

# **STOCHASTIC ANALYSIS OF SHORT-RATE MODELING: WHICH APPROACH YIELDS A BETTER FIT TO DATA?**

A THESIS SUBMITTED TO  
THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE  
OF BILKENT UNIVERSITY  
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR  
THE DEGREE OF  
MASTER OF SCIENCE  
IN  
MATHEMATICS

By  
Mustafa Bulut  
September 2017

Stochastic Analysis of Short-Rate Modeling: Which Approach Yields  
a Better Fit to Data?

By Mustafa Bulut

September 2017

We certify that we have read this thesis and that in our opinion it is fully adequate,  
in scope and in quality, as a thesis for the degree of Master of Science.

---

Azer Kerimov(Advisor)

---

Ali Süleyman Üstünel (Co-Advisor)

---

Nuriye Mefharet Kocatepe

---

Ali Devin Sezer

Approved for the Graduate School of Engineering and Science:

---

Ezhan Karaşan  
Director of the Graduate School

# ABSTRACT

## STOCHASTIC ANALYSIS OF SHORT-RATE MODELING: WHICH APPROACH YIELDS A BETTER FIT TO DATA?

Mustafa Bulut

M.S. in Mathematics

Advisor: Azer Kerimov

Co-Advisor: Ali Süleyman Üstünel

September 2017

This thesis investigates the extent to which the two of the most common one-factor short-rate models are able to describe the market behavior of risk free Turkish treasuries for the post-2005 period. The investigated models are those widely used ones in the literature, which has analytical solutions, namely the Vasicek Model and the Cox-Ingersoll-Ross (CIR) Model. After building the necessary mathematical and financial structure, the thesis discusses the stochastic mechanics of interest rate modeling and in light of it, the zero-coupon bond prices in the models are solved, which are needed to numerically estimate model coefficients. The success of a model depends on how close it estimates the bond price to the market price. In the empirical part, the fitting performance of these two models is compared together with the current benchmark, the Nelson-Siegel (NS) Model, via the standard model fitting statistics. The estimation results reveals two important regularities: Firstly the models yield a poor fitting performance during the financial crisis period and secondly the models' degree of fit to data deteriorates as the maturity raises. Among the alternative models, the CIR in general yields the worst fit to data despite its theoretical complexity while the simple Vasicek Model achieves a high degree of fit to data. Lastly the estimated yield curves for specific dates and the zero rates of varying maturities are provided, which are expected to guide policy makers and practitioners.

*Keywords:* Stochastic Analysis, Short-Rate Modeling, Yield Curve.

## ÖZET

# KISA VADELİ FAİZ MODELLEMESİNİN STOKASTİK ANALİZİ: HANGİ YÖNTEM VERİYE DAHA İYİ UYUYOR?

Mustafa Bulut

Matematik, Yüksek Lisans

Tez Danışmanı: Azer Kerimov

Eş-Tez Danışmanı: Ali Süleyman Üstünel

Eylül 2017

Bu tez, en bilinen tek değişkenli kısa vadeli faiz modellerinden ikisinin 2005 sonrası dönemde risksiz Türk hazine bonolarının piyasa davranışını ne ölçüde tanımlayabildiğini araştırmaktadır. Vasicek Modeli ve Cox-Ingersoll-Ross (CIR) Modeli olarak bilinen söz konusu modeller analitik çözümü olan modeller arasında literatürde en yaygın olarak kullanılanlardır. Gerekli matematiksel ve finansal yapı oluşturulduktan sonra tezde faiz oranı modellemesinin stokastik işleyişi tartışılmakta ve bunun ışığında model katsayılarını hesaplamak için gerekli olan kuponsuz bono fiyatları hesaplanmaktadır. Böyle bir modelin başarısı modelde tahmin edilen bono fiyatlarının piyasa fiyatlarına olabildiğince yakın olmasına bağlıdır. Tezin ampirik kısmında bu iki model ile benchmark model olarak ele alınan Nelson-Siegel (NS) Modeli'nin tahmin performansı standart uyum iyiliği ölçütleri kullanılarak karşılaştırılmıştır. Tahmin sonuçları iki önemli olguyu gün ışığına çıkarmıştır. İlk olarak, modeller finansal kriz döneminde çok kötü performans göstermektedir. İkinci olarak, vade arttıkça modellerin tahmin performansı gerilemektedir. Alternatif modeller arasında CIR Modeli, kapsamlı teorik yapısına rağmen veriyi açıklama düzeyi en düşük olan model olmuştur. Buna karşın basit Vasicek Modeli veriyi daha iyi açıklamıştır. Tezde son olarak, politika yapıcılar ve uygulamacılara yol gösterebileceği düşünülerek belirli tarihlere ait getiri eğrileri ile çeşitli sabit vadelerdeki getirilerin zaman serisi verileri paylaşılmıştır.

*Anahtar sözcükler:* Stokastik Analiz, Kısa Vadeli Faiz Modellemesi, Getiri Eğrisi.

## Acknowledgement

Firstly I would like to express my deepest gratitude to my advisors Prof. Dr. Azer Kerimov and Ali Süleyman Üstünel for their support and patience during the whole of my master studies. It was a privilege for me to be their student. Without their guidance and encouragement this thesis would not come into existence. Besides, I am also grateful to my Thesis Committee Members Prof. Dr. N. Mefharet Kocatepe and Assoc. Prof. Dr. Ali Devin Sezer for their valuable time. Their invaluable comments and suggestions significantly improved this thesis.

Moreover, I would like to thank to the Central Bank of Republic of Turkey (CBRT) for allowing me to do these master studies and letting me to attend classes during working hours.

Lastly, I would like to thank to my family and friends for their endless support through whole of my life. The constant love and encouragement of my family always keep me motivated to achieve more. I have the privilege of having a great family that never imposed anything on me, always respected my decisions and supported me no matter what. I am indebted to Doruk Küçüksaraç for helping me with Matlab codes and helping me to find direction when I am lost in this research. Besides being life-time friends, he and Murat Duran were like mentors to me while learning most of the finance literature in the CBRT.

