Periodic Review Inventory Control and Dynamic Pricing for Perishable Products under Uncertain and Time Dependent Demand

by

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This is to certify that I have examined this copy of a master's thesis by

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and that any and all revisions required by the final
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ABSTRACT

Stored items with fixed finite lifetimes are usually referred to as perishable items in the inventory literature. Fresh foods, groceries, pharmaceuticals, composite materials, sheet metal, packaged chemical products and blood and its derivatives are a few examples of perishable goods found in a wide variety of industries. Due to their common occurrence and importance, a broad literature has developed over the years on management of perishable inventories. However, the existing literature is insufficient in addressing the mostly encountered setting where items have fixed shelf lives, ordering costs are not negligible and age of inventory and pricing are taken into account. We consider coordinated dynamic pricing and inventory decisions for perishable products in a periodically reviewed inventory system with a time and price dependent random demand. We consider the products with a fixed shelf life time and use dynamic programming to model this system in order to determine the optimal price and inventory decisions. In the static pricing case, a stochastic DP model provides optimal policy indicating the proper time to order, optimal amount of ordering size and a single price that should be charged to the products. For the dynamic pricing case, different prices may be chosen at each state. We prove certain structural characteristics of the optimal solution and also analyze the effect of different parameters on the optimal solution through numerical experiments. Moreover, we examined optimal pricing policy properties by analyzing optimal price behavior versus inventory size and shelf lifetime. In addition, we develop simple-to-implement heuristics for the pricing and inventory control and investigate the effectiveness of these heuristics.

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Chapter 1

INTRODUCTION

Global grocery and supermarket are among the largest markets in the world and perishable products such as fresh produce, dairy and meat constitute a large section of these markets. Due to their deterioration over time, the demand for these products depends highly on their freshness and these products become totally obsolete after a certain amount of time causing a high amount of wastage and decreases in grocery profits. In addition, customers are asking for higher product variety in perishable product categories, leading to less predictable demand per product and to more out-dating. Effective management of these perishable products is an important issue since it is observed that billions of dollars' worth of food is expired and wasted every month (Minner and Transchel [16]). Different from non-perishable products, the freshness or age of these products is also a major concern for the managers in addition to the amount of these products at hand. The demand rate for these products will decrease as the products at hand start to age, and the managers need to decide when to order a new batch and what price to charge for the products at hand in order to control the demand and increase the profits. As a result, the time dependency of demand and the perishability of these products make the management of inventory and pricing much harder.

Fresh foods, groceries, pharmaceuticals, composite materials, sheet metal, packaged chemical products and blood and its derivatives are a few examples of perishable goods found in a wide variety of industries. Due to their common occurrence and importance, a broad literature has developed over the years on management of perishable inventories. In this study, we analyze the coordinated inventory control and dynamic pricing decisions for perishable products with a fixed life time, such as dairy products. These products generally have a short shelf life and have a certain expiration date. As the expiration date approaches, the demand for these products start to decline since the customers do not prefer to buy such products that will expire in a very near future. Thus, the demand for these products decrease every day until the shelf is replenished with a new batch of fresh products. In our study, we consider a time and price dependent random demand function in a periodically reviewed inventory system and use dynamic programming in order to determine the best times for replenishment, optimal replenishment quantities and the optimal prices to charge depending on the state of the system. We also develop easy-to-use heuristics for these decisions and analyze the effectiveness of these heuristics under different conditions.

In recent years, dynamic pricing is seen to be increasingly adopted for perishable products in manufacturing and retail industries. For example, perishable products like fruits or eggs are generally priced at a high price when they are fresh. However, their prices are decreased as they start to deteriorate since their quality decreases and there is a higher risk of them not being sold and becoming obsolete in a near future. In the past, companies generally used a fixed price over a long period of time since they faced high costs in changing prices and had insufficient information about customer demand. However, with the improvements in technology today, they have much more data about the customers and can change their prices at relatively low costs suggesting a dynamic pricing policy to be more profitable.

In the literature, there is a stream of research that considers pricing and inventory control under deterministic demand. Benkherouf [6], Mishra and Singh [17], Broekmeulen and Donselaar [8] analyze the inventory control decisions for perishable products with time-dependent demand but they do not consider pricing in their studies. Transchel and Minner [25] consider coordinated pricing and inventory control for non-perishable products under a deterministic demand. Rajan and Steinberg [23], Abad [1, 2, 3, 4], Burwell et al. [9], Wee [26], Mukhopadhyay et al. [19], Sana [24] and P-S You [21] are the studies that consider coordinated pricing and inventory control for perishable products but they all assume a deterministic demand function.

When we look at the literature for the inventory management and pricing of perishable products under random demand, we observe that the inventory and pricing decisions for perishable products are generally studied separately from each other. In the inventory control literature, most of the inventory models assume that inventories can be held in stock indefinitely to satisfy future demands and the demand is independent of the age of the products at hand. Minner and Transchel [16] consider the ordering policies for perishable products in the food retail industry under service level constraints and present a method to determine the order quantities, but they do not consider pricing in their study. Nahmias [20] reviews related literature about determining the ordering policies for perishable inventory systems for products with both fixed life and having exponential decay over time. Rafaat [22], Goyal and Giri [15] and Bakker et al. [5] also present surveys about the inventory management of deteriorating items.

In the dynamic pricing literature, there are many models that consider a system in which a fixed amount of inventory needs to be sold until a certain time, without any replenishment opportunities, as in the airline or hotel management industries. Gallego and van Ryzin [13], Zhao and Zheng [27], Monahan et al. [18], Gayon et al. [14] are a few examples of these studies. Bitran and Caldentey [7] present an extensive review of dynamic pricing policies in the literature. However, in most of these models, the demand is assumed to be independent of time and there is no inventory replenishment decisions.

There are also several studies in literature considering the inventory and pricing problems for perishable products with time dependent demand. Chen et. al. [10] focus on the joint pricing and inventory control problem for perishable inventory systems with stochastic demand and lead times and focus on the structural properties of the optimal solution. Chen and Sapra [11] consider the joint inventory and pricing decisions for perishable products in a periodic review model with a fixed lifetime of two periods. Chew Peng and Chulung [12] develop a discrete time dynamic programming model to determine the optimal inventory allocations and the optimal prices for a perishable product with a two period lifetime and they obtain several optimality properties. They also propose three heuristics to obtain the inventory allocations and the prices when the lifetime of the product is longer than two periods, since the optimality properties for two period lifetime in their model do not hold for longer lifetimes.

In this study, different from the literature, we consider the coordinated inventory control and pricing decisions for perishable products with fixed shelf lives under a time and price dependent random demand function. We consider a periodically reviewed inventory system with fixed ordering costs and use dynamic programming in order to determine the optimal times to order a new batch, optimal quantity to order and the optimal prices to set depending on the state of the inventory at hand. In our model, we do not restrict ourselves to products with only two-period lifetimes but instead we assume a last-in-first-out (LIFO) policy in sales. The customers prefer fresher products and the demand for older products will be very low and they can rarely be sold before they expire when there are fresher products staying just next to them, hence the older products are salvaged when a new batch is ordered. In addition to proving certain structural results about the optimal policy in our model, benefiting from these results, we also develop easy-to-use heuristics to manage inventory and pricing of perishable products. We also analyze the effects of the system parameters on the optimal solution and investigate the effectiveness of the heuristics under different conditions. In the next section, we present our model and provide the analytical results. Then, in section 3, we present the heuristic methods to approximate the optimal solution and in section 4, we present our numerical results. Finally, in section 5, we conclude our study.

1.1 Literature Review

Excellent literature reviews on perishable inventory systems are done by Nahmias [20], Raafat [31], and Goyal and Giri [15]. Until recently, most of the literature on inventory replenishment policies for perishable products in a stochastic environment has focused on policies which do not take into account detailed information about the age of the inventory. Nahmias [32] approximated the original problem with a heuristic that includes detailed age information about newer inventory and aggregated information about older inventory.

Haijema et al. [33] examine age information by using two critical numbers in their replenishment policy: one for the total inventory and one for the young inventory. They also distinguish two demand classes: demand for young items and demand for items of any age.

Rob A.C.M. Broekmeulen and Karel H. van Donselaar [8] studied replenishment policies for a single echelon perishable inventory system with stochastic demand and a fixed life time. Although this type of system has been studied since Van Zyl [36], the number of publications since then has been limited. Nahmias [29] and Fries [28] showed that for a perishable item with fixed life time equal to m periods, incurring outdating costs for units of inventory available in the system at the end of their life time, the optimal replenishment policy in general depends on a $(m1)$-dimensional vector, which describes the age distribution of the inventory in the system.

Another approach to perishable inventory models is to adopt a continuous review policy. With a Poisson demand process, a zero lead time, and constant lifetime, Weiss [30] showed that the optimal ordering policy is an (s, S) policy.

Liming Liu and Zhaotonglian [50] modeled an (s, S) continuous review perishable inventory system with a general renewal demand process and instantaneous replenishments. They study the replenishment problem of an inventory system for a single product with a fixed shelf lifetime with the objective of minimizing the (average) number of outdating units. This paper extends Weiss [30] work by using a general renewal demand process and developing closed-form results for the case with backordering.

Emre Berk and lk Grler [51] investigated a continuous review perishable inventory system with constant shelf lives, fixed ordering costs, and constant lead times under the (Q, r) policy with the restriction $r < Q$, which implies at most one outstanding order at any time. With Poisson demands and lost sales, the system was modeled as an embedded Markov process and the concept of effective shelf life was introduced, which corresponds to the remaining shelf life of items on hand at the instances when the inventory level hits Q.

A different approach is used by Avinadav and Arponen [39] who directly studied the freshness effect of products remaining shelf-life duration on the demand rate.

The literature dealing with pricing policy in inventory models began with Whitin [42] and was subsequently developed by Mills [43], Karlin and Carr [44] and Zabel [45]. Important works on pricing in the context of inventory of perishable items appear in the literature reviews of Silver [46], Nahmias [20], Weatherford and [47] and Gallego and van Ryzin [13].

Using a homogeneous demand model that assumes the demand intensity is a time-invariant function of price, Gallego and van Ryzin [13] derive the following structural properties of the optimal policy:

I. At any given time, the optimal price decreases in the number of items left.

II. With any given number of items, the optimal price decreases over time.

Property I is referred as the inventory-monotonicity property, and Property II as the time-monotonicity property.

TThe primarily negative relationship between the price of a product and the quantity demanded is well established (Kocabiyikoglu and Popescu [38] for examples of some common price-dependent demand functions). Avinadav and Arponen [39] suggested a demand rate that is positively affected by the remaining shelf-life of the product, since higher time-to-expiry indicates better freshness and higher quality.

While the effect of price on demand has been thoroughly analyzed in the literature (Petruzzi and Dada [40]), there is a paucity of research on the effect of the remaining shelf-life duration despite its importance in modeling the demand rate of perishable items (including most grocery items, such as dairy products, bread, beverages and many other industrial food products, as well as ammunition, batteries and printer ink). In general, the age of inventory is likely to negatively affect the demand for perishable goods. Sarker et al. [41] claim that this effect occurs because consumers tend to feel less confident purchasing perishable items whose expiration dates are approaching.

However, there is a paucity of (direct) research on the combined effects of both price and time after replenishment on the demand rate of a perishable item. To our knowledge, the following studies have combined these two effects. You [21] assumed an additive influence of the two factors, studying a case in which the price effect is linear and the time effect is negative-exponential. Valliathal and Uthayakumar [48] assumed a general multiplicative influence of these two factors when demand is convex in price. They introduced an algorithm to obtain optimal pricing and replenishment policies; however, their algorithm, which required iteratively solving a system of three equations, does not guarantee convergence towards the optimum. Maihami and Kamalabadi [49] assumed a special case of multiplicative influence of these two factors, where price has a linear effect and time has an exponential effect. And finally, Wen Zhao and Yu-Sheng Zheng [27] considered a dynamic pricing model for selling a given stock of a perishable product over a finite time horizon. The researchers studied the following optimal pricing model. A given inventory of some product is to be sold in a given time horizon. Demand is stochastic and price sensitive. Items left unsold at the end of the horizon are disposed of at a given salvage value/cost. Inventory is not replenishable, and unsatisfied demand is lost with no penalty cost. The objective is to find a dynamic pricing policy that maximizes expected revenue. The above generic model has been introduced by Kincaid and Darling [37]

We considered two more studies in this area. Yasuo Adachi et al. [34] modeled a perishable inventory system with consideration of different selling prices by the degree of outdating for perishable commodities under stochastic demand by the theory of Markov decision process. In this model, they assumed the discriminating sales prices by m stories m varying different lifetimes for perishable commodities. But Lifetime of goods and Lead time is predefined.

Tal Avinadava et al. [35] formulated a model for determining the optimal pricing, order quantity and replenishment period for perishable items with price-dependent and time-dependent demand. The items have a fixed shelf-life, and the demand rate decreases linearly in the selling price and polynomially over the time after replenishment, until it vanishes either at the reservation price or at expiration time.

Chapter 2

MODEL

In this study, we have perishable products with a fixed shelf lifetime of m periods. We consider a periodically reviewed inventory system that at the beginning of every period we decide whether to order a new batch of products or to continue one more period with the current products. In our model t denotes the remaining shelf lifetime of the products at hand. Products in a new ordering batch have remaining shelf lifetime of m periods and at every period the remaining shelf lifetime of the unsold products decreases by 1 if we decide to continue with the current products at hand. However, if we decide to order a new batch, Q units of new products with a shelf lifetime of m periods will arrive immediately. We assume that there is no lead time in the ordering process. Whenever we order a new batch, there is also a fixed ordering cost, denoted as K in addition to the unit cost of products, denoted as c.

We also assume that all the current products at hand will be salvaged at a unit price s when a new batch of products are ordered. This assumption is analytically critical for us in order to develop a solvable dynamic programming model. We note that, one needs to keep track of the products quantities at all possible ages, hence this is the reason that it is widely assumed the perishable products have a two period lifetime in the inventory control literature (e.g. Chen and Sapra [11], Chew Peng and Chulung [12] etc.). In other words, if a product has an m-period lifetime, then we need an m-dimensional state space, each dimension showing the amount of products at that age. Using a dynamic programming formulation with an m-dimensional state space becomes very difficult to solve as m gets larger than 2. However, in our model, we do not restrict ourselves to two-period lifetimes and instead we assume that the older products are salvaged (or sold in a secondary market) when a new batch arrives. With this assumption, at any time, the system can be modeled using two state variables, the number of products at hand, q, and their remaining lifetime, t, because all the products at hand will have the same age in this model since the older products are discarded when a new batch arrives. Even though this assumption might seem restrictive at first, we believe that it is consistent with many practices in reality. In real life, customers generally prefer fresher products and when they go to the store, if there are two types of the same product, only differing in their freshness, they will prefer the fresh ones, leading the older ones to be unsold. The products on the shelf are generally seen to be sold on a last-come-first-sold basis since the last arrivals are the freshest ones. The older products can only be sold when the fresh products are depleted which takes a certain time and the older products become even more deteriorated during that time. It rarely happens that the new products are sold out, the older ones have still some remaining shelf lifetime and there is a demand for them in that period and even if this happens the demand for them will be generally low and a new batch will be ordered in the next period. For example, if we consider a product with a 7-day shelf life and assume that after 4 days we decide to order a new batch. If the older products are not salvaged, then they need to wait until the new batch is sold out which might take more than 3 days and the older products will already be expired. However, if they are sold out in 2 days, then at that day, some of the older products with a remaining shelf life of 1 day can be sold out but the demand for those products will be very low. Thus, we assume that the sales of the older products will not be significant for our model and salvaging the older products at a certain price when a new batch is ordered seems logical. We also note that this assumption becomes more valid when the product lifetime is relatively short with respect to the times between two orders. In other words, in a longer cycle, products with a shorter lifetime deteriorate more until the new products are depleted, consequently, the possibility of older products being sold becomes less when the time between two orders increase (for example when the fixed order cost is high) or the product lifetime decreases.

In addition to the inventory ordering decisions, at the beginning of every period, we also decide on the price in that period, denoted as $p(q, t)$, depending on the state of the system at that period, (q, t), which shows the number of products at hand and their remaining shelf lifetime. The demand at every period, denoted as $D(p(q, t), t)$, is then a random function of the price $p(q, t)$ and the remaining shelf lifetime of the products, t.

Parameters:

m: Shelf life time; number of periods it takes products to perish

c: Cost per product; this is the cost that retailer pays to buy one product

K: Ordering cost; this is the fixed cost that retailer pays in each ordering

s: Salvage value; this is the value that products will be salvaged with

System States:

q: Inventory size; shows how many products are in the inventory in each period

t: Remaining shelf life time; shows how many periods are remaining for products in the inventory to perish

Decisions Variables:

Q: Order size; how many products retailer should order in each ordering to maximize his profit

$p(q, t)$: Price; value of the products that retailer charge to customers. It may be chosen respects to profit maximization

In addition to Order size and Price values, we control the system by deciding when we should have a new order and when we should continue with current inventory. This decision should be made for all system states; for any inventory size and remaining shelf lifetime.

System Disturbance

$D(p, t)$: Demand; that is Stochastic and dependent on Price and Remaining shelf lifetime of the products in inventory

2.1 *Dynamic Programming Formulation*

In this study, we aim to maximize the total system profit by deciding on when to order a new batch, how much to order and what should be the price to set at every period depending on the system state, (q,t). We consider an infinite planning horizon and use an average cost dynamic programming formulation. The Bellman's equation for our model will be as below:

$$\begin{aligned}
\Pi^* + h(q,t) &= \max\Big\{Continue, Order\Big\} = \max\Big\{f_c(q,t), f_o(q)\Big\} \qquad (2.1.1)\\
f_c(q,t) &= \max_{p(q,t)}\Big\{p(q,t)\; E_D\Big[\min\big\{q, D(p(q,t),t)\big\}\Big] + E_D\Big[h\Big(\big[q - D(p(q,t),t)\big]^+, t-1\Big)\Big]\Big\}\\
f_o(q) &= -K - cQ + sq + f_c(Q,m)
\end{aligned}$$

In the above formulation, Π^* denotes the optimal average profit value, $h(q,t)$ denotes the relative (differential) profit values, $f_c(q,t)$ denotes the profit value if we decide to continue with the current products at hand and $f_o(q)$ denotes the profit value if we decide to order a new batch at state (q,t). Note that if we decide to continue, then we need to pick the best price $p(q,t)$ to maximize the expected profit which includes an immediate expected profit from selling the products in this period, denoted as $p(q,t)\; E_D\Big[\min\big\{q, D(P(q,t),t)\big\}\Big]$ and a future expected profit of the system, denoted as $E_D\Big[h\Big(\big[q - D(p(q,t),t)\big]^+, t-1\Big)\Big]$. Note that, in this case, the next state will be $\Big(\big[q - D(p(q,t),t)\big]^+, t-1\Big)$, in which the products have one less remaining shelf lifetime. If we decide to have a new order, first we have an ordering cost, K, and the cost of the purchased products cQ. Then, we have salvage revenue sq and we come to the state (Q,m) with the new batch whose profit value will be equal to $f_c(Q,m)$.

We note that, in our model, we do not use any inventory holding cost since the perishability of the products and the time dependency of demand make a similar effect in our decisions to the inventory holding costs. The expiration risk of the products and the decrease of demand rate by time prevents us from ordering too much as the inventory holding cost does in the classical inventory control literature. In addition, since we consider short time periods in our model, the price of the product and the other costs will be much more significant than the inventory holding costs per period in our model. Thus, we choose not to include the inventory holding cost in our formulation for the sake of simplicity. However, one can easily include it into the model by subtracting a cost of hq from the function $f_c(q,t)$ where h denotes the inventory holding cost per period and all our results will remain valid.

Chapter 3

ANALYTICAL RESULTS FOR THE INVENTORY CONTROL AND STATIC PRICING

In this section, we focus mostly on the inventory control policy rather than pricing and consider the case in which a single price p is applied at all possible states such that the DP equation is:

$$\begin{aligned}
\Pi^* + h(q,t) &= \max\Big\{Continue, Order\Big\} = \max\Big\{f_c(q,t), f_o(q)\Big\} \\
f_c(q,t) &= p\, E_D\Big[\min\big\{q, D(p,t)\big\}\Big] + E_D\Big[h\Big(\big[q - D(p,t)\big]^+, t-1\Big)\Big] \\
f_o(q) &= -K - cQ + sq + f_c(Q,m)
\end{aligned} \tag{3.0.1}$$

We prove some structural properties about the optimal policy by analyzing the dynamic programming formulation 3.0.1. First, in propositions 1 and 2, we prove that the differential profit function $h(q,t)$ is increasing in q for a given t and it is non-decreasing in t for a given q.

