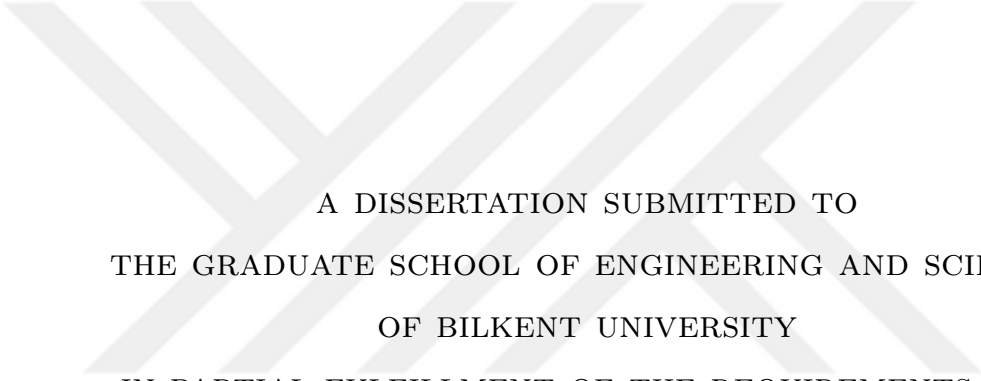


AGE AND LIFETIME BASED POLICIES FOR PERISHABLE ITEMS



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Age and lifetime based policies for perishable items

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October 2018

We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

AGE AND LIFETIME BASED POLICIES FOR PERISHABLE ITEMS

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Many inventory systems hold items which perish after a specific time. Upon perishing, the inventory level falls down to zero which may incur irreparable costs to the system. Therefore, developing a genius control policy for managing such inventories is a crucial task. Since the lifetime of items are now affecting the inventory level, applying the traditional inventory policies which are based only on the stock level causes some shortcomings. The traditional inventory policies lack the information regarding the lifetime of items. On the other hand, the optimal policy for perishable items is known to be a periodic review policy keeping the complete information regarding the remaining lead times of orders, inventory on-hand, and lifetimes of items. Optimal control policy class for continuous review is still an open question. In this regard, we attempt to contribute the remaining lifetime of items into the inventory policy for perishable items with positive lead time and fixed lifetime under a continuous review with a service level constraint. We develop a class of hybrid control policies which utilize the remaining lifetimes of items in addition to stock levels. We study a stochastic single item inventory system where demand follows a Poisson process and unmet demand is lost. The aging process of a new batch starts when it joins the inventories. We provide an exact analytic model by using an embedded Markov chain process to derive the stationary distribution of the effective lifetimes in the presence of both one and more than one outstanding orders assumptions. Operating characteristics of the system are derived using the renewal reward theorem. Additionally, we propose some control policies based on only the remaining lifetime of items. Our results reveal that the hybrid policies consistently outperform the stock level and remaining lifetime-based policies, especially when demand during the lifetime is sufficiently small and unit perishing cost is high. It is observed that the dominance relations among these two policy classes depend on the particular parameter setting. In particular, when the lifetime of items is long enough, the stock level based policy performs very well. Finally, we present our methodology for finding

the optimal solution thorough a heuristic algorithm derived by considering the structure of the objective function and service level constraint, and a sensitivity analysis is performed to evaluate the impact of the key input parameters.



Keywords: Perishable inventory, Inventory policy, Effective lifetime, Embedded Markov chain, Renewal theorem.

ÖZET

TÜRKÇE BAŞLIK

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Birçok envanter sisteminde belirli bir süre sonra bozulan ürünler vardır. Bu ürünlerin bozulması durumunda, stok seviyesi sıfıra iner ve bu durum maliyetlere yol açar. Bu nedenle, bu tür envanterlerin yönetilmesi için akıllıca bir kontrol politikası geliştirilmesi önemlidir. Ürünlerin ömrü artık envanter seviyesini etkilediğinden, sadece stok seviyesine dayalı geleneksel envanter politikalarının uygulanması bazı eksikliklere neden olmaktadır. Geleneksel envanter politikaları, ürünlerin ömrü ile ilgili bilgileri içermez.

Öte yandan, bozulabilir ürünlerin en uygun politikasının, siparişlerin kalan teslim süreleri, eldeki envanter ve ürünlerin ömürleri ile ilgili eksiksiz bilgileri içeren bir politika olacağı bilinmektedir. Bu çalışmada bozulabilir bir ürün için, sürekli gözden geçirilen ve pozitif teslim süresi olan bir envanter modelinde hizmet seviyesi kısıtı altında ürünlerin kalan ömrüne ve mevcut envanter seviyesine dayalı bir sipariş verme politikası öneriyoruz. Talebin bir Poisson sürecini takip ettiğini, karşılanmamış talebin kaybolduğunu ve yeni bir sipariş geldiğinde ürünün bozulma süresinin stoklara katıldıklarında başladığını kabul ediyoruz. Çalışmamızda bir ve birden fazla bekleyen sipari, varsayımıyla hareket ederek, ve gömülü bir Markov zinciri sürecini kullanarak etkin yaşam sürrelerinden oluşan rassal dizinin sabit dağılımını elde etmek için analitik bir model sunuyoruz ve sistemin işletme özelliklerini hesaplıyoruz. Amaç fonksiyonunu yenilenme ödül teoremi yaklaşımını baz alarak oluşturuyoruz. Sonuçlarımız, hibrid politikaların özellikle ömür boyu talep yeterince küçük ve birim maliyeti yüksek olduğunda stok seviyesini ve kalan ömür boyu temelli politikaları sürekli olarak geride bıraktığını ortaya koymaktadır. Bu iki politika sınıfı arasındaki baskınlık ilişkilerinin, özel parametre ayarlarına bağlı olduğu görülmektedir. Özellikle, ürünlerin ömrü yeterince uzun olduğunda, stok seviyesine dayalı politika çok iyi bir performans sergiliyor. Son olarak, çalışmamızda hedef fonksiyonun yapısı ve hizmet seviyesi kısıtlaması göz önünde bulundurularak elde edilen bir sezgisel algoritma ile optimal çözümü bulmak için metodolojimiz sunulmakta ve

anahtar giriş parametrelerinin etkisini değerlendirmek için bir duyarlılık analizi yapılmaktadır.



Anahtar sözcükler: Bozulabilir envanter, Envanter politikası, Etkin ömür, Gömülü Markov zinciri, Yenileme teoremi.

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Chapter 1

Introduction

More than half of the U.S. retail grocery industry consists of perishable products where the damages and spoilages due to perishables loss are up to 15% [1]. Perishable items held in grocery stores comprise up to 55% of the market sales and more than 60% of the profit (Wholefoods, 2013). USDA (U.S. Department of Agriculture) estimates that warehouses lose about \$15 billion per year because of unsold fruits and vegetables. The most distasteful shortcoming in grocery industries is the excess inventory leading to perishables (SN news, 2017). A disastrous perishability is encountered in the food industry. FMI (Food Marketing Institute) reports that there exists up to 71% uneaten food in retailers and restaurants, 25% of which are donated for human consumption and the remaining are counted as a food waste. On the other hand, 40% of food banks demand an emergency food assistance from other parties in food industries implying the shortages in their inventories. Hence, the management of goods with short lifetimes is increasingly becoming a challenging issue in industries where effective ways are lacking for controlling stock levels and balancing between the excess inventory and shortage. Only by a reduction of 15% in food waste, we would be able to feed more than 25 million Americans every year (NRDC news). In the U.S, to prepare any kind of food for a humans consumption, half of the land, 10% of the available energy, and 80% of the fresh water are consumed. Besides, in landfills sector, the second

biggest component is food waste; and landfills are announced to be a large portion of methane emissions in the world which is considered as a major factor in global warming (FDA news). Hence, the waste reduction is now a national goal as growing population will affect the existing natural resources and environmental issues in the near future.

In addition to the grocery and food industries, many other inventory systems hold products that perish at a specific time, after which they become unusable and a burden for the warehouse. Some examples of these items are medical products, fresh fruit, meat, flower, fashion wear, and beverages among which meat, poultry, fish, and fresh products having short lifetime consist of 49.51% of perishables sales share (Statista 2017). Perishability brings also vital risks beyond the operational costs. In the health care industry, for example, shortage of blood products or perishable medication may even result in mortality. As the customer demands for fresh cuisine like vegetables and fruits grow, managing of perishable items becomes an inevitable challenge for the inventory system. In order to comply with this requirement, several inventory systems in the perishable sector are oriented towards an ingenious control policy, especially in the cases with short-lifetime items. In this regard, some companies improve their Enterprise Requirement Planning (ERP) system to track the perishable item information more precisely. As the analysis of perishable inventories is technically challenging in most cases, policies developed for non-perishable products may be used in practice. However, when the perishability aspect is ignored, significant biases may be incurred. In this research, we attempt to find an appropriate policy for perishable items, in which the lifetime of items along with the stock level is taken into account. Although stock level control policies for perishables have been addressed since about fifty years ago, there is still need for new efforts to derive effective policies due to the practical and technical challenges that exist.

In this research, we propose a control policy for perishables that take into account both the stock level and the remaining lifetime of the items. As discussed in detail below, there are other works in the literature that consider both of these aspects. However, they either make simplifying assumptions about perishability or provide merely simulation results. In this work, we provide an effective policy

that dominates the existing ones under similar assumptions and also provide an exact analysis to minimize the expected operating cost. In particular, we consider a single item single location inventory problem in a continuous setting where items in a batch have a fixed lifetime, a positive, constant lead time and the unsatisfied demand is lost. Perishability with positive lead times and lost sales assumption together, make the analysis of the system quite challenging as discussed below. Under the proposed (Q, r, ξ) policy which is based on both stock level and remaining lifetime of the items, we provide an exact analysis built on renewal theory and provide exact expressions for the operating characteristics. Our numerical results indicate that the proposed policy dominates the well-known (Q, r) control policy in significant amounts under certain parameter settings. One of the technically challenging issues in our work is the incorporation of the remaining lifetime distribution at a particular instance to the calculation of the objective function. We provide the exact expression for the objective function under the service level constraint. We next review the related literature.

Chapter 2

Literature Review

In the past few decades, researchers proposed various policies for perishable items based on different characteristics of the lifetime, demand distributions, shortage type, lead time, and time horizon of the policy. We categorize the works on perishable inventory systems into several classes: (i) periodic or continuous review policy, (ii) constant or nonconstant lifetime, (iii) random or deterministic demand, (iv) lost sales or backordering in the case of stock-out, (v) zero, deterministic or non-deterministic lead time, (vi) finite or infinite planning horizon. In Table 2.1, we provide the 34 main studies that have offered a control policy for perishable inventories along with the categories stated above. We consider 34 main papers in the literature among which, 22 of them are continuous, and 12 of them are periodic review policies, in 22 cases, lifetime is assumed to be constant, and in 12 cases nonconstant, in all cases, demand is considered as a random variable, 16 of them are based on lost sales, 14 cases with backordering, and in 4 cases we have both, lead time in 11 cases is zero, in 16 cases is deterministic, in 5 cases is exponential, and in 2 cases has a general distribution, planning horizon is considered finite in 5 cases, infinite in 28 cases, and in 1 case, both. Hence, the constant lifetime and fixed lead time assumptions are not widely used in academic literature. [2] claimed that the problems with the fixed lifetime and continuous review policies were extremely difficult. Specifically, if we add a fixed positive lead time to the model, it makes the analysis of the problem

much harder. Later, [3] repeated their claim and mentioned that it is unlikely to find an optimal policy for controlling the perishable items, in which both fixed lead time and lifetime are considered. Afterwards, based on this consideration, researchers started to propose some heuristic policies for perishables with fixed lead time and lifetime. In the following, we present the main studies that are related to perishable inventories.

Table 2.1: The summary of literature review

Paper/Property	Policy	Lifetime	Demand Dist	Shortage	Lead time	Planning horizon
[4]	Periodic $(S-1, S)$	Constant	Continuous	Lost sales	0	Finite/infinite
[5]	Periodic $(S-1, S)$	Constant	Continuous	Backorder	0	Finite
[6]	Periodic $(S-1, S)$	Constant	Continuous	Backorder	0	Finite
[7]	Periodic (S, r)	Constant	Continuous	Backorder	0	Finite
[8]	Continuous (S, r)	Constant	Poisson	Lost sales/Backorder	0	Infinite
[3]	Continuous $(S-1, S)$	Constant	Poisson	Lost sales	Deterministic	Infinite
[9]	Periodic $(S-1, S)$	Constant	Continuous	Lost sales	0	Infinite
[10]	Continuous (S, r)	Exponential	Poisson	Lost sales	Exponential	Infinite
[11]	Periodic (T, S)	Constant	General	Backorder	Deterministic	Infinite
[12]	Continuous $(S-1, S)$	Exponential	Poisson	Lost sales	General	Infinite
[13]	Continuous (Q, r)	Constant	General	Lost sales/Backorder	Deterministic	Infinite
[14]	Continuous $(S-1, S)$	Exponential	Poisson	Lost sales/Backorder	Exponential	Infinite
[15]	Periodic $(S-1, S)$	Constant	Continuous	Lost sales	Deterministic	Finite
[16]	Continuous (S, r)	Constant	Batch (Geometric)	Backorder	0	Infinite
[17]	Continuous (S, r)	Exponential	Renewal	Backorder	0	Infinite
[18]	Continuous (S, r)	Exponential	Poisson	Backorder	Exponential	Infinite
[19]	Continuous (S, r)	Constant	Renewal	Backorder	0	Infinite
[20]	Continuous $(S-1, S)$	Exponential	Poisson	Lost sales/Backorder	Exponential	Infinite
[21]	Continuous $(S-1, S)$	Exponential	Poisson	Lost sales	General	Infinite
[22]	Continuous (S, r)	Constant	Renewal	Backorder	Deterministic	Infinite
[23]	Continuous (Q, r, T)	Constant	Poisson	Lost sales	Deterministic	Infinite
[24]	Periodic $(S-1, S)$	Constant	Continuous	Lost sales	Deterministic	Finite
[25]	Continuous (S, r)	Exponential	Renewal	Lost sales	Exponential	Infinite
[26]	Continuous (S, r)	General	Renewal	Backorder	0	Infinite
[27]	Continuous (Q, r)	Constant	Poisson	Lost sales	Deterministic	Infinite
[28]	Periodic (nQ, r)	Constant	Gamma	Lost sales	Deterministic	Infinite
[29]	Continuous (S, r)	Exponential	Renewal	Back order	0	Infinite
[30]	Continuous $(S-1, S)$	Constant	Poisson	Back order	Deterministic	Infinite
[31]	Periodic (nQ, r)	Constant	Discrete	Lost sales	Deterministic	Infinite
[32]	Continuous (Q, r)	Constant	General	Backorder	Deterministic	Infinite
[33]	Periodic (T, S)	Exponential	Poisson	Lost sales	Deterministic	Infinite
[34]	Periodic (Q, r, T)	Erlang	Poisson	Lost sales	Deterministic	Infinite
[35]	Continuous (Q, r)	Constant	Continuous	Lost sales	Deterministic	Infinite
[36]	Continuous (Q, r, T)	Constant	Poisson	Lost sales	Deterministic	Infinite

In addition to constant lifetime, the lifetime framework of perishable items are divided into two main categories, the ones which gradually decay as time elapses such as fruits, vegetables, fresh fishes, and flowers; and the ones having finite lifetime, which themselves are classified into two subcategories of fixed and random finite lifetime such as drugs and high-tech industries, respectively. Additionally, there exist different decay frameworks for perishable items. The items with an expiry date that deteriorate over time and the ones which become obsolete after a fixed time, like calendars, yearbooks, and obsolete maps ([37], [38]). Most of researches in this area are studied under deterministic demand.

In the mid-70s and 80s, [39] and independently [40] introduced the first research dealing with perishable items with a lifetime of two periods, $m = 2$. Afterwards, the model was developed for the case with a lifetime of more than two periods, $m > 2$, by Brant Fries [4]. He considered the perishable items with a lifetime of exactly m periods and developed a periodic control policy, namely $(S - 1, S)$ *policy* in which at the beginning of any period, inventory position is raised to S . Here, he assumed that the demand during each period has a general continuous distribution. Since the planning horizon is assumed to be finite, a dynamic programming-base approach is proposed to find an optimal ordering policy for different lifetime length ($m = 1$, $m = 2$, and $m > 2$). Then, this model is extended to the finite horizon and positive lead time by [5]. Having a lifetime of exactly m periods, he established the nature of the optimal ordering policy for perishable items and solved the problem using a multidimensional dynamic programming approach which requires the age of items tracking at the beginning of each period. However, tracking the age of items in the stock and having numerous periods makes the problem computationally very complicated. He also suggested a periodic review policy in which the lifetime of a single item is exactly m periods and the period length is assumed to be arbitrary but fixed. Orders are placed just at the beginning of each period with zero lead time. He presented a dynamic programming for the proposed model; however, a class of heuristics is also proposed by [9]. Afterwards, it was shown that in the cases with a lifetime of more than two periods the problem becomes hard. In this regard, [6] approximated the models with a fixed lifetime of m periods by bounding the expected perishing cost function. Then, [15] dealt with this policy for a single item with a fixed lifetime of two periods and positive lead time up to four periods. Then, [24] studied the same model with one period lead time under different demand distributions and investigated the effect of unit holding cost, ordering, perishing, and shortage cost on the order quantity.

[3] proposed an analysis for the perishable items with fixed lead time under *continuous review* $(S - 1, S)$ *policy*. In the proposed policy, known as lot-for-lot policy, an order is placed immediately whenever a demand occurs. In this policy, the size of the order and the demand size are considered to be the same.

[30] presented the same policy under the backorder assumption for the perishable items. They showed that finding the exact optimal solution is not so easy when backorder is allowed. Afterwards, [12] worked on this policy under a simple assumption of exponential lifetime, and nonindependent and non-Markovian lead times. And a simpler model with exponential lead time was presented by [14]. [20] analyzed a *modified continuous review* $(S - 1, S)$ *policy* in which an order is placed when the inventory level depletes by one unit due to the demand, not perishability. They also worked on this policy with a general distribution for the lead time in [21].

The effect of fixed ordering cost on the structure of optimal *periodic review* (S, r) *policy* is investigated by Steven Nahmias [7]. He considered both the optimal and approximate ordering policy for the perishable items with fixed lifetime and zero lead time. In this policy, the inventory position is monitored periodically with arbitrary equal period length. When the inventory position at the beginning of the period is less than or equal to r , an order is placed so that the inventory position increases to S .

Demand with Poisson distribution under *continuous review* (S, r) *policy* in the infinite planning horizon, zero lead time and fixed lifetime for each item was investigated by [8]. Also, [12] dealt with a simpler model with exponential lifetime and general lead time under this policy which was extended to the case with exponential lead time by [14]. This policy is extended to the case with both exponential lifetime and lead time by [10] and [18]; and to the case with the general renewal demand process by [17]. In addition, this policy is extended by [16] for the perishable items with geometric inter-demand times and batch demands. [19] analyzed this policy to provide a closed-form solution for the steady state probability distribution of the stock level by using a Markov renewal approach. A positive lead time was added to this policy and a heuristic algorithm was proposed by [22]. A new perishing process called discrete point perishability is introduced by [25] under this policy as well. This policy is extended by [26] to the perishable inventory system with general lifetime distribution. [29] analyzed this policy which was based on the Markovian renewal demand process with the general distribution for inter-demand time.

A new order-up-to-level and review interval policy, namely *periodic review* (T, S) *policy* is offered in [11] in which the inventory position is raised to S at the beginning of any period with length T . In this work, the lifetime and lead time are assumed to be fixed. An approximation for the number of items perished and some extended bounds for the total expected cost are presented. The simple version of this model under the exponential lifetime assumption was developed by [33] where he analyzed the lifetime variability leading to a considerable improvement in the total optimal cost.

Continuous review (Q, r) *policy* was introduced by [13] in which the expected total cost and operating characteristics of the system are approximated. This policy works as follows: an order of size Q is placed whenever the inventory position drops to r . This policy does not provide a significant improvement compared with the continuous (S, r) policy. Berk and Gürlü in [27] dealt with the perishable inventory problem under continuous (Q, r) policy with positive lead time and fixed lifetime by introducing the new concept pertaining to the lifetime distribution modeled by the embedded Markov process approach. This policy was extended by [35] to a model with continuous demand distribution, constant lifetime and constant lead time.

Periodic review (nQ, r) *policy* was proposed by [28] in which a number of n batches with size Q are placed when the inventory position at a periodic review moment reaches the reorder level r . Later, [31] investigated this policy by introducing an approximation for all costs under discrete demand. Another policy considered in the literature is *periodic* (Q, r, T) *policy* introduced by [34] with period length T , at which an order of size Q is placed whenever the inventory position crosses r .

The age of items has not been engaged in the policies until [23] introduced an age-based policy for the perishable items, namely *continuous review* (Q, r, T) *policy*. In this policy, an order of size Q is placed whenever T units of time elapses or the inventory level reaches r , whichever occurs first. They developed an age-based policy with at most one order outstanding restriction ($r < Q$), and with a special aging process in which the lifetime of a new order starts when it

is unpacked. That is, whenever a new embedded cycle starts, the new batch has a fixed lifetime.

In [36], a *continuous review* (Q, r, T) policy is proposed in which the lifetime of items is taken into account. In this policy, an order of size Q is placed whenever the inventory level hits r or T units of time is elapsed since the time we receive the batch, whichever occurs first. One of the shortcomings in this policy is that it is not able to analyze the inventory systems with more than one outstanding order. They applied a simulation-based optimization approach and found that their policy dominates the existing policies.

In recent works which are less related to our study, [41] developed an approximation algorithm for perishables in periodic review systems with general product lifetime and nonstationary and correlated demand. [42] dealt with perishable items with a long production lead time and developed a single period stochastic programming model for the system. [43] considered a lot sizing problem with the perishable items and dynamic order quantity. The model is conducted under production capacity, which limits the amount of production in each period. Interested readers are referred to [44] for different categories of the perishable items including the deterministic and stochastic demand, dependent deterioration rate, time-varying demand, stock-dependent demand, and price-dependent demand. The studies stated above ([41], [42], [43], and [44]) take perishable items into account, and they are related to our work only in terms of perishability.

Since tracking the age of perishable items for deriving an intelligent control policy is crucial, continuous review policies seem to be more efficient than the periodic ones. Therefore, with perishable items, the impact of elapsing time is more significant and a continuous review setting provides a more efficient control policy. The time dimension becomes crucial not only in the review structure, but also in keeping track of the remaining lifetime of the items. We believe that this remaining time should be incorporated into the control policy in order to have an effective inventory replenishment policy. On the other hand, the inventory system places orders more frequently because the items are perishable and there are some instances at which the inventory level falls down. Hence, it is more

likely that the system includes multiple outstanding orders, especially when a high service level is required. We analyze our inventory system under multiple outstanding orders assumption (i.e., we relax the $r < Q$ restriction). It makes the model technically more complicated because we need to track the age of multiple items over time. Therefore, in this research, we consider a lifetime-based policy under a continuous review for perishable items with multiple outstanding orders assumption and compare it with existing policies to show the importance of tracking the age of items.

We also assume that the lifetime of items are fixed, but unlike [23] we assume that when the batches join inventories their aging process begins. It implies that at the beginning of any embedded cycle, the lifetime of batches creates a sequence of random variables, as discussed in [27]. Hence, in addition to base stock level for reordering, we need to incorporate the lifetime of items in the control policy framework in the sense that the remaining lifetime of items at the beginning of an embedded cycle is considered as a threshold for reordering as well. Obviously, the lifetime of the oldest batch affects the embedded cycle length and other operating characteristics of the system. Therefore, we can introduce a new policy which works under the following framework. An order of size Q is placed whenever the inventory position crosses a base stock threshold by demand, or the remaining lifetime of the oldest batch crosses its prespecified threshold, whichever occurs first. The order size Q , the inventory position threshold r , and the remaining lifetime threshold ξ are decision variables.

Our proposed policy is similar to Tekin et al. work [23], in the sense that in both of them the aging process is taken into account. In [23] the time is considered in the control policy framework, while in our policy the remaining lifetime of the oldest batch is taken into account. Moreover, the aging process considered in our research is similar to that of Berk and Gürler [27]. They considered the continuous (Q, r) policy, while we associate the remaining lifetime of items into their policy for reordering and propose a new policy. The study of Lowalekar and Ravichandran [36] is also similar to our study in the sense that they take the age of items into account for developing a control policy for perishable items. The difference of their policy with ours is that they count the time for placing

an order since the instance a new batch joins inventories, but we count the time from the instance a current batch is consumed or perished completely. They applied a simulation study to evaluate their proposed policy under one order outstanding restriction. Although the policy proposed in this study is similar to ours, it is not able to deal with the inventory systems with more than one outstanding order. The three studies [23], [27], and [36] are constructed based on one order outstanding restriction. Our result reveals that the policies proposed in these studies are not able to provide any feasible solution with high service level setting if one order outstanding restriction is imposed, while by relaxing this restriction a feasible solution with multiple outstanding orders is attainable. One of the advantages of our proposed policy is that it is able to provide a feasible solution with less number of outstanding orders even in cases with high service level.

The rest of this dissertation is organized as follows. Chapter 3 provides the basic assumptions of our model and introduces the proposed policy. In this chapter, we introduce (Q, r, ξ) policy and its properties. In chapter 4, we provide the concept of the effective lifetime of items and the effective lifetime stationary distribution and its ergodicity properties are derived. The expressions of operating characteristics and the objective function structure for the inventory system are developed in chapter 5. Chapter 6 is reserved to deal with a time-based control policy, namely (Q, r, T) policy, and the special policies as sub-optimal policies for perishable items. We present the effective lifetime stationary distribution for these policies and discuss some properties of them. In chapter 7, we discuss the structure of operating characteristics by which we present a solution approach for the proposed model. Our observations from numerical study are presented in chapter 8. Also, we compare different proposed policies under different parameter settings. Finally, we conclude in chapter 9 with some open issues for future works.

Chapter 3

Model Setup and Proposed Policy

3.1 Assumptions and Preliminaries

In this section, we present the basic assumptions and preliminaries for our model. We discuss the motivation behind our proposed policy and also its properties. We explain how the inventory system works under different control policies. Different realizations of the stochastic process under consideration are investigated and the operating characteristics, together with the objective function are derived.

A single item single location inventory setting in infinite time horizon is considered with perishable products having a fixed lifetime of τ . The lifetime of items starts whenever they join the inventories. Replenishment is done in batches with size Q and procurement lead time $L > 0$ is assumed to be fixed, where $\tau > L$ is assumed. Lifetime of all items in a batch is identical. Items are issued from the stock according to FIFO (first-in, first-out) policy. The ordering cost is K per order, a perished product incurs a cost of p per unit, and one unit of product held in the system incurs a holding cost of h per time unit. Our objective is

to minimize the expected total cost per unit time composed of average ordering, holding and perishing costs subject to a service level constraint α . The service level constraint represents the fraction of lost demands. We assume that the external demands follow a Poisson process with rate λ .

3.2 The (Q, r, ξ) Policy

Considering the assumptions defined above, we introduce the following remaining lifetime and stock level based policy, namely *the (Q, r, ξ) policy* for Q, r, ξ all non-negative. Let $IP(t)$ be the inventory position (on-hand plus on-order) and $IL(t)$ be the on-hand inventory at time $t \geq 0$.

Policy: *An order of size Q is placed whenever the inventory position hits r by demand, or the remaining lifetime crosses ξ , whichever occurs first.*

The (Q, r, ξ) policy considers the time dimension in placing a replenishment order and keeps track of the remaining lifetime of items in the stock.

In the policy under consideration, it is assumed that r and Q can take any non-negative integer value. Our analysis regarding the age of batches in the (Q, r, ξ) policy reveals that in order to characterize the system behavior, we need to trace the age of m batches over the time simultaneously, where $m = \lceil \frac{r}{Q} \rceil$ and $\lceil x \rceil$ denotes the smallest integer strictly greater than x . We investigate the system characteristics by considering the case $r > (m - 1)Q$ and the special region of search space $r = (m - 1)Q$, i.e., when r is an integer multiple of Q . In this special region, the system behaves differently from the case $r > (m - 1)Q$ and some realizations of the system would not occur. Therefore, we deal with the proposed policy with two cases $r > (m - 1)Q$ and $r = (m - 1)Q$, separately. For instance, when $0 \leq r < Q$, then $m = 1$ and the special region is $r = 0$; when $Q \leq r < 2Q$, then $m = 2$ and the special region is $r = Q$, and so on. To illustrate our proposed policy, we depict a sample path of the inventory position and the inventory level for $Q = 5$, $r = 3$, and arbitrary fixed ξ , where $m = 1$ in Figure 3.1, and a sample

path for $Q = 4$, $r = 6$, and arbitrary fixed ξ , where $m = 2$ in Figure 3.2, where the dashed, solid and bold lines represent the inventory position ($IP(t)$) at time t , inventory level ($IL(t)$) at time t , and amount of perished items, respectively. Without loss of generality, for illustration we assume that the process starts with one outstanding order. In this section, we illustrate the system characteristics by considering Figures 3.1 and 3.2. Since we use these figures throughout this chapter, there are some terms which are defined later. For now, in these figures one can follow how the inventory position and inventory level are related to each other, control policy mechanism, and how to count the number of outstanding orders over time. Number of outstanding orders is counted by following the lead times. Any batch which is ordered but has not yet joined the inventories is considered as an outstanding order.

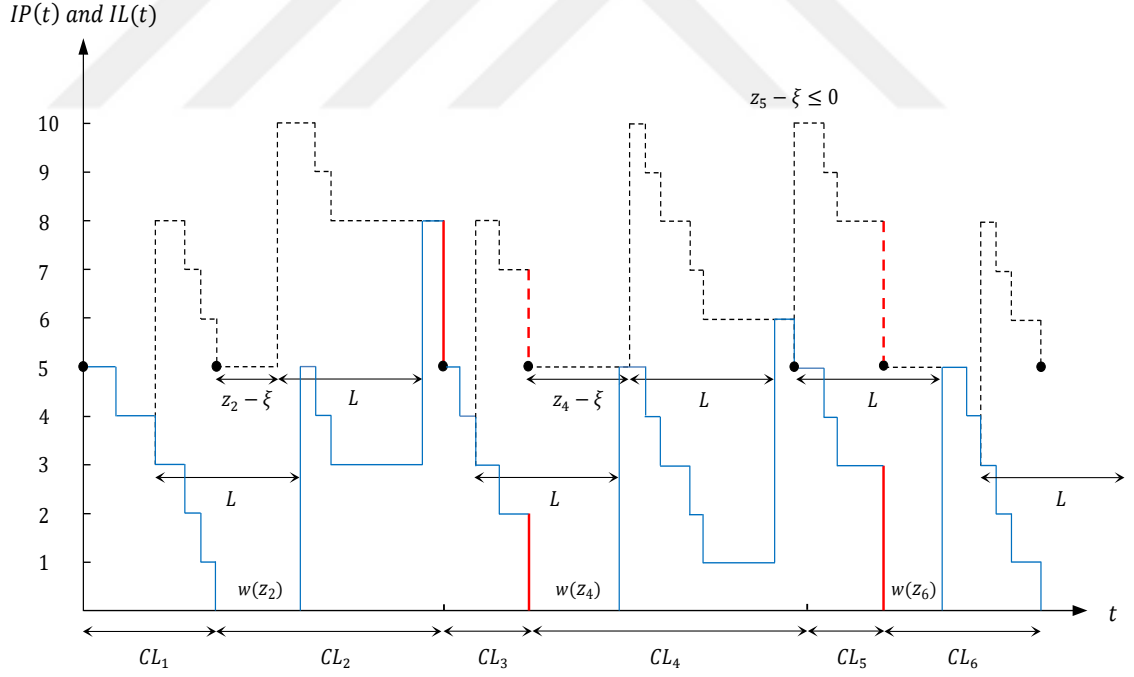


Figure 3.1: A sample path of the inventory position and inventory level for $Q = 5$ and $r = 3$ ($m = 1$) in the (Q, r, ξ) policy

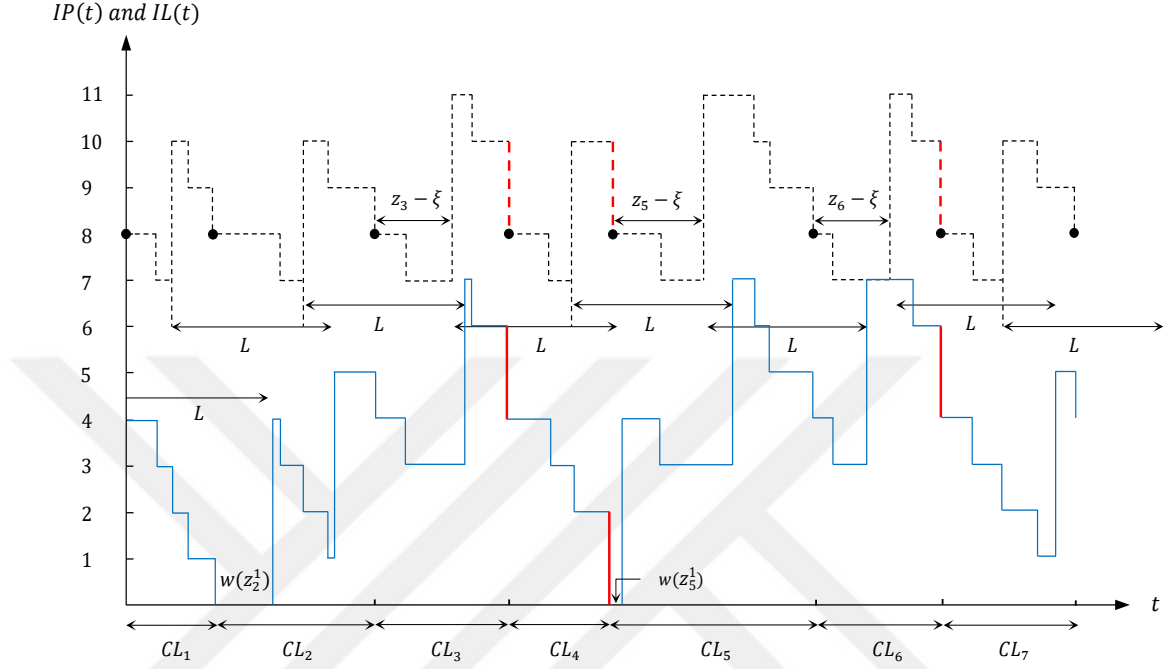


Figure 3.2: A sample path of the inventory position and inventory level for $Q = 4$ and $r = 6$ ($m = 2$) in the (Q, r, ξ) policy

We have the following properties of the system under the proposed policy:

- (i) The inventory position takes values between $r + 1$ and $(m + 1)Q$.
- (ii) There are some embedded cycles at the beginning of which we order immediately; this happens when the remaining lifetime of the oldest batch at the beginning of that embedded cycle is less than ξ .
- (iii) Between two consecutive replenishments, there is always an instance at which the inventory position hits mQ .
- (iv) There may be at most $m + 1$ outstanding orders at any given time in the inventory system. One can follow these observations from Figures 3.1 and 3.2. Hence, we can conclude that the inventory system can be fully characterized by the inventory position and the remaining lifetime of batches at any point of time.

In the inventory system under consideration, the number of outstanding orders, inventory level (or inventory position), the age of items, and the remaining delivery time of outstanding orders are the variables changing over time and can be defined as states of the system. Since analyzing and deriving the operating

characteristics of the system are easily carried out by taking the inventory position and age of items into account, we define the state of the system at time t by considering the inventory position and the remaining lifetime of items at time t .

We base our analysis on *embedded cycles*. An embedded cycle is defined as the time interval between two consecutive instances at which the inventory position hits mQ . By doing so, we only need to trace the age of items at these instances, where the inventory position takes a constant value mQ . In Figure 3.1, the time interval between two consecutive instances at which the inventory position hits $Q = 5$, and in Figure 3.2, the time interval between two consecutive instances at which the inventory position hits $2Q = 8$ represent the embedded cycles lengths. These consecutive instances are shown by bold circles ($\bullet$) in Figures 3.1 and 3.2. Note that the value of m denotes the number of batches whose lifetimes are needed to be traced at the beginning of any embedded cycle. For example, for $m = 1$, we need to trace the age of one batch and for $m = 2$, we need to trace the age of two batches at the beginning of any embedded cycle and so on.

At the beginning of an embedded cycle, the oldest batch among m batches is issued to be consumed during the embedded cycle. We refer to this batch as the *current batch*. Let CL_n denote the length of the n^{th} embedded cycle. An embedded cycle ends when the current batch with size Q is depleted either by demand or perishing. For example, in Figure 3.1, cycles 1, 4, and 6 end by depletion and cycles 2, 3, and 5 end by perishing. In some cases, at the beginning of an embedded cycle, the current batch is still an outstanding order and after its lead time completion it joins the stocks. If this is the case, the current batch joins the inventories during the embedded cycle and upon its arrival it will be a fresh batch. We refer to this batch as the *on-order batch*. In such cases, we have a stock-out period with a length of the remaining lead time at the beginning of the embedded cycle. During the stock-out period any demand is lost. Therefore, stock-out periods occur at the beginning of an embedded cycle, if there is any. For instance, in Figure 3.1, we have a stock-out period at the beginning of cycles 2, 4, and 6; and in Figure 3.2, at the beginning of cycles 2 and 5. At the beginning of any cycle, we refer to the m^{th} outstanding batch as the *youngest batch* which is the batch recently ordered.

We define the *effective lifetime* of a batch at the beginning of an embedded cycle as $\tau + L$ minus the time elapsed since the order time of that particular batch. According to the embedded cycle definition, the effective lifetime of a batch at the beginning of an embedded cycle is the remaining lifetime of the batch if that batch is already in stock and is the remaining lead time plus τ if that batch is still an on-order batch. Based on the embedded cycle and effective lifetime definitions, one can observe that at the beginning of any embedded cycle the inventory position is mQ but the remaining lifetime of the items is still a random variable. Hence, the system under consideration can be fully characterized by an m -dimensional array of the effective lifetimes of m batches at the beginning of the embedded cycle. Effective lifetime of batches over embedded cycle beginnings yields a sequence of random variables referred as the sequence of *effective lifetime vector*. Let Z_n denote the effective lifetime of the current batch at the beginning of embedded cycle n for case $m = 1$. Then, $\{Z_1, Z_2, Z_3, \dots\}$ yields a sequence of random variables. Similarly, let Z_n^i denote the effective lifetime of batch i , $i = 1, \dots, m$ at the beginning of embedded cycle n for case $m > 1$. Then, $\{(Z_1^1, Z_1^2, \dots, Z_1^m), (Z_2^1, Z_2^2, \dots, Z_2^m), (Z_3^1, Z_3^2, \dots, Z_3^m), \dots\}$ yield a sequence of vector of random variables; where Z_n^1 and Z_n^m are the current and the youngest batches at the beginning of the embedded cycle n , respectively. In the following section, we show that this sequence of random variables has the Markovian property by which we can derive the distribution function of the effective lifetimes. For the case $m = 1$, if the effective lifetime of the current batch at the beginning of embedded cycle n is $Z_n = z_n$, a new order is placed whenever the inventory position hits r or $z_n - \xi$ units of time has elapsed since the beginning of the embedded cycle (remaining lifetime crosses ξ), whichever occurs first. For the case $m > 1$, if the effective lifetime of the current batch at the beginning of embedded cycle n is $Z_n^1 = z_n^1$, a new order is placed whenever the inventory position hits r or $z_n^1 - \xi$ units of time has elapsed since the beginning of the embedded cycle (remaining lifetime crosses ξ), whichever occurs first. Note that the policy definition for the cases $m = 1$ and $m > 1$ is identical, the only difference is the effective lifetime distributions. In the case $m = 1$, we only need to track the remaining lifetime of one batch while in the case $m > 1$ we need to track the remaining lifetime of the current batch (oldest batch) among m batches for

placing a new order. The effective lifetime distributions of the current batch in the cases $m = 1$ and $m > 1$ are not identical. For instance, in Figure 3.1 (where $m = 1$), in cycles 1, 3, and 6 reordering is triggered by crossing reorder point r and in cycles 2, 4, and 5 is triggered by crossing the remaining lifetime ξ . Note that when $Z_n - \xi \leq 0$, we should order immediately. For example, $z_5 - \xi \leq 0$, thus we need to reorder immediately at the beginning of the embedded cycle 5 because the remaining lifetime crosses ξ . One can follow the reordering points in Figure 3.2 analogous to Figure 3.1.

Chapter 4

Effective Lifetime Distribution

In this section, we present the Markovian property of the effective lifetime vector. Then, by proving the ergodicity of the process we derive the effective lifetime distribution. Let $\{T_n, n \geq 1\}$ be the sequence of time epochs at which the inventory position hits mQ for the n^{th} time starting with $T_1 = 0$. Assuming that the inventory system starts with m batches in the system at time 0, $IP(T_n) = mQ$ for all $n \geq 1$. The time interval between T_{n+1} and T_n represents the n^{th} embedded cycle length for $n \geq 1$. For each $i \geq 1$, let $\{X_i\}$ be the random variable denoting the arrival time of i^{th} demand since the beginning of an embedded cycle. Since demand is generated by a Poisson process with rate λ , the arrival time for i^{th} demand which is measured from the beginning of the embedded cycles has an Erlang distribution with parameters λ and i . For an Erlang i variable, its distribution function (d.f.) is denoted by $F_i(\cdot)$ and complementary d.f. by $\bar{F}_i(\cdot)$. Furthermore, let $N(t)$ be the counting process of the arrivals in $(0, t]$. Then, $\{T_n, n \geq 1\}$ is a sequence of stopping times for $(N(t))_{t \geq 0}$.

4.1 Effective Lifetime Distribution for Case $m = 1$

In the case $m = 1$, we need to trace the lifetime of one batch at the beginning of any embedded cycle. Let $\{Z_n, n \geq 1\}$ be the sequence of effective lifetimes of the batch in the system at time T_n , and z_n be a particular realization of Z_n , where $0 \leq z_n \leq \tau + L$. It should be noted that when $z_n > \tau$, the current batch is an outstanding batch at the beginning of the embedded cycle. That particular batch will join inventory after $z_n - \tau$ units of time and upon its arrival it is a fresh batch with effective lifetime of τ (cycles 2, 4, and 6 in Figure 3.1). When $z_n \leq \tau$, it implies that at the beginning of the embedded cycle, the current batch is in stock and under consumption (i.e., it is not a fresh batch with effective lifetime less than τ). In this case, we do not have any lost sales at the beginning of the embedded cycle (cycles 1, 3, and 5 in Figure 3.1). The system loses the demands that arrive, if any, during the time segment $w(Z_n)$ which is called *stock-out period*, where $w(Z_n) = \max\{0, Z_n - \tau\}$. That is, in embedded cycle n , $w(Z_n)$ is a period of time over which stock-out occurs.

To derive the effective lifetime distribution of the batches, we need to show the Markovian property of $\{Z_n, n \geq 1\}$. To do so, we address different realizations of the system and the effective lifetime expression by which we can show the Markovian property. In Figures 4.1-4.4, we illustrate possible realizations and provide the corresponding quantities of interest. In these realizations, we assume that we are currently at the beginning of embedded cycle n . Thus, the n^{th} embedded cycle length and the effective lifetime of the current batch at the beginning of embedded cycle $(n+1)^{th}$ are provided for each realization. We refer to these realizations as Events 1 through 4. In all realizations, it is assumed that the lifetime of the current batch at the beginning of embedded cycle n is a given value z_n . We use U_n to denote a period of time measured from the beginning of embedded cycle n to the instance a new order is placed. A new order is placed whenever inventory position hits r or the remaining lifetime of the current batch crosses ξ , whichever occurs first. Thus, we can define $U_n = \min\{z_n - \xi, X_{Q-r}\}$,

where z_n is effective lifetime of the current batch at the beginning of embedded cycle n and X_{Q-r} is the time during which $Q - r$ items are consumed which is measured from the beginning of embedded cycle n . Also, we use CL_n to denote the length of embedded cycle n .

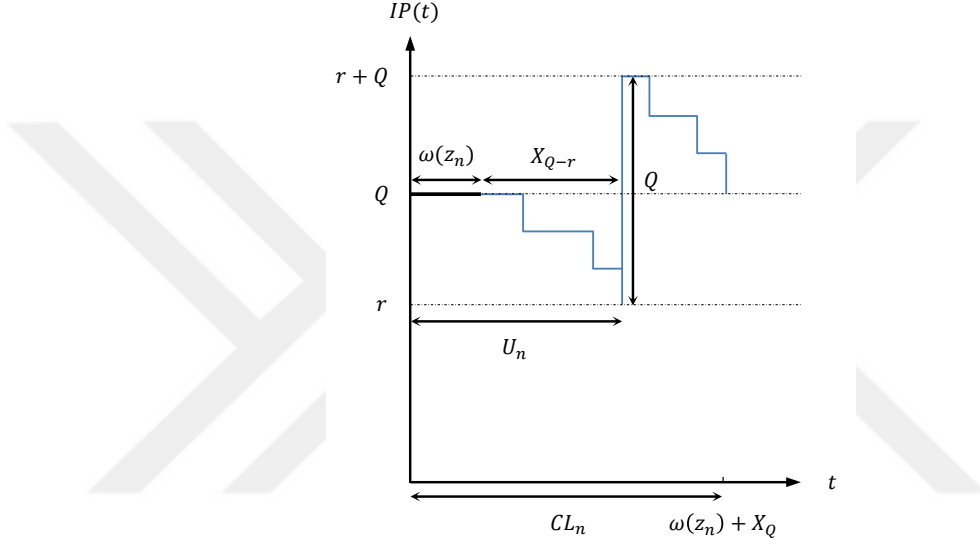


Figure 4.1: Event 1, $w(z_n) + X_{Q-r} < z_n - \xi$, $w(z_n) + X_Q < z_n$, $CL_n = w(z_n) + X_Q$, and $Z_{n+1} = \tau + L - [X_Q - X_{Q-r}]$

In Event 1, as shown in Figure 4.1, $Q - r$ demands have arrived before the effective lifetime crosses ξ (before $z_n - \xi$ units of time is elapsed); the remaining r units are also depleted by demand before perishing. This realization can be characterized by the events $w(z_n) + X_{Q-r} < z_n - \xi$ and $w(z_n) + X_Q < z_n$. In this realization, $U_n = w(z_n) + X_{Q-r}$ and $CL_n = w(z_n) + X_Q$.

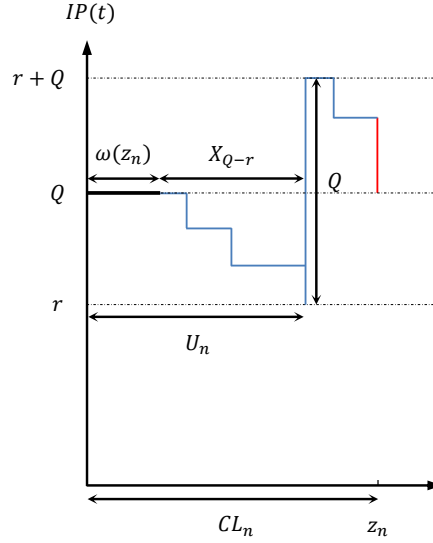


Figure 4.2: Event 2, $w(z_n) + X_{Q-r} < z_n - \xi$, $w(z_n) + X_Q > z_n$, $CL_n = z_n$, and $Z_{n+1} = \tau + L - [z_n - (w(z_n) + X_{Q-r})]$

In Event 2 (Figure 4.2), similar to Event 1, $Q - r$ demands have arrived before the effective lifetime crosses ξ but some of the remaining r units perish before complete consumption. This realization can be characterized by the events $w(z_n) + X_{Q-r} < z_n - \xi$ and $w(z_n) + X_Q > z_n$. In this realization, $U_n = w(z_n) + X_{Q-r}$ and $CL_n = z_n$.

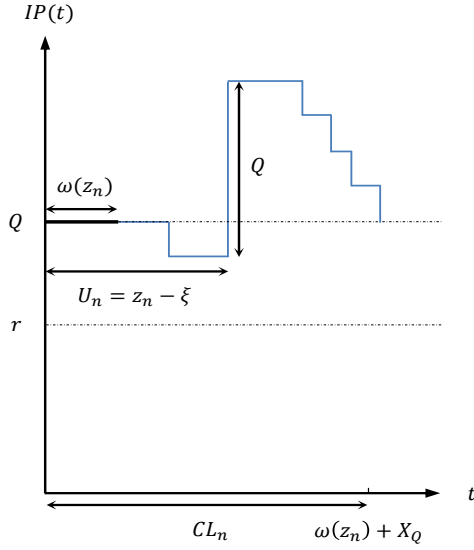


Figure 4.3: Event 3, $w(z_n) + X_{Q-r} > z_n - \xi$, $w(z_n) + X_Q < z_n$, $CL_n = w(z_n) + X_Q$, and $Z_{n+1} = \tau + L - [w(z_n) + X_Q - (z_n - \xi)]$

Event 3 (Figure 4.3) is similar to Event 1; the only difference is that the reorder point is at time $z_n - \xi$. That is, before $Q - r$ demand arrivals the remaining lifetime crosses ξ . This realization can be characterized by the events $w(z_n) + X_{Q-r} > z_n - \xi$ and $w(z_n) + X_Q < z_n$. In this realization, $U_n = z_n - \xi$ and $CL_n = w(z_n) + X_Q$.

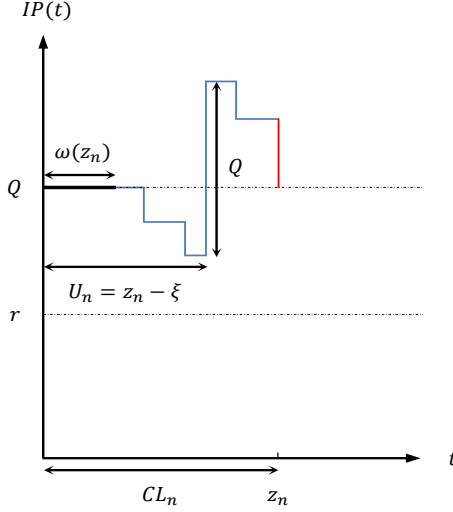


Figure 4.4: Event 4, $w(z_n) + X_{Q-r} > z_n - \xi$, $w(z_n) + X_Q > z_n$, $CL_n = z_n$, and $Z_{n+1} = \tau + L - \xi$

Event 4 (Figure 4.4) is similar to Event 2; the only difference is that the reorder point is at time $z_n - \xi$. That is, before $Q - r$ demand arrivals the remaining lifetime crosses ξ . This realization can be characterized by the events $w(z_n) + X_{Q-r} > z_n - \xi$ and $w(z_n) + X_Q > z_n$. In this realization, $U_n = z_n - \xi$ and $CL_n = z_n$. In Figure 3.1, cycles 1 and 6 represent Event 1; cycle 3 represents Event 2; cycle 4 represents Event 3; and cycles 2 and 5 represent Event 4.

We generalize the system realizations stated above as follows. Any of these realizations can be started by a stock out period of length $w(Z_n)$. The value of $w(Z_n)$ is either $Z_n - \tau$ when $Z_n > \tau$ or zero when $Z_n \leq \tau$. For illustration purpose, in all realizations depicted in Figures 4.1-4.4, we assume that the embedded cycle starts with a stock out period. Starting the embedded cycle n with Q items in the system, $w(Z_n)$ is the time during which system is out of stock (which is either positive or zero) and X_{Q-r} is the time at which the inventory position reduces from Q to r for the first time and $Z_n - \xi$ is the time at which the remaining lifetime of the current batch crosses ξ . Therefore, the reorder point in embedded cycle n , U_n , is expressed as $U_n = \min\{Z_n - \xi, w(Z_n) + X_{Q-r}\}$. Again, the value of U_n is measured from the beginning of embedded cycle n . Regarding

the embedded cycle, when $Z_n > \tau$, the embedded cycle starts with a stock out period with length $w(Z_n) = Z_n - \tau$ and it ends when Q items is either depleted by demand during X_Q units of time or perish before complete consumption at time τ , whichever occurs first. When $Z_n \leq \tau$, the embedded cycle starts without a stock out period ($w(Z_n) = 0$) and it ends when Q items is either depleted by demand during X_Q units of time or perish before complete consumption at time Z_n , whichever occurs first. Therefore, the length of embedded cycle n , CL_n , is expressed as $CL_n = w(Z_n) + \min\{\min\{Z_n, \tau\}, X_Q\} = w(Z_n) + \min\{Z_n, \tau, X_Q\}$.

Considering different events stated above, we can conclude that the $(n + 1)^{th}$ effective lifetime vector is expressed as

$$Z_{n+1} = \tau + L - (CL_n - U_n), \quad (4.1)$$

where $CL_n = w(Z_n) + \min\{Z_n, \tau, X_Q\}$ and $U_n = \min\{Z_n - \xi, w(Z_n) + X_{Q-r}\}$. One can follow that CL_n and U_n are function of Z_n ; and consequently, the effective lifetime of batch at the beginning of embedded cycle $n + 1$, Z_{n+1} , is a function of the effective lifetime of batch at the beginning of embedded cycle n , Z_n . According to (4.1), the effective lifetime vector Z_{n+1} at the beginning of the $(n + 1)^{th}$ embedded cycle is completely determined by Z_n and the Poisson demand arrival process after the stopping time T_n . Therefore, the embedded process $\{Z_n, n \geq 1\}$ has the Markovian property. Clearly, $CL_n - U_n$ can take a minimum value of zero and a maximum value of $\tau + L$. For any $n \geq 1$, $0 \leq Z_n \leq \tau + L$ and $0 \leq w(Z_n) \leq L$. For instance, consider cycle 2 in Figure 3.1 and suppose that the stock-out period $w(z_2)$ is equal to L ($z_2 = \tau + L$). If we reorder very soon (i.e., $z_2 - \xi$ gets very small value) and embedded cycle ends up by perishing, then Z_n approaches to zero. In any event in which the embedded cycle ends immediately after reordering (either by depletion or perishing), Z_n approaches to $\tau + L$. Suppose that we place a new order during the embedded cycle n at any time, and after reordering the embedded cycle n ends very shortly; then $CL_n - U_n \rightarrow 0$ and the effective lifetime of the youngest batch approaches to $\tau + L$ based on Equation (4.1).

Next, we consider the transition probabilities of the sequence $\{Z_1, Z_2, \dots, Z_m\}$

and find the limiting distribution of this sequence. In the remainder of this thesis we have the following notation. Let U be an arbitrary function and F be a d.f. Then, $\int U(x)dF(x) = \int_{\mathfrak{C}} U(x)f(x)dx + \sum_{x_i \in \mathcal{D}} U(x_i)[F(x_i) - F(x_i^-)]$, where $\mathfrak{C}$ is the set of points where F is continuous and differentiable, $\mathcal{D}$ is the set of points where F has jumps, and f is p.d.f of F . Also, $F(x^-) = \lim_{\epsilon \rightarrow 0} F(x - \epsilon)$. The above definition is generalized to higher dimensions if necessary. The state space of the remaining lifetimes is $S = (0, \tau + L]$.

We will consider the sequence $\{Z_n, n \geq 1\}$ as a stochastic process with Markovian property and will derive the limit distribution. To this end, we first consider the probability that the process at time $n + 1$ enters a particular set $\mathcal{S}$, given that at time n it was in state x , where $\mathcal{S} = (0, z]$ and $0 \leq z \leq \tau + L$. Hence, let $\mathbb{P}(\mathcal{S}|x) \equiv \mathbb{P}(Z_{n+1} \leq z|Z_n = x)$, which we will refer to as transition probability function. Then, we have the following result.