# Contents

|          |                                               |           |
|----------|-----------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                           | <b>1</b>  |
| <b>2</b> | <b>Mathematical Preliminaries</b>             | <b>5</b>  |
| <b>3</b> | <b>Modeling Short-Rate</b>                    | <b>15</b> |
| 3.1      | Terminology . . . . .                         | 15        |
| 3.2      | Mechanics of Bond Pricing . . . . .           | 19        |
| 3.3      | Vasicek Model . . . . .                       | 21        |
| 3.4      | Cox, Ingersoll and Ross (CIR) Model . . . . . | 24        |
| 3.5      | Nelson and Siegel (NS) Model . . . . .        | 26        |
| <b>4</b> | <b>Data and Empirical Analysis</b>            | <b>29</b> |
| 4.1      | Data . . . . .                                | 29        |
| 4.2      | Estimation Methodology . . . . .              | 30        |
| 4.3      | Empirical Evidence . . . . .                  | 31        |

|       |                                               |    |
|-------|-----------------------------------------------|----|
| 4.3.1 | In-Sample Fitting Performance . . . . .       | 33 |
| 4.3.2 | Estimated Yield Curves . . . . .              | 38 |
| 4.3.3 | Time Series of Estimated Zero Rates . . . . . | 40 |

|          |                   |           |
|----------|-------------------|-----------|
| <b>5</b> | <b>Conclusion</b> | <b>44</b> |
|----------|-------------------|-----------|

|          |             |           |
|----------|-------------|-----------|
| <b>A</b> | <b>Data</b> | <b>50</b> |
|----------|-------------|-----------|

# List of Figures

|     |                                                     |    |
|-----|-----------------------------------------------------|----|
| 3.1 | Yield Curve Types . . . . .                         | 17 |
| 4.1 | Vasicek Coefficients . . . . .                      | 37 |
| 4.2 | CIR Coefficients . . . . .                          | 37 |
| 4.3 | Upward Sloping Yield Curve (02.01.2013) . . . . .   | 38 |
| 4.4 | Humped Yield Curve (27.01.2014) . . . . .           | 39 |
| 4.5 | Downward Sloping Yield Curve (31.03.2014) . . . . . | 39 |
| 4.6 | Estimated 6-Month Zero Rates . . . . .              | 40 |
| 4.7 | Estimated 2-Year Zero Rates . . . . .               | 41 |
| 4.8 | Estimated 5-Year Zero Rates . . . . .               | 41 |
| 4.9 | Estimated 10-Year Zero Rates . . . . .              | 42 |

# List of Tables

|     |                                                         |    |
|-----|---------------------------------------------------------|----|
| 4.1 | In-Sample Fitting Performance Across Time . . . . .     | 34 |
| 4.2 | In-Sample Fitting Performance Across Maturity . . . . . | 36 |
| A.1 | Daily Bond Market Data . . . . .                        | 52 |

# Chapter 1

## Introduction

Modeling the behavior of interest rates over time and the term-structure of interest rate at a point in time is of great theoretical and practical importance. The uncertainty attached to the future movement of interest rates has major consequences for financial markets. The unexpected movements in interest rates have the potential to cause huge capital losses and cash flow problems for market players, who either holds a debt instrument or have an option written on various financial products. Hence every market participant desires to have a good model that sufficiently describes the market behavior of interest rates. Up until now, a consensus has not reached on a certain model best reflecting the market behavior of interest rates and options written on interest rate products. Therefore, the question that which of the existing interest rate models best describes the real world is still an open issue in the literature and testing the empirical applicability of the existing models has the potential to yield important theoretical and practical insights.

There are various approaches to model interest rate processes in the literature. For instance one strand of literature obtains interest rate dynamics from solving the maximization problems of various economic agents (and a system of equations describing the economic environment) within a general equilibrium framework. Such models are generally known as economic models and the obtained interest rate dynamics in the models is kind of a by-product of the solution, which clears

and is consistent with the whole of the other markets. However the most common approach in interest rate modeling is what is widely known as short-rate approach. A short-rate is defined as the average interest rate applicable to a very short (infinitesimal) time period,  $[t, t + dt]$ . Risk-free interest rate of any maturity should be a combination of short-rates applicable for the small time intervals in the period until the maturity. In these models, the short-rate dynamics is generally described by a stochastic differential equation (SDE) and by solving the SDE one can obtain distributional properties of interest rate process and the resulted term structure. By the term structure, it is meant that at a certain point in time how the (annualized) yield on debt instruments of similar credit quality changes for a continuum of time-to-maturities. Graphical representation of the term structure, known as the yield curve, reveals important insights about economic and financial phenomena. A yield curve in its simplest sense depicts the spot rates (on zero coupon bonds) on a continuum of maturities. On a typical day, there are zero-coupon or fixed-coupon government securities only at a limited number of maturities, which are hardly comparable. Properly estimating the term structure of interest rates firstly makes it possible to get comparable market rates for each and every maturity. Secondly one can easily infer on term premium, inflation expectations, current stance of monetary policy and expectations toward future policy, and last but not least expectations of future economic activity.

In the short-rate modeling approach, the bond prices and options prices are all derived from the short-rate process and the dynamic of the short-rate process under risk neutral measure determines all variables of interest. If there is only one source of the uncertainty (mostly level of  $r(t)$  itself) in the model, the model is called as “one-factor short-rate model”. Models that allow more than one source for uncertainty are also available. Moreover, depending on the form of the short-rate process, such models are also classified as “equilibrium models” or “no-arbitrage models”. The equilibrium models assume an economic relationship and derive the short-rate dynamics from it. After solving the short-rate process (the SDE), one can get prices of bonds and options etc. the Vasicek Model[1], The Rendleman-Bartter Model[2] and the Cox- Ingersoll- Ross (CIR) Model[3] are the main one factor equilibrium models. In addition, Brennan-Schwartz [4]

and Longstaff-Schwartz[5] are the proposed two-factor equilibrium models in the literature. Although they are able to yield good fit to many of the term structures in practice, the equilibrium models may yield poor fit to today's term structure(Hull[6]). This is considered as one of the main drawbacks of them by practitioners and no-arbitrage models are proposed to cure this drawback. In no-arbitrage models, time enters into the SDEs of the short-rate process generally as a factor in drift or standard deviation coefficients so that the current term structure enters into the equation as an input rather than output, as it is the case in equilibrium models. Some of the main no-arbitrage models are the Ho-Lee Model[7], the Black-Karasinski Model[8] and Hull-White Model[9] (both one-factor and two-factor model).The mentioned models (it being equilibrium or no-arbitrage model) have certain limitations. They mostly have only one source of uncertainty and do not allow flexibility in volatility structure, which is highly nonstationary. Later on more complex models are proposed in the literature that take into account this nonstationarity and allow flexibility in volatility structure. The models in this direction are the Heath-Jarrow-Morton Model[10] and the Libor Market Model (due to Brace-Gatarek-Musiela[11]) among others.

The aim of this thesis is to investigate the extent to which the two of the most common one-factor short-rate models are able to describe the market behavior of risk free Turkish treasuries for the post-2005 period. The models analyzed are those widely used ones in the literature, which has analytical solutions. The investigated term-structure models are namely Vasicek Model[1] and CIR Model[3] among equilibrium models. The fitting performance of these parsimonious short-term modeling strategies is then compared to that of Nelson-Siegel[12] (NS) Methodology, which is the benchmark approach for term structure modeling in central banks all around the world. The fitting performance of alternative models is compared based on the standard in-sample model fitting measures of root mean-squared error (RMSE) and mean absolute error (MAE).

