POINTWISE CONVERGENCE OF CONVOLUTION TYPE SINGULAR INTEGRAL OPERATORS

by

KÜBRA USLU

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Prof. Dr. Nihat Çelebi
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Zafer Ercan
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Harun Karslı
Supervisor

Examining Committee Members

Prof. Dr. Zafer Ercan	(AİBÜ)	
Assoc. Prof. Dr. Harun Karslı	(AİBÜ)	
Assist. Prof. Dr. Turhan Köprübaşı	(KU)	

ABSTRACT

Uslu, Kübra

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Supervisor: Assoc. Prof. Dr. Harun Karslı

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This thesis is a survey on some approximation properties of singular integral operators in the approximation theory. One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. It is to be expected that the better behaved functions are to be constructed from the given f by some smoothing operation on f itself. This thesis consists of four chapters. The first chapter is devoted to the introduction. The second chapter contains, concepts and definitions which are needed in the further chapters. In the third chapter, studied convergence of linear convolution type singular integrals. In the fourth chapter, investigate convergence of nonlinear convolution type singular integral operators. And there is a study of Assoc. Prof. Dr. Harun Karslı which deals with the approximation properties of nonlinear singular integral operators of convolution type.

Keywords: Pointwise convergence, Singular integrals, Dirichlet kernel, Modulus of continuity.

ÖZET

KONVOLÜSYON TÍPLÍ SÍNGÜLER ÍNTEGRALLERDE NOKTASAL YAKINSAKLIK

Uslu, Kübra
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Bu tez, yaklaşım teorisindeki singüler integral operatörlerinin yaklaşım özellikleri üzerine yapılan bir çalışmadır. Analizin temel problemlerinden biri f gibi bazı kötü özelliklere sahip bir fonksiyona, daha iyi özellikleri olan başka bir fonksiyonla yaklaşmaktır. Bu tez dört bölümden oluşmaktadır. Birinci kısım giriş bölümüne ayrılmıştır. İkinci bölümde ileri bölümlerde gerekli olan kavramlar ve tanımlar verilmiştir. Üçüncü bölümde lineer konvolüsyon tipli singüler integral operatörlerinde yakınsaklık ve bazı özellikleri verildi. Dördüncü bölümde ise lineer olmayan konvolüsyon tipli singüler integral operatörlerinde yakınsaklık ve bazı özellikleri verildi. Ve Doç. Dr. Harun Karslı' nın çalışması olan "Lineer Olmayan Konvolüsyon Tipli Singüler Integral Operatörlerinde Yakınsaklık" adlı makale incelendi.

Anahtar Kelimeler: Noktasal yakınsaklık, Singüler integral, Dirichlet çekirdeği, Süreklilik modülü.

To my family and my supervisor

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I would like to express sincere appreciation to my supervisor, Assoc. Prof. Dr. Harun Karslı for his motivation, helpful discussions, encouragement, patience and constant guidance during this work. I would also like to thank other examining committee members for their suggestions.

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TABLE OF CONTENTS

LIST OF SYMBOLS

- $|x|$ — Absolute value
- $[|x|]$ — Integral part of the real number x
- $X(\mathbb{R})$ — $X(\mathbb{R})$ denotes one of the spaces C or L_p, $1 \leq p < \infty$
- $C[a,b]$ — Space of continuous functions which is defined on $[a,b]$
- $\mathbb{R}$ — Real line or the field of real numbers
- $||x||$ — Norm of x
- $||.||_{C[a,b]}$ — A norm defined by $||.|| = \max_{a\leq x\leq b} |.|$
- $\omega(f;\delta)$ — Modulus of continuity of f: $\omega(f;\delta) = \sup_{|h|<\delta} ||f(o+h)-f(o)||_{X_{2\pi}}$.
- $f * g$ — Convolution of f and g: $(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u)g(u)du$
- $F_n(x)$ — Kernel: $\sigma_n(f;x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{\sin \frac{n}{2}(t-x)}{\sin \frac{1}{2}(t-x)} \right]^2 dt$
- $D_n(x)$ — Dirichlet Kernel: $D_N(x) = \frac{1}{2\pi} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})}$

CHAPTER 1

INTRODUCTION

One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. It is to be expected that the better be haved functions are to be constructed from the given f by some smoothing operation on f itself.
Let X the set of functions which had bad properties and there exist a set $Y \subset X$. Y is the set of functions with good properties and Y is dense in X ($\overline{Y} = X$). In this case for all points of X are *accumulation point* of Y.
Weierstrass studied with this problem in 1885. He took the set of bad property functions of X as $C[a,b]$. Then he determined the set of good properties functions of Y which is dense in X is $P(x)$. The mean of this is; every function defined on $C[a,b]$ is the limit of $P_n(x)$.

CHAPTER 2

PRELIMINARIES

2.1 Some Important Definitions

Definition 2.1 *($C[a,b]$). As a set X we take the set of all real- valued functions $x, y, \cdots$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = C[a,b]$. Choosing the metric defined by*

$$d(x,y) = \max_{t \epsilon J} |x(t) - y(t)|,$$

where max denotes the maximum, we obtain a metric space which is denoted by $C[a,b]$. This is a function space because every point of $C[a,b]$ is a function.

Definition 2.2 *A polynomial in the variable x is a function of the form*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$ and a_0 are constants and $a_n \neq 0$.
The number n is the degree of the polynomial $P(x)$.

Definition 2.3 *(Accumulation Point) If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point (or accumulation point) of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter.*

Definition 2.4 *We Say that $f(x)$ approaches the limit L as x approaches a, and we write*

$$\lim_{x \to a} f(x) = L$$

if the following condition is satisfied:

for every number $\varepsilon > 0$ *there exists a number* $\delta > 0$*, depending on* ε*, such that*

$$0 < |x - a| < \delta \quad implies \quad |f(x) - L| < \varepsilon.$$

Definition 2.5 *The sequence* f_n *uniformly converge to* f *on* $A \Leftrightarrow \forall \varepsilon > 0 \quad \exists n_0$ *such that* $n \geq n_0$ *and* $\forall x \epsilon A \quad , |f_n(x) - f(x)| < \varepsilon$

Definition 2.6 *(Topological Space) A topology on a set X is a collection* τ *of subsets of X having the following properties:*

1. $\emptyset$ *and X are in* τ*.*

2. *The union of the elements of any subcollection of* τ *is in* τ

3. *The intersection of the elements of any finite subcollection of* τ *is in* τ

A set X for which a topology τ *has been specified is called a topological space*

Definition 2.7 *(Linear Operator). A linear operator* T *is an operator such that*

- *The domain* $D(T)$ *of T is a vector space and the range* $R(T)$ *lies in a vector space over the same field,*

- *for each* $x, y \epsilon D(T)$ *and scalar* α*,*

$$T(x + y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha Tx$$

(Kreyszig, 1989)

Definition 2.8 *(Integral Operator). We can define an integral operator* $T : C[0, 1] \rightarrow C[0, 1]$ *by*

$$y = Tx, \qquad where \qquad y(t) = \int_0^1 x(t)k(t, \tau)d\tau.$$

Here k is a given function which is called the kernel of T and is assumed to be continuous on the closed spuare $G = J \times J$ in the $t\tau -$ plane, where $J[0, 1]$. This operator is linear.

(Kreyszig, 1989)

Definition 2.9 *($L_p(-\infty, \infty)$ Space). L_p is the set of functions which are Lebesgue integrable to the p^{th} power over $\mathbb{R}$, for $1 \leq p < \infty$, and essentially bounded (bounded almost everywhere) on $\mathbb{R}$ if $p = \infty$. For $f \epsilon L_p(-\infty, \infty)$*

$$\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p dt \right\}^{\frac{1}{p}},$$

if $1 \leq p < \infty$ *and in case* $p = \infty$

$$\|f\|_\infty = ess \sup_{x \epsilon \mathbb{R}} |f(t)|.$$

(Butzer, 1971)

Definition 2.10 *($L_p[a, b]$ Space). $L_p[a, b]$ is the set of functions which are Lebesgue integrable to the p^{th} power of $[a, b]$. For $f \epsilon L_p[a, b]$*

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{1\backslash p}.$$

Definition 2.11 *($L_1[a, b]$ Space). $L_1[a, b]$ is the set of functions which are Lebesgue integrable to the first power of the period $[a, b]$. For $f \epsilon L_1[a, b]$*

$$\|f\|_1 = \int_a^b |f(t)| dt.$$

(Butzer, 1971)

Definition 2.12 *($L_p[-\pi, \pi]$ Space). $L_p[-\pi, \pi]$ is the set of functions which are Lebesgue integrable to the p^{th} power of the period $[-\pi, \pi]$. For $f \epsilon L_p[-\pi, \pi]$*

$$\|f\|_p = \left\{ \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{1/p}.$$

Definition 2.13 *($X(\mathbb{R})$ Space). $X(\mathbb{R})$ always denotes one of the spaces C or L_p, $1 \leq p < \infty$. For $f, g \epsilon X(\mathbb{R})$ we write $f(x) = g(x)$ (a.e.) if equality holds for all $x \epsilon X(\mathbb{R})$ in case $X(\mathbb{R}) = C$, and almost everywhere in case $X(\mathbb{R}) = L_p$, $1 \leq p < \infty$, i.e., if $\|f - g\| = 0$. In this event we also write $f = g$ in $X(\mathbb{R})$.*

Definition 2.14 *(Minkowski' s Inequality). Let $f, g \epsilon X(\mathbb{R})$. Then $(f + g) \epsilon X(\mathbb{R})$ and*

$$\|f + g\|_{X(\mathbb{R})} \leq \|f\|_{X(\mathbb{R})} + \|g\|_{X(\mathbb{R})}.$$

If p is such that $1 \leq p \leq \infty$, the conjugate number q is defined through $(1/p)+(1/q) = 1$ in case $1 < p < \infty$; $q = \infty$ if, $p = 1$, and $q = 1$ if $p = \infty$.

Definition 2.15 *(Hölder' s Inequality). Let $f \epsilon L_p$, $1 \leq p \leq \infty$, and $g \epsilon L_q$. Then $fg \epsilon L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$. Namely;*

$$\left| \int fg \right| \leq \left(\int |f|^p \right)^{1/p} . \left(\int |g|^q \right)^{1/q} \tag{2.1}$$

Proof. *We first note that if $\int |f|^p = 0$, then $|f|^p = 0$ almost everywhere; so $f = 0$, and therefore $fg = 0$, almost everywhere. Then fg is integrable, $\int fg = 0$, and (2.1) holds trivially, as it does also in the case where $\int |g|^q = 0$. Thus we may assume that $\int |f|^p > 0$ and $\int |g|^q > 0$. We then have, almost everywhere,*

$$\begin{aligned} \frac{|fg|}{(\int |f|^p)^{1/p} (\int |g|^q)^{1/q}} &= \left(\frac{|f|^p}{\int |f|^p} \right)^{1/p} \left(\frac{|g|^q}{\int |g|^q} \right)^{1/q} \\ &\leq \frac{|f|^p}{p \int |f|^p} + \frac{|g|^q}{q \int |g|^q} \end{aligned}$$

So;

$$|fg| \leq \left(\int |f|^p \right)^{1/p} . \left(\int |g|^q \right)^{1/q} . \frac{|f|^p}{p \int |f|^p} + \frac{|g|^q}{q \int |g|^q} \tag{2.2}$$

Now, fg is measurable and the righthand side of (2.2) is integrable. Hence, fg is *integrable and*

$$\left|\int fg\right| \leq \int |fg| \leq \left(\int |f|^p\right)^{1/p} \cdot \left(\int |g|^q\right)^{1/q} \cdot \left(\frac{1}{p} + \frac{1}{q}\right),$$

from which (2.1) *follows.* □

Let us emphasize that the assumption $f \epsilon L_p$, $1 \leq p \leq \infty$, *means that* f *belongs to* L_p *for some fixed* p *with* $1 \leq p \leq \infty$ *(and not for all* p *of this interval).*

Definition 2.16 *(Hölder- Minkowski Inequality). Let* $f(x,y)$ *be defined and measurable on* $\mathbb{R}^2$. *If* $\|f(x,y)\|_{X(\mathbb{R})} \epsilon L_1$, *then*

$$\left\|\int_{-\infty}^{\infty} f(x,y)dy\right\|_{X(\mathbb{R})} \leq \int_{-\infty}^{\infty} \|f(x,y)\|_{X(\mathbb{R})} dy.$$

This is also known as the generalized Minkowski inequality.

Definition 2.17 *Let* Λ *is an index set and* λ_0 *is accumulation point of the set* Λ. *For* $\lambda \epsilon \Lambda$;

$$L(f;x,\lambda) = \int_D f(t)H(t-x,\lambda)dt, \qquad x \epsilon D$$

This integral is called "the family of convolution type integral operator depends on parameters (x,λ)*".*

Definition 2.18 *Let* Λ *is an index set and* λ_0 *is accumulation point of the set* Λ. *If the function* $K(t,\lambda)$ *is satisfied the conditions;*

- *There exist a number* M *such that* $\forall \lambda \epsilon \Lambda$, $\|K(t,\lambda)\|_1 \leq M < \infty$
- $\lim_{\lambda \to \lambda_0} \int_D K(t,\lambda)dt = 1.$

Definition 2.19 *When the kernel function* $K(t,\lambda)$ *satisfies the* $\lim_{\lambda \to \lambda_0} K(t_0,\lambda) = \infty$ *at the point* t_0, *then we say the kernel function* $K(t,\lambda)$ *is "Singular Kernel" and,*

$$L(f;x,\lambda) = \int_D f(t)K(t-x,\lambda)dt, \qquad x \epsilon D$$

The family of integral operators are called the family of convolution type singular integral operator depends on parameters (x, λ).

Definition 2.20 *If the singular kernel function* $K(t, \lambda)$ *satisfies the condition;*

$$\lim_{\lambda \to \lambda_0} \left[\sup_{\delta \leq |t|} |K(t, \lambda)| \right] = 0 \qquad \forall \delta > 0.$$

then the function is called "Approximation Identity".

Some Examples

Example 2.21 *(Abel-Poisson Kernel) Laplace operator in the plane is defined as,*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $\Delta u = 0$ *equation is called Laplace equation.*

The problem of finding the function u which is provide Laplace equation in the upper half plane and equal to $u(x, 0) = f(x)$ *on the real axis is the Dirichlet problem for the upper half-plane and the solution of this problem;*

$$f_\varepsilon(x) = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1}{\varepsilon^2 + (t - x)^2} dt, \qquad \varepsilon > 0 \tag{2.3}$$

is given by integration as (2.3).

