F *is said to be flat if given any exact sequence* $0 \longrightarrow A \longrightarrow B$ *of right R-modules, the tensored sequence* $0 \longrightarrow A \otimes_R F \longrightarrow B \otimes_R F$ *is exact. An R-module* M *is said to be cotorsion if* $Ext^1(F, M) = 0$ *for all flat R-modules* F. $(R-Flat, R-Cotorsion)$ *is a cotorsion pair. By Proposition 7.4.3 and Theorem 7.4.1 in [4] this cotorsion pair is a complete cotorsion pair. Similar arguments give that* $\mathcal{K}(R-Flat) \longrightarrow \mathcal{K}(R-Mod)$ *has a right adjoint. This was proved also Neeman by other methods in [13] (see also [14]). Starting with the same cotorsion pair of modules we also get that* $\mathcal{K}(R-Cotorsion) \longrightarrow \mathcal{K}(R-Mod)$ *has left adjoint.*

Example 4.11 *In [11] and [15] Gillespie gives another method for constructing complete cotorsion pairs in* $C(R-Mod)$ *from certain cotorsion pairs* (A, B) *in* $R-Mod$*. Each instance of these supplies us with adjoints.*

Example 4.12 *To get examples of complete cotorsion pairs in* $C(R-Mod)$ *that don't necessarily come from pairs in* $R-Mod$ *one could start with a set* S *of complexes closed under taking suspensions and then generating a complete cotorsion pair. For example if* $K \in C(R-Mod)$ *is a Koszul complex we could let* S *consist of all the* $S^n(K)$ *where* $n \geq 0$*. In [16] one can find concrete descriptions of such cotorsion pairs albeit* *in a different setting. But the translation to this setting is easy. In this construction we could also use any set of Koszul complexes.*

4.2 Hovey Pairs

Definition 4.13 *We will say that $(\mathcal{C} \cap \varepsilon, \mathcal{D})$ and $(\mathcal{C}, \mathcal{D} \cap \varepsilon)$ form a Hovey Pair if each of these pairs is a complete cotorsion pair in $\mathcal{C}(R - Mod)$.*

Hovey proved (Theorem 2.2 [9]) that every such pair gives rise to a model structure on $\mathcal{C}(R - Mod)$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{D}$ is the class of fibrant objects and ε is the class of trivial objects of model structure. Here it will be given more examples of Hovey pairs. All cotorsion pairs here will be in the category $\mathcal{C}(R - Mod)$. In Theorem 4.7 [11] Gillespie showed how to use the $\mathcal{B}$ of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $R - Mod$ which is generated by a set to create a Hovey pair in $\mathcal{C}(R - Mod)$. In that article he suggested that there is an analogous way to use the $\mathcal{A}$ to generate Hovey pairs.

Lemma 4.14 *If $(\mathcal{C}, \mathcal{D}')$ is a hereditary cotorsion pair and $(\mathcal{U}, \mathcal{V})$ is a complete and hereditary cotorsion pair where both are in $R - Mod$ or both in $\mathcal{C}(R - Mod)$ and are such that $\mathcal{U} \subseteq \mathcal{C}$, then if $(\mathcal{C} \cap \mathcal{V})^{\perp} = \mathcal{D}$, we have $\mathcal{D}' = \mathcal{D} \cap \mathcal{V}$.*

Proof. Since $\mathcal{C} \cap \mathcal{V} \subseteq \mathcal{C}$, $\mathcal{D}' = \mathcal{C}^{\perp} \subseteq (\mathcal{C} \cap \mathcal{U})^{\perp} = \mathcal{D}$. Since $\mathcal{U} \subset \mathcal{C}$ we have $\mathcal{D}' = \mathcal{C}^{\perp} \subseteq \mathcal{U}^{\perp} = \mathcal{V}$. Let $D \in \mathcal{D} \cap \mathcal{V}$. We want to show that $D \in \mathcal{D}' = \mathcal{C}^{\perp}$, i.e. that $Ext^1(C, D) = 0$ for all $C \in \mathcal{C}$. Since $(\mathcal{U}, \mathcal{V})$ is complete, for any $C \in \mathcal{C}$ we have an exact sequence $0 \longrightarrow C \longrightarrow V \longrightarrow U \longrightarrow 0$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $C \in \mathcal{C}$ and $U \in \mathcal{U} \subseteq \mathcal{C}$, we have $V \in \mathcal{C}$ since $\mathcal{C}$ is closed under extensions. So finally we get $V \in \mathcal{C} \cap \mathcal{V}$. This gives that $Ext^1(V, D) = 0$. Since we have the exact $Ext^1(V, D) \longrightarrow Ext^1(C, D) \longrightarrow Ext^2(U, D)$ and by hereditary condition $Ext^2(U, D) = 0$, $Ext^1(C, D) = 0$. This gives that $D \in \mathcal{D}'$. So we have $\mathcal{D}' = \mathcal{D} \cap \mathcal{V}$. $\square$