The rest of the thesis is organized as follows: Section-II provides the mathematical preliminaries and builds the basic mathematical structure upon which the modeling of short-rate will be developed later on. Section-III firstly lays down the main definitions of bond markets and discusses the mechanics of short-rate modeling; then presents the models with their derivations. Section- IV describes

the data, discusses the estimation strategy and lastly outlines the empirical results as which modeling strategy yield a better fit to data and section-V concludes.



## Chapter 2

# Mathematical Preliminaries

Throughout this section, a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  is always assumed and we will deal with the class of processes defined on time index  $\mathbb{R}_+$  or  $[0, T]$ , where  $T > 0$ .

**Definition 1 (Filtration).** *An increasing family  $(\mathcal{F}_t, t \geq 0)$  of sub-sigma algebras of  $\mathcal{A}$  is called a filtration. The sub-sigma algebra  $\mathcal{F}_t$ , in practice, is considered to reflect the information available in the market up until time- $t$ .*

**Definition 2 (Adapted Process).** *A process  $(X_t)_{t \geq 0}$  is said to be adapted to a filtration  $(\mathcal{F}_t)_{t \geq 0}$  if  $X_t$  is  $\mathcal{F}_t$ -measurable for all  $t \geq 0$ .*

For technical reasons, the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is assumed to be right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \quad (2.1)$$

and that  $\mathcal{F}_t$  contains all  $\mathbb{P}$ -negligible subsets of  $\mathcal{A}$ . This later assumption guarantees that if  $X=Y$   $\mathbb{P}$ -a.s. and  $Y$  is  $\mathcal{F}_t$ -measurable, then  $X$  is  $\mathcal{F}_t$ -measurable too. Moreover  $\mathcal{F}_t = \sigma(X_s, s \leq t)$  is defined as the filtration generated by the process  $(X_t)_{t \geq 0}$ . If  $\mathcal{F}_t$  is augmented by  $\mathcal{N}$ , the sigma algebra of all  $\mathbb{P}$ -null sets of  $\mathcal{A}$ , then the above condition is satisfied by this augmented filtration. This new filtration is called as the natural filtration of the process  $(X_t)_{t \geq 0}$ .

**Definition 3 (Martingale).** Let  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space. An adapted sequence  $(M_t)_{t \geq 0}$  of integrable random variables, i.e.  $\mathbb{E}[|M_t|] < +\infty \forall t$ , is called a martingale (respectively a sub-martingale or super-martingale) if  $\forall s \leq t \mathbb{E}[M_t | \mathcal{F}_s] = M_s$  (respectively  $\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s$  or  $\mathbb{E}[M_t | \mathcal{F}_s] \geq M_s$ ).

**Definition 4 (Brownian Motion).** A real-valued continuous stochastic process  $(W_t)_{t \geq 0}$  is called a Brownian Motion if

- i . The map  $t \rightarrow W_t(\omega)$  is  $\mathbb{P}$ -a.s. continuous.
- ii .  $\forall s \leq t$ ,  $W_t - W_s$  is independent of  $\mathcal{F}_s = \sigma(W_u, u \leq s)$ .
- iii .  $\forall s \leq t$ ,  $W_t - W_s$  and  $W_{t-s} - W_0$  have the same probability law.

Thus, the process  $(W_t)_{t \geq 0}$  is said to have independent and stationary increments. For a Brownian Motion  $(W_t)_{t \geq 0}$ , the process  $(W_t - W_0)$  is distributed normally with mean  $\mu t$  and variance  $\sigma^2 t$ , i.e.  $(W_t - W_0) \sim \mathcal{N}(\mu t, \sigma^2 t)$ . A Brownian Motion is said to be standard if  $W_0 = 0$   $\mathbb{P}$ -a.s. ,  $\mathbb{E}[W_t] = 0$  and  $\mathbb{E}[W_t^2] = t$ . Now, we will provide another characterization of a Brownian Motion:

**Definition 5 (Brownian Motion with Respect to a Filtration).** Given a real-valued continuous stochastic process  $(W_t)_{t \geq 0}$  and a filtration  $(\mathcal{F}_t)_{t \geq 0}$ ,  $(W_t)_{t \geq 0}$  is called an  $\mathcal{F}_t$ -Brownian Motion if

- i .  $(W_t)_{t \geq 0}$  is adapted to  $(\mathcal{F}_t)_{t \geq 0}$ .
- ii .  $\forall s \leq t$ ,  $W_t - W_s$  is independent of  $\mathcal{F}_s$ .
- iii .  $\forall s \leq t$ ,  $W_t - W_s$  and  $W_{t-s} - W_0$  have the same probability law.

Now we are ready to discuss well-known Itô Calculus, which is the backbone of stochastic calculus to analyze any kind of stocks, bonds or options written on them. Itô Calculus allow us to evaluate a twice continuously differentiable

function  $f: t \rightarrow f(W_t)$ , defined on a Brownian Motion  $(W_t)_{t \geq 0}$ . In fact,  $W_t$  can be a more general function containing a Brownian Motion. Since a Brownian Motion is almost nowhere differentiable, the rules of ordinary calculus do not apply here. Hence a new framework is needed, which is the famous *Itô's Formula*.

One can define the Itô Formula for a large class of processes. However, in this thesis, it is defined for certain type of processes that are called as Itô Process.

**Definition 6 (Itô Process).** *Given an  $\mathcal{F}_t$ -Brownian Motion  $(W_t)_{0 \leq t \leq T}$ , a real-valued stochastic process  $(X_t)_{0 \leq t \leq T}$  is called an Itô Process if it can be written as*

$$\forall t \leq T, \quad X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s \quad \mathbb{P}\text{-a.s.} \quad (2.2)$$

where  $X_0$  is  $\mathcal{F}_0$ -measurable;  $K_t$  and  $H_t$  are respectively integrable and square-integrable on  $[0, T]$ , both adapted to  $(\mathcal{F}_t)_{0 \leq t \leq T}$ . Such processes are also widely known as *stochastic integrals*. For an Itô Process  $X_t$ , the equation 2.2 can be written in differential form, which is shorter and more practical, as:

$$dX_t = K_t dt + H_t dW_t \quad (2.3)$$

Then the *Itô's Formula* for  $f(X_t)$ , where  $(X_t)_{0 \leq t \leq T}$  is an Itô Process and  $f$  is again a twice continuously differentiable function, is

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s \\ &= f(X_0) + \int_0^t f'(X_s) K_s ds + \int_0^t f'(X_s) H_s dW_s + \frac{1}{2} \int_0^t f''(X_s) H_s^2 ds \end{aligned} \quad (2.4)$$

In the thesis, this practical version of the formula will be frequently employed.