Theorem 1. (*Transition Probability Function of Z_n*).

$$\begin{aligned} \mathbb{P}(\mathcal{S}|x) = \mathbb{1}_{\{x \geq \tau + L - z + m_\xi\}} & \left[\int_{m_f - \xi}^{m_f} \bar{F}_r(\ell - t) dF_{Q-r}(t) + \bar{F}_{Q-r}(m_f) \right] \\ & + \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{m_f - (\tau + L - z), m_f - \xi\}), \end{aligned} \quad (4.2)$$

where $m_\xi = \max\{0, x - \xi\}$, $m_\tau = \max\{0, x - \tau\}$, $m_f = \min\{x, \tau\}$, $\ell = \tau + L - z + m_\xi - m_\tau$, and $\mathbb{1}_{\{\cdot\}}$ is the indicator function.

Proof. Using the Expression (4.1), we can rewrite $\mathbb{P}(\mathcal{S}|x)$ as follows:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(Z_{n+1} \leq z|Z_n = x) = \mathbb{P}(\tau + L - (CL_n - U_n) \leq z) \\ &= \mathbb{P}(CL_n - U_n \geq \tau + L - z). \end{aligned} \quad (4.3)$$

For simplicity, let's denote X_{Q-r} by t and X_r by u . We consider the realizations of the system when $\xi < \tau$ and $\xi \geq \tau$. Then, these realizations can be easily expressed by Regions A, B, and C as below:

A: $\max\{\xi, \tau\} < x \leq \tau + L$

- | | |
|--|--|
| A1. If $t > \tau$, | $CL_n = x$ and $U_n = x - \xi$. |
| A2. If $\tau - \min\{\xi, \tau\} < t < \tau$, | $CL_n = x - \tau + \min\{\tau, X_Q\}$ and $U_n = x - \xi$. |
| A3. If $t < \tau - \min\{\xi, \tau\}$, | $CL_n = x - \tau + \min\{\tau, X_Q\}$ and $U_n = x - \tau + t$. |

B: $\min\{\xi, \tau\} < x \leq \max\{\xi, \tau\}$

- | | |
|----------------------------|--|
| B1. If $t > m_f$, | $CL_n = x$ and $U_n = m_\xi$. |
| B2. If $m_\xi < t < m_f$, | $CL_n = x - \tau + \min\{m_f, X_Q\}$ and $U_n = m_\xi$. |
| B3. If $t < m_\xi$, | $CL_n = \min\{x, X_Q\}$ and $U_n = t$. |

C: $0 \leq x \leq \min\{\xi, \tau\}$

- | | |
|------------------|---|
| C1. If $t > x$, | $CL_n = x$ and $U_n = 0$. |
| C2. If $t < x$, | $CL_n = \min\{x, X_Q\}$ and $U_n = 0$. |

Using the Expression (4.3) and considering different realizations for different Regions A, B, and C expressed above, we can derive the transition probability function as follows:

A1.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_{Q-r} > \tau; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} > \tau; \xi \geq \tau + L - z) \\
&= \mathbb{1}_{\{z \geq \tau - \xi + L\}} \int_{\tau}^{\infty} dF_{Q-r}(t).
\end{aligned}$$

A2.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{Q-r} < \tau; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{Q-r} < \tau; \min\{\tau, X_Q\} \geq \tau + L - z + \tau - \xi) \\
&= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{Q-r} < \tau; \tau \geq \tau + L - z + \tau - \xi; X_Q \geq \tau + L - z + \tau - \xi) \\
&= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{Q-r} < \tau; z \geq \tau - \xi + L; X_Q \geq \tau + L - z + \tau - \xi) \\
&= \mathbb{1}_{\{z \geq \tau - \xi + L\}} \int_{\tau - \min\{\xi, \tau\}}^{\tau} dF_r(u) \int_{\ell - t}^{\infty} dF_{Q-r}(t),
\end{aligned}$$

where $\ell = \tau + L - z + \tau - \xi$.

A3.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_{Q-r} < \tau - \min\{\xi, \tau\}; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} < \tau - \min\{\xi, \tau\}; \min\{\tau, X_Q\} - t \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} < \tau - \min\{\xi, \tau\}; \tau \geq \tau + L - z + t; X_Q \geq \tau + L - z + t) \\
&= \mathbb{P}(X_{Q-r} < \tau - \min\{\xi, \tau\}; t \leq z - L; X_Q \geq \tau + L - z + t) \\
&= \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{z - L, \tau - \min\{\xi, \tau\}\}).
\end{aligned}$$

Then, we can simply express the transition probability function for Region A as follows:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{1}_{\{z \geq \tau - \xi + L\}} \left[\int_{\tau - \min\{\xi, \tau\}}^{\tau} \bar{F}_r(\ell - t) dF_{Q-r}(t) + \bar{F}_{Q-r}(\tau) \right] \\
&\quad + \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{z - L, \tau - \min\{\xi, \tau\}\}).
\end{aligned} \tag{4.4}$$

Similarly, for Region B:

B1.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_{Q-r} > m_f; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} > m_f; x - m_\xi \geq \tau + L - z) \\
&= \mathbb{1}_{\{x \geq \tau + L - z + m_\xi\}} \int_{m_f}^{\infty} dF_{Q-r}(t).
\end{aligned}$$

B2.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(m_\xi < X_{Q-r} < m_f; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(m_\xi < X_{Q-r} < m_f; \min\{m_f, X_Q\} \geq \tau + L - z + m_\xi - m_\tau) \\
&= \mathbb{P}(m_\xi < X_{Q-r} < m_f; x \geq \tau + L - z + m_\xi; X_Q \geq \tau + L - z + m_\xi - m_\tau) \\
&= \mathbb{1}_{\{x \geq \tau + L - z + m_\xi\}} \int_{m_\xi}^{m_f} \int_{\ell - t}^{\infty} dF_r(u) dF_{Q-r}(t).
\end{aligned}$$

B3.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_{Q-r} < m_\xi; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} < m_\xi; \min\{x, X_Q\} - t \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} < m_\xi; x \geq \tau + L - z + t; X_Q \geq \tau + L - z + t) \\
&= \mathbb{P}(X_{Q-r} < m_\xi; t \leq x - (\tau + L - z); X_Q \geq \tau + L - z + t) \\
&= \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{x - (\tau + L - z), m_\xi\}).
\end{aligned}$$

Then, we can simply express the transition probability function for Region B as follows:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{1}_{\{x \geq \tau + L - z + m_\xi\}} \left[\int_{m_\xi}^{m_f} \bar{F}_r(\ell - t) dF_{Q-r}(t) + \bar{F}_{Q-r}(m_f) \right] \\
&\quad + \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{x - (\tau + L - z), m_\xi\}).
\end{aligned} \tag{4.5}$$

For Region C:

C1.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_{Q-r} > x; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} > x; x \geq \tau + L - z) \\
&= \mathbb{1}_{\{x \geq \tau + L - z\}} \int_x^\infty dF_{Q-r}(t).
\end{aligned}$$

C2.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_{Q-r} < x; CL_n - U_n \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} < x; \min\{x, X_Q\} \geq \tau + L - z) \\
&= \mathbb{P}(X_{Q-r} < x; x \geq \tau + L - z; X_Q \geq \tau + L - z) \\
&= \mathbb{1}_{\{x \geq \tau + L - z\}} \int_0^x \int_{\ell-t}^\infty dF_r(u) dF_{Q-r}(t).
\end{aligned}$$

Then, we can simply express the transition probability function for Region C as follows:

$$\mathbb{P}(\mathcal{S}|x) = \mathbb{1}_{\{x \geq \tau + L - z\}} \left[\int_0^x \bar{F}_r(\ell - t) dF_{Q-r}(t) + \bar{F}_{Q-r}(x) \right]. \tag{4.6}$$

Finally, we can express all regions by a single term as Expression (4.2). Obviously, the transition probability function is a differentiable function. $\square$

4.2 Effective Lifetime Distribution for Case $m > 1$

Let $\{\mathbf{Z}_n, n \geq 1\}$ be the sequence of effective lifetimes of the m batches in the system at time T_n , where $\mathbf{Z}_n = \{Z_n^1, \dots, Z_n^m\}$; and $\mathbf{z}_n = \{z_n^1, \dots, z_n^m\}$ denote a particular realization of $\mathbf{Z}_n$, where $0 \leq z_n^1 \leq z_n^2 \leq \dots \leq z_n^m \leq \tau + L$. Moreover, z_n^1 and z_n^m represent the oldest (current) and youngest batch effective lifetime in embedded cycle n , respectively. It should be noted that when $z_n^1 > \tau$, the current batch is an outstanding batch at the beginning of the embedded cycle. That particular batch will join the inventory after $z_n^1 - \tau$ units of time and upon its arrival it is a fresh batch (cycles 2 and 5 in Figure 3.2). And when $z_n^1 \leq \tau$, it implies that the current batch is in the stock and under consumption at the beginning of the embedded cycle (i.e., it is not a fresh batch). In this case, we do not have any lost sales at the beginning of the embedded cycle. Hence, the system loses the demands that occur, if any, during the segment $w(Z_n^1)$, where $w(Z_n^1) = \max\{0, Z_n^1 - \tau\}$.

Considering the system assumptions and the (Q, r, ξ) policy definition, different realizations that occur for the embedded cycles are illustrated in Figures 4.5-4.8.

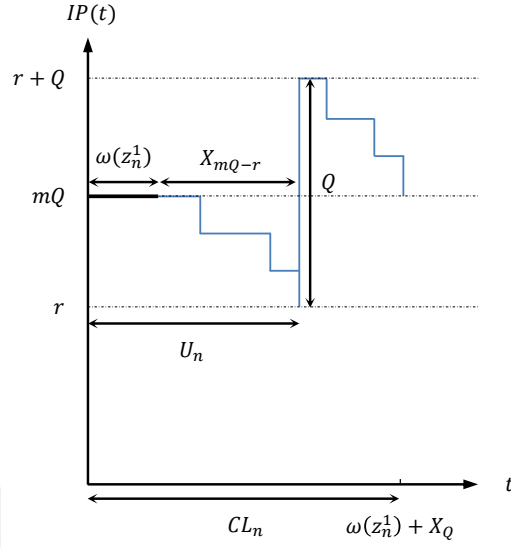


Figure 4.5: Event 1, $w(z_n^1) + X_{mQ-r} < z_n^1 - \xi$, $w(z_n^1) + X_Q < z_n^1$, $CL_n = w(z_n^1) + X_Q$, $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [X_Q - X_{mQ-r}]$

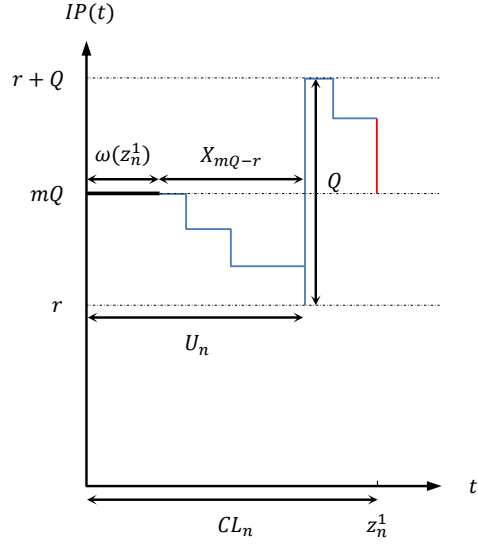


Figure 4.6: Event 2, $w(z_n^1) + X_{mQ-r} < z_n^1 - \xi$, $w(z_n^1) + X_Q > z_n^1$, $CL_n = z_n^1$, $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [z_n^1 - (w(z_n^1) + X_{mQ-r})]$

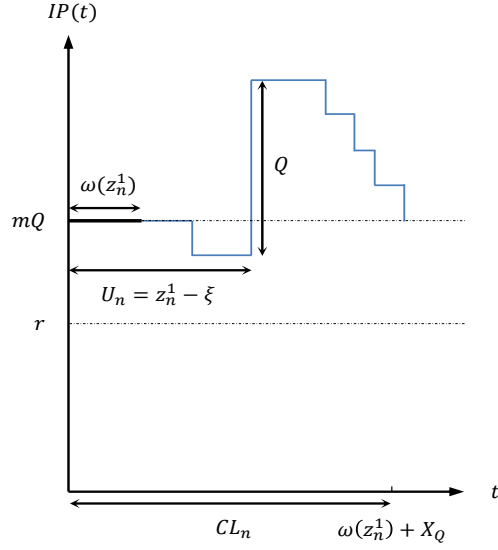


Figure 4.7: Event 3, $w(z_n^1) + X_{mQ-r} > z_n^1 - \xi$, $w(z_n^1) + X_Q < z_n^1$, $CL_n = w(z_n^1) + X_Q$, $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [w(z_n^1) + X_Q - (z_n^1 - \xi)]$

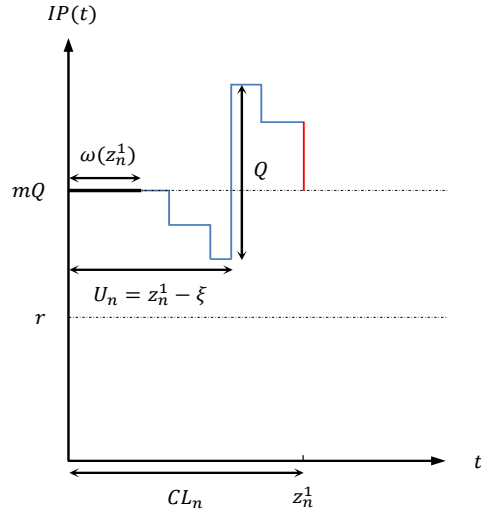


Figure 4.8: Event 4, $w(z_n^1) + X_{mQ-r} > z_n^1 - \xi$, $w(z_n^1) + X_Q > z_n^1$, $CL_n = z_n^1$, $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - \xi$

These realizations are the same with the case $m = 1$ and the events leading to these realizations are also the same. The difference of realizations between cases $m = 1$ and $m > 1$ is the effective lifetime distribution of the current batch at

the beginning of embedded cycles. In the case $m = 1$, we only track the effective lifetime of one batch Z_n , however in case $m > 1$ we track the effective lifetime of the current batch Z_n^1 at the beginning of embedded cycles for placing a new order. In these realizations, we assume that we are currently at the beginning of embedded cycle n . Thus, the n^{th} embedded cycle length and the effective lifetime of the current batch at the beginning of embedded cycle $(n + 1)^{th}$ are provided for each realization. We refer to these realizations as Events 1 through 4. In all realizations, it is assumed that the lifetime of the current batch at the beginning of embedded cycle n is a given value z_n^1 ; and without loss of generality, we assume that the embedded cycle starts with a stock out period and effective lifetime z_n^1 . The value of $w(z_n^1)$ is either $z_n^1 - \tau$ when $z_n^1 > \tau$ or zero when $z_n^1 \leq \tau$. Starting the embedded cycle n with mQ items in the system, $w(Z_n^1)$ is the time during which system is out of stock and X_{mQ-r} is the time at which the inventory position reduces from mQ to r for the first time and $Z_n^1 - \xi$ is the time at which the remaining lifetime of the current batch crosses ξ . In Figure 3.2, cycles 1, 2, and 7 represent Event 1; cycle 4 represents Event 2; cycle 5 represents Event 3; and cycles 3 and 6 represent Event 4.

The $(n + 1)^{th}$ effective lifetime vector can be generally expressed as follows:

$$\begin{aligned} Z_{n+1}^i &= Z_n^{i+1} - CL_n, \text{ for } i = 1, 2, \dots, m-1, \\ Z_{n+1}^m &= \tau + L - (CL_n - U_n), \end{aligned} \tag{4.7}$$

where $U_n = \min\{Z_n^1 - \xi, w(Z_n^1) + X_{mQ-r}\}$, $CL_n = w(Z_n^1) + \min\{\min\{Z_n^1, \tau\}, X_Q\} = w(Z_n^1) + \min\{Z_n^1, \tau, X_Q\}$. According to (4.7), we can conclude that the effective lifetime vector $\mathbf{Z}_{n+1}$ at the beginning of the $(n+1)^{th}$ embedded cycle is completely determined by $\mathbf{Z}_n$ and the Poisson demand arrival process after the stopping time T_n . Therefore, the embedded process $\{\mathbf{Z}_n, n \geq 1\}$ has the Markovian property. Analogous to case $m = 1$, we can easily verify that $0 \leq Z_n^i \leq \tau + L, i = 1, 2, \dots, m$.

The state space of the system is $S = \{(x^1, x^2, \dots, x^m) : 0 \leq x^i \leq \tau + L, i = 1, \dots, m\}$. Let $\mathcal{B}^m$ be the Borel σ -algebra generated by the subsets of S . Without loss of generality, we consider the sets $\mathcal{S} \in \mathcal{B}^m$ which are in the form $\mathcal{S} = (0, z^1] \times (0, z^2] \times \dots \times (0, z^m]$, where $0 \leq z^i \leq \tau + L, i = 1, \dots, m$. Let $\mathbf{x} =$

$(x^1, x^2, \dots, x^m)$, $\mathbf{Z}_n = (Z_n^1, Z_n^2, \dots, Z_n^m)$. Then, considering the definitions above, realizations illustrated in Figures 4.5-4.8, and the Markovian property of the lifetime distribution, we have the following result.

Theorem 2. (*Transition Probability Function of $\mathbf{Z}_n$*).

Let $\mathbb{P}(\mathcal{S}|\mathbf{x}) \equiv \mathbb{P}(Z_{n+1}^i \leq z^i, i = 1, \dots, m | \mathbf{Z}_n = \mathbf{x})$. Then, for $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$ and for $x^1 \geq m_x$ we have:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) = & \mathbb{1}_{\{x^1 \geq \tau + L - z^m + m_\xi\}} \left[\int_{m_f - \xi}^{m_f} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) + \bar{F}_{mQ-r}(m_f) \right] \\ & + \int_0^{\min\{m_f - (\tau + L - z^m), m_f - \xi\}} \bar{F}_{r-(m-1)Q}(\max\{m_x - m_\tau - t, \tau + L - z^m\}) dF_{mQ-r}(t), \end{aligned} \quad (4.8)$$

where $m_\xi = \max\{0, x^1 - \xi\}$, $m_\tau = \max\{0, x^1 - \tau\}$, $m_f = \min\{x^1, \tau\}$, $m_x = \max_{1 \leq i \leq m-1} \{x^{i+1} - z^i\}$, and $\ell = \max\{m_x - m_\tau, \tau + L - z^m + m_\xi - m_\tau\}$.

Proof. Using the Expression (4.7), we can rewrite Z_{n+1}^i for $i = 1, \dots, m-1$ and Z_{n+1}^m as follows:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(Z_{n+1}^1 \leq z^1; Z_{n+1}^2 \leq z^2; \dots; Z_{n+1}^m \leq z^m | \mathbf{Z}_n = (x^1, x^2, \dots, x^m)) \\ &= \mathbb{P}(x^2 - CL_n \leq z^1; x^3 - CL_n \leq z^2; \dots; \tau + L - (CL_n - U_n) \leq z^m) \\ &= \mathbb{P}(CL_n \geq \max_{1 \leq i \leq m-1} \{x^{i+1} - z^i\}; CL_n - U_n \geq \tau + L - z^m) \\ &= \mathbb{P}(CL_n \geq m_x; CL_n - U_n \geq \tau + L - z^m). \end{aligned} \quad (4.9)$$

Using the Expression (4.9) and considering different realizations of CL_n and U_n for different Regions A, B, and C expressed below, we can derive the transition probability function. For simplicity, let's denote X_{mQ-r} by t and $X_{r-(m-1)Q}$ by u .

A: $\max\{\xi, \tau\} < x^1 \leq \tau + L$

A1. If $t > \tau$,

$CL_n = x^1$ and $U_n = x^1 - \xi$.

A2. If $\tau - \min\{\xi, \tau\} < t < \tau$,

$CL_n = x^1 - \tau + \min\{\tau, X_Q\}$ and $U_n = x^1 - \xi$.

A3. If $t < \tau - \min\{\xi, \tau\}$,

$CL_n = x^1 - \tau + \min\{\tau, X_Q\}$ and $U_n = x^1 - \tau + t$.

B: $\min\{\xi, \tau\} < x^1 \leq \max\{\xi, \tau\}$

B1. If $t > m_f$,

$CL_n = x^1$ and $U_n = m_\xi$.

B2. If $m_\xi < t < m_f$,

$CL_n = x^1 - \tau + \min\{m_f, X_Q\}$ and $U_n = m_\xi$.

B3. If $t < m_\xi$,

$CL_n = \min\{x^1, X_Q\}$ and $U_n = t$.

C: $0 \leq x^1 \leq \min\{\xi, \tau\}$

C1. If $t > x^1$,

$CL_n = x^1$ and $U_n = 0$.

C2. If $t < x^1$,

$CL_n = \min\{x^1, X_Q\}$ and $U_n = 0$.

For Region A:

A1.

$$\mathbb{P}(\mathcal{S}|\mathbf{x}) = \mathbb{P}(X_{mQ-r} > \tau; x^1 \geq m_x; \xi \geq \tau + L - z^m) = \mathbb{1}_{\{z^m \geq \tau - \xi + L\}} \int_{\tau}^{\infty} dF_{mQ-r}(t).$$

A2.

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{mQ-r} < \tau; \min\{\tau, X_Q\} \geq m_x - (x^1 - \tau); \\ &\quad \min\{\tau, X_Q\} \geq \tau + L - z^m + m_\xi - (x^1 - \tau)) \\ &= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{mQ-r} < \tau; \tau \geq m_x - (x^1 - \tau); \\ &\quad \tau \geq \tau + L - z^m + m_\xi - (x^1 - \tau); X_Q > \max\{m_x - m_\tau, \tau + L - z^m + m_\xi - m_\tau\}) \\ &= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_{mQ-r} < \tau; x^1 \geq m_x; z^m \geq \tau - \xi + L; \\ &\quad X_{r-(m-1)Q} > \max\{m_x - m_\tau, \tau + L - z^m + m_\xi - m_\tau\} - t) \\ &= \mathbb{1}_{\{z^m \geq \tau - \xi + L\}} \int_{\tau - \min\{\xi, \tau\}}^{\tau} \int_{\ell-t}^{\infty} dF_{r-(m-1)Q}(u) dF_{mQ-r}(t), \end{aligned}$$

where $\ell = \max\{m_x - m_\tau, \tau + L - z^m + m_\xi - m_\tau\}$.

A3.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(X_{mQ-r} < \tau - \min\{\xi, \tau\}; \min\{\tau, X_Q\} \geq m_x - (x^1 - \tau); \\
&\quad \min\{\tau, X_Q\} \geq \tau + L - z^m + t) \\
&= \mathbb{P}(X_{mQ-r} < \tau - \min\{\xi, \tau\}; \tau \geq m_x - (x^1 - \tau); \tau \geq \tau + L - z^m + t; \\
&\quad X_Q > \max\{m_x - m_\tau, \tau + L - z^m + t\}) \\
&= \mathbb{P}(X_{mQ-r} < \tau - \min\{\xi, \tau\}; x^1 \geq m_x; t \leq z^m - L; \\
&\quad X_{r-(m-1)Q} > \max\{m_x - m_\tau - t, \tau + L - z^m\}) \\
&= \int_0^{\min\{z^m-L, \tau-\min\{\xi, \tau\}\}} \int_{\max\{m_x-m_\tau-t, \tau+L-z^m\}}^\infty dF_{r-(m-1)Q}(u) dF_{mQ-r}(t).
\end{aligned}$$

Then, we can simply express the transition probability function for Region A as follows:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{1}_{\{z^m \geq \tau - \xi + L\}} \left[\bar{F}_{mQ-r}(\tau) + \int_{\tau - \min\{\xi, \tau\}}^\tau \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) \right] \\
&\quad + \int_0^{\min\{z^m-L, \tau-\min\{\xi, \tau\}\}} \bar{F}_{r-(m-1)Q}(\max\{m_x - m_\tau - t, \tau + L - z^m\}) dF_{mQ-r}(t).
\end{aligned} \tag{4.10}$$

Similarly, we can derive the transition probability function for Regions B and C as follows:

For Region B:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{1}_{\{x^1 \geq \tau + L - z^m + m_\xi\}} \left[\bar{F}_{mQ-r}(\min\{x^1, \tau\}) + \int_{m_\xi}^{\min\{x^1, \tau\}} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) \right] \\
&\quad + \int_0^{\min\{x^1 - (\tau + L - z^m), m_\xi\}} \bar{F}_{r-(m-1)Q}(\max\{m_x - m_\tau - t, \tau + L - z^m\}) dF_{mQ-r}(t).
\end{aligned} \tag{4.11}$$

For Region C:

$$\mathbb{P}(\mathcal{S}|\mathbf{x}) = \mathbb{1}_{\{x^1 \geq \tau + L - z^m\}} \left[\bar{F}_{mQ-r}(x^1) + \int_0^{x^1} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) \right]. \tag{4.12}$$

Finally, we can express all regions by a single term as Expression (4.8). $\square$

It is worthwhile to note that by setting $m_x = 0$, $z^m = z$, $x^1 = x$, and $\ell = \tau + L - z + m_\xi - m_\tau$ we will achieve the result for $m = 1$.

Proposition 1. (*Boundedness*).

(i) Given $\mathbf{Z}_n = \mathbf{x}$, we have

$$\begin{aligned}\mathbb{E}[Z_{n+1}^i|\mathbf{x}] &= x^{i+1} - \mathbb{E}[CL_n|\mathbf{x}], \text{ for } i = 1, \dots, m-1, \\ \mathbb{E}[Z_{n+1}^m|\mathbf{x}] &= \tau + L - \mathbb{E}[CL_n|\mathbf{x}] + \mathbb{E}[U_n|\mathbf{x}],\end{aligned}\tag{4.13}$$

where

$$\begin{aligned}\mathbb{E}[CL_n|\mathbf{x}] &= \frac{Q}{\lambda} F_{Q+1}(m_f) + w(x^1) F_Q(m_f) + x^1 \bar{F}_Q(m_f), \text{ and} \\ \mathbb{E}[U_n|\mathbf{x}] &= \frac{mQ-r}{\lambda} F_{mQ-r+1}(m_f - \xi) + w(x^1) F_{mQ-r}(m_f - \xi) + (x^1 - \xi) \bar{F}_{mQ-r}(m_f - \xi).\end{aligned}\tag{4.14}$$

(ii) Given $\mathbf{Z}_n = \mathbf{x}$, we have

$$\sum_{i=1}^m \mathbb{E}[Z_{n+1}^i|\mathbf{Z}_n = \mathbf{x}] \leq \sum_{i=1}^m x^i - \epsilon \leq g(\mathbf{x}) - \epsilon,\tag{4.15}$$

where $x^1 = \tau + L$ and $\epsilon = m\mathbb{E}[CL_n|\mathbf{x}] - \mathbb{E}[U_n|\mathbf{x}]$.

Proof. According to Expression (4.7), we arrive at

$$\begin{aligned}\mathbb{E}[Z_{n+1}^i|\mathbf{x}] &= x^{i+1} - \mathbb{E}[CL_n|\mathbf{x}], \text{ for } i = 1, 2, \dots, m-1, \\ \mathbb{E}[Z_{n+1}^m|\mathbf{x}] &= \tau + L - \mathbb{E}[CL_n|\mathbf{x}] + \mathbb{E}[U_n|\mathbf{x}],\end{aligned}\tag{4.16}$$

The $\mathbb{E}[CL_n|\mathbf{x}]$ term is the expected embedded cycle length which is expressed as (5.3) as shown later, and the term $\mathbb{E}[U_n|\mathbf{x}]$ is obtained as follows:

$$[U_n|\mathbf{x}] = \begin{cases} w(x^1) + X_{mQ-r} & \text{if } X_{mQ-r} < x^1 - \xi - w(x^1), \\ x^1 - \xi & \text{if } X_{mQ-r} \geq x^1 - \xi - w(x^1), \end{cases}\tag{4.17}$$

$$\begin{aligned}
\mathbb{E}[U_n|\mathbf{x}] &= \int_0^{m_f-\xi} (y + w(x^1)) dF_{mQ-r}(y) + \int_{m_f-\xi}^{\infty} (x^1 - \xi) dF_{mQ-r}(y) \\
&= \frac{mQ-r}{\lambda} F_{mQ-r+1}(m_f - \xi) + w(x^1) F_{mQ-r}(m_f - \xi) \\
&\quad + (x^1 - \xi) \bar{F}_{mQ-r}(m_f - \xi).
\end{aligned} \tag{4.18}$$

Note that $x^1 - w(x^1) = \min\{x^1, \tau\} = m_f$. $\square$

Theorem 3. (*Ergodicity*). *The process $\{\mathbf{Z}_n, n \geq 1\}$ is ergodic.*

Proof. We use the proof of Berk et al. in [45]. Recall that if there are no closed classes in a Markov chain, except for the entire class of states, the chain is irreducible ([46], p.127). Inspecting (4.8), we see that for any $\mathcal{S} \in \mathcal{B}^m$, there exists some $\mathbf{x} \in \mathcal{S}$ such that $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$. In particular, for any $\mathbf{x}$ that satisfies $x^i \leq z^i, i = 1, \dots, m$ and $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$. That is, $\mathbb{P}(\mathbf{Z}_{n+1} \in \mathcal{S} | \mathbf{Z}_n = \mathbf{x} \in \mathcal{S}) = 0$ and $\mathbf{Z}_{n+1}$ gets out of the subset $\mathcal{S}$. This implies that a set of the form $\mathcal{S}$ given above is not closed. Similarly, it is easy to check that $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 1$ if $z^i = \tau + L, i = 1, \dots, m$, in which case, $\mathcal{S}$ is the entire state space. Hence, irreducibility follows. We now invoke Theorem 2.1 of [47] for a multi-dimensional Euclidean space and take $g(\mathbf{x}) = \sum_{i=1}^m x^i$. We use μ_τ to denote the mean hitting time which is defined as the mean time between two consecutive embedded cycles at the beginning of which the vectors of effective lifetimes are the same. To confirm the ergodicity of the process, we need to show that the mean hitting time is bounded. Proposition 1 implies the boundedness of the mean hitting times of the process, which implies the ergodicity. Note that in the case $m = 1$, the proof of boundedness is more straightforward. If the embedded cycle n starts with a fresh batch ($Z_n = \tau$), the probability that the embedded cycle $n + 1$ starts also with a fresh batch is $p_g = 1 - \bar{F}_r(L)F_{Q-r}(\tau - L) < \infty$ (See proposition 2 in [27]). Now, suppose that an embedded cycle starts with a fresh batch; the number of embedded cycles required to observe for the first time an embedded cycle with a fresh batch has a geometric distribution with probability of success p_g . Then, the expected number of embedded cycles required to reach hitting time is $1/p_g$ (i.e., mean of geometric distribution with parameter p_g). On the other hand, from Expression (5.1), one can see that the maximum length of an embedded cycle is $\tau + L$. Thus, to derive an upper bound for the mean hitting time, we can assume

that all embedded cycle occurring during the hitting time take maximum value of $\tau + L$. Then, we have

$$\mu_\tau \leq \frac{\tau + L}{p_g} < \infty, \quad (4.19)$$

which implies that the mean hitting time is bounded above. Analogously, we can apply this approach to find an upper bound for the mean hitting time in the case $m > 1$, the only difference is the derivation of p_g . $\square$

Theorem 3 ensures that the limiting distribution of the effective lifetime process exists. Hence, we let $\lim_{n \rightarrow \infty} F_{n+1} = \lim_{n \rightarrow \infty} F_n \equiv F$ and obtain the limiting distributions in terms of implicit integral equations as follows.

For $m > 1$,

$$F(\mathbf{z}) = F(z^1, z^2, \dots, z^m) = \int_0^{\tau+L} \dots \int_0^{\tau+L} \mathbb{P}(\mathcal{S}|x^1, \dots, x^m) dF(x^1, \dots, x^m). \quad (4.20)$$

And for the special case of $m = 1$, we have

$$F(z) = \int_0^{\tau+L} \mathbb{P}(\mathcal{S}|x) dF(x). \quad (4.21)$$

These equations are solved for F numerically in our numerical section.

To illustrate the convergence of F_n , we provide some examples below. Consider an instance with $m = 2$, $Q = 5$, $r = 8$, $L = 1$, $\tau = 2$, and $\lambda = 6$. The convergence of the oldest and youngest batch distributions for this instance is depicted in Figures 4.9 and 4.10, respectively.

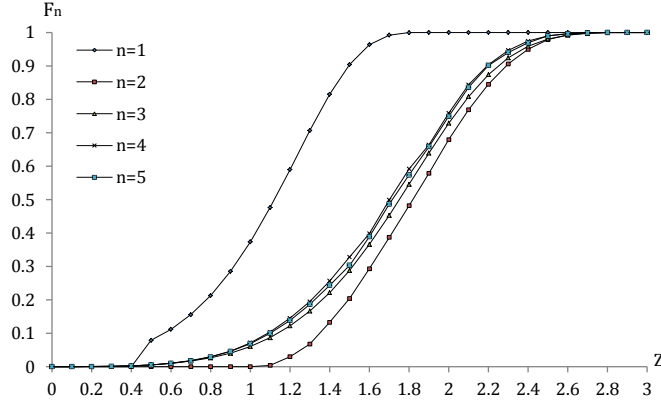


Figure 4.9: The oldest batch effective lifetime distribution function convergence

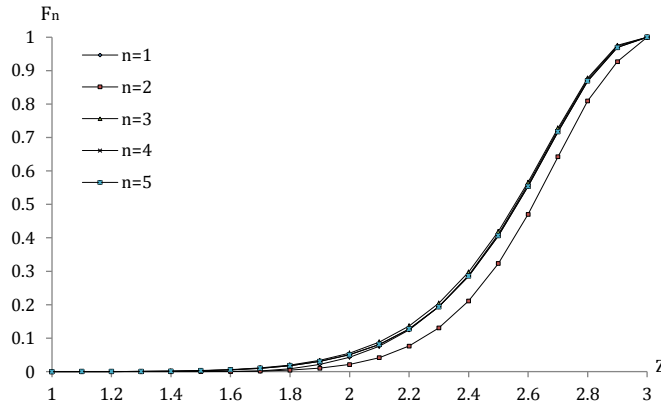


Figure 4.10: The youngest batch effective lifetime distribution function convergence

From the figures above, we can see that the distribution function converges which confirms that the effective lifetime distribution has a stationary distribution. However, we do not use the iterative method to derive the effective lifetime distribution function. Based on our observations, the convergence speed depends on the parameter settings such as λ , L , τ , Q , and r . For example, consider an instance with the following set of data $Q = 7$, $L = 1$, $\tau = 1.5$, and $\lambda = 8$. We consider this instance with $r = 11$ and $r = 8$. The convergence of oldest and youngest batch lifetime distribution function for $r = 11$ are depicted in Figures 4.11 and 4.12, respectively; and for $r = 8$ in Figures 4.13 and 4.14.

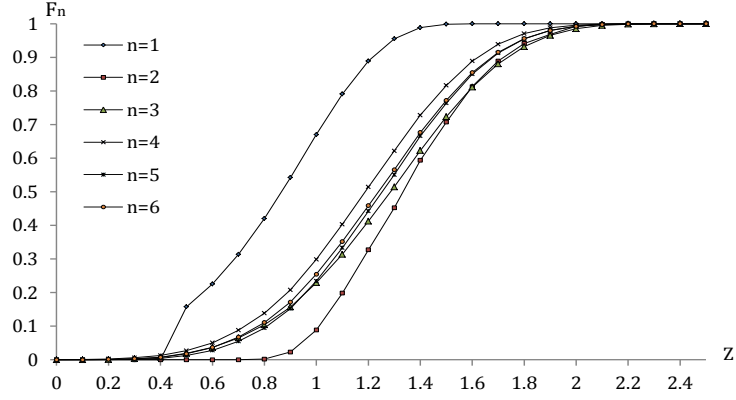


Figure 4.11: The oldest batch effective lifetime distribution function with $r = 11$

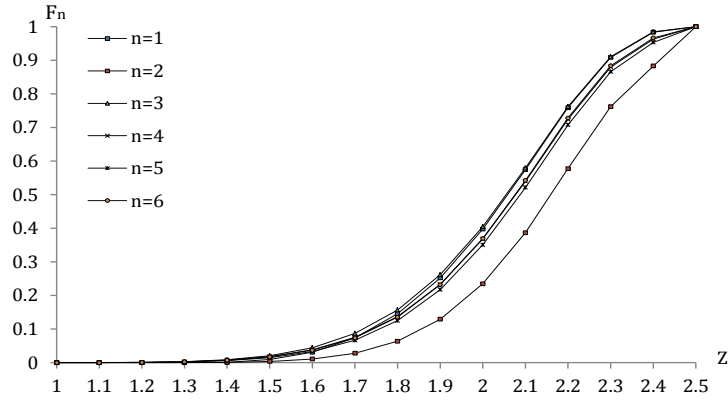


Figure 4.12: The youngest batch effective lifetime distribution function with $r = 11$

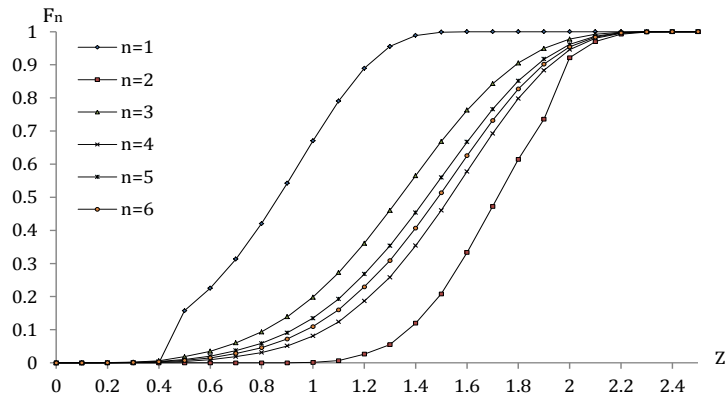


Figure 4.13: The oldest batch effective lifetime distribution function with $r = 8$

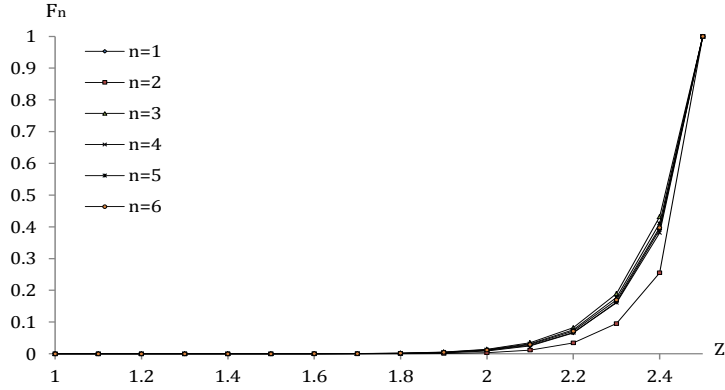


Figure 4.14: The youngest batch effective lifetime distribution function with $r = 8$

It seems that for $r = 11$ we have faster convergence. You can deal with the effect of other input parameters similarly. Moreover, our numerical study reveals that in all instances, the youngest batch distribution function converges faster than the oldest batch distribution function.

4.2.1 The Structure of Effective Lifetime Distribution Function

For the cases with one order outstanding restriction $m = 1$, we have two mass points at $\tau - \xi + L$ and ξ . In the following, we elaborate the events leading us to these mass points. Following the realizations presented in Figures 4.1-4.4, we can express the effective lifetime of the batch in embedded cycle $n + 1$ as $Z_{n+1} = \tau + L - (CL_n - U_n)$, where $CL_n = w(Z_n) + \min\{\min\{Z_n, \tau\}, X_Q\}$ and $U_n = \min\{Z_n - \xi, w(Z_n) + X_{Q-r}\}$; and both depend on the current batch lifetime Z_n . Suppose that we are at the beginning of the embedded cycle n and $Z_n = z_n$. If the remaining lifetime of the current batch crosses ξ and the embedded cycle ends up by perishing, then $CL_n = z_n$ and $U_n = z_n - \xi$ which results in $CL_n - U_n = \xi$. Therefore, $Z_{n+1} = \tau + L - (CL_n - U_n) = \tau - \xi + L$ which implies that we have a mass point at $\tau - \xi + L$. Note that when $Z_n \leq \xi$ we need to order immediately because the remaining lifetime crosses ξ in which case $U_n = 0$. Now, suppose that we are at the beginning of the embedded cycle n and $Z_n = \tau - \xi + L$; and

assume that we order immediately (i.e., $\tau - \xi + L \leq \xi$ and $U_n = 0$) and embedded cycle ends up by perishing which indicates that $CL_n = \tau - \xi + L$. Therefore, $Z_{n+1} = \tau + L - (CL_n - U_n) = \tau + L - (\tau - \xi + L) = \xi$ which implies that we have a mass point at ξ as well. Hence, we can conclude that when $\tau - \xi + L \leq \xi$ (i.e., $(\tau + L)/2 \leq \xi$) we have two mass points at $\tau - \xi + L$ and ξ , otherwise we have one mass point at $\tau - \xi + L$. Similarly, we can easily show that we have only one mass point at $\tau - \xi + L$ over Z^m in the case with $m > 1$.

4.2.2 How to Find the Effective Lifetime Distribution

As stated earlier for the (Q, r, ξ) policy, (i) when $\tau \leq \xi \leq \tau + L$ the effective lifetime distribution in the case $m > 1$ is in interval $[0, \tau + L]$ for $Z^i, i = 1, \dots, m$; and in the case $m = 1$ is in interval $[0, \tau + L]$ for Z , and (ii) when $0 \leq \xi < \tau$ the effective lifetime distribution in the case $m > 1$ is in interval $[0, \tau + L]$ for $Z^i, i = 1, \dots, m-1$ and $[L, \tau + L]$ for Z^m ; and in the case $m = 1$ is in interval $[L, \tau + L]$ for Z . The effective lifetime distribution for the inventory system under study is a hybrid distribution. To obtain the continuous portion of the distribution, we develop a two-step discretization approach as follows.

Step 1. We discretize the domain (support) of the continuous portion of the distribution function into k segments. The selection of k is dictated by the desired accuracy and computational memory burden. To find an appropriate number of segments k for approximation, we tried different values for k as 150, 200, 250, and 300 segments. Our results reveal that by setting $k = 200$, the total cost value is converged to the same value and after increasing the value of k we do not observe considerable change in total cost. By increasing the value of k we have more computational time. Therefore, we choose $k = 200$ to have a reasonable running time and an appropriate approximation, as well. More specifically, for case $m = 1$, we discretize the interval $[L, \tau + L]$ (when $0 \leq \xi < \tau$) or the interval $[0, \tau + L]$ (when $\tau \leq \xi \leq \tau + L$) into k segments, where k is set to be 200. For case $m > 1$, when $0 \leq \xi < \tau$ we discretize the interval $Z^i \in [0, \tau + L], i = 1, \dots, m-1$ into s_1 segments, where s_1 is set to be around 100 and s_2 is chosen in such a way

that the discretization size Δ remains the same; and when $\tau \leq \xi \leq \tau + L$ we discretize the interval $Z^i \in [0, \tau + L]$, $i = 1, \dots, m$ into s_1 segments, where s_1 is set to be 100 ($s_1 = s_2$). We summarize the discretization approach for cases $m = 1$ and $m > 1$ in Tables 4.1 and 4.2, respectively.

Table 4.1: Discretization approach for case $m = 1$

Range of ξ	Z	Δ	k
$0 \leq \xi < \tau$	$[L, \tau + L]$	$\frac{\tau}{k}$	200
$\tau \leq \xi \leq \tau + L$	$[0, \tau + L]$	$\frac{\tau+L}{k}$	200

Table 4.2: Discretization approach for case $m > 1$

Range of ξ	$Z^i, i = 1, 2, \dots, m-1$	Z^m	Δ	s_1 and s_2
$0 \leq \xi < \tau$	$[0, \tau + L]$	$[L, \tau + L]$	$\frac{\tau+L}{s_1} = \frac{\tau}{s_2}$	$s_1 \approx 100, s_2$ accordingly
$\tau \leq \xi \leq \tau + L$	$[0, \tau + L]$	$[0, \tau + L]$	$\frac{\tau+L}{s_1} = \frac{\tau+L}{s_2}$	$s_1 = s_2 = 100$

In Tables above, Δ represents the discretization size, k is the number of segments over vector Z for case $m = 1$, s_1 is the number of segments over Z^i , $i = 1, 2, \dots, m-1$ and s_2 is the number of segments over Z^m . Since for $m > 1$ we have a vector of random variables and the discretization size should be the same, we choose the values of s_1 and s_2 in such a way that the discretization size remains the same over all $Z^i, i = 1, 2, \dots, m$.

Due to limitation in the Matlab memory, as the number of outstanding orders increases we should decrease the number of segments in Z^i . For instance, when $m = 3$ we set k around 40. In the cases where we have a high gap between simulation and exact methods (where $m \geq 3$), we employ a dynamically adjusted discretization that gives more weight on the segments having higher effect on the objective function. In this approach, we ignore parts of the effective lifetime distribution having zero probabilities, then we distribute the discrete points over the part of the effective lifetime distribution having positive probabilities.

Step 2. In our preliminary studies, we observed that the accuracy in values of the computed operating characteristics can be increased as follows. We find

numerically the probabilities of discrete points p_i , for $i = 1, \dots, k$ located in the middle of each segment. Then, we can approximate the probability density function (p.d.f) of the effective lifetime using rectangles which is formed by p_i . That is, each segment forms a rectangle which is a part of the p.d.f with length Δ and height p_i/Δ . By doing so, we distribute the probability of the mid-points (discrete points) over the segments. Thus, we obtain the probability masses p_i located at mid-points of the selected segments approximating the continuous p.d.f. of the effective lifetime. We add the lower and upper level of each vector to the set of discrete points; for example, if $Z \in [L, \tau + L]$, we add points L and $\tau + L$ to the set of discrete points. Additionally, there might be some mass points over the effective lifetime distribution function (let's say we have N_m mass points in total). We add those mass points to the set of discrete points as well. Therefore, we have $k + 2 + N_m$ discrete points in total. For instance, consider the case with one order outstanding $m = 1$ and suppose that $0 \leq \xi < \tau$, where $Z \in [L, \tau + L]$. Then, we need to find the probability of points $\{\{L\} \cup \{L + i\frac{\Delta}{2}, i = 1, 3, 5, \dots, 2k - 1\} \cup \{\tau + L\} \cup \{\tau - \xi + L\}\}$, where k is number of segments over vector Z . Among these points, L and $\tau + L$ are the lower and upper level of vector Z , $L + i\frac{\Delta}{2}$ for $i = 1, 3, 5, \dots, 2k - 1$ are k discrete points located in the middle of each segment, and $\tau - \xi + L$ is the mass point over vector Z .

To clarify our discretization approach, we provide some examples in the following. In the following, we first illustrate *step 1* of the discretization approach through two examples for cases $m = 1$ and $m > 1$. For the case $m = 1$, let's assume that $L = 1$ and $\tau = 2$; then, (i) when $0 \leq \xi < \tau$ we have $\Delta = \tau/k = 2/200 = 0.01$, and segments over vector Z are $\{[1, 1.01], [1.01, 1.02], [1.02, 1.03], \dots, [2.99, 3]\}$, and (ii) when $\tau \leq \xi \leq \tau + L$ we have $\Delta = (\tau + L)/k = 3/200 = 0.015$, and segments over vector Z are $\{[0, 0.015], [0.015, 0.03], \dots, [2.985, 3]\}$. For the case $m > 1$, let's again assume that $L = 1$ and $\tau = 2$; then, (i) when $0 \leq \xi < \tau$ we have $\Delta = (\tau + L)/s_1 = \tau/s_2 = 3/99 = 2/66$ (i.e., $s_1 = 99$ and $s_2 = 66$. We can not set $s_1 = 100$ because the corresponding s_2 making the same Δ value is not attainable), and segments over vector Z^i , $i = 1, 2, \dots, m - 1$ are $\{[0, 1/33], [1/33, 2/33], \dots, [98/33, 3]\}$ and over vector Z^m are $\{[1, 34/33], [34/33, 35/33], \dots, [98/33, 3]\}$, and (ii) when

$\tau \leq \xi \leq \tau + L$ we have $\Delta = (\tau + L)/s_1 = 3/100 = 0.03$, and segments over vector Z^i , $i = 1, 2, \dots, m$ are $\{[0, 0.03], [0.03, 0.06], \dots, [2.97, 3]\}$.

Now, we illustrate *step 2* of the discretization approach by considering the case with one order outstanding $m = 1$ with $L = 1$, $\tau = 2$, and $k = 200$. Suppose that $\xi = 0.8$, then $\{\{1\} \cup \{1.005, 1.015, \dots, 2.995\} \cup \{3\} \cup \{2.2\}\}$, where $\{1.005, 1.015, 1.025, \dots, 2.995\}$ are $L + i\frac{\Delta}{2}$ for $i = 1, 3, 5, \dots, 2k - 1$ which are mid-points of segments $\{[1, 1.01], [1.01, 1.02], [1.02, 1.03], \dots, [2.99, 3]\}$, respectively. The discrete points $\{1\}$ and $\{3\}$ are lower and upper level of vector Z ; and point $\{2.2\}$ is a single mass point at $\tau - \xi + L$ over vector Z .

It should be noted that ξ has to be an integer multiple of Δ in case $m = 1$, otherwise the transition matrix constructed by $\mathbb{P}(\mathcal{S}|\mathbf{x})$ becomes singular and we are not able to find the probability of the discrete points and also not able to approximate p.d.f of the effective lifetime by rectangles. However, for case $m > 1$ we do not have such a restriction for ξ value. The reason is that in case $m > 1$ we only need the p.d.f of Z^1 for deriving the operating characteristics of system (as discussed later) and since mass points are only over vector Z^m , we can approximate p.d.f of Z^1 by rectangles without any concern on location ξ .

In Figures 4.15-4.19, the joint probability density function for $Q = 5$, $r = 7$, $\xi = 0.8$, $L = 1$, $\tau = 2$, and $\lambda = 4.5$ is depicted in different views. In Figures 4.20 and 4.21, we depict a sample of the effective lifetime distribution and probability density function for a given data set based on the discretization method described above. From Figure 4.20, we can see that for case $m > 1$, we have a jump at point $\tau - \xi + L$ over Z^2 which implies a mass point in the effective lifetime distribution of the youngest batch. Similarly, for case $m = 1$ we have a mass point at $\tau - \xi + L$ if $\xi \leq (\tau + L)/2$ and two mass points at $\tau - \xi + L$ and ξ , otherwise. Figure 4.22 depicts the case with one mass point at $\tau - \xi + L$, and Figure 4.23 shows the case with two mass points at $\tau - \xi + L$ and ξ .