Proposition 1. $h(q,t)$ *is increasing in* q *for any given* t*.*

Proof. We can write the Bellman's equation in 2.1.1 as $h = T(h)$ where T is an appropriate transformation function for the vector h. We prove that $h(q+1,t) \geq h(q,t)$ by showing that the transformation function used in the DP equation 2.1.1 preserves the monotonicity of the function h as stated in the equation below.

$$\begin{aligned}
& \Pi^* + h(q,t) \\
= \; & \max\Big\{ -K - cQ + sq + f_c(Q,m), p\, E_D\Big[\min\{q, D(p,t)\}\Big] + E_D\Big[h\Big(\big[q - D(p,t)\big]^+, t-1\Big)\Big]\Big\} \\
\leq \; & \max\Big\{ -K - cQ + sq + f_c(Q,m), p\, E_D\Big[\min\{q+1, D(p,t)\}\Big] + E_D\Big[h\Big(\big[q + 1 - D(p,t)\big]^+, t-1\Big)\Big]\Big\} \\
= \; & \Pi^* + h(q+1,t)
\end{aligned}$$

In the above inequality, it can be easily seen that the second line is greater than the first line due to the assumption that $h(q+1,t) \geq h(q,t)$. As a result, we can say that the transformation function T given in the DP equation 2.1.1 preserves the monotonicity assumption that $h(q+1,t) \geq h(q,t)$ and thus it is proved that $h(q,t)$ is increasing in q for any given t. □

Proposition 2. *$h(q,t)$ is non-decreasing in t for a given q value.*

Proof. We need to show that $h(q,t+1) \geq h(q,t)$ for any q and t. First, note that $\Pi^* + h(q,t) = \max\Big\{f_c(q,t), f_o(q)\Big\}$ and $f_o(q)$ is independent of t. Thus it is enough just to show that $f_c(q,t+1) \geq f_c(q,t)$. To prove this statement, we use a sample path argument. Using the sample path argument, we assume that we have two sets of products:

Set 1: q products in inventory with $t+1$ remaining shelf lifetime

Set 2: q products in inventory with t remaining shelf lifetime

Now, consider the case when the same customer arrives to these two sets and decides to buy or not a product. Three cases can happen in this system:

Case I: Customer decides not to buy from both sets

Case II: Customer decides to buy in both sets

Case III: Customer decides to buy from set 1 and not to buy from set 2

Note that, it is not possible for the customer to buy from set 2 but not from set 1 because we assume that the customers always prefer fresher products and set 1 has fresher products than set 2 and they both have the same price. So, at a certain day after all customers visit the market, three cases can happen:

Case 1: All products in both sets are sold out (the number of customers that decide to buy are more than q for both sets)

Case 2: k number of products are sold out in both conditions ($k \leq q$)

Case 3: k number of products are sold out from set 1 and m number of products are sold from set 2, ($m \leq k \leq q$)

Now, using induction we will show that in all these three cases $h(q, t+1) \geq h(q, t)$ for any q and t values.

First for the boundary condition, for $t = 0$,

$h(q, 1) = \max \Big\{ f_c(q, 1), f_o(q) \Big\} - \Pi^*$

$h(q, 0) = f_o(q) - \Pi^*$

So, it is obvious that $h(q, 1) \geq h(q, 0)$ (in all cases)

For $t \geq 1$, using the induction assumption that $h(q, t) \geq h(q, t-1)$

Case 1:

$f_c(q, t+1) = pq + h(0, t)$ where $h(0, t) = f_o(0) - \Pi^*$

$f_c(q, t) = pq + h(0, t-1)$ where $h(0, t-1) = f_o(0) - \Pi^*$

So, $h(q, t+1) \geq h(q, t)$

Case 2:

$f_c(q, t+1) = pk + h(q-k, t)$

$f_c(q, t) = pk + h(q-k, t-1)$

Due to the induction assumption that $h(q-k, t) \geq h(q-k, t-1)$, we can state that

$f_c(q, t+1) \geq f_c(q, t)$ for any q and t.

Case 3:

$f_c(q, t+1) = pk + h(q-k, t)$

$f_c(q, t) = pm + h(q-m, t-1)$

$f_c(q+1, t) - f_c(q, t) = p(k-m) + h(q-k, t) - h(q-m, t-1) \geq p(k-m) + h(q-k, t) - h(q-m, t)$ due to the induction assumption that $h(q-m, t) \geq h(q-m, t-1)$.

Now, we note the following inequality, since each product at hand will be sold or salvaged until the end of the period and thus the value of having k more products at hand now is at least sk and at most pk.

$$sk \leq h(q+k, t) - h(q, t) \leq pk \tag{3.0.2}$$

Then, due to the above inequality, we have $s(k-m) \leq h(q-m, t) - h(q-k, t) \leq p(k-m)$. Thus, $p(k-m) + h(q-k, t) - h(q-m, t) \geq 0$ and we have $f_c(q+1, t) - f_c(q, t) \geq 0$.

As a result, we can say that in all three cases $h(q, t+1) \geq h(q, t)$ for any q and t and we can conclude that $h(q, t)$ is non-decreasing in t for any q value. □

Using propositions 1 and 2, we now prove the structure about the optimal ordering policy in propositions 3 and 4.

Proposition 3. *There is a critical value $q_c(t)$ for any t that we should give an order if $q < q_c(t)$ and do not order if $q \geq q_c(t)$. In addition, $q_c(t)$ is non-increasing in t.*

Proof. If we decide to order, the profit will be $f_o(q)$ that is an increasing function of q with the slope of s. On the other hand, if we decide to continue, the profit will be $f_c(q, t)$ that is again an increasing function of the inventory size, q. Since, a product in inventory will be sold or salvaged at the end, each product will have either a value of p or S for the system. So, it means that $s < f_c(q+1, t) - f_c(q, t) < p$. Hence, as seen in Figure 3.1, $f_o(q)$ and $f_c(q, t)$ will intersect at a single point and we label it as $q_c(t)$.

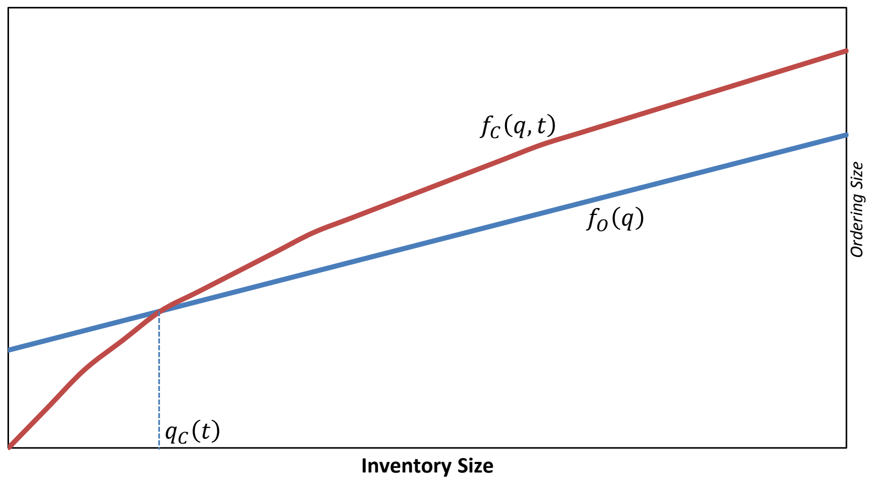

Figure 3.1: Typical profit structure in ordering and non-ordering case vs Inventory Size for a given Remaining Shelf lifetime value

$f_o(q)$ function is independent of t and $f_c(q,t)$ is non-decreasing in remaining shelf lifetime value. So, you can see in figure 3.2 that by increasing the t the intersection point will happen at a lower q value meaning that $q_c(t)$ is non-increasing in t.

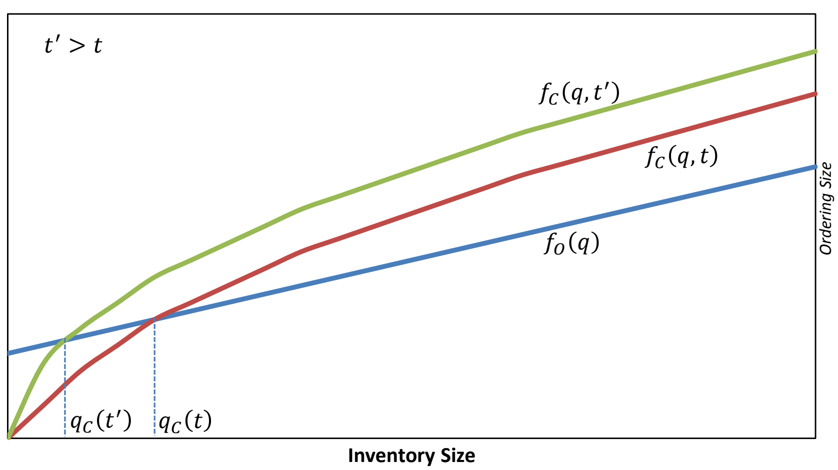

Figure 3.2: Typical structure of ordering and non-ordering profit vs Inventory Size for different Remaining Shelf lifetime

$\square$

Proposition 4. *There is a critical value $t_c(q)$ for any q that we should give an order if $t < t_c(q)$ and do not order if $t \geq t_c(q)$. In addition, $t_c(q)$ is non-increasing in q.*

Proof. We know that $f_o(q)$ is independent of t and $f_c(q,t)$ is non-decreasing in t. So as figure 3.3 shows, for a given q value, f_o and $f_c(t)$ will have exactly one intersection point, denoted as $t_c(q)$. Hence, $t_c(q)$ is the critical value for any q such that we should give an order if $t < t_c(q)$ and do not order if $t \geq t_c(q)$.

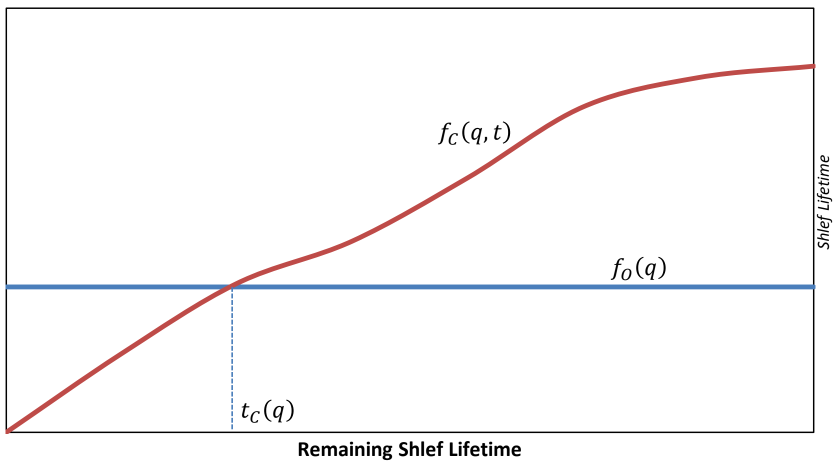

Figure 3.3: Typical profit structure in ordering and non-ordering case vs Remaining Shelf lifetime for a given Inventory size value

As it is obvious in Figure 1, $q_c(t)$ and $t_c(q)$ show the same curve for ordering states. In other words:

$$t_c(q_c(t)) = t \tag{3.0.3}$$

We proved that in proposition 3 that $q_c(t)$ is non-increasing in t. So, for any $t_1 \geq t_2$, $q_c(t_1) \leq q_c(t_2)$. Let $q_1 = q_c(t_1)$ and $q_2 = q_c(t_2)$. Then, since $t_1 = t_c(q_c(t_1))$ and $t_2 = t_c(q_c(t_2))$, we can equivalently write that $t_1 = t_c(q_1)$ and $t_2 = t_c(q_2)$. Since $t_1 \geq t_2$, $t_c(q_1) \geq t_c(q_2)$ for $q_1 \leq q_2$, which means that $t_c(q)$ is non-increasing in q. $\square$

Propositions 3 and 4 essentially show that there is a monotonic structure in the optimal ordering decision policy. Figure 3.4 shows a typical structure of the optimal ordering policy with respect to inventory size and remaining shelf lifetime. As seen in Figure 3.4, there is a non-decreasing curve that separates the ordering and non-ordering states.

Proposition 5 denotes that the optimal order size Q^* is independent of the current state once we decide to order since all the older products will be discarded.

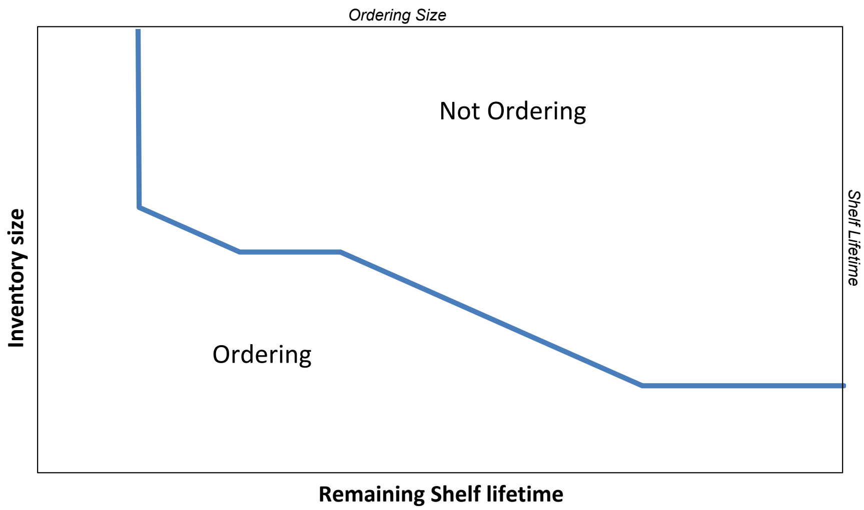

Figure 3.4: Optimal Ordering Decision Policy - Typical Monotone Structure.

Proposition 5. *Optimal order size, Q^* is independent of q and t.*

Proof. In any state, we should choose an ordering size, Q, that maximizes $-K - cQ + sq + f_c(Q, m)$ and the maximizer value will be independent of q and t. So, we can simply say that the optimal order size is same for all q and t values. □

However, we observe that the optimal average profit value is not a concave function of Q. An example of this situation can be seen in Figure 3.5. Thus, in order to determine the value of Q^* we need to do a line search between lower bond L and the upper bound U stated in proposition 6.

Figure 3.5 shows non-concavity of Average Profit function in Ordering Size in a example.

Proposition 6. *For a given Price value, p, the optimal order size Q^* is between $L(Q)$ and $U(Q)$, where $L(Q)$ and $U(Q)$ are denoted as below.*

$$U(Q) = min\{Q|F_{T_m}^{-1}(Q) \geq \frac{p-c}{p-s}\} \quad (3.0.4)$$

$$L(Q) = min\{Q|F_{D_m}^{-1}(Q) \geq \frac{p-c}{p-s}\} \quad (3.0.5)$$

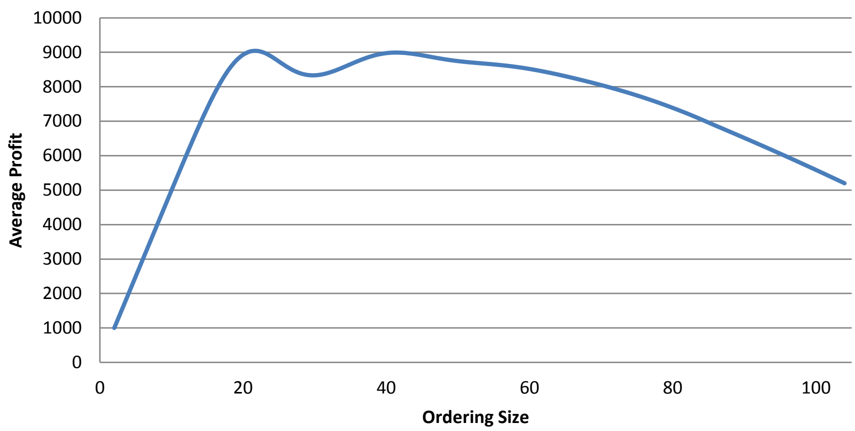

Figure 3.5: Average Profit in different Ordering size values

where

F_{T_m}*: Cumulative Distribution function of total demand of a period (m days)*

F_{D_m}*: Cumulative Distribution function of first day in a period (when remaining shelf lifetime is m)*

Proof. Assume that we have standard Newsvendor problem where the demand is equal to total demand of a period in our model (demand of m days). We know that the optimal ordering quantity for newsvendor problem is equal to

$$min\{Q|F_{D_1}^{-1}(Q) \geq \frac{p-c}{p-s}\}$$

where F_{D_1} is cumulative distribution function of demand in newsvendor problem.
The newsvendor problem is a one period problem that we wait until all demand come and sell the products as much as we can after salvage if there are some remaining products. However, in our model, the difference is that the model infinite horizon problem and we may have a new order before a period finishes. So, in a period, we may not wait for all demand to come and we can stop a period in the middle by having a new order. Hence, it is obvious that the optimal ordering size for our model is always less that Optimal ordering quantity of newsvendor problem. So, the optimal ordering size of newsvendor

problem can be a upper-bond for our perishable inventory model.

Now assume that we have another newsvendor problem where its demand it equal to demand of first day of a period in our model (where the remaining shelf lifetime is equal to m). We know that a period in our model is at least one day. After passing one day in a period, the period may finish or we may have more days in the period. So, the optimal ordering size of a one day one period problem is a lower bond of our model. It means that, the optimal ordering quantity of a newsvendor problem with demand equals to the first day of our problem period is always less than or equal to our model ordering size. So, the value below

$$min\{Q|F_{D_2}^{-1}(Q) \geq \frac{p-c}{p-s}\}$$

where F_{D_2} is cumulative distribution function of demand in this newsvendor problem, is a lower bond for this perishable inventory model.

□

We also need to do a line search for the optimal single price to set in order to maximize the profit, and we also claim that the optimal price value has always a finite value in the following proposition.

Proposition 7. *There is a finite value of price that maximizes the average profit value.*

Proof. To show that this proposition is true, we use this argument that when price value goes to zero, there will be no profit in the system. On the other side, by going to infinity the demand goes to zero and again there will be no profit in the system. Hence, there is a finite value of price meanwhile that maximizes the average profit value of the system.

□

Chapter 4

ANALYTICAL METHOD FOR THE INVENTORY CONTROL AND STATIC PRICING

In the previous section, we prove that the optimal replenishment policy has a monotonic structure such that there exists a critical value of $q_c(t)$ that determines the order or not to order decision. For any remaining shelf life time t, if the inventory at hand is less than $q_c(t)$, then it is optimal to give a new order. In this section, we develop an analytical expression for the average profit to determine the optimal $q_c(t)$ values, without using dynamic programming formulation. This expression will help us to easily calculate the expected average profit values for different values of decision variables and it also helps us to analyze the heuristic algorithms for the replenishment policy.

Note that, in our model, every time a new order of size Q is given, we go to the state (Q, m) and a new cycle starts. Thus, using the renewal theory, we calculate the expected profit per unit time with the following equation.

$$E[\text{profit per unit time}] = \frac{E[\text{profit of a cycle}]}{E[\text{length of a cycle}]}$$

In order to calculate the expected profit per cycle and expected length of a cycle, we first introduce the following notation:

x_i: A random value denoting the demand of the ith day

y_i: A random value denoting the total demand of first i days

$q_i = q_c(m-i)$: Decision variable denoting the critical inventory size for ordering after day i (i.e. when the remaining shelf life time is $m-i$). Note that $q_i > q_{i-1}$ due to Proposition 3.