Furthermore, the following is of great practical use: For a function  $f: (t, x) \rightarrow f(t, x)$  s.t.  $f \in C^{1,2}$  i.e.  $f$  is once and twice continuously differentiable respectively with respect to  $t$  and  $x$ , the Itô's Formula is expressed as:

$$f(t, X_t) = f(0, X_0) + \int_0^t f'_s(s, X_s) ds + \int_0^t f'_x(s, X_s) dX_s + \frac{1}{2} \int_0^t f''_{xx}(s, X_s) d\langle X, X \rangle_s \quad (2.5)$$

**Theorem 7 (Radon-Nikodym Theorem).** *Let  $\mathbb{P}$  and  $\mathbb{Q}$  be two probability measures on  $(\Omega, \mathcal{A})$ .  $\mathbb{Q}$  is said to be absolutely continuous with respect to  $\mathbb{P}$  (denoted as  $\mathbb{Q} \ll \mathbb{P}$ ) if*

$$\forall A \in \mathcal{A} \text{ s.t. } \mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$$

*The Radon-Nikodym Theorem says that: Under the condition  $\mathbb{Q} \ll \mathbb{P}$ ,  $\exists$  a non-negative random variable  $Z$  such that  $\forall A \in \mathcal{A}$*

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

*$Z$  is called Radon-Nikodym derivative (or simply density) of  $\mathbb{Q}$  with respect to  $\mathbb{P}$ .*

Radon-Nikodym Theorem is a standard theorem of measure theory. Hence we do not give a proof here. Interested reader can consult to Cohn[13] for a reference.

The Radon-Nikodym Theorem is of great significance as it is sometimes required to change the probability measure in order to get a more suitable measure under which the random variables of interest satisfy certain properties. For example, absence of arbitrage (no free lunch), which is a fundamental assumption about financial markets, is characterized with existence of a probability measure, equivalent to the original measure, under which the discounted asset prices are martingale. Actually, the change of measure is one of the most important theme in financial mathematics and the issue is addressed by celebrated Girsanov Theorem.

Before giving the Girsanov Theorem and its proof, we state Paul Levy's Theorem without proof, which will be used to prove the Girsanov Theorem. Interested reader may refer to Karatzas and Shreve[14] for a version of the proof of the theorem.

**Theorem 8 (Paul Levy's Theorem).** *Let  $(M_t)_{t \geq 0}$  be a continuous local martingale such that  $M_0 = 0$  and  $\langle M, M \rangle_t = t$  for  $t \geq 0$ . Then  $M$  is a Brownian Motion.*

**Theorem 9 (Girsanov Theorem).** *Given a filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ , let  $(W_t)_{0 \leq t \leq T}$  be an  $\mathcal{F}_t$ -standard Brownian Motion and  $(u_t)_{0 \leq t \leq T}$  be an adapted, measurable process with  $\int_0^T u_s^2 ds < +\infty$ . For  $0 \leq t \leq T$ , define:*

$$Z_t = \exp\left\{-\int_0^t u_s dW_s - \frac{1}{2} \int_0^t u_s^2 ds\right\}$$

*and a new probability measure*

$$\forall A \in \mathcal{A}, \quad \tilde{\mathbb{P}} = \int_A Z_T d\mathbb{P}$$

*Then under  $\tilde{\mathbb{P}}$ , the process  $\tilde{W}_t$ , defined as  $\tilde{W}_t = \int_0^t u_s ds + W_t$  is a Brownian Motion.*

**Remark.**

*i .  $Z_t$  is a martingale. However the following condition, known as Novikov Condition, is required as a sufficient condition for  $Z_t$  to be a martingale:*

$$\mathbb{E}\left[\exp\left\{\frac{1}{2} \int_0^T \sigma_s^2 ds\right\}\right] < +\infty \quad (2.6)$$

*ii .  $\tilde{\mathbb{P}}$  is a probability measure too. Since  $Z_t$  is a martingale with  $Z_0 = 1$ ,  $\mathbb{E}[Z_t] = 1 \forall t \geq 0$ . Hence*

$$\tilde{\mathbb{P}}(\Omega) = \int_{\Omega} Z_T d\mathbb{P} = \mathbb{E}[Z_T] = 1 \quad (2.7)$$

*iii . Letting  $\tilde{\mathbb{E}}$  denote the expectation under  $\tilde{\mathbb{P}}$ , for a random variable  $X$  we have:*

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[Z_T X] \quad (2.8)$$

*Proof.* Let  $(T_n, n \geq 1)$  be an increasing sequence of stopping times. Throughout the proof, by  $\widetilde{W}_t^n$  it is meant that  $\widetilde{W}_{t \wedge T_n}$ . It is required to prove that:

$$\mathbb{E}[Z_t \widetilde{W}_t^n F] = \mathbb{E}[Z_s \widetilde{W}_s^n F], \quad \forall F \in L^\infty(\mathcal{F}_s) \quad (2.9)$$

If this holds then  $Z_t \widetilde{W}_t$  is a  $\mathbb{P}$ -local martingale, which implies  $\widetilde{W}_t$  is a  $\widetilde{\mathbb{P}}$ -local martingale.

$$\begin{aligned} Z_t \widetilde{W}_t &= \int_0^t \widetilde{W}_s dZ_s + \int_0^t Z_s d\widetilde{W}_s + \langle W, Z \rangle_t \\ &= - \int_0^t \widetilde{W}_s u_s dZ_s dW_s + \int_0^t Z_s (dW_s + u_s ds) + \langle W, - \int_0^t Z_s u_s dW_s \rangle_t \end{aligned}$$

Since  $\langle W, - \int_0^t Z_s u_s dW_s \rangle_t = - \int_0^t Z_s u_s d\langle W, W \rangle_s = - \int_0^t Z_s u_s ds$ , we get:

$$Z_t \widetilde{W}_t = - \int_0^t \widetilde{W}_s u_s dZ_s dW_s + \int_0^t Z_s dW_s \quad (2.10)$$

Hence  $Z_t \widetilde{W}_t$  is a continuous local martingale with respect to  $\mathbb{P}$ , which implies  $\widetilde{W}_t$  is a  $\widetilde{\mathbb{P}}$ -continuous local martingale.