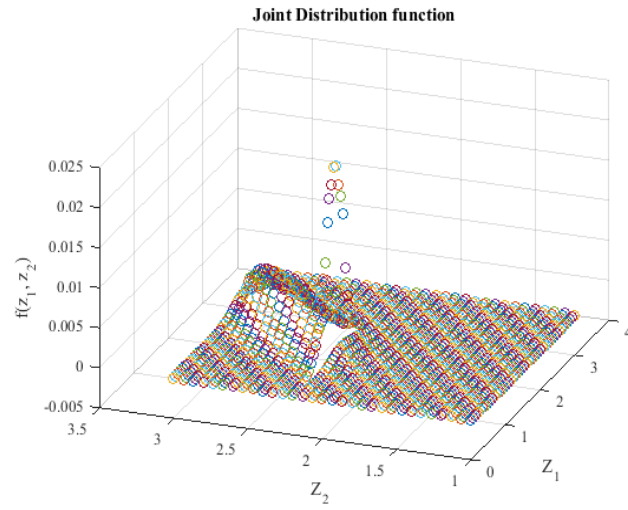


Figure 4.15: Joint distribution function of the effective lifetime in view 1

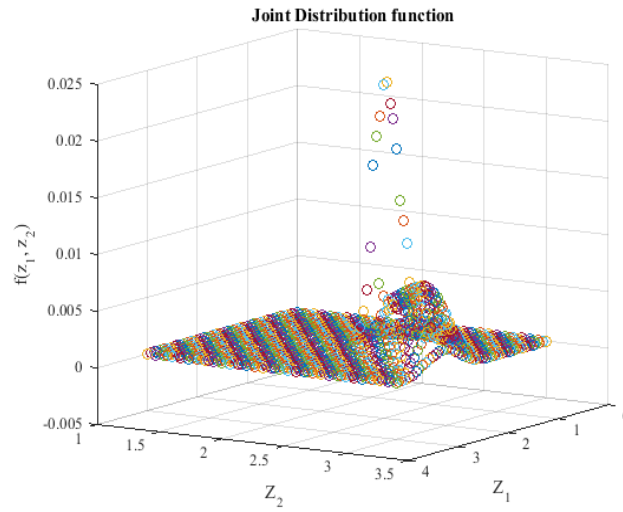


Figure 4.16: Joint distribution function of the effective lifetime in view 2

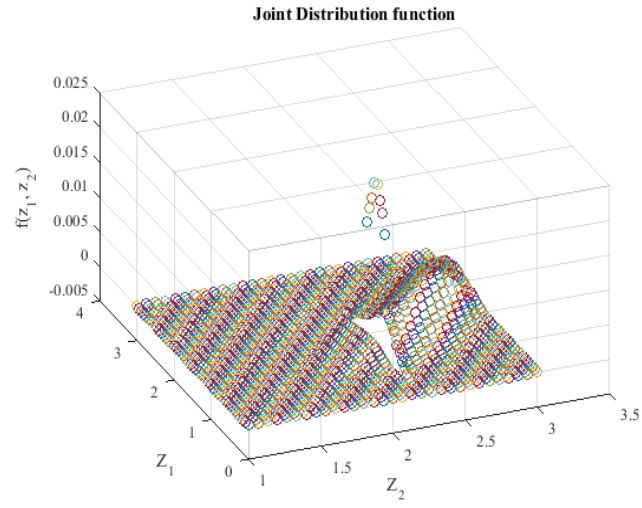


Figure 4.17: Joint distribution function of the effective lifetime in view 3

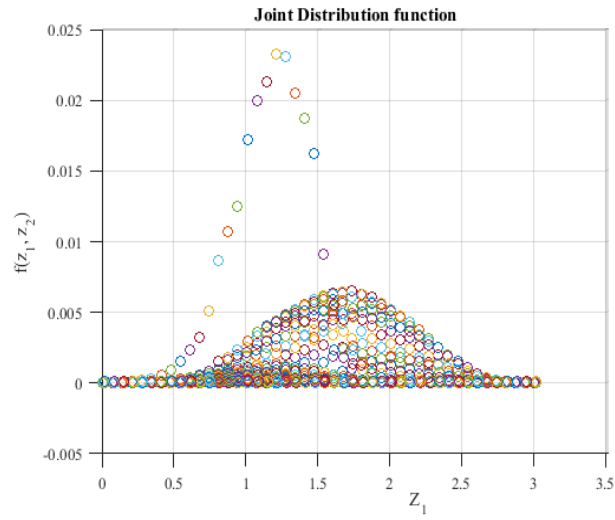


Figure 4.18: Joint distribution function of the effective lifetime in view 4

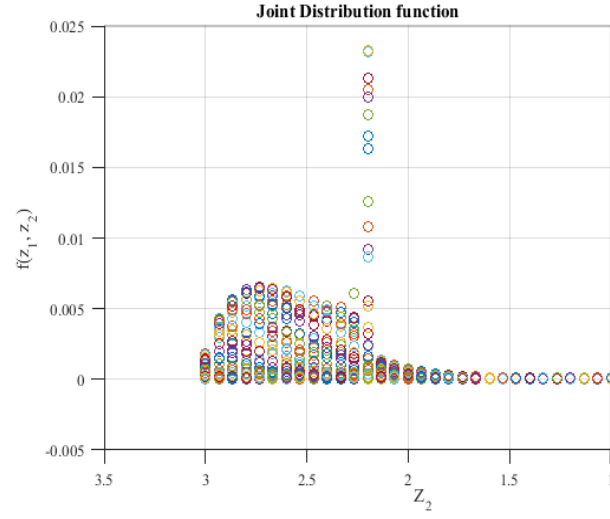


Figure 4.19: Joint distribution function of the effective lifetime in view 5

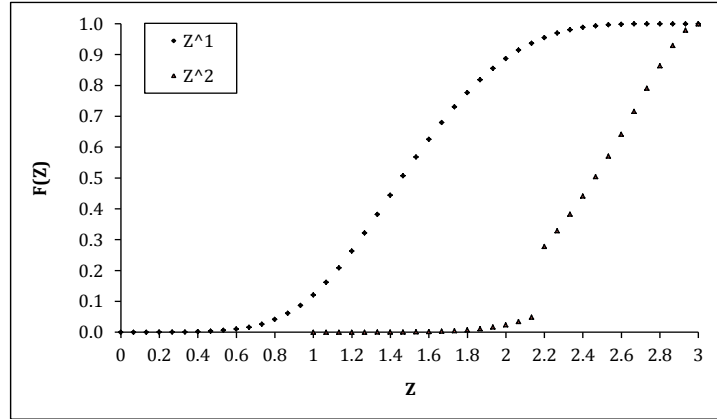


Figure 4.20: Cumulative distribution function of Z^1 and Z^2 for $Q = 5$, $r = 7$, $\xi = 0.8$, $L = 1$, $\tau = 2$, and $\lambda = 4.5$

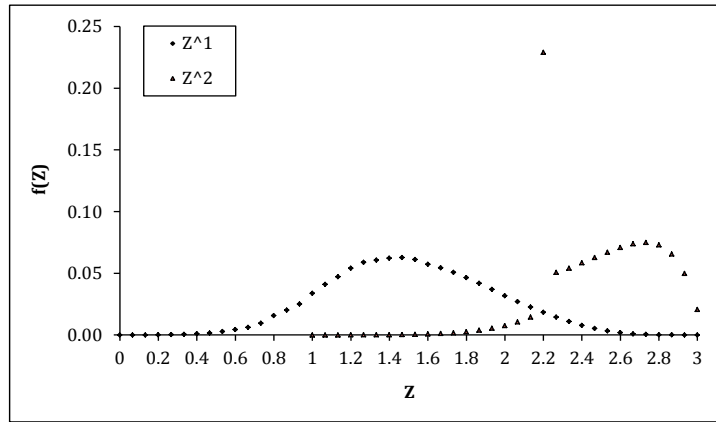


Figure 4.21: Probability density function of Z^1 and Z^2 for $Q = 5$, $r = 7$, $\xi = 0.8$, $L = 1$, $\tau = 2$, and $\lambda = 4.5$

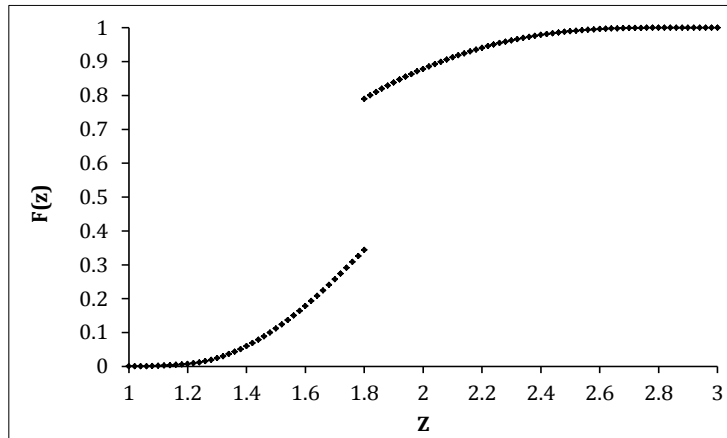


Figure 4.22: Effective lifetime distribution of Z for $Q = 7$, $r = 5$, $\xi = 1.2$, $\lambda = 3$, $L = 1$, and $\tau = 2$

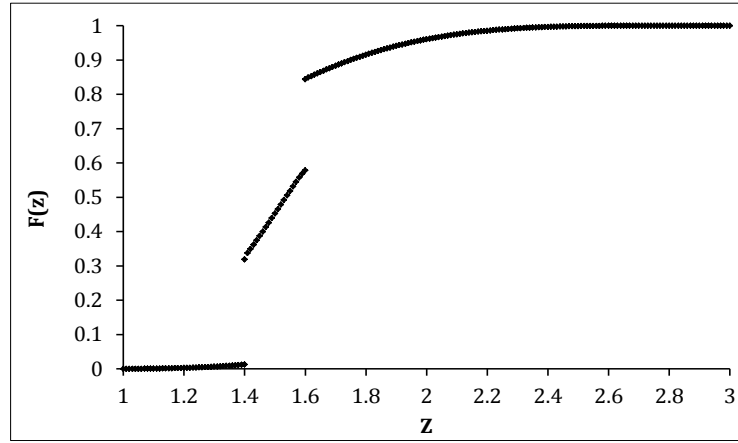


Figure 4.23: Effective lifetime distribution of Z for $Q = 7$, $r = 5$, $\xi = 1.6$, $\lambda = 3$, $L = 1$, and $\tau = 2$

Chapter 5

Operating Characteristics and the Objective Function

Having the steady state effective lifetime distribution, we next derive the operating characteristics of the system, including the expected embedded cycle length, number of lost sales, number of perishing items, as well as on-hand inventory for a given lifetime $\mathbf{Z} = \mathbf{z} = (z^1, z^2, \dots, z^m)$. Our analysis reveals that only the current batch lifetime affects the operating characteristics of the system. Therefore, in the following, we derive the operating characteristics for a given current batch lifetime $Z^1 = z^1$. Since in the case $m = 1$ we have only one batch on hand, the effective lifetime of the current batch $Z = z$ affects the operating characteristics. Therefore, z^1 and z are equivalent and can be used interchangeably so that operating characteristics expressions are the same in both cases $m = 1$ and $m > 1$. In the following, we derive the operating characteristics for a given current batch lifetime $Z = z$.

5.1 Operating Characteristics

Considering the events depicted in Figures 4.5-4.8, we can write the conditional embedded cycle length for a given current batch lifetime $Z = z$:

$$[CL|z] = \begin{cases} \min\{X_Q, z\} & \text{if } z < \tau, \\ z - \tau + \min\{X_Q, \tau\} & \text{if } z \geq \tau. \end{cases} \quad (5.1)$$

After some modification, we can rewrite Equation (5.1) as follows:

$$[CL|z] = \begin{cases} X_Q + w(z) & \text{if } X_Q < z - w(z), \\ z & \text{if } X_Q \geq z - w(z). \end{cases} \quad (5.2)$$

Therefore, the conditional expected embedded cycle length is obtained as follows:

$$\begin{aligned} \mathbb{E}[CL|z] &= \int_0^{z-w(z)} (y + w(z)) dF_Q(y) + \int_{z-w(z)}^{\infty} z dF_Q(y) \\ &= \frac{Q}{\lambda} F_{Q+1}(z - w(z)) + w(z) F_Q(z - w(z)) + z \bar{F}_Q(z - w(z)). \end{aligned} \quad (5.3)$$

Note that $z - w(z) = \min\{z, \tau\} = m_f$. So, we can simplify the Expression (5.3) as follows:

$$\mathbb{E}[CL|z] = \frac{Q}{\lambda} F_{Q+1}(m_f) + w(z) F_Q(m_f) + z \bar{F}_Q(m_f). \quad (5.4)$$

We have a stock out period at the beginning of the embedded cycle with length $z - \tau$ if $z > \tau$. Then, the conditional expected number of lost sales in an embedded cycle is obtained as follows:

$$\mathbb{E}[LS|z] = \lambda w(z). \quad (5.5)$$

The conditional number of perishing items can be calculated as:

$$[P|z] = \begin{cases} 0 & \text{if } X_Q < z - w(z), \\ Q - N(z - w(z)) & \text{if } X_Q \geq z - w(z), \end{cases} \quad (5.6)$$

where $N(z - w(z))$ represents the number of demand arrivals during $z - w(z)$ units of time. Then, the conditional expected number of perishing items is obtained as follows:

$$\mathbb{E}[P|z] = \sum_{n=0}^{Q-1} (Q - n) e^{-\lambda m_f} \frac{(\lambda m_f)^n}{n!}. \quad (5.7)$$

Next, we consider the (conditional) total stock-years (i.e., area under the inventory curve in an embedded cycle), $[OH|z]$. We use two approaches to derive the expected on-hand inventory for given lifetime, $\mathbb{E}[OH|z]$. In the first approach, for convenience, we define the following events:

$$\begin{aligned} E_{1a} &: \{X_Q < X_{mQ-r} + L; X_Q < z_1 - w(z_1)\}, \\ E_{1b} &: \{X_{mQ-r} + L < X_Q < z_1 - w(z_1)\}, \\ E_{2a} &: \{z_1 - w(z_1) - L < X_{mQ-r} < z_1 - w(z_1); X_Q > z_1 - w(z_1)\}, \\ E_{2b} &: \{X_{mQ-r} < z_1 - w(z_1) - L; X_Q > z_1 - w(z_1)\}, \\ E_3 &: \{X_{mQ-r} > z_1 - w(z_1)\}. \end{aligned}$$

Event E_{1a} corresponds to the case where the oldest batch is depleted by demand without perishing, an order is given during the embedded cycle but this order does not arrive before the embedded cycle terminates. Event E_{1b} corresponds to the case where the oldest batch is depleted with demand and the order given during the embedded cycle arrives before the embedded cycle is completed. Event E_{2a} describes the case where the oldest batch perishes after reordering and the last ordered batch does not join the stock before the embedded cycle ends; Event E_{2b} corresponds to the case where the oldest batch perishes after reordering and the ordered batch arrives before perishing; and, finally, Event E_3 corresponds to the

case where the oldest batch perishes before the reorder point is reached. Then,

$$[OH|\mathbf{z}] = \begin{cases} \{\sum_{i=1}^Q X_i\} + \{\sum_{i=2}^m Q[X_Q - (w(z_i) - w(z_1))]^+\} + \{0\} & \text{if } E_{1a}, \\ \{\sum_{i=1}^{N(z_1-w(z_1))} X_i + (z_1 - w(z_1))[Q - N(z_1 - w(z_1))]\} \\ + \{\sum_{i=2}^m Q[z_1 - w(z_i)]^+\} + \{0\} & \text{if } E_{2a}, \\ \{\sum_{i=1}^Q X_i\} + \{\sum_{i=2}^m Q[X_Q - (w(z_i) - w(z_1))]^+\} \\ + \{Q[X_Q - X_{mQ-r} - L]\} & \text{if } E_{1b}, \\ \{\sum_{i=1}^{N(z_1-w(z_1))} X_i + (z_1 - w(z_1))[Q - N(z_1 - w(z_1))]\} \\ + \{\sum_{i=2}^m Q[z_1 - w(z_i)]^+\} + \{Q[z_1 - w(z_1) - X_{mQ-r} - L]\} & \text{if } E_{2b}, \\ \{\sum_{i=1}^{N(z_1-w(z_1))} X_i + (z_1 - w(z_1))[Q - N(z_1 - w(z_1))]\} \\ + \{\sum_{i=2}^m Q[z_1 - w(z_i)]^+\} + \{0\} & \text{if } E_3. \end{cases}$$

The expression for $[OH|\mathbf{z}]$ for each event consists of three components (within curly brackets): The stock-years computed for the items in the batch that is currently being used, for those in the previously placed $m - 1$ orders which are already in stock or may join stock during the current cycle, and finally, for the most recently placed order. For example, for Event E_{1a} , the oldest batch will be carried in stock within an embedded cycle for the time until Q units are sold (one by one); the remaining batches ($i = 2$ through m) of size Q each will be carried in inventory for the time segment during which they are on hand; the most recently placed will result in zero positive stock since the embedded cycle ends before the lead time does. The stock-years are computed similarly for the other events.

The stock dynamics under the model herein is such that the stock-years computed for the oldest batch and the most recently placed order are identical to those obtained for the model with $m = 1$ and the same effective lifetimes for the oldest batch in [27]. Then, the expected stock-years expression for multiple outstanding orders can be written as follows.

$$\begin{aligned} \mathbb{E}[OH|\mathbf{z}] &= \mathbb{E}[\tilde{OH}|z_1 - w(z_1)] + Q \sum_{i=2}^m \int_{w(z_i)-w(z_1)}^{z_1-w(z_1)} [u + w(z_1) - w(z_i)] dH_Q(u) \\ &\quad + Q \sum_{i=2}^m [z_1 - w(z_i)]^+ \bar{H}_Q(z_1 - w(z_1)), \end{aligned}$$

(5.8)

where $\mathbb{E}[\tilde{O}H|x]$ is the corresponding expression in [27] for $m = 1$ (modified to allow for $m > 1$) with the oldest batch having an effective lifetime of $z_1 = x$,

$$\begin{aligned} \mathbb{E}[\tilde{O}H|x] = & Q[\eta(Q, r, x) + x\bar{H}_Q(x) - \frac{mQ - r}{\lambda}H_{mQ-r+1}(x - L) \\ & + \frac{(Q+1)}{2\lambda}H_{Q+1}(x) + \gamma(Q, r, x)] - \frac{\lambda x^2}{2}\bar{H}_{Q-1}(x), \end{aligned} \quad (5.9)$$

and

$$\gamma(Q, r, x) = H_{mQ-r}(x - L)[x - L\bar{H}_{r-(m-1)Q}(L) - \frac{r}{\lambda}H_{r-(m-1)Q+1}(L)], \quad (5.10)$$

$$\begin{aligned} \eta(Q, r, x) = & \int_0^{x-L} \left[\frac{r - (m-1)Q}{\lambda}H_{r-(m-1)Q+1}(x - u) \right. \\ & \left. - (x - u)H_{r-(m-1)Q}(x - u) \right] dH_{mQ-r}(u). \end{aligned} \quad (5.11)$$

Now, we deal with the second approach for deriving $\mathbb{E}[OH|z]$. For finding the expected on-hand inventory we use the following lemma.

Lemma 1. *The joint distribution of arrival time of Q demands, $(X_1, X_2, \dots, X_Q)$, is given by*

$$f(x_1, x_2, \dots, x_Q) = \lambda^Q e^{-\lambda x_Q}, \quad 0 < x_1 < x_2 < \dots < x_Q. \quad (5.12)$$

Proof. The sequence of $(X_1, X_2, \dots, X_Q) = (x_1, x_2, \dots, x_Q)$ can be represented equivalently by $(x_1, x_2 - x_1, \dots, x_Q - x_{Q-1})$. Then, we have

$$f(x_1, x_2, \dots, x_Q) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda(x_2 - x_1)} \dots \lambda e^{-\lambda(x_Q - x_{Q-1})} = \lambda^Q e^{-\lambda x_Q}. \quad (5.13)$$

□

To find the area below inventory level (the expected on-hand inventory) we use the idea of Little's law stating that the horizontal area below the inventory level is the same with vertical area. To find the conditional expected on-hand inventory,

we find the average waiting time of items (horizontal area). The waiting time of an item is a period of time during which that particular item spends in stock before depletion. If $z > \tau$ and assume that n demands arrive to the system during an embedded cycle, the waiting time of item n is X_n . When we have n demand arrivals, $Q - n$ items will be left at the end of the embedded cycle which means that they spend τ units of time in the stock. Thus, the waiting time of the $Q - n$ items is τ . If $z < \tau$ and assume that n demands arrive to the system during an embedded cycle, all of n items wait for $\tau - z$ units of time before the embedded cycle starts; and item n wait X_n units of time during the embedded cycle to be depleted by demand. Similarly, the waiting time of the $Q - n$ items is τ . Therefore, the expected waiting time of products for a given effective lifetime vector $\mathbf{Z} = \mathbf{z} = (z^1, z^2, \dots, z^m)$ and given n are found as follows:

$$\mathbb{E}(H_n|z) = \int_{0 < x_1 < \dots < x_n < m_f < x_{n+1} < \dots < x_Q} [n \cdot m_l + (x_1 + x_2 + \dots + x_n) + (Q - n)\tau] \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_Q, \quad (5.14)$$

where $m_l = \max(0, \tau - z)$. Than, by using Lemma 1 we can find the conditional expected waiting time (on-hand inventory) as follows.

For $0 \leq z \leq \tau$,

$$\begin{aligned} \mathbb{E}[OH|z] &= \sum_{n=0}^Q \mathbb{E}(H_n|z) = Q(\tau - z) + \int_0^z \sum_{k=0}^{Q-1} (Q - k) e^{-\lambda z} \frac{(\lambda z)^k}{k!} dz \\ &= Q(\tau - z) + \int_0^z \mathbb{E}[P|z] dz \\ &= Q(\tau - z) + \frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q - k)}{k!} \gamma(k + 1, \lambda z), \end{aligned} \quad (5.15)$$

For $\tau < z \leq \tau + L$,

$$\begin{aligned} \mathbb{E}[OH|z] &= \sum_{n=0}^Q \mathbb{E}(H_n|z) = \int_0^\tau \sum_{k=0}^{Q-1} (Q - k) e^{-\lambda z} \frac{(\lambda z)^k}{k!} dz \\ &= \frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q - k)}{k!} \gamma(k + 1, \lambda \tau), \end{aligned} \quad (5.16)$$

where

$$\gamma(k+1, \lambda z) = \int_0^{\lambda z} x^k e^{-x} dx. \quad (5.17)$$

For the sake of clarity, we illustrate the derivation of $\mathbb{E}[OH|z]$ expression for different values of Q in Appendix A.

Finally, we can find the expected value of operating characteristics as $\mathbb{E}(CL) = \int_z \mathbb{E}[CL|z]dF(z)$, $\mathbb{E}(LS) = \int_z \mathbb{E}[LS|z]dF(z)$, $\mathbb{E}(P) = \int_z \mathbb{E}[P|z]dF(z)$, and $\mathbb{E}(OH) = \int_z \mathbb{E}[OH|z]dF(z)$.

5.2 The Objective Function

The approach to construct the objective function is similar to [27] and is motivated by the results of [48] and [49].

Let $C_i \equiv C_i(\mathbf{Z}_i, \mathbf{X})$ and $L_i \equiv L_i(\mathbf{Z}_i, \mathbf{X})$ be the cost and the length of the i^{th} embedded cycle ($i = 1, 2, \dots$), where $\mathbf{X}$ denotes the array of inter-arrival times of Poisson demands within the i^{th} embedded cycle, independent of $(\mathbf{Z}_1, \dots, \mathbf{Z}_i)$.

Also, let $\mathcal{C}(\mathbf{z}) = \mathbb{E}[C_i(\mathbf{Z}_i, \mathbf{X})|\mathbf{Z}_i = \mathbf{z}]$ and $\mathcal{L}(\mathbf{z}) = \mathbb{E}[L_i(\mathbf{Z}_i, \mathbf{X})|\mathbf{Z}_i = \mathbf{z}]$ for $i \geq 1$. The expectations are independent of the index i when $\mathbf{Z}_i = \mathbf{z}$ is given and are calculated with respect to the inter-arrival times of Poisson demands $\mathbf{X}$. Let $F(\cdot)$ be the limiting distribution function of $\{\mathbf{Z}_i, i \geq 1\}$. Then, we use the following objective function:

$$\phi = \frac{\int_{\mathbf{z}} \mathcal{C}(\mathbf{z})dF(\mathbf{z})}{\int_{\mathbf{z}} \mathcal{L}(\mathbf{z})dF(\mathbf{z})}. \quad (5.18)$$

That is, we use the ratio of the expected cost of an embedded cycle in the limit divided by the length of such a cycle in the limit. We do not provide a rigorous proof here but the usage of this objective function is motivated by the results of (i) Ross [48] where it is shown that under a mild regularity condition which

states that transitions in the embedded Markov process do not occur too quickly, it holds that

$$\phi_1(\mathbf{x}) = \overline{\lim}_{t \rightarrow \infty} \mathbb{E}\left[\frac{C(t)}{t} | \mathbf{Z}_1 = \mathbf{x}\right] = \phi_2(\mathbf{x}) = \overline{\lim}_{n \rightarrow \infty} \frac{\mathbb{E}[\sum_{i=1}^n \mathbf{C}_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{x}]}{\mathbb{E}[\sum_{i=1}^n \mathbf{L}_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{x}]}, \quad (5.19)$$

where $C(t)$ is the cost incurred in interval $[0, t]$.

(ii) As we have shown $\{Z_n, n \geq 1\}$ converges in distribution, and we have $Z_n \rightarrow Z$ and $F_n \rightarrow F$. Since $\{Z_n, n \geq 1\}$ is a bounded sequence, we have $\mathbb{E}(Z_n) \rightarrow \mathbb{E}(Z)$ by bounded convergence theorem.

(iii) Finally, since both the cost and the length of the embedded cycles are continuous and bounded we assume that $\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n C_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{z}\right] \rightarrow \mathbb{E}[C(\mathbf{Z}, \mathbf{X})] = \mathbb{E}[\mathbb{E}[C(\mathbf{Z}, \mathbf{X}) | \mathbf{Z}]] = \mathbb{E}[\mathcal{C}(\mathbf{Z})] = \int_{\mathbf{z}} \mathcal{C}(\mathbf{z}) dF(\mathbf{z})$.

The objective function given in Equation (5.18) implicitly involves the policy parameters (Q, r, ξ) . Hence, we explicitly write now the objective function of our policy as

$$\phi = TC(Q, r, \xi) = \frac{K + \int_{\mathbf{z}} (hE[OH | \mathbf{Z} = \mathbf{z}] + pE[P | \mathbf{Z} = \mathbf{z}] + \pi E[LS | \mathbf{Z} = \mathbf{z}]) dF(\mathbf{z})}{\int_{\mathbf{z}} E[CL | \mathbf{Z} = \mathbf{z}] dF(\mathbf{z})}, \quad (5.20)$$

and the optimization problem is

$$\min_{Q, r, \xi} TC(Q, r, \xi)$$

s.t:

$$\frac{\mathbb{E}(LS)}{\lambda \mathbb{E}(CL)} \leq \alpha,$$

where $\mathbb{E}(LS)/\lambda \mathbb{E}(CL)$ represents the fraction of demands lost over a long time which is denoted by $FL(\xi)$.

Chapter 6

Another Time-Based Policy and Special Policies

In this chapter, we consider another time-based control policy and some special policies. They are (Q, r, T) , (Q, r) , (Q, ξ) , and (Q, T) policies. The (Q, r, T) policy is a time-based policy which is based on the elapsed time and inventory level. This policy is different from the (Q, r, ξ) policy, in the sense that in the (Q, r, T) policy the time is considered for reordering, while in the (Q, r, ξ) policy the lifetime of the oldest batch is taken into account. The (Q, r) , (Q, ξ) , and (Q, T) policies are special policies which are achieved by three-parameter policies. The (Q, r) policy is attained by setting $T = \infty$ and $\xi = 0$ in the (Q, r, T) and (Q, r, ξ) policies, respectively. The reason is that by setting $T = \infty$ and $\xi = 0$, we never trigger an order by reaching the time threshold T and lifetime threshold ξ in the (Q, r, T) and (Q, r, ξ) policies, respectively. The (Q, T) and (Q, ξ) policies are the special case of the (Q, r, T) and (Q, r, ξ) policies, respectively which are achieved by setting $r = -\infty$ in both. The reason is that by setting $r = -\infty$, we never trigger an order by reaching the inventory level threshold r . It should be noted that the definition of U_n and CL_n are the same in all of the policies proposed here. That is, U_n is a period of time started from the beginning of embedded cycle n till reorder point, and CL_n is the length of embedded cycle n .

At the end of this chapter, we deal with the special region of search space where r is an integer multiple of Q . In this special region, the inventory system behaves in different manner. Hence, we need to consider this special region separately.

6.1 The (Q, r, T) Policy

In this section we consider the (Q, r, T) policy and derive the system characteristics under this policy analogous to the (Q, r, ξ) policy. The (Q, r, T) policy associates the time (regardless of the age of items) into the control policy which works as follows:

Policy: *An order of size Q is placed whenever the inventory position crosses r by demand or perishing, or T units of time is elapsed since the beginning of the embedded cycle, whichever occurs first.*

The embedded cycle definition, the youngest and oldest batch definitions are identical to the (Q, r, ξ) policy. Different realizations (events) in the (Q, r, T) policy is also the same as (Q, r, ξ) policy. The only difference is that the reorder point is either when stock level crosses r or T units of time is elapsed since the beginning of the embedded cycles. This policy is also different from the continuous (Q, r, T) policy proposed by Lowalekar and Ravichandran in [36], in the sense that the parameter T defined here is the time elapsed since the beginning of the embedded cycle, while the parameter T in [36] is the time elapsed since the instance we order a new batch.

In Figure 6.1, we depict a sample path of the inventory position and inventory level of the (Q, r, T) policy. In the (Q, r, T) policy there is a possibility that the batch perishes before reordering (cycle 7 in Figure 6.1), although in the (Q, r, ξ) policy there is not such a possibility. In this event, Z^m takes a value of $\tau + L$. Therefore, there is a mass point at $\tau + L$ in the effective lifetime distribution under the (Q, r, T) policy.

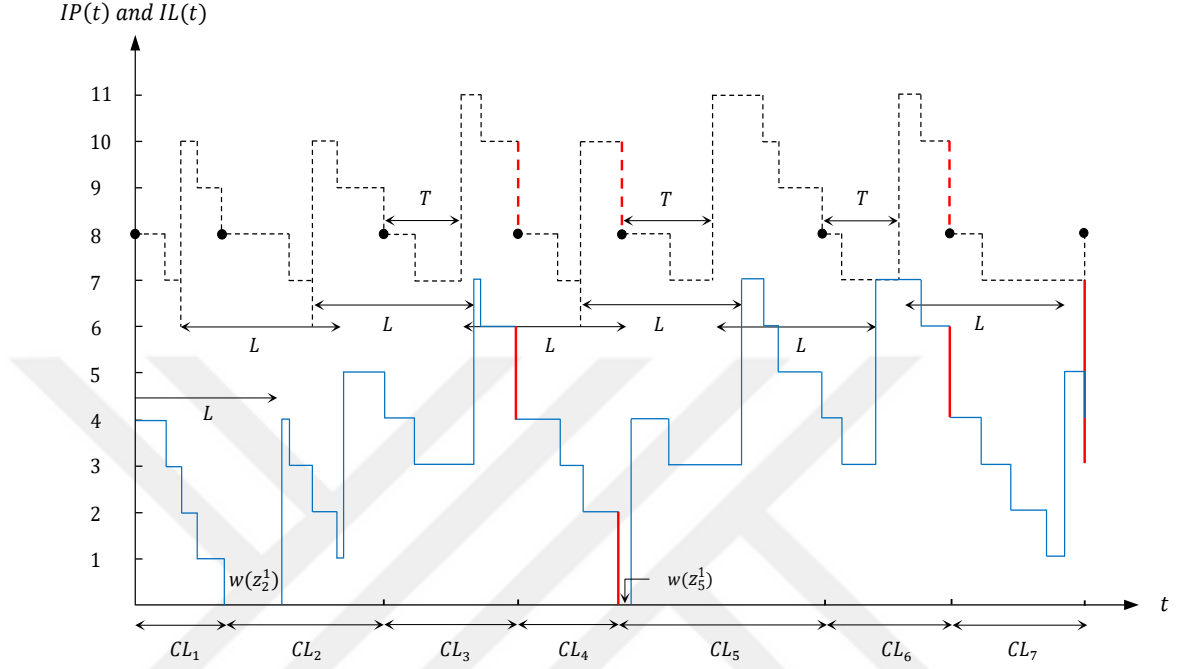


Figure 6.1: A sample path of the inventory position and inventory level for $Q = 4$ and $r = 6$ ($m = 2$) in the (Q, r, T) policy

Considering different events, we can conclude that the $(n+1)^{th}$ effective lifetime vector can be generally expressed as follows:

$$\begin{aligned} Z_{n+1}^i &= Z_n^{i+1} - CL_n, \text{ for } i = 1, 2, \dots, m-1, \\ Z_{n+1}^m &= \tau + L - (CL_n - U_n), \end{aligned} \quad (6.1)$$

where $U_n = \min\{\min\{T, w(Z_n^1) + X_{mQ-r}\}, Z_n^1\}$, $CL_n = w(Z_n^1) + \min\{\min\{Z_n^1, \tau\}, X_Q\}$ and $0 \leq Z_n^i \leq \tau + L$, $i = 1, 2, \dots, m$. According to (6.1), we can conclude that the effective lifetime vector $\mathbf{Z}_{n+1}$ at the beginning of the $(n+1)^{th}$ embedded cycle is completely determined by $\mathbf{Z}_n$ and the Poisson demand arrival process after the stopping time T_n . Therefore, the embedded process $\{\mathbf{Z}_n, n \geq 1\}$ has the Markovian property. The state space of the system is identical to the (Q, r, ξ) policy. In Figures 6.2-6.6, we provide the realization of the system under the (Q, r, T) policy. Note that realization 5 occurs in the (Q, r, T) policy but in the (Q, r, ξ) policy does not.

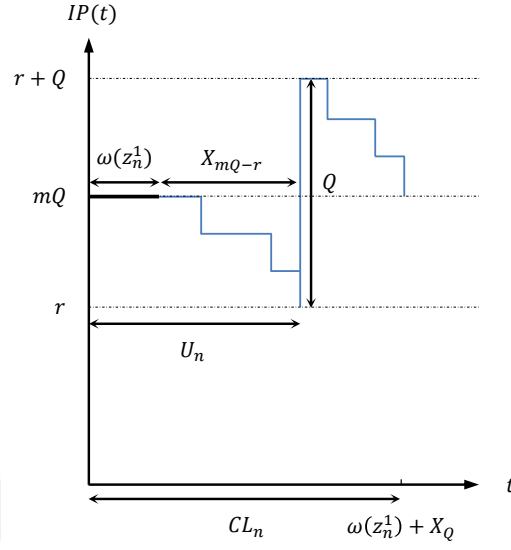


Figure 6.2: Event 1, $w(z_n^1) + X_{mQ-r} < T$, $w(z_n^1) + X_Q < z_n^1$, $CL_n = w(z_n^1) + X_Q$, $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [X_Q - X_{mQ-r}]$

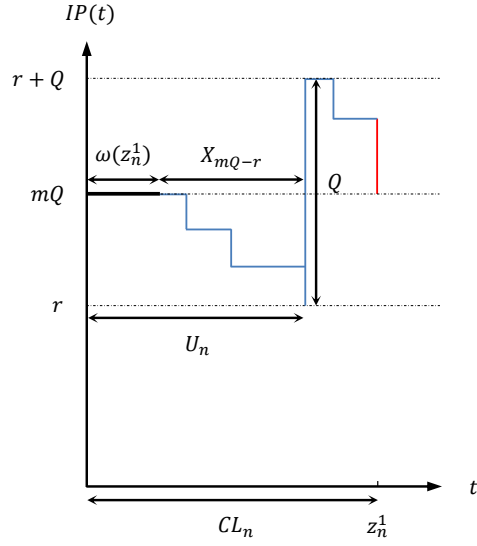


Figure 6.3: Event 2, $w(z_n^1) + X_{mQ-r} < T$, $w(z_n^1) + X_Q > z_n^1$, $CL_n = z_n^1$, $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [z_n^1 - (w(z_n^1) + X_{mQ-r})]$

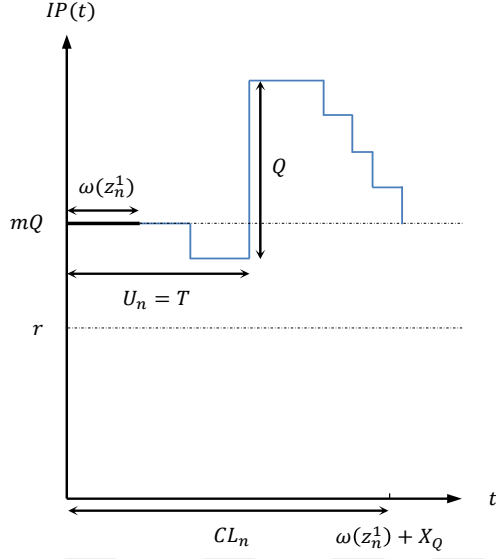


Figure 6.4: Event 3, $w(z_n^1) + X_{mQ-r} > T$, $w(z_n^1) + X_Q < z_n^1$, $CL_n = w(z_n^1) + X_Q$, $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [w(z_n^1) + X_Q - T]$

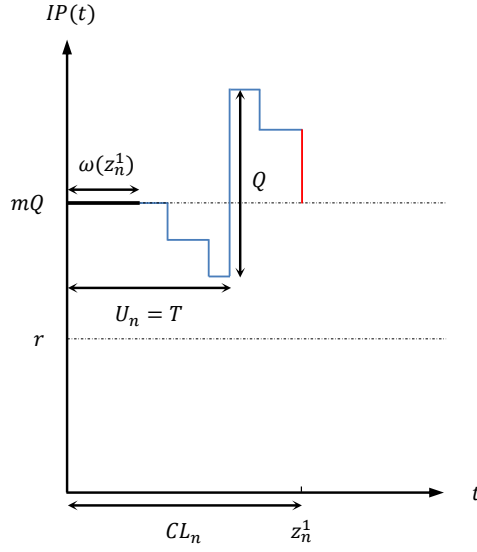


Figure 6.5: Event 4, $w(z_n^1) + X_{mQ-r} > T$, $w(z_n^1) + X_Q > z_n^1$, $CL_n = z_n^1$, $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - (z_n^1 - T)$

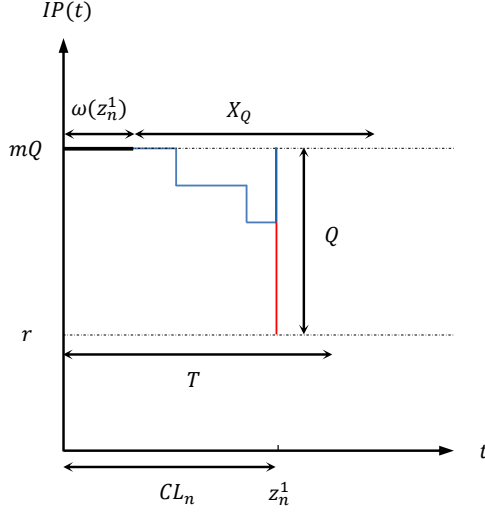


Figure 6.6: Event 5, $w(z_n^1) + X_{mQ-r} > z_n^1$, $z_n^1 < T$, $CL_n = z_n^1$, $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L$

Then, considering the realizations of the system, and the Markovian property of the lifetime distribution, we have the following result.

Theorem 4. (*Transition Probability Function of $\mathbf{Z}_n$*).

Let $\mathbb{P}(\mathcal{S}|\mathbf{x}) \equiv \mathbb{P}(Z_{n+1}^i \leq z^i, i = 1, \dots, m | \mathbf{Z}_n = \mathbf{x})$. Then, for $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$ and for $x^1 \geq m_x$ we have:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) = & \mathbb{1}_{\{m_T \geq \tau + L - z^m\}} \left[\int_{m_c - m_\tau}^{m_f} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) + \bar{F}_{mQ-r}(m_f) \right] \\ & + \int_0^{\min\{m_f - (\tau + L - z^m), m_c - m_\tau\}} \bar{F}_{r-(m-1)Q}(\max\{m_x - m_\tau - t, \tau + L - z^m\}) dF_{mQ-r}(t), \end{aligned} \quad (6.2)$$

where $m_T = \max\{0, x^1 - T\}$, $m_c = \min\{x^1, T\}$, and $\ell = \max\{m_x - m_\tau, \tau + L - z^m - m_\tau + m_c\}$.

Proof. We can rewrite Z_{n+1}^i for $i = 1, \dots, m-1$ and Z_{n+1}^m as follows:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(Z_{n+1}^1 \leq z^1; Z_{n+1}^2 \leq z^2; \dots; Z_{n+1}^m \leq z^m | \mathbf{Z}_n = (x^1, x^2, \dots, x^m)) \\
&= \mathbb{P}(x^2 - CL_n \leq z^1; x^3 - CL_n \leq z^2; \dots; \tau + L - (CL_n - U_n) \leq z^m) \\
&= \mathbb{P}(CL_n \geq \max_{1 \leq i \leq m-1} \{x^{i+1} - z^i\}; CL_n - U_n \geq \tau + L - z^m) \\
&= \mathbb{P}(CL_n \geq m_x; CL_n - U_n \geq \tau + L - z^m).
\end{aligned} \tag{6.3}$$

Using the Expression (6.3) and considering different realizations of CL_n and U_n for different regions A, B, and C expressed below, we can derive the transition probability function. Let's denote X_{mQ-r} by t and $X_{r-(m-1)Q}$ by u .

A: $\max\{T, \tau\} < x^1 \leq \tau + L$

- | | |
|--|--|
| A1. If $t < T - (x^1 - \tau)$, | $CL_n = x^1 - \tau + \min\{\tau, X_Q\}$ and $U_n = x^1 - \tau + t$. |
| A2. If $T - (x^1 - \tau) < t < \tau$, | $CL_n = x^1 - \tau + \min\{\tau, X_Q\}$ and $U_n = T$. |
| A3. If $\tau < t$, | $CL_n = x^1$ and $U_n = T$. |

B: $\min\{T, \tau\} < x^1 \leq \max\{T, \tau\}$

- | | |
|--------------------------------------|---|
| B1. If $t < \min\{T, \tau\}$, | $CL_n = m_\tau + \min\{m_f, X_Q\}$ and $U_n = m_\tau + t$. |
| B2. If $\min\{T, \tau\} < t < m_f$, | $CL_n = \min\{x^1, X_Q\}$ and $U_n = T$. |
| B3. If $m_f < t$, | $CL_n = x^1$ and $U_n = m_c$. |

C: $0 \leq x^1 \leq \min\{T, \tau\}$

- | | |
|--------------------|---|
| C1. If $t < x^1$, | $CL_n = \min\{x^1, X_Q\}$ and $U_n = t$. |
| C2. If $t > x^1$, | $CL_n = x^1$ and $U_n = x^1$. |

For Region A:

A1.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(X_{mQ-r} < T - (x^1 - \tau); x^1 - \tau + \min\{\tau, X_Q\} \geq m_x; \\
&\quad \min\{\tau, X_Q\} - t \geq \tau + L - z^m) \\
&= \mathbb{P}(X_{mQ-r} < T - (x^1 - \tau); x^1 \geq m_x; \tau \geq \tau + L - z^m + t; \\
&\quad X_Q \geq m_x - m_\tau; X_Q \geq \tau + L - z^m + t) \\
&= \int_0^{\min\{z^m - L, T - (x^1 - \tau)\}} \int_{\max\{m_x - m_\tau - t, \tau + L - z^m\}}^\infty dF_{r-(m-1)Q}(u) dF_{mQ-r}(t).
\end{aligned}$$

A2.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(T - (x^1 - \tau) < X_{mQ-r} < \tau; x^1 - \tau + \min\{\tau, X_Q\} \geq m_x; \\
&\quad x^1 - \tau + \min\{\tau, X_Q\} - T \geq \tau + L - z^m) \\
&= \mathbb{P}(T - (x^1 - \tau) < X_{mQ-r} < \tau; x^1 \geq m_x; \tau \geq \tau + L - z^m - m_\tau + T; \\
&\quad X_Q \geq m_x - m_\tau; X_Q \geq \tau + L - z^m - m_\tau + T) \\
&= \mathbb{1}_{\{x^1 - T \geq \tau + L - z^m\}} \int_{T - (x^1 - \tau)}^{\tau} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t),
\end{aligned}$$

where $\ell = \max\{m_x - m_\tau, \tau + L - z^m - m_\tau + T\}$.

A3.

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(X_{mQ-r} > \tau; x^1 \geq m_x; x^1 - \tau \geq \tau + L - z^m) \\
&= \mathbb{1}_{\{x^1 - T \geq \tau + L - z^m\}} \int_{\tau}^{\infty} dF_{mQ-r}(t).
\end{aligned}$$

Then, we can simply express the transition probability function for Region A as follows:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{1}_{\{x^1 - T \geq \tau + L - z^m\}} \left[\int_{T - (x^1 - \tau)}^{\tau} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) + \bar{F}_{mQ-r}(\tau) \right] \\
&\quad + \int_0^{\min\{z^m - L, T - (x^1 - \tau)\}} \int_{\max\{m_x - m_\tau - t, \tau + L - z^m\}}^{\infty} dF_{r-(m-1)Q}(u) dF_{mQ-r}(t).
\end{aligned} \tag{6.4}$$

Similarly, we can derive the transition probability function for Regions B and C. Finally, we can express all regions by a single term as Expression (6.2). Also, note that any negative term should be equalized to zero, if there is any. For example, the term $T - (x^1 - \tau)$ can take a negative value in which case it is set to zero. $\square$

Setting $m_x = 0$, $z^m = z$, $x^1 = x$, $\ell = \tau + L - z + m_c - m_\tau$ we will achieve the transition probability function for the case $m = 1$ as follows:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{1}_{\{m_T \geq \tau + L - z\}} \left[\int_{m_c - m_\tau}^{m_f} \bar{F}_r(\ell - t) dF_{Q-r}(t) + \bar{F}_{Q-r}(m_f) \right] \\
&\quad + \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{m_f - (\tau + L - z), m_c - m_\tau\}).
\end{aligned} \tag{6.5}$$

Considering different regions for the case $m = 1$ stated above, we can conclude that the $(n + 1)^{th}$ effective lifetime vector can be generally expressed as

$$Z_{n+1} = \tau + L - (CL_n - U_n), \quad (6.6)$$

where $U_n = \min\{\min\{T, w(Z_n) + X_{Q-r}\}, Z_n\}$, $CL_n = w(Z_n) + \min\{\min\{Z_n, \tau\}, X_Q\}$ and $0 \leq Z \leq \tau + L$. Our results show that we have two mass points at T and $\tau + L$. In Event 5, the effective lifetime of a new cycle turns out to be $\tau + L$. On the other hand, if a cycle starts with effective lifetime of $\tau + L$ and reordering is triggered by T and the cycle ends up by perishing, then $Z_{n+1} = \tau + L - (CL_n - U_n) = Z_{n+1} = \tau + L - (\tau + L - T) = T$ implying that we have a mass point at T . Therefore, we have two mass points at T and $\tau + L$. Figure 6.7 depicts a sample of effective lifetime distribution for $Q = 9$, $r = 7$, $T = 1.2$, $L = 0.5$, $\tau = 1.5$, and $\lambda = 1$.

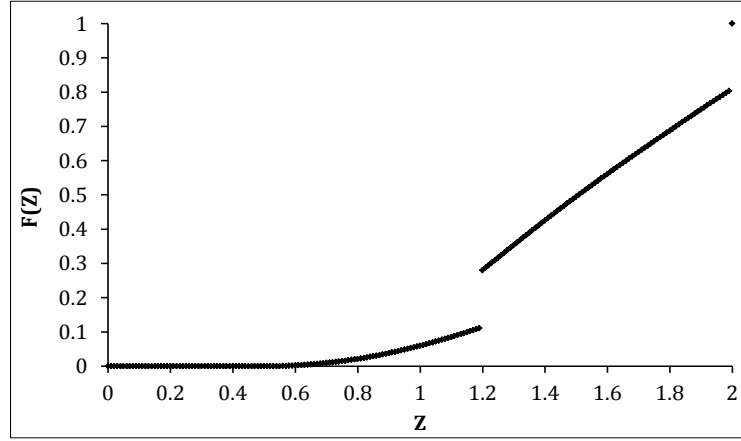


Figure 6.7: Effective lifetime distribution for $Q = 9$, $r = 7$, $T = 1.2$, $L = 0.5$, $\tau = 1.5$, and $\lambda = 1$

Figure 6.8 depicts the (Q, r, T) policy with two outstanding orders, where we have a mass point at $\tau + L$.

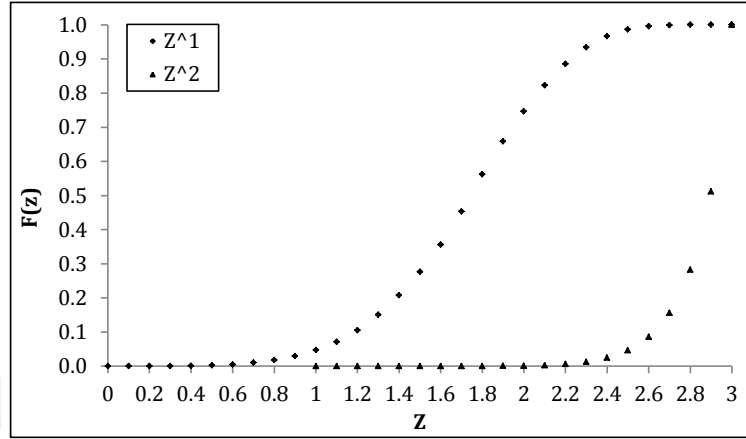


Figure 6.8: Effective lifetime distribution for $Q = 7$, $r = 8$, $T = 1.7$, $L = 1$, $\tau = 2$, and $\lambda = 6$

We can easily prove the ergodicity of the process for the (Q, r, T) policy analogous to the (Q, r, ξ) policy. Moreover, the operating characteristics for a given lifetime z for the (Q, r, T) policy is the same as the (Q, r, ξ) policy as well. Therefore, the effective lifetime distributions in the (Q, r, T) and (Q, r, ξ) policies make the expected value of the operating characteristics different from each other. It implies that the optimal ordering policy is the one which is able to tune the operating characteristics in such a way that the total cost per unit time is minimized. Tuning the operating characteristic values is done by the effective lifetime distribution function.

6.2 The (Q, r) Policy

The (Q, r, ξ) policy has a special case namely (Q, r) policy which works as follows:

Policy: *An order of size Q is placed whenever the inventory position crosses r by demand, or perishing.*

This policy is analyzed by [27] for $m = 1$. Berk et al. in [45] generalized their results to allow for more than one outstanding orders ($r \geq Q$). They

provided the operating characteristics of the inventory system and construct the exact expected cost rate expression using a renewal theoretic approach. Through a numerical study they indicated that allowing for multiple outstanding orders ($r \geq Q$) may result in significant savings in the expected cost rate, compared to the case with $r < Q$. In particular, when the fixed lifetimes are short and the ordering costs are low, expected costs can be reduced by more than half.

For the general case of $m \geq 1$, the (Q, r) policy can be achieved by setting $\xi = 0$ in the (Q, r, ξ) policy or $T = \tau + L$ in the (Q, r, T) policy. To see how the policy operates, let m be a positive integer defined as $m := \lceil \frac{r}{Q} \rceil$ for a (Q, r) pair, where $\lceil x \rceil$ denotes the smallest integer greater than x . We have $(m-1)Q \leq r < mQ$. Under the above policy, items join stock in batches of Q units and all items in a batch have the same lifetime. So long as inventory depletion occurs through demand, inventory position decreases incrementally by one unit and a reorder decision is triggered when the inventory position hits r . However, it is also possible that items in the oldest batch simultaneously perish; then, the reorder decision is triggered either when the inventory position hits r or crosses r momentarily depending on whether or not r is an integer multiple of Q . If the reorder point is an integer multiple of the lot size ($r = (m-1)Q$), the inventory position hits r at instances of order placement by perishing; otherwise, reordering is triggered as the inventory position *crosses* r . In both cases, the inventory position drops exactly to the level $(m-1)Q$. Furthermore, we make the following observations: (i) At steady state, the inventory position takes on values in the interval $[r+1, Q+r]$ in both cases $(m-1)Q \leq r < mQ$ and $r = (m-1)Q$. (ii) There may be at most $m+1$ different batches if $(m-1)Q \leq r < mQ$ and m different batches if $r = (m-1)Q$ in the inventory system. (iii) Between two consecutive order placements, there is always an instance when the inventory position hits mQ for the first time. This point will become important when we discuss the embedded cycles in the subsequent sections. (iv) The inventory system can be fully characterized by the inventory position at any point in time and the times elapsed since each of the batches in the system was ordered.

To illustrate the above particulars of the policy, we provide the following examples. Suppose $Q = 5$ and $r = 3$. Then, $m = \lceil \frac{r}{Q} \rceil = 1$. At any instance at most

one order may be outstanding. The policy orders 5 units when the inventory position drops to 3 by demand or to 0 ($= (m - 1)Q$) by perishing, whichever occurs first. At steady state, the inventory position takes values in the interval $[4, 8]$. Suppose now, $Q = 5$ and $r = 5$ so that $m = 2$ which implies that there may be at most two outstanding orders at any time. The policy orders 5 units whenever the inventory position drops to 5 by demand or perishing. At steady state, the inventory position takes on values between $[6, 10]$. Lastly, suppose that $Q = 5$ and $r = 7$ so that $m = 2$ again. There may be at most two outstanding orders at any time. An order of 5 units is ordered when the inventory drops to 7 by demand or crosses 7 dropping momentarily to 5 by perishing of the oldest batch, whichever occurs first. At steady state, the inventory position takes on values between $[8, 12]$.

A sample path of the (Q, r) policy is depicted in Figure 6.9. You can follow the embedded cycles as well as inventory position and inventory level in this figure. We use this sample path to depict different realizations of the inventory system under the (Q, r) policy. These realizations are discussed later. Note that the embedded cycle definition, the relationship between inventory level and inventory position, the stock-out period definition in all policies proposed here including (Q, r) , (Q, T) , and (Q, ξ) policies are the same. We use the black, blue and red graphs to represent the inventory position, inventory level and amount of perished items in all sample path of these policies.

Regarding the reorder point, at which a new order is placed, the (Q, r) policy is based on the stock level; however, the (Q, r, ξ) policy is based on both stock level and remaining lifetime of the items. Therefore, it is expected that the (Q, r, ξ) policy outperforms the (Q, r) policy defined above. Similarly, we expect that the (Q, r, T) policy outperforms the (Q, r) policy. Since the proposed (Q, r, ξ) policy has three control parameters, finding the optimal solution needs more search space consideration. Hence, for cases in which two-parameter policy like (Q, r) performs good enough, we are more inclined to use (Q, r) policy rather than three-parameter policy like (Q, r, ξ) or (Q, r, T) policy.

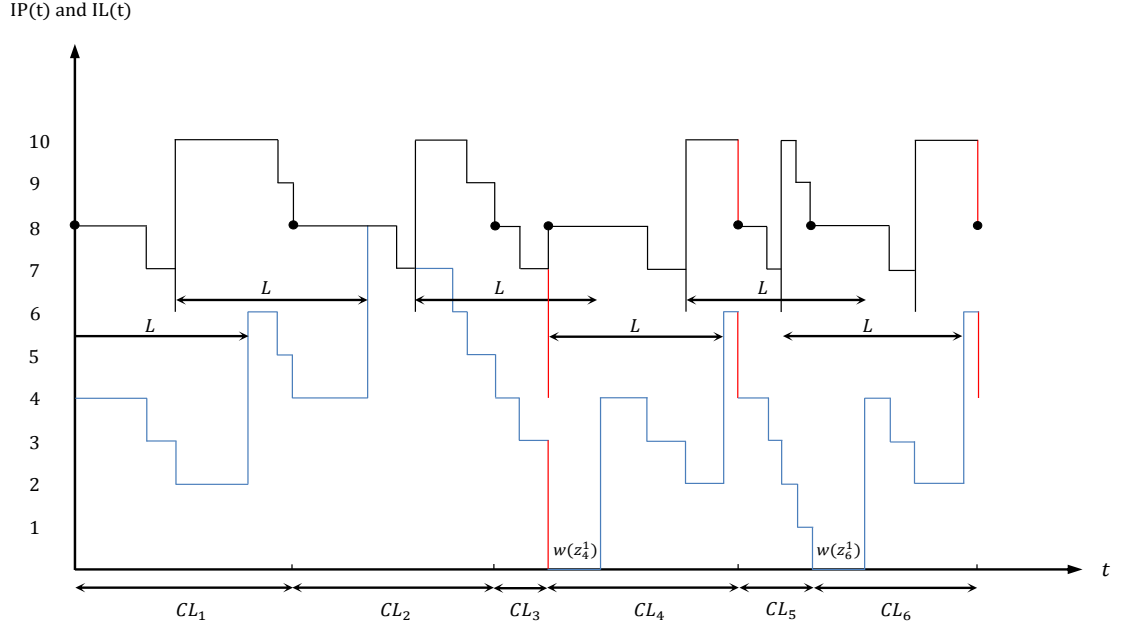


Figure 6.9: A sample path of the inventory position and inventory level for $Q = 4$ and $r = 6$ in the (Q, r) policy

Realizations of the system in the (Q, r) policy are as follows.

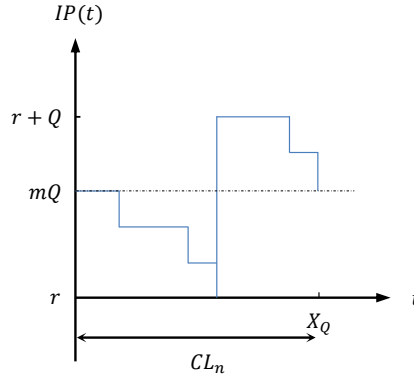


Figure 6.10: Event 1, $X_Q < z_n^1$ and $CL_n = X_Q$

In this event, the embedded cycle starts with mQ items. When the inventory position hits the reorder point r , it is raised to mQ and after depletion again to mQ the embedded cycle ends. Cycles 1, 2, and 5 in Figure 6.9 represent this event.

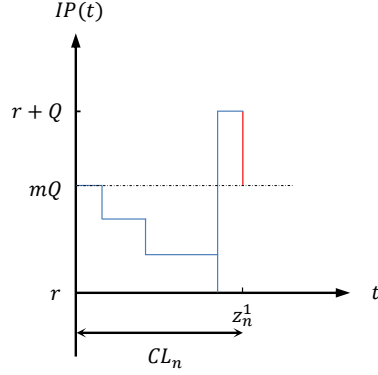


Figure 6.11: Event 2, $X_{mQ-r} < z_n^1$; $X_Q > z_n^1$ and $CL_n = z_n^1$

This event is similar to Event 1, but after reordering the inventory position hits mQ by perishing and embedded cycle ends. Cycles 4 and 6 in Figure 6.9 represent this event.

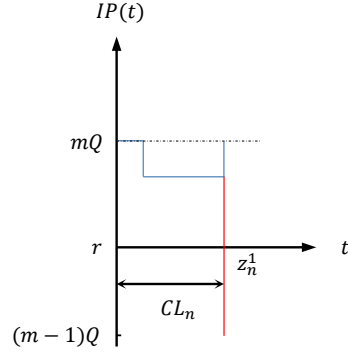


Figure 6.12: Event 3, $X_{mQ-r} > z_n^1$ and $CL_n = z_n^1$

In this event, before hitting the reorder point some of items are perished so that the inventory position drops to $(m-1)Q$ by crossing r . In this event, upon placing an order the inventory position is raised to mQ and the embedded cycle ends. Cycle 3 in Figure 6.9 represents this event.

The transition probability function for the cases $m = 1$ and $m > 1$ in the (Q, r) policy are achieved by setting $\xi = 0$ in the that of the (Q, r, ξ) policy.