$f_D^t(x)$: Probability that demand is x when remaining shelf lifetime is t

$f_T^t(y)$: Probability that the total demand, from the beginning of the cycle until the remaining shelf life time is t, is y

Then, using the above notation, we can write the expected length of a cycle as below. Note that, since the maximum life of the product is m, the length of a cycle can be between 1 and m. The length of a cycle is 1, if the inventory goes under q_1 after satisfying the first day demand. In general, the length of a cycle is i if the inventory stays above q_{i-1} after first $i-1$ days, but goes under q_i after day i.

$$\begin{aligned}
E & \quad [\text{Length of a cycle}] \qquad (4.0.1)\\
= & \quad \sum_{x_1=Q-q_1}^{\infty} f_D^m(x_1)\\
+ & \quad \sum_{i=2}^{m-1}\left[i * \left(\sum_{y_{i-1}=Q-q_i}^{Q-q_{i-1}-1} f_T^{m-i+2}(y_{i-1}) + \sum_{y_{i-1}=0}^{Q-q_i-1} f_T^{m-i+2}(y_{i-1}) \sum_{x_i=Q-y_{i-1}-q_i}^{\infty} f_D^{m-i+1}(x_i) \right)\right]\\
+ & \quad m * \sum_{y_{m-1}=0}^{Q-q_{m-1}-1} f_T^2(y_{m-1})
\end{aligned}$$

Then, we calculate the expected profit of one cycle. In each cycle, we first have a fixed ordering cost and the purchasing cost of products. Then, depending on the demand and the critical ordering thresholds q_i, we obtain a revenue from the sale of the products at a unit price p and the rest are salvaged at the price s.

$$
\begin{aligned}
E\ &[\text{Profit of a cycle}] \qquad\qquad (4.0.2)\\
=\ & -K - cQ + \sum_{x_1=Q-q_1}^{Q} f_D^m(x_1)\big(px_1 + s(Q-x_1)\big) + \sum_{x_1=Q+1}^{\infty} f_D^m(x_1)pQ\\
+\ & \sum_{i=2}^{m-1}\Bigg[\sum_{y_{i-1}=Q-q_i}^{Q-q_{i-1}-1} f_T^{m-i+2}(y_{i-1}) \sum_{x_i=0}^{Q-y_{i-1}} f_D^{m-i+1}(x_i)\big(p(+y_{i-1}+x_i) + s(Q-y_{i-1}-x_i)\big)\\
+\ & \sum_{y_{i-1}=0}^{Q-q_i-1} f_T^{m-i+2}(y_{i-1}) \sum_{x_i=Q-y_{i-1}-q_i}^{Q-y_{i-1}} f_D^{m-i+1}(x_i)\big(p(+y_{i-1}+x_i) + s(Q-y_{i-1}-x_i)\big)\\
+\ & \sum_{y_{i-1}=0}^{Q-q_{i-1}-1} f_T^{m-i+2}(y_{i-1}) \sum_{x_i=Q-q_i}^{\infty} f_D^{m-i+1}(x_i)pQ\Bigg]\\
+\ & \sum_{y_{m-1}=0}^{Q-q_{m-1}-1} f_T^{2}(y_{m-1}) \sum_{x_m=0}^{Q-y_{m-1}} f_D^{1}(x_m)\big(p(+y_{m-1}+x_m) + s(Q-y_{m-1}-x_m)\big)\\
+\ & \sum_{y_{m-1}=0}^{Q-q_{m-1}-1} f_T^{2}(y_{m-1}) \sum_{x_m=Q-y_{m-1}+1}^{\infty} f_D^{1}(x_m)pQ
\end{aligned}
$$

Using the equations above, we need to determine the optimal values of threshold points q_i for $i = 1, 2..m-1$ ($q_m = Q$ since we have to order a new batch when the remaining shelf lifetime is 0), in addition to the optimal order size Q and the optimal price p, which makes a total of $m+1$ unknowns. However, due to the complexity and non-concavity of the functions above, we are not able to find a closed form expression for these decision variables. Instead, we can solve this problem to optimality either by using a commercial nonlinear solver or by searching through all feasible solutions. However, both methods might fail to provide the optimal solution in a reasonable time as the number of unknowns, m, gets larger, even though we can determine the optimal solutions when m is small. For the cases when m is large, we present two simple heuristics below.

4.1 H1: Single Inventory Ordering Level(r) Heuristic

In this heuristic, we consider a single threshold level r instead of different q_i values for different i as stated in the previous section. This heuristic is like a modified (s, S) policy in the classical inventory literature such that if the inventory at hand at the beginning of any period is less than r (s in the classical inventory literature), we order a new batch of Q units and since we discard the old ones, our inventory level goes up to Q (S in the classical inventory literature). If the inventory never goes below r in m periods, then we order a new batch after m periods since all the products at hand will be expired at that time. In our model, also note that the demand is time dependent and we also need to decide on the optimal price, p, to set in this system.

In this heuristic, since it is a simplified version of the model in the previous section, we use the same equations as above but use the single unknown r instead of different q_i values.

$$\begin{aligned}
E\ & [\text{Length of a cycle}] \qquad (4.1.1)\\
= & \sum_{x_m=Q-r}^{\infty} f_D^m(x_m) + \sum_{i=2}^{m-1}(m-i+1)\sum_{y_{i+1}=0}^{Q-r-1}\sum_{x_i=Q-r-y_{i+1}}^{\infty} f_T^{i+1}(y_{i+1})f_D^i(x_i)\\
+ & \ m\sum_{y_2=0}^{Q-r-1} f_T^2(y_2)\\
= & \ 1-F_D^m(Q-r-1)\\
+ & \sum_{i=2}^{m-1}(m-i+1)\sum_{y_{i+1}=0}^{Q-r-1} f_T^{i+1}(y_{i+1})\big(1-F_D^i(Q-r-y_{i+1}-1)\big)\\
+ & \ mF_T^2(Q-r-1)
\end{aligned}$$

$$
\begin{aligned}
E &\quad [\text{Profit of one period}] &(4.1.2)\\
= &\quad -K - cQ + \sum_{x_m=Q-r}^{Q} f_D^m(x_m)\big(px_m + s(Q-x_m)\big) + \sum_{x_m=Q+1}^{\infty} f_D^m(x_m)pQ \\
+ &\quad \sum_{i=2}^{m-1}\Bigg[\sum_{y_{i+1}=0}^{Q-r-1}\sum_{x_i=Q-r-y_{i+1}}^{Q-y_{i+1}} f_T^{i+1}(y_{i+1})f_D^i(x_i)\big(p(x_i+y_{i+1}) + s(Q-x_i-y_{i+1})\big) \\
+ &\quad \sum_{y_{i+1}=0}^{Q-r-1}\sum_{x_i=Q-y_{i+1}+1}^{\infty} f_T^{i+1}(y_{i+1})f_D^i(x_i)pQ\Bigg] \\
+ &\quad \sum_{y_2=0}^{Q-r-1}\sum_{x_1=0}^{Q-y_2} f_T^2(y_2)f_D^1(x_1)\big(p(x_1+y_2)+s(Q-x_1-y_2)\big) \\
+ &\quad \sum_{y_2=0}^{Q-r-1}\sum_{x_1=Q-y_2+1}^{\infty} f_T^2(y_2)f_D^1(x_1)pQ
\end{aligned}
$$

which can be simplified to

$$
\begin{aligned}
= &\quad -K - cQ + SQ\big(F_D^m(Q) - F_D^m(Q-r-1)\big) &(4.1.3)\\
+ &\quad (p-s)\sum_{x_m=Q-r}^{Q} x_m f_D^m(x_m) + pQ(1-F_D^m(Q)) \\
+ &\quad \sum_{i=2}^{m-1}\Bigg[\sum_{y_{i+1}=0}^{Q-r-1}\sum_{x_i=Q-r-y_{i+1}}^{Q-y_{i+1}} f_T^{i+1}(y_{i+1})f_D^i(x_i)\big((p-s)(x_i+y_{i+1})+sQ\big) \\
- &\quad pQ\sum_{y_{i+1}=0}^{Q-r-1} f_T^{i+1}(y_{i+1})F_D^i(Q-y_{i+1}-1) + pQF_T^{i+1}(Q-r-1)\Bigg] \\
+ &\quad \sum_{y_2=0}^{Q-r-1} f_T^2(y_2)F_D^1(Q-y_2)\big((p-s)(y_2+x_1)+sQ\big) \\
- &\quad pQ\sum_{y_2=0}^{Q-r-1} f_T^2(y_2)F_D^1(Q-y_2-1) + pQF_T^2(Q-r-1)
\end{aligned}
$$

Then,

$$
E[\text{Profit per unit time}] = PI(p,Q,r) = \frac{E[\text{Profit of a cycle}]}{E[\text{Length of a cycle}]} \tag{4.1.4}
$$

Complexity of $PI(p, Q, r)$ function prevents us to find a closed form expression for the decision variables. To solve this function to optimality, we use the methods that are introduced in Appendix 1. The tables 6.1 and 6.2 in Numerical Result Section compares the results with this heuristic with the optimal DP solution.

4.2 H2: Single Remaining shelf lifetime Ordering Level(l) Heuristic

In this heuristic, similar to the previous one, we again consider a single threshold level but this threshold level is based on the remaining shelf lifetime of the products, instead of the remaining quantity of the products. As stated in proposition 4, the optimal policy requires us to find the critical time threshold levels, denoted as $t_c(q)$ for all possible q. However, in this heuristic, we consider a single time threshold level l such that we give a new order when the remaining lifetime of the products becomes less than l, independent of the quantity at hand. We also note that if all the products are sold before l periods, we also give a new order at that time even though l periods has not passed yet. This heuristic can be considered as a time-based replenishment policy.

Since this heuristic is also a simplified version of the model in the previous sections, we use similar equations as stated below. In this model, the length of a cycle can be less than $m - l$ if all the products are finished up before remaining shelf lifetime of the products reach l, and the length of a cycle will be $m - l$ otherwise.

$$
\begin{aligned}
E & \quad [\text{Length of a cycle}] && (4.2.1)\\
&= \sum_{x_m=Q}^{\infty} f_D^m(x_m) + \sum_{i=m-1}^{l+2} (m-i+1) \sum_{y_{i+1}=0}^{Q-1} f_T^{i+1}(y_{i+1}) \sum_{x_i=Q-y_{i+1}}^{\infty} f_D^i(x_i) + (m-l) \sum_{y_{l+2}=0}^{Q-1} f_T^{l+2}(y_{l+2})\\
&= 1 - F_D^m(Q-1) + \sum_{i=m-1}^{l+2} (m-i+1) \sum_{y_{i+1}=0}^{Q-1} f_T^{i+1}(y_{i+1})\big(1 - F_D^i(Q - y_{i+1} - 1)\big) + (m-l) F_T^{l+2}(Q-1)
\end{aligned}
$$

For the expected profit in a cycle, there are two possibilities: either all products are sold out before products reach a remaining shelf lifetime of l, or some of the products are not sold until that time and remains in inventory. Below is the equation for the expected profit per cycle.

$$\begin{aligned} E\ \ & [\text{Profit of a cycle}] \qquad (4.2.2) \\ = \ \ & -K - cQ + \sum_{y_{l+1}=Q}^{\infty} f_T^{l+1}(y_{l+1})pQ + \sum_{y_{l+1}=0}^{Q-1} f_T^{l+1}(y_{l+1})\big(py_{l+1} + s(Q - y_{l+1})\big) \\ = \ \ & -K - cQ + pQ + (s-p)Q\big(F_T^{l+1}(Q-1)\big) + (p-s)\sum_{y_{l+1}=0}^{Q-1} y_{l+1} f_T^{l+1}(y_{l+1}) \end{aligned}$$

Then, we can write expected profit per unit time as below:

$$E[\text{Profit per unit time}] = PS(p,Q,l) = \frac{E[\text{Profit of a cycle}]}{E[\text{Length of a cycle}]} \qquad (4.2.3)$$

We compare the results of this heuristic with the optimal DP solution in Tables 6.1 and 6.2 in the Numerical Results Section.

Chapter 5

ANALYTICAL RESULTS FOR THE COORDINATED INVENTORY CONTROL AND DYNAMIC PRICING MODEL

In this section, instead of using a single price value, we allow different prices, $p(q,t)$, to be used at different states (q,t). The DP equation for this model will be the one as stated in equation 2.1.1. Firstly, we note that all the propositions stated for the single price model related to the inventory policy, propositions 1 to 5, are also valid in this case. We don't repeat them here for the sake of space and instead we focus on the pricing decisions in this section.

When we look at the optimal pricing decisions, first we can say that for any state in which the optimal inventory decision is to order a new batch, the price to set at all of those states will be the same as stated in proposition 8.

Proposition 8. *If the optimal policy is to order a new batch at any two states* (q_1,t_1) *and* (q_2,t_2)*, then the optimal prices for these states are the same, i.e.* $p(q_1,t_1) = p(q_2,t_2)$.

Proof. If the optimal policy is ordering, so:

$$\Pi^* + h(q,t) = f_o(q) \tag{5.0.1}$$

$$f_o(q) = -K - cQ + sq + f_c(Q,m) \tag{5.0.2}$$

$$f_c(Q,m) = \max_p \left\{ p\, E_D\Big[\min\big\{Q, D(p,m)\big\}\Big] + E_D\Big[h\Big(\big[Q - D(p,m)\big]^+, m-1\Big)\Big]\right\}$$

So, for any q and t value, the optimal price is the maximizer of $f_c(Q,m)$ that is independent of q and t. □

When we look at the prices in the non-ordering states, we observe that these prices are non-decreasing in the remaining shelf life time if the demand has a Poisson distribution as stated in Observation 1 (without proof), meaning that higher prices should be charged for fresher products. However, when we look at the relationship between the prices and the inventory size, we observe that they do not have such a monotonic structure in the inventory size as seen in the example in Figure 6.30.

Observation 1. *The optimal price values, $p^*(q,t)$,in states with not ordering decision, are non-decreasing in Remaining Shelf Lifetime.*

Chapter 6

NUMERICAL RESULTS

In this section, we present our numerical results for both Static and Dynamic Pricing models. We analyze the effects of different parameters on the system and investigate the benefit of dynamic pricing. We also analyze the efficiencies of the heuristics presented above. In all our experiments, we assume a demand function that has a Poisson distribution with rate $\lambda(p,t) = a\left(\frac{t}{m}\right)^n e^{\beta(1-\frac{p}{c})}$ where t is the remaining lifetime, p is the price, c is the unit cost of the product, a is a constant, β denotes the price sensitivity of demand and n represents the time sensitivity of demand.

6.1 Numerical Results for the Inventory Control and Static Pricing

First, we analyze the static pricing model and as a base case for our numerical experiments, we use the parameters $c = 1000$, $K = 20000$, $a = 60$, $m = 7$, $s = 400$, $\beta = 2$ and $n = 1$. In some parts we also analyze another example with same parameters value except that it is assumed there is no ordering cost.
For these parameters, using the DP equation 3.0.1, we find out that the optimal price is 1497 and the optimal order quantity is 78.

Figure 6.1 shows the effect of using different ordering size value on the optimal average profit in two different lines. In the blue line each point shows the optimal average profit for a given ordering size value where the price has its optimal value while in the red line the price has a fixed value for all ordering size values.

Similarly, figure 6.2 shows the effect of price value on the optimal average profit in two different cases where in one the Ordering Size is fixed for all points and the other ordering size has its optimal value.

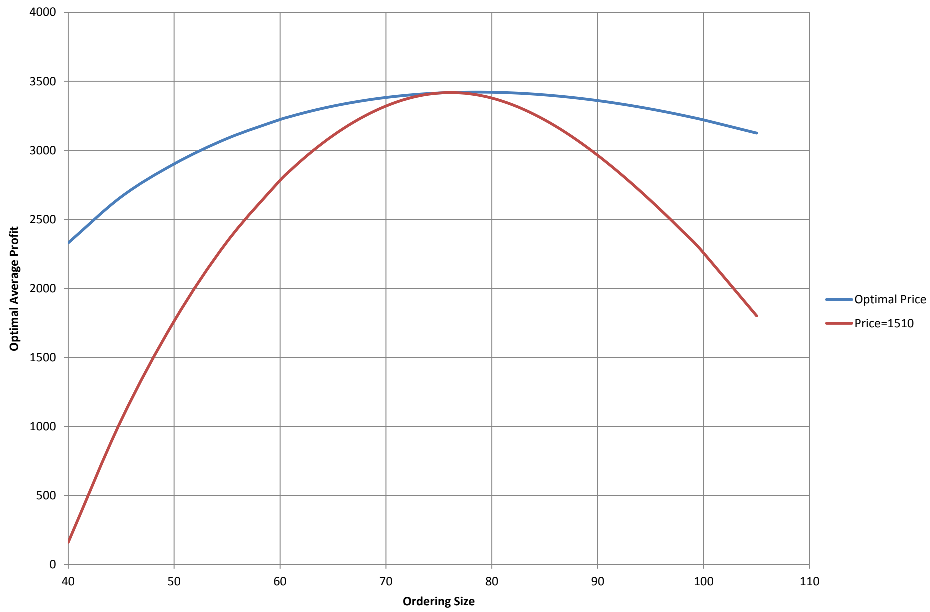

Figure 6.1: Optimal Average Profit for different Ordering Size value where price is constant and optimal

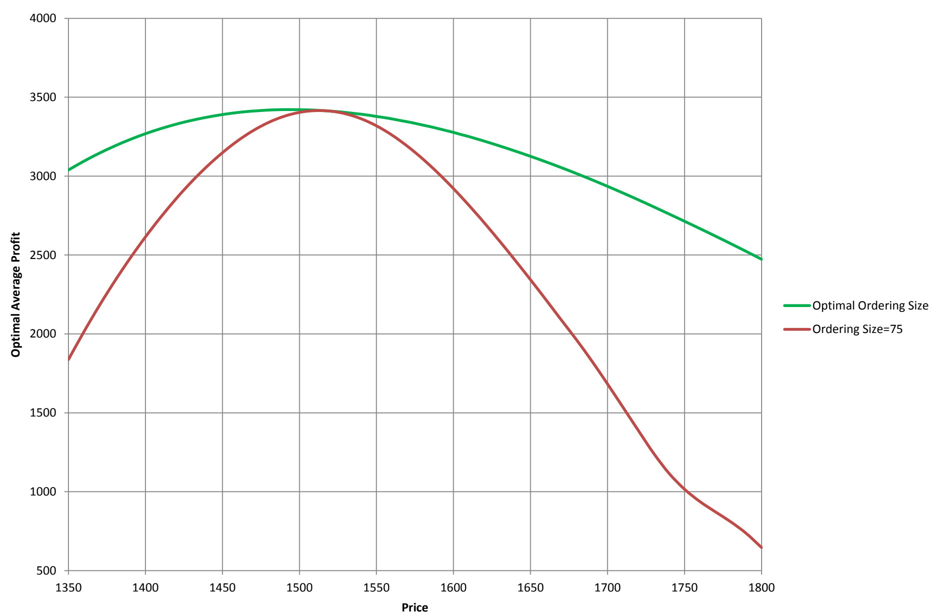

Figure 6.2: Optimal Average Profit for different Price value where Ordering Size is constant and optimal

Figure 6.3 shows both the effect of price on optimal ordering quantity and effect of ordering size on optimal price value. In this chart we see a monotone structure in these relations. As an observation, optimal solution of price and ordering quantity should have this characteristic that both price and ordering size value should be the optimal one for each other, in other words, optimal solution should be a point that is an intersection of two lines in the figure 6.3. In this example, since it seems there is just one intersection, we know that this point is the optimal solution.

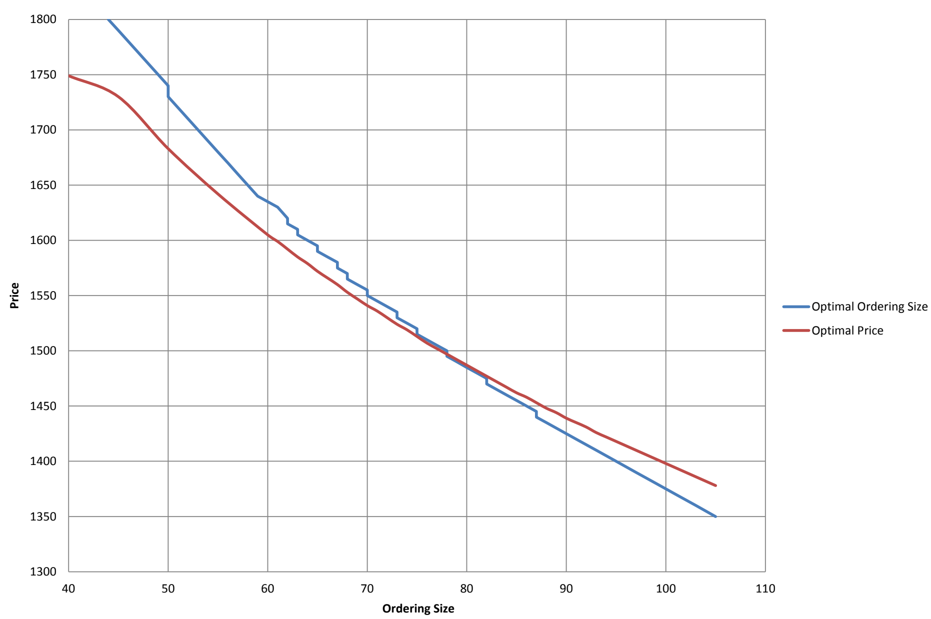

Figure 6.3: How Optimal Ordering Size and Optimal Price changes respects to each other in the analyzing example

In aforementioned charts, we observe a monotone or concave structure with respect to Optimal Average profit, Price and Ordering Quantity. However, in another example in which the ordering cost is decreased to 0, we see a different and more complicated structure. As we see in Figures 6.4 and 6.5, there may be more than one local maximum point in the average profit function versus Price or Ordering Quantity.