By taking derivatives;

$$\frac{\partial^2 f_\varepsilon(x)}{\partial x^2} + \frac{\partial^2 f_\varepsilon(x)}{\partial \varepsilon^2} = 0$$

equation can be achieved. Now show this;

$$\frac{\partial f_\varepsilon(x)}{\partial x} = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{2(t - x)}{[\varepsilon^2 + (t - x)^2]^2} dt$$

$$\frac{\partial^2 f_\varepsilon(x)}{\partial x^2} = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{-2[\varepsilon^2 + (t - x)^2]^2 + 2[\varepsilon^2 + (t - x)^2].2(t - x).2(t - x)}{[\varepsilon^2 + (t - x)^2]^4} dt$$

$$= \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{8(t-x)^2 - 2[\varepsilon^2 + (t-x)^2]}{[\varepsilon^2 + (t-x)^2]^3} \tag{2.4}$$

$$\frac{\partial f_\varepsilon(x)}{\partial \varepsilon} = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1}{\varepsilon^2 + (t-x)^2} dt + \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{-2\varepsilon}{[\varepsilon^2 + (t-x)^2]^2} dt$$

$$\frac{\partial^2 f_\varepsilon(x)}{\partial \varepsilon^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{-2\varepsilon}{[\varepsilon^2 + (t-x)^2]^2} dt + \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{-2\varepsilon}{[\varepsilon^2 + (t-x)^2]^2} dt$$

$$+ \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{-2[\varepsilon^2 + (t-x)^2]^2 + 2\varepsilon.2[\varepsilon^2 + (t-x)^2].2\varepsilon}{[\varepsilon^2 + (t-x)^2]^4} dt$$

$$= \frac{-4\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{dt}{[\varepsilon^2 + (t-x)^2]^2} + \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{8\varepsilon^2 - 2[\varepsilon^2 + (t-x)^2]}{[\varepsilon^2 + (t-x)^2]^3} dt \tag{2.5}$$

is obtained. Is (2.4) *and(2.5) are gathered;*

$$\frac{\partial^2 f_\varepsilon(x)}{\partial x^2} + \frac{\partial^2 f_\varepsilon(x)}{\partial \varepsilon^2} = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{8[\varepsilon^2 + (t-x)^2] - 4[\varepsilon^2 + (t-x)^2]}{[\varepsilon^2 + (t-x)^2]^3} dt$$

$$- \frac{4\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{dt}{[\varepsilon^2 + (t-x)^2]^2}$$

$$= \frac{4\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{dt}{[\varepsilon^2 + (t-x)^2]^2} - \frac{4\varepsilon}{\pi} \int_{-\infty}^{\infty} f(t) \frac{dt}{[\varepsilon^2 + (t-x)^2]^2} = 0$$

is found.

(2.3) *is called a Abel-Poisson integral;*

$$A_\varepsilon(x) = \frac{\varepsilon}{\pi} \frac{1}{\varepsilon^2 + x^2} \tag{2.6}$$

expression is called the Abel-Poisson kernel. $A_\varepsilon(x)$ *is a non-negative and even function.*

Example 2.22 *(The Poisson Kernel a in the Unit Circle)*

One of the summability techniques is "Poisson Summability Technique". According to *this technique, to analyse the convergence of the series;*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \qquad 0 < r < 1$$

Now, take the series at the fixed point $x = x_0$;

$$P(r, x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{2.7}$$

When $P(r, x_0) = L$, the series (2.7) *is summable to L at x_0 by the Poisson technique. Privately, when $L = f(x_0)$, the series* (2.7) *is converge to $f(x_0)$ by the Poisson technique.*

Now; let find the summation of the series (2.7)*, when $f(x)$ is $2\pi-$periodic and integrable on $[-\pi, \pi]$. The Fourier coefficients at $f(x)$ converges zero as $k \to \infty$. Namely;*

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$$

According to this;

$$|a_k| \leq M, \quad |b_k| \leq M, \quad k = 0, 1, 2, \cdots$$

there exist a number M satisfies this.

Also, $|\sin kx| \leq 1$, $|\cos kx| \leq 1$;

$$|P(r, x)| \leq M + \sum_{k=1}^{\infty} 2Mr^k = M\left(1 + 2\sum_{k=1}^{\infty} r^k\right)$$

So the series (2.7) *is always convergent.*

Now, if we denote the series with the integral benefit from this convergence,

$$P(r, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{\infty} r^k \int_{-\pi}^{\pi} \cos k(t - x) f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos k(t - x)\right] f(t)dt \tag{2.8}$$

Let say $t - x = \theta$ *and edit the series* $\frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos k(\theta)$, *when* $0 < r < 1$ *and* $z = re^{i\theta}$*;*

$$\begin{aligned}
\sum_{k=0}^{\infty} z^k &= \frac{1}{1-z} \\
&= \frac{1}{1 - r\cos\theta - ir\sin\theta} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{(1 - r\cos\theta)^2 - (ir\sin\theta)^2} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2}
\end{aligned}$$

$$= \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} + i\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} \tag{2.9}$$

Also for $z = re^{i\theta}$*;*

$$\sum_{k=0}^{\infty} z^k = 1 + \sum_{k=1}^{\infty} r^k(\cos k\theta + i\sin k\theta) \tag{2.10}$$

equations (2.30) *and* (2.10) *are equal.*

Therefore, from the equivalence of real parts;

$$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} = 1 + \sum_{k=1}^{\infty} r^k \cos k\theta$$

is obtained.

$$\begin{aligned}
\frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos k\theta &= \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} - \frac{1}{2} \\
&= \frac{2 - 2r\cos\theta - 1 + 2r\cos\theta - r^2}{2(1 - 2r\cos\theta + r^2)} \\
&= \frac{1}{2}\frac{1 - r^2}{1 - 2r\cos\theta + r^2}
\end{aligned}$$

If we substitute this equivalence in the (2.7),

$$P(r,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(t-x)+r^2}f(t)dt \tag{2.11}$$

therefore we find "Poisson Integral".

Example 2.23 *(Cesàro summability techniques) An interesting application of Cesàro summability appears in the context of Fourier series.*
We know that the Dirichlet kernels fail to belong to the family of good kernels. Quite surprisingly, their averages are very well behaved functions, in the sense that they do form a family of good kernels.
To see this, we form the N^{th} Cesàro mean of the Fourier series, which by definition is

$$\sigma_N(f)(x) = \frac{S_0(f)(x)+\cdots+S_{N-1}(f)(x)}{N}.$$

Since $S_n(f) = f * D_n$, we find that

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where $F_N(x)$ is the N^{th} Fejèr kernel given by

$$F_N(x) = \frac{D_0(x)+\cdots+D_{N-1}(x)}{N}.$$

Example 2.24 *(Fejer Kernel) Partial sum of Fourier series of 2π periodic function f is;*

$$S_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)$$

in the expression

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt, \qquad k = 0,1,2,\cdots \\ b_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt, \qquad k = 0,1,2,\cdots \end{aligned}$$

If Fourier coefficients are used,

$$S_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(t-x)}{2\sin\frac{t-x}{2}}dt, \qquad n = 0, 1, 2, \cdots$$

obtained. Fejer took arithmetic mean of these was the partial sums, $\sigma_n(f;x) = \frac{s_0(f;x)+s_1(f;x)+\cdots+s_{a-1}(f;x)}{n}$. *This average;*

$$\sigma_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt \tag{2.12}$$

is equal to this integral which is called Fejer integral;

$$F_n(t) = \frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}t}{\sin\frac{1}{2}t}\right]^2$$

$F_n(t)$ *function is called Fejer kernel. Fejer kernel is a non-negative and even function.*

Example 2.25 *(Singular Integral of Gauss-Weierstrass) Heat equation for the entire real axis is;*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

and to be a given function f;

$$u(x,0) = f(x)$$

a function that satisfies the initial condition,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\pi}^{\pi} f(\zeta)e^{-\frac{(\zeta-x)^2}{4t}}d\zeta$$

is of the form. Here $1/2\sqrt{t} = \lambda$, $u(x,t) = u_\lambda(x)$ is written,

$$u_\lambda(x) = \frac{\lambda}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(\zeta)e^{-\lambda^2(\zeta-x)^2}d\zeta, \qquad \lambda > 0 \tag{2.13}$$

is obtained as an integral. This integral is called the Gauss-Weierstrass integral and,

$$w_\lambda(x) = \frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2} \tag{2.14}$$

function is called the Gauss-Weierstrass kernel.The kernel $w_\lambda(x)$ is a non-negative and even function.

Comparing $L_p(D)$ spaces when D is finite interval

The properties of $L_p(D)$ can change connected with the interval D is finite of infinite. Let $a < b$ are finite numbers and $D = (a, b)$.

In this case, if we use Hölder inequality in $g(x) \equiv 1$;

$$\begin{aligned}\int_D |f(x)|dx &\leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} . \left(\int_a^b dx\right)^{\frac{1}{q}} \\ &= \|f\|_{L_p(D)}.(b-a)^{\frac{1}{q}}\end{aligned}$$

is found. So we can write;

$$\|f\|_{L_1(a,b)} \leq \|f\|_{L_p(a,b)}(b-a)^{\frac{1}{q}}.$$

If right hand side of this inequality is finite (i.e. $f\epsilon L_p(a, b)$), then left hand side is also finite (i.e. $f\epsilon L_p(a, b)$). This shows for all $p \geq 1$;

$$L_p(a, b) \subset L_1(a, b) \tag{2.15}$$

Namely;

$$\begin{aligned}\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} &\leq \left(ess \sup_{x\epsilon[a,b]} |f(x)|\right) . \left(\int_a^b dx\right)^{\frac{1}{q}} \\ &= (b-a)^{\frac{1}{q}}.\|f\|_{L_\infty(a,b)}\end{aligned}$$

from the inequality for all $p \geq 1$, we see;

$$L_\infty(a, b) \subset L_p(a, b) \tag{2.16}$$

Finally we get from (2.15) and (2.16). ;

$$L_\infty(a,b) \subset L_p(a,b) \subset L_1(a,b) \tag{2.17}$$

The equation (2.28) shows that when (a,b) is a finite interval; the largest space is $L_1(a,b)$ and the smallest space is $L_\infty(a,b)$ in $L_p(a,b)$ spaces.

Now, let assume $p_1 \geq p_2 \geq 1$. In this case suppose that $\frac{p_1}{p_2} \geq 1$ and $p = \frac{p_1}{p_2}$, we get $\frac{1}{p} = \frac{p_2}{p_1}$ and $\frac{1}{q} = 1 - \frac{p_2}{p_1} = \frac{p_1 - p_2}{p_1}$.

Let assume $\varphi \epsilon L_{p_2}(a,b)$.

If we use Hölder inequality;

$$|f(x)| = |\varphi(x)|^{p_2}, \qquad g(x) \equiv 1$$

$$p = \frac{p_1}{p_2} \qquad and \qquad \frac{1}{q} = 1 - \frac{p_2}{p_1}$$

$$\begin{aligned}\int_a^b |\varphi(x)|^{p_2} dx &\leq \left(\int_a^b |\varphi(x)|^{p_2 \frac{p_1}{p_2}} dx\right)^{\frac{p_2}{p_1}} (b-a)^{\frac{p_1-p_2}{p_1}} \\ &= \left(\int_a^b |\varphi(x)|^{p_1 dx}\right)^{\frac{p_2}{p_1}} (b-a)^{\frac{p_1-p_2}{p_1}}\end{aligned}$$

is found and then we see;

$$\left(\int_a^b |\varphi(x)|^{p_2} dx\right)^{\frac{1}{p_2}} \leq (b-a)^{\frac{p_1-p_2}{p_1 p_2}} \left(\int_a^b |\varphi(x)|^{p_1} dx\right)^{\frac{1}{p_1}}$$

Namely if;

$$p_1 \geq p_2 \geq 1$$

$$||\varphi||_{L_{p_2}(a,b)} \leq ||\varphi||_{L_{p_1}(a,b)}.(b-a)^{\frac{p_1-p_2}{p_1 p_2}}$$

with another term if;

$$p_1 \geq p_2 \geq 1$$

$$L_{p_1(a,b)} \subset L_{p_2(a,b)}$$

This shows that if (a, b) is afinite interval, then $L_p(a, b)$ spaces extend with p' s be reduced from the equation (2.28). As;

$$L_\infty(a, b) \subset L_{p_1}(a, b) \subset L_{p_2}(a, b) \subset L_1(a, b), \qquad p_1 > p_2. \tag{2.18}$$

Comparing $L_p(D)$ spaces when $D = (-\infty, \infty)$

We researched that there is the relation (2.18) with $L_p(D)$ spaces in finite interval $D(a, b)$. Namely, $L_1(D)$ space contains all properties of $L_p(D)$, $1 < p \leq \infty$. But unfortunately, there is not any relation as if (2.18), when D is an infinite interval. This shows the properties of $L_p(D)$ spaces can be change with respect to p in infinite intervals. For example it is obvious that, for the function $f_1(x) \equiv 1$ is member of $L_\infty(-\infty, \infty)$, but it is not the member of $L_p(-\infty, \infty)$, $L_\infty(a, \infty)$ or $L_\infty(-\infty, b)$. That is to say;

$$L_\infty(-\infty, \infty) \not\subset L_p(-\infty, \infty).$$

Also for $\alpha > 0$ is a number;

$$f_2(x) = \begin{cases} 0 \leq x \leq \alpha, & \frac{1}{x^{\frac{1}{p+1}}}; \\ \text{otherwise}, & 0. \end{cases}$$

$f_2(x)$ is member of $L_p(-\infty, \infty)$, but the neighborhood of $f_2(x)$ at $x = 0$ is infinite and increasing, so $f_2 \notin L_\infty(-\infty, \infty)$, namely;

$$L_p(-\infty, \infty) \not\subset L_\infty(-\infty, \infty).$$

Let find some examples for the other cases, if $p_1 > p_2$, take $f_3(x)$ as;

$$f_3(x) = \begin{cases} 0 \leq x \leq \alpha;, & \frac{1}{\sqrt[p_1]{x}} \\ otherwise, & 0. \end{cases}$$

In this case $\frac{p_2}{p_1} < 1$, so;

$$\|f_3\|_{L_{p_2}(-\infty,\infty)} = \left(\int_0^{\alpha} \frac{1}{|x|^{\frac{p_2}{p_1}}} dx\right)^{\frac{1}{p_2}} < \infty$$

Namely, $f_3 \epsilon L_{p_2}(-\infty,\infty)$. Besides;

$$\|f_3\|_{L_{p_1}(-\infty,\infty)} = \left(\int_0^{\alpha} \frac{1}{x} dx\right)^{\frac{1}{p_1}} = \infty$$

and $f_3 \notin L_{p_1}(-\infty,\infty)$. This shows that, when $p_1 > p_2$;

$$L_{p_1}(-\infty,\infty) \not\subset L_{p_2}(-\infty,\infty)$$

Also , again take $f_4(x)$, if $\alpha > 0$ and $p_1 > p_2$,

$$f_4(x) = \begin{cases} \alpha \leq x < \infty, & \frac{1}{\sqrt[p_2]{x}}; \\ otherwise, & 0. \end{cases}$$

We know $\frac{p_1}{p_2} > 1$, so;

$$\|f_4\|_{L_{p_1}(-\infty,\infty)} = \int_{\alpha}^{\infty} \left(\frac{1}{x^{\frac{p_1}{p_2}}}\right)^{\frac{1}{p_1}} < \infty$$

Namely $f_4 \epsilon L_{p_1}(-\infty,\infty)$, but;

$$\left(\int_{-\infty}^{\infty} |f_4(x)|^{p_2} dx\right)^{\frac{1}{p_2}} = \left(\int_{\alpha}^{\infty} \frac{1}{x} dx\right)^{\frac{1}{p_2}} = \infty$$

inequality shows that;

$$f_4 \notin L_{p_2}(-\infty,\infty).$$

Because of $p_1 > p_2$;

$$L_{p_2}(-\infty,\infty) \not\subset L_{p_1}(-\infty,\infty)$$

This examples shows that; there is not a relation in $L_p(-\infty,\infty)$ spaces like equation (2.18). Similar examples are valid for $L_p(\alpha,\infty)$ and $L_p(-\infty,b)$ spaces.