Transition probability function of $\mathbf{Z}_n$ for case $r > (m-1)Q$:

For $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$ and for $x^1 \geq m_x$ we have:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) &= \int_0^{m_f - (\tau + L - z^m)} \bar{F}_{r-(m-1)Q}(\max\{m_x - m_\tau - t, \tau + L - z^m\}) dF_{mQ-r}(t) \\ &\quad + \mathbb{1}_{\{z^m = \tau + L\}} \bar{F}_{mQ-r}(m_f). \end{aligned} \quad (6.7)$$

Transition probability function of Z_n for case $m = 1$:

$$\mathbb{P}(\mathcal{S}|x) = \mathbb{1}_{\{z = \tau + L\}} \bar{F}_{Q-r}(m_f) + \bar{F}_r(\tau + L - z) F_{Q-r}(m_f - (\tau + L - z)). \quad (6.8)$$

Transition probability function of $\mathbf{Z}_n$ for special region $r = (m-1)Q$:

For $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$ and for $x^1 \geq m_x$ we have:

$$\mathbb{P}(\mathcal{S}|\mathbf{x}) = \mathbb{1}_{\{z^m = \tau + L\}} \bar{F}_Q(m_x - m_\tau). \quad (6.9)$$

Transition probability function of Z_n for special region $r = 0$:

$$\mathbb{P}(\mathcal{S}|x) = \mathbb{1}_{\{z = \tau + L\}}. \quad (6.10)$$

For more detail regarding the poof ot transition probability function for different regions above, we refer the readers to [45].

Remark. The operating characteristics for a given lifetime z in the (Q, r) policy, the (Q, r, ξ) policy, and the (Q, r, T) policy are identical. It implies that the effective lifetime distribution affects the total cost per unit time.

A sample of the effective lifetime distribution in the case $m = 1$ is depicted in Figure 6.13.

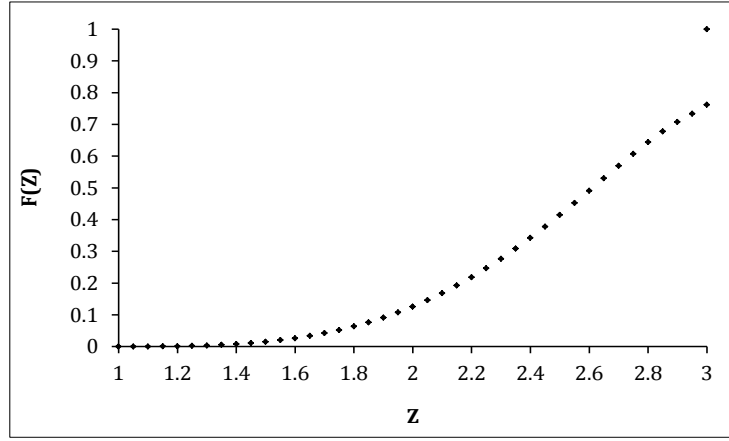


Figure 6.13: Effective lifetime distribution for $Q = 5$, $r = 2$, $L = 1$, $\tau = 2$, and $\lambda = 2$ for the (Q, r) policy, ($m = 1$)

A sample of the effective lifetime distribution in the case $m = 2$ is depicted in Figure 6.14.

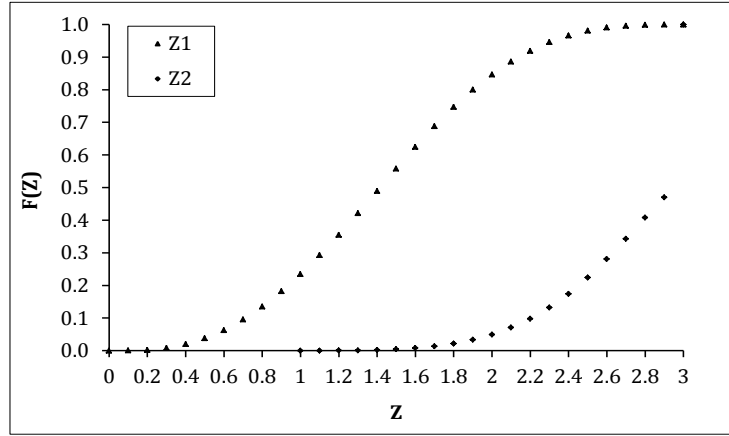


Figure 6.14: Effective lifetime distribution for $Q = 5$, $r = 7$, $L = 1$, $\tau = 2$, and $\lambda = 2$ for the (Q, r) policy, ($m = 2$)

A sample of the effective lifetime distribution in the case $m = 2$ with special region $r = Q$ is depicted in Figure 6.15.

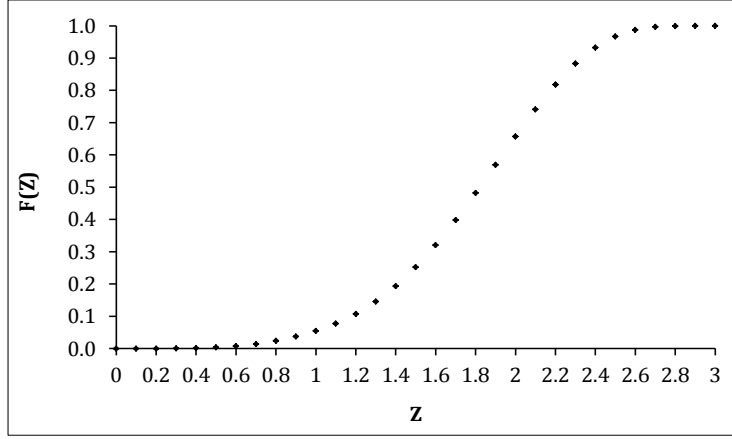


Figure 6.15: Effective lifetime distribution for $Q = 6$, $r = 6$, $L = 1$, $\tau = 2$, and $\lambda = 5$. Z_2 takes a value of $\tau + L$ always

6.2.1 Comparison of $m = 1$ and $m > 1$ in the (Q, r) Policy

In this subsection, we provide some examples to illustrate the sensitivity of the optimal values of the policy parameters w.r.t. operating environment parameters and, the restrictiveness of the $r < Q$ assumption and benefits ensuing from alleviating this restriction. These example are based on the data set provided in Berk et al. [45]. In this study, they fix $L = 1$, $h = 1$, $\lambda = 10$ and $\pi = 40$, where π is the unit lost sales cost. We vary the lifetime ($\tau = 2, 2.5$, and 3), the fixed ordering cost ($K = 5, 10, 50$, and 100) and the unit perishing cost ($p = 10$ and 50). Let's denote the expected fraction of stock-out time within an embedded cycle by FR defined as $FR = \int_{\mathbf{z}} w(z_1) dF(\mathbf{z}) / \int_{\mathbf{z}} \mathbb{E}[CL|\mathbf{Z} = \mathbf{z}] dF(\mathbf{z})$. The optimization problem has the objective function with cost rate including holding, shortage, perishing, and lost sale costs without any service level constraint. Also, similar to other policies proposed here, for optimization, we used exhaustive search over a broad range of policy parameter values, and observed unimodality in all of the cases considered.

We present our results in Tables 6.1 and 6.2. The optimal lotsize-reorder point pairs, the corresponding expected total cost rate and stock-out fractions

are shown, respectively, by (Q^*, r^*) , $\mathbb{E}(TC^*)$ and FR^* ; the corresponding maximum number of outstanding orders is m^* . Their counterparts under the $r < Q$ restriction ($m = 1$) are shown by $(Q, r)_1$, $\mathbb{E}(TC)_1$, and FR_1 . The percentage deviation of the $\mathbb{E}(TC)_1$ from the optimal is denoted by gap%.

Table 6.1: Comparison of the optimal and one-outstanding-order-restricted lotsize-reorder policies; $p = 10$, $L = 1$, $h = 1$, $\lambda = 10$ and $\pi = 40$.

Parameters	K	$(Q, r)_1$	$\mathbb{E}(TC)_1$	FR_1	(Q^*, r^*)	$\mathbb{E}(TC^*)$	FR^*	m^*	gap%
$\tau = 2$	5	(15, 14)	28.53	98.55	(8, 16)	21.72	99.40	3	31.35
	10	(15, 14)	32.07	98.55	(9, 16)	27.91	99.37	2	14.91
	50	(15, 14)	60.58	98.52	(15, 14)	60.58	98.52	1	0.00
	100	(17, 13)	94.63	97.01	(17, 13)	94.63	97.01	1	0.00
$\tau = 2.5$	5	(15, 14)	22.50	98.73	(9, 16)	19.37	99.45	2	16.16
	10	(15, 14)	25.86	98.73	(11, 16)	24.42	99.48	2	5.90
	50	(17, 14)	51.46	98.76	(17, 14)	51.46	98.76	1	0.00
	100	(19, 14)	79.54	98.62	(19, 14)	79.54	98.62	1	0.00
$\tau = 3$	5	(16, 15)	20.10	99.35	(10, 16)	18.63	99.50	2	7.89
	10	(16, 15)	23.23	99.35	(13, 16)	22.84	99.58	2	1.71
	50	(20, 14)	45.95	98.95	(20, 14)	45.96	98.95	1	0.00
	100	(23, 14)	69.48	98.77	(23, 14)	69.48	98.77	1	0.00

Table 6.2: Comparison of the optimal and one-outstanding-order-restricted lotsize-reorder policies; $p = 50$, $L = 1$, $h = 1$, $\lambda = 10$ and $\pi = 40$.

Parameters	K	$(Q, r)_1$	$\mathbb{E}(TC)_1$	FR_1	(Q^*, r^*)	$\mathbb{E}(TC^*)$	FR^*	m^*	gap%
$\tau = 2$	5	(13, 12)	42.75	95.90	(7, 15)	26.49	98.81	3	61.38
	10	(13, 12)	46.55	95.90	(8, 14)	34.57	98.06	2	34.65
$\tau = 2.5$	5	(14, 13)	28.69	97.71	(7, 16)	21.16	99.35	3	35.59
	10	(14, 13)	32.22	97.71	(9, 15)	27.09	99.00	2	18.94
$\tau = 3$	5	(15, 14)	22.38	98.76	(9, 16)	19.13	99.45	2	16.99
	10	(15, 14)	25.68	98.76	(11, 16)	24.11	99.52	2	6.51

The sensitivity of the optimal values of policy parameters (Q^*, r^*) w.r.t. the parameters of the operating environment is as follows. Both Q^* and r^* are non-decreasing in lifetime τ , but they are non-increasing in unit perishing cost p . But the maximum number of outstanding orders, m is non-increasing in τ whereas it is also non-decreasing in p , which implies that for systems with higher unit perishing costs such as pharmaceuticals the $r < Q$ restriction is more costly. As the fixed ordering cost K increases, Q^* is non-decreasing and r^* is non-increasing such that m is overall non-increasing.

We observe that the savings can be significant when the restriction is alleviated.

The deviations are increasing as the optimal number of outstanding orders (m^*) increases, as the fixed ordering cost decreases, as the unit perishing cost increases, and as lifetime decreases.

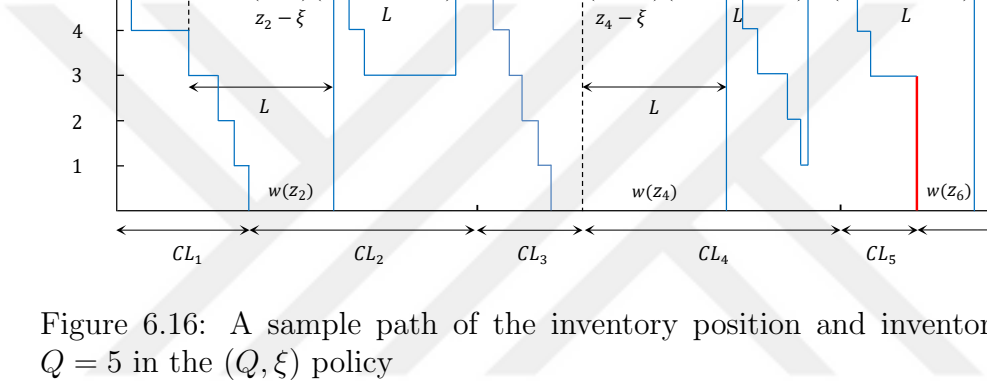
In summary, from our numerical study, we observed that the contribution of the generalization to allow more than one outstanding orders becomes more pronounced when ordering costs and lifetimes are small, perishing and lost sales costs are high. Even in systems holding items with the long lifetime, small fixed ordering cost enforces the system to have multiple outstanding orders. Among all parameters, lifetime and fixed ordering cost are the most important ones playing very important rule in sensitivity of other parameters and the number of outstanding orders. The batch size is affected directly by unit perishing cost and reorder point by unit lost sales cost. The number of orders outstanding is very sensitive to any increase in lead time.

6.3 The (Q, ξ) Policy

Another special case of the (Q, r, ξ) policy is the (Q, ξ) *policy* which works as follows:

Policy: *An order of size Q is placed whenever the remaining lifetime of the oldest batch crosses ξ .*

The (Q, ξ) policy can be achieved by setting $r = -\infty$ in the (Q, r, ξ) policy. In this policy, we do not have $m > 1$. So, we deal with the case with $m = 1$. Note that it is possible to have two outstanding orders in the system, but for analyzing the system characteristics we only need to track the effective lifetime of the current batch at the beginning of embedded cycles. In Figure 6.16, a sample path of the inventory position and inventory level of this policy for $Q = 5$ and given ξ is depicted.



Theorem 5. (*Transition Probability Function of Z_n*).

$$\mathbb{P}(\mathcal{S}|x) = \mathbb{1}_{\{z=\tau+L\}} F_Q(m_f - \xi) + \mathbb{1}_{\{x \geq \tau+L-z+m_\xi\}} \bar{F}_Q(m_f) + F_Q(\max\{\ell, m_f\}) - F_Q(\ell), \quad (6.12)$$

where $\ell = \tau + L - z + m_\xi - m_\tau$.

Proof. Similar to other policies, we can derive different regions:

A: $\max\{\xi, \tau\} < x \leq \tau + L$

- | | |
|--|---|
| A1. If $X_Q > \tau$, | $CL_n = x$ and $U_n = x - \xi$. |
| A2. If $\tau - \min\{\xi, \tau\} < X_Q < \tau$, | $CL_n = x - \tau + X_Q$ and $U_n = x - \xi$. |
| A3. If $X_Q < \tau - \min\{\xi, \tau\}$, | $CL_n = x - \xi$ and $U_n = x - \xi$. |

B: $\min\{\xi, \tau\} < x \leq \max\{\xi, \tau\}$

- | | |
|------------------------------|---|
| B1. If $X_Q > m_f$, | $CL_n = x$ and $U_n = m_\xi$. |
| B2. If $m_\xi < X_Q < m_f$, | $CL_n = m_\tau + X_Q$ and $U_n = m_\xi$. |
| B3. If $X_Q < m_\xi$, | $CL_n = x - \xi$ and $U_n = x - \xi$. |

C: $0 \leq x \leq \min\{\xi, \tau\}$

- | | |
|--------------------|------------------------------|
| C1. If $X_Q > x$, | $CL_n = x$ and $U_n = 0$. |
| C2. If $X_Q < x$, | $CL_n = X_Q$ and $U_n = 0$. |

For Region A:

A1.

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(X_Q > \tau; CL_n - U_n \geq \tau + L - z) = \mathbb{P}(X_Q > \tau; \xi \geq \tau + L - z) \\ &= \mathbb{1}_{\{z \geq \tau - \xi + L\}} \int_{\tau}^{\infty} dF_Q(y). \end{aligned}$$

A2.

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_Q < \tau; CL_n - U_n \geq \tau + L - z) \\ &= \mathbb{P}(\tau - \min\{\xi, \tau\} < X_Q < \tau; X_Q + \xi - \tau \geq \tau + L - z) \\ &= \int_{\max\{\tau+L-z+\tau-\xi, \tau-\min\{\xi, \tau\}\}}^{\tau} dF_Q(y). \end{aligned}$$

A3.

$$\begin{aligned}\mathbb{P}(\mathcal{S}|\mathbf{x}) &= \mathbb{P}(X_Q < \tau - \min\{\xi, \tau\}; CL_n - U_n \geq \tau + L - z) \\ &= \mathbb{1}_{\{z=\tau+L\}} \int_0^{\tau - \min\{\xi, \tau\}} dF_Q(y).\end{aligned}$$

Then, we can similarly express the transition probability function for Regions B and C and finally we can express all regions by a single term as Expression (6.12). $\square$

For a given effective lifetime of current batch at the beginning of embedded cycle z , the operating characteristics of the (Q, ξ) policy are the same with other policies, except $\mathbb{E}(CL)$ and $\mathbb{E}(LS)$ when $\xi \leq \tau$. In the following, we provide $\mathbb{E}(CL)$ and $\mathbb{E}(LS)$ when $\xi \leq \tau$. Considering the regions above, we can derive the $\mathbb{E}(CL)$ as following.

$$\begin{aligned}\mathbb{E}(CL) &= \int_L^{\tau+L} z dF(z) + \int_L^{\max\{\xi, L\}} \left[\frac{Q}{\lambda} F_{Q+1}(z) - z F_Q(z) \right] dF(z) \\ &\quad + \int_{\max\{\xi, L\}}^{\tau} \left[(z - \xi) F_Q(z - \xi) - z F_Q(z) + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(z - \xi)] \right] dF(z) \\ &\quad + \int_{\tau}^{\tau+L} \left[(\tau - \xi) F_Q(\tau - \xi) - \tau F_Q(\tau) + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(\tau - \xi)] \right] dF(z).\end{aligned}\tag{6.13}$$

For the (Q, ξ) policy, we can not say that $\mathbb{E}(LS|z) = \lambda(z - \tau)$ and the reason is because of the event depicted in embedded cycle 3 in Figure 6.16. Thus, for $\mathbb{E}(LS)$, we consider the regions below

A: $\max\{\xi, \tau\} < x \leq \tau + L$

- A1. If $X_Q > \tau$, $\mathbb{E}[LS|x] = \lambda(x - \tau).$
A2. If $\tau - \min\{\xi, \tau\} < X_Q < \tau$, $\mathbb{E}[LS|x] = \lambda(x - \tau).$
A3. If $X_Q < \tau - \min\{\xi, \tau\}$, $\mathbb{E}[LS|x] = \lambda[(x - \tau) + (\tau - \xi - X_Q)].$

B: $\min\{\xi, \tau\} < x \leq \max\{\xi, \tau\}$

- B1. If $X_Q > m_f$, $\mathbb{E}[LS|x] = \lambda m_\tau.$
B2. If $m_\xi < X_Q < m_f$, $\mathbb{E}[LS|x] = \lambda m_\tau.$
B3. If $X_Q < m_\xi$, $\mathbb{E}[LS|x] = \lambda[x - \xi - X_Q].$

C: $0 \leq x \leq \min\{\xi, \tau\}$

- C1. If $X_Q > x$, $\mathbb{E}[LS|x] = 0.$
C2. If $X_Q < x$, $\mathbb{E}[LS|x] = 0.$

Then, we have:

$$\begin{aligned}
\mathbb{E}(LS) &= \int_{\tau}^{\tau+L} \left[\lambda(z - \xi)F_Q(\tau - \xi) - QF_{Q+1}(\tau - \xi) + \lambda(z - \tau)\bar{F}_Q(\tau - \xi) \right] dF(z) \\
&\quad + \int_{\max\{\xi, L\}}^{\tau} \left[\lambda(z - \xi)F_Q(z - \xi) - QF_{Q+1}(z - \xi) \right] dF(z) \\
&= \lambda \int_{\tau}^{\tau+L} z dF(z) + \left[\lambda(\tau - \xi)F_Q(\tau - \xi) - QF_{Q+1}(\tau - \xi) - \lambda\tau \right] \int_{\tau}^{\tau+L} dF(z) \\
&\quad + \int_{\max\{\xi, L\}}^{\tau} \left[\lambda(z - \xi)F_Q(z - \xi) - QF_{Q+1}(z - \xi) \right] dF(z).
\end{aligned} \tag{6.14}$$

When $\xi > \tau$, all operating characteristics including $\mathbb{E}(CL)$ and $\mathbb{E}(LS)$ are the same with other policies including (Q, r) , (Q, r, ξ) , and (Q, r, T) policies. Note that $\mathbb{E}(P)$ and $\mathbb{E}(OH)$ in (Q, ξ) policy is the same with other policies either $\xi > \tau$ or $\xi \leq \tau$.

A sample of the effective lifetime distribution when $\xi < (\tau + L)/2$ is depicted in Figure 6.17.

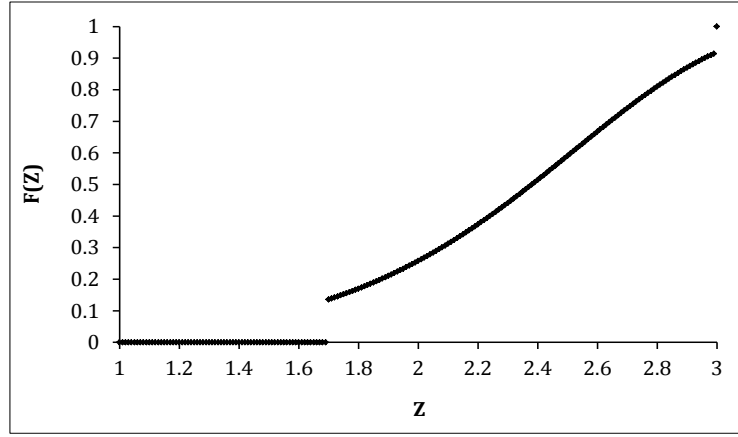


Figure 6.17: Effective lifetime distribution for $Q = 6$, $\xi = 1.3$, $L = 1$, $\tau = 2$, and $\lambda = 4.5$ for the (Q, ξ) policy

A sample of the effective lifetime distribution when $\xi > (\tau + L)/2$ is depicted in Figure 6.18.

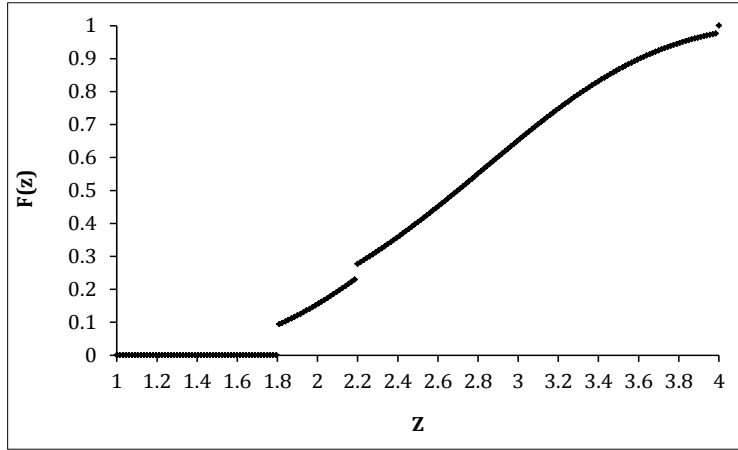


Figure 6.18: Effective lifetime distribution for $Q = 6$, $\xi = 2.2$, $L = 1$, $\tau = 3$, and $\lambda = 3.3$ for the (Q, ξ) policy

Analogous to the (Q, r, ξ) policy, we may have three mass points at $\tau - \xi + L$, ξ , and $\tau + L$ in the (Q, ξ) policy as well. In the following, we discuss the realizations affecting the range of effective lifetime vector. Suppose that the effective lifetime vector is $\mathbf{x} = (x^1, \dots, x^m)$ at the beginning of the embedded cycle n . To find the range of random variable Z^m , consider a case in which the reorder point is

$x^1 - \xi$ and the embedded cycle ends up by perishing. In this case, $CL_n - U_n = \xi$ which results in $Z^m = \tau + L - \xi$. This realization is depicted in Figure 6.19. Note that if reordering is triggered by remaining lifetime threshold, $x^1 - \xi$, then $0 \leq CL_n - U_n \leq \xi$ and if is triggered by stock level threshold, r , then $0 \leq CL_n - U_n \leq \tau$. Considering this case and the fact that $0 \leq \xi \leq \tau + L$ and $Z^m = \tau + L - [CL_n - U_n]$, if $\xi < \tau$, then $Z^m \in [L, \tau + L]$, otherwise $Z^m \in [0, \tau + L]$. Hence, Z^m in (Q, r, ξ) and (Q, ξ) policies is in interval $[L, \tau + L]$ if $\xi < \tau$ and in interval $[0, \tau + L]$, otherwise.

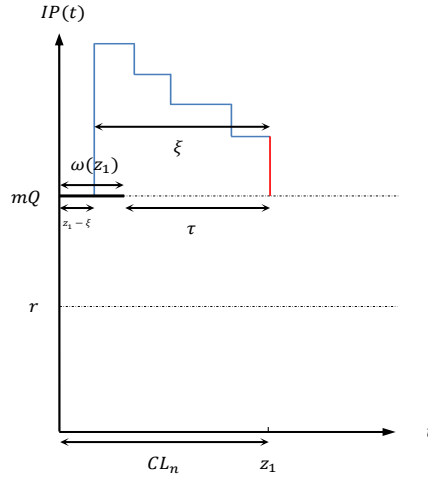


Figure 6.19: When $\xi > \tau$ and embedded cycle ends by perishing

6.4 The (Q, T) Policy

The (Q, r, T) policy has a special case namely (Q, T) policy which works as follows:

Policy: *An order of size Q is placed whenever T units of times is elapsed since the beginning of the embedded cycle.*

The (Q, T) policy can be achieved by setting $r = -\infty$ in the (Q, r, T) policy. In this policy, we do not have $m > 1$. So, we deal with the case with $m = 1$. Analogous to the (Q, ξ) policy, it is possible to have two outstanding orders in

the system, but for analyzing the system characteristics we only need to track the effective lifetime of the current batch at the beginning of embedded cycles. In Figure 6.20, a sample path of the inventory position and inventory level is depicted.

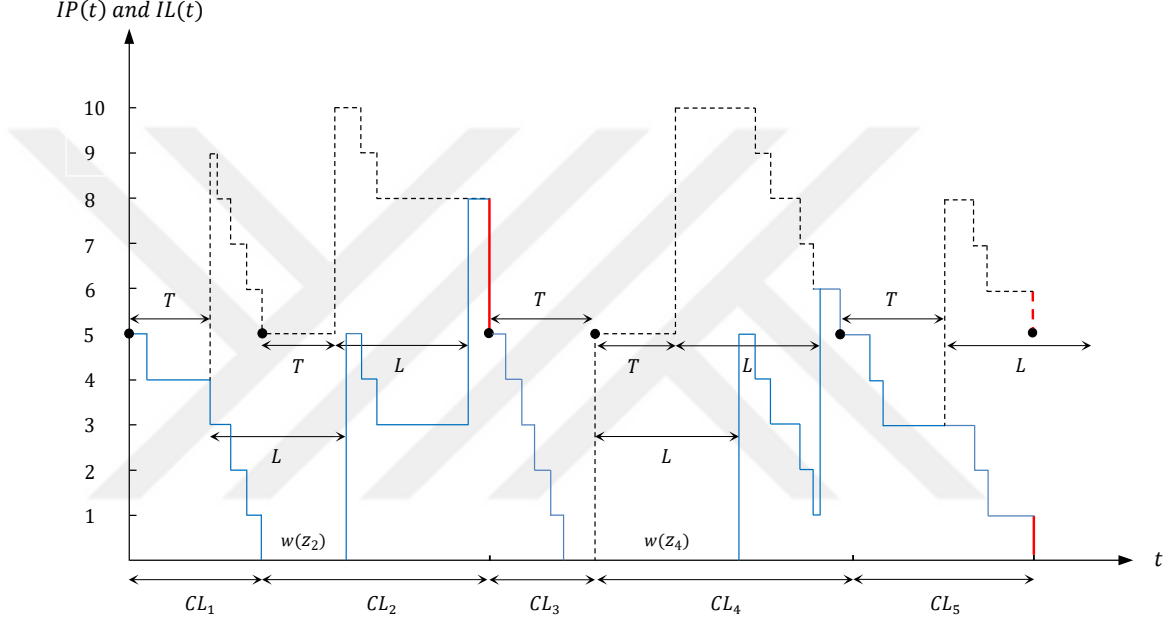


Figure 6.20: A sample path of the inventory position and inventory level for $Q = 5$ in the (Q, T) policy

There exist three realizations in this policy. In Event 1, after reordering Q items are depleted (cycles 1 and 4 in Figure 6.20). In Event 2, after reordering some portion of Q items perish (cycles 2 and 5 in Figure 6.20). In Event 3, Q items are either depleted or perish before reordering (cycle 3 in Figure 6.20).

Considering different events for the case $m = 1$ stated above, we can conclude that the $(n + 1)^{th}$ effective lifetime vector can be generally expressed as

$$Z_{n+1} = \tau + L - (CL_n - U_n), \quad (6.15)$$

where $U_n = T$ which is defined as a period of time started from the beginning of embedded cycle n till reorder point, and $CL_n = \max\{w(Z_n) + \min\{\min\{Z_n, \tau\}, X_Q\}, T\}$; $0 \leq Z \leq \tau + L$ which is the n^{th} embedded cycle

length. Considering difference events stated above, we can achieve the following result.

Theorem 6. (*Transition Probability Function of Z_n*).

$$\begin{aligned}\mathbb{P}(\mathcal{S}|x) &= \mathbb{1}_{\{z=\tau+L\}}F_Q(m_c - m_\tau) + \mathbb{1}_{\{m_T \geq \tau+L-z\}}\bar{F}_Q(m_f) \\ &\quad + F_Q(m_f) - F_Q(\tau + L - z - m_\tau + m_c).\end{aligned}\tag{6.16}$$

Proof. Similar to other policies, we can derive different regions:

A: $\max\{T, \tau\} < x \leq \tau + L$

- | | |
|--|---|
| A1. If $X_Q < T - (x - \tau)$, | $CL_n = T$ and $U_n = T$. |
| A2. If $T - (x - \tau) < X_Q < \tau$, | $CL_n = x - \tau + X_Q$ and $U_n = T$. |
| A3. If $\tau < X_Q$, | $CL_n = x$ and $U_n = T$. |

B: $\min\{T, \tau\} < x \leq \max\{T, \tau\}$

- | | |
|--|----------------------------------|
| B1. If $X_Q < \min\{T, \tau\}$, | $CL_n = T$ and $U_n = T$. |
| B2. If $\min\{T, \tau\} < X_Q < m_f$, | $CL_n = X_Q$ and $U_n = T$. |
| B3. If $m_f < X_Q$, | $CL_n = T + m_T$ and $U_n = T$. |

C: $0 \leq x \leq \min\{T, \tau\}$

- | | |
|--------------------|----------------------------|
| C1. If $X_Q < x$, | $CL_n = T$ and $U_n = T$. |
| C2. If $X_Q > x$, | $CL_n = T$ and $U_n = T$. |

For Region A:

A1.

$$\begin{aligned}\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(X_Q < T - (x - \tau); CL_n - U_n \geq \tau + L - z) \\ &= \mathbb{P}(X_Q < T - (x - \tau); 0 \geq \tau + L - z) \\ &= \mathbb{1}_{\{z=\tau+L\}} \int_0^{T-(x-\tau)} dF_Q(y).\end{aligned}$$

A2.

$$\begin{aligned}\mathbb{P}(\mathcal{S}|x) &= \mathbb{P}(T - (x - \tau) < X_Q < \tau; CL_n - U_n \geq \tau + L - z) \\ &= \mathbb{P}(T - (x - \tau) < X_Q < \tau; x - \tau + X_Q - T \geq \tau + L - z) \\ &= \mathbb{P}(T - (x - \tau) < X_Q < \tau; X_Q \geq \tau + L - z - m_\tau + T) \\ &= \int_{\tau+L-z-m_\tau+T}^{\tau} dF_Q(y).\end{aligned}$$

A3.

$$\mathbb{P}(\mathcal{S}|\mathbf{x}) = \mathbb{P}(X_Q > \tau; x - T \geq \tau + L - z) = \mathbb{1}_{\{x-T \geq \tau+L-z\}} \int_{\tau}^{\infty} dF_Q(y).$$

Then, we can simply express the transition probability function for Region A as follows:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|x) &= \mathbb{1}_{\{z=\tau+L\}} F_Q(T - (x - \tau)) + \mathbb{1}_{\{x-T \geq \tau+L-z\}} \bar{F}_Q(\tau) \\ &\quad + F_Q(\tau) - F_Q(\tau + L - z - m_{\tau} + T). \end{aligned} \quad (6.17)$$

Similarly, we can derive the transition probability function for Regions B and C. Finally, we can express all regions by a single term as Expression (6.16). $\square$

Considering the regions above, we can derive the $\mathbb{E}(CL)$ as following.

$$\begin{aligned} \mathbb{E}(CL) &= \int_0^{\min\{T,\tau\}} T dF(z) + \int_{\min\{T,\tau\}}^{\max\{T,\tau\}} \left[\int_0^{\min\{T,\tau\}} T dF_Q(y) \right. \\ &\quad \left. + \int_{\min\{T,\tau\}}^{m_f} y dF_Q(y) + \int_{m_f}^{\infty} (T + m_T) dF_Q(y) \right] dF(z) \\ &\quad + \int_{\max\{T,\tau\}}^{\tau+L} \left[\int_0^{T-m_{\tau}} T dF_Q(y) + \int_{T-m_{\tau}}^{\tau} (m_{\tau} + y) dF_Q(y) + \int_{\tau}^{\tau+L} z dF_Q(y) \right] dF(z). \end{aligned} \quad (6.18)$$

After simplification, we have

$$\begin{aligned} \mathbb{E}(CL) &= \int_0^{\min\{T,\tau\}} T dF(z) + \int_{\min\{T,\tau\}}^{\max\{T,\tau\}} \left[T F_Q(\min\{T, \tau\}) \right. \\ &\quad \left. + \frac{Q}{\lambda} [F_{Q+1}(m_f) - F_{Q+1}(\min\{T, \tau\})] + (T + m_T) \bar{F}_Q(m_f) \right] dF(z) \\ &\quad + \int_{\max\{T,\tau\}}^{\tau+L} \left[T F_Q(T - m_{\tau}) + m_{\tau} [F_Q(\tau) - F_Q(T - m_{\tau})] \right. \\ &\quad \left. + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - m_{\tau})] + z \bar{F}_Q(\tau) \right] dF(z). \end{aligned} \quad (6.19)$$

And similarly for $\mathbb{E}(LS)$, we can consider the same regions:

A: $\max\{T, \tau\} < x \leq \tau + L$

- A1. If $X_Q < T - (x - \tau)$, $\mathbb{E}[LS|x] = \lambda(T - X_Q)$.
A2. If $T - (x - \tau) < X_Q < \tau$, $\mathbb{E}[LS|x] = \lambda(x - \tau)$.
A3. If $\tau < X_Q$, $\mathbb{E}[LS|x] = \lambda(x - \tau)$.

B: $\min\{T, \tau\} < x \leq \max\{T, \tau\}$

- B1. If $X_Q < \min\{T, \tau\}$, $\mathbb{E}[LS|x] = \lambda(T - X_Q)$.
B2. If $\min\{T, \tau\} < X_Q < m_f$, $\mathbb{E}[LS|x] = 0$.
B3. If $m_f < X_Q$, $\mathbb{E}[LS|x] = \lambda(T - \min\{T, \tau\})$.

C: $0 \leq x \leq \min\{T, \tau\}$

- C1. If $X_Q < x$, $\mathbb{E}[LS|x] = \lambda(T - X_Q)$.
C2. If $X_Q > x$, $\mathbb{E}[LS|x] = \lambda(T - x)$.

Then, we have:

$$\begin{aligned} \mathbb{E}(LS) &= \int_0^{\min\{T, \tau\}} \left[\int_0^z \lambda(T - y) dF_Q(y) + \int_z^\infty \lambda(T - z) dF_Q(y) \right] dF(z) \\ &\quad + \int_{\min\{T, \tau\}}^{\max\{T, \tau\}} \left[\int_0^{\min\{T, \tau\}} \lambda(T - y) dF_Q(y) + \int_{m_f}^\infty \lambda(T - \min\{T, \tau\}) dF_Q(y) \right] dF(z) \\ &\quad + \int_{\max\{T, \tau\}}^{\tau+L} \left[\int_0^{T-m_\tau} \lambda(T - y) dF_Q(y) + \int_{T-m_\tau}^\infty \lambda(z - \tau) dF_Q(y) \right] dF(z). \end{aligned} \quad (6.20)$$

After simplification, we have

$$\begin{aligned} \mathbb{E}(LS) &= \int_0^{\min\{T, \tau\}} \left[\lambda T - QF_{Q+1}(z) - \lambda z \bar{F}_Q(z) \right] dF(z) \\ &\quad + \int_{\min\{T, \tau\}}^{\max\{T, \tau\}} \left[\lambda T F_Q(\min\{T, \tau\}) - QF_{Q+1}(\min\{T, \tau\}) \right. \\ &\quad \left. + \lambda(T - \min\{T, \tau\}) \bar{F}_Q(m_f) \right] dF(z) \\ &\quad + \int_{\max\{T, \tau\}}^{\tau+L} \left[\lambda T F_Q(T - m_\tau) - QF_{Q+1}(T - m_\tau) + \lambda(z - \tau) \bar{F}_Q(T - m_\tau) \right] dF(z). \end{aligned} \quad (6.21)$$

Note that in the (Q, T) policy, $T - m_\tau$ can take negative value. In this case,

we set it to zero. For instance, suppose that $T \leq \tau$ in the (Q, T) policy. Then, we need to consider $\mathbb{E}(CL)$ and $\mathbb{E}(LS)$ in two cases $T < L$ and $T \geq L$.

If $T < L$,

$$\begin{aligned}
\mathbb{E}(CL) = & \int_0^T T dF(z) + \int_T^\tau \left[TF_Q(T) + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(T)] + z\bar{F}_Q(z) \right] dF(z) \\
& + \int_\tau^{\tau+T} \left[TF_Q(T - z + \tau) + (z - \tau)[F_Q(\tau) - F_Q(T - z + \tau)] \right. \\
& + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - z + \tau)] + z\bar{F}_Q(\tau) \left. \right] dF(z) \\
& + \int_{\tau+T}^{\tau+L} \left[z - \tau F_Q(\tau) + \frac{Q}{\lambda} F_{Q+1}(\tau) \right] dF(z).
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
\mathbb{E}(LS) = & \int_0^T \left[\lambda T - QF_{Q+1}(z) - \lambda z\bar{F}_Q(z) \right] dF(z) + \int_T^\tau \left[\lambda TF_Q(T) - QF_{Q+1}(T) \right] dF(z) \\
& + \int_\tau^{\tau+T} \left[\lambda TF_Q(T - z + \tau) - QF_{Q+1}(T - z + \tau) + \lambda(z - \tau)\bar{F}_Q(T - z + \tau) \right] dF(z) \\
& + \int_{\tau+T}^{\tau+L} \lambda(z - \tau) dF(z).
\end{aligned} \tag{6.23}$$

If $T \geq L$,

$$\begin{aligned}
\mathbb{E}(CL) = & \int_0^T T dF(z) + \int_T^\tau \left[TF_Q(T) + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(T)] + z\bar{F}_Q(z) \right] dF(z) \\
& + \int_\tau^{\tau+L} \left[TF_Q(T - z + \tau) + (z - \tau)[F_Q(\tau) - F_Q(T - z + \tau)] \right. \\
& + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - z + \tau)] + z\bar{F}_Q(\tau) \left. \right] dF(z).
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
\mathbb{E}(LS) = & \int_0^T \left[\lambda T - QF_{Q+1}(z) - \lambda z\bar{F}_Q(z) \right] dF(z) + \int_T^\tau \left[\lambda TF_Q(T) - QF_{Q+1}(T) \right] dF(z) \\
& + \int_\tau^{\tau+L} \left[\lambda TF_Q(T - z + \tau) - QF_{Q+1}(T - z + \tau) + \lambda(z - \tau)\bar{F}_Q(T - z + \tau) \right] dF(z).
\end{aligned}$$

(6.25)

You can analyze the case $T > \tau$ similarly.

Note that $\mathbb{E}(P)$ and $\mathbb{E}(OH)$ in (Q, T) policy is the same with (Q, r, T) and (Q, r, ξ) policies. Figures 6.21 and 6.22 depict two samples of the effective lifetime distribution. Similar to the (Q, r, T) policy, in the (Q, T) policy we can have at most two mass points at T and $\tau + L$ as well.

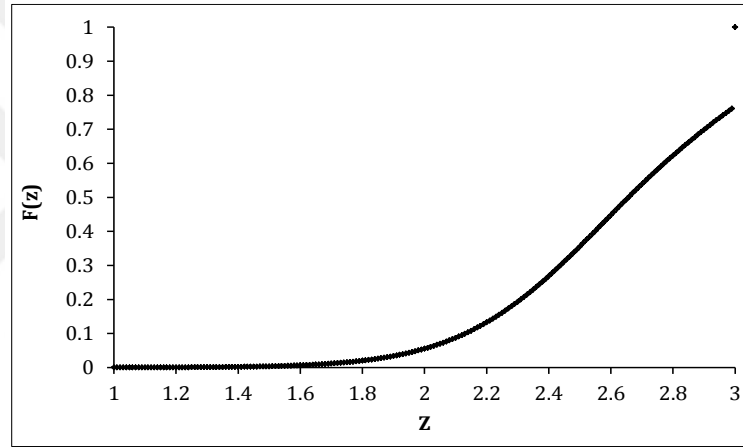


Figure 6.21: Effective lifetime distribution with $\lambda = 10$, $L = 1$, $\tau = 2$, $Q = 8$, and $T = 1.1$

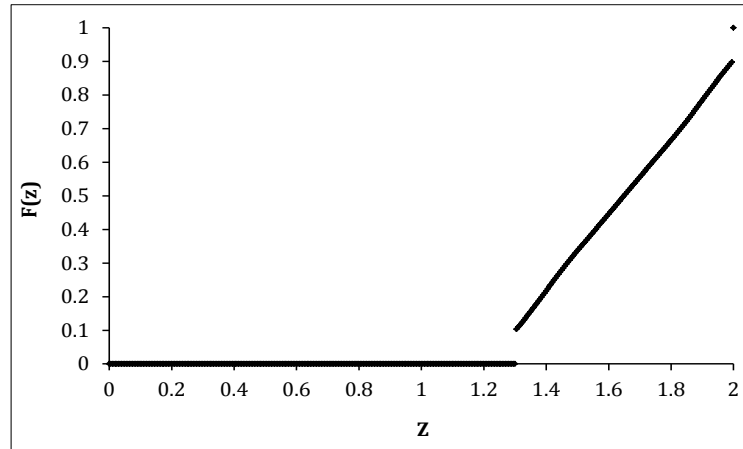


Figure 6.22: Effective lifetime distribution with $\lambda = 0.5$, $L = 0.5$, $\tau = 1.5$, $Q = 10$, and $T = 1.3$

Finally, we can summarize the structure of the effective lifetime distribution in all policies as follows.

A. For the case with $m = 1$, one outstanding order:

1. In the (Q, r, ξ) policy when $r < Q$:
 - 1.1. If $0 < \xi \leq \frac{\tau+L}{2}$, we have one mass point at $\tau - \xi + L$; and $Z \in [L, \tau + L]$.
 - 1.2. If $\frac{\tau+L}{2} < \xi < \tau$, we have two mass points at $\tau - \xi + L$ and ξ ; and $Z \in [L, \tau + L]$.
 - 1.3. If $\tau \leq \xi \leq \tau + L$, we have two mass points at $\tau - \xi + L$ and ξ ; and $Z \in [0, \tau + L]$.
2. In the (Q, r, ξ) policy when $r = 0$:
 - 2.1. If $0 < \xi \leq \frac{\tau+L}{2}$, we have two mass points at $\tau - \xi + L$ and $\tau + L$; and $Z \in [L, \tau + L]$.
 - 2.2. If $\frac{\tau+L}{2} < \xi < \tau$, we have three mass points at $\tau - \xi + L$, ξ , and $\tau + L$; and $Z \in [L, \tau + L]$.
 - 2.3. If $\tau \leq \xi \leq \tau + L$, we have three mass points at $\tau - \xi + L$, ξ , and $\tau + L$; and $Z \in [0, \tau + L]$.

B. For the case with $m > 1$, more than one outstanding order:

1. In the (Q, r, ξ) policy when $r \neq (m - 1)Q$:
 - 1.1. If $0 < \xi < \tau$, we have one mass point at $\tau - \xi + L$ in Z^m ; and $Z^i \in [0, \tau + L]$, for $i = 1 \dots m - 1$, and $Z^m \in [L, \tau + L]$.
 - 1.2. If $\tau \leq \xi \leq \tau + L$, we have one mass point at $\tau - \xi + L$ in Z^m ; and $Z^i \in [0, \tau + L]$, for $i = 1 \dots m$.
2. In the (Q, r, ξ) policy when $r = (m - 1)Q$:
 - 2.1. If $0 < \xi < \tau$, we have two mass points at $\tau - \xi + L$ and $\tau + L$ in Z^m ; and $Z^i \in [0, \tau + L]$, for $i = 1 \dots m - 1$, and $Z^m \in [L, \tau + L]$.
 - 2.2. If $\tau \leq \xi \leq \tau + L$, we have two mass points at $\tau - \xi + L$ and $\tau + L$ in Z^m ; and $Z^i \in [0, \tau + L]$, for $i = 1 \dots m$.

- In the (Q, r) policy either it is $m = 1$ or $m > 1$, we have one mass point at $\tau + L$. More specifically, in the $(Q, 0)$ policy, the effective lifetime is $\tau + L$ with probability of 1. For the (Q, r) policy when $m > 1$ and $r = (m - 1)Q$, the random variable Z^m is always $\tau + L$. In the case $r > (m - 1)Q$,

$Z^i = [0, \tau + L]$ for $i = 1, \dots, m - 1$ and $Z^m = [L, \tau + L]$.

- In the (Q, ξ) policy, if $0 < \xi \leq \frac{\tau+L}{2}$, we have two mass points at $\tau - \xi + L$ and $\tau + L$; and if $\frac{\tau+L}{2} < \xi < \tau + L$, we have three mass points at $\tau - \xi + L$, ξ , and $\tau + L$. If $\xi < \tau$, then $Z \in [L, \tau + L]$, otherwise $Z \in [0, \tau + L]$.
- In both (Q, T) policy and (Q, r, T) policies, we have at most two mass points at T and $\tau + L$; and $0 \leq Z \leq \tau + L$.

It should be noted here that the structure of the objective function and service level constraint for all proposed policies are the same.

6.5 Special Region, $r = (m - 1)Q$

6.5.1 Special Region of the (Q, r, ξ) Policy

Considering the realizations of the (Q, r, ξ) policy, we see that when $r = (m - 1)Q$ for $m \geq 1$ some of the realizations would not occur. In special region of search space (i.e., when $r = (m - 1)Q$) an embedded cycle starts when the inventory position hits mQ and whenever Q items are depleted by demand or partially perish a new order is placed which brings the inventory position to mQ again and the embedded cycle ends. Thus, in special regions, when the reorder point is triggered by r , the inventory position hits mQ immediately after reordering and the replenishment instances coincide with the embedded cycle endings. This realization is illustrated in Figure 6.23 for the case $m = 1$ (where $r = 0$) and in Figure 6.24 for the case $m > 1$ (where $r = (m - 1)Q$). However, this case would not happen when the reorder point is triggered by the remaining time threshold ξ . Hence, we have three realizations depicted in Figures 4.3, 4.4, and 6.23 for the case $m = 1$; and three realizations depicted in Figures 4.7, 4.8, and 6.24 for the case $m > 1$ in the special regions.

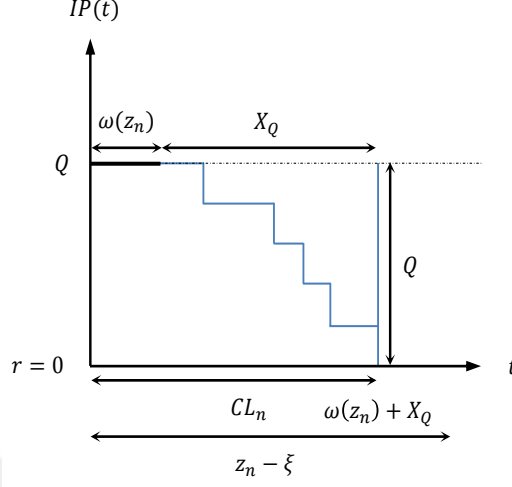


Figure 6.23: Special region, $r = (m - 1)Q$, $m = 1$, $w(z_n) + X_Q < z_n - \xi$, $CL_n = w(z_n) + X_Q$, and $Z_{n+1} = \tau + L$

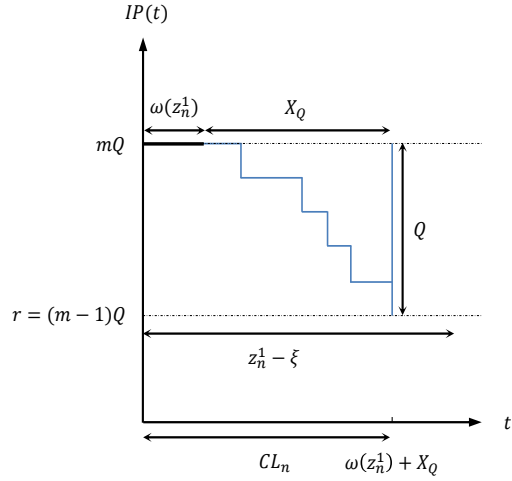


Figure 6.24: Special region, $r = (m - 1)Q$, $m > 1$, $w(z_n^1) + X_Q < z_n^1 - \xi$, $CL_n = w(z_n^1) + X_Q$, $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m - 1$, and $Z_{n+1}^m = \tau + L$

In the following, we provide some main results for the special region $r = (m - 1)Q$ analogous to the case $r > (m - 1)Q$. Noting that the state space for the region $r = (m - 1)Q$ and $r > (m - 1)Q$ is identical, we have the following result.

Theorem 7. (*Transition Probability Function of Z_n*).

Suppose $m = 1$ and $r = (m - 1)Q$. Then,

$$\begin{aligned}\mathbb{P}(\mathcal{S}|x) &= F_Q(\max\{\ell, m_f\}) - F_Q(\max\{\ell, m_f - \xi\}) + \mathbb{1}_{\{x \geq \tau + L - z + m_\xi\}} \bar{F}_Q(m_f) \\ &\quad + \mathbb{1}_{\{z = \tau + L\}} F_Q(m_f - \xi).\end{aligned}\tag{6.26}$$

Proof. Similar to Theorem 1, omitted. $\square$

Theorem 8. (*Transition Probability Function of $\mathbf{Z}_n$*).

Suppose $m > 1$ and $r = (m - 1)Q$. For $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$ and for $x^1 \geq m_x$ we have:

$$\begin{aligned}\mathbb{P}(\mathcal{S}|\mathbf{x}) &= F_Q(\max\{\ell, m_f\}) - F_Q(\max\{\ell, m_f - \xi\}) + \mathbb{1}_{\{x^1 \geq \tau + L - z^m + m_\xi\}} \bar{F}_Q(m_f) \\ &\quad + \mathbb{1}_{\{z^m = \tau + L\}} \left[F_Q(\max\{m_x - m_\tau, m_f - \xi\}) - F_Q(m_x - m_\tau) \right].\end{aligned}\tag{6.27}$$

Proof. Similar to Theorem 2, omitted. $\square$

Proposition 2. (*Boundedness*).

(i) Given $\mathbf{Z}_n = \mathbf{x}$, we have

$$\begin{aligned}\mathbb{E}[Z_{n+1}^i|\mathbf{x}] &= x^{i+1} - \mathbb{E}[CL_n|\mathbf{x}], \text{ for } i = 1, \dots, m-1, \\ \mathbb{E}[Z_{n+1}^m|\mathbf{x}] &= \tau + L - \mathbb{E}[CL_n|\mathbf{x}] + \mathbb{E}[U_n|\mathbf{x}],\end{aligned}\tag{6.28}$$

where

$$\begin{aligned}\mathbb{E}[CL_n|\mathbf{x}] &= \frac{Q}{\lambda} F_{Q+1}(m_f) + w(x^1) F_Q(m_f) + x^1 \bar{F}_Q(m_f), \text{ and} \\ \mathbb{E}[U_n|\mathbf{x}] &= \frac{Q}{\lambda} F_{Q+1}(m_f - \xi) + w(x^1) F_Q(m_f - \xi) + (x^1 - \xi) \bar{F}_Q(m_f - \xi).\end{aligned}\tag{6.29}$$

(ii) Given $\mathbf{Z}_n = \mathbf{x}$, we have

$$\sum_{i=1}^m \mathbb{E}[Z_{n+1}^i|\mathbf{Z}_n = \mathbf{x}] \leq \sum_{i=1}^m x^i - \epsilon,\tag{6.30}$$

where $x^1 = \tau + L$ and $\epsilon = m\mathbb{E}[CL_n|\mathbf{x}] - \mathbb{E}[U_n|\mathbf{x}]$.

Proof. The $\mathbb{E}[CL_n|\mathbf{x}]$ term is the same as Proposition 1, and the term $\mathbb{E}[U_n|\mathbf{x}]$ is obtained as follows:

$$[U_n|\mathbf{x}] = \begin{cases} w(x^1) + X_Q & \text{if } X_Q < x^1 - \xi - w(x^1), \\ x^1 - \xi & \text{if } X_Q \geq x^1 - \xi - w(x^1), \end{cases} \quad (6.31)$$

$$\begin{aligned} \mathbb{E}[U_n|\mathbf{x}] &= \int_0^{m_f - \xi} (y + w(x^1)) dF_Q(y) + \int_{m_f - \xi}^{\infty} (x^1 - \xi) dF_Q(y) \\ &= \frac{Q}{\lambda} F_{Q+1}(m_f - \xi) + w(x^1) F_Q(m_f - \xi) + (x^1 - \xi) \bar{F}_Q(m_f - \xi). \end{aligned} \quad (6.32)$$

□

Theorem 9. (*Ergodicity*). *The process $\{\mathbf{Z}_n, n \geq 1\}$ is ergodic.*

Proof. Similar to Theorem 3, omitted. □

All operating characteristics for a given $Z = z$ and the objective function structure for the cases $r = (m - 1)Q$ and $r > (m - 1)Q$ are identical. Thus, the only difference between them is the effective lifetime distribution function.

In special regions, $r = (m - 1)Q$ for $m > 1$, and $r = 0$ for $m = 1$, if a new batch is placed when inventory position crosses r , then the inventory position raises to mQ which means that the new embedded cycle starts with $z_m = \tau + L$. Thus, we can infer that in the special regions, we have one more mass point at $\tau + L$ in addition to $\tau - \xi + L$ and ξ .

Two samples of the effective lifetime distribution for $r = 0$ are shown in Figures 6.25 and 6.26.