Then applying Paul Levy's Theorem with a decreasing sequence of stopping times

$$\begin{aligned} \langle \widetilde{W}, \widetilde{W} \rangle_t &= \lim_{|\pi_n| \rightarrow 0} \sum_{t_i \in \pi_n} (\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i})^2 \\ &= \lim_{|\pi_n| \rightarrow 0} \sum_{t_i \in \pi_n} (\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}) \left( \int_{t_i}^{t_{i+1}} u_s ds + W_{t_{i+1}} + W_{t_i} \right) \\ &\leq \sup_i |\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}| \int_0^t |u_s| ds + \lim_{|\pi_n| \rightarrow 0} \sum_{t_i \in \pi_n} (\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}) (W_{t_{i+1}} + W_{t_i}) \end{aligned}$$

The first term in the last equation goes to zero due to the size of the partition  $|\pi_n| \rightarrow 0$ .

$$\begin{aligned} &\leq \lim_{|\pi_n| \rightarrow 0} \left( \sup_i |W_{t_{i+1}} - W_{t_i}| \int_0^t |u_s ds| + \sum_{t_i \in \pi_n} (W_{t_{i+1}} - W_{t_i})^2 \right) \\ &= \langle W, W \rangle_t = t \end{aligned}$$

Therefore  $\widetilde{W}_t$  is a Brownian Motion under  $\widetilde{\mathbb{P}}$ . □

**Theorem 10 (Ito Representation Theorem).** *Let  $(W_t)_{0 \leq t \leq T}$  be an  $\mathcal{F}_t$ -standard Brownian Motion and  $(\mathcal{F}_t)_{0 \leq t \leq T}$  be its natural filtration, i.e.  $\mathcal{F}_t = \sigma(W_s : s \leq t) \cup \mathcal{N}$  where  $\mathcal{N}$  is the sigma-algebra generated by all  $\mathbb{P}$ -negligible sets of  $\mathcal{A}$ . Suppose further that  $\mathcal{F} = \bigcup_{t > 0} \mathcal{F}_t$ . Then for any square integrable and  $\mathcal{F}$ -measurable random variable  $F$  (i.e.  $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ ), there exists a unique adapted, square-integrable process  $H$  (i.e.  $H \in L_a^2(dt \times d\mathbb{P})$ ) such that*

$$F = \mathbb{E}[F] + \int_0^T H_s dW_s \quad (2.11)$$

*Proof.* For a proof see Øksendal [15] pp. 51-53. □

**Theorem 11 (Martingale Representation Theorem).** *Let  $(W_t)_{0 \leq t \leq T}$  be an  $\mathcal{F}_t$ -standard Brownian Motion and  $(\mathcal{F}_t)_{0 \leq t \leq T}$  be its natural filtration. Suppose  $(M_t)_{0 \leq t \leq T}$  is an  $\mathcal{F}_t$ -martingale with  $M_t \in L^2(\mathbb{P}) \forall t \geq 0$ . Then there exists a unique adapted, square-integrable process  $(H_t)_{0 \leq t \leq T}$  (i.e.  $\mathbb{E}[H] = \int_0^T H_s^2 ds$ ) such that*

$$\forall t \leq T, \quad M_t = M_0 + \int_0^t H_s dW_s \quad \mathbb{P}\text{-a.s.} \quad (2.12)$$

*Proof.* Apply Ito Representation Theorem (IRT) for  $T = t$  and  $F = M_t$

$$M_t = \mathbb{E}[M_0] + \int_0^t H_s^t dW_s \quad (2.13)$$

where  $H^t$  is the unique process corresponding to time- $t$ . Take  $0 \leq t_1 \leq t_2 \leq T$ :

$$M_{t_1} = \mathbb{E}[M_{t_2} | \mathcal{F}_{t_1}] = \mathbb{E}[M_0] + \mathbb{E}\left[\int_0^{t_2} H_s^{t_2} dW_s \middle| \mathcal{F}_{t_1}\right] = \mathbb{E}[M_0] + \int_0^{t_1} H_s^{t_2} dW_s \quad (2.14)$$

Moreover, writing 2.13 for  $t = t_1$ ,  $M_{t_1}$  also satisfies:

$$M_{t_1} = \mathbb{E}[M_0] + \int_0^{t_1} H_s^{t_1} dW_s \quad (2.15)$$

2.14 and 2.15 together with Ito Isometry imply:

$$\begin{aligned} \int_0^{t_1} H_s^{t_1} dW_s &= \int_0^{t_1} H_s^{t_2} dW_s \Rightarrow \int_0^{t_1} (H_s^{t_1} - H_s^{t_2}) dW_s = 0 \\ &\Rightarrow \int_0^{t_1} (H_s^{t_1} - H_s^{t_2})^2 ds = 0 \Rightarrow H_s^{t_1} = H_s^{t_2} \text{ a.s.} \end{aligned}$$

As a result, write almost surely  $H(t, \omega) = H^\tau(t, \omega)$  for  $t \in [0, \tau]$ . Then

$$M_t = \mathbb{E}[M_0] + \int_0^t H_s^t dW_s = \mathbb{E}[M_0] + \int_0^t H_s dW_s$$

□

**Definition 12 (Stochastic Differential Equation).** *A general equation of the form:*

$$X_t = \xi + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \quad (2.16)$$

where again  $(W_t)_{0 \leq t \leq T}$  is a (standard) Brownian Motion, is called a Stochastic Differential Equation (SDE). A solution to this equation is called as a diffusion. The equation in differentail form is:

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t \quad (2.17)$$

The coefficients  $\mu(t, X_t)$  and  $\sigma(t, X_t)$  are respectively indicators of short-term growth (drift term) and short-term variability (volatility term) of the process  $X_t$ .

The equation 2.16 has a unique solution, if the coefficients of the equation satisfy Lipschitz continuity in the state variables, i.e.

$$\begin{aligned} |\mu(t, x) - \mu(t, y)| &\leq K|x - y| \\ |\sigma(t, x) - \sigma(t, y)| &\leq K|x - y| \end{aligned}$$

for all  $t, x, y$  and the spatial growth condition:

$$|\mu(t, x)|^2 + |\sigma(t, x)|^2 \leq K(1 + |x|)$$

for all  $t, x$ . If the above conditions on coefficients are satisfied and given the initial condition  $X_0 = x$ , a solution of equation-2.16 is a continuous adapted process  $(X_t)_{0 \leq t \leq T}$ , uniformly bounded in  $L^2(\mathbb{P})$ , that satisfies:

$$\begin{aligned} X_0 &= x \\ X_t &= x + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \end{aligned} \quad (2.18)$$

A special kind of process known as Ornstein-Uhlenbeck Process is of particular importance for solution of certain type of short-rate models (for example the Vasicek Model). An Ornstein-Uhlenbeck Process is the unique solution of the following SDE:

$$\begin{aligned} X_0 &= x \\ dX_t &= -\alpha X_t dt + \sigma dW_t \end{aligned} \quad (2.19)$$

where  $\alpha$  and  $\sigma$  are positive constants. As it is obvious, the SDE has a drift term  $(-\alpha X_t dt)$  proportional to the level of the process itself. Furthermore, the process displays mean-reversion property (toward 0), i.e. if  $X_t > 0$ , then  $-\alpha X_t dt < 0$  and the drift term pulls  $X_t$  towards zero (the opposite when  $X_t < 0$ ). In order to solve the equation 2.19, let  $Y_t = e^{\alpha t} X_t$  and integrate  $Y_t$  by parts:

$$dY_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t + d\langle e^{\alpha \cdot}, X \rangle_s \quad (2.20)$$