Characteristic Points of $f \epsilon L_1$

Definition 2.26 *(D- Point). A point* $x \epsilon \mathbb{R}$ *is called a* $D-$ *point of the function* f *if*

$$\lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} [f(t) - f(x)] dt = 0. \tag{2.19}$$

D-points stand for differentiability; for the D-points of an integrable function f *are precisely those points at which the indefinite integral of* f *is differentiable to the value* $f(x)$.

Proof. If we substitute $h = -h$ in the equation (2.19),

$$\lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) dt = f(x). \tag{2.20}$$

Then sum equations (2.19) and (2.20)

$$\lim_{h \to 0} \frac{1}{2h} \int_{x-h}^{x+h} [f(t) - f(x)] dt = 0. \tag{2.21}$$

(2.21) is valid (a. e). Namely, if (2.19) is exists, then (2.20) and (2.21) exist.

We can write (2.19) and (2.20) as;

$$\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} [f(x+t) - f(x)] dx = 0. \tag{2.22}$$

$$\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} [f(x-t) - f(x)] dx = 0. \tag{2.23}$$

and sum the equations (2.22) and (2.23);

$$\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} [f(x+t) - f(x-t) - 2f(x)] dx = 0. \tag{2.24}$$

Namely, if x is d- point of function f, at this point (2.22), (2.23) and (2.24) are valid. □

(Bary, 1964)

Definition 2.27 *(Lebesgue Point). A point $x\epsilon\mathbb{R}$ is called a Lebesgue point of the function f if*

$$\lim_{h\to 0}\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt = 0.$$

(Bary, 1964)

Evidently, any Lebesgue point of f is a D-point of f but not conversely, and any point of continuity of an integrable function f is a Lebesgue point. Indeed; let $L(f)$ be the set of all Lebesgue points of $f\epsilon L_1$ and $D(f)$ be the set of all points of all D-points of $f\epsilon L_1$. Then

$$\left|\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt\right| \leq \frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt.$$

The inequality shows that every Lebesgue point is also D-point. Thus,

$$L(f)\subset D(f)$$

Now, its clear that any point, where $f(x)$ is continuous, is a Lebesgue point; in fact, if there exists any $\varepsilon > 0$ and the function $f(x)$ is continuous at the point x, it can be found a $\delta > 0$ such that

$$|f(t)-f(x)| < \varepsilon \qquad for \qquad |h| \leq \delta$$

and then

$$\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt < \varepsilon,$$

or, we can write

$$\lim_{h\to 0}\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt = 0.$$

This shows that any continuity point of the function $f\epsilon L_1$ is also a Lebesgue point. Let $C(f)$ be the set of continuity points of $f\epsilon L_1$. Then we can write

$$C(f)\subset L(f)\subset D(f).$$

(Bary, 1964)

2.2 Some Density Theorems

Definition 2.28 *(Chebyshev Systems). $C[a,b]$ denotes the space of continuous functions on $[a,b]$ and $C^* = C^*[-\pi,\pi]$, the space of $2\pi-$ periodic and continuous functions on the line. Both of these spaces are equipped with the supremum norm over the appropriate interval. For example, for $f \epsilon C[a,b]$*

$$\|f\| = \sup_{a \leq x \leq b} |f(x)|$$

We will use the notation $\|.\|_{[a,b]}$ to indicate that the supremum is taken over $[a,b]$ whenever it is necessary to make it clear which interval the norm is taken over.

A set of functions $u_0, u_1, \cdots u_n \subseteq C[a,b]$ is called a Chebyshev system on $[a,b]$ if each function $u \epsilon span(u_0, u_1, \cdots u_n)$ has at most n zeros unless it is identically zero on $[a,b]$. For the periodic case, $u_0, u_1, \cdots u_n \subseteq C^*$ and any $u \epsilon span(u_0, u_1, \cdots u_n)$ has at most n zeros unless it is identically zero on $[-\pi,\pi]$, unless it is identically zero. The space $u_n = span(u_0, u_1, \cdots u_n)$ is called a Chebyshev space. P_n, the space of algebraic polynomials of degree $\leq$ n is a Chebyshev space for any $[a,b]$ and T_n the space of trigonometric polynomials of degree $\leq$ n is a Chebyshev subspace of C^*.

A Chebyshev space on $[a,b]$ can also be characterized by the following interpolation property.

Some Properties of Chebyshev System

If $a \leq x_0 < x_1 < \cdots < x_n \leq b$ and $(y_i)_{i=0}^n$ are real numbers then there is a unique $u \epsilon sp(u_0, u_1, \cdots u_n)$ such that

$$u(x_i) = y_i \qquad i = 0, 1, \cdots, n.$$

If U_n is a Chebyshev subspace of $C[a,b]$ with each non-zero $u \epsilon U_n$ having n continuous derivatives on $[a,b]$ and at most n zeros on $[a,b]$ counting multiplicity then we say U_n

is an extended Chebyshev space. If for each $k = 0, \cdots, n$, $span(u_0, u_1, \cdots u_k)$ is an extended Chebyshev system, then we say U_n is an extended complete Chebyshev space. For an extended complete Chebyshev space there is a canonical basis $u_0, u_1, \cdots u_n$ described by

$$\begin{aligned} u_0(t) &= w_0(t) \\ u_1(t) &= w_0(t) \int_a^b w_1(x_1) dx_1 \\ u_n(t) &= w_0(t) \int_a^b w_1(x_1) \int_a^{x_1} w_2(x_2) \cdots \int_a^{x_{n-1}} w_n(x_n) \cdots dx_1 \end{aligned}$$

where each w_i is a continuous strictly positive function on $[a, b]$, $w_i \epsilon C^{n-i}[a, b]$.
Let $f \epsilon C[a, b](C^*)$ and U_n be a Chebyshev subspace of $C[a, b](C^*)$. A function $u^* \epsilon U_n$ if

$$\|f - u^*\| = \inf_{u^* \epsilon U_n} \|f - u\|$$

The following theorem of Chebyshev gives the existence, uniqueness and characterization of u^*.

Theorem 2.29 *Let $f \epsilon C[a, b](C^*[-\pi, \pi])$ and $u_0, u_1, \cdots u_n$ be a Chebyshev system on $[a, b]([-\pi, \pi])$. Then the best approximation u^* to f exists and is unique. If u is any function in $sp(u_0, u_1, \cdots u_n)$ then u is the best approximation to f if and only if there exist points*

$$a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b(-\pi \leq x_0 < x_1 < \cdots < x_{n+1} < \pi)$$

such that $f - u$ alternately takes on the values $\pm \|f - u\|$ at the points x_i, $i = 0, \cdots, n+1$.

Theorem 2.30 *(D. F. Egorov). Let a sequence of measurable and almost everywhere finite functions*

$$f_1(x),\ f_2(x),\ f_3(x), \cdots$$

be defined and finite almost everywhere on a measurable set E. Suppose that $f_n(x)$ converges almost everywhere to the measurable function $f(x)$ which is finite almost *everywhere:*

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{2.25}$$

Then, for every $\delta > 0$, there exist a measurable set $E_\delta \subset E$ such that:

1. *$mE_\delta > mE - \delta$;*

2. *The limit* (2.25) *is uniform on the set E_δ.*

Proof. We know from *Lebesgue Theorem*;

$$mR_n(\sigma) \to 0, \qquad as \quad n \to \infty \tag{2.26}$$

as $n \to \infty$ for all $\sigma > 0$, where

$$R_n(\sigma) = \sum_{k=n}^{\infty} E(|f_k - f| \geq \sigma).$$

With this in mind we consider a convergent series of positive terms

$$\eta_1 + \eta_2 + \eta_3 + \cdots \qquad (\eta_i > 0)$$

and a sequence of positive numbers tending to zero:

$$\sigma_1 > \sigma_2 > \sigma_3 > \cdots, \qquad \lim \sigma_i = 0.$$

In view of (2.26), we can find for every natural number i another natural number n_i such that

$$mR_{n_i}(\sigma_i) < \eta_i.$$

Having done this, we select an i_0 such that

$$\sum_{i=i_0}^{\infty} \eta_i < \delta$$

(δ is the number appearing in the statement of the theorem), and we set

$$e = \sum_{i=i_0}^{\infty} R_{n_i}(\sigma_i).$$

It is obvious that

$$me < \delta.$$

Let

$$E_\delta = E - e.$$

we now show that E_δ is the required set. The inequality

$$mE_\delta > mE - \delta$$

is clear, so we need only verify that the limit

$$f_n(x) \to f(x)$$

is uniform on the set E_δ.

Choose an arbitrary $\varepsilon > 0$. We find an i such that

$$i \geq i_0, \qquad \sigma_i < \varepsilon,$$

and show that for $k \geq n_i$ and for all $x \epsilon E_\delta$,

$$|f_k(x) - f(x)| < \varepsilon,$$

If $x \epsilon E_\delta$, then $x \notin e$. This implies in particular that

$$x \notin R_{n_i}(\sigma_i).$$

In other words,

$$x \notin E(|f_k - f| \geq \sigma_i),$$

for $k \geq n_i$, so that

$$|f_k(x) - f(x)| < \sigma_i \quad k \geq n_i,$$

and hence,

$$|f_k(x) - f(x)| < \varepsilon \quad k \geq n_i.$$

the theorem is proved, because n_i depends only on ε and not on x. □

Theorem 2.31 *(M. FRÉCHET). For every measurable function $f(x)$ which is defined on $[a, b]$ and is finite almost everywhere, there exists a sequence of polynomials converging to $f(x)$ almost everywhere.*

Proof. Let $\varphi_n(x)$ be a sequence of continuous functions converging almost everywhere to $f(x)$. If $P_n(x)$ is a polynomial such that

$$|P_n(x) - \varphi_n(x)| < \frac{1}{n},$$

for all $x \epsilon [a, b]$, the sequence $P_n(x)$ converges to $f(x)$ at every point x for which

$$\varphi_n(x) \to f(x),$$

This proves the theorem. □

Theorem 2.32 *(N. N. Luzin). Let $f(x)$ be a measurable function defined on $[a, b]$ which is finite almost everywhere. For every $\delta > 0$, there exists a continuous function $\varphi(x)$ such that*

$$mE(f \neq \varphi) < \delta.$$

If $|f(x)| \leq K$, then

$$|\varphi(x)| \leq K$$

also.

Proof. Let

$$\varphi_1(x), \varphi_2(x), \varphi_3(x), ...$$

be a sequence of continuous functions having the property described in $Frechet's$ $Theorem$. Using $Egorov's$ $Theorem$, we can find a set E_δ such that

$$mE_\delta > b - a - \frac{\delta}{2},$$

and such that

$$\varphi_n(x) \to f(x)$$

uniformly with respect to x on the set E_δ. A basic theorem of analysis states that the function $f(x)$ is continuous on the set E_δ (It does not follow that the function $f(x)$, considered on $[a, b]$, is continuous at every point of the set E_δ). We find a closed subset F of the set E_δ such that

$$mF > mE_\delta - \frac{\delta}{2}.$$

If the function $f(x)$ is considered only the set F, it is obviously continuous on this set. We produce a continuous function $\varphi(x)$, defined on $[a, b]$ and coinciding with $f(x)$ on the set F. Thus

$$E(f \neq \varphi) \subset [a, b] - F,$$

and the measure of this set is $< \delta$, so that $\varphi(x)$ is a function meeting all requirements of the present theorem. If $|f(x)| \leq K$, this inequality is valid also for x in F, and then, we have

$$|\varphi(x)| \leq K.$$

This completes the proof. □

Theorem 2.33 *(WEIERSTRASS). Let $f(x)$ be a continuous function defined on the* *closed interval* $[a, b]$. *For every* $\varepsilon > 0$, *there exist a polynomial* $P(x)$ *such that*

$$|f(x) - P(x)| < \varepsilon$$

for all $x \epsilon [a, b]$

Proof. If $[a, b] = [0, 1]$, the theorem follows directly from Bernstein' s theorem. Suppose that $[a, b] \neq [0, 1]$. Consider the following function of the argument y:

$$f[a + y(b - a)].$$

This function is defined and continuous on the segment $[0, 1]$. We can find a polynomial $\varphi(y)$ such that for all $y \epsilon [0, 1]$,

$$|f[a + y(b - a)] - \varphi(y)| < \varepsilon$$

If $x \epsilon [a, b]$, then $\frac{x-a}{b-a} \epsilon [0, 1]$ and hence

$$\left| f(x) - \varphi\left(\frac{x-a}{b-a}\right) \right| < \varepsilon$$

The polynomial $P(x) = \varphi\left(\frac{x-a}{b-a}\right)$ therefore satisfies our requirements. □

Definition 2.34 *Any function of the form*

$$T(x) = A + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

is called a trigonometric polynomial of the n^{th} *order. If* $b_1 = \ldots = b_n = 0$, *the trigonometric polynomial* $T(x)$ *is said to be even.*

Lemma 2.35

1. *The function* $\cos kx$ *can be written as an even trigonometric polynomial.*
2. *If* $T(x)$ *is a trigonometric polynomial, then* $T(x) \sin x$ *is also a trigonometric polynomial.*

3. *If $T(x)$ is a trigonometric polynomial, then $T(x+a)$ also is a trigonometric polynomial.*

Lemma 2.36 *If the function $f(x)$ is defined and continuous on the closed interval $[0,\pi]$, then for every $\varepsilon > 0$, there exists an even trigonometric polynomial $T(x)$ such that*

$$|f(x) - T(x)| < \varepsilon$$

for all $x \epsilon [0,\pi]$.