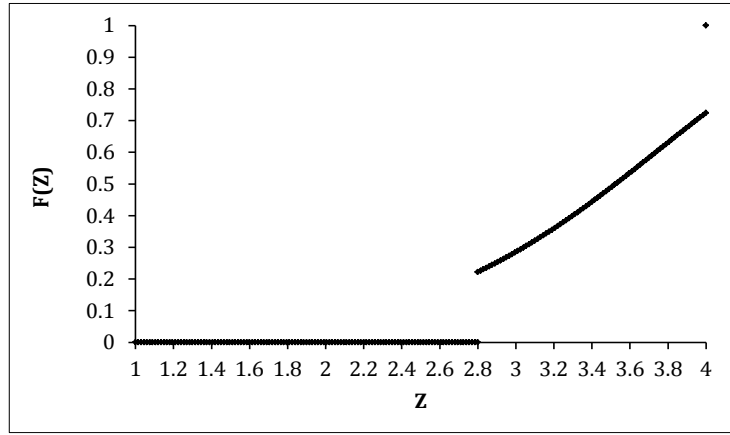


Figure 6.25: Effective lifetime distribution for $Q = 7$, $r = 0$, $\xi = 1.2$, $L = 1$, $\tau = 3$, and $\lambda = 3$

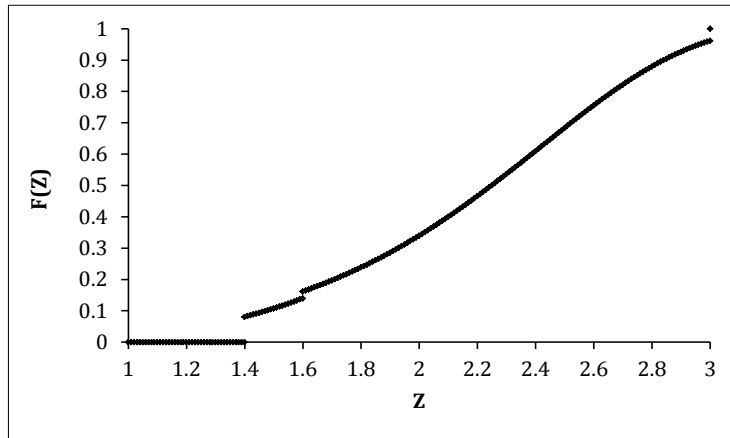


Figure 6.26: Effective lifetime distribution for $Q = 4$, $r = 0$, $\xi = 1.6$, $L = 1$, $\tau = 2$, and $\lambda = 3.5$

A sample of the effective lifetime distribution for $r = (m - 1)Q$ is shown in Figure 6.27.

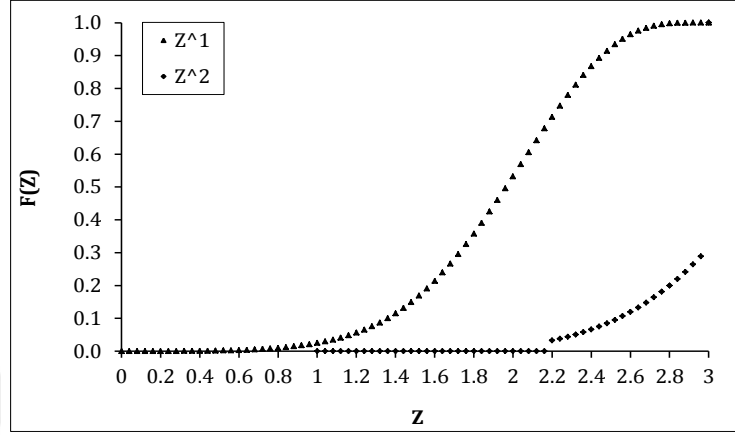


Figure 6.27: Effective lifetime distribution for $Q = 5$, $r = 5$, $\xi = 0.8$, $L = 1$, $\tau = 2$, and $\lambda = 6$

6.5.2 Special Region of the (Q, r, T) Policy

In the following, we provide some main results for the special region $r = (m - 1)Q$ analogous to the case $r > (m - 1)Q$ for the (Q, r, T) policy. Noting that the state space for the region $r = (m - 1)Q$ and $r > (m - 1)Q$ is identical, we have the following result.

Theorem 10. (*Transition Probability Function of $\mathbf{Z}_n$*).

For $x^1 < m_x$, $\mathbb{P}(\mathcal{S}|\mathbf{x}) = 0$ and for $x^1 \geq m_x$ we have:

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) = & F_Q(m_f) - F_Q(\max\{m_x - m_\tau, \tau + L - z^m - m_\tau + m_c\}) \\ & + \mathbb{1}_{\{m_T \geq \tau + L - z^m\}} \bar{F}_Q(m_f) + \mathbb{1}_{\{z^m = \tau + L\}} \left[F_Q(m_c - m_\tau) - F_Q(m_x - m_\tau) \right]. \end{aligned} \quad (6.33)$$

Proof. Similar to Theorem 8, omitted. □

And for the case $m = 1$ (i.e., the $(Q, 0, T)$ policy), we have

$$\begin{aligned} \mathbb{P}(\mathcal{S}|\mathbf{x}) = & F_Q(m_f) - F_Q(\tau + L - z - m_\tau + m_c) \\ & + \mathbb{1}_{\{m_T \geq \tau + L - z\}} \bar{F}_Q(m_f) + \mathbb{1}_{\{z = \tau + L\}} F_Q(m_c - m_\tau). \end{aligned} \quad (6.34)$$

All operating characteristics for a given $Z = z$ and the objective function structure for the cases $r = (m - 1)Q$ and $r > (m - 1)Q$ are identical.

An example of effective lifetime distribution for the (Q, r, T) policy is depicted in Figure 6.28.

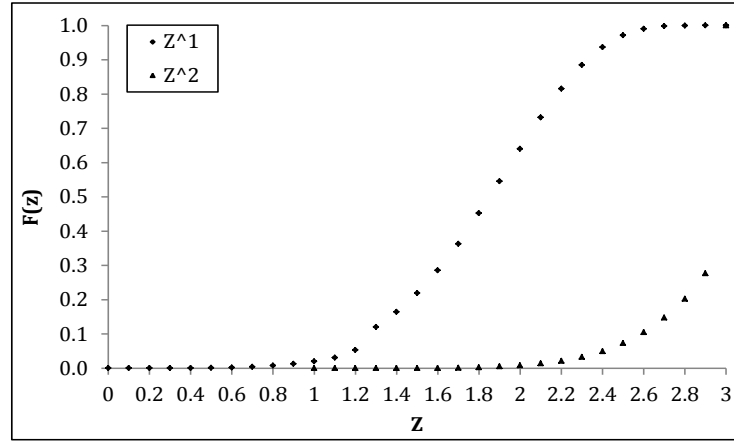


Figure 6.28: Effective lifetime distribution for $Q = 7$, $r = 7$, $T = 1.2$, $L = 1$, $\tau = 2$, and $\lambda = 7$

Chapter 7

Analytical Results

In this chapter, we discuss some structural properties regarding the operational cost components, the objective function, and the service level constraint. Recall that the operating characteristics for a given effective lifetime z are the same in all policies and also $FL(\xi)$ is the service level constraint expression (i.e., $FL(\xi) = \mathbb{E}[LS]/\lambda\mathbb{E}[CL]$).

Proposition 3. *The conditional expectation of the embedded cycle length for given lifetime $Z = z$, $\mathbb{E}[CL|z]$, is increasing and concave in z if $0 \leq z \leq \tau$; and an increasing and linear in z if $\tau < z \leq \tau + L$.*

Proof. Considering the conditional expectation of the embedded cycle length for given lifetime $Z = z$ expression (Equation (5.4)), we have

For $0 \leq z \leq \tau$,

$$\frac{d\mathbb{E}[CL|z]}{dz} = zf_Q(z) + \int_z^\infty f_Q(y)dy + z(-f_Q(z)) = \bar{F}_Q(z) > 0. \quad (7.1)$$

$$\frac{d^2\mathbb{E}[CL|z]}{dz^2} = -f_Q(z) < 0. \quad (7.2)$$

For $\tau < z \leq \tau + L$,

$$\frac{d\mathbb{E}[CL|z]}{dz} = 1 > 0. \quad (7.3)$$

□

Proposition 4. *The conditional expectation of number of lost sales for given lifetime $Z = z$, $\mathbb{E}[LS|z]$, is zero if $0 \leq z \leq \tau$; and is increasing and linear in z if $\tau < z \leq \tau + L$.*

Proof. Knowing that $\mathbb{E}[LS|z] = \lambda z$ for $\tau < z \leq \tau + L$, we have

$$\frac{d\mathbb{E}[LS|z]}{dz} = \lambda > 0. \quad (7.4)$$

which means that expected number of lost sales for a given lifetime $Z = z$ is an increasing linear function of z with slope of λ . □

Proposition 5. *The expected number of perishing items for given lifetime $Z = z$, $\mathbb{E}[P|z]$, is decreasing and convex in z if $0 \leq z \leq \tau$; and is constant if $\tau < z \leq \tau + L$.*

Proof. First, we simplify the $\mathbb{E}[P|z]$ expression (Equation (5.7)),

$$\begin{aligned} \mathbb{E}[P|z] &= \sum_{n=0}^{Q-1} (Q-n) e^{-\lambda z} \frac{(\lambda z)^n}{n!} \\ &= Q e^{-\lambda z} + (Q-1) e^{-\lambda z} \lambda z + (Q-2) e^{-\lambda z} \frac{(\lambda z)^2}{2!} + \dots + e^{-\lambda z} \frac{(\lambda z)^{Q-1}}{(Q-1)!} \\ &= e^{-\lambda z} \left[Q + (Q-1)\lambda z + (Q-2) \frac{(\lambda z)^2}{2!} + \dots + 2 \frac{(\lambda z)^{Q-2}}{(Q-2)!} + \frac{(\lambda z)^{Q-1}}{(Q-1)!} \right]. \end{aligned}$$

Then, after taking derivative for $0 \leq z \leq \tau$, we have:

$$\begin{aligned} \frac{d\mathbb{E}[P|z]}{dz} &= -\lambda e^{-\lambda z} \left[Q + (Q-1)\lambda z + (Q-2) \frac{(\lambda z)^2}{2!} + \dots + 2 \frac{(\lambda z)^{Q-2}}{(Q-2)!} + \frac{(\lambda z)^{Q-1}}{(Q-1)!} \right] \\ &\quad + e^{-\lambda z} \left[(Q-1)\lambda + (Q-2)\lambda^2 z + (Q-3) \frac{\lambda^3 z^2}{2!} \dots + 2 \frac{\lambda^{Q-2} z^{Q-3}}{(Q-3)!} + \frac{\lambda^{Q-1} z^{Q-2}}{(Q-2)!} \right] \\ &= -\lambda e^{-\lambda z} \left[1 + \lambda z + \frac{(\lambda z)^2}{2!} + \frac{(\lambda z)^3}{3!} + \dots + \frac{(\lambda z)^{Q-2}}{(Q-2)!} + \frac{(\lambda z)^{Q-1}}{(Q-1)!} \right] \\ &= -\lambda e^{-\lambda z} \sum_{n=0}^{Q-1} \frac{(\lambda z)^n}{n!} < 0. \end{aligned}$$

(7.5)

Similarly, we can easily verify that:

$$\frac{d^2\mathbb{E}[P|z]}{dz^2} = e^{-\lambda z} \frac{(\lambda z)^{Q-1}}{(Q-1)!} > 0. \quad (7.6)$$

which means that number of perishing items is a decreasing and convex function of z for $0 \leq z \leq \tau$; and for $\tau < z \leq \tau + L$ it takes a constant value $\sum_{n=0}^{Q-1} (Q-n)e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$. $\square$

Proposition 6. *The expected on-hand inventory for given lifetime $Z = z$, $\mathbb{E}[OH|z]$, is decreasing and concave in z if $0 \leq z \leq \tau$; and is constant if $\tau < z \leq \tau + L$.*

Proof. By taking derivative from the expected on-hand inventory expression (Equation (5.15)) we have,

$$\frac{d\mathbb{E}[OH|z]}{dz} = e^{-\lambda z} \left[Q + (Q-1)\lambda z + (Q-2)\frac{(\lambda z)^2}{2!} + \dots + 2\frac{(\lambda z)^{Q-2}}{(Q-2)!} + \frac{(\lambda z)^{Q-1}}{(Q-1)!} \right] - Q.$$

We multiply the last term with $e^{-\lambda z}e^{\lambda z}$. Then, we have

$$\begin{aligned} \frac{d\mathbb{E}[OH|z]}{dz} &= e^{-\lambda z} \left[Q + (Q-1)\lambda z + (Q-2)\frac{(\lambda z)^2}{2!} + \dots + 2\frac{(\lambda z)^{Q-2}}{(Q-2)!} + \frac{(\lambda z)^{Q-1}}{(Q-1)!} \right] \\ &\quad - Qe^{-\lambda z}e^{\lambda z}. \end{aligned}$$

Substituting Maclaurin series of $e^{\lambda z}$, we have

$$\begin{aligned} \frac{d\mathbb{E}[OH|z]}{dz} &= e^{-\lambda z} \left[Q + (Q-1)\lambda z + (Q-2)\frac{(\lambda z)^2}{2!} + \dots + 2\frac{(\lambda z)^{Q-2}}{(Q-2)!} + \frac{(\lambda z)^{Q-1}}{(Q-1)!} \right] \\ &\quad - e^{-\lambda z} \left[Q + Q\lambda z + Q\frac{(\lambda z)^2}{2!} + \dots \right] \\ &= e^{-\lambda z} \left[-\lambda z - 2\frac{(\lambda z)^2}{2!} - 3\frac{(\lambda z)^3}{3!} - \dots - Q \sum_{n=Q}^{\infty} \frac{(\lambda z)^n}{n!} \right] < 0. \end{aligned}$$

And,

$$\frac{d^2 \mathbb{E}[OH|z]}{dz^2} = -\lambda e^{-\lambda z} \sum_{n=0}^{Q-1} \frac{(\lambda z)^n}{n!} < 0. \quad (7.7)$$

For $\tau < z \leq \tau + L$, $m_f = \tau$, $m_l = 0$; thus, $\mathbb{E}[OH|z]$ is not a function of z and it takes a constant fixed value $\frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q-k)}{k!} \gamma(k+1, \lambda\tau)$. $\square$

Proposition 7. *Suppose that the operational costs include holding and perishing costs. Then, the total expected cost per unit time for a given lifetime $Z = z$ is a decreasing function of z in range $0 \leq z \leq \tau + L$.*

Proof. Considering the structure of the objective function, since $\mathbb{E}[OH|z]$ and $\mathbb{E}[P|z]$ are decreasing functions of z and $\mathbb{E}[CL|z]$ is an increasing function of z , the total expected cost per unit time is a decreasing function of z . $\square$

Proposition 8. *The $FL(\xi)$ function takes value of zero in range $0 \leq z \leq \tau$; and is an increasing and concave function of z in range $\tau < z \leq \tau + L$.*

Proof. For $\tau < z \leq \tau + L$, the expected cycle length for given $Z = z$ is $\mathbb{E}[CL|z] = \frac{Q}{\lambda} F_{Q+1}(\tau) + (z - \tau) F_Q(\tau) + z \bar{F}_Q(\tau)$ and the expected lost sales for given $Z = z$ is $\mathbb{E}[LS|z] = \lambda(z - \tau)$. Then,

$$\begin{aligned} \frac{d[FL(\xi)|z]}{dz} &= \frac{1}{\lambda} \frac{\frac{d\mathbb{E}[LS|z]}{dz} \mathbb{E}[CL|z] - \frac{d\mathbb{E}[CL|z]}{dz} \mathbb{E}[LS|z]}{\mathbb{E}[CL|z]^2} \\ &= \frac{1}{\lambda} \frac{\lambda \left(\frac{Q}{\lambda} F_{Q+1}(\tau) + (z - \tau) F_Q(\tau) + z \bar{F}_Q(\tau) \right) - \lambda(z - \tau)}{\mathbb{E}[CL|z]^2} \\ &= \frac{\frac{Q}{\lambda} F_{Q+1}(\tau) + \tau \bar{F}_Q(\tau)}{\mathbb{E}[CL|z]^2} > 0. \end{aligned}$$

And,

$$\frac{d^2[FL(\xi)|z]}{dz^2} = -2 \times \frac{\frac{Q}{\lambda} F_{Q+1}(\tau) + \tau \bar{F}_Q(\tau)}{\mathbb{E}[CL|z]^3} < 0.$$

$\square$

In the following, we draw some plots representing the propositions above by considering the data set $h = 1$, $p = 10$, $K = 50$, $\lambda = 5$, $L = 0.5$, $\tau = 2$, $Q = 8$, $r = 6$, and $\xi = 1.1$.

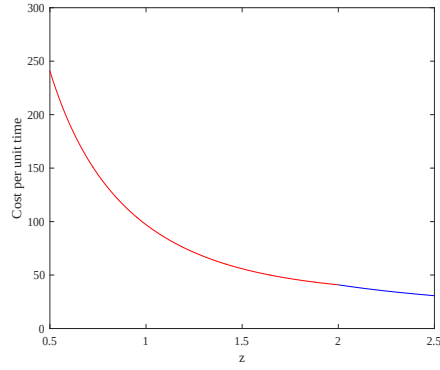


Figure 7.1: Total cost per unit time for a given z

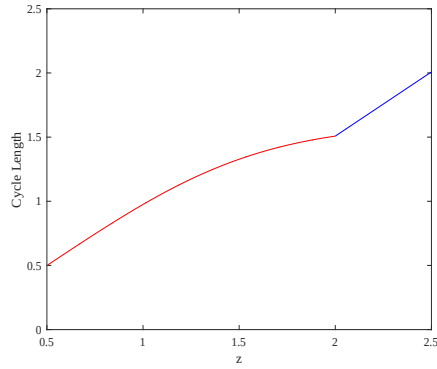


Figure 7.2: Expected cycle length for a given z

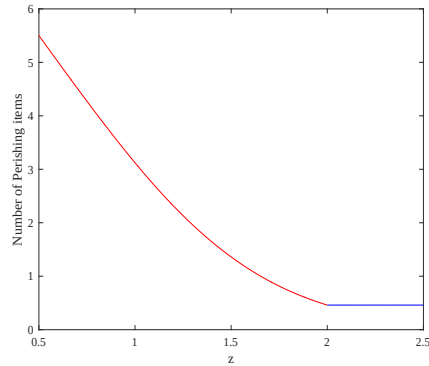


Figure 7.3: Expected number of perishing items for a given z

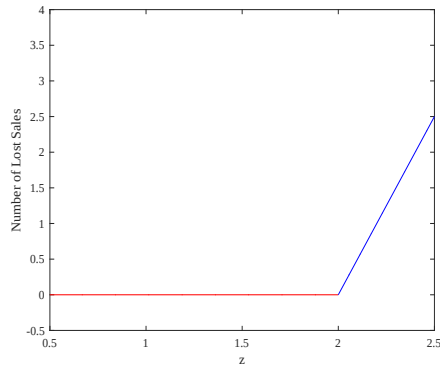


Figure 7.4: Expected number of lost sales for a given z

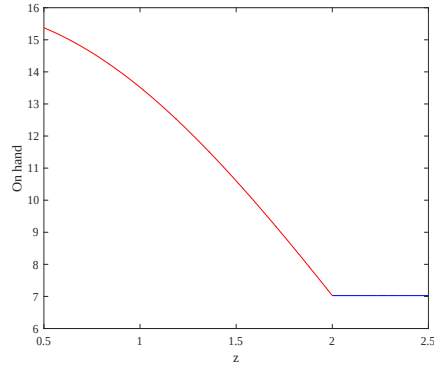


Figure 7.5: Expected on-hand inventory for a given z

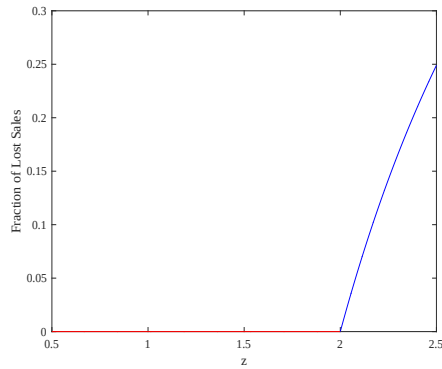


Figure 7.6: Fraction of lost sales for a given z

In order to develop an effective heuristic algorithm for the (Q, r, ξ) policy, we investigate the pattern of the objective function and service level constraint

with respect to the remaining lifetime. To find the optimal parameters of the (Q, r, ξ) policy, we first analyze the behavior of the expected total cost (ETC) and the fraction of lost demand ($FL(\xi)$) w.r.t ξ for the fixed values of Q and r . To illustrate these patterns in ETC and $FL(\xi)$, some of our experiments are depicted in Figure 7.7 and Figure 7.8 for the following set $Q = 8$, $\lambda = 6$, $L = 1$, $\tau = 2$, $p = 50$, $K = 100$, and $h = 1$. According to Figure 7.8, by fixing $FL(\xi)$ and Q , we need to decrease r as ξ increases to bind the service level constraint. This is because of the fact that by increasing ξ we likely have more service level and holding cost. To compensate this increase in holding cost, we can fall down r to have less holding cost so that the service level is not violated. Moreover, for a fixed value of Q , and for any r , $FL(\xi)$ and ETC converge to a fixed value as $\xi \rightarrow \tau$, as shown in Figure 7.7 and Figure 7.8. This limit is the same value as we achieve in (Q, Q) policy. In this case, as $\xi \rightarrow \tau$, the effect of r becomes negligible and it does not affect the expected total cost and service level because ξ is large enough and plays a role for reordering not r . Furthermore, when $\xi = 0$, the values in Figure 7.7 and Figure 7.8 represent the corresponding (Q, r) policy. It is obvious that for a fixed Q , and a fixed ξ where it is not close to τ , the higher r results in higher ETC and lower $FL(\xi)$.

According to Figure 7.7 and Figure 7.8, we observe that for a given Q and r , ETC is an increasing function of ξ , and $FL(\xi)$ is a decreasing function of ξ for any $\xi \in [0, \tau + L]$. The reason is because of the fact that by fixing Q and r and increasing ξ , we expect to have higher expected on-hand because we most likely order earlier than before. Earlier reordering causes the higher expected on-hand inventory. Therefore, by increasing ξ we expect to have more holding costs and consequently more operating costs. On the other hand, earlier reordering also causes receiving the new batch earlier and less number of lost demand during the lead time, which results in a higher service level (lower $FL(\xi)$) in the system.

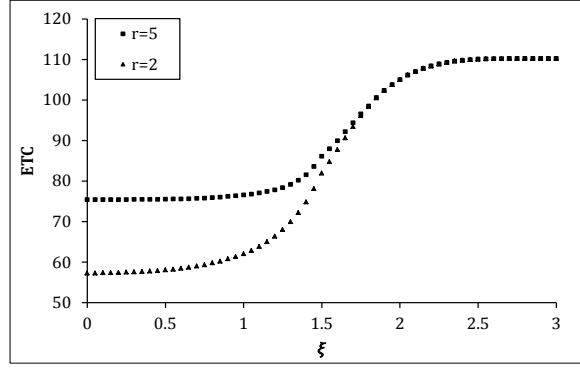


Figure 7.7: The expected total cost w.r.t ξ for the given Q and r

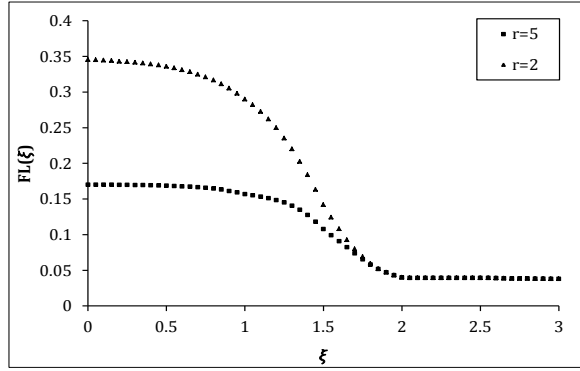


Figure 7.8: The fraction of lost demand w.r.t ξ for the given Q and r

Focusing on Figure 7.7, we observe that the objective function is an increasing function of ξ on $[0, \tau + L]$. Therefore, to achieve the minimum objective value, we can decrease ξ as much as possible. Adversely, by decreasing ξ , $FL(\xi)$ increases and the service level constraint will be violated accordingly. Therefore, for a given Q and r , we start from $\xi = \tau + L$, then by decreasing the value of ξ until the service level constraint binds, we attain the best ξ . Hence, we have the following conjecture.

Conjecture 1. *The optimal value of ξ for a fixed value of Q and r is obtained by solving $FL(\xi) = \alpha$.*

Since the implicit expression of $FL(\xi)$ is not attainable, the conjecture above is only a claim which is argued as follows. By fixing Q and r and increasing ξ ,

we expect to have higher expected on-hand because we most likely order earlier than before. Earlier reordering causes the higher expected on-hand inventory. Therefore, by increasing ξ we expect to have more holding costs and consequently more operating costs. On the other hand, earlier reordering also causes receiving the new batch earlier and less number of lost sales during the lead time, which results in the higher service level in the system. We observe that the objective function is an increasing function of ξ on $[0, \tau + L]$. Therefore, to achieve the minimum objective value, we can decrease ξ as much as possible. Adversely, by decreasing ξ , $FL(\xi)$ increases and the service level constraint will be violated accordingly. Therefore, for a fixed Q and r , the optimal ξ satisfies $FL(\xi) = \alpha$. We utilize the result of this conjecture to propose a heuristic algorithm for finding the optimal ξ^* in the (Q, r, ξ) policy. In the next chapter, we argue about this algorithm and its efficiency in detail.

Chapter 8

Numerical Study

In this section, we first provide the research questions which gives the outline of our numerical study. Then, we examine the performance of control policies under different parameter settings and deal with the sensitivity of policy parameters (Q , r , T , or ξ) and total cost rate with respect to fixed ordering cost, unit perishing cost, lead time, lifetime, demand rate, and service level. Our main goal in numerical study is to propose an efficient control policy for a specific parameter setting. We do not impose any restriction on Q and r values. That is, r and Q can take any integer value resulting in m outstanding orders, where $m \geq 1$. However, we deal with the effect of one order outstanding restriction on optimal solution and the corresponding objective function. Moreover, we consider the lifetime-based policy without a service level constraint to discover the effect of fraction of lost sales on the performance of policy. The structure of the objective function and service level constraint is also studied by which we elicit a heuristic algorithm for finding the optimal solution. Finally, we consider the effect of effective lifetime on finding the optimal control policy parameters. We apply the discretization approach argued in subsection 4.2.2 through all numerical studies.

8.1 Research Questions for Numerical Study

In this section, we carry out a numerical study to address our research questions which are as follows:

- Q1. How to find the optimal solution in each policy
- Q2. What are appropriate bounds for the policy parameters
- Q3. Is it possible to design a heuristic algorithm using the structure of objective function and service level constraint
- Q4. What is the behavior of total cost with respect to policy parameters
- Q5. Effect of the input parameters on selection of the control policy
- Q6. When and which policy is the best choice for an inventory system
- Q7. How do the input parameters affect the operating characteristics
- Q8. How does the number of outstanding orders affect the performance of policies
- Q9. How does the service level constraint affect the performance of policies
- Q10. How does the effective lifetime distribution affect the performance of policies

Our base numerical set is an extended version of that studied by Tekin et al. in [23]; it is provided below in Table 8.1. In our sensitivity analyses, we use various combinations of the base experiment instances as discussed in the relevant sections.

Table 8.1: Test parameters

Notation	Values
K	50, 100
p	1, 10, 50, 100
L	0.5, 1, 1.5
τ	2, 4, 6
λ	2, 5
h	1
α	0.005, 0.01, 0.02, 0.05, 0.1

We establish different parameter settings to address a number of research questions stated above as reported in Table 8.2.

Table 8.2: Test sets

Notation	Set 1	Set 2	Set 3
K	50, 100	50, 100	50
p	10, 100	1, 10, 50	1, 10, 50
L	1	1	0.5, 1, 1.5
τ	2, 4	2, 4, 6	2
λ	2, 5	5	5

We vary the fraction of lost sales threshold as $\alpha = 0.005, 0.01, 0.02, 0.05, 0.1$ and fix the value of unit holding cost to 1 ($h = 1$) in all data settings. The Set 1 is designed to deal with the effect of demand rate on the efficiency of policies. We also vary fixed ordering cost, unit perishing cost, and lifetime in this set to observe the effect of demand in different settings and the sensitivity of policy parameters with respect to various parameters. To investigate the effect of effective lifetime distribution and the lack of lifetime-based parameter ξ in control policy, we design another set in which we vary the fixed ordering cost and unit perishing cost which result in Set 2. Set 3 is provided to consider the effect of lead time on both optimal solution and total cost rate. Both Set 2 and Set 3 are introduced by Tekin et al. [23].

8.2 Numerical Search Approach

In this subsection, we consider the methodology of finding the effective lifetime distribution and the search method for finding the optimal parameters in each control policy. Then, we provide some bounds for control policy parameters. Utilizing the structure of the objective function and service level constraint we organize a heuristic algorithm. Our numerical results reveal that the proposed heuristic algorithm performs extremely well in comparison with the search method employed herein.

The operating characteristics and the ensuing objective function are computed in Matlab R2016a by the server of the Industrial Engineering Department of Bilkent University with a speed of 3.70 GHz. All results tabulated in this chapter are as computed by the expressions provided herein in earlier sections. The average computation times for a given parameters in the (Q, r, ξ) and (Q, r, T) policies are 274 seconds and 122 seconds for the cases $r > (m - 1)Q$ and $r = (m - 1)Q$, respectively. For the (Q, ξ) and (Q, T) policies is 19 seconds on average, and for the (Q, r) policy is 87 seconds on average. The running time depends on the value of Q and r ; the higher Q and r generally needs higher running time. The average time is calculated by considering all experiment instances. Note that the computation time for the (Q, r, ξ) and (Q, r, T) policies, and for the (Q, ξ) and (Q, T) policies are almost the same.

In order to identify and eliminate possible sources of coding error, we checked our computations against simulation as well. A simulation study was conducted with the Arena 14.5 student version using SIMAN simulation language on a personal computer with a speed of 3.20 GHz. Ten replications have been generated to obtain the simulation averages. The corresponding run time of a representative set of ten replications in Arena is 8 seconds on average and it does not vary discernibly across all policies. As we only use simulation to check our analytical results, simulations results were not reported herein.

8.2.1 Search Approach and Bounds

For finding the optimal values of the control policy parameters for all proposed policies, we implemented an exhaustive search algorithm over an appropriately selected search space. Let Q^* , r^* , T^* , and ξ^* denote the optimal value of control policy parameters. We search over the range $[Q_L, Q_U]$ for Q and the range $[0, r_U]$ for r , where Q_L , Q_U , and r_U are set to an initial value. The search is conducted on an increment size of one unit. If the found optimal r or Q (or both) hit the initial boundaries, we extend our search space and continue our search in this fashion until the found optimal lies inside the search space. The policy parameters ξ or T are searched over all possible values, i.e., $T = [0, \tau + L]$ and $\xi = [0, \tau + L]$ with a search increment of Δ . However, our numerical results show that $\xi^* \leq \tau$ and $T^* \leq \tau$. Thus, faster searches are possible by taking this observation into account. Noting that the effective lifetime distribution is different in the special region $r = (m-1)Q$, we search the special case for $m \geq 1$ in all policies, separately. Then, the optimal solution is obtained by comparing the solutions attained for $m = 1$ and $m > 1$.

As part of a preliminary study, we selected in addition 100 random solutions from a set of feasible solutions and compared them with the optimal solution obtained from exhaustive search. Our results show that the optimal solution attained by exhaustive search is never dominated by random solutions. We only do it for some instances and we select our random solutions outside of the ranges $[Q_L, Q_U]$ and $[0, r_U]$. To select those random solutions, we define an other range which is wider than $[Q_L, Q_U]$ and $[0, r_U]$. For instance, for a given set of data we start our search space with $[Q_L, Q_U] = [2, 10]$ and $[0, r_U] = [0, 10]$. We found the solution $(Q, r, \xi) = (10, 4, 0.2)$. Since the optimal solution hits the boundary Q_U , we only extend the search space for Q as $[Q_L, Q_U] = [10, 20]$ and then after implementing the exhaustive search we obtain the optimal solution $(Q, r, \xi) = (15, 7, 0.52)$ which is the global optimal solution because we do not hit the boundaries. In this case, we select the random solutions from a wider range such as $[Q_L, Q_U] = [2, 30]$ and $[0, r_U] = [0, 20]$, randomly.

To have a larger search space for Q and r , we can discretize the range of T or ξ to 0.1. That is, we search over $Q = [Q_L, Q_U]$, $r = [0, r_U]$ and $\xi = [0, \tau + L]$ with 0.1 increment for ξ . By doing so, we can search over the more extended range for Q and r , i.e., we can have smaller Q_L and larger Q_U and r_U . The optimal solution achieved by this approach gives us the correct direction of search to optimality. Then, using that solution and decreasing increment size of 0.01 we can obtain the global optimal value. For instance, suppose that $\tau = 2$, $L = 1$ and $(Q^*, r^*, \xi^*) = (14, 8, 0.6)$ is the optimal solution attained by increment size of 0.1 for ξ , where $[Q_L, Q_U] = [10, 20]$, $[0, r_U] = [0, 10]$, and $\xi \in \{0, 0.1, 0.2, 0.3, \dots, 3\}$. Now, the optimal solution gives us a clue for the global optimal solution for Q and r . Then, by reducing the increment size of ξ to 0.01 ($\xi \in \{0, 0.01, 0.02, 0.03, \dots, 3\}$) and reducing the search space of Q and r to $[Q_L, Q_U] = [12, 16]$, $[r_L, r_U] = [6, 10]$, we will reach the global optimal solution of $(Q^*, r^*, \xi^*) = (15, 7, 0.53)$. By implementing this approach we will get the same optimal solution as the exhaustive search algorithm with increment size of 0.01. Indeed, by selecting 0.1 as the increment size we do not change our search method (exhaustive algorithm), we can only search for a wider range of Q and r . This approach avoids the local optimality as well because we are searching more extensive search space. In result, the exhaustive search algorithm is the main methodology for finding the optimal solution for each policy (Q1 in research questions). The pseudo code of exhaustive search method with bounded variables Q and r is provided in the following.

Algorithm 1 The pseudo code of exhaustive search method

Initialization: Set $\lambda, L, \tau, h, p, K, \alpha, \Delta, MinCost \leftarrow \infty$;
for $Q = Q_L : Q_U$ **do**
 for $r = 0 : r_U$ **do**
 for $\xi = 0 : \Delta : \tau + L$ **do**
 Derive effective lifetime distribution using discretization approach;
 Calculate $\mathbb{E}[CL]$ and $\mathbb{E}[LS]$;
 if $\frac{\mathbb{E}[LS]}{\lambda \mathbb{E}[CL]} \leq \alpha$ **then**
 Calculate $\mathbb{E}[P]$ and $\mathbb{E}[OH]$;
 Calculate $\mathbb{E}[TC]$;
 if $\mathbb{E}[TC] < MinCost$ **then**
 $Q^* \leftarrow Q$;
 $r^* \leftarrow r$;
 $\xi^* \leftarrow \xi$;
 $\mathbb{E}[TC^*] \leftarrow \mathbb{E}[TC]$;
 $MinCost \leftarrow \mathbb{E}[TC]$;
 end if
 end if
 end for
 end for
end for
Return Q^*, r^*, ξ^* , and $\mathbb{E}[TC^*]$;

As mentioned earlier, for the three-parameter policies, the search space is large. Consider the (Q, r, ξ) policy and suppose that $[Q_L, Q_U]$ and $[0, r_U]$ are search spaces for Q and r , respectively; moreover, the search space of ξ is $[0, \tau + L]$ with $k+2$ discrete points (k discrete points over $(0, \tau + L)$ and two points 0 and $\tau + L$). Then, we need to evaluate $(Q_U - Q_L + 1) \times (r_U + 1) \times (k + 2)$ solutions for finding the global solution. However, we need to evaluate $(Q_U - Q_L + 1) \times (r_U + 1)$ solutions in the (Q, r) policy. It implies that the search space of the (Q, r, ξ) policy is $k + 2$ times larger than that of the (Q, r) policy. Therefore, for the three-parameter policies, in order to find the optimal solutions efficiently, we reduce our search space by finding some bounds for the control parameters as well as the objective value (Q2 in research questions). In the following, we introduce a number of bounds for control parameters and the objective function value (total cost). Since the implicit expression of the effective lifetime distribution is not attainable, all bounds provided here are dealt with intuitively and observed

through our numerical study.

(i) In order to achieve a lower bound for the optimal Q in all policies, we propose a special policy which is the special case of (Q, r, T) policy when $T = 0$ or special case of (Q, r, ξ) policy when $\xi = \tau + L$, in which the optimal Q gives us an appropriate lower bound. The reason is that in this special policy, we order immediately after receiving the new batch, which leads to more holding cost and higher service level in the system. Hence, in this policy, we have to reduce Q to have less holding cost so that the desired service level is not violated. In the cases with small α , these bounds are close to the optimal solution in most of which they hit the optimal solution.

(ii) The optimal r obtained for the (Q, r) policy is always greater than or equal to the optimal r in the (Q, r, T) policy for a given pair of Q and T . Similarly, the optimal r obtained in the (Q, r) policy is always greater than or equal to the optimal r in the (Q, r, ξ) policy for a given pair of Q and ξ . The reason is that when the reorder point is found based on only r , we have to reorder by considering the stock level. Therefore, the optimal reorder point r^* would be an integer value. Reordering at point $r^* - 1$ (at time X_{Q-r+1}) may violate the service level constraint and at point $r^* + 1$ (at time X_{Q-r-1}) may increase the holding cost. Hence, by incorporating the time ξ (or T) into the reorder level, we can have reordering at any time epoch between X_{Q-r-1} and X_{Q-r+1} . By doing so, we can bind the service level constraint and decrease r in order to have the lower holding cost. Therefore, the optimal r^* in the (Q, r) policy can be considered as an upper bound for the optimal r in the three-parameter policies for a given Q and ξ (or T). By the same interpretation, the optimal T obtained in the (Q, T) policy is less than or equal to the optimal T in the (Q, r, T) policy, which provides a lower bound for this policy. Also, the optimal ξ obtained in the (Q, ξ) policy is greater than or equal to the optimal ξ in the (Q, r, ξ) policy which can be considered as an upper bound for this policy.

(iii) Regarding the objective value of the (Q, r, ξ) policy, in the cases with a short lifetime τ , the optimal objective value of the (Q, ξ) policy can be considered as an upper bound; and in the cases with a long lifetime τ , the optimal

objective value of the (Q, r) policy can be considered as an upper bound for the objective value of the (Q, r, ξ) policy. Similarly, the objective function value obtained by (Q, T) and (Q, r) policies can be considered as an upper bound for the objective value of the (Q, r, T) policy in the cases with a short and long lifetime, respectively.

(iv) Consider the Tekin et al. model in [23] in which it is assumed that the new batch are always fresh at the beginning of the embedded cycles. Considering the (Q, r, ξ) policy under this assumption, we can easily find the optimal value of ξ by solving $\mathbb{E}(LS) - \alpha\lambda\mathbb{E}(CL) = 0$ (i.e., $FL(\xi) = \alpha$) for a fixed Q and r (applying Conjecture 1). Let us denote this optimal value by ξ_f^* , therefore $FL(\xi_f^*) = \alpha$. Let ξ^* denote the optimal value of ξ in the (Q, r, ξ) policy by considering the effective lifetime distribution at the beginning of embedded cycles (our model). To find the optimal control parameters in the (Q, r, ξ) policy, we use an exhaustive search algorithm as well, however we can reduce the search space of ξ by utilizing the result of the following conjecture.

Conjecture 2. ξ_f^* is an upper bound for the optimal value of ξ^* for a fixed value of Q and r in the (Q, r, ξ) policy.

From Conjecture 1, we can infer that the optimal value of ξ for a fixed value of Q and r binds the service level constraint. Since the effective lifetime distribution in the Tekin et al. approach is consistently greater than or equal to the effective lifetime distribution in our approach, we are willing to order early (higher ξ) in Tekin et al. model in order to have the same service level value. Hence, $\xi_f^* \geq \xi^*$.

The implicit expression of $FL(\xi)$ function w.r.t ξ is unavailable in our model in which we have a random effective lifetime at the beginning of the embedded cycles. Thus, we need to search for a value of ξ satisfying $FL(\xi) = \alpha$. In the following section, we derive a heuristic algorithm utilizing the result of Conjecture 1 and Conjecture 2 for reducing the search space of the (Q, r, ξ) policy (Q3 in research questions).

8.2.2 Heuristic Algorithm

To reduce the search space of ξ , we apply the result of Conjecture 2. Since the implicit expression of $FL(\xi)$ function w.r.t ξ is unavailable, we apply the interpolation method over the interval $[0, \xi_f^*]$ to find a ξ value satisfying $FL(\xi) = \alpha$. In the first iteration of the interpolation method, we set the new value of ξ to the midpoint of this interval, $\xi = \xi_f^*/2$. If $\xi = \xi_f^*/2$ satisfies the service level constraint, then the new interval for search will be $[0, \xi_f^*/2]$, otherwise $[\xi_f^*/2, \xi_f^*]$. Then, we select ξ as the midpoint of new interval and continue this procedure while the desired range for the lower bound ξ_L and upper bound ξ_U is attained. Then, by doing an exhaustive search over the interval $[\xi_L, \xi_U]$ we can find ξ^* . The desired range for lower bound and upper bound is set such that the interval $[\xi_L, \xi_U]$ contains 10 mass points. That is, $\xi_U - \xi_L = 10 \times \Delta$. The interval $[\xi_L, \xi_U]$ might contain higher number of mass points. With higher number of mass points, the algorithm converges faster but it needs to search higher number of solutions after convergence. Note also that in any iteration, ξ is set to be the nearest multiple of Δ . Then, we can develop a heuristic algorithm as Algorithm 1. In this heuristic algorithm, $ETC(Q, r, \xi)$ and $FL(Q, r, \xi)$ denote the expected total cost and the fraction of lost sales for a given Q , r and ξ , respectively.

Algorithm 2 Heuristic Algorithm

Initialization: Set $\lambda, L, \tau, h, p, K, \alpha, MinCost \leftarrow \infty$;

for $Q = Q_L : Q_U$ **do**

for $r = 0 : r_U$ **do**

 Find ξ_f^* ;

$\xi_L \leftarrow 0$;

$\xi_U \leftarrow \xi_f^*$;

while $\xi_U - \xi_L \geq 10 \times \Delta$ **do**

$\xi = \lceil \frac{\xi_L + \xi_U}{2} \rceil \times \Delta$;

if $FL(Q, r, \xi) > \alpha$ **then**

$\xi_L \leftarrow \xi$;

else

$\xi_U \leftarrow \xi$;

end if

end while

for $\xi = \xi_L : \xi_U$ **do**

if $FL(Q, r, \xi) \leq \alpha$ **then**

$\xi_1^* \leftarrow \xi$;

end if

end for

 Calculate $ETC(Q, r, \xi_1^*)$;

if $ETC(Q, r, \xi_1^*) < MinCost$ **then**

$MinCost \leftarrow ETC(Q, r, \xi_1^*)$;

$Q^* \leftarrow Q$;

$r^* \leftarrow r$;

$\xi^* \leftarrow \xi_1^*$;

end if

end for

end for

Return Q^*, r^* , and ξ^* ;

Our numerical results indicate that the optimal solution obtained by implementing the heuristic algorithm (Algorithm 2) is the same as the ones obtained by the exhaustive search algorithm. The computation time of the heuristic algorithm depends on the shape of $FL(Q, r, \xi)$, especially its shape close to optimal ξ . There are some cases in which the heuristic algorithm returns the optimal ξ for a given Q and r after five iterations and some cases after 20 iterations. The proposed heuristic algorithm saves time for finding the optimal solution because for a given Q and r , we do not need to search over all possibilities of ξ . Hence,

the heuristic algorithm is faster than the exhaustive search algorithm for finding the global optimal solution.

It is worthwhile to note that in the (Q, r, T) policy, we have also properties like the ones mentioned above for the (Q, r, ξ) policy. In particular, for a given Q and r the total cost per unit time is a decreasing function of T and fraction of lost sales is an increasing function of T . Therefore, we can apply a similar heuristic algorithm for the (Q, r, T) policy for obtaining the optimal solution. We are not able to prove this argument because the implicit expression of the service level constraint is not attainable. We only observed this behavior in our numerical study.

8.3 Numerical Results

In this section, we deal with the sensitivity of key parameters in our model in the cases with and without service level constraint. Then, we investigate the effect of effective lifetime distribution on the optimal solution and how it affects the structure of optimal solutions. Moreover, we demonstrate that the number of outstanding orders can drastically affects the expected total cost. We also discuss some observations elicited from the numerical study without an analytic proof. Finally, we check the consistency of the proposed analytical results and our optimization results through a simulation study.

8.3.1 Model with Service Level Constraint

In this subsection, we present the performance of our proposed policies which are (Q, r, ξ) , (Q, r, T) , (Q, ξ) , (Q, T) , and (Q, r) under the service level constraint and respond Q4, Q5, and Q6 in research questions. Our sensitivity analysis results are based on different values of τ , p , λ and K . The numerical results for Set 1 are reported in Tables 8.3-8.6, where $\mathbb{E}(TC_i)$ denotes the expected cost per unit time for policy i . The index of $i = 1, 2, 3, 4$, and 5 are for (Q, r, ξ) , (Q, r, T) ,

(Q, ξ) , (Q, T) , and (Q, r) policies, respectively. We select a small and large value for λ to illustrate the efficiency of the proposed policies in low demand rate; other settings are designed to demonstrate how sensitive the policies are to the key input parameters. We also compute the percentage deviation between two policies in Tables 8.7-8.10, where $\%gap_{ij}$ represents the percentage deviation between two policies i and j and is computed as $\%gap_{ij} = \frac{\mathbb{E}(TC_j) - \mathbb{E}(TC_i)}{\mathbb{E}(TC_i)} \times 100$, where $i, j = 1, 2, 3, 4, 5$ which represent (Q, r, ξ) , (Q, r, T) , (Q, ξ) , (Q, T) , and (Q, r) policies, respectively. In Tables below “**Ave**” and “**St.dev**” represents the average and the standard deviation values, respectively. We can find the number of outstanding orders using the $m = \lceil \frac{r}{Q} \rceil$ formulation. For instance, for $(Q, r) = (4, 11)$ pair, the number of outstanding orders is $m = \lceil \frac{11}{4} \rceil = 3$. And similarly, for the (Q, r, ξ) and (Q, r, T) policies, we have also the same formulation. For instance, for $(Q, r, \xi) = (5, 9, 0.64)$, we have $m = \lceil \frac{9}{5} \rceil = 2$ outstanding orders. Although in the (Q, ξ) and (Q, T) policies there might be at most two outstanding orders in the system, we only need to track the effective lifetime of the current batch at the beginning of embedded cycles for analyzing the system characteristics.

In Tables 8.3-8.6, we compare five proposed policies by varying the input parameters λ , p , τ , K , and α in order to find the range of parameters in which the proposed (Q, r, ξ) policy performs well. Moreover, we can discover the effect of inventory position threshold r and remaining lifetime threshold ξ on expected total cost rate. Also, the effect of input parameters on optimal solutions is studied.

Table 8.3: Comparison of policies for $\lambda = 5$ and $p = 100$, ($L = 1$, $h = 1$)

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, T)	$\mathbb{E}(TC_4)$	(Q, r)	$\mathbb{E}(TC_5)$
$\tau = 2$ $K = 50$	0.005	(4, 10, 0.31)	138.29	(4, 10, 0.40)	145.77	(10, 1.61)	219.64	(11, 0.06)	278.01	(4, 11)	159.84
	0.01	(5, 9, 0.64)	119.85	(4, 9, 0.62)	125.79	(9, 1.76)	170.17	(10, 0.07)	221.19	(4, 10)	137.01
	0.02	(5, 8, 0.58)	101.41	(5, 8, 0.46)	107.27	(8, 1.89)	128.12	(9, 0.10)	168.06	(4, 9)	118.02
	0.05	(5, 7, 0.04)	84.90	(5, 7, 1.90)	84.91	(7, 1.85)	89.05	(7, 0.05)	100.06	(6, 7)	90.96
	0.1	(6, 5, 1.79)	64.21	(5, 5, 1.30)	67.87	(6, 1.87)	65.04	(6, 0.12)	73.63	(5, 6)	73.44
$K = 100$	0.005	(6, 9, 0.84)	196.01	(6, 10, 0.39)	210.30	(10, 1.61)	253.35	(11, 0.06)	311.27	(5, 11)	222.68
	0.01	(6, 9, 0.40)	174.94	(5, 9, 0.22)	182.04	(9, 1.76)	204.76	(10, 0.07)	254.87	(4, 10)	206.56
	0.02	(6, 8, 0.32)	153.79	(6, 8, 0.90)	159.08	(8, 1.89)	164.08	(9, 0.10)	202.29	(5, 9)	174.02
	0.05	(7, 6, 1.71)	123.70	(6, 6, 1.21)	130.35	(7, 1.85)	126.33	(7, 0.05)	138.07	(6, 7)	134.18
	0.1	(7, 5, 1.32)	100.29	(7, 5, 0.56)	106.35	(6, 1.87)	104.35	(6, 0.12)	113.63	(7, 6)	111.73
$\tau = 4$ $K = 50$	0.005	(10, 9, 2.40)	44.48	(10, 9, 0.42)	44.72	(10, 3.54)	48.07	(10, 0.12)	46.80	(8, 10)	47.74
	0.01	(10, 8, 2.54)	40.79	(10, 8, 0.54)	40.96	(9, 3.68)	42.17	(9, 0.08)	42.10	(9, 9)	43.52
	0.02	(11, 7, 2.16)	37.14	(11, 7, 1.16)	37.42	(8, 3.82)	40.27	(9, 0.24)	39.74	(10, 8)	39.35
	0.05	(11, 6, 0.00)	32.35	(11, 6, 4.00)	32.35	(9, 3.26)	37.15	(9, 0.52)	36.37	(11, 6)	32.35
	0.1	(12, 4, 1.62)	27.67	(12, 4, 2.54)	27.77	(9, 3.08)	33.50	(10, 1.10)	32.36	(11, 5)	29.48
$K = 100$	0.005	(11, 9, 1.00)	69.28	(11, 9, 1.06)	69.45	(10, 3.54)	73.58	(10, 0.12)	72.26	(10, 10)	75.70
	0.01	(12, 8, 1.00)	63.95	(11, 8, 0.86)	65.07	(9, 3.68)	70.00	(10, 0.24)	69.33	(11, 9)	69.32
	0.02	(12, 7, 1.68)	58.59	(12, 7, 1.66)	58.86	(9, 3.48)	67.98	(10, 0.42)	65.55	(11, 8)	63.32
	0.05	(13, 5, 1.89)	52.45	(13, 5, 1.90)	53.26	(10, 3.04)	62.93	(12, 1.12)	60.68	(12, 6)	53.29
	0.1	(13, 4, 0.70)	45.36	(13, 4, 3.00)	45.74	(12, 2.38)	53.53	(11, 1.32)	53.56	(14, 4)	45.84

Table 8.4: Comparison of policies for $\lambda = 5$ and $p = 10$, ($L = 1$, $h = 1$)

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, T)	$\mathbb{E}(TC_4)$	(Q, r)	$\mathbb{E}(TC_5)$
$\tau = 2$ $K = 50$	0.005	(11, 9, 1.21)	58.51	(9, 9, 1.33)	60.78	(10, 1.61)	60.77	(11, 0.06)	67.26	(11, 11)	69.49
	0.01	(11, 8, 1.15)	53.68	(8, 9, 1.44)	56.93	(9, 1.76)	55.50	(10, 0.07)	60.77	(10, 10)	63.03
	0.02	(11, 7, 1.05)	48.79	(9, 8, 0.35)	52.16	(8, 1.89)	51.42	(9, 0.10)	54.73	(10, 9)	55.99
	0.05	(9, 6, 1.04)	42.72	(9, 7, 1.14)	45.73	(10, 1.25)	44.88	(8, 0.20)	48.76	(8, 7)	46.72
	0.1	(10, 3, 1.02)	37.88	(9, 5, 1.10)	39.31	(10, 1.08)	38.67	(9, 0.62)	44.30	(9, 6)	40.85
$K = 100$	0.005	(14, 8, 1.06)	87.47	(9, 9, 1.33)	97.03	(14, 1.18)	92.87	(11, 0.06)	100.53	(11, 11)	103.37
	0.01	(12, 8, 1.00)	81.73	(10, 9, 0.24)	91.14	(14, 1.09)	85.99	(10, 0.07)	94.45	(10, 10)	97.39
	0.02	(11, 7, 1.05)	76.98	(10, 8, 0.38)	85.66	(14, 1.00)	79.94	(9, 0.10)	88.96	(10, 9)	88.16
	0.05	(11, 4, 1.09)	70.53	(10, 7, 0.92)	75.70	(12, 1.02)	70.74	(10, 0.44)	81.69	(10, 8)	80.23
	0.1	(10, 3, 1.02)	63.70	(10, 6, 1.80)	67.32	(11, 0.97)	64.31	(10, 0.75)	73.65	(9, 6)	69.71
$\tau = 4$ $K = 50$	0.005	(14, 9, 0.88)	33.06	(13, 9, 1.46)	33.33	(10, 3.54)	36.90	(12, 0.42)	35.62	(14, 10)	35.25
	0.01	(14, 8, 0.90)	31.03	(14, 8, 1.82)	31.30	(11, 3.12)	36.12	(12, 0.58)	34.14	(14, 9)	32.98
	0.02	(14, 7, 0.84)	29.13	(14, 7, 2.22)	29.29	(11, 3.00)	34.88	(13, 0.94)	32.41	(15, 8)	30.91
	0.05	(16, 5, 0.96)	25.98	(15, 5, 2.52)	26.46	(14, 2.20)	30.43	(14, 1.48)	29.71	(15, 6)	27.09
	0.1	(16, 3, 1.06)	23.64	(15, 4, 4.00)	23.96	(14, 1.98)	26.06	(14, 1.98)	26.47	(15, 4)	23.96
$K = 100$	0.005	(18, 9, 0.96)	50.57	(17, 9, 1.72)	52.23	(20, 1.76)	58.92	(14, 0.72)	55.82	(14, 10)	54.42
	0.01	(18, 7, 1.40)	47.46	(16, 8, 1.98)	48.83	(18, 1.88)	54.58	(15, 1.04)	53.50	(15, 9)	50.81
	0.02	(18, 6, 1.24)	44.53	(16, 7, 2.40)	45.98	(20, 1.50)	50.20	(16, 1.40)	51.07	(16, 8)	47.53
	0.05	(18, 5, 0.86)	41.16	(17, 5, 2.92)	41.66	(19, 1.38)	43.89	(17, 1.98)	46.83	(16, 6)	42.76
	0.1	(17, 3, 0.90)	37.74	(18, 3, 3.56)	38.06	(19, 1.10)	38.93	(17, 2.56)	41.66	(16, 4)	38.48

Table 8.5: Comparison of policies for $\lambda = 2$ and $p = 100$, ($L = 1$, $h = 1$)