A dual argument gives the next result:

Lemma 4.15 *If* $(\mathcal{C}', \mathcal{D})$ *is a hereditary cotorsion pair and* $(\mathcal{U}, \mathcal{V})$ *is a complete and hereditary cotorsion pair and if* $\mathcal{V} \subset \mathcal{D}$ *where* $\mathcal{C} = ^{\perp}(\mathcal{D} \cap \mathcal{U})$*, then* $\mathcal{C}' = \mathcal{C} \cap \mathcal{U}$*.*

Proposition 4.16 *If* (A, B) *is a cotorsion pair generated by a set in* $R - Mod$*, then if* $\mathcal{D} = (C(A) \cap \varepsilon)^{\perp}$ *we have that* $(C(A) \cap \varepsilon, \mathcal{D})$ *and* $(C(A), \mathcal{D} \cap \varepsilon)$ *are a Hovey pair.*

Proof. By Proposition 2.6 $^{\perp}\varepsilon$ is the class of DG-Projective complexes. By Proposition 2.11 we can say that $(^{\perp}\varepsilon, \varepsilon)$ is a cotorsion pair. By Corollary 2.14 and Lemma 2.33 we understand that $(^{\perp}\varepsilon, \varepsilon)$ is a complete cotorsion pair. By Remark 2.8 we can say that $(^{\perp}\varepsilon, \varepsilon)$ is a hereditary complete cotorsion pair. In Lemma 4.14 let $(^{\perp}\varepsilon, \varepsilon)$ be the $(\mathcal{U}, \mathcal{V})$. Since (A, B) is a cotorsion pair generated by a set in $R - Mod$, by the explanations above we say that $(C(A) \cap \varepsilon, \mathcal{D})$ is a cotorsion pair generated by a set, so it is a complete cotorsion pair. Since the class of DG-projective complexes is generated by the complex $\cdots \longrightarrow O \longrightarrow R \longrightarrow 0 \longrightarrow \cdots$ (with R in the 0-th place) and all its suspensions (both positive and negative) and also A contains all the projective objects, we have $R \in A$ and so all these suspensions are in $C(A)$ and also $^{\perp}\varepsilon \subseteq C(A)$. Now let $(C, D') = (C(A), C(A)^{\perp})$ in Lemma 4.14 and we see that the claim holds. □

REFERENCES

[1] Rotman, Joseph J. *An Introduction to Homological Algebra*, Springer, New York (2009).

[2] Avramov, L., Foxby, H. B. *Homological Dimensions of Unbounded Complexes,* J. Pure and Applied Algebra, 71 (1991), 129-155.

[3] Dold, A. *Zur Homtietheorie der Ketten Komplexe*, Math. Ann. 140 (1960), 278-298.

[4] Enochs, E.E., Jenda, O.M.G. *Relative Homological Algebra*, de Gruyter Ex. Math. Volume 30, Walter de Gruyter and Co., Berlin (2000).

[5] Spaltenstein, N. *Resolutions of Unbounded Complexes*, Composito Math. 65 (1988), 121-154.

[6] Enochs, E.E., Jenda, O.M.G., Xu, J. *Covers and envelopes over Gorenstein rings,*Tsukuba J. Math. 20 (1966), 487-503.

[7] Enochs, E.E., Rozas, J.R. Garcia *Exact Envelopes of Complexes*, Communication in Algebra 27 (4) , (1999), 1615-1627.

[8] Foxby, H.B. *A Homological Theory Of Complexes Of Modules* ,Preprint Series No.19, Dept. of Mathematics, Univ. Copenhagen(1981).

[9] Hovey, M. *Cotorsion Theories, Model Category Structures and Representation Theory* , Math. Zeit 241 (2002), 553-592.

[10] Eklof, P.C. and Trlifaj, J. *How to make Ext vanish* , Bull. London Math Soc. 33, (2001), 41 - 51.

[11] Gillespie, J. *Cotorsion pairs and degreewise homological model structures, Homology, Homotopy and Aplications 10* , (2008), 283-304.

[12] Stovicek, J., Trlifaj, J. *Generalized Hill Lemma, Kaplansky Theorem for cotorsion pairs and some applications* , Rocky Mt. J. Math. 39 (2009), 305-324.

[13] Neeman A. *The Homotopy Category of a Flat Modules and Grothendieck Duality* , Invent. Math. 174 (2008), 255-308.

[14] Neeman A. *Some Adjoints in Homotopy Categories* , to Annals of Math.

[15] Gillespie, J. *The Flat Model Structure on Complexes of Sheaves* , Trans. Amer. Math. Soc. 358 (2006), 2855-2874.

[16] Dwyer, W.G., Greenlees, J.P.C. *Complete and Torsion modules* , Amer.J. Math. 124 (2002), 199-220.