Proof. Consider the function

$$f(\arccos y).$$

This function is defined and continuous on the segment $[-1,+1]$. Therefore, we can produce a polynomial $\sum_{k=0}^{n} a_k y^k$ such that for all $y\epsilon[-1,+1]$,

$$\left| f(\arccos y) - \sum_{k=0}^{n} a_k y^k \right| < \varepsilon.$$

Now let $x\epsilon[0,\pi]$. Then $\cos x\epsilon[-1,+1]$ and, hence,

$$\left| f(x) - \sum_{k=0}^{\infty} a_k \cos^k x \right| < \varepsilon.$$

It remains to note that in view of Lemma (4.22), the function $\sum_{k=0}^{n} a_k \cos^k x$ is an even trigonometric polynomial. □

Corollary 2.37 *If the even function $f(x)$ is defined for all x, has period 2π, and is continuous everywhere, then for every $\varepsilon > 0$, we can find a trigonometric polynomial such that*

$$|f(x) - T(x)| < \varepsilon$$

for all real x.

In fact, on the segment $[0,\pi]$, this inequality can be satisfied by an even trigonometric polynomial $T(x)$ so that the inequality is automatically satisfied for $x\epsilon[-\pi,0]$, and, *then, since the difference $f(x) - T(x)$ has period 2π, the inequality is satisfied everywhere.*

Theorem 2.38 *(WEIERSTRASS). Let $f(x)$ be a continuous periodic function on period 2π. For every $\varepsilon > 0$, there exist o trigonometric polynomial $T(x)$ such that*

$$|f(x) - T(x)| < \varepsilon$$

for all x.

Proof. By the corollary to Lemma (2.36), for the even functions

$$f(x) + f(-x), \quad [f(x) - f(-x)]\sin x$$

there exist trigonometric polynomials $T_1(x)$ and $T_2(x)$ such that

$$f(x) + f(-x) = T_1(x) + a_1(x), \qquad [f(x) - f(-x)]\sin x = T_2(x) + a_2(x),$$

where

$$|a_1(x)| < \frac{\varepsilon}{2}, \qquad |a_2(x)| < \frac{\varepsilon}{2}.$$

Multiplying the first of these equalities by $\sin^2 x$ and the second by $\sin x$, the adding them and dividing by 2, we obtain

$$f(x)\sin^2 x = T_3(x) + \beta(x) \qquad \left(|\beta(x)| < \frac{\varepsilon}{2}\right),$$

where $T_3(x)$ is again some trigonometric polynomial. Here, the function $f(x)$ is an arbitrary continuous periodic function. Therefore, a similar equality is true of the function $f(x - \frac{\pi}{2})$:

$$f\left(x - \frac{\pi}{2}\right)\sin^2 x = T_4(x) + \gamma(x) \qquad \left(|\gamma(x)| < \frac{\varepsilon}{2}\right).$$

Replacing x by $x + \frac{\pi}{2}$, we obtain

$$f(x)\cos^2(x) = T_5(x) + \delta(x) \qquad \left(|\delta(x)| < \frac{\varepsilon}{2}\right),$$

hence

$$f(x) = T_3(x) + T_5(x) + \beta(x) + \delta(x),$$

i. e., the polynomial $T_3(x) + T_5(x)$ satisfies all of our requirements. □

Lemma 2.39 *For every x;*

$$\sum_{k=0}^{n} c_n^k x^k (1-x)^{n-k} = 1 \tag{2.27}$$

Here c_n^k is the binomial $coefficient$ $\frac{n!}{k!(n-k)!}$ In fact, setting $a = x$ and $b = 1 - x$ in Newton' s binomial formula,

$$(a+b)^n = \sum_{k=0}^{n} c_n^k a^k b^{n-k} \tag{2.28}$$

we obtain formula (2.28).

Lemma 2.40 *For all real x,*

$$\sum_{k=0}^{n} c_n^k (k-nx)^2 x^k (1-x)^{n-k} \leq \frac{n}{4}$$

.

Proof. Differentiate the identity with respect to z;

$$\sum_{k=0}^{n} c_n^k z^k = (1+z)^n \tag{2.29}$$

then multiply the result by z:

$$\sum_{k=0}^{n} k c_n^k z^k = nz(1+z)^{n-1} \tag{2.30}$$

Differentiating (2.30) and multiplying the result by z, we obtain

$$\sum_{k=0}^{n} k^2 c_n^k z^k = nz(1+nz)(1+z)^{n-2} \tag{2.31}$$

Set

$$z = \frac{x}{1-x}$$

in (2.29), (2.30), (2.31), and multiply the identities thus obtained by $(1-x)^n$. This gives us

$$\sum_{k=0}^{n} c_n^k x^k (1-x)^{n-k} = 1 \tag{2.32}$$

$$\sum_{k=0}^{n} k c_n^k x^k (1-x)^{n-k} = nx \tag{2.33}$$

and

$$\sum_{k=0}^{n} k^2 c_n^k x^k (1-x)^{n-k} = nx(1-x+nx) \tag{2.34}$$

Multiply (2.32) by n^2x^2, (2.33) by $-2nx$, (2.34) by 1 and add together the resulting equalities. We obtain

$$\sum_{k=0}^{n} (k-nx)^2 c_n^k x^k (1-x)^{n-k} = nx(1-x)$$

To prove the lemma, it is sufficient to note that for all real x,

$$x(1-x) \leq \frac{1}{4}$$

Because let

$$f(x) = x(1-x)$$

then

$$f'(x) = 1-2x$$

and root

$$x = \frac{1}{2}$$

So $\frac{1}{2}$ is maximum point of $f(x)$.

Finally we can see

$$x(1-x) \leq \frac{1}{4}.$$

□

Definition 2.41 *Let $f(x)$ be a finite function defined on the closed interval $[0,1]$. The polynomial*

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) c_n^k x^k (1-x)^{n-k} \tag{2.35}$$

is called the Bernstein polynomial of degree n for the function $f(x)$.

Theorem 2.42 *(S. N. BERNSTEIN) If the function $f(x)$ is continuous on the segment $[0,1]$, then*

$$B_n(x) \to f(x) \tag{2.36}$$

uniformly y with respect to x, as $n \to \infty$.

Proof. Let

$$M = \max |f(x)|.$$

Let ε be any number > 0. Since f is continuous on the bounded closed interval $[0,1]$, there exist a $\delta > 0$ such that

$$|f(x^{''}) - f(x^{'})| < \varepsilon,$$

whenever $|x^{''} - x^{'}| < \delta$. Now select an arbitrary $x \epsilon [0,1]$.

It follows from (2.35) that

$$f(x) = \sum_{k=0}^{n} f(x) c_n^k x^k (1-x)^{n-k},$$

and therefore

$$|B_n(x) - f(x)| \leq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| c_n^k x^k (1-x)^{n-k} \tag{2.37}$$

Divide all the numbers $k = 0, 1, 2, ..., n$ into two categories A and B according to the rules

$$k \epsilon A \quad if \quad \left|\frac{k}{n} - x\right| < \delta \tag{2.38}$$
$$k \epsilon B \quad if \quad \left|\frac{k}{n} - x\right| \geq \delta \tag{2.39}$$

If $k\epsilon A$, then

$$\left| f\left(\frac{k}{n}\right) - f(x) \right| < \epsilon,$$

and on the basis of Lemma (2.39),

$$\sum_{A} \left| f\left(\frac{k}{n}\right) - f(x) \right| c_n^k x^k (1-x)^{n-k} \leq \varepsilon \sum_{A} c_n^k x^k (1-x)^{n-k} \leq \varepsilon \sum_{k=0}^{n} c_n^k x^k (1-x)^{n-k} = \varepsilon \tag{2.40}$$

If $k\epsilon B$, then

$$\frac{(k - nx)^2}{n^2\delta^2} \geq 1,$$

and, by Lemma (2.40),

$$\sum_{B} \left| f\left(\frac{k}{n}\right) - f(x) \right| c_n^k x^k (1 - x)^{n-k} \leq \frac{2M}{n^2\delta^2} \sum_{B} (k - nx)^2 c_n^k x^k (1 - x)^{n-k} \tag{2.41}$$

$$\leq \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (k - nx)^2 c_n^k x^k (1 - x)^{n-k} < \frac{M}{n\delta^2} \tag{2.42}$$

Assembling (2.40) and , (2.42) we find that for all $x\epsilon[0, 1]$,

$$|B_n(x) - f(x)| < \varepsilon + \frac{M}{n\delta^2}.$$

Consequently, if

$$n > \frac{M}{2\varepsilon\delta^2}$$

we have

$$|B_n(x) - f(x)| < 2\varepsilon,$$

and this completes the proof. □

The Approximation Problem

We mentioned that one of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have better properties than f. Suppose that a given function f can be shown that the limit of set of functions $\varphi_n(x)$ at the point which has good properties at x_0. Namely,

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0)$$

be as described above. In this case the series of $\varphi_n(x)$ approximates or converges to f. The second fundamental problem of the approaching theorem is finding the rate of approach to the problem.

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0)$$

while $n \to \infty$, the series $(\alpha_n) = (f(x_0) - \varphi_n(x_0))$ is infinite decreasing. If we find the series β_n when $n \to \infty, \beta_n \to 0$ and $\alpha_n = o(\beta_n)$. In this case;

$$f(x_0) - \varphi_n(x_0) = o(\beta_n)$$

and this formula shows that the $f(x_0) - \varphi_n(x_0)$ is approaches to zero more rapidly than β_n, when $n \to \infty$.

Definition 2.43 *A norm,*

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_X = 0$$

If,

$$\lim_{n\to\infty} \frac{\|f_n(x) - f(x)\|_X}{\alpha_n} = K$$

the rate of (f_n) to f is α_n. Also if;

$$\|f_n(x) - f(x)\|_X = o(\alpha_n)$$

we say the rate of convergence of $f_n \to f$ is more rapidly than α_n. Finding the speed of convergence α_n to zero determines the speed of convergence of f_n to f. It is enough to *compare the sequence α_n with another zero sequence. Because providing the equality of $\alpha_n = o(\beta_n)$ is the proof of α_n goes to zero more rapidly than β_n.*

2.3 Modulus of Continuity and Some Properties

Definition 2.44 *(Modulus of Continuity) For $f \epsilon X_{2\pi}$ the modulus of continuity is defined for $\delta \geq 0$ by*

$$\omega(X_{2\pi}; f; \delta) = \sup_{|h|<\delta} \| f(o+h) - f(o) \|_{X_{2\pi}}.$$

In connection with even kernels we shall need the following generalization.

(Butzer, 1971)

Definition 2.45 *For $f \epsilon X_{2\pi}$ the generalized modulus of continuity is defined for $\delta \geq 0$ by*

$$\omega^*(X_{2\pi}; f; \delta) = \sup_{|h|<\delta} \| f(o+h) - f(o-h) - 2f(o) \|_{X_{2\pi}}.$$

(Butzer, 1971)

Lemma 2.46 *$\omega(f; \delta)$ is a monotonically increasing function.*

Proof. Let $0 < \delta_1 \leq \delta_2$

$$\{|f(x) - f(t)| : a \leq x, t \leq b, \quad |x-t| \leq \delta_1\} \subset$$

$$\{|f(x) - f(t)| : a \leq x, t \leq b, \quad |x-t| \leq \delta_2\}$$

So we say

$$\omega(f; \delta_1) = \sup\{|f(x) - f(t)| : a \leq x, t \leq b, \quad |x-t| \leq \delta_1\} \leq$$

$$\sup\{|f(x) - f(t)| : a \leq x, t \leq b, \quad |x-t| \leq \delta_2\} = \omega(f; \delta_2)$$

□

Lemma 2.47 *If f is continuous on $[a,b]$, then* $\lim_{\delta\to 0}\omega(f;\delta) = 0$

Proof. If f is continuous on $[a,b]$ then $\forall\varepsilon > 0$ $\exists x, y\epsilon[a,b]$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. In case $\delta \to 0$, then $x \to y$. So $|f(x) - f(y)| < \varepsilon$ Therefore $\lim_{\delta\to 0}\omega(f;\delta) = 0$. □

Lemma 2.48 *f is uniformly continuous* $\Leftrightarrow \lim_{\delta\to 0}\omega(f;\delta) = 0$.

Proof. ($\Rightarrow$): is obviously from lemma (2.48).
($\Leftarrow$): $\forall\varepsilon > 0$ $\exists\eta$ such that $\delta < \eta$ $\omega(f;\delta) < \varepsilon$. Therefore f is a uniformly continuous function. □

Lemma 2.49 *f is continuous on $[a,b]$ and there is a number $m > 1$ and $m\epsilon\mathbb{R}$ such that*

$$\omega(f;m\delta) \leq m\omega(f;\delta)$$

Proof. The definition of modulus of continuity is;

$$\omega(f;\delta) = \sup_{|t-x|<\delta}{}_{a\leq t,x\leq b}|f(t) - f(x)|.$$

Let $t = x + h$ in the equation then we get;

$$\omega(f;\delta) = \sup_{|h|<\delta}{}_{a\leq x,x+h\leq b}|f(x+h) - f(x)|$$

We know from Telescoping Series;

$$|f(t+mh) - f(t)| = \left|\sum_{k=0}^{m-1} f(t+(k+1)h) - f(t+kh)\right|$$

So;

$$\begin{aligned}
\omega(f;m\delta) &= \sup_{a\le t,t+u\le b} {}_{|u|\le m\delta}|f(t+u)-f(t)| \\
&= \sup_{a\le t,t+mu\le b} {}_{|u|\le \delta}|f(t+mu)-f(t)| \\
&\le \sup_{a\le t,t+mu\le b} {}_{|u|\le \delta}\left|\sum_{k=0}^{m-1} f(t+(k+1)h)-f(t+ku)\right| \\
&\le m\omega(f;\delta)
\end{aligned}$$

□

Lemma 2.50 *If f is continuous on $[a,b]$ and $\forall\lambda>0$ and $\lambda\epsilon\mathbb{R}$ such that*

$$\omega(f;\lambda\delta)\le(\lambda+1)\omega(f;\delta)$$

Proof. We know the modules of continuity is a monotonically increasing function from Lemma (2.46). So $m<\lambda<m+1$ and $m\epsilon\mathbb{R}$. In this case

$$\begin{aligned}
\omega(f;\lambda\delta) &\le \omega(f;(m+1)\delta) \\
&\le (m+1)\omega(f;\delta) \\
&\le (\lambda+1)\omega(f;\delta)
\end{aligned}$$

□

Lemma 2.51 *If f is continuous on $[a,b]$, then*

$$|f(t)-f(x)|\le\omega(f;|t-x|)$$

Proof. Let,

$$\begin{aligned}
\omega(f;|t-x|) &= \sup_{a\le x,t\le b} {}_{|t-x|\le|t-x|}|f(t)-f(x)| \\
&= \sup_{a\le x,t\le b} |f(t)-f(x)| \\
&\ge |f(t)-f(x)|.
\end{aligned}$$

□

Lemma 2.52 *If f is continuous on $[a,b]$ and $\delta > 0$, then*

$$|f(t)-f(x)| \le \left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta)$$

Proof. From Lemma (2.51);

$$|f(t)-f(x)| \le \left(f;\frac{|t-x|}{\delta}\delta\right) \le \left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta)$$