Since  $d\langle e^{\alpha \cdot}, X \rangle_s = 0$ , and using  $dX_t = -\alpha X_t dt + \sigma dW_t$ , we get  $dY_t = \sigma e^{\alpha t} dW_t$ . Consequently, the solution to the equation-2.19 is obtained as:

$$X_t = x e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s \quad (2.21)$$

From the solution in 2.21, firstly observe that the effect of the initial value ( $x$ ) decays very rapidly (exponentially) as  $t \rightarrow \infty$ . Secondly, one can easily obtain the mean and the variance of the process ( $X_t$ ):

$$\mathbb{E}[X_t] = \mathbb{E}[xe^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s] = xe^{-\alpha t} + \sigma e^{-\alpha t} \mathbb{E}[\int_0^t e^{\alpha s} dW_s] \quad (2.22)$$

Since  $W_t$  is a martingale null at zero,  $\mathbb{E}[X_t] = xe^{-\alpha t}$ , which also goes to zero exponentially as  $t \rightarrow \infty$ . To compute the variance, we firstly calculate  $X_t - \mathbb{E}[X_t] = \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$ . Then

$$Var(X_t) = \sigma^2 e^{-2\alpha t} \mathbb{E}[\int_0^t e^{\alpha s} dW_s]^2 = \frac{\sigma^2}{2\alpha} \{1 - e^{-2\alpha t}\} \quad (2.23)$$

which converges to  $\frac{\sigma^2}{2\alpha}$  as  $t \rightarrow \infty$ . Moreover, the integral term in 2.21 is a  $dW_t$  integral of a deterministic process, which implies that an Ornstein-Uhlenbeck Process is Gaussian. Furthermore, as  $t \rightarrow \infty$ ,  $X_t$  converges (in distribution) to a Gaussian Process with mean 0 and variance  $\frac{\sigma^2}{2\alpha}$ .

# Chapter 3

## Modeling Short-Rate

We first begin with some terminology of government securities market and interest rates in order to make the arguments and proofs accessible to non-finance readers.

### 3.1 Terminology

**Definition 13 (Bond).** *A bond is a debt instrument by which a lender provides a certain amount of cash to a borrower in exchange for a future payment of money plus some interest. Generally a bond contract specifies following terms and conditions: The issuer, the face value, coupon rate and frequency, maturity (with dates of coupon payments) etc. Independent of the type of the bond, face value (also known as par value) is the amount payed to the lender at the maturity. There are a huge variety of bonds traded in the markets, depending on the above terms and conditions. Here we will cover just the most relevant ones for our case.*

**Definition 14 (Zero-Coupon Bond).** *A zero-coupon bond is a debt instrument that pays its face value at maturity without any intermediate coupon payment. The bond is bought at a value below its face value and so that it is also called as a discount bond. At time- $t$ , the price of a zero-coupon bond, with a maturity  $T$ , is denoted as  $P(t, T)$ , for  $t \leq T$ . We will assume throughout that a discount bond*

pays \$1 at the maturity  $T$ . This is just normalization and does not cause any change in substance of our analysis. Obviously  $P(T, T) = 1$  due to no arbitrage condition.

**Definition 15 (Fixed-Coupon Bond).** *A fixed-coupon bond is a type of bond that makes periodical payments (generally semiannual) in fixed proportion of its face value in addition to the final payment (at the maturity) of the principal.*

Whether it is a zero-coupon or fixed-coupon bond, the price at time-  $t$  of bond with maturity  $T$  is expressed as follows:

$$P(t, T) = \sum_{i=1}^N \frac{M_i}{(1 + R)^{t_i}} \quad (3.1)$$

where  $t_1, t_2, \dots, t_N$  are times that payments are made (assuming there are  $N$  of such payments) and  $M_1, M_2, \dots, M_n$  are the corresponding amount of payments (including the principal or the face value). Here  $R$  is considered as an average annualized interest rate for the period  $[t, T]$  and the valuation is based on discrete time compounding. In that case,  $\frac{1}{(1 + R)^{t_i}}$  can be considered as the discount factor corresponding to the cash flow  $M_i$ . Of course, one can also do a continuous compounding, the approach we will follow throughout the thesis for the sake of simplicity and compactness. We furthermore have compelling theoretical reasons to do so, since we are working in continuous time in all of the models throughout the thesis. In that case, the discount factor will be  $e^{-(1+R)t_i}$ , while the bond pricing formula becomes:

$$P(t, T) = \sum_{i=1}^N M_i e^{-(1+R)t_i} \quad (3.2)$$

**Remark.** *Note that above the average annualized yield,  $R$ , for the period  $[t, T]$  is assumed to be known (with certainty) and constant throughout whole period, for the sake of simplicity. This is far away from capturing the reality. In section-3.2, we firstly relax the assumption that  $R$  is fixed, then secondly model the interest rate as a stochastic process in a world where future rates are unknown. This later approach is closer to the reality and the whole purpose of short-rate modeling is to describe interest rate as a stochastic process in an uncertain world.*

**Definition 16 (Yield Curve).** A yield curve in its simplest sense is the graphical illustration of relationship between yield on bonds of the same credit risk and their maturities.

**Remark.** As yield on coupon paying bonds is hardly comparable, we solely consider yield on zero-coupon bonds that only pay the face value in their maturities. Nevertheless, we will also make use of cash flows of coupon-paying bonds to get a rich enough set of observations to estimate zero-coupon bonds yield curve. In that case each cash flow of a coupon-paying bond is treated as a separate zero-coupon bond.

Figure 3.1: Yield Curve Types

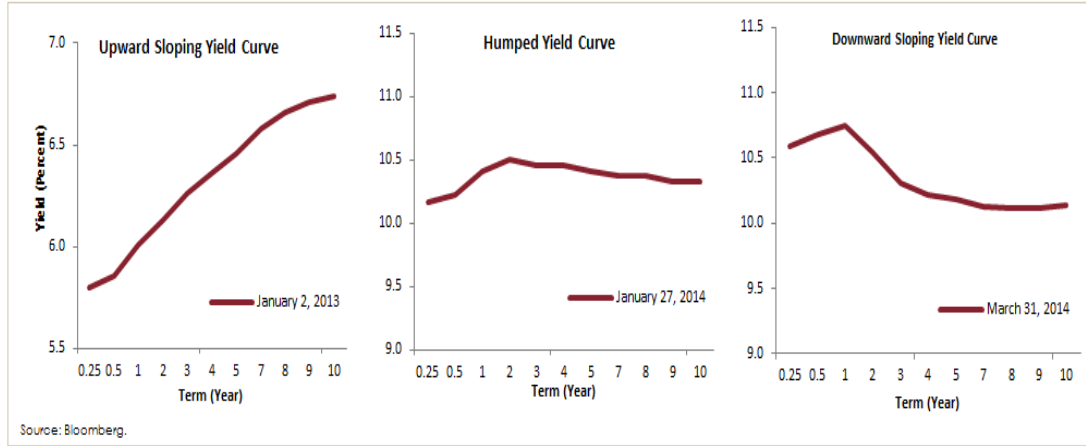


Figure 3.1 depicts the various shapes a yield curve can take. Other than these general ones, which are mostly referred in the literature, a yield curve can take many alternative shapes depending on the term structure of interest rates. In the first panel, an upward sloping yield curve is depicted, in which yield increases as the term-to-maturity raises. This is the normal shape that a yield curve is expected to take. Since the term premium is positive, i.e. term premium increases as time-to-maturity raises, an upward sloping yield curve is the shape of the term-structure that is expected to be observed in most of the time. In the middle panel, a flat yield curve is depicted, in which the interest rates changes little as the term-to-maturity raises. The specific date chosen there is just a few days before the Central Bank of Republic of Turkey's (CBRT) unexpected