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, T)	$\mathbb{E}(TC_4)$	(Q, r)	$\mathbb{E}(TC_5)$
$\tau = 2$ $K = 50$	0.005	(3, 5, 1.02)	164.70	(5, 5, 1.28)	203.10	(6, 1.43)	221.06	(8, 0.10)	358.94	(4, 8)	267.56
	0.01	(3, 5, 0.56)	147.45	(3, 5, 0.10)	172.94	(5, 1.61)	177.04	(7, 0.12)	292.52	(3, 6)	182.33
	0.02	(3, 4, 0.56)	125.04	(4, 4, 1.41)	140.83	(5, 1.43)	158.86	(6, 0.13)	231.02	(3, 6)	182.33
	0.05	(2, 3, 1.40)	103.74	(3, 3, 1.09)	107.51	(4, 1.50)	113.46	(4, 0.04)	133.04	(4, 4)	136.00
	0.1	(3, 2, 1.75)	78.59	(2, 3, 1.39)	87.79	(3, 1.82)	79.64	(3, 0.05)	92.50	(3, 3)	94.93
$K = 100$	0.005	(3, 5, 1.02)	215.96	(5, 5, 1.28)	239.13	(6, 1.43)	252.91	(8, 0.10)	391.25	(4, 8)	318.71
	0.01	(3, 4, 0.90)	195.87	(4, 5, 0.88)	217.23	(5, 1.61)	210.67	(7, 0.12)	324.73	(3, 6)	236.02
	0.02	(3, 4, 0.56)	170.36	(4, 4, 1.41)	177.78	(5, 1.43)	190.71	(6, 0.13)	263.41	(3, 6)	236.02
	0.05	(4, 3, 1.28)	136.78	(3, 3, 1.09)	149.55	(4, 1.50)	146.79	(4, 0.04)	168.49	(4, 4)	171.90
	0.1	(4, 2, 1.08)	110.72	(4, 3, 1.32)	125.03	(3, 1.82)	116.36	(3, 0.05)	131.03	(3, 3)	134.09
$\tau = 4$ $K = 50$	0.005	(4, 5, 0.52)	59.95	(4, 5, 1.44)	64.40	(6, 3.00)	82.88	(6, 0.22)	81.30	(4, 6)	71.27
	0.01	(5, 4, 2.66)	56.24	(4, 5, 4.00)	58.37	(5, 3.40)	60.72	(5, 0.12)	61.57	(4, 5)	58.37
	0.02	(4, 3, 3.96)	47.30	(4, 3, 0.02)	48.03	(4, 4.00)	47.30	(4, 0.00)	48.44	(4, 4)	48.44
	0.05	(4, 3, 1.86)	37.84	(4, 3, 1.08)	38.56	(4, 3.38)	42.03	(4, 0.34)	42.75	(4, 4)	48.44
	0.1	(5, 2, 0.82)	30.84	(5, 2, 2.86)	31.27	(4, 3.04)	36.48	(3, 0.24)	38.93	(4, 3)	37.27
$K = 100$	0.005	(6, 5, 0.96)	91.81	(6, 5, 0.66)	95.58	(6, 3.00)	104.02	(6, 0.22)	102.34	(4, 6)	100.63
	0.01	(5, 4, 2.66)	78.85	(5, 4, 0.22)	82.48	(5, 3.40)	83.73	(5, 0.12)	84.66	(4, 5)	86.29
	0.02	(5, 4, 0.64)	71.87	(5, 4, 1.62)	72.16	(5, 3.12)	79.10	(4, 0.00)	75.11	(4, 4)	75.11
	0.05	(5, 3, 0.64)	59.30	(5, 3, 2.32)	59.66	(4, 3.38)	67.39	(4, 0.34)	68.19	(5, 4)	71.73
	0.1	(5, 2, 0.82)	49.71	(5, 2, 2.86)	50.14	(6, 2.02)	58.66	(4, 0.76)	60.88	(5, 3)	58.98

Table 8.6: Comparison of policies for $\lambda = 2$ and $p = 10$, ($L = 1$, $h = 1$)

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, T)	$\mathbb{E}(TC_4)$	(Q, r)	$\mathbb{E}(TC_5)$
$\tau = 2$ $K = 50$	0.005	(7, 5, 1.00)	52.52	(5, 6, 1.26)	59.67	(8, 1.09)	54.63	(8, 0.10)	72.85	(8, 8)	75.98
	0.01	(6, 4, 1.17)	48.64	(5, 5, 2.36)	54.45	(7, 1.15)	50.77	(7, 0.12)	64.91	(6, 6)	60.95
	0.02	(6, 3, 1.10)	44.60	(6, 5, 0.34)	54.74	(7, 1.01)	45.76	(6, 0.13)	57.78	(6, 6)	60.95
	0.05	(6, 0, 1.00)	40.08	(5, 4, 0.72)	45.81	(6, 1.00)	40.01	(4, 0.04)	48.77	(4, 4)	49.54
	0.1	(5, 0, 1.00)	35.34	(5, 3, 1.20)	39.02	(5, 1.01)	35.36	(4, 0.35)	43.59	(5, 4)	42.93
$K = 100$	0.005	(8, 4, 1.00)	77.86	(5, 6, 1.26)	96.52	(8, 1.09)	80.81	(8, 0.10)	105.16	(8, 8)	109.36
	0.01	(7, 4, 1.00)	74.73	(5, 5, 2.36)	88.85	(8, 1.00)	76.18	(7, 0.12)	97.12	(6, 6)	94.68
	0.02	(7, 1, 1.00)	70.55	(6, 5, 0.34)	85.94	(7, 1.01)	70.89	(6, 0.13)	90.16	(6, 6)	94.68
	0.05	(6, 0, 1.00)	65.12	(5, 4, 0.72)	76.06	(6, 1.00)	65.01	(5, 0.25)	81.33	(4, 4)	85.44
	0.1	(5, 0, 1.00)	60.48	(5, 3, 1.20)	65.74	(5, 1.01)	60.48	(5, 0.58)	73.11	(5, 4)	71.51
$\tau = 4$ $K = 50$	0.005	(8, 4, 1.84)	30.48	(6, 5, 0.66)	31.84	(6, 3.00)	33.63	(6, 0.22)	33.19	(6, 6)	35.16
	0.01	(7, 4, 1.36)	27.32	(6, 4, 0.74)	30.01	(7, 2.34)	32.71	(6, 0.42)	31.60	(5, 5)	32.90
	0.02	(7, 3, 1.72)	26.03	(7, 4, 1.80)	27.57	(7, 2.20)	29.90	(5, 0.34)	30.48	(6, 5)	30.76
	0.05	(7, 2, 1.40)	22.78	(7, 3, 2.80)	23.83	(8, 1.60)	24.94	(6, 1.12)	26.97	(7, 4)	25.98
	0.10	(7, 1, 1.10)	20.23	(7, 1, 2.78)	21.44	(8, 1.24)	21.30	(7, 2.12)	23.50	(8, 3)	22.75
$K = 100$	0.005	(11, 4, 0.98)	43.61	(8, 5, 1.12)	50.50	(12, 1.34)	49.64	(8, 0.74)	53.31	(6, 6)	56.89
	0.01	(10, 3, 1.20)	42.11	(8, 4, 1.40)	47.26	(12, 1.16)	46.13	(8, 0.96)	50.80	(5, 5)	56.35
	0.02	(10, 2, 1.12)	40.04	(8, 4, 1.94)	44.21	(11, 1.16)	42.68	(8, 1.22)	48.14	(6, 5)	50.91
	0.05	(10, 1, 0.92)	37.27	(8, 3, 2.88)	38.86	(10, 1.04)	37.67	(7, 1.46)	43.81	(7, 4)	42.48
	0.10	(9, 0, 0.84)	33.95	(8, 2, 3.90)	34.39	(9, 0.96)	34.05	(8, 2.50)	37.89	(8, 3)	36.56

Table 8.7: Percentage deviation between policies with $\lambda = 5$ and $p = 100$ in Table 8.3

Parameters		α	%gap ₁₂	%gap ₁₃	%gap ₁₄	%gap ₁₅	%gap ₂₃	%gap ₂₄	%gap ₂₅	%gap ₃₄	%gap ₃₅	%gap ₄₅
$\tau = 2$	$K = 50$	0.005	5.41	58.83	101.03	15.58	50.68	90.72	9.65	26.58	-27.23	-42.51
		0.01	4.96	41.99	84.56	14.32	35.28	75.84	8.92	29.98	-19.49	-38.06
		0.02	5.78	26.34	65.72	16.38	19.44	56.67	10.02	31.17	-7.88	-29.78
		0.05	0.00	4.89	17.85	7.13	4.89	17.85	7.13	12.36	2.14	-9.10
		0.1	5.70	1.30	14.67	14.38	-4.16	8.49	8.22	13.21	12.92	-0.26
		Ave	4.37	26.67	56.77	13.56	21.22	49.91	8.79	22.66	-7.91	-23.94
	$K = 100$	0.005	7.29	29.25	58.80	13.61	20.47	48.01	5.89	22.86	-12.11	-28.46
		0.01	4.06	17.05	45.69	18.07	12.48	40.01	13.47	24.47	0.88	-18.95
		0.02	3.44	6.69	31.54	13.15	3.14	27.16	9.39	23.29	6.06	-13.97
		0.05	5.38	2.13	11.62	8.47	-3.08	5.92	2.94	9.29	6.21	-2.82
		0.1	6.04	4.05	13.30	11.41	-1.88	6.85	5.06	8.89	7.07	-1.67
		Ave	5.24	11.83	32.19	12.94	6.23	25.59	7.35	17.76	1.62	-13.18
$\tau = 4$	$K = 50$	0.005	0.54	8.07	5.21	7.34	7.48	4.64	6.76	-2.64	-0.67	2.02
		0.01	0.40	3.38	3.21	6.68	2.97	2.80	6.26	-0.17	3.19	3.36
		0.02	0.76	8.45	7.00	5.95	7.63	6.19	5.15	-1.33	-2.30	-0.98
		0.05	0.00	14.81	12.42	0.00	14.81	12.42	0.00	-2.08	-12.90	-11.05
		0.1	0.35	21.08	16.95	6.55	20.66	16.53	6.17	-3.42	-12.00	-8.89
		Ave	0.41	11.16	8.96	5.30	10.71	8.52	4.87	-1.93	-4.94	-3.11
	$K = 100$	0.005	0.24	6.20	4.29	9.26	5.94	4.04	9.00	-1.80	2.88	4.76
		0.01	1.76	9.47	8.41	8.40	7.58	6.54	6.53	-0.97	-0.98	-0.01
		0.02	0.47	16.03	11.89	8.08	15.48	11.36	7.58	-3.57	-6.85	-3.40
		0.05	1.55	19.99	15.69	1.61	18.16	13.93	0.06	-3.58	-15.32	-12.17
		0.1	0.85	18.02	18.09	1.06	17.03	17.10	0.21	0.06	-14.37	-14.43
		Ave	0.97	13.94	11.68	5.68	12.84	10.59	4.67	-1.97	-6.93	-5.05

Table 8.8: Percentage deviation between policies with $\lambda = 5$ and $p = 10$ in Table 8.4

Parameters		α	%gap ₁₂	%gap ₁₃	%gap ₁₄	%gap ₁₅	%gap ₂₃	%gap ₂₄	%gap ₂₅	%gap ₃₄	%gap ₃₅	%gap ₄₅
$\tau = 2$	$K = 50$	0.005	3.88	3.87	14.95	18.77	-0.01	10.66	14.34	10.67	14.35	3.32
		0.01	6.06	3.38	13.21	17.42	-2.52	6.74	10.72	9.50	13.58	3.72
		0.02	6.91	5.39	12.18	14.77	-1.42	4.94	7.35	6.44	8.90	2.30
		0.05	7.04	5.04	14.12	9.35	-1.87	6.62	2.16	8.64	4.10	-4.18
		0.1	3.77	2.08	16.95	7.84	-1.63	12.70	3.92	14.57	5.65	-7.79
		Ave	5.53	3.95	14.28	13.63	-1.49	8.33	7.70	9.97	9.31	-0.52
	$K = 100$	0.005	10.93	6.18	14.94	18.18	-4.29	3.61	6.54	8.25	11.31	2.83
		0.01	11.51	5.21	15.56	19.16	-5.65	3.63	6.86	9.83	13.26	3.12
		0.02	11.27	3.84	15.56	14.51	-6.68	3.85	2.91	11.28	10.28	-0.90
		0.05	7.33	0.29	15.83	13.75	-6.56	7.91	5.98	15.49	13.42	-1.79
		0.1	5.67	0.96	15.61	9.42	-4.46	9.40	3.55	14.51	8.39	-5.35
		Ave	9.34	3.30	15.50	15.01	-5.53	5.68	5.17	11.87	11.33	-0.42
$\tau = 4$	$K = 50$	0.005	0.79	11.59	7.73	6.60	10.72	6.89	5.76	-3.46	-4.47	-1.06
		0.01	0.87	16.40	10.03	6.30	15.40	9.08	5.38	-5.47	-8.68	-3.39
		0.02	0.56	19.73	11.28	6.10	19.06	10.66	5.51	-7.06	-11.38	-4.65
		0.05	1.85	17.12	14.37	4.28	14.99	12.29	2.38	-2.34	-10.96	-8.83
		0.1	1.36	10.24	11.99	1.36	8.76	10.49	0.00	1.59	-8.06	-9.49
		Ave	1.08	15.01	11.08	4.93	13.79	9.88	3.81	-3.35	-8.71	-5.48
	$K = 100$	0.005	3.28	16.50	10.39	7.61	12.80	6.88	4.20	-5.25	-7.63	-2.51
		0.01	2.89	14.98	12.71	7.04	11.76	9.55	4.04	-1.98	-6.91	-5.03
		0.02	3.25	12.74	14.68	6.74	9.19	11.07	3.37	1.72	-5.33	-6.92
		0.05	1.20	6.62	13.77	3.88	5.36	12.41	2.65	6.70	-2.57	-8.69
		0.1	0.83	3.15	10.37	1.95	2.29	9.46	1.10	7.01	-1.16	-7.64
		Ave	2.29	10.80	12.38	5.44	8.28	9.87	3.07	1.64	-4.72	-6.16

Table 8.9: Percentage deviation between policies with $\lambda = 2$ and $p = 100$ in Table 8.5

Parameters		α	%gap ₁₂	%gap ₁₃	%gap ₁₄	%gap ₁₅	%gap ₂₃	%gap ₂₄	%gap ₂₅	%gap ₃₄	%gap ₃₅	%gap ₄₅
$\tau = 2$	$K = 50$	0.005	23.32	34.22	117.94	62.45	8.84	76.73	31.74	62.37	21.04	-25.46
		0.01	17.29	20.07	98.39	23.66	2.37	69.15	5.43	65.23	2.99	-37.67
		0.02	12.63	27.05	84.76	45.82	12.80	64.04	29.47	45.42	14.77	-21.08
		0.05	3.63	9.37	28.24	31.10	5.53	23.75	26.50	17.26	19.87	2.22
		0.1	11.70	1.33	17.69	20.78	-9.29	5.36	8.13	16.15	19.20	2.63
		Ave	13.71	18.41	69.40	36.76	4.05	47.81	20.25	41.29	15.57	-15.87
	$K = 100$	0.005	10.73	17.11	81.17	47.58	5.76	63.61	33.28	54.70	26.02	-18.54
		0.01	10.91	7.56	65.79	20.50	-3.02	49.49	8.65	54.14	12.03	-27.32
		0.02	4.36	11.95	54.62	38.54	7.27	48.17	32.76	38.12	23.76	-10.40
		0.05	9.34	7.32	23.18	25.68	-1.85	12.66	14.94	14.78	17.11	2.02
		0.1	12.92	5.09	18.34	21.11	-6.93	4.80	7.25	12.61	15.24	2.34
		Ave	9.65	9.80	48.62	30.68	0.25	35.75	19.38	34.87	18.83	-10.38
	$K = 50$	0.005	7.43	38.25	35.61	18.89	28.69	26.23	10.66	-1.91	-14.00	-12.33
		0.01	3.80	7.97	9.47	3.80	4.03	5.47	0.00	1.39	-3.87	-5.18
		0.02	1.54	0.00	2.42	2.42	-1.52	0.87	0.87	2.42	2.42	0.00
		0.05	1.91	11.07	12.99	28.03	8.99	10.87	25.64	1.73	15.28	13.32
		0.1	1.40	18.26	26.20	20.84	16.63	24.47	19.18	6.72	2.18	-4.25
		Ave	3.21	15.11	17.34	14.80	11.36	13.58	11.27	2.07	0.40	-1.69
$\tau = 4$	$K = 100$	0.005	4.11	13.30	11.47	9.60	8.83	7.07	5.28	-1.62	-3.26	-1.67
		0.01	4.60	6.18	7.36	9.43	1.51	2.63	4.62	1.11	3.07	1.93
		0.02	0.41	10.06	4.52	4.52	9.62	4.10	4.10	-5.03	-5.03	0.00
		0.05	0.62	13.66	14.99	20.97	12.96	14.29	20.23	1.17	6.43	5.19
		0.1	0.86	18.01	22.47	18.65	17.00	21.42	17.63	3.78	0.54	-3.12
		Ave	2.12	12.24	12.16	12.63	9.98	9.90	10.37	-0.12	0.35	0.47

Table 8.10: Percentage deviation between policies with $\lambda = 2$ and $p = 10$ in Table 8.6

Parameters		α	%gap ₁₂	%gap ₁₃	%gap ₁₄	%gap ₁₅	%gap ₂₃	%gap ₂₄	%gap ₂₅	%gap ₃₄	%gap ₃₅	%gap ₄₅
$\tau = 2$	$K = 50$	0.005	13.60	4.01	38.70	44.65	-8.44	22.10	27.34	33.35	39.07	4.29
		0.01	11.95	4.39	33.46	25.32	-6.75	19.22	11.94	27.85	20.04	-6.10
		0.02	22.74	2.61	29.55	36.67	-16.40	5.55	11.35	26.26	33.19	5.49
		0.05	14.29	-0.16	21.69	23.60	-12.65	6.48	8.15	21.89	23.80	1.57
		0.1	10.41	0.05	23.36	21.48	-9.38	11.73	10.02	23.29	21.41	-1.53
		Ave	14.60	2.18	29.35	30.34	-10.72	13.02	13.76	26.53	27.50	0.74
	$K = 100$	0.005	23.96	3.79	35.06	40.45	-16.27	8.96	13.31	30.13	35.33	3.99
		0.01	18.89	1.94	29.97	26.71	-14.26	9.31	6.57	27.49	24.29	-2.51
		0.02	21.81	0.48	27.80	34.21	-17.51	4.91	10.18	27.19	33.57	5.02
		0.05	16.80	-0.16	24.90	31.21	-14.52	6.93	12.33	25.09	31.42	5.05
		0.1	8.70	0.00	20.88	18.24	-8.00	11.21	8.78	20.88	18.23	-2.19
		Ave	18.03	1.21	27.72	30.16	-14.11	8.27	10.23	26.16	28.57	1.87
	$K = 50$	0.005	4.48	10.35	8.91	15.37	5.62	4.24	10.42	-1.30	4.55	5.93
		0.01	9.86	19.74	15.67	20.42	8.99	5.29	9.62	-3.40	0.57	4.11
		0.02	5.92	14.87	17.08	18.16	8.45	10.53	11.55	1.92	2.86	0.93
		0.05	4.60	9.48	18.40	14.06	4.67	13.20	9.05	8.15	4.18	-3.67
		0.1	6.00	5.26	16.16	12.43	-0.70	9.58	6.07	10.35	6.81	-3.21
		Ave	6.17	11.94	15.24	16.09	5.41	8.57	9.34	3.14	3.80	0.82
$\tau = 4$	$K = 100$	0.005	15.80	13.81	22.24	30.44	-1.72	5.56	12.64	7.41	14.61	6.71
		0.01	12.23	9.56	20.65	33.82	-2.38	7.50	19.24	10.12	22.15	10.92
		0.02	10.41	6.59	20.23	27.15	-3.46	8.89	15.16	12.80	19.29	5.76
		0.05	4.26	1.07	17.55	13.99	-3.06	12.75	9.33	16.31	12.79	-3.03
		0.1	1.29	0.29	11.60	7.67	-0.98	10.18	6.30	11.27	7.36	-3.52
		Ave	8.80	6.26	18.45	22.61	-2.32	8.98	12.53	11.58	15.24	3.37

We classify the proposed policies into two-parameter policies, including (Q, r) , (Q, T) and (Q, ξ) , as well as three-parameter policies, counting (Q, r, T) and (Q, r, ξ) . Obviously, the search space for the two-parameter policies is smaller than that of the three-parameter policies. It results in less running time for finding the optimal solution in the two-parameter policies. Hence, in the cases in which we observe a negligible difference between the two and three-parameter approaches, the two-parameter policies are recommended to implement. In the following, we discuss about our findings from the numerical results reported in Tables 8.3-8.6.

Effect of parameters on the optimal order quantity Q^* : According to Tables 8.3-8.6, in all policies, the order quantity, Q , increases in K as expected. This increase is significant in the case with large τ . For example, in the case with $\tau = 2$, any increase in K does not affect Q , but in the case with $\tau = 4$, increase in K raises Q more significantly. The reason is that when the lifetime is short,

increase in Q may incur huge perishing cost to the system. Therefore, we would prefer to increase Q when the lifetime is long enough and p is small. As you can see in Table 8.3, increase in K has almost no effect in Q , but in Table 8.4, this increase has significant influence even in the short lifetime because p takes small value. We have a similar pattern in Tables 8.5 and 8.6.

For small α , we do not observe necessarily larger Q . The reason is due to the fact that, when α is small, we have to increase r in order to raise the service level. When r raises, the holding cost of the system increases as well. To prevent this increase in the cost, we may be willing to decrease Q leading to the higher ordering cost and fewer perishing cost. Hence, depending on parameters of K , p , h , and amount of increase in r we may increase or decrease Q . For instance, in Table 8.4, when $\tau = 4$ and $K = 50$ for $\alpha = 0.02$ and 0.05 the value of Q is 14 and 16, respectively.

As τ increases, the amount of order quantity Q increases because the number of items perished decreases drastically. Therefore, we increase Q such that a trade-off between the ordering and holding cost is made. The high value of p in the system definitely stimulates us to reduce Q (compare Tables 8.3 & 8.4 and Tables 8.5 & 8.6). Obviously, increase in λ raises the order quantity Q to satisfy the excess demands (compare Tables 8.3 & 8.5 and Tables 8.4 & 8.6).

Effect of parameters on the optimal reorder point: By increasing τ , the effect of perishing cost disappears and Q increases. Since holding costs raise by increase in Q , the reorder point tends to be smaller in order to decrease the holding costs. You can follow this change in all Tables 8.3-8.6 so that increase in τ decreases the reorder point r in the (Q, r) policy and increases T in the (Q, T) policy. Note that we should have also reduction in ξ when τ increases. But by increase in τ , the remaining lifetime increases as well. Thus, the value of ξ depends on the value of τ and the amount of increase in τ .

From Tables 8.3-8.6, we can observe that the smaller α causes the higher optimal reorder value r^* . Intuitively, the higher service level requires the higher reorder point to ensure that during the lead time we would not encounter too

much lost sales.

By increasing λ , since demand during the lead time increases, we have to raise r to satisfy the desired service level. The amount of increase in r depends on the amount of increase in Q and reduction in service level which arises from increase in λ . However, interestingly, there is no noticeable change of r^* regarding the changes in p and K .

Effect of parameters on the expected total cost: Let ETC denote the expected total cost per unit time regardless of policy class. The expected total cost, ETC , increases in p especially when the demand rate is low. In Tables 8.5 & 8.6, an increase in p raises ETC significantly in comparison with the cases in Tables 8.3 & 8.4. For instance, in the (Q, r, ξ) policy by increasing p from 10 to 100, the total cost rate increases by 75.34% on average for $\lambda = 5$ and the total cost rate increases by 127.47% on average for $\lambda = 2$. The reason is that in the system with a low demand rate, we have the larger amount of items perished during the lifetime τ .

As τ decreases, the number of items perished in the system becomes larger which results in an increase in ETC . Therefore, when the lifetime of the items is short and p is high enough, the increase in ETC becomes more significant as it is shown in Tables 8.3 & 8.5. For instance, in Table 8.3 for the (Q, r, ξ) policy and $K = 50$, as τ decreases from 4 to 2 the total cost rate increases 64.13% on average. Additionally, an increase in K raises the order quantity Q which leads to the larger amount of items perished and larger holding cost, and consequently an increase in ETC . For example, in Table 8.3 for the (Q, r, ξ) policy and $\tau = 2$, as K increases from 50 to 100 the total cost rate increases 47.20% on average. A decrease in α requires the higher r which causes an increase in holding costs and in ETC . But this time there is no traceable relation between the expected cost and the demand rate.

When and which policy: From Tables 8.3-8.6, we can find that as the demand rate decreases, the policies incorporating the age of stocks outperform the (Q, r) policy. The reason is related to the perishability of items and the fact that (Q, r)

policy only controls the system based on the stock level. More specifically, when the demand rate is low, the time between two consecutive orders is long enough and we need to incorporate the time and age of the items into the model in order to find the reorder point more accurately. Thus, in such a condition, we expect that the (Q, r, T) and (Q, r, ξ) policies perform well. However, as the lifetime of items increases, it is not necessary to track the age of items in stocks. That is, only by tracking the stock level, we can have almost accurate results. Hence, the (Q, r) policy can be offered as a reasonable policy for the system with long lifetime, however it is not efficient in the case with short lifetime.

Based on our numerical results presented in Tables 8.3-8.6, we can compare policies as follows: (i) *The (Q, r, ξ) and (Q, r) policies*: the (Q, r, ξ) policy outperforms (Q, r) policy in all instances. As the unit perishing cost increases, the gap between these two policies increases. Moreover, the gap between these two policies increases when τ and λ decreases. Therefore, the maximum gap occurs when $\tau = 2$, $\lambda = 2$, $p = 100$, and $\alpha = 0.005$ with amount of 62.45%. (ii) *The (Q, r, ξ) and (Q, r, T) policies*: the (Q, r, ξ) policy outperforms (Q, r, T) policy in all instances. As the unit perishing cost decreases, the gap between these two policies increases. Moreover, the gap between these two policies increases when τ and λ decreases. Therefore, the maximum gap occurs when $\tau = 2$, $\lambda = 2$, $p = 10$, and $\alpha = 0.005$ with amount of 23.96%. (iii) *The (Q, r, ξ) and (Q, ξ) policies*: the (Q, r, ξ) policy outperforms (Q, ξ) policy in all instances. As the unit perishing cost increases, the gap between these two policies increases. The maximum gap occurs when $\tau = 2$, $\lambda = 5$, $p = 100$, and $\alpha = 0.005$ with amount of 58.83%. (iv) *The (Q, r, T) and (Q, r) policies*: as the unit perishing cost increases, the gap between these two policies increases. Moreover, the gap between these two policies increases when τ and λ decreases. Therefore, the maximum gap occurs when $\tau = 2$, $\lambda = 2$, $p = 100$, and $\alpha = 0.005$ with amount of 33.28%. (v) *The (Q, r, T) and (Q, T) policies*: by increasing unit perishing cost and shorten lifetime, the gap between these two policies increases. The maximum gap occurs when $\tau = 2$, $\lambda = 5$, $p = 100$, and $\alpha = 0.005$ with amount of 90.72%. (vi) *The (Q, ξ) and (Q, T) policies*: in short lifetimes, the (Q, ξ) outperforms (Q, T) policy. The gaps tends to be higher as the unit perishing cost increases and λ decreases. The maximum

gap occurs when $\tau = 2$, $\lambda = 2$, $p = 100$, and $\alpha = 0.01$ with amount of 65.23%. Since tracking the system by using the elapsed time, T , is easier than using the remaining lifetime, ξ , we recommend to employ the (Q, T) policy in the cases with negligible gap between these two policies. Moreover, as the lifetime decreases the (Q, ξ) policy considerably outperforms the (Q, T) policy, which even pressures us more to utilize the (Q, ξ) policy in general. We can not find an obvious difference in performance of (Q, T) and (Q, ξ) policies in comparison with (Q, r) policy.

Regarding the (Q, r, ξ) and (Q, r, T) policies, an increase in the unit perishing cost p results in an increase in the percentage deviation in the expected cost between these two policies. In all aspects, including control parameters and the objective value, the (Q, r, ξ) and (Q, r, T) policies converge to the (Q, r) policy as the lifetime grows, as shown in Tables 8.3-8.6. Moreover, it can be inferred from Tables 8.3-8.6 that the performance of the (Q, r, ξ) policy is always better than the others. Specifically, when we have a high service level (small α), high unit perishing cost, low demand rate, and short lifetime, the (Q, r, ξ) policy outperforms the other policies more considerably.

Our results show that the (Q, r, ξ) policy is the best policy among all policies proposed here. Only in two instances of Table 8.6, where $\alpha = 0.05$ and $K = 50, 100$ the (Q, ξ) policy can outperform the (Q, r, ξ) policy. In these instances, the solution $(6, 1.00)$ in the (Q, ξ) policy can outperform the solution $(6, 0, 1.00)$ in the (Q, r, ξ) policy. It should be noted that if we have smaller increment size for ξ ($\Delta = 0.005$), then the solution $(6, 0, 0.995)$ in the (Q, r, ξ) policy can outperform the solution $(6, 1.00)$ in the (Q, ξ) policy. Hence, we can still conclude that the (Q, r, ξ) policy is the best policy.

The question is that what is the best control policy after the (Q, r, ξ) policy (i.e, the second-best policy). Now, we exclude the (Q, r, ξ) policy from the set of proposed policies and consider the second-best policy. Table 8.11 represents how many times a policy is the second-best policy in Tables 8.3-8.6.

Table 8.11: The second-best policy

Data	(Q, r, T)	(Q, ξ)	(Q, T)	(Q, r)
Table 8.7, $\lambda = 5, p = 100$	17	3	0	0
Table 8.8, $\lambda = 5, p = 10$	10	10	0	0
Table 8.9, $\lambda = 2, p = 100$	15	5	0	0
Table 8.10, $\lambda = 2, p = 10$	4	16	0	0
Total	46	34	0	0

Therefore, the second-best policy is the (Q, r, T) policy, especially when unit perishing cost is large. It seems that with a high demand rate, the (Q, r, T) policy performs well. Among two-parameter policies (the (Q, ξ) , (Q, T) , and (Q, r) policies) we can have similar ranking as well. This result is summarized in Table 8.12.

Table 8.12: The third-best policy

Data	(Q, ξ)	(Q, T)	(Q, r)
Table 8.7, $\lambda = 5, p = 100$	6	3	11
Table 8.8, $\lambda = 5, p = 10$	10	0	10
Table 8.9, $\lambda = 2, p = 100$	16	0	4
Table 8.10, $\lambda = 2, p = 10$	18	2	0
Total	50	5	25

The result reveals that the (Q, ξ) policy is the third-best policy, specially for small λ . However, for large λ , it seems that the (Q, r) policy performs well and for small λ it is not an efficient policy. The (Q, T) policy outperforms the other two-parameter policies rarely. The (Q, T) policy performs well when α value is small ($\alpha = 0.005$ or 0.01) and fixed lifetime is large ($\tau = 4$).

Considering the fact that the lifetime being long or short depends completely on the demand rate and lead time, and no efficient policy can be proposed for a specific range of lifetime, we can have a general guideline discussed in the following. For inventory systems with large demand rate and long lifetime, we recommend the (Q, r) policy and for the cases with short lifetime we recommend the (Q, ξ) policy among two-parameter policies. Since in all cases the (Q, r, ξ) policy is the most efficient policy regardless of the existing parameter settings, in delicate situations like health care applications and chemical industries, utilizing the (Q, r, ξ) policy might be the best choice. Moreover, in the case with negligible

deviation, the (Q, T) and (Q, r, T) policies are more practical in comparison with the (Q, ξ) and (Q, r, ξ) policies because tracking of the time elapsed is easier than tracking of the remaining lifetime of items.

By eliminating the stock-based parameter r in the (Q, r, ξ) policy we achieve the (Q, ξ) policy, where we lose 12.94% saving in inventory system on average (the higher unit perishing cost, the higher lost in saving we have). Additionally, by eliminating the lifetime-based parameter ξ in the (Q, r, ξ) policy we achieve the (Q, r) policy, where we lose 20.17% saving in inventory system on average (the lower demand rate, the higher lost in saving we have). As expected, incorporating the parameter ξ in a control policy is significant when the demand rate is low. Moreover, the effect of lifetime-based parameter on total cost is more considerable than the stock-based parameter.

Number of outstanding orders m^* : In our numerical study tabulated in Tables 8.3-8.6, we do not impose any restriction for number of outstanding orders. Therefore, we discuss here the effect of parameters on number of outstanding orders under this assumption. Later, in subsection 8.3.2 we deal with the effect of one order outstanding restriction assumption on the optimal solution and the objective function.

Considering Tables 8.3-8.6, one can observe that in instances with low lifetime ($\tau = 2$), high service level (low α), high demand rate, and high unit perishing cost, we have $m > 1$. For instance, when $\tau = 2$, $p = 100$, $\lambda = 5$, and $\alpha = 0.005$, we have $m = 3$. The reason is that when the unit perishing cost is large, we are willing to reduce the batch size to incur fewer perishing cost to the system. Thus, by reducing the batch size, we have more frequent replenishment which leads to higher amount of outstanding orders as well. When τ gets larger, then number of outstanding orders becomes smaller. It is due to the fact that by increasing τ , we have larger batch size and low probability of perishing which reduces the replacement frequency and, consequently, the number of outstanding orders. As service level increases (α decreases), the reorder point should get larger in order to satisfy the desired service level. It results in higher amount of outstanding orders. Increase in fixed ordering cost K leads to larger batch size and fewer amount of

outstanding orders. Clearly, number of outstanding orders in the (Q, r, ξ) policy is smaller than that of the (Q, r) policy; and the reason is because of lifetime based control parameter incorporated into the (Q, r, ξ) policy. By associating the reorder point by lifetime threshold, ξ , we can reduce the stock level threshold, r , which leads to the smaller amount of outstanding orders. Moreover, as number of outstanding orders increases, number of random variables representing the effective lifetime of batches raises which makes the problem computationally harder.

Service level: Regarding the service level, all policies, but (Q, r) policy bind the service level constraint. That is one of the reason that hybrid policies such as the (Q, r, T) and (Q, r, ξ) can outperform the (Q, r) policy. On the other hand, in all policies, but (Q, r) policy, we have a set of solutions whose objective value is close to the optimal value. It is worthwhile to note that in the (Q, r, T) and (Q, r, ξ) policies number of these solutions is more than that of the two-parameter policies. For instance, in Table 8.6 for $\tau = 2$, $K = 100$, and $\alpha = 0.01$, the optimal solution in $(7, 4, 1.00)$ with total cost of 74.73; there are some solutions whose objective values are close to 74.73 which are $(7, 3, 1.08)$ and $(8, 2, 1.00)$ with total cost of 75.08 and 76.25, respectively.

Sensitivity of operating characteristics: We also deal with the sensitivity of operating characteristics in the (Q, r, ξ) policy (Q7 in research questions). In Tables 8.13-8.16, the operating characteristics value for the optimal solution of the (Q, r, ξ) policy are reported. The optimal solutions are reported from Tables 8.3-8.6.

Table 8.13: Operating characteristics for the (Q, r, ξ) policy with $\lambda = 5$ and $p = 100$ in Table 8.3

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	$\mathbb{E}(CL)$	$\mathbb{E}(P)$	$\mathbb{E}(LS)$	$\mathbb{E}(OH)$
$\tau = 2$	$K = 50$	0.005 (4, 10, 0.31)	138.29	0.7157	0.4395	0.0181	5.0281
		0.01 (5, 9, 0.64)	119.85	0.9043	0.5240	0.0453	5.9762
		0.02 (5, 8, 0.58)	101.41	0.9393	0.3978	0.0942	5.4702
		0.05 (5, 7, 0.04)	84.90	0.9904	0.2922	0.2443	4.8672
		0.1 (6, 5, 1.79)	64.21	1.2732	0.2644	0.6328	5.3128
	$K = 100$	0.005 (6, 9, 0.84)	196.01	1.0208	0.9209	0.0248	7.9961
		0.01 (6, 9, 0.40)	174.94	1.0557	0.7733	0.0515	7.3495
		0.02 (6, 8, 0.32)	153.79	1.0977	0.6199	0.1082	6.8202
		0.05 (7, 6, 1.71)	123.70	1.3481	0.5944	0.3373	7.3252
		0.1 (7, 5, 1.32)	100.29	1.4651	0.4042	0.7321	6.5058
	$K = 50$	0.005 (10, 9, 2.40)	44.48	1.9717	0.1876	0.0496	18.9423
		0.01 (10, 8, 2.54)	40.79	1.9921	0.1356	0.0999	17.7053
		0.02 (11, 7, 2.16)	37.14	2.2157	0.1370	0.2217	18.5016
		0.05 (11, 6, 0.00)	32.35	2.2848	0.0762	0.5043	16.3066
		0.1 (12, 4, 1.62)	27.67	2.6502	0.0648	1.3206	16.8456
	$K = 100$	0.005 (11, 9, 1.00)	69.28	2.1543	0.2787	0.0539	21.3774
		0.01 (12, 8, 1.00)	63.95	2.3653	0.2897	0.1207	22.2902
		0.02 (12, 7, 1.68)	58.59	2.4069	0.2022	0.2412	20.7938
		0.05 (13, 5, 1.89)	52.45	2.6938	0.1981	0.6719	21.4700
		0.1 (13, 4, 0.70)	45.36	2.8665	0.1087	1.4466	19.1386

Table 8.14: Operating characteristics for the (Q, r, ξ) policy with $\lambda = 5$ and $p = 10$ in Table 8.4

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	$\mathbb{E}(CL)$	$\mathbb{E}(P)$	$\mathbb{E}(LS)$	$\mathbb{E}(OH)$
$\tau = 2$	$K = 50$	0.005 (11, 9, 1.21)	58.51	1.6156	2.9595	0.0404	14.9354
		0.01 (11, 8, 1.15)	53.68	1.6885	2.6393	0.0850	14.2454
		0.02 (11, 7, 1.05)	48.79	1.7733	2.3068	0.1764	13.4431
		0.05 (9, 6, 1.04)	42.72	1.6595	1.1139	0.4143	9.7588
		0.1 (10, 3, 1.02)	37.88	1.9361	1.2810	0.9650	10.5291
	$K = 100$	0.005 (14, 8, 1.06)	87.47	1.8867	4.6094	0.0464	18.9289
		0.01 (12, 8, 1.00)	81.73	1.7934	3.1159	0.0863	15.4187
		0.02 (11, 7, 1.05)	76.98	1.7733	2.3068	0.1764	13.4431
		0.05 (11, 4, 1.09)	70.53	1.8864	2.0241	0.4597	12.8128
		0.1 (10, 3, 1.02)	63.70	1.9361	1.2810	0.9650	10.5291
	$K = 50$	0.005 (14, 9, 0.88)	33.06	2.6543	0.7927	0.0691	29.8361
		0.01 (14, 8, 0.90)	31.03	2.7056	0.6028	0.1358	27.9222
		0.02 (14, 7, 0.84)	29.13	2.7648	0.4475	0.2763	26.0594
		0.05 (16, 5, 0.96)	25.98	3.2491	0.5559	0.8071	28.8526
		0.1 (16, 3, 1.06)	23.64	3.4599	0.4338	1.7395	27.4465
	$K = 100$	0.005 (18, 9, 0.96)	50.57	3.2021	2.0626	0.0787	41.3067
		0.01 (18, 7, 1.40)	47.46	3.2964	1.6760	0.1639	39.6991
		0.02 (18, 6, 1.24)	44.53	3.3937	1.3671	0.3416	37.4515
		0.05 (18, 5, 0.86)	41.16	3.5525	1.1033	0.8722	35.1934
		0.1 (17, 3, 0.90)	37.74	3.6300	0.6471	1.8035	30.5424

Table 8.15: Operating characteristics for the (Q, r, ξ) policy with $\lambda = 2$ and $p = 100$ in Table 8.5

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	$\mathbb{E}(CL)$	$\mathbb{E}(P)$	$\mathbb{E}(LS)$	$\mathbb{E}(OH)$
$\tau = 2$	$K = 50$	0.005 (3, 5, 1.02)	164.70	0.9754	1.0597	0.0097	4.6723
		0.01 (3, 5, 0.56)	147.45	1.0262	0.9693	0.0209	4.3959
		0.02 (3, 4, 0.56)	125.04	1.1034	0.8381	0.0440	4.1540
		0.05 (2, 3, 1.40)	103.74	0.8598	0.3674	0.0863	2.4562
		0.1 (3, 2, 1.75)	78.59	1.3676	0.5403	0.2748	3.4524
	$K = 100$	0.005 (3, 5, 1.02)	215.96	0.9754	1.0597	0.0097	4.6723
		0.01 (3, 4, 0.90)	195.87	1.0270	0.9666	0.0199	4.5052
		0.02 (3, 4, 0.56)	170.36	1.1034	0.8381	0.0440	4.1540
		0.05 (4, 3, 1.28)	136.78	1.5462	1.0632	0.1548	5.1641
		0.1 (4, 2, 1.08)	110.72	1.7357	0.8752	0.3457	4.6549
	$K = 50$	0.005 (4, 5, 0.52)	59.95	1.7755	0.4680	0.0177	9.6379
		0.01 (5, 4, 2.66)	56.24	2.2110	0.6239	0.0449	11.9515
		0.02 (4, 3, 3.96)	47.30	1.8857	0.3061	0.0767	8.2794
		0.05 (4, 3, 1.86)	37.84	2.0072	0.1885	0.2020	7.0970
		0.1 (5, 2, 0.82)	30.84	2.6505	0.2341	0.5340	8.3396
	$K = 100$	0.005 (6, 5, 0.96)	91.81	2.4617	1.1030	0.0253	15.2819
		0.01 (5, 4, 2.66)	78.85	2.2110	0.6239	0.0449	11.9515
		0.02 (5, 4, 0.64)	71.87	2.2815	0.5298	0.0919	10.9852
		0.05 (5, 3, 0.64)	59.30	2.4456	0.3547	0.2447	9.5432
		0.1 (5, 2, 0.82)	49.71	2.6505	0.2341	0.5340	8.3396

Table 8.16: Operating characteristics for the (Q, r, ξ) policy with $\lambda = 2$ and $p = 10$ in Table 8.6

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	$\mathbb{E}(CL)$	$\mathbb{E}(P)$	$\mathbb{E}(LS)$	$\mathbb{E}(OH)$
$\tau = 2$	$K = 50$	0.005 (7, 5, 1.00)	52.52	1.8029	3.4079	0.0129	10.6195
		0.01 (6, 4, 1.17)	48.64	1.7369	2.5626	0.0355	8.8498
		0.02 (6, 3, 1.10)	44.60	1.8457	2.3828	0.0734	8.4865
		0.05 (6, 0, 1.00)	40.08	1.9970	2.1968	0.1899	8.0708
		0.1 (5, 0, 1.00)	35.34	1.9887	1.4112	0.3875	6.1661
	$K = 100$	0.005 (8, 4, 1.00)	77.86	1.9648	4.0876	0.0163	12.1081
		0.01 (7, 4, 1.00)	74.73	1.9063	3.2176	0.0294	10.2786
		0.02 (7, 1, 1.00)	70.55	1.9970	3.0864	0.0796	10.0292
		0.05 (6, 0, 1.00)	65.12	1.9970	2.1968	0.1899	8.0708
		0.1 (5, 0, 1.00)	60.48	1.9887	1.4112	0.3875	6.1661
	$K = 50$	0.005 (8, 4, 1.84)	30.48	3.0112	2.0087	0.0297	21.6867
		0.01 (7, 4, 1.36)	27.32	2.9090	1.2423	0.0589	17.0426
		0.02 (7, 3, 1.72)	26.03	2.9949	1.1305	0.1190	16.6553
		0.05 (7, 2, 1.40)	22.78	3.2304	0.8640	0.3233	14.9481
		0.10 (7, 1, 1.10)	20.23	3.4982	0.7043	0.6992	13.7313
	$K = 100$	0.005 (11, 4, 0.98)	43.61	3.7698	3.5189	0.0569	29.2227
		0.01 (10, 3, 1.20)	42.11	3.6494	2.7759	0.0732	25.9131
		0.02 (10, 2, 1.12)	40.04	3.7747	2.6018	0.1496	25.1192
		0.05 (10, 1, 0.92)	37.27	3.9873	2.4332	0.4059	24.2735
		0.10 (9, 0, 0.84)	33.95	4.0515	1.7105	0.8115	20.4641

The sensitivity of operating characteristics result is summarized in Table 8.17

with respect to increasing in the input parameters τ , K , p , λ , and α .

Table 8.17: Operating characteristics sensitivity results

Increase	$\mathbb{E}(CL)$	$\mathbb{E}(P)$	$\mathbb{E}(LS)$	$\mathbb{E}(OH)$
τ	Increasing	Decreasing	Increasing	Increasing
K	Increasing	Increasing	Non-monotonic	Increasing
p	Decreasing	Decreasing	Decreasing	Decreasing
λ	Non-monotonic	Non-monotonic	Increasing	Increasing
α	Non-monotonic	Non-monotonic	Increasing	Non-monotonic

Finally, we observe the following facts in our numerical results:

- There is no stochastic dominance in effective lifetime at optimal solutions.
- Optimal ξ and T in (Q, r, ξ) and (Q, r, T) policies are close to the expected value of the effective lifetime.
- Let TC denote the total cost including holding and perishing costs. Then, $\mathbb{E}(TC)/\mathbb{E}(CL)$ is close to $\mathbb{E}(TC/CL)$ in some cases and in some cases not.

8.3.2 Effect of m

We compare the results of the instances by considering the model with $m = 1$ and $m > 1$ assumptions to discover how the number of outstanding orders, m , affects the system characteristics (Q8 in research questions). As stated before, number of outstanding orders m is calculated by $\lceil \frac{r}{Q} \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than x . In Tables 8.18 and 8.19, we present the results of Tables 8.3 and 8.6 under one order outstanding restriction. The results show that the (Q, r) policy is not able to reach any feasible solution to satisfy the service level constraint specially when demand rate is low. Number of outstanding orders in the (Q, r) policy is higher than other policies. Number of outstanding orders in the (Q, r, ξ) policy is lower than other policies; the reason is due to the fact that lifetime parameters ξ and T decrease the number of outstanding orders. Lifetime parameters cause earlier reordering and it can reduce the r value and number of outstanding orders. In the (Q, T) and (Q, ξ) policies, we

have only $m = 1$ in all cases regardless of the input parameters. As τ decreases, p increases, λ increases, and α decreases the number of outstanding orders increases. Therefore, the maximum gap occurs for the (Q, r) policy when $\tau = 2$, $p = 100$, $\lambda = 5$, and $\alpha = 0.005$ so that we do not have any feasible solution under one order outstanding restriction. In the (Q, r, ξ) policy, the maximum gap between two solutions obtained under one and multiple outstanding orders assumptions is 49.97% which occurs when $\tau = 2$, $p = 100$, $\lambda = 5$, and $\alpha = 0.005$; and the maximum gap in the (Q, r, T) policy occurs when $\tau = 2$, $p = 100$, $\lambda = 5$, and $\alpha = 0.005$ with amount of 82.57%. In all policies, as fixed ordering cost K increases number of outstanding orders decreases because batch size Q increases in K .

8.3.3 Model without Service Level Constraint

To investigate the effect of service level constraint, we introduce the model without the service level constraint (Q9 in research questions). In this model, we associate the lost sales cost to the objective function. So, the optimization problem is conducted by an objective function including the holding, perishing, and lost sales costs and without service level constraint. Let π denote the unit lost sales cost and where c the unit purchasing cost. Then, we have the following optimization problem.

$$\min TC(Q, r, \xi) = \frac{K + cQ + h\mathbb{E}(OH) + p\mathbb{E}(P) + \pi\mathbb{E}(LS)}{\mathbb{E}(CL)}. \quad (8.1)$$

The (Q, r) policy under this model is considered by [27]. Now, we consider the (Q, r, ξ) policy to investigate the effect of service level and see how the service level constraint affects the performance of this policy. Tables 8.20 and 8.21 tabulate the results for the data set proposed by [27], where β is the real value of fraction of lost sales. We fixed $L = 1$, $h = 1$, and $\tau = 3$ and vary the demand rate ($\lambda = 5$ and $\lambda = 10$). Note that here we set the number of segments $k = 300$ so that ξ can be a multiple of 0.01.

Table 8.18: Comparison of policies for $\lambda = 5$ and $p = 100$ with one order outstanding restriction

Parameters	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, T)	$\mathbb{E}(TC_4)$	(Q, r)	$\mathbb{E}(TC_5)$
$\tau = 2$ $K = 50$	0.005	(10, 9, 1.43)	208.11	(11, 10, 0.16)	266.14	(10, 1.61)	219.64	(11, 0.06)	278.01	-	-
	0.01	(9, 8, 1.54)	164.95	(10, 9, 0.26)	204.65	(9, 1.76)	170.17	(10, 0.07)	221.19	-	-
	0.02	(8, 7, 1.82)	126.51	(9, 8, 0.38)	151.97	(8, 1.89)	128.12	(9, 0.10)	168.06	(10, 9)	194.26
	0.05	(7, 6, 1.71)	86.44	(7, 6, 0.06)	99.34	(7, 1.85)	89.05	(7, 0.05)	100.06	(8, 7)	106.18
	0.1	(6, 5, 1.79)	64.21	(6, 5, 0.20)	69.95	(6, 1.87)	65.04	(6, 0.12)	73.63	(7, 6)	76.10
$K = 100$	0.005	(10, 9, 1.43)	241.27	(11, 10, 0.16)	298.89	(10, 1.61)	253.35	(11, 0.06)	311.27	-	-
	0.01	(9, 8, 1.54)	199.27	(10, 9, 0.26)	237.54	(9, 1.76)	204.76	(10, 0.07)	254.87	-	-
	0.02	(8, 7, 1.82)	162.37	(9, 8, 0.38)	185.35	(8, 1.89)	164.08	(9, 0.10)	202.29	(10, 9)	226.42
	0.05	(7, 6, 1.71)	123.70	(7, 6, 0.06)	137.30	(7, 1.85)	126.33	(7, 0.05)	138.07	(8, 7)	140.17
	0.1	(7, 5, 1.32)	100.29	(7, 5, 0.56)	106.35	(6, 1.87)	104.35	(6, 0.12)	113.63	(7, 6)	111.73
$\tau = 4$ $K = 50$	0.005	(10, 9, 2.40)	44.48	(10, 9, 0.42)	44.72	(10, 3.54)	48.07	(10, 0.12)	46.80	(11, 10)	53.22
	0.01	(10, 8, 2.54)	40.79	(10, 8, 0.54)	40.96	(9, 3.68)	42.17	(9, 0.08)	42.10	(10, 9)	43.92
	0.02	(11, 7, 2.16)	37.14	(11, 7, 1.16)	37.42	(8, 3.82)	40.27	(9, 0.24)	39.74	(10, 8)	39.35
	0.05	(11, 6, 0.00)	32.35	(11, 6, 4.00)	32.35	(9, 3.26)	37.15	(9, 0.52)	36.37	(11, 6)	32.35
	0.1	(12, 4, 1.62)	27.67	(12, 4, 2.54)	27.77	(9, 3.08)	33.50	(10, 1.10)	32.36	(11, 5)	29.48
$K = 100$	0.005	(11, 9, 1.00)	69.28	(11, 9, 1.06)	69.45	(10, 3.54)	73.58	(10, 0.12)	72.26	(11, 10)	76.74
	0.01	(12, 8, 1.00)	63.95	(11, 8, 0.86)	65.07	(9, 3.68)	70.00	(10, 0.24)	69.33	(11, 9)	69.32
	0.02	(12, 7, 1.68)	58.59	(12, 7, 1.66)	58.86	(9, 3.48)	67.98	(10, 0.42)	65.55	(11, 8)	63.32
	0.05	(13, 5, 1.89)	52.45	(13, 5, 1.90)	53.26	(10, 3.04)	62.93	(12, 1.12)	60.68	(12, 6)	53.29
	0.1	(13, 4, 0.70)	45.36	(13, 4, 3.00)	45.74	(12, 2.38)	53.53	(11, 1.32)	53.56	(14, 4)	45.84

Table 8.19: Comparison of policies for $\lambda = 2$ and $p = 100$, ($L = 1$, $h = 1$) with one order outstanding restriction

Parameters		α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, T)	$\mathbb{E}(TC_4)$	(Q, r)	$\mathbb{E}(TC_5)$
$\tau = 2$	$K = 50$	0.005	(6, 5, 1.11)	203.43	(7, 6, 0.04)	305.04	(6, 1.43)	221.06	(8, 0.10)	358.94	-	-
		0.01	(5, 4, 1.49)	173.93	(6, 5, 0.00)	247.54	(5, 1.61)	177.04	(7, 0.12)	292.52	-	-
		0.02	(5, 4, 1.00)	145.90	(6, 5, 0.36)	213.90	(5, 1.43)	158.86	(6, 0.13)	231.02	-	-
		0.05	(4, 3, 1.27)	103.93	(4, 3, 0.06)	131.64	(4, 1.50)	113.46	(4, 0.04)	133.04	-	-
		0.1	(3, 2, 1.75)	78.59	(3, 2, 0.06)	92.16	(3, 1.82)	79.64	(3, 0.05)	92.50	(5, 4)	136.31
$\tau = 4$	$K = 100$	0.005	(6, 5, 1.11)	233.95	(7, 6, 0.04)	338.09	(6, 1.43)	252.91	(8, 0.10)	391.25	-	-
		0.01	(5, 4, 1.49)	207.28	(6, 5, 0.00)	281.27	(5, 1.61)	210.67	(7, 0.12)	324.73	-	-
		0.02	(5, 4, 1.00)	176.62	(6, 5, 0.36)	244.94	(5, 1.43)	190.71	(6, 0.13)	263.41	-	-
		0.05	(4, 3, 1.28)	136.78	(4, 3, 0.06)	166.88	(4, 1.50)	146.79	(4, 0.04)	168.49	-	-
		0.1	(4, 2, 1.08)	110.72	(4, 3, 1.32)	125.03	(3, 1.82)	116.36	(3, 0.05)	131.03	(5, 4)	164.87
$\tau = 4$	$K = 50$	0.005	(5, 4, 3.83)	63.96	(6, 5, 0.70)	74.77	(6, 3.00)	82.88	(6, 0.22)	81.30	-	-
		0.01	(5, 4, 2.66)	56.24	(5, 4, 0.22)	59.53	(5, 3.40)	60.72	(5, 0.12)	61.57	-	-
		0.02	(4, 3, 3.96)	47.30	(4, 3, 0.02)	48.03	(4, 4.00)	47.30	(4, 0.00)	48.44	(6, 5)	70.92
		0.05	(4, 3, 1.86)	37.84	(4, 3, 1.08)	38.56	(4, 3.38)	42.03	(4, 0.34)	42.75	(5, 4)	49.82
		0.1	(5, 2, 0.82)	30.84	(5, 2, 2.86)	31.27	(4, 3.04)	36.48	(3, 0.24)	38.93	(4, 3)	37.27
$\tau = 4$	$K = 100$	0.005	(6, 5, 0.96)	91.81	(6, 5, 0.66)	95.58	(6, 3.00)	104.02	(6, 0.22)	102.34	-	-
		0.01	(5, 4, 2.66)	78.85	(5, 4, 0.22)	82.48	(5, 3.40)	83.73	(5, 0.12)	84.66	-	-
		0.02	(5, 4, 0.64)	71.87	(5, 4, 1.62)	72.16	(5, 3.12)	79.10	(4, 0.00)	75.11	(6, 5)	91.04
		0.05	(5, 3, 0.64)	59.30	(5, 3, 2.32)	59.66	(4, 3.38)	67.39	(4, 0.34)	68.19	(5, 4)	71.73
		0.1	(5, 2, 0.82)	49.71	(5, 2, 2.86)	50.14	(6, 2.02)	58.66	(4, 0.76)	60.88	(5, 3)	58.98

Table 8.20: Percentage deviation between (Q, r, ξ) and (Q, r) policies without service level constraint, $\lambda = 5$, ($L = 1$, $h = 1$, $\tau = 3$)

π	p	K	c	(Q, r, ξ)	$\mathbb{E}(TC_1)$	β	(Q, r)	$\mathbb{E}(TC_5)$	β	%gap ₁₅	
20	10	50	5	(12, 5, 1.01)	60.85	0.0634	(11, 5)	61.80	0.0876	1.56	
20	10	100	5	(13, 3, 1.00)	80.00	0.1094	(13, 4)	81.15	0.1345	1.44	
40	10	50	5	(12, 6, 1.07)	65.36	0.0375	(10, 7)	66.74	0.0308	2.11	
40	10	100	5	(14, 6, 1.00)	85.45	0.0298	(12, 6)	89.22	0.0594	4.41	
20	10	50	15	(11, 0, 0.00)	105.02	0.3176	(11, 0)	105.02	0.3176	0.00	
20	10	100	15	(12, 0, 0.00)	120.73	0.3019	(12, 0)	120.73	0.3019	0.00	
40	10	50	15	(11, 6, 1.00)	116.03	0.0425	(10, 6)	116.85	0.0542	0.71	
40	10	100	15	(13, 5, 1.00)	136.92	0.0558	(12, 5)	138.84	0.0901	1.40	
20	100	50	5	(9, 4, 1.26)	69.17	0.1325	(9, 4)	69.36	0.1420	0.27	
20	100	100	5	(10, 2, 1.00)	91.16	0.2150	(10, 3)	91.54	0.1871	0.42	
40	100	50	5	(8, 6, 1.36)	78.26	0.0576	(8, 6)	78.54	0.0600	0.36	
40	100	100	5	(10, 5, 1.00)	103.78	0.0797	(10, 5)	104.73	0.0885	0.91	
20	100	50	15	(9, 0, 0.36)	107.96	0.3538	(9, 0)	107.96	0.3589	0.00	
20	100	100	15	(10, 0, 0.00)	126.06	0.3366	(10, 0)	125.06	0.3366	0.00	
40	100	50	15	(9, 5, 1.12)	125.32	0.0878	(9, 5)	125.73	0.0932	0.33	
40	100	100	15	(10, 4, 1.00)	149.50	0.1222	(10, 4)	150.21	0.1340	0.47	
										St.dev	1.14
										Min	0.00
										Max	4.41
										Ave	0.90

Table 8.21: Percentage deviation between (Q, r, ξ) and (Q, r) policies without service level constraint, $\lambda = 10$, ($L = 1$, $h = 1$, $\tau = 3$)

π	p	K	c	(Q, r, ξ)	$\mathbb{E}(TC_1)$	β	(Q, r)	$\mathbb{E}(TC_5)$	β	$\%gap_{15}$	
20	10	50	5	(21, 12, 1.00)	92.69	0.0248	(20, 12)	93.00	0.0274	0.33	
20	10	100	5	(25, 10, 1.00)	113.75	0.0444	(24, 11)	114.82	0.0419	0.94	
40	10	50	5	(20, 14, 1.12)	96.15	0.0093	(20, 14)	96.43	0.0103	0.29	
40	10	100	5	(24, 13, 1.06)	118.94	0.0130	(22, 13)	120.33	0.0177	1.17	
20	10	50	15	(21, 8, 1.00)	188.18	0.1031	(21, 8)	188.21	0.1075	0.02	
20	10	100	15	(23, 0, 0.00)	205.89	0.3052	(23, 0)	205.89	0.3052	0.00	
40	10	50	15	(19, 13, 1.18)	196.36	0.0167	(19, 13)	196.47	0.0175	0.06	
40	10	100	15	(23, 11, 1.02)	219.82	0.0343	(22, 12)	220.49	0.0273	0.30	
20	100	50	5	(17, 11, 1.18)	98.12	0.0467	(17, 11)	98.16	0.0471	0.04	
20	100	100	5	(20, 9, 1.22)	123.13	0.0804	(20, 10)	123.36	0.0604	0.19	
40	100	50	5	(16, 13, 1.72)	103.62	0.0193	(16, 13)	103.67	0.0199	0.05	
40	100	100	5	(18, 12, 1.48)	132.04	0.0280	(18, 12)	132.21	0.0293	0.13	
20	100	50	15	(19, 7, 0.98)	190.37	0.1450	(19, 7)	190.37	0.1469	0.00	
20	100	100	15	(20, 0, 0.32)	209.09	0.3310	(20, 0)	209.10	0.3341	0.00	
40	100	50	15	(17, 12, 1.46)	201.96	0.0299	(17, 12)	202.01	0.0305	0.02	
40	100	100	15	(19, 11, 1.24)	229.07	0.0417	(19, 11)	229.22	0.0431	0.06	
										St.dev	0.34
										Min	0.00
										Max	1.17
										Ave	0.23

The results reveal that the hybrid policy and stock based policy performs almost the same and as the demand rate increases they converge to each other. The percentage deviation gets large value when the unit lost sales cost is high. To attain a higher gap between the hybrid policy and the stock-based policy, we

add $\pi = 100$ and 200 to the data set in [27] resulting in Table 8.22, where $L = 1$, $h = 1$, and $\tau = 3$, $\lambda = 5$. The result shows that in the model without service level constraint the higher π results in the higher gap between (Q, r, ξ) policy and (Q, r) policy.