□

Lemma 2.53 *If f is continuous on $[a,b]$ and $0 < \delta_1 < \delta_2$ then,*

$$\frac{\omega(f;\delta_2)}{\delta_2} \le \frac{2\omega(f;\delta_1)}{\delta_1}$$

Proof. We can easily write,

$$\omega(f;\delta_2) = \omega\left(f;\delta_1\frac{\delta_2}{\delta_1}\right) \le \left(1+\frac{\delta_2}{\delta_1}\right)\omega(f;\delta_1) \le 2\frac{\delta_2}{\delta_1}\omega(f;\delta_1)$$

□

Lemma 2.54 *If f is continuous on $[a,b]$ and $\delta_1 < \delta_2$ then,*

$$\frac{\omega(f;\delta_2)}{\delta_2} \le \frac{2\omega(f;\delta_1)}{\delta_1}$$

Proof.

$$\omega(f;\delta_2) = \omega\left(f;\delta_1\frac{\delta_2}{\delta_1}\right) \le \left(1+\frac{\delta_2}{\delta_1}\right)\omega(f;\delta_1) \le 2\frac{\delta_2}{\delta_1}\omega(f;\delta_1)$$

□

Theorem 2.55 *(T. Popoviciu). If $f(x)$ is continuous and $\omega(f;\delta)$ the modulus of continuity of $f(x)$, then*

$$|f(x) - B_n(x)| \le \frac{5}{4}\omega\left(f;\frac{1}{\sqrt{n}}\right) \tag{2.43}$$

Proof. For arbitrary x_1, x_2 in $[0,1]$ and a $\delta > 0$ we denote by $\lambda = \lambda(x_1, x_2;\delta)$ the integer $[|x_1, x_2|\delta^{-1}]$; the difference $f(x_1) - f(x_2)$ is then a sum of $\lambda + 1$ differences of $f(x)$ on intervals of length $< \delta$. Therefore

$$|f(x_1) - f(x_2)| \le (\lambda+1)\omega(f;\delta), \tag{2.44}$$

and it follows that

$$\begin{aligned}
|f(x) - B_n(x)| &\le \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right| c_n^k(x)\\
&\le \omega(f;\delta)\sum_{k=0}^{n}\left\{1+\lambda\left(x,\frac{k}{n};\delta\right)\right\}c_n^k = \omega(f;\delta)\left\{1+\sum_{\lambda\ge1}^{n}\lambda\left(x,\frac{k}{n};\delta\right)c_n^k\right\}\\
&\le \omega(f;\delta)\left\{1+\delta^{-1}\sum_{\lambda\ge1}\left|x-\frac{k}{n}\right|c_n^k\right\} \le \omega(f;\delta)\left\{1+\delta^{-2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 c_n^k\right\}\\
&\le \omega(f;\delta)\{1+(4n\delta^2)^{-1}\},
\end{aligned}$$

Then we obtain (2.43) by choosing $\delta = n^{-\frac{1}{2}}$.

The smallest value of the constant K for which

$$|f(x) - B_n(x)| \le K\omega(f;n^{-\frac{1}{2}}) \tag{2.45}$$

is true for each function f and each n, is not known; in any case we have $K \le 5/4$. Moreover, $K \ge 1$, as is seen from the following example. Let $\delta_n = o(n^{-\frac{1}{2}})$ and suppose that $f_n(x)$ is the function which is equal to zero at x_0 , $0 < x_0 < 1$, equal to 1 in $[0, x_0 - \delta_n]$ and $[x_0 + \delta_n, 1]$ and linear in the rest of $[0, 1]$. For large n, we have $\omega(f; n^{-\frac{1}{2}}) = 1$ for the function f_n; also,

$$|f(x_0) - B_n(x_0)| = B_n(x_0) \geq \sum_{\left|\frac{k}{n} - x_0\right| \geq \delta_n} c_n^k(x_0) = 1 - \varepsilon_n, \quad \varepsilon_n \to 0,$$

Therefore (2.45) can not be true if $K < 1$.

The function $\omega(f; \delta)$ cannot therefore be replaced in (2.45) by any other function decreasing to zero more rapidly. This may be shown also by the example of *one single* continuous function. The function $f(x) = |x - x_0|^a$ with fixed $0 < x_0 < 1$, $0 < a \leq 1$, belongs to the class $Lipa$. We have $\omega(f; \delta) = \delta^a$ and $\delta = n^{-\frac{1}{2}}$,

$$\begin{aligned} |f(x_0) - B_n(x_0)| &= B_n(x_0) = \sum_{k=0}^{n} \left|\frac{k}{n} - x_0\right|^a c_n^k(x_0) \\ &\geq n^{-\frac{1}{2}a} \sum_{\left|\frac{k}{n} - x_0\right| \geq n^{-1/2}} c_n^k(x_0) \cong K n^{-\frac{1}{2}a} \end{aligned}$$

for a certain constant $K > 0$. □

2.4 Modulus of Continuity and Some Characteristic Points on L_1

Definition 2.56 *(Modulus of Continuity on L_1). Let f is a function on L_1, the modulus of continuity of f on $L_1[a, b]$ is defined by,*

$$\omega_{L_1}(f; \delta) = \sup_{|t| \leq \delta} \int_{-\infty}^{\infty} |f(x + t) - f(x)| dx \tag{2.46}$$

We can see the function ω_{L_1} is non-negative, and monotonically increasing function.

Theorem 2.57 *If $f \epsilon L_1$, then*

$$\lim_{\delta \to 0} \omega_{L_1}(f; \delta) = 0 \tag{2.47}$$

Proof. If $|t| \leq \delta$ then we take the equation (2.46).

If $f \epsilon L_1[a,b]$ then, for all $\varepsilon > 0$ there exist a such that,

$$\int_{-\infty}^{-a} |f(x)|dx < \frac{\varepsilon}{4}, \qquad \int_{a}^{\infty} |f(x)|dx < \frac{\varepsilon}{4} \tag{2.48}$$

in addition for all $\delta > 0$ from the equation (2.48)

$$\int_{-\infty}^{-a-\delta} |f(x)|dx < \frac{\varepsilon}{4}, \qquad \int_{a+\delta}^{\infty} |f(x)|dx < \frac{\varepsilon}{4}$$

So, in case $|t| \leq \delta$, as $-\delta \leq t \leq \delta$,

$$\int_{a+\delta}^{\infty} |f(x+t)|dx = \int_{a+\delta+t}^{\infty} |f(x)|dx < \int_{a}^{\infty} |f(x)|dx \frac{\varepsilon}{4} \tag{2.49}$$

Because, $\delta + t \geq 0$ and $a + \delta + t > a$. Similarly, $t - \delta \leq 0$ and $-a - \delta + t \leq a$, then

$$\int_{-\infty}^{-a-\delta} |f(x+t)|dx = \int_{-\infty}^{-a-\delta+t} |f(x)|dx < \int_{-\infty}^{-a} |f(x)|dx \frac{\varepsilon}{4} \tag{2.50}$$

inequalities found. We know $a + \delta > a$ and $-a - \delta < -a$, and so, from the equations (2.48), (2.49), and (2.50)

$$\sup_{|t| \leq \delta} \left\{ \int_{-\infty}^{-a-\delta} |f(x+t) - f(x)|dx + \int_{a+\delta}^{\infty} |f(x+t) - f(x)|dx \right\} < \varepsilon \tag{2.51}$$

is found.

Then,

$$\sup_{|t| \leq \delta} \int_{-\infty}^{\infty} |f(x+t) - f(x)|dx \leq \sup_{|t| \leq \delta} \int_{-a-\delta}^{a+\delta} |f(x+t) - f(x)|dx + \varepsilon \tag{2.52}$$

can be written. *Luzin Theorem* says that, if $f \epsilon L_1(a,b)$, for all $\varepsilon > 0$ there exists a continuous function $\varphi(x)$ such that,

$$||f - \varphi||_{L_1(a,b)} < \varepsilon.$$

From this theorem, let find a continuous function φ on the interval $[-a-2\delta, a+2\delta]$ and, write the inequality,

$$\int_{-a-2\delta}^{a+2\delta} |f(x) - \varphi(x)| dx < \varepsilon$$

In this case,

$$\begin{aligned}
\int_{-a-\delta}^{a+\delta} |f(x+t) - f(x)| dx \quad &\leq \quad \int_{-a-\delta}^{a+\delta} |f(x+t) - \varphi(x+t)| dx \\
&+ \quad \int_{-a-\delta}^{a+\delta} |\varphi(x+t) - \varphi(x)| dx \\
&+ \quad \int_{-a-\delta}^{a+\delta} |\varphi(x) - f(x)| dx
\end{aligned}$$

and

$$\begin{aligned}
\sup_{|t|\leq\delta} \int_{-a-\delta}^{a+\delta} |f(x+t) - \varphi(x+t)| dx \quad &\leq \quad \int_{-a-2\delta}^{a+2\delta} |f(x) - \varphi(x)| dx \\
&< \quad \varepsilon \\
\int_{-a-\delta}^{a+\delta} |f(x) - \varphi(x)| dx \quad &\leq \quad \int_{-a-2\delta}^{a+2\delta} |f(x) - \varphi(x)| dx \\
&< \quad \varepsilon
\end{aligned}$$

inequalities can written, then,

$$\sup_{|t|\leq\delta} \int_{-a-\delta}^{a+\delta} |f(x+t) - f(x)| dx \leq 2\varepsilon + \sup_{|t|\leq\delta} \int_{-a-\delta}^{a+\delta} |\varphi(x+t) - \varphi(x)| dx \qquad (2.53)$$

is found.

Moreover, we know φ is a continuous function, so we can choose a δ_1 when $|t| < \delta_1$,

$$|\varphi(x+t) - \varphi(x)| < \frac{\varepsilon}{2(a+\delta)}$$

is satisfied. And when $\delta < \delta_1$,

$$\sup_{|t|\leq\delta} \int_{-a-\delta}^{a+\delta} |\varphi(x+t) - \varphi(x)| dx < \varepsilon$$

if this inequality used in the equation (2.53),

$$\sup_{|t|\leq\delta}\int_{-a-\delta}^{a+\delta}|f(x+t)-f(x)|dx \leq 3\varepsilon$$

If we use this inequality in the equation (2.52), then the theorem is proved. □

Theorem 2.58 *Let $m\epsilon\mathbb{N}$, then*

$$\omega_{L_1}(f;m\delta) \leq m\omega_{L_1}(f;\delta)$$

Proof. From the definition of the modulus of continuity in equation (2.46)

$$\omega_{L_1}(f;m\delta) = \sup_{|t|\leq m\delta}\int_{-\infty}^{\infty}|f(x+t)-f(x)|dx$$

In this equation sup is on the interval $[-m\delta, m\delta]$.

If take $t = my$, then we see,

$$\begin{aligned}\omega_{L_1}(f;m\delta) &= \sup_{|y|\leq\delta}\int_{-\infty}^{\infty}|f(x+my)-f(x)|dx\\ &= \sup_{|y|\leq\delta}\int_{-\infty}^{\infty}|f(x+my)-f(x+(m-1)y)+f(x+(m-1)y)-f(x+(m-2)y)\\ &\qquad +\cdots+f(x+y)-f(x)|dx\end{aligned}$$

That is,

$$\omega_{L_1}(f;m\delta) \leq \sup_{|y|\leq\delta}\int_{-\infty}^{\infty}\sum_{k=1}^{m}|f(x+ky)-f(x+(k-1)y)|dx$$

a Telescoping Series. In this equation substitute $x+(k-1)y = z$ and then again change z with x, x and y with t.

Therefore;

$$\begin{aligned}\omega_{L_1}(f;m\delta) &\leq \sup_{|t|\leq\delta}\int_{-\infty}^{\infty}\sum_{k=1}^{m}|f(x+t)-f(x)|dx\\ &= m\omega_{L_1}(f;\delta)\end{aligned}$$

theorem is proved. □

Corollary 2.59 *Let* $\lambda > 0$ *be any real number*

$$\omega_{L_1}(f;\lambda\delta) \leq (1+\lambda)\omega_{L_1}(f;\delta).$$

Proof. Let denote the integer of λ with $[|\lambda|]$. In this case, $\lambda < [|\lambda|]+1$ is obvious. And $\omega_{L_1}(f;\delta)$ is monotonically increasing function. So;

$$\omega_{L_1}(f;\lambda\delta) \leq \omega_{L_1}(f;([|\lambda|]+1)\delta)$$

$[|\lambda|]+1$ is an integer, so we can use the theorem (2.58) and finally if use the property $[|\lambda|] < \lambda$;

$$\begin{aligned} \omega_{L_1}(f;\lambda\delta) &\leq \omega_{L_1}(f;([|\lambda|]+1)\delta) \\ &\leq ([|\lambda|]+1)\omega_{L_1}(f;\delta) \\ &\leq (\lambda+1)\omega_{L_1}(f;\delta) \end{aligned}$$

is found and proof is complete. □

CHAPTER 3

CONVERGENCE OF LINEAR CONVOLUTION TYPE SINGULAR INTEGRAL OPERATORS

3.1 Convergence on L_1 Norm

One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. It is to be expected that the better behaved functions are to be constructed from the given f by some smoothing operation on f itself. Problems of approximation theory can be solved on the class of integrable functions. In this case we can not always find the value of $f(\frac{k}{n})$ by the Bernstein polynomials. Because Lebesgue integrable function is defined on a set which except the measure zero. So, we should take as an integral the series which converges to a function. We conclude that Bernstein polynomials study on $C[a,b]$, but a function cannot be continuous every time, in this case we can use the Lebesgue integral.
Now denote the family of integral operators A_λ converge to the function f on the space $L_1[a,b]$ of norm.

Theorem 3.1 *Let take the family of integral operators A_λ which transforms on the space $L_1(-\infty,\infty)$, and $K_\lambda(x-t)$ is defined as in definition (1. 20);*

$$A_\lambda(f;x) = \int_{-\infty}^{\infty} f(t)K_\lambda(x-t)dt, \qquad -\infty < x < \infty \tag{3.1}$$

In this case,

$$\lim_{\lambda\to\lambda_0} \|A_\lambda(f;x) - f(x)\|_1 = 0$$

Proof. Obviously,

$$|A_\lambda(f;x) - f(x)| \leq \int_{-\infty}^{\infty} |f(x+t) - f(x)| K_\lambda(t)dt$$

and $K_\lambda(t)$ is an even function, so;

$$|A_\lambda(f;x) - f(x)| \leq \int_{0}^{\infty} (|f(x+t) - f(x)| + |f(x-t) - f(x)|) K_\lambda(t)dt.$$

If take the integral of both sides;

$$\begin{aligned} \|A_\lambda(f;x) - f(x)\|_1 &\leq \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} (|f(x+t) - f(x)| + |f(x-t) - f(x)|) K_\lambda(t)dt \right) dx \\ &= \int_{0}^{\infty} \left(\int_{-\infty}^{\infty} |f(x+t) - f(x)| dx + \int_{-\infty}^{\infty} |f(x-t) - f(x)| dx \right) K_\lambda(t)dt \end{aligned}$$

and then for any $\delta > 0$;

$$\begin{aligned} |A_\lambda(f;x) - f(x)| &\leq \int_{0}^{\delta} \left(\int_{-\infty}^{\infty} |f(x+t) - f(x)| dx + \int_{-\infty}^{\infty} |f(x-t) - f(x)| dx \right) K_\lambda(t)dt \\ &+ \int_{\delta}^{\infty} \left(\int_{-\infty}^{\infty} |f(x+t) - f(x)| dx + \int_{-\infty}^{\infty} |f(x-t) - f(x)| dx \right) K_\lambda(t)dt \end{aligned}$$

inequality is found.