Figure 6.6 illustrates an example where the Optimal price and Optimal Ordering size have a non-monotone relation. The reason of these jumps in this chart is directly related

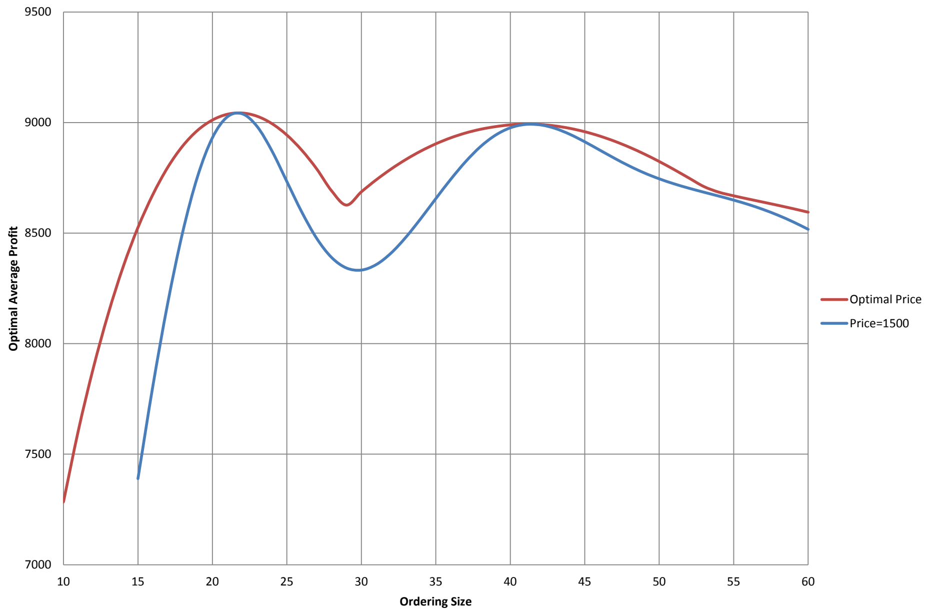

Figure 6.4: Optimal Average Profit in different Ordering Size values where price is a constant given or has its optimal value

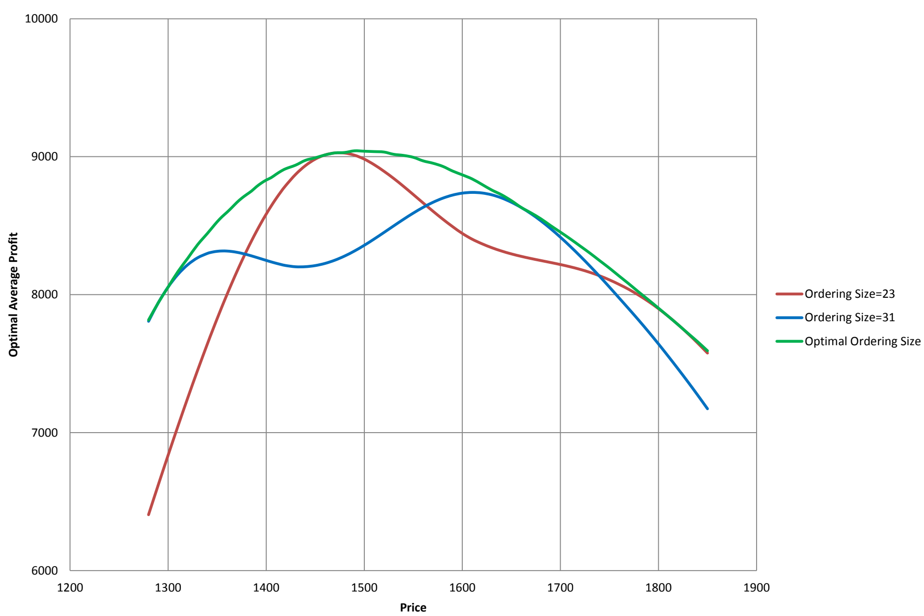

Figure 6.5: Optimal Average Profit in different Price values where Ordering Size is a constant given or has its optimal value

to non-concavity of Average profit function in Price and Ordering Sizes. In some cases, by changing the ordering size or price values, the optimal solution will move from one local maximum point to another and causes these jumps in figure 6.6.

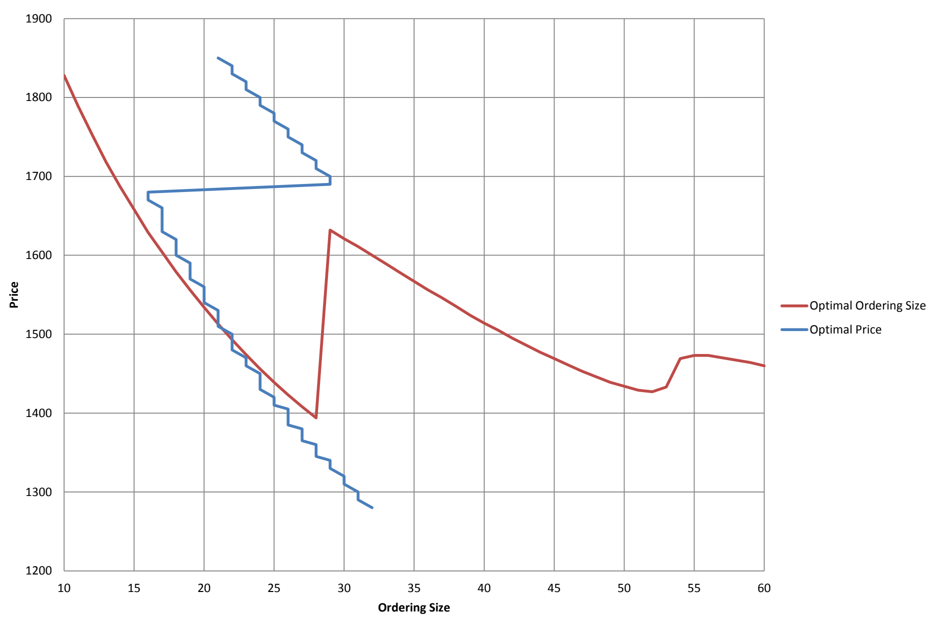

Figure 6.6: Optimal Ordering Size and Optimal Price relations in the example without fixed ordering cost

In Figures 6.7 and 6.8, we illustrate the structure of the function $h(q,t)$, that is also stated in propositions 1 and 2. You can see in figure 6.7 that for any remaining shelf lifetime value, relative profit function $h(q,t)$ is increasing in inventory size. Similarly, in Figure 6.8, for any inventory size value, $h(q,t)$ is non-decreasing in remaining shelf lifetime.

In Figure 6.9, we present the optimal inventory replenishment strategy denoting at which states, we should order a new batch and at which states we should continue with the current inventory at hand. Observe that Figure 6.9 is consistent with the structure stated in propositions 3 and 4.

In the following charts we consider the effect of different parameters in the problem on Optimal Average profit, Optimal Price and Optimal Ordering Size. Note that in all

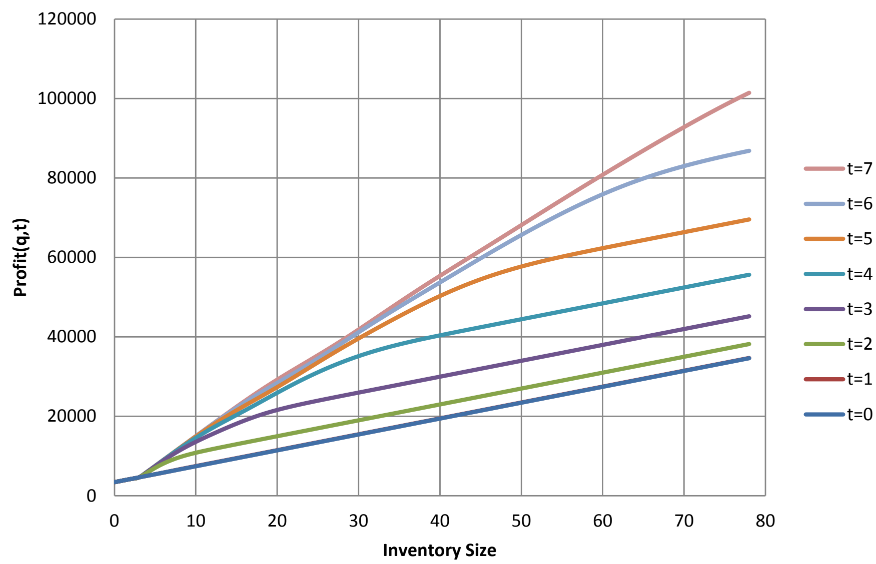

Figure 6.7: Profit Function versus Inventory Size for any remaining shelf lifetime values

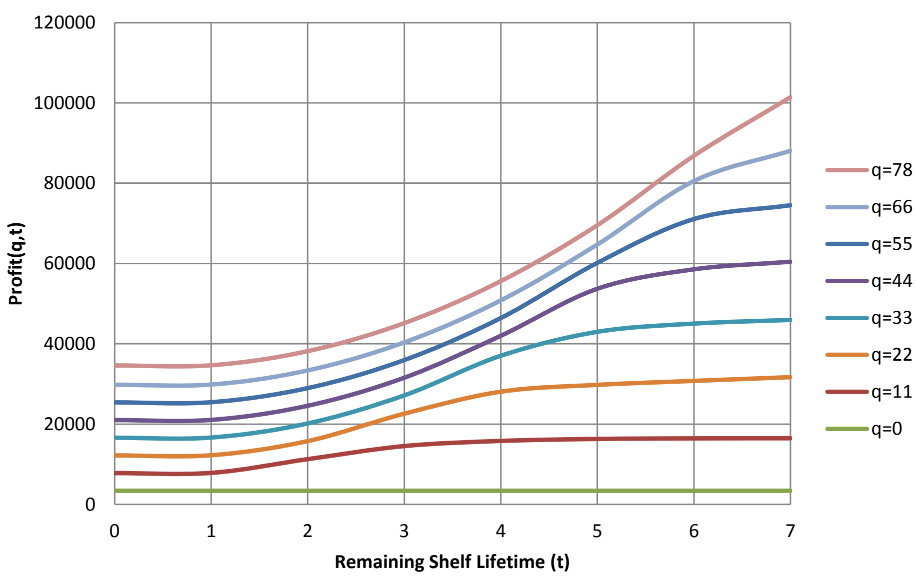

Figure 6.8: Profit Function versus Remaining Shelf lifetime for different Inventory Size values

charts each point shows a solution that Price and Ordering Size have their optimal values.

Figure 6.10 shows that as the the shelf life time of the product increases, the optimal order size also increases. It means that when products have more time before expiration,

Figure 6.9: Optimal Decision Policy structure for any Inventory size and Remaining Shelf Lifetime

we can order more. But we also observe that this increase is in a concave manner and after a while, even though the product's shelf lifetime increases, the order quantity stays constant.

In figure 6.11 we see how optimal price value changes versus Shelf lifetime of the products. We observe that the optimal price value goes up and down but in a small range of values. It indicates that shelf lifetime of the products does not have a significant and monotone effect on Optimal Price value.

A similar structure is also seen in Figure 6.12 in which we present the average profit value as opposed to the shelf lifetime of the products denoting that products with a short lifetime are more riskier and bring less profit.

Figures 6.13 and 6.15 show the effect of the demand rate coefficient on the order size and the average profit values, respectively and we observe that both values increase almost linearly with the demand rate. These small ups and downs in the figure 6.14 illustrates same thing as I mentioned about the shelf lifetime effect on Optimal Price value.

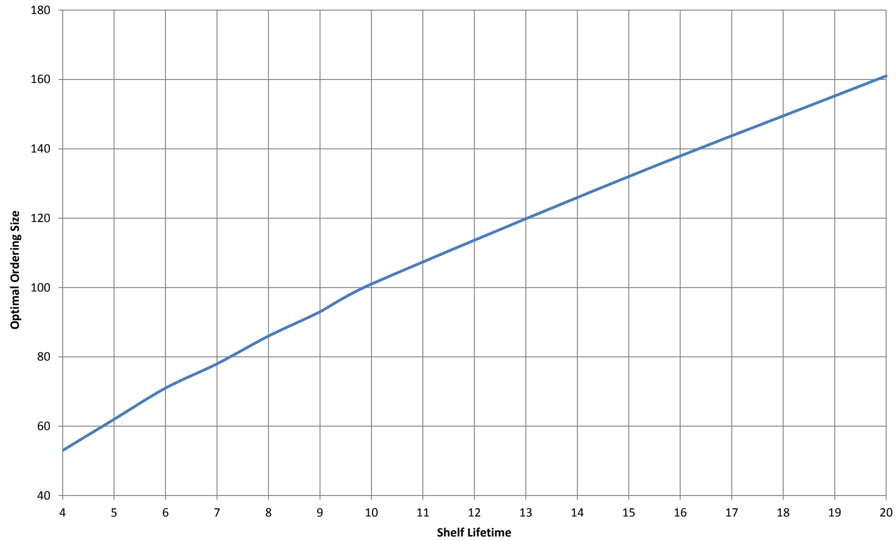

Figure 6.10: Products Shelf Lifetime effect on Optimal Ordering Size

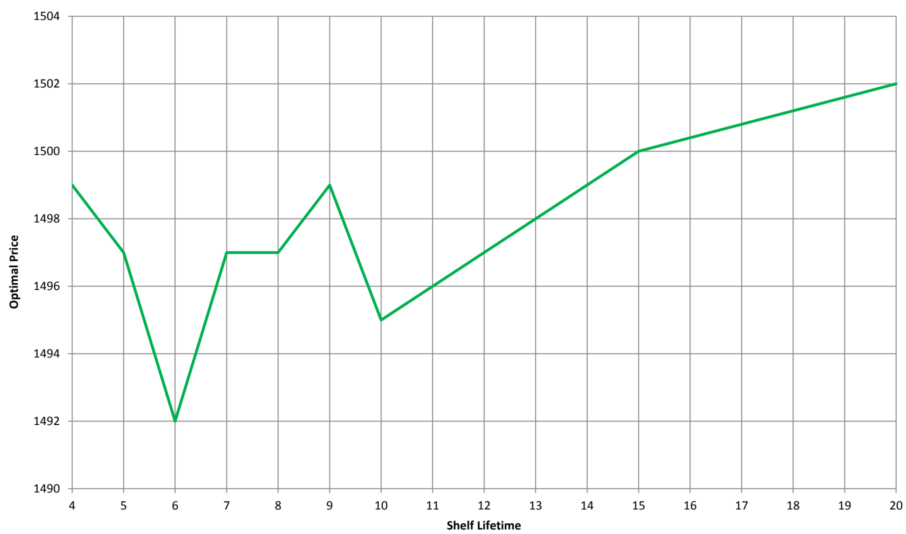

Figure 6.11: Products Shelf Lifetime effect on Optimal Price

Figures 6.16 and 6.18 show the effect of the fixed ordering cost on the order size and the average profit values, respectively. We observe that the order size increases with the

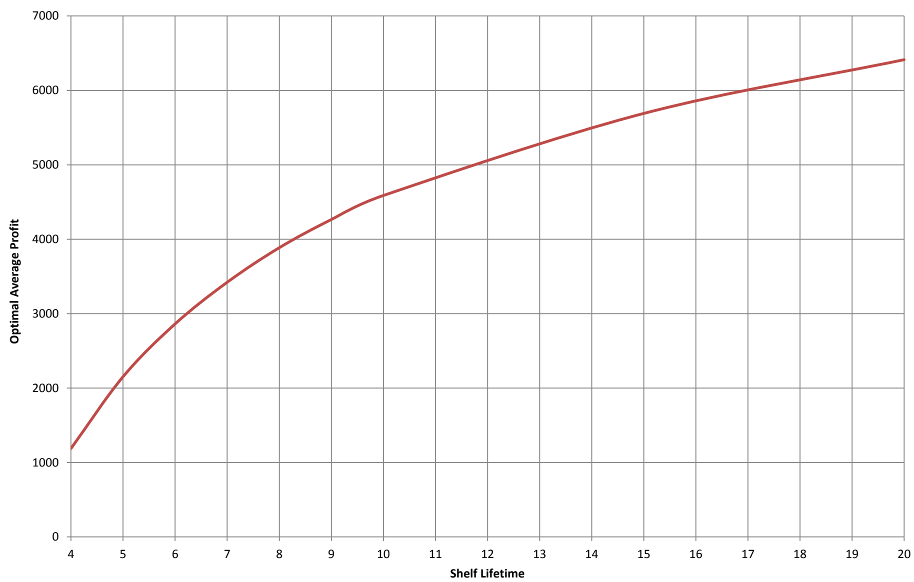

Figure 6.12: Products Shelf Lifetime effect on Optimal Average Profit

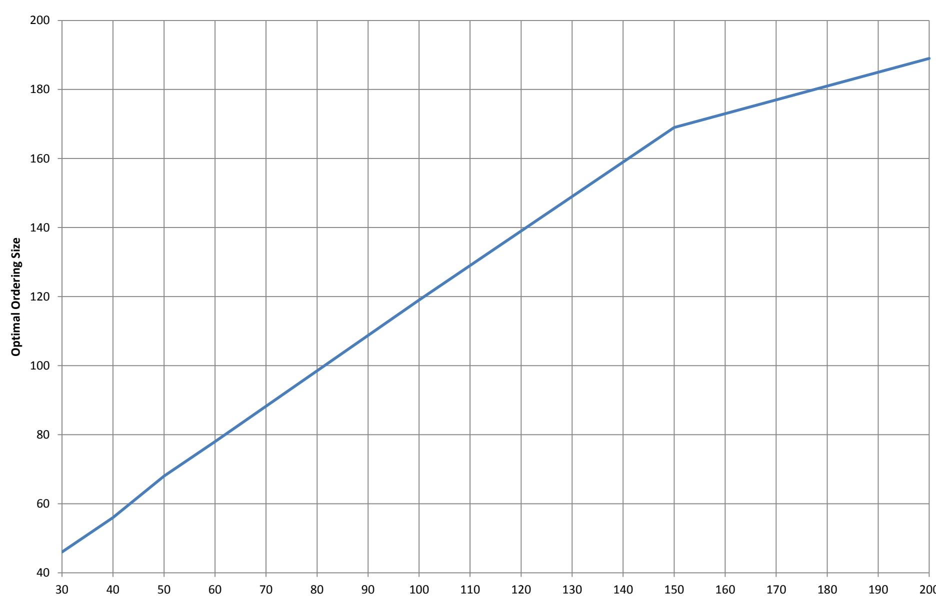

Figure 6.13: Demand rate coefficient effect on Optimal Ordering Size

fixed cost in order to decrease the order frequencies to avoid high fixed costs. But, the order size converges concavely to a certain value as the fixed cost goes beyond a certain level because ordering more than a certain amount will not be profitable since they can

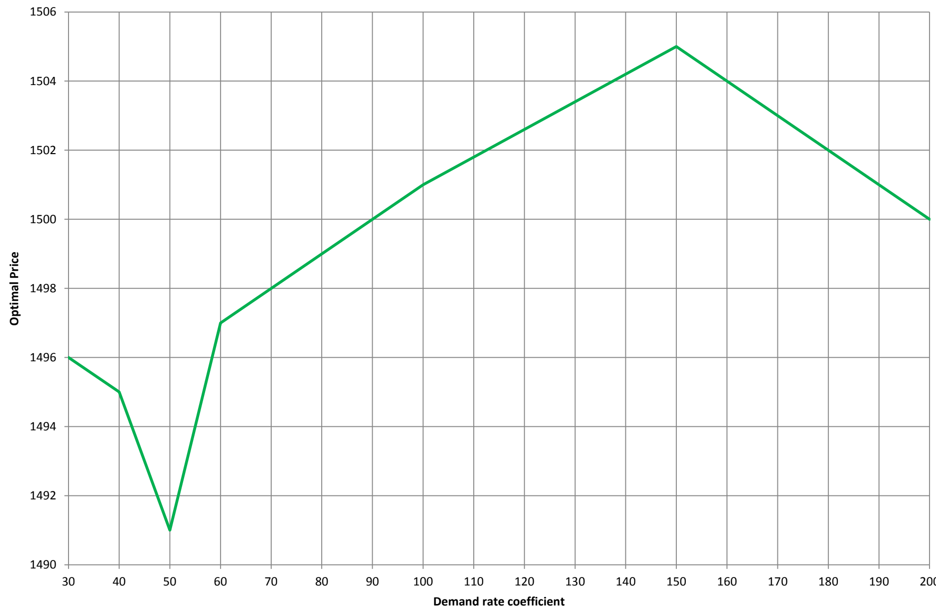

Figure 6.14: Demand rate coefficient effect on Optimal Price

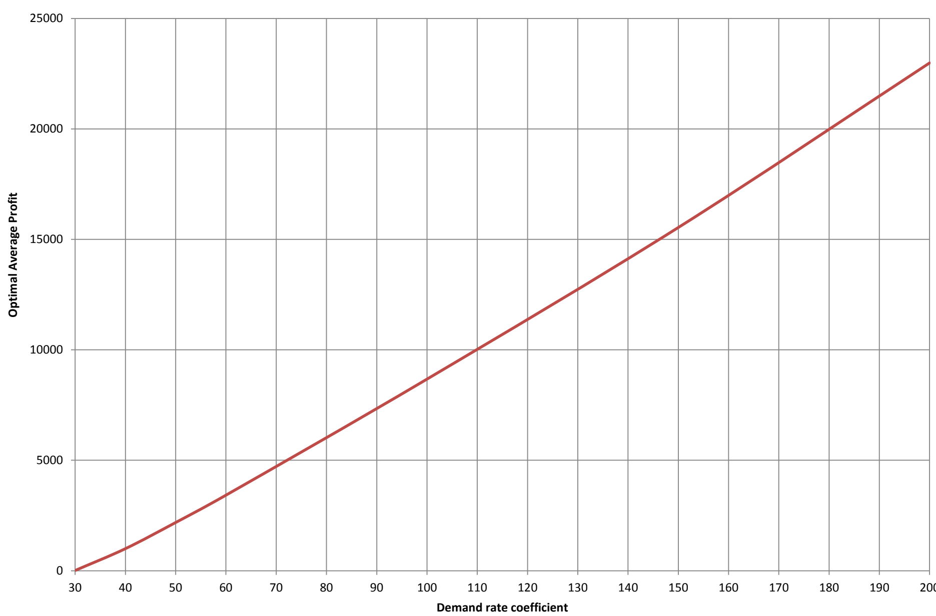

Figure 6.15: Demand rate coefficient effect on Optimal Average Profit

not be sold before their expiration dates and even though the fixed cost is too high, the order size will never be more than a certain amount. We may observe that this amount that Ordering size eventually converges to is equal to upper-bond that we introduced in proposition 6 .When we consider the average profit value, we observe that it decreases as the fixed cost increases, as expected.