Table 8.22: Percentage deviation between (Q, r, ξ) and (Q, r) policies without service level constraint with high π , ($L = 1$, $h = 1$, $\tau = 3$, $\lambda = 5$)

π	p	K	c	(Q, r, ξ)	$\mathbb{E}(TC_1)$	β	(Q, r)	$\mathbb{E}(TC_5)$	β	$\%gap_{15}$	
100	10	50	5	(11, 8, 1.42)	70.11	0.0106	(10, 9)	71.62	0.0072	2.15	
100	10	100	5	(14, 7, 1.00)	91.74	0.0167	(11, 8)	97.44	0.0186	6.21	
200	10	50	5	(11, 9, 1.32)	73.39	0.0050	(10, 10)	74.95	0.0026	2.12	
200	10	100	5	(14, 9, 1.00)	96.38	0.0040	(10, 9)	102.33	0.0072	6.17	
100	100	50	5	(8, 7, 1.88)	89.53	0.0293	(8, 7)	90.29	0.0323	0.85	
100	100	100	5	(9, 7, 1.52)	119.15	0.0268	(9, 7)	120.25	0.0307	0.92	
200	100	50	5	(7, 8, 0.44)	97.26	0.0144	(6, 9)	98.69	0.0095	1.47	
200	100	100	5	(9, 8, 1.46)	130.45	0.0135	(8, 8)	131.86	0.0148	1.08	
										St.dev	2.25
										Min	0.85
										Max	6.21
										Ave	2.62

Following the real value of fraction of lost sales β in Tables 8.20-8.22, we can observe pretty high values for that. It is one of the shortcomings in the model without service level constraint. By considering this data set subject to the service level constraint with $\alpha = 0.01$, we observed that the proposed hybrid policy performs well, especially when we have high unit perishing and low demand rate. Hence, in a model with the service level constraint we are willing to apply a hybrid control policy instead of a stock-based policy.

8.3.4 Tekin et al. Data Set

We can consider our proposed policy under the *freshness assumption* in which the new batch is always fresh and the effective lifetime of batch is equal to τ with probability of 1 at the beginning of embedded cycles. Under this assumption the new batch is kept packed and whenever the previous batch is completely consumed the new batch is unpacked and its lifetime starts. Therefore, the freshness assumption implies that the lifetime of the new batch at the beginning of embedded cycles is always equal to τ . This approach is dealt with by Tekin et

al. in [23] by considering the (Q, r, T) policy. Now, we address our model by considering the data set in [23] (Set 2). The results are summarized in Tables 8.23 and 8.24 with $L = 1$, $h = 1$, and $\lambda = 5$. In all tables below, the bold numbers are the average of gaps. Our goal for proposing data Set 2 is to investigate the effect of lifetime-based parameter ξ and the sensitivity of optimal solution to input parameters such as K , p , and τ . However we carried out the data Set 1 for such findings, we apply data Set 2 to have more results and confirm our previous findings.

Table 8.23: Comparison of policies for data Set 2 with $K = 50$, ($L = 1$, $h = 1$, $\lambda = 5$)

	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, r)	$\mathbb{E}(TC_5)$	%gap ₁₃	%gap ₁₅	%gap ₃₅	
$p = 1$	$\tau = 2$	0.005	(15, 8, 1.00)	38.69	(16, 1.01)	39.30	(11, 11)	47.28	1.57	22.19	20.31
		0.01	(14, 7, 1.00)	37.18	(15, 1.01)	37.81	(10, 10)	45.91	1.71	23.48	21.40
		0.02	(13, 6, 1.00)	35.70	(14, 1.00)	36.11	(10, 9)	42.20	1.16	18.20	16.85
		0.05	(12, 3, 1.00)	33.52	(12, 1.02)	33.76	(10, 8)	38.92	0.73	16.13	15.28
		0.1	(11, 2, 0.95)	31.48	(11, 0.97)	31.57	(11, 7)	34.17	0.26	8.53	8.25
	$\tau = 4$	0.005	(20, 8, 1.08)	28.31	(24, 1.34)	32.18	(14, 10)	31.74	13.67	12.10	-1.38
		0.01	(20, 7, 1.02)	27.10	(24, 1.20)	30.18	(15, 9)	29.72	11.39	9.67	-1.54
		0.02	(19, 6, 1.06)	25.92	(24, 1.06)	28.37	(16, 8)	27.87	9.47	7.53	-1.77
		0.05	(19, 4, 1.00)	24.06	(22, 1.00)	25.35	(16, 6)	25.23	5.36	4.89	-0.45
		0.1	(18, 2, 0.98)	22.33	(20, 0.96)	22.93	(16, 4)	22.74	2.67	1.82	-0.83
	$\tau = 6$	0.005	(24, 8, 1.02)	26.30	(28, 2.13)	32.44	(21, 9)	27.07	23.35	2.95	-16.53
		0.01	(24, 7, 1.08)	25.30	(27, 2.13)	30.64	(22, 8)	25.98	21.12	2.70	-15.21
		0.02	(23, 6, 1.11)	24.15	(28, 1.83)	28.55	(22, 7)	24.92	18.23	3.18	-12.73
		0.05	(24, 4, 1.17)	22.61	(27, 1.65)	25.18	(21, 5)	22.86	11.38	1.12	-9.21
		0.1	(20, 3, 0.00)	20.92	(26, 1.44)	22.32	(20, 3)	20.92	6.70	0.00	-6.28
								8.58	8.97	1.08	
$p = 10$	$\tau = 2$	0.005	(12, 9, 1.00)	58.28	(10, 1.61)	60.77	(11, 11)	69.49	4.27	19.23	14.35
		0.01	(11, 8, 1.15)	53.68	(9, 1.76)	55.50	(9, 10)	60.01	3.38	11.80	8.14
		0.02	(11, 7, 1.05)	48.79	(8, 1.89)	51.42	(8, 9)	56.06	5.39	14.92	9.04
		0.05	(9, 6, 1.05)	42.78	(9, 1.38)	45.35	(8, 7)	46.72	6.02	9.22	3.01
		0.1	(10, 3, 1.02)	37.88	(10, 1.08)	38.67	(9, 6)	40.85	2.08	7.84	5.65
	$\tau = 4$	0.005	(14, 9, 0.90)	33.07	(10, 3.54)	36.90	(14, 10)	35.25	11.57	6.58	-4.47
		0.01	(14, 8, 0.92)	31.03	(11, 3.12)	36.12	(14, 9)	32.98	16.38	6.29	-8.68
		0.02	(15, 7, 0.84)	29.09	(17, 1.90)	34.79	(15, 8)	30.91	19.58	6.24	-11.16
		0.05	(16, 5, 0.96)	25.98	(17, 1.68)	29.53	(15, 6)	27.09	13.66	4.28	-8.25
		0.1	(16, 3, 1.08)	23.69	(17, 1.42)	25.48	(15, 4)	23.90	7.57	0.89	-6.21
	$\tau = 6$	0.005	(21, 8, 2.04)	27.48	(13, 4.77)	33.30	(19, 9)	27.88	21.17	1.43	-16.29
		0.01	(21, 7, 2.07)	26.33	(14, 4.44)	32.54	(19, 8)	26.62	23.60	1.10	-18.21
		0.02	(22, 6, 1.56)	24.89	(14, 4.32)	31.14	(19, 7)	25.41	25.11	2.09	-18.40
		0.05	(20, 5, 0.00)	23.09	(21, 2.70)	26.89	(20, 5)	23.09	16.41	0.00	-14.10
		0.1	(20, 3, 0.00)	21.03	(22, 2.19)	22.89	(20, 3)	21.03	8.86	0.00	-8.14
								12.34	6.13	-4.91	
$p = 50$	$\tau = 2$	0.005	(6, 9, 0.83)	101.36	(10, 1.61)	131.38	(5, 11)	115.35	29.62	13.80	-12.20
		0.01	(5, 9, 0.65)	90.90	(9, 1.76)	106.46	(5, 10)	104.10	17.11	14.52	-2.22
		0.02	(6, 8, 0.32)	80.00	(8, 1.89)	85.51	(6, 9)	89.60	6.88	12.00	4.79
		0.05	(7, 6, 1.72)	64.72	(7, 1.85)	65.92	(6, 7)	69.77	1.85	7.80	5.84
		0.1	(7, 5, 1.32)	52.36	(6, 1.88)	54.56	(7, 6)	58.30	4.20	11.34	6.85
	$\tau = 4$	0.005	(11, 9, 1.12)	39.61	(10, 3.54)	41.86	(11, 10)	43.77	5.70	10.53	4.57
		0.01	(11, 8, 2.16)	36.80	(9, 3.68)	39.36	(10, 9)	39.40	6.94	7.07	0.11
		0.02	(12, 7, 1.72)	33.66	(9, 3.48)	38.11	(11, 8)	36.14	13.21	7.36	-5.17
		0.05	(13, 5, 1.90)	30.28	(9, 3.26)	35.51	(12, 6)	30.46	17.28	0.59	-14.23
		0.1	(13, 4, 0.80)	26.04	(10, 2.84)	31.36	(14, 4)	26.49	20.44	1.73	-15.53
	$\tau = 6$	0.005	(16, 9, 0.00)	29.28	(12, 5.01)	34.06	(16, 9)	29.28	16.34	0.00	-14.04
		0.01	(16, 8, 0.00)	27.78	(12, 4.89)	33.33	(16, 8)	27.78	20.01	0.00	-16.67
		0.02	(17, 7, 0.00)	26.28	(12, 4.77)	32.26	(17, 7)	26.28	22.77	0.00	-18.55
		0.05	(18, 5, 0.00)	23.57	(13, 4.35)	29.23	(18, 5)	23.57	24.01	0.00	-19.36
		0.1	(19, 3, 1.35)	21.43	(15, 3.69)	25.09	(20, 3)	21.54	17.10	0.51	-14.17
								14.90	5.82	-7.33	

Table 8.24: Comparison of policies for data Set 2 with $K = 100$, ($L = 1$, $h = 1$, $\lambda = 5$)

	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, r)	$\mathbb{E}(TC_5)$	%gap ₁₃	%gap ₁₅	%gap ₃₅
$p = 1$	$\tau = 2$	0.005 (16, 5, 1.00)	64.09	(16, 1.01)	64.42	(11, 11)	81.16	0.52	26.63	25.98
		0.01 (15, 4, 1.00)	62.62	(15, 1.01)	62.94	(10, 10)	80.27	0.52	28.19	27.53
		0.02 (14, 2, 1.00)	61.13	(14, 1.00)	61.11	(10, 9)	74.36	-0.03	21.65	21.68
		0.05 (12, 3, 1.00)	58.66	(13, 0.97)	58.82	(10, 8)	69.08	0.27	17.77	17.45
		0.1 (12, 2, 0.89)	55.53	(12, 0.90)	55.56	(11, 7)	60.22	0.04	8.43	8.39
	$\tau = 4$	0.005 (24, 7, 1.02)	42.18	(28, 1.00)	45.03	(14, 10)	50.92	6.75	20.71	13.08
		0.01 (23, 6, 1.02)	40.89	(26, 1.02)	42.89	(15, 9)	47.46	4.89	16.06	10.65
		0.02 (22, 5, 1.00)	39.46	(24, 1.06)	41.06	(16, 8)	44.31	4.06	12.28	7.90
		0.05 (20, 4, 0.94)	37.46	(22, 1.00)	37.85	(19, 7)	39.67	1.03	5.89	4.81
		0.1 (19, 1, 0.96)	35.28	(20, 0.96)	35.30	(20, 5)	35.98	0.06	1.96	1.90
	$\tau = 6$	0.005 (26, 8, 0.99)	36.82	(34, 1.47)	42.09	(21, 9)	39.21	14.34	6.50	-6.85
		0.01 (27, 7, 0.96)	35.67	(35, 1.20)	39.91	(22, 8)	37.50	11.90	5.15	-6.03
		0.02 (26, 6, 0.90)	34.47	(33, 1.20)	37.49	(24, 7)	35.55	8.75	3.12	-5.18
		0.05 (26, 4, 0.87)	32.39	(31, 1.08)	33.98	(26, 5)	32.80	4.91	1.27	-3.47
		0.1 (26, 1, 0.96)	30.33	(28, 1.08)	30.81	(26, 3)	30.68	1.61	1.17	-0.43
								3.97	11.79	7.83
	$\tau = 2$	0.005 (12, 9, 1.00)	87.33	(14, 1.18)	92.87	(11, 11)	103.37	6.35	18.37	11.31
		0.01 (12, 8, 1.00)	81.73	(14, 1.09)	85.99	(10, 10)	97.39	5.21	19.16	13.26
		0.02 (11, 7, 1.05)	76.98	(14, 1.00)	79.94	(10, 9)	88.16	3.84	14.51	10.28
		0.05 (11, 5, 1.00)	69.02	(12, 1.02)	70.74	(10, 8)	80.23	2.49	16.24	13.42
		0.1 (10, 3, 1.02)	63.70	(11, 0.97)	64.31	(9, 6)	69.71	0.96	9.42	8.39
	$\tau = 4$	0.005 (20, 8, 1.08)	49.56	(20, 1.76)	58.92	(14, 10)	54.42	18.88	9.81	-7.63
		0.01 (20, 7, 1.02)	47.06	(18, 1.88)	54.58	(15, 9)	50.81	15.97	7.97	-6.91
		0.02 (19, 6, 1.06)	44.36	(20, 1.50)	50.20	(16, 8)	47.53	13.17	7.14	-5.33
		0.05 (19, 4, 1.00)	40.90	(19, 1.40)	44.33	(16, 6)	42.76	8.40	4.56	-3.54
		0.1 (18, 2, 0.98)	37.71	(19, 1.10)	38.93	(16, 4)	38.48	3.24	2.04	-1.16
	$\tau = 6$	0.005 (24, 8, 1.02)	38.80	(26, 2.40)	49.62	(21, 9)	40.29	27.88	3.85	-18.79
		0.01 (24, 7, 1.02)	37.25	(27, 2.13)	46.23	(22, 8)	38.56	24.12	3.52	-16.60
		0.02 (24, 6, 0.96)	35.76	(28, 1.83)	42.80	(24, 7)	36.89	19.70	3.18	-13.80
		0.05 (24, 4, 1.17)	33.41	(27, 1.65)	37.13	(24, 5)	34.03	11.14	1.84	-8.36
		0.1 (24, 2, 0.93)	31.05	(26, 1.44)	32.74	(24, 3)	31.49	5.45	1.44	-3.80
								11.12	8.20	-1.95
$p = 50$	$\tau = 2$	0.005 (7, 9, 0.52)	146.32	(10, 1.61)	165.09	(5, 11)	174.75	12.83	19.43	5.85
		0.01 (7, 8, 0.46)	135.22	(9, 1.76)	141.05	(5, 10)	161.65	4.31	19.55	14.60
		0.02 (8, 7, 1.83)	120.77	(8, 1.89)	121.47	(7, 9)	133.87	0.58	10.85	10.21
		0.05 (8, 6, 1.34)	99.95	(7, 1.85)	103.20	(8, 7)	107.22	3.25	7.27	3.90
		0.1 (8, 5, 0.96)	84.32	(8, 1.39)	89.83	(8, 6)	93.30	6.53	10.65	3.87
	$\tau = 4$	0.005 (12, 9, 0.88)	62.12	(10, 3.54)	67.37	(12, 10)	67.23	8.46	8.24	-0.21
		0.01 (13, 8, 0.92)	57.62	(10, 3.38)	66.01	(12, 9)	62.13	14.55	7.82	-5.87
		0.02 (13, 7, 0.94)	53.42	(10, 3.24)	64.05	(13, 8)	57.53	19.91	7.71	-10.18
		0.05 (15, 5, 1.26)	47.38	(15, 2.02)	57.67	(14, 6)	49.39	21.71	4.23	-14.36
		0.1 (14, 4, 0.00)	42.75	(15, 1.78)	47.17	(14, 4)	42.75	10.34	0.00	-9.37
	$\tau = 6$	0.005 (20, 8, 2.34)	43.34	(13, 4.77)	53.72	(18, 9)	43.82	23.94	1.10	-18.43
		0.01 (20, 7, 2.34)	41.35	(14, 4.44)	52.32	(19, 8)	41.68	26.52	0.79	-20.34
		0.02 (21, 6, 1.89)	38.76	(14, 4.32)	50.27	(19, 7)	39.72	29.68	2.48	-20.97
		0.05 (21, 5, 0.00)	35.91	(17, 3.48)	45.09	(21, 5)	35.91	25.55	0.00	-20.35
		0.1 (21, 3, 0.00)	32.72	(18, 3.03)	37.10	(21, 3)	32.72	13.41	0.00	-11.82
								14.77	6.67	-6.23

Again, the (Q, r, ξ) policy is the best policy. As p increases and τ increases, the gap between the (Q, r, ξ) and the (Q, ξ) policies (%gap₁₃) increases. It implies that the (Q, ξ) policy performs well when unit perishing cost and fixed lifetime are small. For the small value of service level (large α), the gap between these

two policies decreases. We do not observe a significant effect by changing the fixed ordering cost K . Again, note that in one instance (Q, ξ) dominates (Q, r, ξ) policy, where $p = 1$, $\tau = 2$, $K = 100$, and $\alpha = 0.02$ (Table 8.24). Then, we observed that solution $(14, 2, 0.997)$ in the (Q, r, ξ) policy dominates solution $(14, 1.00)$ in the (Q, ξ) policy.

As p increases and τ increases, the gap between the (Q, r, ξ) and the (Q, r) policies ($\%gap_{15}$) decreases. It implies that the (Q, r) policy performs well when unit perishing cost and fixed lifetime are large. Similarly, for the small value of service level (large α), the gap between these two policies decreases as well. Hence, for small τ , the (Q, ξ) policy performs well and for large τ , the (Q, r) policy. Moreover, by eliminating the stock-based parameter r and the lifetime-based parameter ξ in the (Q, r, ξ) policy we lose 4.37% and 5.51% saving in inventory system on average, respectively. In 55 cases, the (Q, r) policy is the third-best policy and in 35 cases, the (Q, ξ) policy is the third-best policy. In summary, our result shows that the (Q, r, T) policy is the second-best policy on both data Set 1 and 2, but the third-best policy is (Q, ξ) policy in data Set 1 and (Q, r) policy in data Set 2. The (Q, T) policy is not an efficient policy in comparison with other policies. Hence, we can rank the policies in Set 1 and Set 2 as follows:

In Set 1: (Q, r, ξ) , (Q, r, T) , (Q, ξ) , (Q, r) , and (Q, T) .

In Set 2: (Q, r, ξ) , (Q, r, T) , (Q, r) , (Q, ξ) , and (Q, T) .

In result, the (Q, r, ξ) policy is always the best, the (Q, r, T) policy is always the second-best policy, and the (Q, T) policy is always the worst policy, but for the (Q, ξ) and (Q, r) policies we can not have a certain dominance. However, we can conclude that in the cases with short lifetime the (Q, ξ) policy performs well and as lifetime gets larger, we are willing to apply the (Q, r) policy.

In the following, we provide a number of observations found from numerical study for data Set 2. Let's consider the (Q, r, ξ) policy and assume that the optimal solution obtained under freshness assumption (the approach in Tekin et

al. [23]) is denoted by (Q_f^*, r_f^*, ξ_f^*) with corresponding optimal total cost $\mathbb{E}(TC_f^*)$ and (Q^*, r^*, ξ^*) denotes the optimal solution obtained by our approach with corresponding optimal total cost $\mathbb{E}(TC_1)$. Then, we observe that

- For a given Q and r , $\xi_f^* \geq \xi^*$ and for a given r and ξ , $Q_f^* \geq Q^*$; and in optimal solution, $r_f^* \geq r^*$.
- $\mathbb{E}(TC_f^*) < \mathbb{E}(TC_1)$ and as the service level decrease (α increases), these bounds become close to each other. As τ increases and the unit perishing cost p decreases, these bounds become close to each other as well.
- In Tekin et al. approach, the (Q, r, ξ) policy and the (Q, r, T) policy are equivalent to each other so that $\xi = \tau - T$ and $\mathbb{E}(TC_1) = \mathbb{E}(TC_2)$, but in our approach there is no such an equivalence.
- For small τ and large p we have multiple outstanding orders.

In Table 8.25, we deal with the effect of lead time by using data setting in Tekin et al. [23]: $\tau = 2$, $h = 1$, $K = 50$, $\lambda = 5$ (Set 3). The result shows that the higher lead time stimulates us for using a time-based or lifetime-based policy. It implies that the (Q, r, ξ) and (Q, r, T) policies outperform the (Q, r) policy as lead time increases. By increasing lead time the reorder point also increases which generally results in increase of r in the (Q, r) policy and increase of ξ in the (Q, ξ) policy. The reason is that the uncertainty during the higher lead time is larger causing the larger amount of lost sales. Thus, we are willing to have earlier reordering (larger r or ξ) to satisfy the desired service level. Higher lead time also leads to higher number of outstanding orders. For instance, when $p = 50$ and $\alpha = 0.005$, for lead time of 0.5, 1, and 1.5 we have 1, 2, and 3 outstanding orders, respectively.

We also consider the effect of demand rate so that we decrease the demand rate to $\lambda = 0.25$. The results are tabulated in Table 8.26. We use data setting in Tekin et al. [23]: $L = 1$, $h = 1$, $K = 50$, $\lambda = 0.25$ (Table 4 in [23]). We can conclude that when the demand rate is small, the effect of remaining lifetime in control policy becomes more considerable so that 54.41% gap between the

(Q, r, ξ) and the (Q, r) policy, 45.83% gap between the (Q, r, T) and the (Q, r) policy, and 28.47% gap between the (Q, r, ξ) and the (Q, r, T) policy are observed in our results. It is due to the fact that in cases where demand rate is low, the time between two consecutive demand arrivals is long and by tracking only stock level we may lose the correct point of reordering. Hence, by incorporating the time or lifetime of items in reordering point we are more flexible regarding the replacements.



Table 8.25: Sensitivity of lead time for Set 2 by fixing $\tau = 2$ and $K = 50$, ($h = 1$, $\lambda = 5$)

	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, ξ)	$\mathbb{E}(TC_3)$	(Q, r)	$\mathbb{E}(TC_5)$	%gap ₁₃	%gap ₁₅	%gap ₃₅		
$p = 1$	$L = 0.5$	0.005	(14, 4, 0.50)	37.35	(16, 0.51)	39.30	(7, 7)	53.37	5.21	42.89	35.82	
		0.01	(12, 4, 0.50)	36.48	(15, 0.51)	37.82	(8, 7)	49.52	3.65	35.74	30.96	
		0.02	(12, 3, 0.50)	35.24	(14, 0.50)	36.11	(8, 6)	45.85	2.46	30.10	26.97	
		0.05	(12, 1, 0.48)	33.60	(12, 0.52)	33.76	(8, 4)	39.56	0.49	17.74	17.17	
		0.1	(11, 1, 0.37)	31.48	(11, 0.47)	31.57	(11, 4)	33.80	0.28	7.38	7.08	
								2.42	26.77	23.60		
	$L = 1$	0.005	(15, 8, 1.00)	38.69	(16, 1.01)	39.30	(11, 11)	47.28	1.57	22.19	20.31	
		0.01	(14, 7, 1.00)	37.18	(15, 1.01)	37.81	(10, 10)	45.91	1.71	23.48	21.40	
		0.02	(13, 6, 1.00)	35.70	(14, 1.00)	36.11	(10, 9)	42.20	1.16	18.20	16.85	
		0.05	(12, 3, 1.00)	33.52	(12, 1.02)	33.76	(10, 8)	38.92	0.73	16.13	15.28	
		0.1	(11, 2, 0.95)	31.48	(11, 0.97)	31.57	(11, 7)	34.17	0.26	8.53	8.25	
								1.09	17.71	16.42		
	$L = 1.5$	0.005	(16, 6, 1.51)	39.30	(16, 1.52)	39.57	(8, 16)	56.93	0.69	44.87	43.88	
		0.01	(15, 5, 1.51)	37.81	(15, 1.51)	37.81	(8, 16)	56.93	0.00	50.56	50.56	
		0.02	(14, 5, 1.50)	36.11	(14, 1.50)	36.11	(9, 16)	52.06	0.00	44.16	44.16	
		0.05	(12, 5, 1.51)	33.56	(12, 1.52)	33.77	(12, 11)	37.03	0.61	10.35	9.68	
		0.1	(11, 4, 1.46)	31.44	(11, 1.47)	31.57	(11, 9)	32.91	0.41	4.67	4.25	
								0.34	30.92	30.51		
	$p = 10$	$L = 0.5$	0.005	(11, 5, 0.50)	51.63	(10, 1.04)	62.69	(6, 6)	60.83	21.41	17.81	-2.96
			0.01	(11, 4, 0.62)	49.78	(10, 0.97)	57.40	(7, 6)	57.25	15.30	15.01	-0.26
0.02			(9, 4, 0.49)	45.42	(10, 0.90)	52.77	(7, 5)	51.63	16.16	13.66	-2.15	
0.05			(10, 2, 0.55)	40.94	(10, 0.75)	44.88	(8, 4)	44.63	9.64	9.01	-0.57	
0.1			(9, 1, 0.53)	36.84	(10, 0.58)	38.67	(9, 3)	39.48	4.94	7.15	2.10	
							13.49	12.53	-0.77			
$L = 1$		0.005	(12, 9, 1.00)	58.28	(10, 1.61)	60.77	(11, 11)	69.49	4.27	19.23	14.35	
		0.01	(11, 8, 1.15)	53.68	(9, 1.76)	55.50	(9, 10)	60.01	3.38	11.80	8.14	
		0.02	(11, 7, 1.05)	48.79	(8, 1.89)	51.42	(8, 9)	56.06	5.39	14.92	9.04	
		0.05	(9, 6, 1.05)	42.78	(9, 1.38)	45.35	(8, 7)	46.72	6.02	9.22	3.01	
		0.1	(10, 3, 1.02)	37.88	(10, 1.08)	38.67	(9, 6)	40.85	2.08	7.84	5.65	
							4.23	12.60	8.04			
$L = 1.5$		0.005	(13, 11, 1.78)	63.65	(13, 1.82)	63.77	(8, 16)	75.60	0.20	18.78	18.55	
		0.01	(12, 9, 1.90)	57.73	(12, 1.90)	57.73	(7, 14)	68.35	0.00	18.40	18.40	
		0.02	(11, 9, 1.92)	51.96	(11, 1.92)	51.96	(7, 13)	62.81	0.00	20.87	20.87	
		0.05	(11, 8, 1.60)	44.64	(10, 1.75)	44.88	(12, 11)	52.22	0.56	16.98	16.34	
		0.1	(10, 7, 1.50)	38.29	(10, 1.58)	38.67	(11, 9)	42.40	0.98	10.72	9.65	
							0.35	17.15	16.76			
$p = 50$		$L = 0.5$	0.005	(6, 5, 1.66)	84.38	(6, 1.82)	85.89	(6, 6)	87.96	1.79	4.24	2.41
			0.01	(6, 5, 0.98)	74.47	(6, 1.66)	82.26	(6, 6)	87.96	10.46	18.11	6.93
	0.02		(7, 4, 0.93)	67.81	(5, 1.82)	71.88	(6, 5)	73.46	6.01	8.34	2.20	
	0.05		(7, 3, 0.73)	55.88	(5, 1.63)	65.93	(6, 4)	62.00	17.99	10.95	-5.96	
	0.1		(7, 2, 0.60)	47.74	(7, 1.06)	57.10	(6, 3)	52.90	19.60	10.81	-7.35	
								11.17	10.49	-0.36		
	$L = 1$	0.005	(6, 9, 0.83)	101.36	(10, 1.61)	131.38	(5, 11)	115.35	29.62	13.80	-12.20	
		0.01	(5, 9, 0.65)	90.90	(9, 1.76)	106.46	(5, 10)	104.10	17.11	14.52	-2.22	
		0.02	(6, 8, 0.32)	80.00	(8, 1.89)	85.51	(6, 9)	89.60	6.88	12.00	4.79	
		0.05	(7, 6, 1.72)	64.72	(7, 1.85)	65.92	(6, 7)	69.77	1.85	7.80	5.84	
		0.1	(7, 5, 1.32)	52.36	(6, 1.88)	54.56	(7, 6)	58.30	4.20	11.34	6.85	
								11.93	11.89	0.61		
	$L = 1.5$	0.005	(5, 13, 0.78)	111.69	(13, 1.82)	162.57	(5, 15)	132.25	45.55	18.41	-18.65	
		0.01	(7, 12, 0.90)	99.41	(12, 1.90)	136.21	(5, 14)	117.79	37.02	18.49	-13.52	
		0.02	(6, 11, 1.12)	85.07	(11, 1.92)	111.15	(6, 12)	96.33	30.66	13.24	-13.33	
		0.05	(6, 9, 0.95)	69.22	(10, 1.75)	83.23	(5, 10)	77.70	20.24	12.25	-6.65	
		0.1	(6, 7, 0.82)	58.65	(9, 1.73)	62.09	(6, 9)	65.00	5.87	10.82	4.68	
								27.87	14.64	-9.50		

Table 8.26: Effect of demand rate (low demand rate, $\lambda = 0.25$), $L = 1$, $h = 1$, $K = 50$

		α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	(Q, r)	$\mathbb{E}(TC_5)$	%gap ₁₂	%gap ₁₅	%gap ₂₅
$p = 1$	$\tau = 12$	0.005	(4, 1, 1.02)	8.29	(4, 1, 4.56)	10.65	(2, 2)	12.60	28.47	51.99	18.31
		0.01	(4, 1, 0.90)	8.23	(4, 1, 6.72)	9.49	(2, 2)	12.60	15.31	53.10	32.77
		0.02	(4, 1, 0.72)	8.16	(4, 1, 8.94)	8.64	(2, 2)	12.60	5.88	54.41	45.83
		0.05	(4, 0, 0.60)	7.25	(4, 0, 11.28)	7.47	(3, 1)	8.25	3.03	13.79	10.44
		0.1	(3, 0, 0.00)	6.87	(3, 0, 12.00)	6.87	(3, 0)	6.87	0.00	0.00	0.00
	$\tau = 15$	0.005	(4, 1, 1.50)	7.63	(4, 1, 6.975)	8.95	(2, 2)	11.40	17.30	49.41	27.37
		0.01	(4, 1, 0.90)	7.49	(4, 1, 9.45)	8.18	(2, 2)	11.40	9.21	52.20	39.36
		0.02	(4, 1, 0.60)	7.43	(4, 1, 13.125)	7.53	(3, 2)	9.45	1.35	27.19	25.50
		0.05	(4, 0, 0.75)	6.52	(4, 0, 14.55)	6.57	(4, 1)	7.32	0.77	12.27	11.42
		0.1	(3, 0, 0.00)	6.34	(3, 0, 15.00)	6.34	(4, 0)	6.34	0.00	0.00	0.00
	$\tau = 20$	0.005	(5, 1, 1.00)	6.96	(5, 1, 12.70)	7.75	(2, 2)	10.35	11.35	48.71	33.55
		0.01	(4, 1, 0.80)	6.90	(4, 1, 14.70)	7.10	(3, 2)	8.71	2.90	26.23	22.68
		0.02	(4, 1, 0.20)	6.84	(4, 1, 20.00)	6.84	(3, 1)	7.27	0.00	6.29	6.29
		0.05	(4, 0, 0.80)	5.83	(4, 0, 20.00)	5.83	(4, 1)	6.83	0.00	17.15	17.15
		0.1	(4, 0, 0.00)	5.74	(4, 0, 20.00)	5.74	(4, 0)	5.74	0.00	0.00	0.00
									6.37	27.52	19.38
$p = 10$	$\tau = 12$	0.005	(4, 1, 1.02)	9.61	(3, 1, 3.00)	12.21	(2, 2)	13.71	27.06	42.66	12.29
		0.01	(3, 1, 1.02)	9.44	(3, 1, 5.34)	10.75	(2, 2)	13.71	13.88	45.23	27.53
		0.02	(3, 1, 0.72)	9.34	(3, 1, 8.34)	9.70	(2, 2)	13.71	3.85	46.79	41.34
		0.05	(3, 0, 1.60)	8.04	(4, 0, 11.28)	8.60	(3, 1)	9.12	6.97	13.43	6.05
		0.1	(3, 0, 0.00)	7.45	(3, 0, 12.00)	7.45	(3, 0)	7.45	0.00	0.00	0.00
	$\tau = 15$	0.005	(4, 1, 1.50)	8.50	(4, 1, 6.975)	10.14	(2, 2)	12.09	19.29	42.24	19.23
		0.01	(4, 1, 0.90)	8.33	(3, 1, 7.575)	9.12	(2, 2)	12.09	9.48	45.14	32.57
		0.02	(4, 1, 0.60)	8.26	(4, 1, 13.125)	8.38	(3, 2)	10.36	1.45	25.42	23.63
		0.05	(4, 0, 0.75)	7.14	(4, 0, 14.55)	7.22	(4, 1)	8.13	1.12	13.87	12.60
		0.1	(3, 0, 0.00)	6.66	(3, 0, 15.00)	6.66	(3, 0)	6.66	0.00	0.00	0.00
	$\tau = 20$	0.005	(5, 1, 1.00)	7.58	(4, 1, 9.30)	8.46	(2, 2)	10.67	11.61	40.77	26.12
		0.01	(4, 1, 0.80)	7.32	(4, 1, 14.70)	7.56	(3, 2)	9.18	3.28	25.41	21.43
		0.02	(4, 1, 0.20)	7.26	(4, 1, 20.00)	7.26	(3, 1)	7.52	0.00	3.58	3.58
		0.05	(4, 0, 0.80)	6.10	(4, 0, 20.00)	6.10	(4, 1)	7.23	0.00	18.52	18.52
		0.1	(4, 0, 0.00)	6.00	(4, 0, 0.00)	6.00	(4, 0)	6.00	0.00	0.00	0.00
									6.53	24.20	16.33
$p = 50$	$\tau = 12$	0.005	(4, 1, 1.02)	15.50	(2, 1, 0.60)	17.62	(2, 2)	18.64	13.68	20.26	5.79
		0.01	(3, 1, 1.02)	13.49	(2, 1, 1.92)	15.94	(2, 2)	18.64	18.16	38.18	16.94
		0.02	(2, 1, 0.72)	13.03	(2, 1, 7.02)	13.34	(2, 2)	18.64	2.38	43.05	39.73
		0.05	(3, 0, 1.62)	11.03	(2, 0, 0.00)	11.60	(2, 1)	12.87	5.17	16.68	10.95
		0.1	(2, 0, 0.96)	9.39	(2, 0, 11.16)	9.52	(3, 0)	10.06	1.38	7.14	5.67
	$\tau = 15$	0.005	(4, 1, 1.50)	12.33	(3, 1, 4.275)	14.23	(2, 2)	15.15	15.41	22.87	6.47
		0.01	(3, 1, 0.975)	10.88	(3, 1, 7.575)	12.14	(2, 2)	15.15	11.58	39.25	24.79
		0.02	(3, 1, 0.525)	10.78	(3, 1, 12.675)	10.90	(3, 2)	14.40	1.11	33.58	32.11
		0.05	(3, 0, 3.225)	9.25	(3, 0, 9.825)	9.93	(3, 1)	10.69	7.35	15.57	7.65
		0.1	(3, 0, 0.00)	8.12	(3, 0, 15.00)	8.12	(3, 0)	8.12	0.00	0.00	0.00
	$\tau = 20$	0.005	(4, 1, 4.60)	10.01	(3, 1, 5.60)	10.68	(2, 2)	12.10	6.69	20.88	13.30
		0.01	(3, 1, 0.90)	8.68	(3, 1, 12.60)	8.94	(3, 2)	11.24	3.00	29.49	25.73
		0.02	(3, 1, 0.00)	8.64	(3, 1, 19.70)	8.64	(3, 1)	8.64	0.00	0.00	0.00
		0.05	(4, 0, 0.80)	7.30	(4, 0, 20.00)	7.30	(3, 1)	8.63	0.00	18.22	18.22
		0.1	(3, 0, 0.00)	6.62	(3, 0, 20.00)	6.62	(3, 0)	6.62	0.00	0.00	0.00
									5.87	20.34	13.69

Now, we again consider data Set 2 to compare the (Q, r, ξ) and (Q, r, T) policies and investigate the effect of effective lifetime distribution (Q10 in research questions). The result is summarized in Tables 8.27 and 8.28, where $(Q, r, T)_T$ is the optimal solution obtained by assuming that the current batch is always fresh

at the beginning of embedded cycles (Tekin et al. [23], Table 2).

Table 8.27: Effect of effective lifetime for data Set 2 with $K = 50$, ($L = 1$, $h = 1$, $\lambda = 5$)

	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	$(Q, r, T)_T$	$\mathbb{E}(TC_6)$	%gap ₁₂	%gap ₂₆	
$p = 1$	$\tau = 2$	0.005	(15, 8, 1.00)	38.69	(11, 10, 0.17)	45.18	(13, 9, 1.00)	37.24	16.78	-17.57
		0.01	(14, 7, 1.00)	37.18	(11, 9, 0.29)	43.13	(12, 8, 0.94)	36.22	16.00	-16.01
		0.02	(13, 6, 1.00)	35.70	(11, 8, 0.47)	40.65	(12, 7, 1.00)	35.22	13.87	-13.36
		0.05	(12, 3, 1.00)	33.52	(11, 7, 0.90)	36.62	(11, 5, 0.98)	33.45	9.25	-8.65
		0.1	(11, 2, 0.95)	31.48	(11, 5, 1.29)	32.50	(11, 2, 1.05)	31.48	3.22	-3.13
	$\tau = 4$	0.005	(20, 8, 1.08)	28.31	(16, 9, 1.68)	30.10	(21, 8, 2.95)	27.56	6.32	-8.44
		0.01	(20, 7, 1.02)	27.10	(16, 8, 1.98)	28.62	(20, 7, 2.81)	26.70	5.62	-6.70
		0.02	(19, 6, 1.06)	25.92	(17, 7, 2.44)	27.02	(20, 6, 3.01)	25.61	4.23	-5.20
		0.05	(19, 4, 1.00)	24.06	(18, 5, 3.00)	24.68	(19, 4, 2.99)	23.96	2.60	-2.93
		0.1	(18, 2, 0.98)	22.33	(18, 3, 3.56)	22.52	(18, 2, 3.02)	22.32	0.86	-0.90
	$\tau = 6$	0.005	(24, 8, 1.02)	26.30	(22, 8, 3.27)	26.72	(24, 8, 4.79)	26.04	1.60	-2.54
		0.01	(24, 7, 1.08)	25.30	(21, 7, 3.30)	25.66	(24, 7, 4.57)	25.17	1.42	-1.90
		0.02	(23, 6, 1.11)	24.15	(22, 6, 3.93)	24.46	(23, 6, 4.61)	24.08	1.26	-1.54
		0.05	(24, 4, 1.17)	22.61	(22, 4, 4.23)	22.83	(24, 4, 4.76)	22.57	0.99	-1.16
		0.1	(20, 3, 0.00)	20.92	(20, 3, 6.00)	20.92	(20, 3, 6.00)	20.92	0.00	0.00
								5.60	-6.00	
$p = 10$	$\tau = 2$	0.005	(12, 9, 1.00)	58.28	(9, 10, 1.47)	60.86	(10, 9, 0.23)	45.01	4.42	-26.04
		0.01	(11, 8, 1.15)	53.68	(8, 9, 1.44)	56.93	(9, 8, 0.11)	43.28	6.05	-23.97
		0.02	(11, 7, 1.05)	48.79	(8, 8, 2.00)	53.14	(9, 7, 0.30)	42.09	8.92	-20.79
		0.05	(9, 6, 1.05)	42.78	(9, 7, 1.12)	45.75	(9, 6, 0.84)	39.53	6.96	-13.60
		0.1	(10, 3, 1.02)	37.88	(9, 5, 1.08)	39.43	(9, 5, 1.16)	36.90	4.10	-6.43
	$\tau = 4$	0.005	(14, 9, 0.90)	33.07	(13, 9, 1.46)	33.33	(16, 9, 4.00)	29.63	0.78	-11.09
		0.01	(14, 8, 0.92)	31.03	(14, 8, 1.80)	31.32	(16, 8, 4.00)	28.54	0.93	-8.88
		0.02	(15, 7, 0.84)	29.09	(14, 7, 2.20)	29.31	(16, 7, 4.00)	27.41	0.74	-6.48
		0.05	(16, 5, 0.96)	25.98	(16, 5, 2.78)	26.47	(16, 5, 3.02)	25.34	1.87	-4.25
		0.1	(16, 3, 1.08)	23.69	(15, 4, 4.00)	23.90	(16, 3, 2.89)	23.58	0.89	-1.34
	$\tau = 6$	0.005	(21, 8, 2.04)	27.48	(20, 8, 2.76)	27.63	(22, 8, 3.65)	26.48	0.53	-4.17
		0.01	(21, 7, 2.07)	26.33	(20, 7, 3.00)	26.39	(22, 7, 3.70)	25.61	0.22	-2.95
		0.02	(22, 6, 1.56)	24.89	(21, 6, 3.69)	24.98	(22, 6, 4.10)	24.47	0.34	-2.03
		0.05	(20, 5, 0.00)	23.09	(20, 5, 6.00)	23.09	(22, 4, 4.05)	22.97	0.00	-0.54
		0.1	(20, 3, 0.00)	21.03	(20, 3, 6.00)	21.03	(20, 3, 6.00)	21.02	0.00	-0.05
								2.45	-8.84	
$p = 50$	$\tau = 2$	0.005	(6, 9, 0.83)	101.36	(6, 10, 0.40)	108.86	(10, 9, 0.23)	73.46	7.40	-32.52
		0.01	(5, 9, 0.65)	90.90	(7, 9, 0.98)	98.86	(9, 8, 0.11)	62.42	8.76	-36.86
		0.02	(6, 8, 0.32)	80.00	(7, 8, 1.39)	83.88	(8, 7, 0.05)	54.60	4.85	-34.91
		0.05	(7, 6, 1.72)	64.72	(6, 6, 1.21)	67.84	(7, 6, 0.10)	49.18	4.83	-27.51
		0.1	(7, 5, 1.32)	52.36	(7, 5, 0.55)	55.67	(7, 5, 0.50)	45.79	6.31	-17.75
	$\tau = 4$	0.005	(11, 9, 1.12)	39.61	(11, 9, 0.98)	39.76	(13, 9, 4.00)	31.78	0.39	-20.07
		0.01	(11, 8, 2.16)	36.80	(11, 8, 0.86)	37.16	(13, 8, 4.00)	30.69	0.99	-17.42
		0.02	(12, 7, 1.72)	33.66	(12, 7, 1.52)	34.04	(13, 7, 4.00)	29.52	1.12	-13.28
		0.05	(13, 5, 1.90)	30.28	(12, 6, 4.00)	30.44	(13, 5, 1.79)	27.81	0.55	-8.65
		0.1	(13, 4, 0.80)	26.04	(13, 4, 3.26)	26.09	(13, 4, 3.19)	25.48	0.20	-2.33
	$\tau = 6$	0.005	(16, 9, 0.00)	29.28	(16, 9, 6.00)	29.28	(20, 8, 2.88)	27.16	0.00	-7.23
		0.01	(16, 8, 0.00)	27.78	(16, 8, 6.00)	27.78	(20, 7, 2.99)	26.29	0.00	-5.35
		0.02	(17, 7, 0.00)	26.28	(17, 7, 6.00)	26.28	(20, 6, 3.30)	25.15	0.00	-4.28
		0.05	(18, 5, 0.00)	23.57	(18, 5, 6.00)	23.57	(19, 5, 6.00)	23.32	0.00	-1.08
		0.1	(19, 3, 1.35)	21.43	(19, 3, 4.98)	21.46	(19, 3, 4.61)	21.37	0.16	-0.43
								2.37	-15.31	

Table 8.28: Effect of effective lifetime for data Set 2 with $K = 100$, ($L = 1$, $h = 1$, $\lambda = 5$)

	α	(Q, r, ξ)	$\mathbb{E}(TC_1)$	(Q, r, T)	$\mathbb{E}(TC_2)$	$(Q, r, T)_T$	$\mathbb{E}(TC_6)$	%gap ₁₂	%gap ₂₆	
$p = 1$	$\tau = 2$	0.005	(16, 5, 1.00)	64.09	(15, 10, 0.38)	76.82	(13, 9, 1.00)	62.96	19.87	-18.05
		0.01	(15, 4, 1.00)	62.62	(13, 9, 0.40)	73.50	(14, 7, 1.00)	61.92	17.38	-15.75
		0.02	(14, 2, 1.00)	61.13	(13, 8, 0.57)	69.70	(13, 6, 1.00)	60.67	14.02	-12.96
		0.05	(12, 3, 1.00)	58.66	(13, 7, 0.94)	63.14	(12, 2, 0.99)	58.59	7.65	-7.21
		0.1	(12, 2, 0.89)	55.53	(13, 5, 1.34)	56.79	(12, 4, 1.15)	55.49	2.26	-2.29
	$\tau = 4$	0.005	(24, 7, 1.02)	42.18	(19, 8, 1.78)	46.14	(21, 8, 2.95)	40.89	9.39	-11.38
		0.01	(23, 6, 1.02)	40.89	(19, 8, 2.14)	44.17	(21, 7, 3.00)	40.01	8.03	-9.42
		0.02	(22, 5, 1.00)	39.46	(20, 7, 2.56)	41.93	(20, 6, 3.01)	39.06	6.26	-6.85
		0.05	(20, 4, 0.94)	37.46	(20, 5, 3.16)	38.39	(20, 4, 3.05)	37.36	2.47	-2.68
		0.1	(19, 1, 0.96)	35.28	(20, 3, 3.74)	35.38	(19, 2, 3.13)	35.26	0.28	-0.35
	$\tau = 6$	0.005	(26, 8, 0.99)	36.82	(24, 8, 3.57)	37.88	(27, 8, 5.03)	36.09	2.88	-4.72
		0.01	(27, 7, 0.96)	35.67	(26, 7, 4.11)	36.48	(27, 7, 5.05)	35.14	2.28	-3.67
		0.02	(26, 6, 0.90)	34.47	(25, 6, 4.50)	34.93	(26, 6, 5.11)	34.15	1.32	-2.23
		0.05	(26, 4, 0.87)	32.39	(26, 4, 5.22)	32.58	(26, 4, 5.13)	32.29	0.58	-0.88
		0.1	(26, 1, 0.96)	30.33	(26, 2, 5.94)	30.38	(26, 1, 5.04)	30.33	0.17	-0.16
								6.32	-6.57	
$p = 10$	$\tau = 2$	0.005	(12, 9, 1.00)	87.33	(10, 10, 2.00)	97.33	(10, 9, 0.23)	73.44	11.45	-24.54
		0.01	(12, 8, 1.00)	81.73	(10, 9, 0.23)	91.28	(10, 8, 0.38)	72.29	11.69	-20.80
		0.02	(11, 7, 1.05)	76.98	(10, 8, 0.39)	85.48	(10, 7, 0.57)	70.72	11.04	-17.27
		0.05	(11, 5, 1.00)	69.02	(10, 7, 0.92)	75.70	(10, 6, 1.03)	67.47	9.69	-10.88
		0.1	(10, 3, 1.02)	63.70	(10, 5, 1.22)	66.32	(10, 3, 0.97)	63.35	4.11	-4.48
	$\tau = 4$	0.005	(20, 8, 1.08)	49.56	(15, 9, 1.62)	51.67	(18, 8, 2.14)	44.92	4.25	-13.06
		0.01	(20, 7, 1.02)	47.06	(16, 8, 1.98)	48.83	(18, 8, 3.20)	43.87	3.77	-10.16
		0.02	(19, 6, 1.06)	44.36	(16, 7, 2.40)	45.98	(18, 6, 2.54)	42.57	3.64	-7.41
		0.05	(19, 4, 1.00)	40.90	(17, 5, 2.92)	41.66	(17, 5, 3.11)	40.20	1.86	-3.50
		0.1	(18, 2, 0.98)	37.71	(17, 3, 3.40)	38.04	(17, 3, 3.10)	37.63	0.86	-1.07
	$\tau = 6$	0.005	(24, 8, 1.02)	38.80	(23, 8, 3.48)	39.53	(25, 8, 5.02)	37.19	1.89	-5.92
		0.01	(24, 7, 1.02)	37.25	(23, 7, 3.72)	38.00	(25, 7, 5.01)	36.18	2.03	-4.80
		0.02	(24, 6, 0.96)	35.76	(23, 6, 4.20)	36.20	(24, 6, 5.03)	35.06	1.24	-3.15
		0.05	(24, 4, 1.17)	33.41	(24, 4, 4.83)	33.69	(25, 4, 5.04)	33.19	0.85	-1.49
		0.1	(24, 2, 0.93)	31.05	(24, 2, 5.55)	31.18	(24, 2, 5.05)	31.06	0.41	-0.37
								4.59	-8.59	
$p = 50$	$\tau = 2$	0.005	(7, 9, 0.52)	146.32	(7, 10, 0.70)	159.35	(10, 9, 0.23)	101.90	8.91	-36.05
		0.01	(7, 8, 0.46)	135.22	(8, 9, 1.43)	139.63	(9, 8, 0.11)	92.58	3.26	-33.70
		0.02	(8, 7, 1.83)	120.77	(8, 8, 2.00)	127.29	(8, 7, 0.05)	87.11	5.40	-31.57
		0.05	(8, 6, 1.34)	99.95	(8, 7, 2.00)	107.18	(8, 6, 0.44)	83.37	7.23	-22.21
		0.1	(8, 5, 0.96)	84.32	(8, 5, 0.88)	87.95	(8, 5, 1.03)	78.13	4.30	-11.17
	$\tau = 4$	0.005	(12, 9, 0.88)	62.12	(12, 9, 1.32)	62.31	(15, 9, 4.00)	49.92	0.32	-19.89
		0.01	(13, 8, 0.92)	57.62	(13, 8, 1.66)	58.11	(15, 8, 4.00)	48.74	0.84	-16.12
		0.02	(13, 7, 0.94)	53.42	(13, 7, 1.98)	53.81	(15, 7, 4.00)	47.41	0.73	-11.89
		0.05	(15, 5, 1.26)	47.38	(14, 5, 2.22)	48.41	(15, 5, 2.57)	44.82	2.16	-7.41
		0.1	(14, 4, 0.00)	42.75	(14, 4, 4.00)	42.75	(15, 3, 2.55)	42.04	0.00	-1.66
	$\tau = 6$	0.005	(20, 8, 2.34)	43.34	(19, 8, 2.52)	43.39	(22, 8, 3.65)	39.13	0.12	-9.82
		0.01	(20, 7, 2.34)	41.35	(20, 7, 3.00)	41.43	(22, 7, 3.70)	38.19	0.20	-7.82
		0.02	(21, 6, 1.89)	38.76	(21, 6, 3.66)	39.08	(22, 6, 4.10)	36.92	0.82	-5.53
		0.05	(21, 5, 0.00)	35.91	(21, 5, 6.00)	35.91	(22, 4, 4.05)	35.05	0.00	-2.40
		0.1	(21, 3, 0.00)	32.72	(21, 3, 6.00)	32.72	(21, 3, 6.00)	32.64	0.00	-0.23
								2.29	-14.50	

We also consider the effect of effective lifetime in another way. Let $(Q^*, r^*, T^*)_T$ be the optimal solution obtained by assuming that the current batch is always fresh at the beginning of embedded cycles (freshness assumption) and let's call it the optimal solution under freshness. If we put the optimal solution under

freshness in our model, in which we have an effective lifetime distribution, we have a different objective value and different service level as well. Table 8.29 summarizes these results from which we can observe that the optimal solution under freshness is not a feasible solution in most of the cases and the total cost is always less than that of our model. As the unit perishing cost increases, the total cost gap increases. Recall that $\mathbb{E}(TC_6)$ is the total cost value under freshness assumption and $\mathbb{E}(TC_2)$ is the total cost under our model assumption, and β is the real value of fraction of lost sales. Note that the optimal solutions obtained under our model assumption are different from the ones obtained under freshness assumption. Therefore, a wrong assumption about the effective lifetime distribution of batches in the inventory system causes higher operational costs with wrong optimal control parameters as well.