Now clearly;

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x \pm t) - f(x)| dx &\leq \int_{-\infty}^{\infty} |f(x \pm t)| dx + \int_{-\infty}^{\infty} |f(x)| dx \leq 2\|f\|_1 \\ \sup_{|t| \leq \delta} \int_{-\infty}^{\infty} |f(x \pm t) - f(x)| dx &= \omega_{L_1}(f;\delta) \end{aligned}$$

According to these equations;

$$\begin{aligned} \|A_\lambda(f;x) - f(x)\|_1 &\leq 2\omega_{L_1}(f;\delta) \int_{0}^{\delta} K_\lambda(t)dt + 4\|f\|_1 \int_{\delta}^{\infty} K_\lambda(t)dt \leq & (3.2) \\ &\leq 2\omega_{L_1}(f;\delta) + 4\|f\|_1 \int_{\delta}^{\infty} K_\lambda(t)dt. & (3.3) \end{aligned}$$

From the property of kernel $K_\lambda(t)$, for any $\delta > 0$;

$$\lim_{\lambda\to\lambda_0}\int_\delta^\infty K_\lambda(t)dt = 0.$$

And from the property of modulus of continuity;

$$\lim_{\delta\to 0}\omega_{L_1}(f;\delta) = 0.$$

Firstly take the limit when $\lambda \to \lambda_0$, and then take $\delta \to 0$

$$\lim_{\lambda\to\lambda_0}\|A_\lambda(f;x) - f(x)\|_1 = 0$$

So the theorem is proved. □

Theorem 3.2 *The family of integral operators B_λ transforms of $2\pi-$periodic functions on $L_1(-\pi,\pi)$ as;*

$$B_\lambda(f;x) = \int_{-\pi}^{\pi} f(t)K_\lambda(x-t)dt. \tag{3.4}$$

Assume in the equation (3.4)*, $K_\lambda(t)$ is the family of periodic kernels and $f\epsilon L_1(-\pi,\pi)$. In this case*

$$\lim_{\lambda\to\lambda_0}\|B_\lambda(f;x) - f(x)\|_1 = 0$$

Proof. Obviously;

$$|B_\lambda(f;x) - f(x)| \leq \int_{-\pi}^{\pi}|f(x+t) - f(x)|K_\lambda(t)dt$$

and,

$$\|B_\lambda(f;x) - f(x)\|_1 \leq \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|f(x+t) - f(x)|dx\right)K_\lambda(t)dt$$

We know modulus of continuity on $L_1(-\pi,\pi)$;

$$\omega_{L_1}(f;\delta) = \sup_{|t|\leq\delta}\int_{-\pi}^{\pi}|f(x+t) - f(x)|dx$$

So we can say;

$$\int_{-\pi}^{\pi} |f(x+t) - f(x)|dx \leq \omega_{L_1}(f;|t|)$$

In this case;

$$\|B_\lambda(f;x) - f(x)\|_1 \leq \int_{-\pi}^{\pi} \omega_{L_1}(f;|t|)K_\lambda(t)dt.$$

If we take a sequence of δ_λ from the property of the lemma (2.54);

$$\omega_{L_1}(f;|t|) = \omega_{L_1}\left(f;\frac{|t|}{\delta_\lambda}\delta_\lambda\right) \leq \left(\frac{|t|}{\delta_\lambda}+1\right)\omega_{L_1}(f;\delta_\lambda)$$

is can be written. So;

$$\|B_\lambda(f;x) - f(x)\|_1 \leq \omega_{L_1}(f;\delta_\lambda)\left(\int_{-\pi}^{\pi} \frac{|t|}{\delta_\lambda}K_\lambda(t)dt + \int_{-\pi}^{\pi} K_\lambda(t)dt\right) \tag{3.5}$$

$$= \omega_{L_1}(f;\delta_\lambda)\left(\frac{1}{\delta_\lambda}\int_{-\pi}^{\pi} |t|K_\lambda(t)dt + 1\right)$$

Now if take the function $g(t) = |t|$, $g(t+2\pi) = g(t)$, we see $g\epsilon L_1(-\pi,\pi)$ and;

$$B_\lambda(g,0) = \int_{-\pi}^{\pi} |t|K_\lambda(t)dt$$

From the function g is continuous at $x_0 = 0$, then

$$\lim_{\lambda\to\lambda_0} B_\lambda(g,0) = g(0) = 0.$$

So;

$$\lim_{\lambda\to\lambda_0} \int_{-\pi}^{\pi} |t|K_\lambda(t)dt = 0.$$

Now if take the sequence δ_λ,

$$\delta_\lambda = \int_{-\pi}^{\pi} |t|K_\lambda(t)dt$$

from the equation (3.5);

$$\|B_\lambda(f;x) - f(x)\|_1 \leq 2\omega_{L_1}(f;\delta_\lambda)$$

as $\lambda \to \lambda_0$ then, $\delta_\lambda \to 0$. So $\omega_{L_1}(f;\delta_\lambda) \to 0$ (from lemma (2.47)).

Thus the theorem is complete. □

Corollary 3.3 *If $f \epsilon L_1(-\pi,\pi)$ then the $Fejer$ Integral is defined by;*

$$\sigma_n(f;x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt \tag{3.6}$$

From the equation (3.6)*, we say;*

$$\lim_{n\to\infty}\|\sigma_n(f;x) - f(x)\|_1 = 0.$$

Corollary 3.4 *If $f \epsilon L_1(-\pi,\pi)$ then the $Poisson$ Integral is defined by;*

$$P_r(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(t-x)+r^2} f(t)dt, \qquad 0<r<1 \tag{3.7}$$

From the equation (3.7)*, we say;*

$$\lim_{r\to 1}\|P_r(f;x) - f(x)\|_1 = 0.$$

Corollary 3.5 *For $f \epsilon X(\mathbb{R})$ the singular integral of Gauss-Weierstrass exists for all $x \epsilon \mathbb{R}$ and $t > 0$ and defines a function in $X(\mathbb{R}) \cap C$ satisfying*

$$\|W(f;\circ;t)\|_{X(\mathbb{R})} \leq \|f\|_{X(\mathbb{R})} \qquad (t>0) \tag{3.8}$$

$$\lim_{t\to 0}\|W(f;\circ;t) - f(\circ)\|_{X(\mathbb{R})} = 0. \tag{3.9}$$

The periodic kernel of Weierstrass and the nonperiodic kernel of Gauss-Weierstrass are also related. Indeed

$$\sum_{k=-\infty}^{\infty} e^{-k^2 t}\cos kx = \sqrt{\frac{\pi}{t}}\sum_{k=-\infty}^{\infty} e^{-(x+2k\pi)^2/4t} \qquad (t>0), \tag{3.10}$$

Let us mention another important aspect of the singular integral of Gauss-Weierstrass.

$W(f;x;t)\epsilon C^{\infty}$ for every $f\epsilon X(\mathbb{R})$ and $t > 0$. Thus, in view of (3.9), the set of functions $\{W(f;x;t)\}$ forms a well-behaved, dense subset of $X(\mathbb{R})$. Moreover, as is easily checked, the integral (2.26) is a solution of the initial value problem of the heat equation for an infinite rod, namely of

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (x\epsilon\mathbb{R}, t > 0), \\ \lim_{t\to 0} \|u(\circ,t) - f(\circ)\|_{X(\mathbb{R})} &= 0, \end{aligned}$$

$f\epsilon X(\mathbb{R})$ being the prescribed initial temperature distribution.

3.2 Convergence on $\mathbf{L_p}$ Norm

Theorem 3.6 *Let take the family of integral operators A_λ which transforms on the space $L_p(-\infty,\infty)$, and $K_\lambda(x-t)$ is defined as in definition (1. 20);*

$$A_\lambda(f;x) = \int_{-\infty}^{\infty} f(t)K_\lambda(x-t)dt, \qquad -\infty < x < \infty \tag{3.11}$$

In this case,

$$\lim_{\lambda\to\lambda_0} \|A_\lambda(f;x) - f(x)\|_p = 0$$

Proof. We know,

$$A_\lambda(f;x) - f(x) = \int_0^{\infty} [f(x+t) + f(x-t) - 2f(x)]K_\lambda(t)dt$$

here $x\epsilon(-\infty,\infty)$.

Then if we take the L_p norm with get by the absolute value of both sides.

$$\|A_\lambda(f;x) - f(x)\|_p \leq \left(\int_{-\infty}^{\infty}\left(\int_0^{\infty} |f(x+t) + f(x-t) - 2f(x)|K_\lambda(t)dt\right)^p dx\right)^{\frac{1}{p}}$$

is found and from the Minkowsky Inequality,

$$\|A_\lambda(f;x) - f(x)\|_p \le \left(\int_0^\infty \left(\int_{-\infty}^\infty |f(x+t) + f(x-t) - 2f(x)|^p dx \right)^{\frac{1}{p}} \right) K_\lambda(t) dt$$

For any $\delta > 0$, take the interval $[0, \delta]$ and $[\delta, \infty)$ instead of $(0, \infty)$;

$$\begin{aligned} \|A_\lambda(f;x) - f(x)\|_p &\le \left(\int_0^\delta \left(\int_{-\infty}^\infty |f(x+t) + f(x-t) - 2f(x)|^p dx \right)^{\frac{1}{p}} \right) K_\lambda(t) dt \\ &+ \left(\int_\delta^\infty \left(\int_{-\infty}^\infty |f(x+t) + f(x-t) - 2f(x)|^p dx \right)^{\frac{1}{p}} \right) K_\lambda(t) dt \end{aligned}$$

$$= \iota' + \iota'' \tag{3.12}$$

Now take the term ι';

$$\begin{aligned} \left(\int_{-\infty}^\infty |f(x+t) + f(x-t) - 2f(x)|^p dx \right)^{\frac{1}{p}} &\le \left(\int_{-\infty}^\infty |f(x+t) - f(x)|^p dx \right)^{\frac{1}{p}} \\ &+ \left(\int_{-\infty}^\infty |f(x-t) - f(x)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

and then from the definition of the modulus of continuity on L_p, we can write the inequality;

$$\sup_{|t| \le \delta} \left(\int_{-\infty}^\infty |f(x+t) + f(x-t) - f(x)|^p dx \right)^{\frac{1}{p}} \le 2\omega_{L_p}(f;\delta)$$

If write this inequality in the integral ι'_λ

$$\iota'_\lambda \le 2\omega_{L_p}(f;\delta) \int_0^\delta K_\lambda(t) dt \le 2\omega_{L_p}(f;\delta) \tag{3.13}$$

Now take the term ι_λ''. Again from Minkowsky Inequality;

$$\begin{aligned}\left(\int_{-\infty}^{\infty}|f(x-t)+f(x-t)-2f(x)|^p dx\right)^{\frac{1}{p}} &\le \left(\int_{-\infty}^{\infty}|f(x+t)|^p dx\right)^{\frac{1}{p}}\\ &+ \left(\int_{-\infty}^{\infty}|f(x-t)|^p dx\right)^{\frac{1}{p}}+2\left(\int_{-\infty}^{\infty}|2f(x)|^p dx\right)^{\frac{1}{p}}\\ &\le 4\|f\|_p\end{aligned}$$

From here;

$$\iota_\lambda'' \le 4\|f\|_p\int_{\delta}^{\infty}K_\lambda(t)dt \tag{3.14}$$

If we write the inequalities (3.13) and (3.14) in (3.12);

$$\|A_\lambda(f;x)-f(x)\|_p \le 2\omega_{L_p}(f;\delta)+4\|f\|_p\int_{\delta}^{\infty}K_\lambda(t)dt$$

is found. We know for any $\delta > 0$;

$$\lim_{\lambda\to\lambda_0}\int_{\delta}^{\infty}K_\lambda(t)dt = 0$$

So,

$$\lim_{\lambda\to\lambda_0}\|A_\lambda(f;x)-f(x)\|_p \le 2\omega_{L_p}(f;\delta)$$

Now when $\delta\to 0$;

$$\lim_{\lambda\to\lambda_0}\|A_\lambda(f;x)-f(x)\|_p = 0$$

So theorem is proved. □

Corollary 3.7 *If $f\epsilon L_p(-\infty,\infty)$, then Gauss-Weierstrass integral is;*

$$\lim_{\lambda\to\infty}\|W_\lambda(f;x)-f(x)\|_p = 0$$

Corollary 3.8 *If $f\epsilon L_p(-\infty,\infty)$, then Abel-Poisson integral is;*

$$\lim_{\lambda\to\infty}\|\alpha_\lambda(f;x)-f(x)\|_p = 0$$

Definition 3.9 *Modulus of continuity on $L_p(-\pi, \pi)$ space is defined by;*

$$\omega_{L_p}(f;\delta) = \sup_{|t|\leq\delta}\left(\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}}$$

Theorem 3.10 *Let take the family of integral operators A_λ which transforms on the space $L_p(-\pi, \pi)$, and $K_\lambda(x-t)$ is defined as in definition (1. 20);*

$$A_\lambda(f;x) = \int_{-\infty}^{\infty} f(t)K_\lambda(x-t)dt, \qquad -\infty < x < \infty \tag{3.15}$$

In this case,

$$\lim_{\lambda\to\lambda_0} \|B_\lambda(f;x) - f(x)\|_p = 0$$

Here $\|.\| = \|.\|_{L_p(-\pi,\pi)}$.