Figure 6.17 tells us that again the fixed ordering cost can not be a influential parameter on optimal price value.

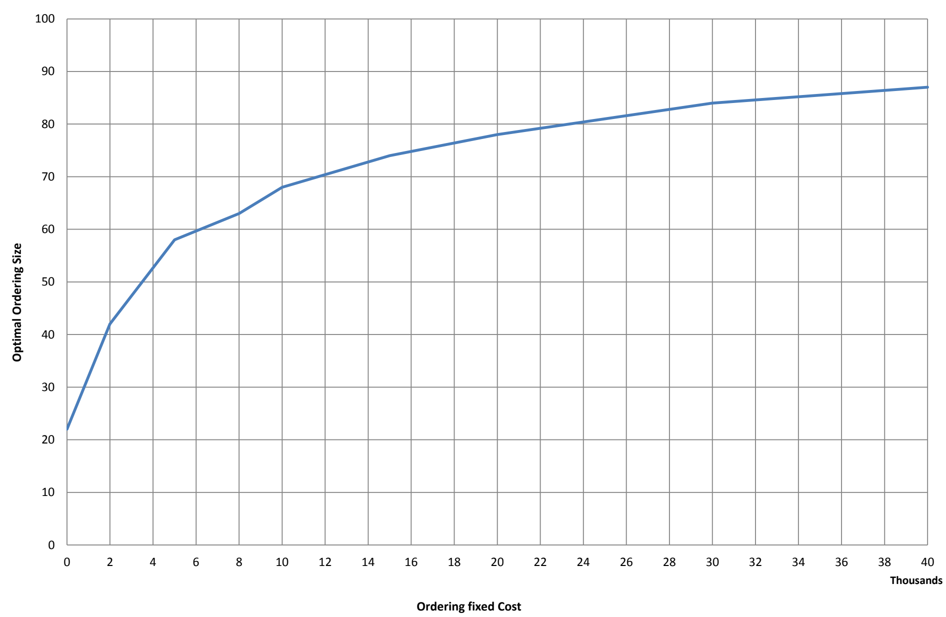

Figure 6.16: Fixed Ordering Cost effect on Optimal Ordering Size

Figures 6.19, 6.21 and 6.20 show the effect of the salvage value on the optimal ordering size , price and the average profit values, respectively. When we can salvage the products with higher prices, there is less risk to order more. So, it is reasonable that by increasing the salvage value, the optimal ordering size increases. Furthermore, we can simply conclude that, if salvage value gets close enough to the unit cost value, the optimal ordering size goes to infinity, since there is no risk in ordering more when we can salvage the product with same value that we paid for them. This behavior is also true for the optimal average profit. When salvage values increase and the difference between product cost and salvage value decreases, it is obvious that we can make more profit in the system. About the optimal price value, again we can not conclude any meaningful and considerable relation between salvage and optimal price values.

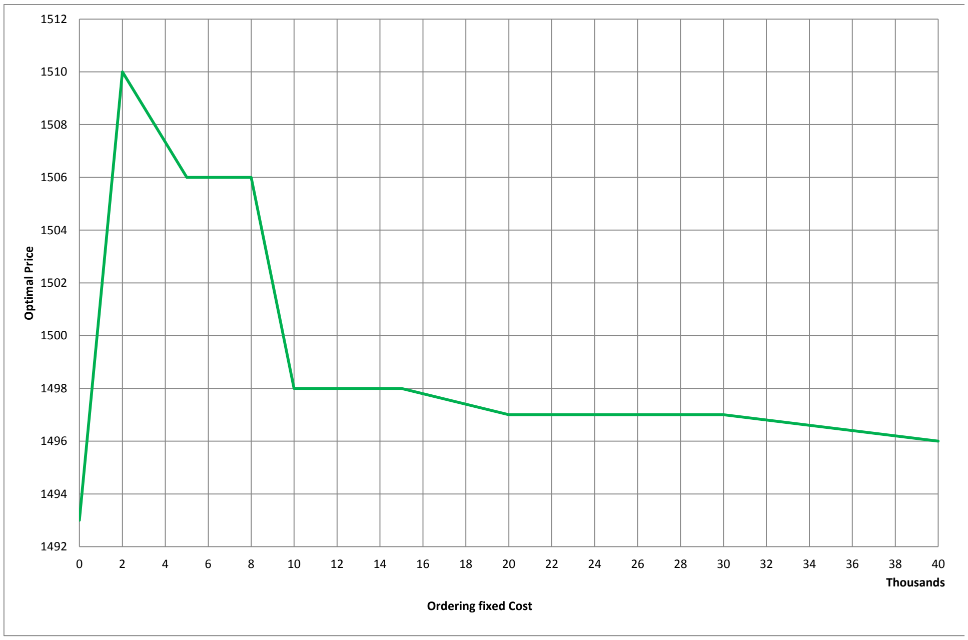

Figure 6.17: Fixed Ordering Cost effect on Optimal Price

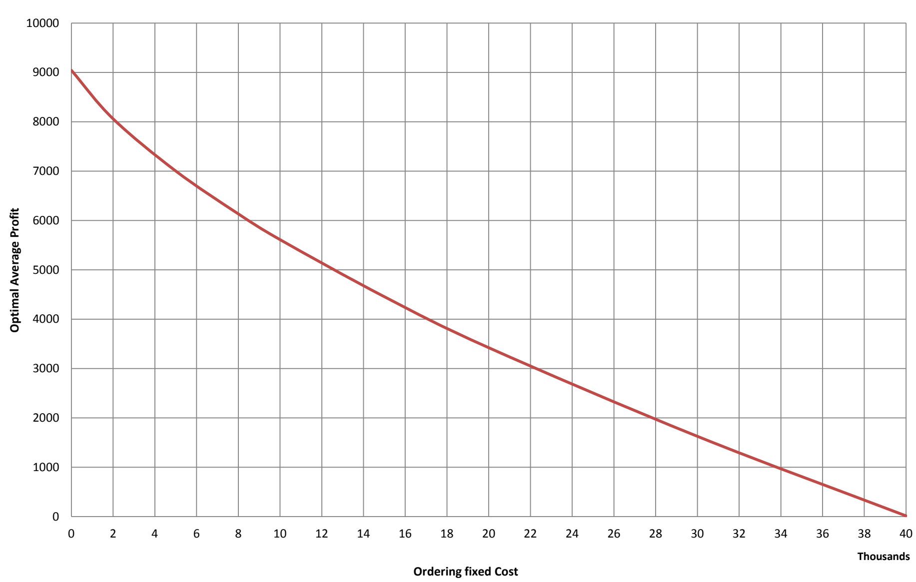

Figure 6.18: Fixed Ordering Cost effect on Optimal Average Profit

Figures 6.22 and 6.24 show the effect of the price sensitivity of demand, β, on the ordering size and the average profit values, respectively. For a given price, when β in-

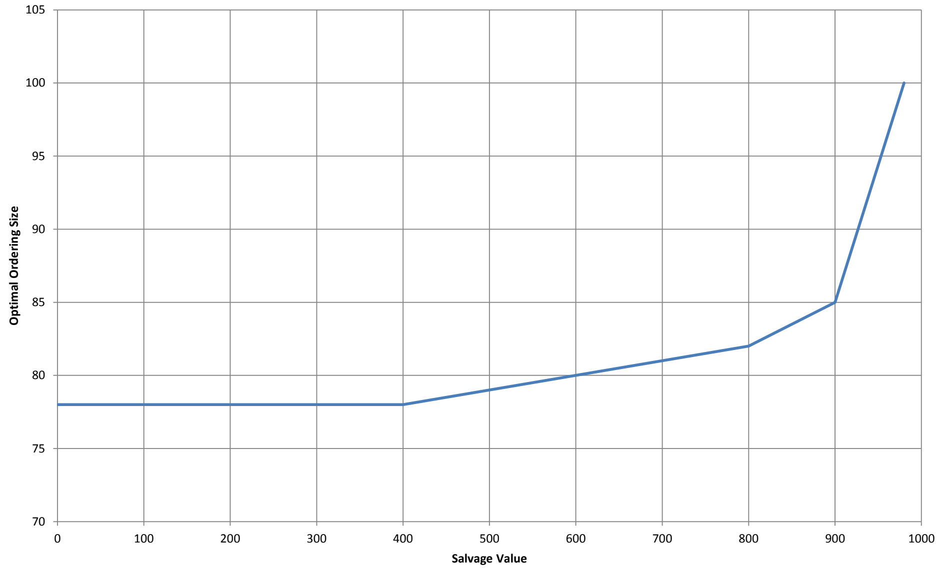

Figure 6.19: Salvage value effect on Optimal Ordering Size

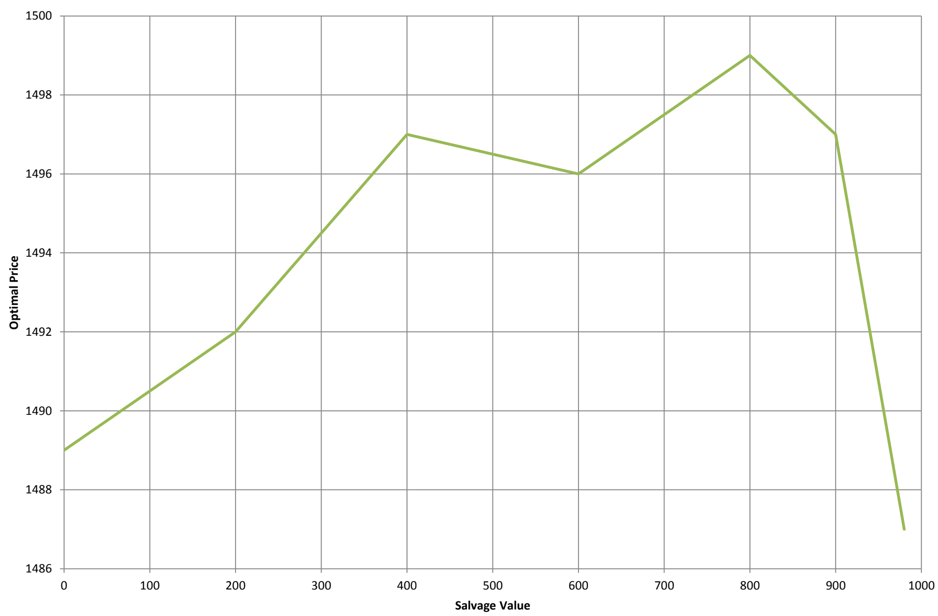

Figure 6.20: Salvage value effect on Optimal Price

creases, demand decreases. So having less demand for products, it is reasonable to order less number of products and it obviously decreases the optimal average profit in the system.

Figure 6.23 indicates that β is the only parameter that we can numerically observe a

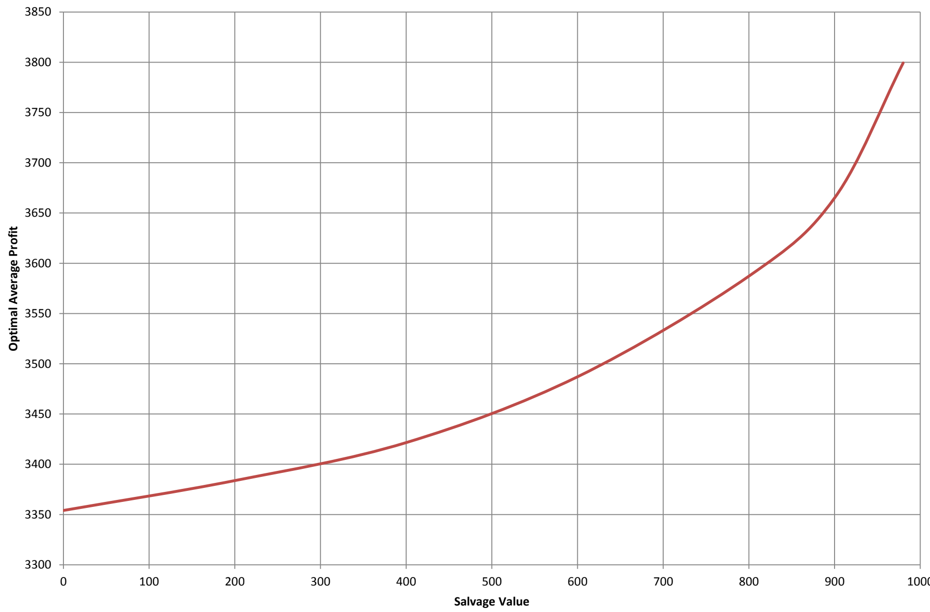

Figure 6.21: Salvage value effect on Optimal Average Profit

noticeable structure in its effect on optimal price values. As Beta increases, the optimal price decreases relatively fast. When the demand becomes more sensitive to price, we should charge lower prices to not lose the demand.

In our model, we assume that the salvage value of the products are not affected by their remaining shelf lifetimes when they are salvaged. Here, we consider the system behavior numerically when we have a salvage value that depends on the freshness of the product. In the example below you can see that when the salvage value is different for products with different age, the system shows a different behavior regarding the optimal ordering decision policy and the monotonic structure no longer holds. In this example, we assume that the salvage value decreases linearly with the remaining shelf lifetime such that $s = \frac{800t}{m}$, which depends on the remaining shelf lifetime instead of being a constant value. In Figure 6.25, you can see that the optimal ordering decision policy has a different structure than the figure given in Figure 6.9 and the structural results given in propositions 1 to 4 do not hold in this condition.

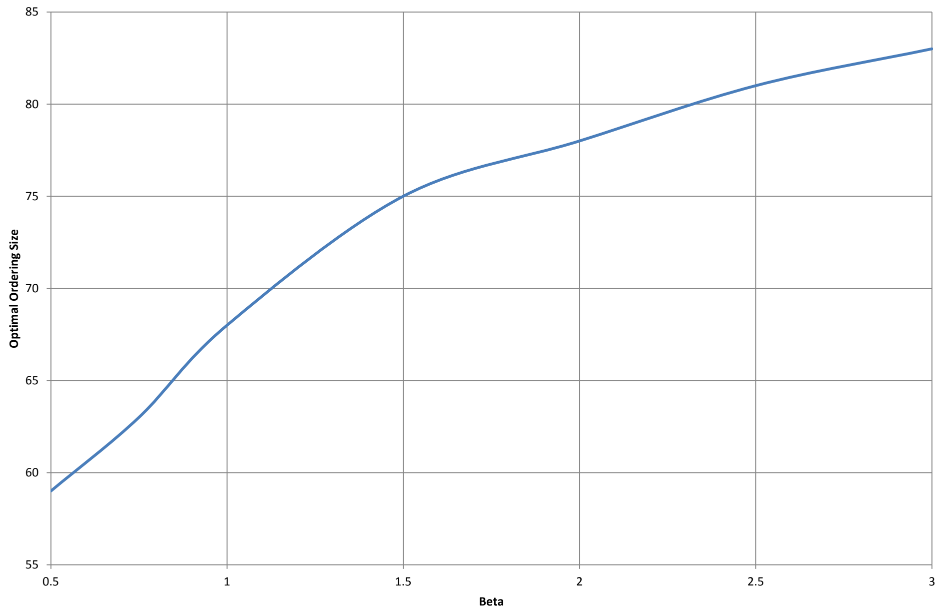

Figure 6.22: Price Sensitivity factor effect on Optimal Ordering Size

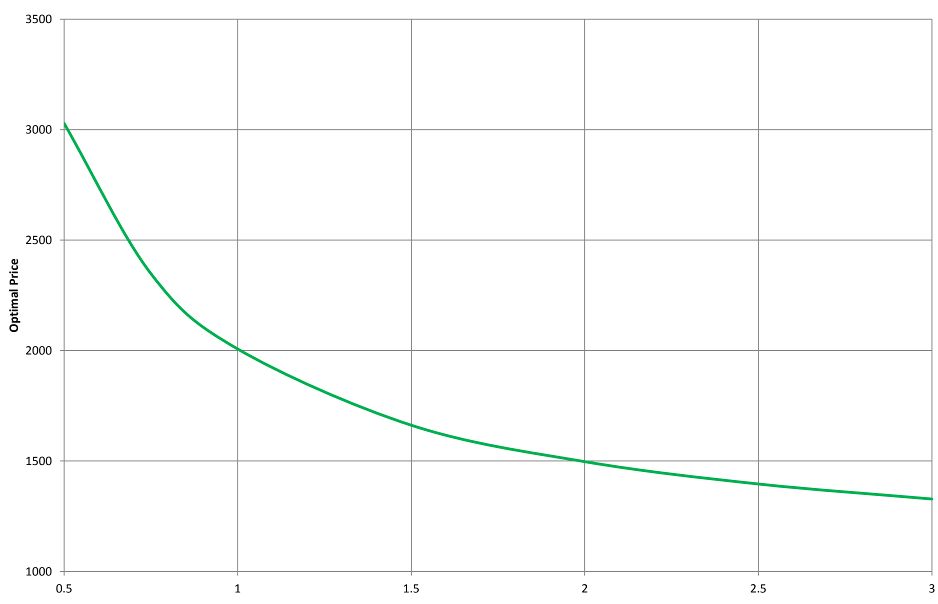

Figure 6.23: Price Sensitivity factor effect on Optimal Price

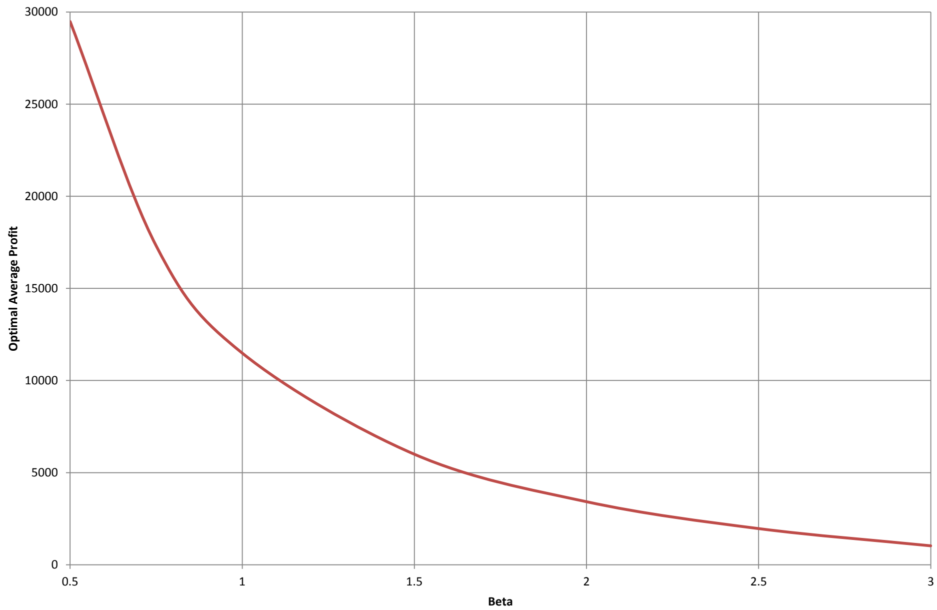

Figure 6.24: Price Sensitivity factor effect on Optimal Average Profit

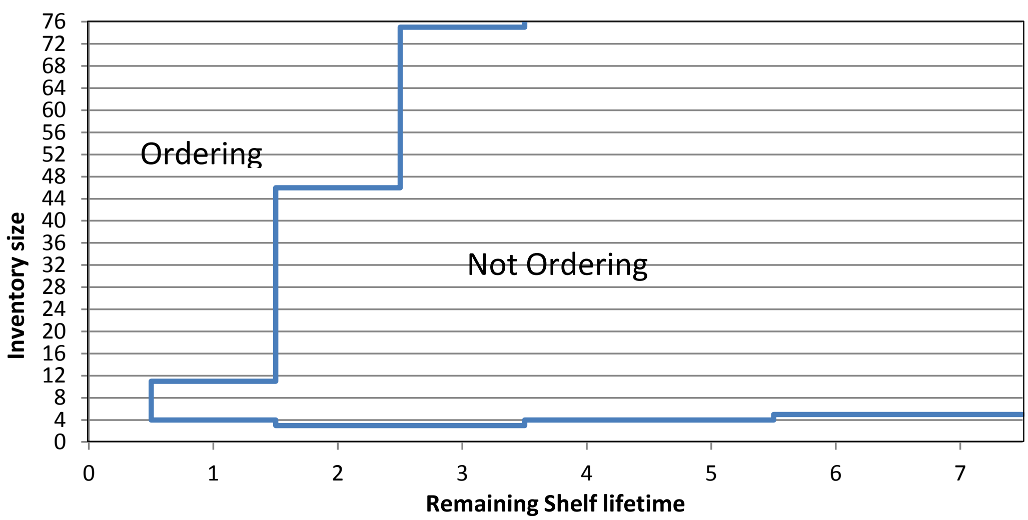

Figure 6.25: Optimal Ordering Decision Policy for an example with salvage value linearly increasing in remaining shelf lifetime

6.2 Analysis of Heuristics

In this section, we compare the introduced heuristics solutions with original DP model solutions through numerical examples. In table 6.1 we present a comparison of DP and two heuristics solutions for different sets of parameters.