We observe that the optimal reorder point r^* obtained from the (Q, r, T) policy under freshness assumption is less than or equal to that of the (Q, r, T) policy under our model assumption. Therefore, we can utilize it as a lower bound in the exhaustive search to reduce the search space. But, there is no a relationship between optimal order quantity Q^* .

We also observe the following findings, however they are observed in data Set 2 as well.

- As p increases, the gap between the (Q, r, ξ) and (Q, r, T) policies decreases and for longer lifetime, we have less gap as well.
- The optimal reorder point r^* in the (Q, r, T) policy is consistently greater than or equal to that of the (Q, r, ξ) policy.
- As p increase and τ gets smaller, the number of outstanding orders increases.

Table 8.29: Effect of effective lifetime by using the optimal solution under freshness for data Set 2 with $K = 50$, ($L = 1$, $h = 1$, $\lambda = 5$)

		α	$(Q, r, T)_T$	$\mathbb{E}(TC_6)$	$\mathbb{E}(TC_2)$	β	%gap
$p = 1$	$\tau = 2$	0.005	(13, 9, 1.00)	37.24	38.60	0.0508	3.65
		0.01	(12, 8, 0.94)	36.22	37.56	0.0467	3.70
		0.02	(12, 7, 1.00)	35.22	36.08	0.0556	2.44
		0.05	(11, 5, 0.98)	33.45	34.59	0.0788	3.41
		0.1	(11, 2, 1.05)	31.48	33.65	0.1216	6.89
	$\tau = 4$	0.005	(21, 8, 2.95)	27.56	28.02	0.0332	1.67
		0.01	(20, 7, 2.81)	26.70	27.13	0.0278	1.61
		0.02	(20, 6, 3.01)	25.61	26.03	0.0380	1.64
		0.05	(19, 4, 2.99)	23.96	24.67	0.0577	2.96
		0.1	(18, 2, 3.02)	22.32	23.62	0.0915	5.82
	$\tau = 6$	0.005	(24, 8, 4.79)	26.04	26.14	0.0113	0.38
		0.01	(24, 7, 4.57)	25.17	25.24	0.0144	0.28
		0.02	(23, 6, 4.61)	24.08	24.17	0.0225	0.37
		0.05	(24, 4, 4.76)	22.57	22.89	0.0487	1.40
		0.1	(20, 3, 6.00)	20.92	20.94	0.0987	0.10
	2.42						
	$\tau = 2$	0.005	(10, 9, 0.23)	45.01	58.29	0.0099	29.50
		0.01	(9, 8, 0.11)	43.28	54.87	0.0164	26.78
		0.02	(9, 7, 0.30)	42.09	50.65	0.0294	20.34
		0.05	(9, 6, 0.84)	39.53	43.11	0.0647	9.06
		0.1	(9, 5, 1.16)	36.90	38.97	0.1026	5.61
	$\tau = 4$	0.005	(16, 9, 4.00)	29.63	33.31	0.0137	12.42
		0.01	(16, 8, 4.00)	28.54	31.07	0.0198	8.86
		0.02	(16, 7, 4.00)	27.41	29.02	0.0296	5.87
		0.05	(16, 5, 3.02)	25.34	26.07	0.0548	2.88
		0.1	(16, 3, 2.89)	23.58	24.78	0.0908	5.09
	$\tau = 6$	0.005	(22, 8, 3.65)	26.48	27.45	0.0059	3.66
		0.01	(22, 7, 3.70)	25.61	26.33	0.0108	2.81
		0.02	(22, 6, 4.10)	24.47	24.93	0.0207	1.88
		0.05	(22, 4, 4.05)	22.97	23.63	0.0471	2.87
		0.1	(20, 3, 6.00)	21.02	21.33	0.0919	1.47
	9.27						
$p = 50$	$\tau = 2$	0.005	(10, 9, 0.23)	73.46	123.99	0.0099	68.79
		0.01	(9, 8, 0.11)	62.42	105.21	0.0164	68.55
		0.02	(8, 7, 0.05)	54.60	88.05	0.0266	61.26
		0.05	(7, 6, 0.10)	49.18	69.90	0.0523	42.13
		0.1	(7, 5, 0.50)	45.79	56.67	0.0966	23.76
	$\tau = 4$	0.005	(13, 9, 4.00)	31.78	42.37	0.0063	33.32
		0.01	(13, 8, 4.00)	30.69	37.85	0.0123	23.33
		0.02	(13, 7, 4.00)	29.52	34.03	0.0227	15.28
		0.05	(13, 5, 1.79)	27.81	31.26	0.0469	12.41
		0.1	(13, 4, 3.19)	25.48	26.10	0.0990	2.43
	$\tau = 6$	0.005	(20, 8, 2.88)	27.16	30.57	0.0052	12.56
		0.01	(20, 7, 2.99)	26.29	28.91	0.0100	9.97
		0.02	(20, 6, 3.30)	25.15	26.88	0.0197	6.88
		0.05	(19, 5, 6.00)	23.32	23.65	0.0452	1.42
		0.1	(19, 3, 4.61)	21.37	21.62	0.0958	1.17
	25.55						

8.3.5 The Summary of Numerical Study

We can summarize our main findings from the numerical study as follows:

- The effect of the key input parameters on optimal solution, $\%gap_{15}$, and the expected cost rate of the (Q, r, ξ) policy is summarized in Table 8.30. The results are reported by considering an increase in input parameters K , λ , τ , α , and p . For instance, increase in K causes increase in value of Q and increase in expected cost rate.

Table 8.30: Effect of the key input parameters on optimal solution

Increase	Q	r	$\%gap_{15}$	$\mathbb{E}(TC_1)$
K	$\nearrow$	-	-	$\nearrow$
λ	$\nearrow$	$\nearrow$	$\searrow$	-
τ	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
α	-	$\searrow$	$\searrow$	$\searrow$
p	$\searrow$	-	$\nearrow$	$\nearrow$

- The effect of the key input parameters on gaps is summarized in Table 8.31. Again, the results are reported by considering an increase in input parameters. The first two rows represent the minimum and maximum gap observed from numerical study. For instance, the minimum gap observed between policy 2 and policy 4 is 0.87%, and the maximum gap observed between policy 3 and policy 5 is 39.07%.

Table 8.31: Effect of the key input parameters on gaps

Increase	gap_{12}	gap_{13}	gap_{15}	gap_{24}	gap_{25}	gap_{34}	gap_{35}	gap_{45}
min	0	0	0	0.87	0	-5.03	-27.23	-42.51
max	23.96	58.83	62.45	90.72	33.28	65.23	39.07	13.32
p	$\searrow$	$\nearrow$	-	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$
λ	$\searrow$	$\nearrow$	$\searrow$	-	$\searrow$	$\searrow$	$\searrow$	$\searrow$
τ	$\searrow$	-	$\searrow$	-	$\searrow$	$\searrow$	$\searrow$	$\nearrow$
α	-	-	-	-	-	$\searrow$	-	-

- The decision variable ξ becomes more effective when lifetime τ is short.
- The decision variable r becomes more effective when lifetime τ is long.
- The (Q, r) policy performs well when τ is large.
- The (Q, r) policy performs well when λ is large.
- Three-parameter policies dominate two-parameter policies more and more as p increases.

- The notation $policy1 \succeq policy2$ means that policy 1 outperforms policy 2. Then, we have $(Q, r, \xi) \succeq (Q, r, T) \succeq (Q, \xi) \cong (Q, r) \succeq (Q, T)$.
- Implementation of the (Q, r, T) policy is easier than (Q, r, ξ) policy because tracking the time T is easier than tracking the effective lifetime of current batch ξ .
- Number of outstanding orders in the (Q, ξ) and (Q, T) policies is at most 2, but we only need to track the effective lifetime of current batch in these two policies.
- Number of outstanding orders: $(Q, r) \geq (Q, r, T) \geq (Q, r, \xi)$.
- α , λ , K , and p are the most effective parameters on m .
- With small fixed ordering cost, we need more number of outstanding orders.
- As demand rate increases, we need more number of outstanding orders.
- In the cases with short lifetimes, we need more number of outstanding orders.
- With high unit perishing cost, we need more number of outstanding orders. In this case, the saving value is quite high (61.38% gap, in the (Q, r) policy).
- The (Q, r) policy is the only one which is not able to provide any feasible solution when service level is high (α is small).
- The higher lead time stimulates us for using a time-based or lifetime-based policy, i.e., three parameters policies dominates two-parameters policies.
- The higher lead time, the higher m .
- Under freshness assumption, we obtain the optimal solution different from our models. The optimal total cost under freshness assumption is always less than the optimal total cost in our model.
- If we put the optimal solution obtained under freshness assumption in our model, we will not get a feasible solution because of violating service level constraint.

8.4 Consistency Check with Simulation

In this section, we address the simulation framework for each control policy and examine the convergence of simulation model. We compare the simulation results with the exact analytical results implemented by Matlab code to confirm the accuracy of the simulation and the discretization approach as well as the optimal solutions.

We specify a warm-up period at the beginning of each replication, after which the statistical accumulators are all cleared to wear off the effect of initial conditions. This warm-up period is taken out of the beginning of our replication length. Therefore, statistics are collected for the last time units (replication length minus warm-up period). In our research, we set the warm up period to be 1000 days after some trial and error. That is, we run the simulation model for some time periods and record the total cost rate. After 600 days, we observed that total cost converges to a steady state. However, this warm-up period depends on the policy class. For instance, the total cost rate of (Q, r, T) and (Q, r, ξ) policies converge after 800 days. Therefore, to be more exact, we set the warm up period to 1000 days. Again, we run some instances in different policies and observed that after 100,000 days we have small variance in total cost rate (around 0.03). Thus, we set the replication length to be 100,000 days after running some trials. Note that the replication length is long enough. So, we will reach to steady state regardless of initial states.

The “ HW ” column in Arena report denotes the “Half Width” value which is the “ $\pm$ ” part of a 95% confidence interval on the expected total cost in system, found by taking the standard deviation of the N replications, dividing by the square root of N , and multiplying by the student t critical value for a 95% confidence interval with $N - 1$ degrees of freedom. That is,

$$HW = \frac{s}{\sqrt{N}} \times t_{N-1, 1-\alpha/2} \quad (8.2)$$

To find the number of replication, N , we set half width to a specific value, hw .

Then, we can find N by solving the following equation:

$$hw = \frac{s}{\sqrt{N}} \times t_{N-1, 1-\alpha/2} \quad (8.3)$$

Since $t_{N-1, 1-\alpha/2}$ is also a function of N , finding N is sophisticated. Therefore, we use normal approximation as $t_{N-1, 1-\alpha/2} = Z_{1-\alpha/2}$. Then, we have

$$hw = \frac{s}{\sqrt{N}} \times Z_{1-\alpha/2} \quad \longrightarrow \quad N = \frac{s^2}{hw^2} \times Z_{1-\alpha/2}^2 \quad (8.4)$$

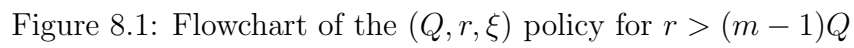
Firstly, we approximate the standard deviation value, s , by taking some runs with 10 replications. Our result shows that $s = 0.028$. Our goal is to have $hw = 0.01$. Therefore,

$$N = \frac{s^2}{hw^2} \times Z_{1-\alpha/2}^2 = \frac{0.028^2}{0.01^2} \times 1.96^2 \approx 10.$$

Hence, we set $N = 10$. In summary, we set warm up period, replication length, number of replications to be 1000 days, 100,000 days, and 10, respectively. Another method for running the simulation is that we run the model for a long time and we divide run length to some sub runs each of which are considered as one replication.

Figures 8.1-8.8 illustrate the implementation algorithm of each policy through flowcharts. In “decision module”, the “Yes” part is depicted in flowchart, and the other part is “No”. The “Sep module” makes a copy from the original entity in flowchart. The “Disposal module” disposes the entity from the flowchart forever. The “Blue modules” represent a delay in the entity through the flowchart. Input parameters (global variables) such as demand rate, lifetime, lead time, initial inventory, unit perishing and holding costs are also defined in “Variable” in “Basic Process” palette in Arena. We use some notations in flowcharts which are listed below.

τ	Constant lifetime
LeadTime	Lead time, L
Q	Order quantity
r	Inventory threshold for reordering
T	Time threshold for reordering in the (Q, r, T) policy
ξ	Remaining lifetime threshold for reordering in the (Q, r, ξ) policy
m	Maximum number of outstanding orders
$Z(i)$	Effective lifetime of batch i
$U(L, \tau)$	A random variable generated from a uniform distribution in the interval (L, τ)
$\text{Exp}(\lambda)$	Arrival rate with exponential distribution with rate λ
$\text{First}(\lambda)$	First arrival occurs in $\text{exp}(\lambda)$
Max_Arr=1	Just one entity enters to the system
IL	Inventory level at time t
IP	Inventory position at time t
Total lost	Number of lost sales during simulation run
Perish	Number of perished items during an embedded cycle
Total Perish	Total number of perished items during simulation run
Dcycle	Number of items consumed before perishing, and $(Q - \text{Dcycle})$ is number of items perished during an embedded cycle
Stat	Indicator of ordering, 1 if an order is placed, 0 otherwise
Order Time	The time at which we place an order
CumCycle	Represents cumulative cycle length during simulation run
Cycle Length	The length of an embedded cycle
TNOW	Current time in simulation run (Set Base Time Units as days)
NumCycle	Number of embedded cycles during simulation run
Current Cycle	A variable in order to track whether we are in the current cycle or not
Expire Time	Effective lifetime
Wait Time	The value of $z - \xi$ in the (Q, r, ξ) policy
Tr	Reorder point in multiple orders outstanding cases



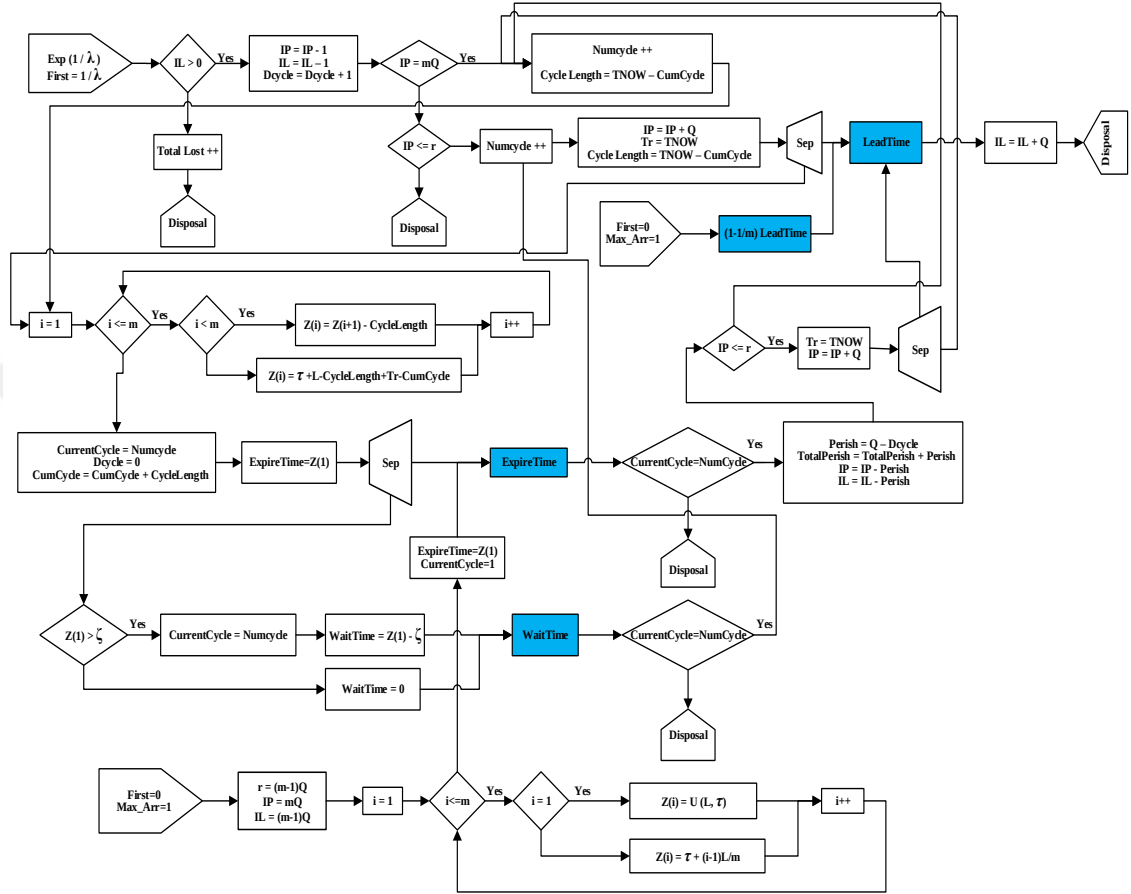


Figure 8.2: Flowchart of the (Q, r, ξ) policy for $r = (m - 1)Q$

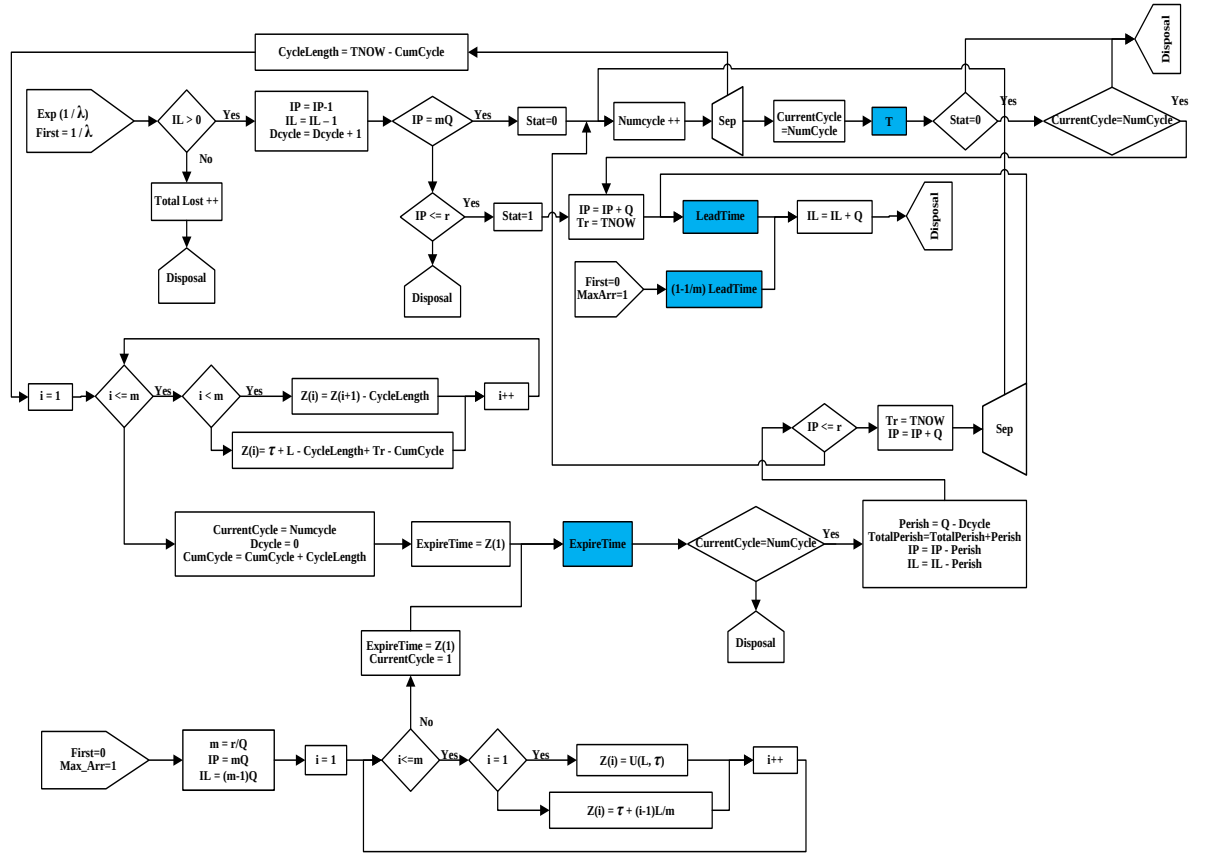
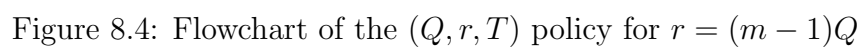


Figure 8.3: Flowchart of the (Q, r, T) policy for $r > (m - 1)Q$



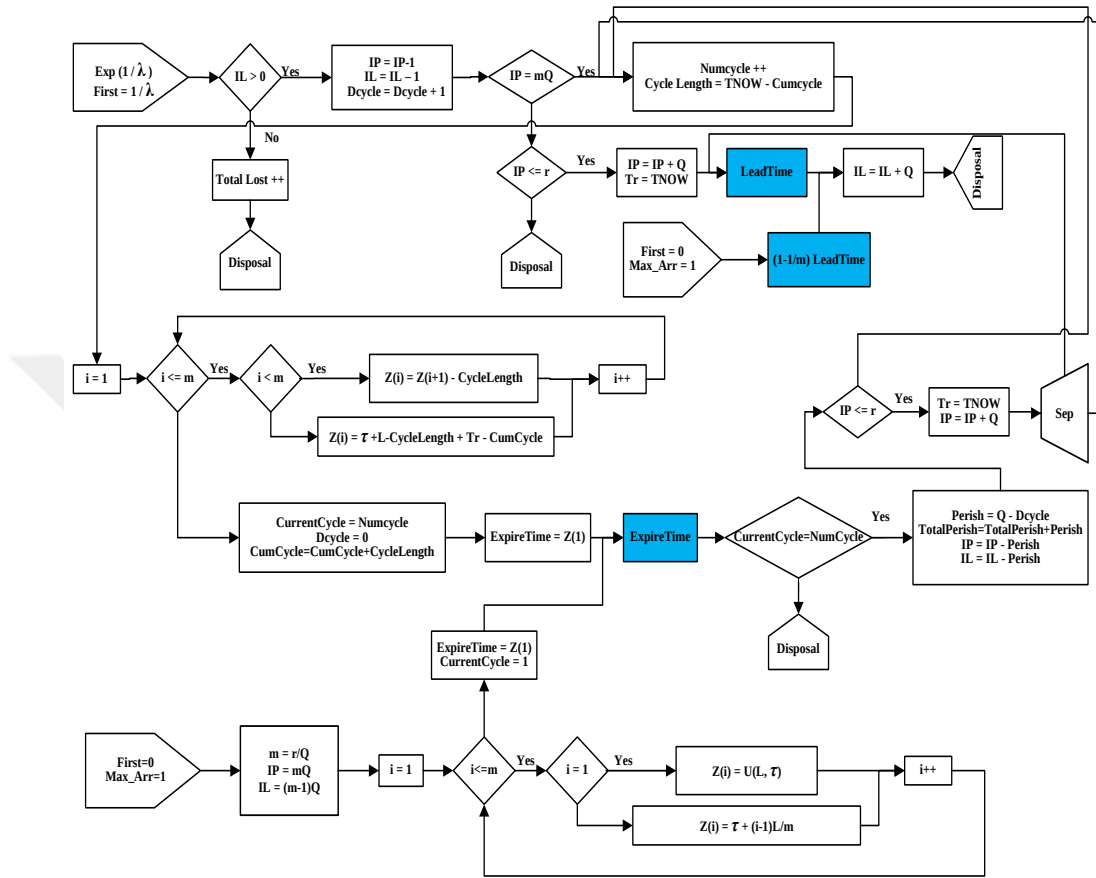


Figure 8.5: Flowchart of the (Q, r) policy for $r > (m - 1)Q$

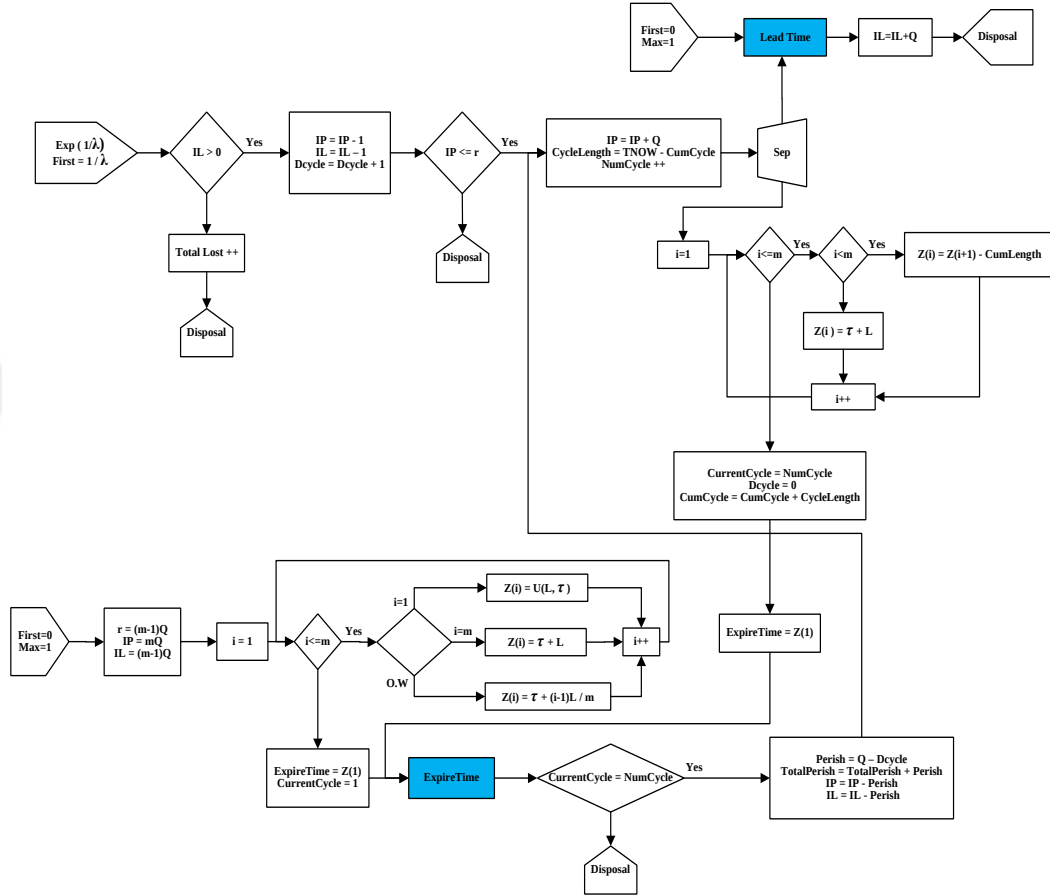


Figure 8.6: Flowchart of the (Q, r) policy for $r = (m - 1)Q$

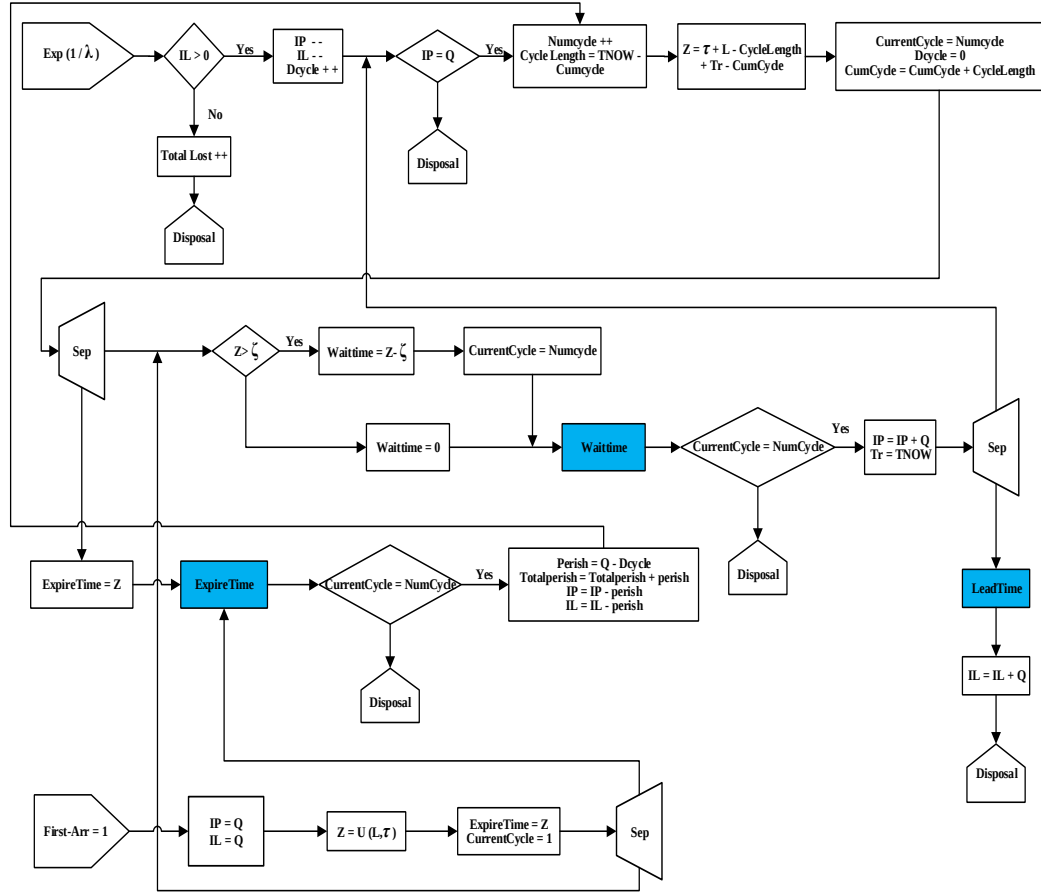


Figure 8.7: Flowchart of the (Q, ξ) policy

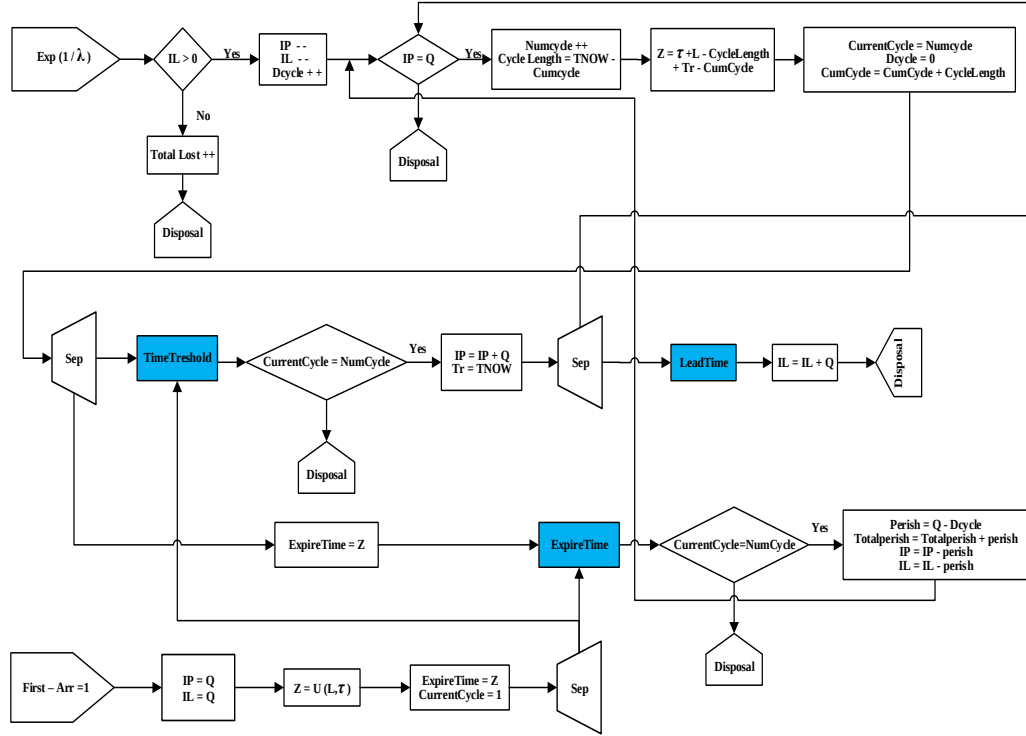


Figure 8.8: Flowchart of the (Q, T) policy

Next, we describe different events may occur in simulation framework. In the following, we consider different events in the (Q, r, ξ) policy where $r > (m - 1)Q$ (Figure 8.1). For other policies we have similar description for events.

(Event 1) Initialization: Without loss of generality, we suppose that the inventory level is $(m - 1)Q$ and the inventory position is mQ (we have one order outstanding) at the beginning of the first embedded cycle. Furthermore, we set the current batch effective lifetime $Z(1)$ to be a random number generated from a uniform distribution, $U(L, \tau)$, and $Z(i)$ to be $\tau + (i - 1)L/m$ for $2 \leq i \leq m$. By doing so, we guarantee that $Z_1 \leq Z_2 \leq \dots \leq Z_m$. This initialization is depicted in Figure 8.9 which is a part of Figure 8.1.

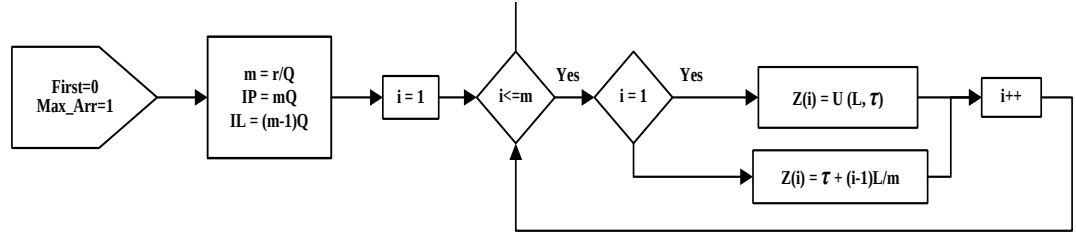


Figure 8.9: Initialization scheme

We only need one entity for initialization created at time zero. We use “Max_Arr=1” and “First=0” to represent this issue.

(Event 2) Demand arrival and no reordering is triggered: In this event, upon arrival of a demand both inventory level and the inventory position decrease one unit. Moreover, the amount of variable “Dcycle” representing the number of demand arrival in one cycle increases one unit. This event is depicted in Figure 8.10 which is a part of Figure 8.1.

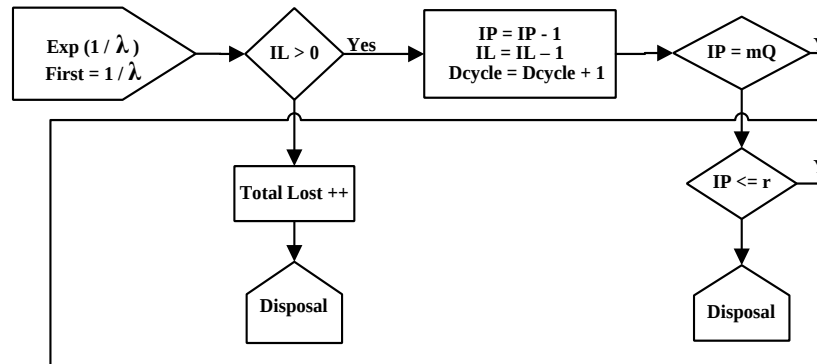


Figure 8.10: Event 2 scheme

In this figure, “First= $1/\lambda$ ” and “Exp($1/\lambda$)” indicate that the first demand arrival does not occur at time zero, and the time interval between two consecutive demand is an exponential distribution with rate λ , respectively.

(Event 3) Demand arrival and reordering is triggered: This event is the same

with Event 2. In addition to actions in Event 2, in this event we have a replacement resulting in Q units increase in inventory position. Furthermore, we need to track the replacement time by variable “Tr”. After reordering, the inventory level increases Q units after lead time L is passed.

(Event 4) Demand arrival and embedded cycle ending: This event is the same with Event 2. In addition to actions in Event 2, in this event the embedded cycle ends upon demand arrival. When event 4 occurs, the number of embedded cycle is increased by 1 and the length of new cycle is calculated as well. At the end of embedded cycle, the effective lifetime vector is updated. Additionally, the variable “Dcycle” is set to zero, and the variable “CumCycle” representing cumulative cycle length is also updated by adding the new cycle length.

When the embedded cycle ends (Event 4 occurs), we make a copy from the original entity using “Separate module” by which the original entity wait for effective lifetime of current batch (Z_1 units of time) to check whether we have perishing or not, and the duplicated entity wait for $Z_1 - \xi$ units of time to check whether we need a replacement or not. After delaying the duplicated entity for $Z_1 - \xi$ units of time, we need to place an order if inventory position has not yet crossed r . Note that either the inventory position crosses r or the remaining lifetime crosses $Z_1 - \xi$ we set the variable “Stat” to 1, whichever occurs first. By using this variable, we can guarantee that reordering occurs once in each embedded cycle. At the beginning of each embedded cycle we set this variable to zero to allow the system for new replacement.

(Event 5) Perishing: When less than Q units are consumed during the effective lifetime, $Q - \text{Dcycle}$ items are perished at the end of embedded cycle. The inventory level and inventory position are decreased by $Q - \text{Dcycle}$ items as well. Whenever event 5 occurs, the embedded cycle ends up and we need to update the effective lifetime vector.

Figures 8.11 and 8.12 depict a sample path of the inventory position and inventory level in the (Q, r, ξ) policy for the $Q = 5$, $r = 7$, $L = 1$, $\lambda = 10$, $\tau = 3$, and $\xi = 2.3$, respectively.

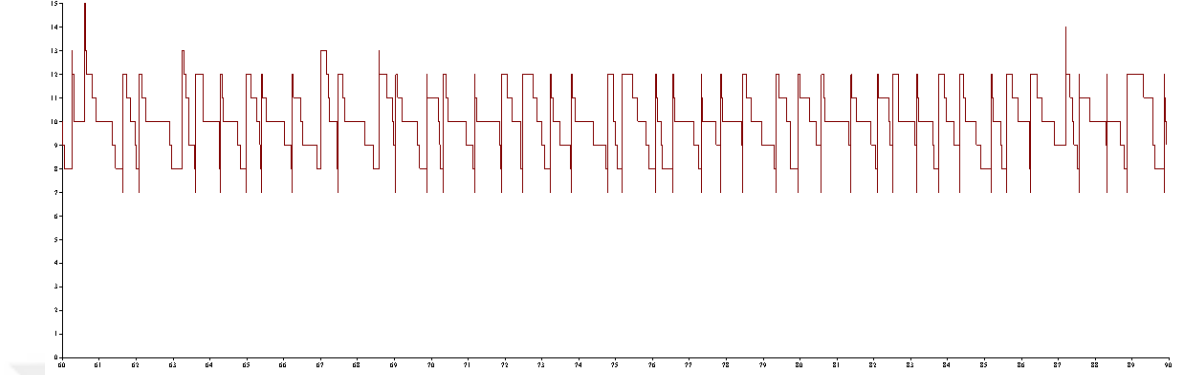


Figure 8.11: A sample path of the inventory position for the (Q, r, ξ) policy

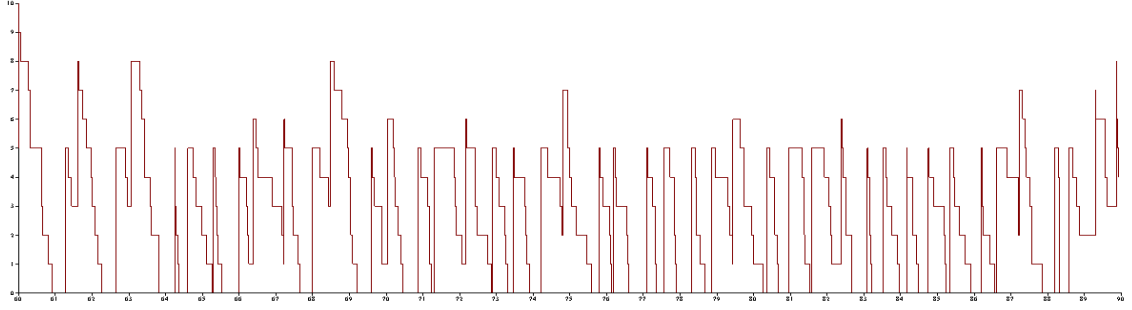


Figure 8.12: A sample path of the inventory level for the (Q, r, ξ) policy

In the following, we attempt to check the consistency of numerical results using simulation. We only consider the objective function value for comparison and consistency check. In Tables 8.32 and 8.33, we provide some results obtained from both simulation and exact method (Matlab results) for cases where $r < Q$ (that is $m = 1$); and in Table 8.34, for cases where $r \geq Q$ (that is $m > 1$). In these tables, $\mathbb{E}(TC_S)$ represents the total cost per unit time obtained from simulation, $\mathbb{E}(TC_A)$ is the total cost per unit time obtained from exact method, numbers in brackets are the minimum and maximum value of the objective function (expected total cost) among 10 replications, hw is the half width value with 95% confidence interval, and $\%gap$ shows the percentage deviation between simulation and exact method given by

$$\%gap = \frac{|\mathbb{E}(TC_A) - \mathbb{E}(TC_S)|}{\mathbb{E}(TC_S)} \times 100.$$

Results show that as the number of outstanding orders increases, the approximation of the effective lifetime becomes less accurate and the gap between the simulation and the exact method increases. The reason is that as the number of batches increases, we need higher memory in Matlab to track a vector of random variables. Due to lack of memory in Matlab, we decrease number of segments in vector of random variables resulting in less accuracy in the approximation of effective lifetime distribution.

The hw value reported in tables below are calculated by Arena with 10 replications. Based on t-test, the null hypothesis ($H_0 : \mu = \mu_0$) with α significance level is accepted if

$$\left| \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{N}}} \right| \leq t_{N-1, 1-\alpha/2},$$

where $\bar{x}$ and s are the sample mean and sample standard deviation, respectively. Therefore, an analytical result is accepted if $\mathbb{E}(TC_A) \in [\mathbb{E}(TC_S) \pm hw]$. For example, the solution (7, 5, 1.22) reported in Table 8.32 is acceptable because $40.8109 \in [40.8112 \pm 0.0123]$. It implies the consistency of the analytical method.

Table 8.32: Example 1, simulation accuracy with $h = 1$, $p = 10$, $K = 50$, $\lambda = 5$, $L = 1$, $\tau = 2$

(Q, r, ξ)	$\mathbb{E}(TC_S)$	[min, max]	hw	$\mathbb{E}(TC_A)$	%gap
(7, 5, 1.22)	40.8112	[40.7810, 40.8211]	0.0123	40.8109	0.0007
(8, 6, 0.91)	42.5944	[42.5634, 42.7245]	0.0224	42.6014	0.0164
(10, 6, 1.00)	43.3138	[43.2903, 43.3276]	0.0177	43.3256	0.0272

Table 8.33: Example 2, simulation accuracy with $h = 1$, $p = 10$, $K = 50$, $\lambda = 7$, $L = 1$, $\tau = 4$

(Q, r, ξ)	$\mathbb{E}(TC_S)$	[min, max]	hw	$\mathbb{E}(TC_A)$	%gap
(8, 5, 0.88)	37.7332	[37.7311, 37.7505]	0.0111	37.7309	0.0061
(6, 3, 0.62)	36.9854	[36.9806, 36.9933]	0.0091	36.9778	0.0205
(7, 6, 1.04)	44.4814	[44.4739, 44.4956]	0.0134	44.4754	0.0135
(9, 8, 0.54)	42.5063	[42.5035, 42.5235]	0.0103	42.4992	0.0167
(10, 5, 0.80)	33.1923	[33.1804, 33.2063]	0.0155	33.1807	0.0349
(11, 3, 0.60)	27.6946	[27.6949, 27.7057]	0.0083	27.6939	0.0025
(13, 7, 0.82)	32.4025	[32.3935, 32.4047]	0.0115	32.4049	0.0074
(14, 5, 0.30)	28.2025	[28.1943, 28.2107]	0.0093	28.1891	0.0475
(14, 3, 0.44)	25.2766	[25.2609, 25.2734]	0.0116	25.2649	0.0463

Table 8.34: Example 3, simulation accuracy with $h = 1$, $p = 10$, $K = 100$, $\lambda = 5$, $L = 1$, $\tau = 2$

(Q, r, ξ)	$\mathbb{E}(TC_S)$	[min, max]	hw	$\mathbb{E}(TC_A)$	%gap
(5, 8, 0.72)	117.1824	[117.0627, 117.2813]	0.0542	117.2316	0.0420
(6, 9, 1.14)	124.8746	[124.7498, 124.9022]	0.0441	124.9139	0.0315
(7, 10, 1.05)	132.2607	[132.2525, 132.4449]	0.0494	132.2427	0.0136

In order to confirm the optimization results obtained by analytical method, we apply the “OptQuest” tool in Arena by which we can find a set of solutions which have service level close to a predefined α value. By doing so, we can find a set of solutions for that given α , then by using the analytical method we can find the global optimal solution from that set. In “OptQuest” tool, a scatter search algorithm is used, but we can tune the number of solutions evaluated. By increasing the number of evaluated solutions in the search space, we allow “OptQuest” to search all solutions having a service level close to predefined α among which the best solution is selected. Again, the optimal solution found by “OptQuest” is the same with the one found by the exhaustive search method. It infers the consistency of our model.

Chapter 9

Conclusion and Future Extensions

In this research, we propose a policy based on both the inventory level and the elapsed time since a particular instance, namely (Q, r, ξ) policy, for the perishable items in inventory systems. Our main contribution to literature is to develop a control policy considering both the remaining lifetime of items and inventory level, while the previous studies have been developed based on either inventory level or both inventory level and elapsed time. Since the items in the inventory system under consideration are perishable, it is expected that tracking the lifetime of items improves the control policy. We also relaxed one order outstanding restriction ($r < Q$) in our model and derived analytical expressions for the operating characteristics under the multiple outstanding orders assumption. It was the second contribution of this research. Analyzing an inventory system with multiple outstanding orders is technically more complicated than the case with one order outstanding restriction; that is the reason that most of the existing models are restricted to one order outstanding assumption.

Our proposed hybrid policy performs very well under the service level constraint when unit perishing cost is high, service level is high, and demand rate is low. Our results show that this policy performs almost the same with a stock

based control policy without the service level constraint when the unit lost sales cost is low. But as the unit lost sale cost increases, still the hybrid policy performs well without the service level constraint. In instances with high unit perishing cost, short lifetime, high service level, and high demand rate we need to have more than one outstanding order most likely. There are some cases with high service level in which the stock based control policy is not able to return any feasible solution.

To show the performance of this policy and evaluate the impact of lifetime of items on control policy performance, we introduced four other policies (Q, r, T) , (Q, T) , (Q, ξ) , and (Q, r) . In all of the policies proposed here, we do not have any restriction regarding the number of order outstanding. In all policies, it is assumed that the lifetime of the new batch starts when it joins the system. Since the issuing policy is FIFO, the effective lifetimes of items at the beginning of embedded cycles construct a sequence of random variables. The effective lifetime was considered and its stationary distribution was obtained through a continuous valued state space embedded Markov chain and its ergodicity property. The policies of (Q, T) , (Q, ξ) , and (Q, r) are the two-parameter policies, while the policies of (Q, r, ξ) and (Q, r, T) are the three-parameter policies which are referred as hybrid policies. Considering the fact that lifetime being long or short depends completely on the demand rate and lead time, and no efficient policy can be proposed for a specific range of lifetime, we can have a general guideline. For inventory systems with perishable items, we recommend (Q, r) policy among the two-parameter policies for systems with a long lifetime. In the cases with short lifetimes, the (Q, ξ) policy as a two-parameter policy and the (Q, r, T) policy as a three-parameter policy perform well. Our numerical results indicate that the (Q, r, ξ) policy outperforms the others considerably when we have a high unit perishing cost, high service level and short lifetime. In the cases with the low service level and long lifetime, the (Q, r, ξ) and (Q, r, T) policies perform almost the same. In these cases, applying (Q, r, T) policy is recommended because tracking of the time elapsed since the last order is easier than tracking the remaining lifetime. Since in all cases the (Q, r, ξ) policy has been the most efficient policy regardless of the existing parameter settings, in delicate situations like health care

applications and chemical industries, utilizing the (Q, r, ξ) policy may be the best choice, however the other policies have a desirable performance. For optimization, an efficient heuristic algorithm is proposed by considering the structure of objective function and service level constraint which reaches the optimal solution.

Finally, this work can be extended by considering the model with a general demand distribution and different issuing policies. Without considering the FIFO issuing policy, when the new batch joins the inventories, the demand rate for the old and new items would be different. On the other hand, by reducing the old items price we can incentive the customers. Therefore, extending our model by combining the pricing concepts can be an open issue as well.

Regarding the methodology of getting optimal solution, we can approach a branch and bound algorithm. The bounds obtained from different stock or lifetime based policies can prune by bound the tree; and the service level constraint also can prune by infeasibility some other parts of the tree in the branch and bound algorithm.

Expected profit or equivalently expected cost is a commonly used performance measure in the traditional inventory systems working under a stochastic demand. In systems with perishable items, especially the ones which are composed of crucial and risky items, not only the operating costs but also the probability of stock-out is an important issue for the system evaluation. Empirical studies on the performance of traditional inventory systems indicate that the expected profit as a single objective and the service level as a constraint do not reveal an appropriate performance, especially when we have perishable items along with high variations in demand. Therefore, we can employ a productive replenishment policy by considering the multi-attribute utility theory approach.

Moreover, we observed that the operating characteristics of the system for given lifetime value are the same for hybrid and stock based policy. Therefore, only the effective lifetime distribution affects the operating characteristics of the system. Thus, we can employ a new hybrid policy with two lifetime based control

parameters to modify the effective lifetime of items in such a way that the minimum total cost is achieved. In this regard, we can propose a control policy in which a new order is placed when the inventory position crosses r , or the remaining lifetime crosses ξ_i , whichever occurs first, where $\xi_i = \xi_1$ if $Z < \eta$ and $\xi_i = \xi_2$ if $Z \geq \eta$ and Z denotes the effective lifetime. Hence, we will have $(Q, r, \xi_1, \xi_2, \eta)$ policy.



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Appendix A

Illustration of the Expected On-hand Inventory Expression

To illustrate the derivation of Expressions (5.15) and (5.16), we derive $\mathbb{E}[OH|z]$ for different value of Q when $0 \leq z \leq \tau$.

For $Q = 1$,

$$\mathbb{E}[OH|z] = \int_z^\infty \tau \lambda e^{-\lambda x} dx + \int_0^z ((\tau - z) + x) \lambda e^{-\lambda x} dx = \tau + \int_0^z (e^{-\lambda z} - 1) dz.$$

For $Q = 2$,

$$\begin{aligned} \mathbb{E}[OH|z] &= \int_z^\infty \int_z^{x_2} 2\tau \lambda^2 e^{-\lambda x_2} dx_1 dx_2 + \int_z^\infty \int_0^z ((\tau - z) + x_1 + \tau) \lambda^2 e^{-\lambda x_2} dx_1 dx_2 \\ &\quad + \int_0^z \int_0^{x_2} (2(\tau - z) + x_1 + x_2) \lambda^2 e^{-\lambda x_2} dx_1 dx_2 \\ &= 2\tau + \int_0^z (e^{-\lambda z} (2 + \lambda z) - 2) dz. \end{aligned}$$

For $Q = 3$,

$$\begin{aligned}
\mathbb{E}[OH|z] &= \int_z^\infty \int_z^{x_3} \int_z^{x_2} 3\tau\lambda^3 e^{-\lambda x_3} dx_1 dx_2 dx_3 \\
&+ \int_z^\infty \int_z^{x_3} \int_0^z ((\tau - z) + x_1 + 2\tau)\lambda^3 e^{-\lambda x_3} dx_1 dx_2 dx_3 \\
&+ \int_z^\infty \int_0^z \int_0^{x_2} (2(\tau - z) + x_1 + x_2 + \tau)\lambda^3 e^{-\lambda x_3} dx_1 dx_2 dx_3 \\
&+ \int_0^z \int_0^{x_3} \int_0^{x_2} (3(\tau - z) + x_1 + x_2 + x_3)\lambda^3 e^{-\lambda x_3} dx_1 dx_2 dx_3 \\
&= 3\tau + \int_0^z (e^{-\lambda z}(3 + 2\lambda z + \frac{(\lambda z)^2}{2!}) - 3)dz.
\end{aligned}$$

For $Q = 4$,

$$\begin{aligned}
\mathbb{E}[OH|z] &= \int_z^\infty \int_z^{x_4} \int_z^{x_3} \int_z^{x_2} 4\tau\lambda^4 e^{-\lambda x_4} dx_1 dx_2 dx_3 dx_4 \\
&+ \int_z^\infty \int_z^{x_4} \int_z^{x_3} \int_0^z ((\tau - z) + x_1 + 3\tau)\lambda^4 e^{-\lambda x_4} dx_1 dx_2 dx_3 dx_4 \\
&+ \int_z^\infty \int_z^{x_4} \int_0^z \int_0^{x_2} (2(\tau - z) + x_1 + x_2 + 2\tau)\lambda^4 e^{-\lambda x_4} dx_1 dx_2 dx_3 dx_4 \\
&+ \int_z^\infty \int_0^z \int_0^{x_3} \int_0^{x_2} (3(\tau - z) + x_1 + x_2 + x_3 + \tau)\lambda^4 e^{-\lambda x_4} dx_1 dx_2 dx_3 dx_4 \\
&+ \int_0^z \int_0^{x_4} \int_0^{x_3} \int_0^{x_2} (4(\tau - z) + x_1 + x_2 + x_3 + x_4)\lambda^4 e^{-\lambda x_4} dx_1 dx_2 dx_3 dx_4 \\
&= 4\tau + \int_0^z (e^{-\lambda z}(4 + 3\lambda z + 2\frac{(\lambda z)^2}{2!} + \frac{(\lambda z)^3}{3!}) - 4)dz.
\end{aligned}$$

Then, we can generalize the expected on-hand inventory for an arbitrary Q as follows:

$$\begin{aligned}
\mathbb{E}[OH|z] &= Q\tau + \int_0^z \sum_{k=0}^{Q-1} ((Q-k)e^{-\lambda z} \frac{(\lambda z)^k}{k!} - Q) dz \\
&= Q(\tau - z) + \int_0^z \sum_{k=0}^{Q-1} (Q-k)e^{-\lambda z} \frac{(\lambda z)^k}{k!} dz \\
&= Q(\tau - z) + \sum_{k=0}^{Q-1} \int_0^z (Q-k)e^{-\lambda z} \frac{(\lambda z)^k}{k!} dz \\
&= Q(\tau - z) + \frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q-k)}{k!} \gamma(k+1, \lambda z).
\end{aligned}$$

And similarly when $\tau < z \leq \tau + L$, we can show that

$$\mathbb{E}[OH|z] = \int_0^\tau \sum_{k=0}^{Q-1} (Q-k)e^{-\lambda z} \frac{(\lambda z)^k}{k!} dz = \frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q-k)}{k!} \gamma(k+1, \lambda \tau).$$