Proof. We know,

$$|B_\lambda(f;x) - f(x)| \leq \int_{-\pi}^{\pi} |f(x+t) - f(x)|K_\lambda(t)dt$$

So,

$$\|B_\lambda(f;x) - f(x)\| \leq \left(\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |f(x+t) - f(x)|K_\lambda(t)dt\right)^p dx\right)^{\frac{1}{p}}$$

Let apply the Minkowsky Inequality to the right-hand side integral from the definition of modulus of continuity on L_p;

$$\|B_\lambda(f;x) - f(x)\| \leq \int_{-\pi}^{\pi} W_{L_p}(f;|t|;K_\lambda(t)dt)$$

is found. For any δ_λ;

$$W_{L_p}(f;|t|) \leq \left(\frac{|t|}{\delta_\lambda} + 1\right) W_{L_p}(f;\delta_\lambda)$$

So we can write the inequality;

$$\|B_\lambda(f;x) - f(x)\| \leq W_{L_p}(f;\delta_\lambda)\left(\frac{1}{\delta_\lambda}\int_{-\pi}^{\pi} |t|K_\lambda(t)dt + 1\right)$$

and choose,

$$\delta_\lambda = \int_{-\pi}^{\pi} |t| K_\lambda(t) dt$$

Then

$$\|B_\lambda(f;x) - f(x)\| \leq 2W_{L_p}(f;\delta_\lambda)$$

Now let denote $\lim_{\lambda\to\lambda_0} \delta_\lambda = 0$ $K_\lambda(t)$ is an even function so,

$$\delta_\lambda = \int_{-\pi}^{\pi} |t| K_\lambda(t) dt = 2\int_{0}^{\pi} tK_\lambda)(t) dt$$

Next for any $\beta > 0$;

$$\begin{aligned}
\delta_\lambda &= 2\int_{0}^{\beta} tK_\lambda(t) dt + 2\int_{\beta}^{\pi} tK_\lambda(t) dt \\
&\leq 2\beta \int_{-\pi}^{\pi} K_\lambda(t) dt + 2\sup_{t\geq\beta} K_\lambda(t)(\pi - \beta) \\
&\leq 2\beta + 2\pi \sup_{t\geq\beta} K_\lambda(t).
\end{aligned}$$

Property of the kernel $K_\lambda(t)$, for any $\beta > 0$;

$$\lim_{\lambda\to\lambda_0} \left(\sup_{t\geq\beta} K_\lambda(t) \right) = 0$$

is obvious. Then,

$$0 \leq \lim_{\lambda\to\lambda_0} \delta_\lambda \leq 2\beta$$

Now when $\beta \to 0$;

$$\lim_{\lambda\to\lambda_0} \delta_\lambda = 0$$

We know that if $\delta_\lambda \to 0$, then $W_{L_p}(f;\delta_\lambda) \to 0$.

So the theorem is proved. □

Corollary 3.11 *If $f \epsilon L_p(-\pi,\pi)$, then Poisson integral $P_r(f;x)$ is;*

$$\lim_{r\to 1} \|P_r(f;x) - f(x)\|_p = 0$$

Corollary 3.12 *If $f \epsilon L_p(-\pi,\pi)$, then Fejer integral $\sigma_n(f;x)$ is;*

$$\lim_{r\to 1} \|\sigma_n(f;x) - f(x)\|_p = 0$$

CHAPTER 4

CONVERGENCE OF NONLINEAR CONVOLUTION TYPE SINGULAR INTEGRAL OPERATORS

The theory of approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [9] and widely developed in [12]. He considers nonlinear integral operators, replacing linearity assumption by Lipschitz condition for kernel functions generating the operators and satisfying suitable singularity assumptions. After this discovery, several mathematicians have undertaken the program of extending approximation by nonlinear integral operators in many ways and to several settings, including modular function spaces, pointwise and uniform convergence of operators, Korovkin type theorems, abstract function spaces, sampling series and so on. Especially, this kind of operators were extensively studied by C. Bardaro, J. Musielak and G. Vinti [23]-[24] in connection with the theory of Modular space. Operators of type (1) and its special cases were studied by Swiderski and Wachnicki [11], Karsli [10]-[35] and Karsli-Ibikli [40] in some Lebesgue spaces. Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators. We also refer the reader to [19]-[20], [25]-[29] as well as the monographs [6] and [12] where other kinds of convergence results concerning the pointwise convergence and rate of pointwise convergence of linear and nonlinear singular integral operators in the Lebesgue and Musielak-Orlicz spaces.

The approach of Musielak on nonlinear integral operators can be given as follows;

In the theory of Fourier series there appears the problem of approximation of 2π-periodic functions f, integrable in the interval $[-\pi, \pi]$ by means of linear, singular integral operators K_n of the form $\int_{-\pi}^{\pi} K_n(t-s) f(t)dt$. For example, in case of the first arithmetic means of partial sums of Fourier series, $K = (K_n)_{n=1}^{\infty}$; is the Fejer kernel

$$K_n(u) = \frac{1}{2n\pi}\left(\frac{\sin(1/2)nu}{\sin(1/2)u}\right)^2,$$

and the singularity conditions mean that $\int_{-\pi}^{-\delta} K_n(u)du + \int_{\delta}^{\pi} K_n(u)du \to 0$ as $n \to \infty$ for every $0 < \delta < \pi$ and $\int_{-\pi}^{\pi} K_n(u)du = 1$ for $n = 1, 2, \cdots$.

These problems are extended in three directions:

1. The interval $[-\pi, \pi]$ is replaced by a compact abelian group G with Haar measure,
2. the set $N = 1, 2, \cdots$ of indices with convergence $\to^{\infty}$ generated by a filter m of subsets of w,
3. the kernel $K = (K_n)_{n=1}^{\infty}$ is replaced by a kernel $K = (K_u)_{u\epsilon w}$ where $K_u : G \times R \to$; the last means that we take nonlinear integral operators T_u

$$(T_u f)(s) = \int_G K_u(t-s, f(t))dt = \int_G K_u(t, f(t+s))dt \tag{4.1}$$

in place of the above considered linear operators.

The assumption of linearity of the operators was replaced in [1] by an assumption of a Lipschitz condition for K_u with respect to the second variable.

Now we will investigate a study of Karslı, which is deal with the pointwise convergence of nonlinear singular integral operators, namely; Approximation Properties of Nonlinear Singular Integral Operators of Convolution Type.

Let Λ be a nonempty set of indices and λ_0 be an accumulation point of Λ. We take a family $\mathcal{K}$ of functions $K : R\text{x}\Lambda\,\text{x}\,R \to R$, where $K(t, \lambda, 0) = 0$ for all $t \in R$ and $\lambda \in \Lambda$, such that $K(t, \lambda, u)$ are integrable over R in the sense of Lebesgue measure with respect to the first variable for all values of the third variable, for every $\lambda \in \Lambda$. The family $\mathcal{K}$ will be called a kernel.

In papers [14] Taberski has shown that the pointwise convergence at which the points are Lebesgue point of integrable functions in $L_1(-\pi, \pi)$, by the family of convolution type singular integral operators depending on two parameters of the form

$$U(f;x,\lambda)=\int_{-\pi}^{\pi} f(t)\,K(t-x,\lambda)\,dt \quad , \quad x\in(-\pi,\pi) \tag{4.2}$$

where $K(t,\lambda)$ is a kernel which satisfies suitable assumptions. Moreover Taberski has shown also that in [14] approximation properties of the derivatives of this operators.

After this fundamental study of Taberski [14], this operators were studied by some other authors. Gadjiev [15] and Rydzewska [16] obtained the pointwise convergence of this operators at which the points are respectively, generalized Lebesgue point and $\mu-$generalized Lebesgue point of integrable functions in $L_1(-\pi,\pi)$. In 1984 Bardaro and Gori Cocchieri [17] estimated the degree of pointwise approximation of Féjer-type singular integrals, which are the special case of (4.2), in the class of functions $f\in L_1(R)$ at which the points are generalized Lebesgue point. Very recently Karsli and İbikli [18], [13] studied both pointwise convergence and pointwise convergence of the *rth* derivatives of the operators $T(f;x,\lambda)$, which are defined as

$$T(f;x,\lambda)=\int_{a}^{b} f(t)\,K(t-x,\lambda)\,dt \quad , \quad x\in(a,b).$$

In approximation theory, however, applications we limited to linear integral operators because the notion of singularity of an integral operators was closely connected with its linearity. In 1981 Musielak [9] used the nonlinear integral operators, which were defined as

$$T_w f(s)=\int_{G} K_w(t-s,f(t))\,dt \quad , \quad s\in G \tag{4.3}$$

by the assumption of linearity of the operators was replaced by an assumption of a Lipschitz condition for K_w with respect to the second variable. This study allows us that using the classical method for linear integral operators [6] to obtain the convergence of the nonlinear integral operators, although the notion of singularity of an integral operators was closely connected with its linearity [12]. Recently Swiderski and Wachnicki [11] estimated the pointwise convergence of the operators (4.3) in the case f belongs to $L_p(-\pi,\pi)$ and $L_p(R)$ on the points, where x_0 is a continuous point and a Lebesgue point. Further paper on the subject were written by Karsli [10].

The purpose of this note is to study both the pointwise approximation and the degree of pointwise approximation of the operators

$$T(f;x,\lambda) = \int_a^b K(t-x,\lambda,f(t))\,dt \quad , \quad x \in <a,b>, \tag{4.4}$$

on a generalized Lebesgue point of $f \in L_1 <a,b>$ as $(x,\lambda) \to (x_0,\lambda_0)$, where $<a,b>$ is an arbitrary interval in R.

Let $L_1 <a,b>$ be the class of functions Lebesgue integrable in the interval $<a,b>$, where $<a,b>$ is an arbitrary interval in R.
Denote the auxiliary function $\tilde{f} \in L_1(R)$ as below

$$\tilde{f}(t) := \begin{cases} f(t) & , \quad t \in <a,b> \\ 0 & , \quad t \notin <a,b> \end{cases} . \tag{4.5}$$

We assume that the function $K : R \text{x} \Lambda \text{ x } R \to R$ satisfies;

a) Let $L(t,\lambda)$ be a integrable function with respect to t such that

$$|K(t,\lambda,u) - K(t,\lambda,v)| \le L(t,\lambda)\,|u-v|\,,$$

for every $t,u,v \in R$ and for any $\lambda \in \Lambda$.

b) $\lim\limits_{\lambda\to\lambda_0} \int\limits_{R\setminus U} L(t,\lambda)dt = 0$, for every $U \in \mathcal{U}(0)$. By $\mathcal{U}(0)$ we denote the family of all neighbourhoods of zero in R.

c) $\lim\limits_{\lambda\to\lambda_0} [\sup\limits_{|t|\ge\delta} L(t,\lambda)] = 0$, for every $\delta > 0$.

d) $\lim\limits_{\lambda\to\lambda_0} \int\limits_R K(t,\lambda,u)\,dt = u$, for every $u \in R$.

e) There exists a $\delta_0 > 0$, such that $L(t,\lambda)$ is non-decreasing on $(-\delta_0, 0]$ and non-increasing on $[0,\delta_0)$ as a function of t, for each $\lambda \in \Lambda$.

The condition $a)$ implies that the class of all functions $K(t,\lambda,u)$ is a kernel.

Theorem A. [9] Let $1 \le p < \infty$ and assume that a function $K_\lambda(t,u)$ satisfies condition $a)$. If $f \in L_p < a,b >$, then $T(f) \in L_p < a,b >$ and

$$\|T(f)\|_{L_p<a,b>} \le \; H(\lambda)\,\|f\|_{L_p<a,b>}$$

for each $\lambda \in \Lambda$.

Pointwise Convergence

Firstly, we assume that $< a,b >= [a,b]$. This means, $< a,b >$ is a finite interval of R.

Theorem 2. Suppose that a kernel function $K(t,\lambda,u)$ satisfies the condition $a),\ b),\ c),\ d)$ and $e)$. Then at each point x_0 for which

$$\lim_{h\to 0^+}\frac{1}{h^{\alpha+1}}\int\limits_0^h |f(x_0+t)-f(x_0)|\,dt = 0, \quad 0 \le \alpha < 1 \tag{4.6}$$

holds we have

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T(f;x,\lambda)-f(x_0)| = 0,$$

on any planar set Z on which the function

$$\int\limits_{x_0-\delta}^{x_0+\delta} L(t-x,\lambda)\ |t-x_0|^{\alpha}\,dt + 2L(0,\lambda)\,|x-x_0|^{\alpha+1}\,, \tag{4.7}$$

is bounded, where $0 < \delta < \delta_{0.}$

Proof. We assume that

$$x_0+\delta < b,\ \ x_0-\delta > a \ \text{ and } \ 0 < x_0 - x < \frac{\delta}{2}, \tag{4.8}$$

for any $0 < \delta < \delta_0$. From (4.5) and d), we can write;

$$\begin{aligned} T(f;x,\lambda) - f(x_0) &= \int_a^b K(t-x,\lambda,f(t))dt - f(x_0) \\ &= \int_R K(t-x,\lambda,\tilde{f}(t))dt - \int_R K(t-x,\lambda,\tilde{f}(x_0))dt \\ &\quad + \int_R K(t-x,\lambda,\tilde{f}(x_0))dt - f(x_0). \end{aligned}$$

Here we take absolute value of the bothside of the last equality, we get

$$\begin{aligned} |I(x,\lambda)| &\leq \int_R \left| K(t-x,\lambda,\tilde{f}(t)) - K(t-x,\lambda,\tilde{f}(x_0)) \right| dt \\ &\quad + \left| \int_R K(t-x,\lambda,\tilde{f}(x_0))dt - f(x_0) \right|. \end{aligned} \tag{4.9}$$

According to a),

$$\left| K(t-x,\lambda,\tilde{f}(t)) - K(t-x,\lambda,\tilde{f}(x_0)) \right| \leq L(t-x,\lambda) \left| \tilde{f}(t) - \tilde{f}(x_0) \right|. \tag{4.10}$$

holds for any $\lambda \in \Lambda$. According to (4.10) we can rewrite (4.9) as follows

$$\begin{aligned} |I(x,\lambda)| &\leq \int_R L(t-x,\lambda) \left| \tilde{f}(t) - \tilde{f}(x_0) \right| dt \\ &\quad + \left| \int_R K(t-x,\lambda,\tilde{f}(x_0))dt - f(x_0) \right|. \end{aligned}$$

For using the property of the point, we shall split the above inequality in six terms as;

$$\begin{aligned}|T(f;x,\lambda)-f(x_0)| \leq & \int\limits_{t\notin<a,b>} \left|\tilde{f}(t)-\tilde{f}(x_0)\right| L(t-x,\lambda)dt \\ & +\left\{\int\limits_a^{x_0-\delta}+\int\limits_{x_0-\delta}^{x_0}+\int\limits_{x_0}^{x_0+\delta}+\int\limits_{x_0+\delta}^{b}\right\}\left|\tilde{f}(t)-\tilde{f}(x_0)\right| L(t-x,\lambda)dt \\ & +\left|\int\limits_R K(t-x,\lambda,\tilde{f}(x_0))dt - f(x_0)\right| \\ = \; & I_0(x,\lambda)+I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda)+I_4(x,\lambda)+I_5(x,\lambda)\end{aligned} \tag{4.11}$$

Thus, it is sufficient to show that the first five terms on the right hand side of (4.11) tends to zero as $(x,\lambda)\to(x_0,\lambda_0)$.

Firstly we consider $I_5(x,\lambda)$.

From the condition d), it is easy to see that

$$I_5(x,\lambda)=\left|\int\limits_R K(t-x,\lambda,\tilde{f}(x_0))dt - f(x_0)\right| \to 0 \tag{4.12}$$

as $(x,\lambda)\to(x_0,\lambda_0)$.

Next we consider $I_1(x,\lambda)$.