We observe that the Single Inventory Level Heuristics gives us better solutions than Single Remaining Shelf Lifetime in this example. However, the running times indicate that in average the second heuristic is almost 10 times faster than first heuristic.

Generally, with respect to running time and gaps in DP and heuristics solutions, both of them seems to be efficient to use. Beside computational efforts, in the sense of application these two heuristics are easy to use. In other words, to implement these heuristic solutions in the system, it is needed to just control three parameters, Ordering quantity, price and Single ordering level.

Table 6.1: Differences between two introduced Heuristics and DP solutions and running times. To calculate the heuristics and DP solutions lag, we use equation $\frac{\Pi_{DP}-\Pi_{H}}{\Pi_{DP}}$

Parameters						DP Solution		H1 Solution			H2 Solution			Heuristics & DP lag %	
s	K	m	a	β	p	Q^*_{DP}	$RT_{DP}(ms)$	Q^*_{H1}	r^*	$RT_{H1}(ms)$	Q^*_{H2}	l^*	$RT_{H2}(ms)$	$Def._{H1}$	$Def._{H2}$
0	20000	7	60	2	1500	76	39332	76	2	3258	76	0	474	0.00001	1.38208
400	20000	7	60	2	1500	78	43850	78	3	3336	78	1	845	0.06107	1.61263
800	20000	7	60	2	1500	82	69368	80	6	3703	81	2	765	1.04099	2.37149
900	20000	7	60	2	1500	84	90422	82	8	3965	86	2	773	1.37538	0.94387
600	30000	7	50	2	1500	73	98896	73	0	3537	73	0	1065	0.00044	0.00044
600	20000	7	50	2	1500	68	35416	68	2	2444	68	1	627	0.10828	1.26883
600	10000	7	50	2	1500	59	16208	58	5	2128	61	2	449	0.22202	3.00364
600	0	7	50	2	1364	19	2650	19	8	1999	35	5	91	0.00000	2.55550
600	15000	4	60	2.75	1364	54	20102	54	0	635	54	0	154	0.00309	0.00309
600	15000	7	60	2.75	1364	78	42914	78	3	3470	79	1	954	0.07871	1.57605
600	15000	10	60	2.75	1364	101	173688	100	4	13011	102	2	3489	0.12636	2.24477
600	15000	15	60	2.75	1364	133	895106	133	5	66973	133	6	15978	0.00376	2.15274
600	15000	20	60	2.75	1500	160	3589120	160	6	497512	162	9	68851	0.00012	1.94809
500	15000	6	40	2	1500	48	26264	48	1	884	48	0	135	0.00002	0.89436
500	15000	6	70	2	1500	76	28970	75	5	3158	74	2	600	0.21745	3.48010
500	15000	6	100	2	1500	99	72310	99	9	8987	107	2	1663	0.00907	3.21636

500	15000	6	150	2	1500	140	230478	140	16	93884	138	3	2303	0.00012	0.90563
500	15000	6	200	2	3000	135	2174908	135	23	419406	135	4	1294	0.00000	0.14417
500	22000	6	80	0.5	2000	61	20676	61	16	7621	60	4	600	0.00079	0.39590
500	22000	6	80	1	1667	80	29688	80	11	7531	77	3	741	0.07354	2.05374
500	22000	6	80	1.5	1500	87	47578	86	7	4831	88	2	895	0.20555	2.69431
500	22000	6	80	2	1333	91	67740	91	5	4851	92	1	1026	0.08948	2.14183
500	22000	6	80	3	1625	96	102722	96	2	8773	96	0	1195	0.00009	0.78793
0	8000	10	75	1.6	1625	93	144132	93	7	24969	97	5	4671	0.00000	3.29732
200	8000	10	75	1.6	1625	93	115386	93	8	26084	90	6	4148	0.00000	2.72141
500	8000	10	75	1.6	1625	92	111446	92	10	26161	94	6	4328	0.00004	2.21755
700	8000	10	75	1.6	1625	84	87382	84	13	26686	97	6	4646	0.00002	1.83826
850	8000	10	75	1.6	1625	84	105328	84	16	27510	102	6	4660	0.00149	1.58620
850	0	9	75	1.6	1625	32	1934	32	21	38175	58	7	2508	0.00000	3.70430
850	4000	9	75	1.6	1625	59	29702	59	17	61763	58	7	3046	0.00191	0.27504
850	8000	9	75	1.6	1625	82	52400	82	16	18375	81	6	2951	0.00547	0.73076
850	16000	9	75	1.6	1625	102	153594	101	13	19182	100	5	3122	0.05474	1.34502
850	9000	10	30	1.6	1625	48	20456	46	5	2081	48	5	430	0.65030	2.32333
850	9000	10	60	1.6	1625	79	68650	78	12	14327	82	6	2402	0.02356	1.23336
850	9000	10	90	1.6	1625	99	114470	99	19	131915	98	7	7557	0.00368	0.53335
850	9000	10	130	1.6	1625	138	452762	138	30	1233251	139	7	40422	0.00055	0.20717
800	11000	11	90	0.75	1800	98	190988	98	24	318276	101	8	26135	0.00972	0.29881

800	11000	11	90	1.25	1571	100	491984	100	20	214025	123	7	28351	0.00202	1.28756
800	11000	11	90	1.75	1400	120	336288	120	17	121325	121	7	28282	0.00045	1.04188
800	11000	11	90	2.5	1667	132	726338	132	13	121414	139	6	31466	0.00366	1.82078
800	10000	4	70	1.5	1667	52	6810	51	11	1467	50	2	150	0.13356	1.21778
800	10000	6	70	1.5	1667	68	26804	68	12	3400	70	3	541	0.08310	0.84789
800	10000	8	70	1.5	1667	76	48638	76	13	9301	74	5	1447	0.01095	1.34390
800	10000	10	70	1.5	1667	89	161674	89	13	22076	94	6	3791	0.01645	1.29466
800	10000	12	70	1.5	1500	97	182798	97	14	45217	97	8	7588	0.00145	1.28141
500	1000	10	75	2	1500	53	31272	53	11	39565	52	8	4374	0.00000	0.79288
500	2000	10	75	2	1500	76	88494	76	10	39416	52	8	4368	0.00000	1.14337
500	3000	10	75	2	1500	76	57700	76	10	39220	74	7	4423	0.00000	1.78259
500	4000	10	75	2	1625	77	62086	77	10	53976	74	7	5526	0.00000	1.99548
500	5000	10	75	1.6	1625	78	62422	78	11	54764	75	7	5670	0.00000	1.67413
500	6000	10	75	1.6	1625	79	164860	79	11	54368	94	6	4705	0.00000	2.82867
500	7000	10	75	1.6	1500	80	162690	80	10	39789	94	6	4734	0.00000	2.42711
900	3000	3	40	2	1500	28	10826	26	8	509	29	1	117	1.95109	0.55831
900	3000	4	50	2	1500	37	55506	36	11	515	38	2	101	0.65100	0.28492
900	4000	5	60	2	1500	46	7346	46	14	1731	46	3	205	0.06417	0.23130
900	4000	6	70	2	1500	54	16030	54	16	5123	54	4	473	0.00921	0.22360
900	5000	3	30	2	1500	24	910	22	3	262	22	0	42	0.62052	4.22230
900	5000	3	40	2	1500	29	34100	29	7	344	29	1	67	1.52454	0.20420

900	4000	4	50	2	1500	37	80826	37	11	487	38	2	92	0.44876	0.17865
800	5000	3	60	2	1500	39	37284	39	10	243	40	1	88	0.85576	0.33864
800	4000	4	70	2	1500	48	8610	48	13	1667	49	2	156	0.07121	0.26797
Average Values							**203759.54**			**65653.38**			**5782.79**	**0.17731**	**1.46524**
Maximum Values														**1.95109**	**4.22230**

As we mentioned before the demand is Poisson where its rate is:

$$\lambda(p,t) = a\left(\frac{t}{m}\right)^n e^{\beta(1-\frac{p}{c})}$$

In the table 6.1 we assume that the rate function has a linear relation with remaining shelf lifetime. It means that n is equal to one. To have a deeper analysis, we examine these two heuristics for three different cases where this relation is linear, convex and concave. By considering the average lag values and maximum differences in our examples in table 6.2, we can conclude that when the demand rate has a concave relation with remaining shelf lifetime ($n < 1$), the Single Inventory Ordering Level Heuristics (H1) gives us much better solutions and when this relation goes to convexity ($n > 1$) the Single Remaining Shelf lifetime Ordering Level (H2) solutions becomes more reliable. In other words, when demand decreases mainly at the beginning of the period, it is more efficient to use the Single Remaining Shelf Lifetime Ordering Level Heuristic, and when the main demand decreasing happens at the end of the period, using Single Inventory Ordering Level gives us a more efficient solution.

Table 6.2: Differences in DP and heuristics solutions for different set of parameters in three cases. For the Linear case, we used $n = 1$, Convex $n = 0.25$ and Concave $n = 4$

Parameters						**Heuristics & DP lag %**					
						Linear		**Convex**		**Concave**	
s	K	m	a	β	p	$H1$	$H2$	$H1$	$H2$	$H1$	$H2$
900	20000	7	60	2	1500	1.37538	0.94387	11.86367	0.04373	0.00000	1.89958
600	10000	7	50	2	1500	0.22202	3.00364	10.28137	1.58484	0.00000	2.48250
600	0	7	50	2	1500	0.00000	2.55550	0.00110	0.06409	0.00000	2.40313
600	15000	10	60	2.75	1364	0.12636	2.24477	13.15032	0.00002	0.00000	1.73552
600	15000	15	60	2.75	1364	0.00376	2.15274	4.82320	1.46706	0.00000	1.56419
600	15000	20	60	2.75	1364	0.00012	1.94809	1.07206	2.07217	1.18104	2.63441
500	15000	6	70	2	1500	0.21745	3.48010	10.43401	1.39119	0.00000	1.99745
500	15000	6	100	2	1500	0.00907	3.21636	3.01737	4.61746	0.00000	2.64601
500	15000	6	150	2	1500	0.00012	0.90563	0.44892	0.33873	0.00000	0.58289

500	15000	6	200	2	1500	0.00000	0.14417	0.00000	0.19880	0.00000	0.09782
500	22000	6	80	0.5	3000	0.00079	0.39590	0.00057	0.05534	0.00088	1.98775
500	22000	6	80	1	2000	0.07354	2.05374	3.15208	0.43031	0.00001	2.98126
500	22000	6	80	1.5	1667	0.20555	2.69431	10.46016	1.41006	0.00000	2.05674
500	22000	6	80	2	1500	0.08948	2.14183	9.34242	0.00000	0.00000	1.51254
0	8000	10	75	1.6	1625	0.00000	3.29732	0.06142	4.72533	0.00000	2.07280
200	8000	10	75	1.6	1625	0.00000	2.72141	0.05062	3.67086	0.00000	2.13777
500	8000	10	75	1.6	1625	0.00004	2.21755	0.09458	1.93520	0.00003	2.10963
700	8000	10	75	1.6	1625	0.00002	1.83826	0.11622	0.66300	0.00000	1.92440
850	8000	10	75	1.6	1625	0.00149	1.58620	0.51787	0.20329	0.00101	1.00071
850	0	9	75	1.6	1625	0.00000	3.70430	0.00000	0.00000	0.00000	0.13692
850	4000	9	75	1.6	1625	0.00191	0.27504	0.00000	0.01230	0.00010	1.04780
850	8000	9	75	1.6	1625	0.00547	0.73076	0.83378	0.27040	0.00071	1.16238
850	16000	9	75	1.6	1625	0.05474	1.34502	2.44381	1.76303	0.00304	2.02797
850	9000	10	30	1.6	1625	0.65030	2.32333	15.66802	1.27360	0.04934	1.91866
850	9000	10	60	1.6	1625	0.02356	1.23336	0.49375	0.96530	0.00260	1.38384
850	9000	10	90	1.6	1625	0.00368	0.53335	0.37631	0.14411	0.00088	0.83412
850	9000	10	130	1.6	1625	0.00055	0.20717	0.00000	0.52833	0.00000	0.39499
800	11000	11	90	0.75	2333	0.00972	0.29881	0.00000	0.00703	0.00001	0.49483
800	11000	11	90	1.25	1800	0.00202	1.28756	0.21715	0.11361	0.00042	0.82180
800	11000	11	90	1.75	1571	0.00045	1.04188	0.04114	0.85455	0.02757	1.20476
800	11000	11	90	2.5	1400	0.00366	1.82078	0.30915	1.85772	0.55219	2.96812
800	10000	4	70	1.5	1667	0.13356	1.21778	1.23960	2.72316	0.00016	2.65202
800	10000	6	70	1.5	1667	0.08310	0.84789	3.23977	0.84496	0.00293	1.33412
800	10000	8	70	1.5	1667	0.01095	1.34390	1.23174	0.30376	0.00273	2.74589
800	10000	10	70	1.5	1667	0.01645	1.29466	0.21702	0.56490	0.00001	1.50622
800	10000	12	70	1.5	1667	0.00145	1.28141	0.09623	1.35026	0.00001	1.26849
500	1000	10	75	2	1500	0.00000	0.79288	0.00001	0.20177	0.00002	1.89323
500	2000	10	75	2	1500	0.00000	1.14337	0.00001	0.46858	0.00000	2.28975
500	3000	10	75	2	1500	0.00000	1.78259	0.00001	0.95391	0.00000	2.11745
500	4000	10	75	2	1500	0.00000	1.99548	0.04178	1.41900	0.00008	1.99104
500	5000	10	75	1.6	1625	0.00000	1.67413	0.03381	0.97029	0.00000	1.88411
500	6000	10	75	1.6	1625	0.00000	2.82867	0.06095	1.16687	0.00008	1.86582
500	7000	10	75	1.6	1625	0.00000	2.42711	0.06996	1.48319	0.00000	2.20961
900	3000	3	40	2	1500	1.95109	0.55831	0.00002	0.00000	0.13165	3.77339

900	3000	4	50	2	1500	0.65100	0.28492	0.00000	0.00000	0.03101	0.68468
900	4000	5	60	2	1500	0.06417	0.23130	0.00000	0.00000	0.00786	1.06592
900	4000	6	70	2	1500	0.00921	0.22360	0.00000	0.00000	0.00483	2.37541
900	5000	3	30	2	1500	0.62052	4.22230	21.94707	0.05441	0.78040	4.82083
900	5000	3	40	2	1500	1.52454	0.20420	0.00000	0.00000	1.17290	3.81098
900	4000	4	50	2	1500	0.44876	0.17865	0.00000	0.00000	0.00591	0.90004
800	5000	3	60	2	1500	0.85576	0.33864	0.00000	0.00000	0.19439	2.29084
800	4000	4	70	2	1500	0.07121	0.26797	0.00000	0.00000	0.00000	4.03407
Average Values						**0.18313**	**1.52851**	**2.45094**	**0.86859**	**0.07990**	**1.87956**
Maximum Values						**1.95109**	**4.22230**	**21.94707**	**4.72533**	**1.18104**	**4.82083**

6.3 Numerical Results for the Coordinated Inventory Control and Dynamic Pricing

In this section, we present the numerical results for the same parameters that we use in static pricing section but now we use dynamic pricing and the price is not fixed for all states. For any inventory size and Remaining Shelf lifetime, the optimal price is chosen. As we stated before, the propositions 1 and 2 also hold for dynamic pricing model. You can see in the charts 6.26 and 6.27 that the profit of states are increasing in inventory size and non-decreasing in remaining shelf lifetime in this dynamic pricing example.

In the charts below, you can see how optimal price values change with respect to inventory size and remaining shelf lifetime. For the sake of illustration, in figure 6.28, each range of price value is indicated with a color. As you can see, for the states that the optimal ordering decision is to have a new order, the optimal price value is same for all (Yellow section). For other states, you see a monotone behavior of optimal price values in inventory size and remaining shelf lifetime. However, although we know that it is always true for remaining shelf lifetime (Observation 1), it may not be true for inventory size. For this example, showing the optimal price values in different ranges causes the relation to seem monotone.

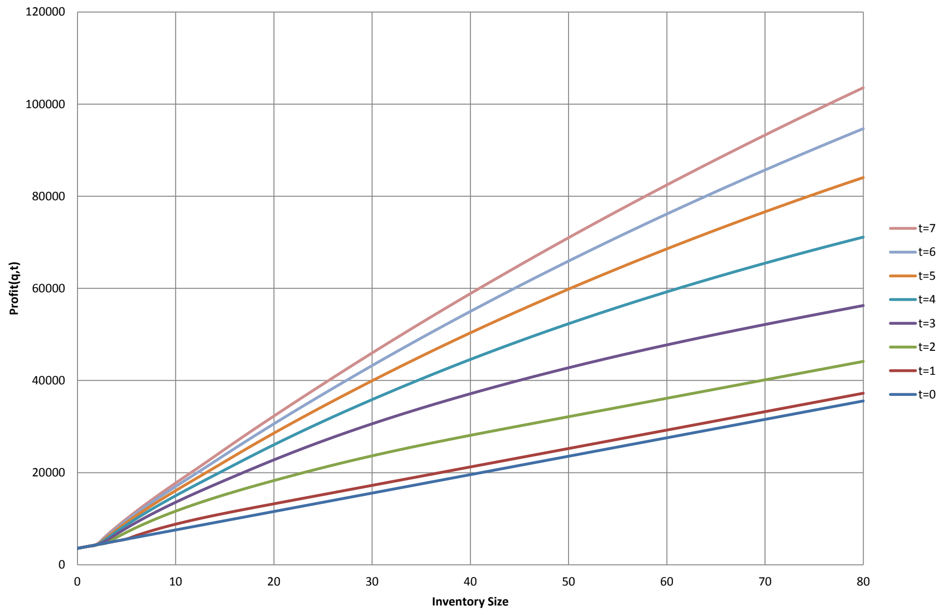

Figure 6.26: Profit of states, $h(q,t)$ versus inventory size for any remaining shelf lifetime values.