and radical monetary tightening. However, before that specific date, the central bank has been implementing tight monetary policy for a long time. Since the monetary policy has the power only to affect short-term rates, as a result of monetary tightening at first the short end of yield curve goes up and the yield curve gets flat. In the panel on the right, a downward sloping yield curve is depicted, in which yield falls as the time-to-maturity increases. The specific day chosen there is the day far away from the initial monetary tightening. That is when the enough time passed after the initial monetary tightening so that the CBRT kept its tightening course and the market participants are convinced that it will defend its inflation-targeting mandate. Hence, the short end of yield curve is still high due to the tight CBRT policy. But, in addition, long end of yield curve goes down due to tight monetary policy anchoring long-term inflation expectations down. Other than intentional monetary policy to do so, a downward sloping yield curve is also inferred as a sign of economic recessions. This is due to the fact that the market participants expect policy rates and market rates to go down in the future when their forecasted economic downturn arrives.

**Definition 17 (Spot Rate).** *One may call the annualized interest rate on an  $n$ -year zero-coupon bond as  $n$ -year spot rate. But more generally, the spot rate (for a given maturity) is the interest rate realized on a bond with the same maturity in the market on a typical day. It is the rate at which the same day-valued trade is done in the market. By graphing the spot rate of each maturity (yield to maturity) against time-to-maturity (term), we obtain the spot yield curve on a given day.*

**Definition 18 (Forward Rate).** *In contrast to spot rates, a forward rate is the interest rate for a period of time in the future. It is the interest rate prevailed for an exchange being made in the future, that is contracted today. One can also graph the forward yield curve that shows the yield-to-maturity against time-to-maturity (term) relationship that is expected to emerge in the future.*

The spot rates and forward rates are mirror images of each other. Due to no arbitrage condition, one can easily recover forward rates from spot rates or spot rates from forward rates. For any  $t \leq S \leq T$ , from no arbitrage condition, the

relationship between spot and forward rates is as follows:

$$F(S, T) = \frac{(T - t)R(t, T) - (S - t)R(t, S)}{T - S} \quad (3.3)$$

where  $F(S, T)$  is the average annualized forward rate in time-period  $[S, T]$ , while  $R(t, S)$  and  $R(t, T)$  are the average annualized interest rates respectively for the periods  $[t, S]$  and  $[t, T]$ .

**Definition 19 (No Arbitrage Condition).** *We have been talking about “no arbitrage condition” repeatedly above. It is one of the cornerstones of modern finance theory. Arbitrage is defined as buying an asset cheap and selling it (at the same time) high without incurring any risk. In efficiently functioning markets, it is assumed that no arbitrage (also called as no free lunch) exists. Any arbitrage opportunity i.e. price differential is eliminated by arbitrageurs within seconds hence the pricing mechanism in the market is always efficient.*

## 3.2 Mechanics of Bond Pricing

Throughout this section, we will be following the notation in Lamberton and Lapeyre[16] in order to be consistent and to make the discussion easy to follow for the reader. Let someone who borrows \$1 at time- $t$  has to pay an amount of  $B(t, T)$  at maturity, i.e. the bond has a face value of  $B(t, T)$ . The borrower pays an average interest rate of  $R(t, T)$  over the period  $[t, T]$  for this transaction. Then

$$B(t, T) = e^{(T-t)R(t, T)} \quad (3.4)$$

Due to no arbitrage condition, in a world where all future interest rates  $(R(t, T))_{t \leq T}$  are known, the following condition holds:

$$\forall t \leq S \leq T, \quad B(t, S) \cdot B(S, T) = B(t, T) \quad (3.5)$$

If  $B$  is a smooth function, then there exists an adapted process  $(r_t)_{t \leq T}$  such that

$$\forall t \leq T, \quad B(t, T) = e^{\int_t^T r_s ds} \quad (3.6)$$

This implies that, in a certain world where all future rates are known, the price at time- $t$  of zero-coupon bond that pays \$1 at its maturity  $T$  can be expressed as:

$$P(t, T) = e^{-\int_t^T r_s ds} \quad (3.7)$$

The process  $(r_t)_{t \leq T}$  is called the the instantaneous interest rate (or short-rate), which is the interest rate for a very short period of time  $(t, t + dt)$ . This is a theoretical construct, but the interest rate on very short maturity contracts such as overnight interest rate is considered as a good proxy for the instantaneous interest rate.

The zero-coupon bond price obtained above is conditional on the assumption that at time- $t$  all the future interest rates  $R(u, T)$  for  $t < u \leq T$  are known. However in reality, one cannot know all these future rates. Hence, we need to obtain a new mechanics for bond price in an uncertain world. In a world where the future interest rates are unknown, the short-rate  $(r(t))_{0 \leq t \leq T}$  is assumed to follow a random process. Therefore as usual we assume a filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ , where  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is assumed to be the natural filtration of a Brownian Motion  $(W_t)_{0 \leq t \leq T}$ .

In an arbitrage free (no free lunch) world, there exists a probability measure  $\tilde{\mathbb{P}}$ , equivalent to  $\mathbb{P}$ , under which discounted zero-coupon bond prices are martingale (Harrison and Kreps[17]). We denote the discounted bond price by  $\tilde{P}(t, T)$  and

$$\tilde{P}(t, T) = e^{-\int_0^t r(s) ds} P(t, T) \quad (3.8)$$

Using the martingale property of  $\tilde{P}(t, T)$  under  $\tilde{\mathbb{P}}$  and the fact that  $P(T, T) = 1$ , we get:

$$\begin{aligned} \tilde{P}(t, T) &= \tilde{\mathbb{E}}[\tilde{P}(T, T) | \mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[e^{-\int_0^T r(s) ds} P(T, T) | \mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[e^{-\int_0^T r(s) ds} | \mathcal{F}_t] \end{aligned}$$

Inserting the expression in 3.8 for  $\tilde{P}(t, T)$  and getting rid of discounting, the

zero-coupon bond price under uncertainty is expressed as follows:

$$P(t, T) = \tilde{\mathbb{E}}[e^{-\int_0^t r(s)ds} | \mathcal{F}_t] \quad (3.9)$$

The new probability measure  $\tilde{\mathbb{P}}$ , under which the discounted asset prices are martingale is called as “risk neutral measure” and the bond price depends only on how the short-rate process  $r(t)$  behaves under this risk neutral measure.