From (4.5), it is seen that for $t\in[a,b]$, $\tilde{f}(t)=f(t)$. Hence we can write,

$$I_1(x,\lambda)=\int\limits_a^{x_0-\delta}|f(t)-f(x_0)|\,L(t-x,\lambda)dt.$$

According to (4.8) we obtain

$$\begin{aligned}I_1(x,\lambda) & \leq \sup_{x_0-\delta<t<a} L(t-x,\lambda)\int\limits_a^{x_0-\delta}|f(t)-f(x_0)|\,dt \\ & \leq \sup_{|u|>\frac{\delta}{2}} L(u,\lambda)\left(\|f\|_{L_1<a,b>}+|f(x_0)|\,(b-a)\right).\end{aligned}$$

In the same way, we find

$$I_4(x,\lambda) \leq \sup_{|u|>\frac{\delta}{2}} L(u,\lambda)\left(\|f\|_{L_1<a,b>} + |f(x_0)|\,(b-a)\right).$$

Thus, we get

$$I_1(x,\lambda) + I_4(x,\lambda) \leq 2\sup_{|u|>\frac{\delta}{2}} L(u,\lambda)\left(\|f\|_{L_1<a,b>} + |f(x_0)|\,(b-a)\right), \tag{4.13}$$

which in view of c) tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Next we consider $I_2(x,\lambda)$.

Denote

$$F(t) := \int_t^{x_0} |f(y) - f(x_0)|\,dy. \tag{4.14}$$

According to (4.6), to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$F(t) \leq \varepsilon\ (x_0 - t)^{\alpha+1}. \tag{4.15}$$

We now fix this δ. Evaluation of $I_2(x,\lambda)$ is done the following way.

$$I_2(x,\lambda) = \int_{x_0-\delta}^{x_0} |f(t) - f(x_0)|\,L(t-x,\lambda)dt$$

By (4.14) we can write

$$|I_2(x,\lambda)| = \left| -\int_{x_0-\delta}^{x_0} L(t-x,\lambda)dF(t)\right|.$$

Integration by parts gives

$$|I_2(x,\lambda)| \leq |F(x_0-\delta)|\,L(x_0-\delta-x,\lambda) + \int_{x_0-\delta}^{x_0} |F(t)|\,\frac{\partial}{\partial t}L(t-x,\lambda)\,dt.$$

According to (4.15) we obtain

$$|I_2(x,\lambda)| \leq \varepsilon\,\delta^{\alpha+1}\,L(x_0-\delta-x,\lambda)+\varepsilon\int\limits_{x_0-\delta}^{x_0}(x_0-t)^{\alpha+1}\frac{\partial}{\partial t}L(t-x,\lambda)dt.$$

Integration by parts again, then

$$|I_2(x,\lambda)| \leq \varepsilon\,(\alpha+1)\int\limits_{x_0-\delta}^{x_0}(x_0-t)^{\alpha}L(t-x,\lambda)\,dt.$$

According to d) and (4.8) (see [13]), we obtain

$$|I_2(x,\lambda)| \leq \varepsilon\left\{(\alpha+1)\int\limits_{x_0-\delta}^{x_0}L(t-x,\lambda)(x_0-t)^{\alpha}dt+2L(0,\lambda)\,|x-x_0|^{\alpha+1}\right\}. \tag{4.16}$$

We can use a similar method for evaluating $I_3(x,\lambda)$. Then, we find the following inequality;

$$|I_3(x,\lambda)| \leq \varepsilon\int\limits_{x_0}^{x_0+\delta}L(t-x,\lambda)(t-x_0)^{\alpha}dt. \tag{4.17}$$

Combining (4.16) and (4.17) we obtain

$$\begin{aligned}|I_2(x,\lambda)|+|I_3(x,\lambda)| \quad &\leq \quad \varepsilon(\alpha+1)\int\limits_{x_0-\delta}^{x_0+\delta}L(t-x,\lambda)\,|t-x_0|^{\alpha}\,dt\\ &\quad +\varepsilon 2L(0,\lambda)\,|x-x_0|^{\alpha+1}\,, \end{aligned} \tag{4.18}$$

which in view of (4.7) tends to zero as $(x,\lambda)\to(x_0,\lambda_0)$.

Finally, we consider $I_0(x,\lambda)$.

$$I_0(x,\lambda)=\int\limits_{t\notin<a,b>}\left|\tilde{f}(t)-\tilde{f}(x_0)\right|L(t-x,\lambda)dt$$

From (4.5), $\tilde{f}(t) = 0$ for $t \notin < a, b >$. Hence

$$I_0(x, \lambda) = |f(x_0)| \int_{t \notin <a,b>} L(t - x, \lambda) dt.$$

According to (4.6), when $t \notin [a, b]$, we have $t < a$ or $t > b$. This implies that

$$t - x < a - x < x_0 - x - \delta < -\frac{\delta}{2} < 0$$

and

$$t - x > b - x > x_0 - x + \delta > \delta > 0.$$

Thus for any $\delta > 0$ there exists a neighborhood U of zero such that

$$\int_{t \notin <a,b>} L_\lambda(t - x) dt \leq \int_{|t-x| > \frac{\delta}{2}} L_\lambda(t - x) dt = \int_{R \setminus U} L_\lambda(u) du.$$

Hence we get

$$I_0(x, \lambda) \leq |f(x_0)| \int_{R \setminus U} L_\lambda(u) du. \tag{4.19}$$

which in view of b) tends to zero as $(x, \lambda) \to (x_0, \lambda_0)$.

Combining (4.12), (4.13), (4.18) and (4.19), we get the required result. From the proof given it is evident that the proof remains in force even when $0 < x_0 - x < \frac{\delta}{2}$.
This completes the proof of the theorem.

Secondly, we consider the case $< a, b >= R$, we have the following theorem.

Theorem 3. Suppose that the condition of Theorem 2 satisfies. Let x_0 be a generalized-Lebesgue point of function $f \in L_1(R)$. Then

$$\lim_{(x,\lambda) \to (x_0, \lambda_0)} |T(f; x, \lambda) - f(x_0)| = 0,$$

if (x, λ) tends to (x_0, λ_0) on any planar set Z on which (4.8) satisfies.

Proof. Analogously to the proof of Theorem 2.

Remark 1. Now, we consider the cases $< a, b >= (-\pi, \pi)$,

$$K(t, \lambda, u) = L_\lambda(t)\, u,$$

$f(t)$ and $L_\lambda(t)$ are $2\pi-$periodic functions with respect to t. This case was widely used in Approximation Theory (See [6]).

Rate of Pointwise Convergence

In this section we shall estimate the rate of pointwise convergence, which we have obtained in previous section, respectively.

Theorem 4. Suppose that a kernel function $K(t, \lambda, u)$ satisfies the condition $a)$, $b)$, $c)$, $d)$ and $e)$. Let

$$\Delta(x, \lambda, \delta) = \int_{x_0-\delta}^{x_0+\delta} L(t - x, \lambda)\ |t - x_0|\, dt,$$

such that for every $|u| > 0$ and for arbitrary $0 < \alpha < 1$,

$$L(u, \lambda) = o(\Delta^\alpha(x, \lambda, \delta)),$$

$$\int_{R \setminus U} L(u, \lambda) du = o(\Delta^\alpha(x, \lambda, \delta))$$

and

$$\left| \int_R K(t - x, \lambda, \tilde{f}(x_0)) dt - f(x_0) \right| = o(\Delta^\alpha(x, \lambda, \delta))$$

as $(x, \lambda) \to (x_0, \lambda_0)$. Then we get

$$|T(f; x, \lambda) - f(x_0)| = o(\Delta^\alpha(x, \lambda, \delta)).$$

Proof. In Theorem 2 we have obtained the following inequality;

$$
\begin{aligned}
|T(f;x,\lambda)-f(x_0)| \;\le\; & |f(x_0)|\int\limits_{R\setminus U} L(t-x,\lambda)dt \\
& +\sup_{|u|>\frac{\delta}{2}} L(u,\lambda)\left[\|f\|_{L_1<a,b>}+|f(x_0)|(b-a)\right]. \\
& +\varepsilon(\alpha+1)\int\limits_{x_0-\delta}^{x_0+\delta} |t-x_0|^{\alpha} L(t-x,\lambda)\,dt \\
& +\left|\int\limits_{R} K(t-x,\lambda,\tilde{f}(x_0))dt-f(x_0)\right|.
\end{aligned}
$$

Here, we take $p=\frac{1}{\alpha}$ and using Hölder inequality, we get

$$
\int\limits_{x_0-\delta}^{x_0+\delta} |t-x_0|^{\alpha} L(t-x,\lambda)\,dt=\int\limits_{x_0-\delta}^{x_0+\delta} |t-x_0|^{\alpha} L(t-x,\lambda)^{1-\alpha+\alpha}\,dt
$$

$$
\begin{aligned}
&= \int\limits_{x_0-\delta}^{x_0+\delta} [|t-x_0|\,L(t-x,\lambda)]^{\alpha} L(t-x,\lambda)^{1-\alpha}\,dt \\
&\le \left(\int\limits_{x_0-\delta}^{x_0+\delta}\left[L(t-x,\lambda)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}dt\right)^{1-\alpha}\left(\int\limits_{x_0-\delta}^{x_0+\delta}([|t-x_0|\,L(t-x,\lambda)]^{\alpha})^{\frac{1}{\alpha}}\,dt\right)^{\alpha} \\
&\le \left(\int\limits_{a}^{b} L(t-x,\lambda)dt\right)^{1-\alpha}\Delta^{\alpha}(x,\lambda,\delta) \\
&= \Delta^{\alpha}(x,\lambda,\delta)
\end{aligned}
$$

The same inequality we also obtain for $-\frac{\delta}{2}<x_0-x<0$.

Therefore our theorem now follows, from the conditions b) and c).

$$
|T(f;x,\lambda)-f(x_0)|=o(\Delta^{\alpha}(x,\lambda,\delta)).
$$

Thus the proof of the theorem is complete.

Theorem 5. Suppose that the condition of Theorem 3 satisfies together with

$$\int\limits_{|t-x_0|>\delta} L(t-x,\lambda)dt = o(\Delta^{\alpha}(x,\lambda,\delta)).$$

Then

$$|T(f;x,\lambda) - f(x_0)| = o(\Delta^{\alpha}(x,\lambda,\delta)).$$

Proof. Proof is the analogous to the proof of Theorem 4.

Example. Setting

$$K(t,\lambda,u) = \begin{cases} \lambda\, u + \sin\frac{\lambda u}{2} & , \quad t \in [0,\frac{1}{\lambda}] \\ 0 & , \quad t \notin [0,\frac{1}{\lambda}] \end{cases}, \tag{4.20}$$

where $\Lambda = [1,\infty)$ be a nonempty set of indices with natural topology and $\lambda_0 = \infty$ is an accumulation point of Λ in this topology. For (4.20), it is seen that for $u, v \in R$ and $t \in [0,\frac{1}{\lambda}]$,

$$|K(t,\lambda,u) - K(t,\lambda,v)| \le \lambda\ |u-v| \tag{4.21}$$

and for $u, v \in R$ and $t \notin [0,\frac{1}{\lambda}]$,

$$|K(t,\lambda,u) - K(t,\lambda,v)| = 0. \tag{4.22}$$

Combining (4.21) and (4.22), we get

$$L(t,\lambda) = \begin{cases} \lambda & , \quad t \in [0,\frac{1}{\lambda}] \\ 0 & , \quad t \notin [0,\frac{1}{\lambda}] \end{cases}. \tag{4.23}$$

Moreover

$$\int\limits_{R} L(t,\lambda)dt = \int\limits_{[0,\frac{1}{\lambda}]} \lambda dt = 1 < \infty$$

and

$$\int_R K(t-x,\lambda,u)\,dt = \int_{[0,\frac{1}{\lambda}]} \left[\lambda\,u + \sin\frac{\lambda\,u}{2}\right] dt = \left[\lambda u + \sin\frac{\lambda\,u}{2}\right]\frac{1}{\lambda}$$
$$= u + \frac{1}{\lambda}\sin\frac{\lambda\,u}{2} \to u \ \text{ as } (x,\lambda) \to (x_0,\infty).$$

Also

$$K(t-x,\lambda,u) = \begin{cases} \lambda\,u + \sin\frac{\lambda\,u}{2} & , \quad t-x \in [0,\frac{1}{\lambda}] \\ 0 & , \quad t-x \notin [0,\frac{1}{\lambda}] \end{cases},$$

Hence, the kernel function K satisfies the condition a), b), c), d) and e).

If we use this kernel function in Theorem 3 and Theorem 4, we find

$$\int_{x_0-\delta}^{x_0+\delta} |L(t-x,\lambda)|\;|t-x_0|^{\alpha}\,dt + 2\,|L(0,\lambda)|\,|x-x_0|^{\alpha+1} = \int_{[x,x+\frac{1}{\lambda}]} \lambda\;|t-x_0|^{\alpha}\,dt + 2\,\lambda\;|x-x_0|^{\alpha+1}.$$

The first part of the right hand side of this equality is finite. Besides

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \lambda\;|x-x_0|^{\alpha+1} = M < \infty,$$

if and only if $|x-x_0| = o(\lambda^{\frac{1}{\alpha+1}})$ or the rates of convergence of $|x-x_0|$ and $(\lambda^{\frac{1}{\alpha+1}})$ are equivalent.

If we use this kernel function in Theorem 5 and Theorem 6, for sufficiently large λ, we find that

$$\Delta(x,\lambda) = \int_{x_0-\delta}^{x_0+\delta} L(t-x,\lambda)\;|t-x_0|\;dt$$
$$= \int_{[x,x+\frac{1}{\lambda}]} \lambda\;|t-x_0|\,dt.$$

From Theorem 2, we obtain $\lim\limits_{(x,\lambda)\to(x_0,\infty)} \Delta(x,\lambda) = 0$. Moreover the rate of convergence of $\Delta(x,\lambda) \to 0$ is equivalent to $\frac{1}{\lambda^{\frac{1}{\alpha+1}}} \to 0$.
According to (4.23) it is seen that $L(u,\lambda) = o(\frac{1}{\lambda})$.

Since $(\frac{1}{\lambda}) = o(\frac{1}{\lambda^{\frac{1}{\alpha+1}}})$, $L(u,\lambda) = o(\frac{1}{\lambda^{\frac{1}{\alpha+1}}})$ is true.

Finally

$$\int_R K(t-x,\lambda,u)\,dt = u + \frac{1}{\lambda}\sin\frac{\lambda u}{2} \to u \text{ as } (x,\lambda) \to (x_0,\infty).$$

This implies $\lim\limits_{(x,\lambda)\to(x_0,\infty)} \frac{1}{\lambda}\sin\frac{\lambda u}{2} = 0$ and $\frac{1}{\lambda}\sin\frac{\lambda u}{2} = o(\frac{1}{\lambda^{\alpha}})$ $(0 < \alpha < 1)$.

Thus, we find

$$|T(f;x,\lambda) - f(x_0)| = o(\frac{1}{\lambda^{\frac{\alpha}{\alpha+1}}}).$$

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