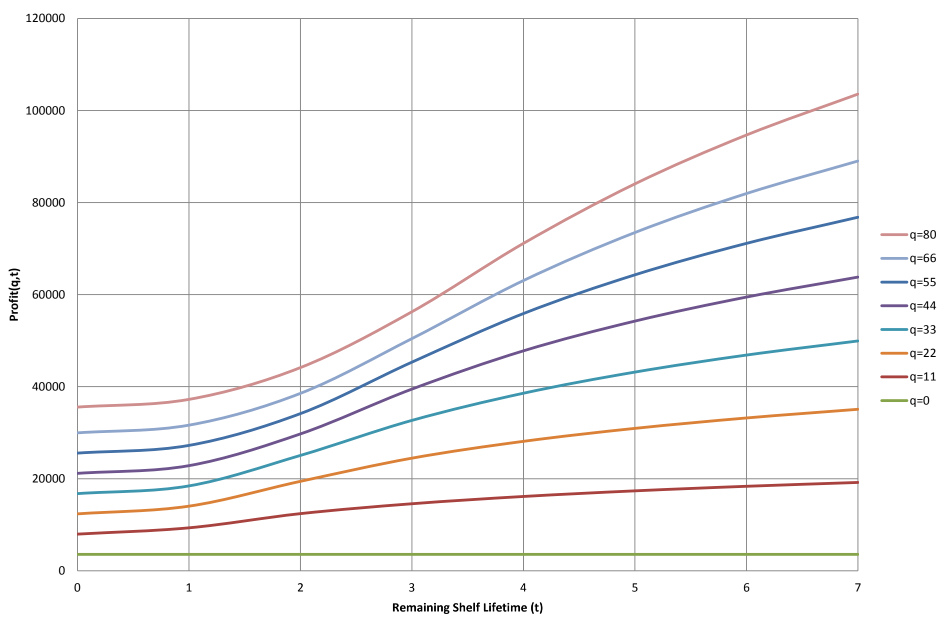

Figure 6.27: Profit of states, $h(q,t)$ versus remaining shelf lifetime for different inventory size values.

Figure 6.29 and 6.30 show how exactly optimal price values change in inventory size and remaining shelf lifetime in this example.

As we mentioned in Observation 1 , you can see in figure 6.29 that optimal price value

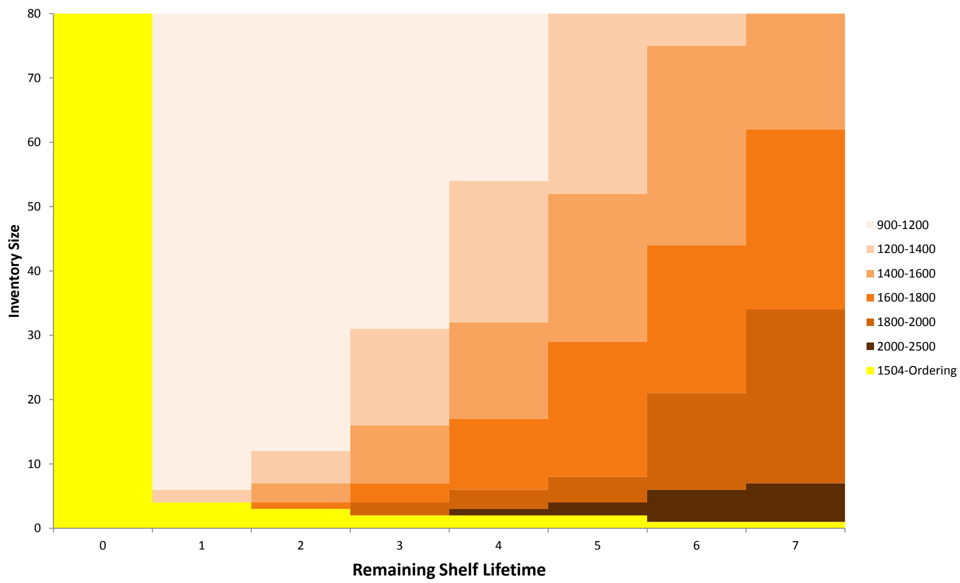

Figure 6.28: Optimal Ordering Decision Policy and Optimal Price value for any Inventory size and Remaining Shelf Lifetime. In the static price model, the optimal price value for same set of parameters is 1497

increases in remaining shelf lifetime. It means that for newer product we should always charge higher prices.

Figure 6.30 indicates that the optimal price value has not a monotone relation with inventory size. In a classical single period pricing problem like flight ticket selling or Hotel room booking, we know that when inventory size (remaining seats or rooms) decreases, we should charge higher prices. We may expected same thing in this model, but you can see in figure 6.30 that this proposition is not held in this model. When the inventory size decreases, first the optimal price increases. Then, there is a sudden decline and again optimal price increases. Since this model is infinite horizon, the Ordering decision policy and optimal price value in each state are dependent on both current and future profit. So, the reason of this sudden decline in optimal price at the end of cycle is to increase demand and to sell more and reach ordering level faster to get rid of old inventory since we can make more profit by having a new order sooner.

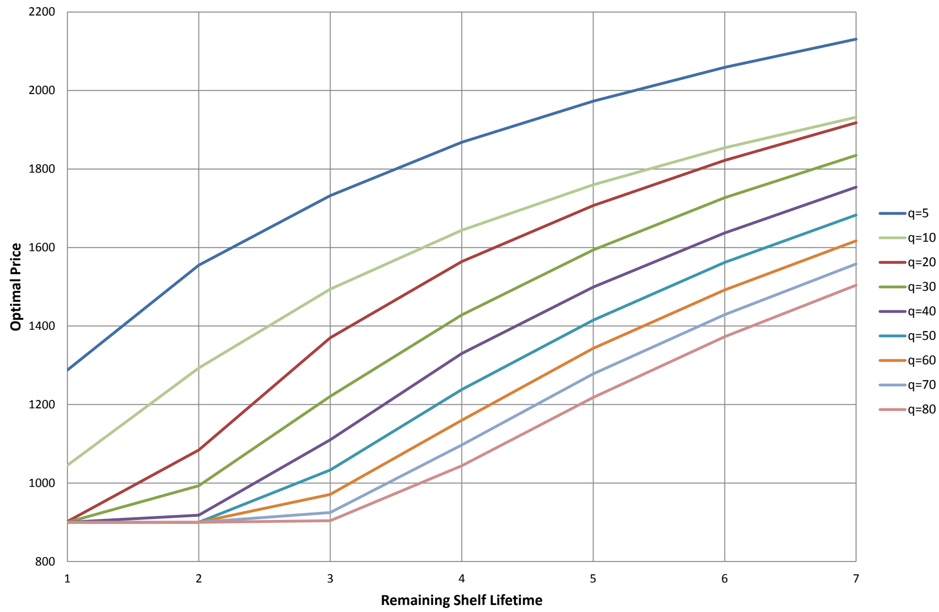

Figure 6.29: Optimal Price value versus Remaining Shelf lifetime for different Inventory Size values

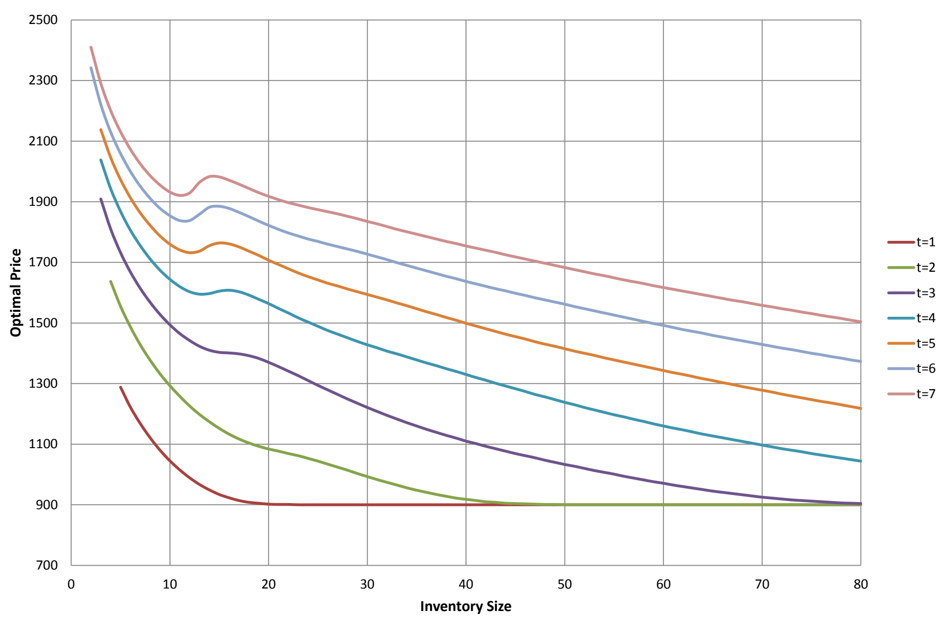

Figure 6.30: Optimal Price value versus Inventory Size for any Remaining Shelf lifetime values

6.4 Comparison of Static Pricing and Dynamic Pricing

In this section we consider the effect of dynamic pricing policy in system outputs. To have a better comparison, in Static pricing model, we should choose the best price and find the optimal average profit for this price value.
We analyze the effect of Dynamic Pricing policy with respect to two factors, Average profit, and Average number of salvaged products (Wastage quantities). By solving the models we can not find the exact value of average wastage quantities. However, we use a method that can help us to compare average number of salvaged products indirectly. The set of ordering states tells us that in average how many products will be salvaged in each cycle. We introduced critical values, $q_c(t)$ in proposition 3 that defines when we should have a new order and when we should continue. It is obvious that as much these critical values in a problem is smaller, we will have less number of products to be salvaged. In other words, if we observe that in the Dynamic Pricing solution, ordering states set is a subset of ordering states in the static pricing solution where its Ordering size is less than or equal to Dynamic Pricing model's, we may conclude that we will have less number of products to be salvaged in average in the Dynamic Pricing model. You can see in figures 6.32 and 6.31 the Dynamic Pricing Ordering States is a subset of Static Pricing Solution's.

Besides analyzing the Ordering states sets, we can compare Dynamic and Static Pricing Solutions using a mathematical expression. Expression below can be a decent factor for Static and Dynamic Pricing comparison in regard to Wastage quantities. We labeled this factor as "Ordering States Factor"

$$\frac{2\sum_{i=1}^{m} iq_c(i)}{m(m+1)Q^*}$$

We know this equation is always between 0 and 1 and closer to 1 means more wastage quantities. In table 6.3 you see the Static and Dynamic Pricing solutions differences in Ordering Size, Average profit value and Ordering States Factor for different set of parameters.

Table 6.3: Differences in Static and Dynamic Pricing models solutions respect to Ordering Size, Ordering States Factor and Average Profit value for different sets of parameters

Parameters					Static Price Solution		Dynamic Price Solution		Differences in
S	**K**	**m**	**a**	**Beta**	**Q**	**Ordg. Sts. Factr.**	**Q**	**Ordg. Sts. Factr.**	**Average Profit (%)**
0	20000	7	60	2	78	0.03215	80	0.01384	4.99194
200	20000	7	60	2	78	0.02956	80	0.01563	4.58880
400	20000	7	60	2	78	0.03984	80	0.02098	4.05538
600	20000	7	60	2	80	0.08585	81	0.06437	3.10729
800	20000	7	60	2	82	0.16496	83	0.09541	3.87863
900	20000	7	60	2	85	0.20953	87	0.12299	4.12839
980	20000	7	60	2	100	0.26492	102	0.19541	5.71182
400	0	7	60	2	22	0.43669	23	0.34317	0.95638
400	2000	7	60	2	42	0.30174	42	0.17602	2.33719
400	5000	7	60	2	58	0.29415	57	0.11779	1.91238
400	8000	7	60	2	63	0.19194	64	0.08929	1.55697
400	10000	7	60	2	68	0.11979	69	0.08437	1.76978
400	15000	7	60	2	74	0.05198	75	0.03476	2.34323
400	30000	7	60	2	84	0.01959	86	0.00415	13.36260
400	40000	7	60	2	87	0.00922	90	0.00259	83.66346
400	20000	4	60	2	53	0.00945	54	0.00185	23.69536
400	20000	5	60	2	62	0.03219	64	0.01667	9.78698
400	20000	6	60	2	71	0.06492	72	0.01653	5.90199
400	20000	8	60	2	86	0.04300	87	0.02942	6.11258
400	20000	9	60	2	93	0.16192	94	0.12182	8.50546
400	20000	10	60	2	101	0.26219	101	0.19816	7.42220
400	20000	15	60	2	132	0.28912	133	0.21192	19.79646
400	20000	20	60	2	161	0.31500	163	0.26500	4.35240
400	20000	7	30	2	46	0.02611	46	0.01220	98.34987
400	20000	7	40	2	56	0.01085	57	0.00125	17.81965
400	20000	7	50	2	68	0.02942	69	0.01501	7.03310
400	20000	7	100	2	119	0.09616	120	0.06494	6.47541
400	20000	7	150	2	169	0.26912	171	0.16492	11.11966
400	20000	7	200	2	189	0.31192	191	0.27129	13.84298
400	20000	7	60	0.5	59	0.48129	59	0.33354	0.92186
400	20000	7	60	0.75	63	0.26459	62	0.19700	0.97751
400	20000	7	60	1	68	0.19128	69	0.11212	1.26341
400	20000	7	60	1.5	75	0.10941	75	0.07143	2.03405
400	20000	7	60	2.5	81	0.09518	83	0.04149	8.01164
400	20000	7	60	3	83	0.06293	85	0.01195	19.56594
Average Values						**0.15366**		**0.10112**	

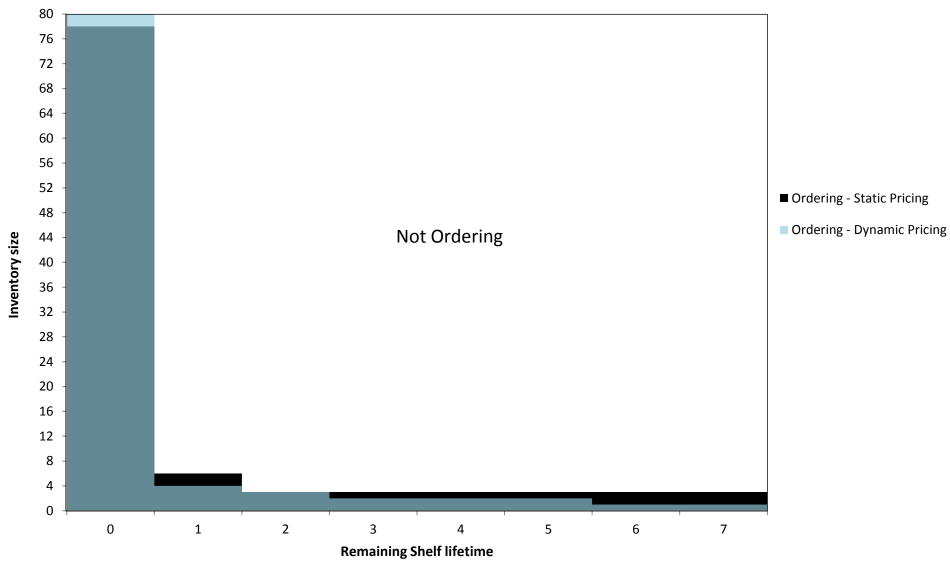

Figure 6.31: Ordering States for Static and Dynamic Pricing Model for the analyzed example

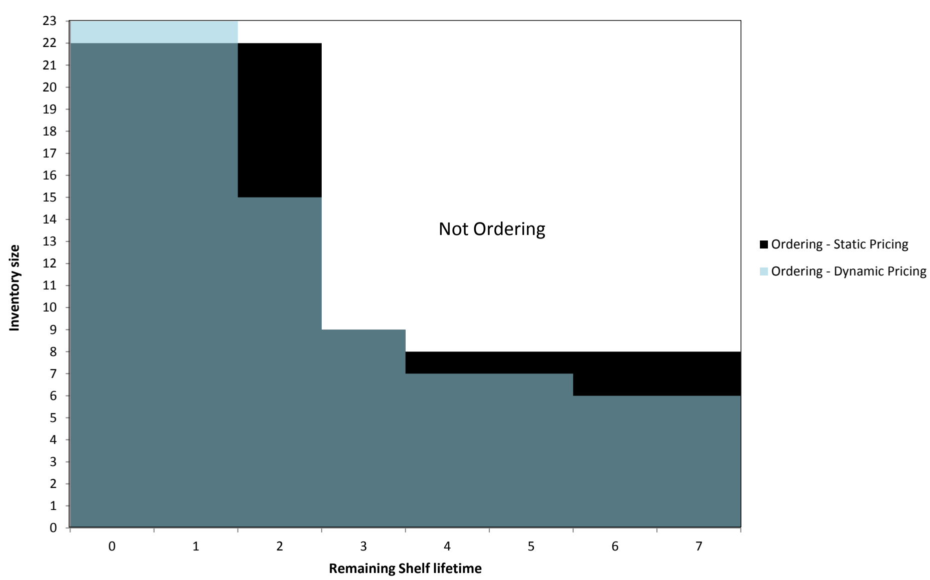

Figure 6.32: Ordering States for Static and Dynamic Pricing Model for an example with same set of parameters as analyzed example except there is no fixed Ordering Cost

The improvement in average profit using Dynamic Pricing model in some examples is totally considerable. In addition, note that the optimal ordering size value in the most of the examples has higher value in Dynamic Pricing model. The reason is that by changing the price through the period, we are able to control the demand better and manage to have higher ordering size value to decreases the fixed ordering cost effect on Average profit value. Moreover, the Ordering States Factor indicates that in the all examples we have less number of salvaged products in average in the Dynamic Pricing model. Usually by decreasing the price at the end of the period, we control the demand in the way that we have a new order later and salvage less number of products.

Generally, numerically we are able to conclude that in addition to increasing average profit by using Dynamic Pricing model, we control the system to waste less number of products.

Chapter 7

CONCLUSION

We model a coordinated inventory control and dynamic pricing problem for perishable products with fixed shelf lifetime by considering a general stochastic demand function that is dependent on both price and remaining shelf lifetime of the products. We design a dynamic programming model for this system that gives us the optimal ordering decision policy, optimal quantity to order and price that should be charged to the products. Under the given assumptions, we proved that the optimal ordering decision scenario has a monotone structure in both static and dynamic pricing models. In the dynamic pricing model, we observe that the optimal price value decreases as time passes in a period. However, the optimal price changes in a non-monotonic structure with respect to inventory size. The proven properties of monotone structure of optimal ordering decision policy and ordering quantity and also the behavior of the optimal dynamic price versus inventory size and remaining shelf lifetime facilitate the use of the model outputs in perishable inventory systems and gives valuable insights to the managers of these systems. To have a more efficient and easy to use method to solve this model, we also designed two heuristics that give us near optimal solutions with considerably less computational effort. In addition, since in these heuristics the number of variables are less than the original problem, it is easier for the management to implement these heuristic outputs in the system.

In addition to analytical analysis for both models, we examine the system numerically for static and dynamic pricing cases. Through the numerical performances, we have different observations in system properties and effects of parameters on the optimal solutions. Moreover, we test the introduced heuristics with different set of parameters to examine differences between heuristics and DP solutions and also check to what extent they are computationally more efficient than original problem. We compared these two heuristics in different cases numerically to see which one is more reliable and efficient to use in each case.

As possible future studies for this research, we may generalize some of the assumptions that we used in our model. Here, by considering that old products can not have any more demand while being next to new products after a new order, we assumed that we are able to salvage all old products with a fixed salvage value after a new order. As an extension, we may assume that the demand for old products can have a finite value that may be sensitive to the age of the products. In other words, in some cases we may not be able to salvage all old products if there is not enough demand for them and the value that we earn from salvaging of those will depend on their ages. This new assumption can be applicable for systems in which there is discount for old products as an attraction for different demand market sections. For this new assumption we need to decide for how many periods we should keep old products in the inventory to be sold.

Chapter 8

APPENDIX

8.1 Numerical Solving Methodology of Fixed Ordering Level Heuristic Problem

In this section, we introduce some numerical methods to solve the Single Inventory Size Ordering Level Heuristics efficiently.

8.1.1 Case 1: When Ordering Size is given

In this case, by observation we know that $PI(Q, r)$ is a quasi-concave function of r (without proof). Hence, since we have a quasi-concave function, we are able to use these two methods below to find best Ordering level value for a given Ordering Size.

Dichotomous line search

One of the method that we used to find the optimal ordering level is Dichotomous line searching. Since we have a discrete function, we do some modification in this method to have a better result. For stopping policy for any problem, the best allowable final length should be equal to 2 (The variable is discrete value). For this problem, the interval of uncertainty is between 0 to Ordering size value.

$$IoU : [0, Q)$$

This is the Algorithm:

Initialize

Choose $\epsilon > 0$ such that ϵ is sufficiently small,but large enough to make function evaluations distinguishable.
Choose $l = 2$, allowable final length of uncertainty.
Set $k = 1$, $a_1 = 0$, $b_1 = Q$.
Step k
If $b_k - a_k = l$, and $b_k = 2$
(in this case we should also consider the value of zero as a candidate for maximum point)
If $PI(0) < PI(1)$, $r^* = 0$
Otherwise $r^* = 1$
If $b_k - a_k = l$, and $b_k \neq 2$ STOP. $r^* = \frac{(a_k+b_k)}{2}$
Otherwise, set $\lambda_k = \left\lfloor \frac{(a_k+b_k)}{2} \right\rfloor$ and $\mu_k = \left\lceil \frac{(a_k+b_k)}{2} + \epsilon \right\rceil$
If $PI(\lambda_k) < PI(\mu_k)$, set $a_{k+1} = a_k$, $b_{k+1} = \mu_k$.
Otherwise, set $a_{k+1} = \lambda_k$, $b_{k+1} = b_k$.
Set $k = k + 1$ and go to Step k.
In the worst case, it takes

$$v = 2 \lceil log_2 Q \rceil$$

number of function evaluations to reach optimal solution.

Discrete Forward line search

In this method, we have an easier methodology to implement. We start from the beginning of Interval of Uncertainty (here is 0) and we evaluate the $PI(Q, r)$ function for this r value. Then, we go forward one by one until we reach the maximum point.
This method is a very simple method and in some examples it can be very efficient (better than Dichotomous line search), but in the worst case, it may takes $2(Q - 1)$ function evaluations to reach optimal solution, that for Q values large enough will be ever worse than Dichotomous method.

8.1.2 Case 2: When Inventory Ordering Level is given

In this case, when the Inventory Ordering level is given, $PI(Q, r)$ will not be a quasi-concave function of Ordering Size value. In other words, there may be more than one local maximum point for $PI(Q, r)$ in feasible set of Q. So, the only methods we have in this case is searching for all feasible values of Ordering Size and pick the best one as the optimal solution. But, we observe that for this function, we will have at most two local maximum points. So, whenever we reach second maximum local point, we may stop the searching before end of the feasible set.

8.1.3 Case 3: Finding Optimal (Q, r) simultaneously

To find the optimal set of (Q, r) values, we will have a more complicated problem. Below, I introduce three methods that we may use to solve this problem.

Finding Best r value for different Q:

In this method, we need to evaluate $PI(Q, r)$ function for all feasible values of Ordering Size (Q) and for each Q value, we use one of the abovementioned methods introduces in Case 1 to find the best Ordering level value (r^*) for this Q value. Then, we pick the set of (Q, r) with maximum function value. This (Q, r) value will be the optimal solution.

Finding Best Q value for different r:

This method is similar to previous one with one difference. In this case, we need to evaluate $PI(Q, r)$ function for all feasible Inventory size Ordering Level values and for each value of r we find the best ordering size value with the method introduced in Case 2. Then, we pick the set of (Q, r) with maximum function value. This (Q, r) value will be the optimal solution.

Multiple Optimum Points Searching Method

This method is more complicated than two previous ones. Both of the previous methods seem to be inefficient. So, we try to design a more efficient method to solve this problem. In this method designing, I use this observation that the function of $PI(Q, r)$ has at most two maximum point in Q.
This is the Algorithm:

Initialize
Start with an initial value of Inventory size Ordering Level.
We may use $r = 0$ to start.
Step k ,
Find all local maximum points of Ordering size (Q) for this r_k value. Q^*_{k1}, Q^*_{k2}
(It may be one or two local maximum points)
Find Optimal Inventory size Ordering Level for each $Q^*_{k1}, Q^*_{k2} : r^*_{k1}, r^*_{k2}$
If for any $i = 1, 2$ $r_k = r^*_{ki}$,keep (Q^*_{ki}, r^*_{ki}) as a condidate for optimal solution.
If for all $i = 1, 2$ $r_k = r^*_{ki}$, STOP
Otherwise, $r_{k+1} = r^*_{ki}$
Set $k = k + 1$ and go to Step k.
Evaluate all optimal solution candidate (Q, r) sets,pick one with maximum function value as Optimal Solution

This algorithm gives a (Q, r) set that will be the optimal solution for this problem. The reason why this method gives us the optimal solution can be shown simply by contradiction. By this method, we find all the (Q, r) set values that Q is the best value if the Inventory size Ordering level is equal to r, and r is the best value if the Ordering size is equal to Q. We claim that the optimal solution for $PI(Q, r)$ are among these sets. By contradiction suppose that the optimal solution, (Q_c, r_c), does not satisfy the condition that is mentioned above (The condition that this method finds (Q, r) values).
Then, if Q_c is optimal solution when the Inventory size Ordering level is equal to r_c , and

r_c is optimal solution when the Ordering size is equal to Q_c, so (Q_c, r_c) satisfies the condition. Otherwise, there will be another value for Ordering size (or Inventor size Ordering Level) that improves the sub-problem that means it also improves the $PI(Q, r)$ function value. So, (Q_c, r_c) cannot be an optimal solution.
Because of difference cases that can happen in this problem in in these three methods, we cannot analytically compare these three methods regard to computational effort. So, in table 8.2 it is shown for different set of parameters which method is faster.

8.1.4 Numerical Results

In this section, we show some numerical result about methods that introduced in previous section.
Fist, about $PI(Q, r)$ function structure, I mentioned some properties that we got by observation about this function and here I am going to illustrate them in some charts. As I mentioned before, $PI(Q, r)$ function is a quasi-concave function r for a given Q value. Figure 8.1 is an example of this function that shows this behavior. Generally, this function has similar structure for different parameters.

In the case that the Inventory size Ordering Level is given, we have non-concave function of $PI(Q, r)$ in inventory size. In the chart below you can see an example that there is two local maximum points for Q values. We know that by observation, $PI(Q, r)$ has at most two local maximum points.

In this part, we compare two methods that we introduced in Case 2 of previous section numerically in some practical examples. We showed analytically that in the worst case, Dichotomous Method shows better result than Discrete Forward line search. In table 8.1 you can see for different parameters how much it takes for each of these methods to find the optimal Inventory size Ordering Level. As you can see, in the most of the examples, Discrete forward method shows better result. So, practically, since Optimal Inventory

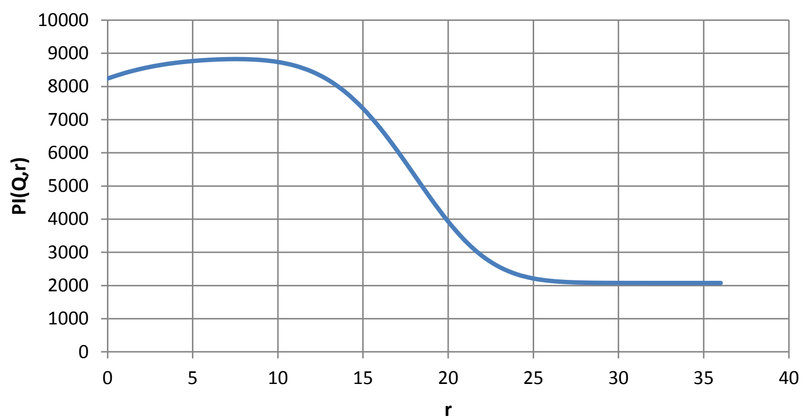

Figure 8.1: Heuristic Expected Profit function versus Inventory Size Ordering Level for a given Ordering Size value

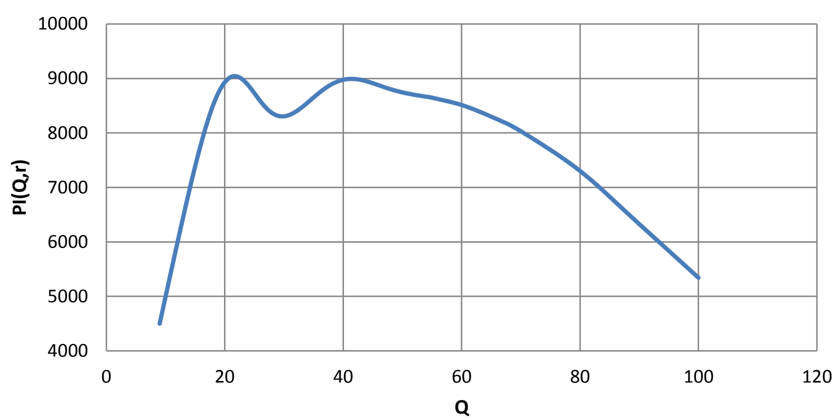

Figure 8.2: Heuristic Expected Profit function versus Ordering Size for a given Inventory Size Ordering Level value

size Ordering level (r^*) usually is small in compared with Ordering size value (Q^*), the Discrete forward method can solve the problem faster than Dichotomous method.

In the table 8.2, you can see different examples that are solved with three different methods that are introduced in Case 3 of previous section. These three methods are

Table 8.1: Dichotomous and Discrete Forward Line search Methods comparison in finding the optimal Inventory size Ordering Level

C	K	m	S	AD	P	Q	r*	Dichotomous RT	Discrete Fwd. RT
1000	0	4	0	100	1500	36	10	44	47
1000	0	7	100	80	1500	53	8	227	183
1000	0	8	200	60	1500	41	6	168	105
1000	0	8	300	50	1500	35	6	104	80
1000	10000	5	400	70	1600	51	5	127	74
1000	10000	5	500	90	1600	64	8	197	165
1000	10000	7	600	110	1600	89	12	758	698
1000	10000	7	700	60	1600	72	1	493	434
1000	30000	6	600	70	1600	72	2	374	103
1000	30000	6	500	80	1500	96	3	588	234
1000	30000	10	400	70	1500	123	4	2269	872
1000	30000	4	300	90	1500	79	1	308	54
1000	2000	5	500	80	1500	30	10	38	47
1000	2000	6	700	100	1500	50	16	136	243

compared to each other computationally. As you can see, in the most of the examples method 3 is faster than others. So, here we have a result that we expected before. Since, in method 3 we use more information about the function structure and we eliminate the solutions that cannot be a candidate for the optimal solution, we have a faster method to solve this problem.

Table 8.2: Comparison of three methods that are introduced in Case 3 for solving the $PI(Q, r)$ function to optimality respects to Computational efforts

C	**K**	**m**	**S**	**AD**	**P**	**Q***	**r***	**M1 RT**	**M2 RT**	**M3 RT**
1000	0	4	0	100	1500	36	10	2481	15161	2512
1000	0	7	100	80	1500	53	8	45386	6213	2233
1000	0	8	200	60	1500	41	6	32219	10214	1441
1000	0	8	300	50	1500	35	6	18185	6046	971
1000	10000	5	400	70	1600	51	5	3975	4064	989
1000	10000	5	500	90	1600	64	8	8218	7214	1881
1000	10000	7	600	110	1600	89	12	69271	62529	11445
1000	10000	7	700	60	1600	72	1	34341	6164	2170
1000	30000	6	600	70	1600	72	2	27210	103326	4236
1000	30000	6	500	80	1500	96	3	41831	222543	7027
1000	30000	10	400	70	1500	123	4	145385	972654	23016
1000	30000	4	300	60	1500	79	1	27155	107933	3942
			Average					38954.727	128738.91	5155.25

BIBLIOGRAPHY

[1] Abad, P. L. (1996). *Optimal pricing and lot sizing under conditions of perishability and partial backordering*, Management Science, vol.42, pp. 1093–1104.

[2] Abad, P. L. (1997). *Optimal policy for a reseller when the supplier offers a temporary reduction in price*, Decision Sciences, vol.28, pp. 637.

[3] Abad, P. L. (2001). *Optimal price and order size for a reseller under partial backordering*,Computers and Operations Research, vol.28, pp. 53–65.

[4] Abad, P. L. (2003). *Optimal pricing and lot sizing under conditions of perishability and partial backordering and lost sale*,European Journal of Operational Research, vol.144, pp. 677–685.

[5] Bakker, M., Riezebos, J. & Teunter, H. R. (2012). *Review of inventory systems with deterioration since 2001*, European Journal of Operational Research, vol. 221, pp. 275–284.

[6] Benkherouf, L. (1995). *On an inventory model with deteriorating items and decreasing time-varying demand and shortages*, European Journal of Operational Research, vol. 86, pp. 293–299.

[7] Bitran, G. C & Caldentey, R. *An Overview of Pricing Models for Revenue Management*, Manufacturing and Service Operations Management, vol. 5(3), pp. 203–229.

[8] Broekmeulen, RACM & van Donselaar, K.H. (2009). *A heuristic to manage perishable inventory with batch ordering, positive lead-times, and time-varying demand*, Computers and Operations Research, vol. 36, pp. 3013–3018.

[9] Burwell, T. H., Dinesh, D. S., Fitzpatrick, K. E. & Melvin, R. (1991). *An Inventory Model with Planned Shortages and Price Dependent Demand*, Decision Sciences, vol.22, pp.1187.

[10] Chen, X., Pang, Z. & Pan, L. (2014). "*Coordinating inventory control and pricing strategies for perishable products*, Operations Research, 62(2), pp. 284–300.

[11] Chen, L. M. & Sapra, A. (2013). *Joint inventory and pricing decisions for perishable products with two-period lifetime*, Naval Research Logistics, 60(5), pp. 343–366.

[12] Chew Peng, E. & Chulung, L. (2009). *Joint inventory allocation and pricing decisions for perishable products*, International Journal of Production Economics, vol.120, pp. 139–150.

[13] Gallego, G. & van Ryzin, G. (1994). *Optimal Dynamic Pricing of Inventories with Stochastic Demand over Finite Horizons*, Management Science, vol. 45, pp. 24–41.

[14] Gayon, J. P., Talay-Değirmenci, I., Karaesmen, F. & Ormeci, E. L. (2009). *Dynamic pricing and replenishment in a production/inventory system with Markov-modulated demand*, Probability in the Engineering and Informational Sciences, vol. 23, pp. 205–230.

[15] Goyal, S. K. & Giri, B.C. (2001). *Recent trends in modeling of deteriorating inventory*, European Journal of Operational Research, vol. 134, pp. 1–16.

[16] Minner, S. & Transchel, S. (2010). *Periodic review inventory-control for perishable products under service-level constraints*, OR Spectrum, vol.32(4), pp.979-996.

[17] Mishra, V. K. & Singh, S. L. (2010). *Deteriorating Inventory Model with Time Dependent Demand and Partial Backlogging*, Applied Mathematical Sciences, vol. 4, no. 72, pp. 3611–3619.

[18] Monahan, G. E., Petruzzi, N. C. & Zhao, W. (2004). *The Dynamic Pricing Problem from a Newsvendor's Perspective*, Manufacturing and Service Operations Management, vol. 6, pp. 73–91.

[19] Mukhopadhyay, S., Mukherjeea, R.N. & Chaudhuri, K.S. (2004). *Joint pricing and ordering policy for a deteriorating inventory*, Computers and Industrial Engineering, vol.47, pp. 339–349.

[20] Nahmias, S. (1982). *Perishable Inventory Theory: A Review.*

[21] P-S You. (2005). *Inventory policy for products with price and time-dependent demands*, 2005, Journal of the Operational Research Society, vol.56, pp.870–873.

[22] Rafaat, F. (1991). *Survey of literature on continuously deteriorating inventory models*, The Journal of the Operational Research Society, vol. 42, pp. 22–37.

[23] Rajan, R. A. & Steinberg, R. (1992). *Dynamic pricing and ordering decisions by a monopolist*, Management Science, vol. 38, no.2, pp. 240–262.

[24] Sana, S. S. (2010). *Optimal selling price and lotsize with time varying deterioration and partial backlogging*, Applied Mathematics and Computation, vol. 217, pp. 185–194.

[25] Transchel, S. & Minner, S. (2009). *The impact of dynamic pricing on the economic order decision*, European Journal of Operational Research, vol. 198, pp. 773–789.

[26] Wee, H. M. (1999). *Deteriorating inventory model with quantity discount, pricing and partial backordering*, International Journal of Production Economics, vol.59, pp. 511–519.

[27] Zhao, W. & Zheng, Y. (2000). *Optimal Dynamic Pricing for Perishable Assets with Nonhomogenous Demand*, Management Science, vol. 46(3), pp. 375–388.

[28] Fries BE. *Optimal ordering policy for a perishable commodity with fixed lifetime.* Operations Research, 1975

[29] Nahmias S. *Optimal ordering policies for perishable inventoryII.* Operations Research, 1975

[30] Weiss, H. J. *Optimal ordering policies for continuous review perishable inventory models.* Operations Research, 1980.

[31] Raafat F. *Survey of literature on continuously deteriorating inventory models.* Operations Research, 1991

[32] Nahmias S. *Higher order approximation for the perishable inventory problem.* Operations Research, 1977

[33] Haijema R, van der Wal J, van Dijk NM. *Blood platelet production: optimization by dynamic programming and simulation.* Computers & Operations Research, 2007

[34] Yasuo Adachi, Toyokazu Nose, Sennosuke Kuriyama, *Optimal inventory control policy subject to different selling prices of perishable commodities*, 1999

[35] Tal Avinadava, Avi Herbona, Uriel Spiegel, *Optimal Inventory Policy for a Perishable Item with Demand Function Sensitive to Price and Time*, 2013

[36] Van Zyl G. *Inventory control for perishable commodities.* Unpublished PhD dissertation, University of North-Carolina, Chapel Hill, NC, 1964.

[37] Kincaid, W. M., D. Darling. 1963. *An inventory pricing problem.* J. Math. Anal. Appl. 7 183208.

[38] Kocabiyikoglu, A., Popescu, I. 2011. *An elasticity approach to the newsvendor with price sensitive demand.* Operations Research, 59(2), 301-312.

[39] Avinadav, T., Arponen, T., 2009. *An EOQ model for items with a fixed shelf-life and declining demand rate based on time-to-expiry technical note.* Asia-Pacific Journal of Operational Research, 26(6), 759-767.

[40] Petruzzi, N.C., Dada, M., 1999. *Pricing and the newsvendor problem: a review with extensions.* Operations Research, 47(2), 183-194.

[41] Sarker, B.R., Mukherjee, S., Balan, C.V., 1997. *An order-level lot size inventory model with inventory-level dependent demand and deterioration*, International Journal of Production Economics, 48, 227-236.

[42] Wu, K.S., Ouyang, L.Y., Yang, C.T., 2006. *An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging.* International Journal of Production Economics, 101, 369-384.

[43] Mills, E.S., 1959. *Uncertainty and price theory.* Quarterly Journal of Economics, 73, 116-130.

[44] Karlin, S., Carr, C.R., 1962. *Prices and optimal inventory policy.* In Arrow, K.J., Karlin, S., and H. Scarf (Eds.), Studies in Applied Probability and Management Science. Stanford University Press, Stanford, CA, 159-172.

[45] Zabel, E., 1970. *Monopoly and uncertainty.* Review of Economic Studies, 37, 205-219.

[46] Silver, E.A., 1981. *A simple inventory replenishment decision rule for a linear trend in demand.* Journal of the Operational Research Society, 30, 71-75.

[47] Weatherford. L.R., Bodily, S.E., 1992. *A taxonomy and research overview of perishable-asset revenue management: yield management, overbooking, and pricing.* Operations Research, 40(5), 831-844.

[48] Valliathal, M., Uthayakumar, R., 2011. *Optimal pricing and replenishment policies of an EOQ model for non-instantaneous deteriorating items with shortages.* International Journal of Advanced Manufacturing Technology, 54, 361-371.

[49] Maihami, R., Kamalabadi, I. N., 2012. *Joint pricing and inventory control for non-instantaneous deteriorating items with partial backlogging and time and price dependent demand.* International Journal of Production Economics, 136, 116-122.

[50] Liming Liu and Zhaotong Lian, *(s, S) Continuous review models for products with fixed lifetimes*, 1998

[51] Emre Berk and UlkuGu rler, *Analysis of the (Q,r) Inventory Model for Perishables with Positive Lead Times and Lost Sales*, 2008