### 3.3 Vasicek Model

The model is due to Vasicek[1] and is one of first models of term structure of interest rates. In the model, the instantaneous interest rate is assumed to satisfy the following SDE:

$$dr(t) = a(b - r(t))dt + \sigma dW_t \quad (3.10)$$

where  $a$ ,  $b$ , and  $\sigma$  are non-negative constants,  $W_t$  is assumed to be a standard Brownian Motion under risk neutral measure. Letting  $X_t = (r(t) - b)$ , the equation 3.10 becomes:

$$dX_t = -aX_t dt + \sigma dW_t \quad (3.11)$$

This means that the process  $X_t$  follows the well-known Ornstein-Uhlenbeck Process (with initial value equal to  $X_0 = (r(0) - b)$ ). Using the properties of Ornstein-Uhlenbeck Process, the equation 3.11 has a solution in the form of:

$$X_t = X_0 e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s \quad (3.12)$$

This process is normally distributed (in fact  $X_t$  is Gaussian) with mean  $\mathbb{E}[X_t] = X_0 e^{-at}$  and variance  $Var(X_t) = \frac{\sigma^2}{2a} \{1 - e^{-2at}\}$  (See Chapter-2 for properties of Ornstein-Uhlenbeck Process).

By substituting  $X_t = (r(t) - b)$  (and also  $X_0 = (r(0) - b)$ ) into equation 3.12, the short-rate process  $r(t)$  is solved as:

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW_s \quad (3.13)$$

Since the initial time is arbitrary, for any time  $s \leq t$ , the  $r(t)$  in general is:

$$r(t) = r(s)e^{-a(t-s)} + ab \int_s^t e^{-a(t-\tau)} d\tau + \sigma \int_s^t e^{-a(t-\tau)} dW_\tau \quad (3.14)$$

Since  $X_t$  is normally distributed,  $r(t)$  is normally distributed as well and has mean  $\mathbb{E}[r(t)] = r(0)e^{-at} + b(1 - e^{-at})$  and variance  $\text{Var}(r(t)) = \frac{\sigma^2}{2a}\{1 - e^{-2at}\}$ . Furthermore, the interest rate process  $r(t)$  satisfies Markov Property in the Vasicek Model since the model coefficients are time-homogeneous.

**Remark.** *As the short-rate process is normally distributed, the interest rate in Vasicek Model can take negative values with (a small but) positive probability. This is considered as a major drawback of the model due to practical reasons.*

**Remark.** *If we let  $t \rightarrow \infty$ , short-rate process  $r(t)$  tends to a normal random variable with mean  $b$  and variance  $\frac{\sigma^2}{2a}$ . The model parameter  $b$  is considered as the long-term mean of the short-rate process and the model exhibits a mean-reversion property (toward  $b$ ) in the sense that short-rates do not increase or decrease indefinitely but moves around its long-term mean of  $b$ .*

In the empirical part, the difference between market price of a bond and model-estimated price will be minimized. Hence an expression for zero-coupon bond price in the Vasicek Model is needed. Let  $P(r, t, T)$  be the price at time- $t$  of zero-coupon bond that matures at time- $T$  and pays \$1 at maturity. Firstly let  $P(r(0), 0, T)$  be the bond price at time-0.

$$P(r(0), 0, T) = \mathbb{E}[e^{-\int_0^T r(t)dt} | \mathcal{F}_0] = \mathbb{E}[e^{-\int_0^T r(t)dt}] \quad (3.15)$$

To study  $P(r(0), 0, T)$ , we analyze the process  $\int_0^T r(t)dt$ . As  $r(t)$  is a Gaussian process,  $\int_0^T r(t)dt$  (as the Riemann sum of a Gaussian random variable) is Gaussian as well. Then the process  $\int_0^T r(t)dt$  is expressed as:

$$\int_0^T r(t)dt = \int_0^T \{r(0)e^{-at} + abe^{-at} \int_0^t e^{as}ds\}dt + \int_0^T \{\sigma e^{-at} \int_0^t e^{as}dW_s\}dt \quad (3.16)$$

The Laplace Transform of a Gaussian random variable  $X$  is:

$$\mathbb{E}[e^{zX}] = \exp\{z\mathbb{E}[X] + \frac{z^2}{2}\text{Var}[X]\} \quad (3.17)$$

Using the Laplace transform of the Gaussian random variable  $\int_0^T r(t)dt$ , the zero-coupon bond price at time-0 is expressed as follows:

$$\begin{aligned} P(r(0), 0, T) &= \mathbb{E} \left[ e^{-\int_0^T r(t)dt} \right] \\ &= \exp \left\{ (-1)\mathbb{E} \left[ \int_0^T r(t)dt \right] + \frac{1}{2}(-1)^2 \text{Var} \left( \int_0^T r(t)dt \right) \right\} \\ &= \exp \left\{ -r(0) \int_0^T e^{-at} dt - \int_0^T \int_0^t abe^{-at} e^{as} ds dt \right. \\ &\quad \left. + \frac{1}{2} \int_0^T e^{2at} \sigma^2 \left( \int_t^T e^{-au} du \right)^2 dt \right\} \\ &= \exp \left\{ -r(0) \int_0^T e^{-at} dt - \int_0^T abe^{at} \left( \int_t^T e^{-au} du \right) dt \right. \\ &\quad \left. + \frac{1}{2} \int_0^T e^{2at} \sigma^2 \left( \int_t^T e^{-au} du \right)^2 dt \right\} \end{aligned}$$

As a result, the zero-coupon bond price at time-0 becomes:

$$P(r(0), 0, T) = B(0, T) \exp\{-r(0)A(0, T)\} \quad (3.18)$$

where

$$\begin{aligned} A(0, T) &= \int_0^T e^{-at} dt = \frac{1 - e^{-aT}}{a} \\ B(0, T) &= \exp \left\{ \int_0^T \left[ ae^{at} b(t) \left( \int_t^T e^{-au} du \right) - \frac{1}{2} \int_0^T e^{2at} \sigma^2 \left( \int_t^T e^{-au} du \right)^2 \right] dt \right\} \\ &= \exp \left\{ \left( b - \frac{\sigma^2}{2a^2} \right) [A(0, T) - T] - \frac{\sigma^2 A(0, T)^2}{4a} \right\} \end{aligned}$$

Now consider the bond price at time- $t$ ,  $P(r(t), t, T)$ :

$$P(r(t), t, T) = \mathbb{E}[e^{-\int_t^T r(u)du} | \mathcal{F}_t] \quad (3.19)$$

Since  $r(t)$  is Markov, the price function this time becomes:

$$P(r(t), t, T) = B(t, T) \exp\{-r(t)A(t, T)\} \quad (3.20)$$

where

$$\begin{aligned} A(t, T) &= e^{at} \int_t^T e^{-au} du = \frac{1 - e^{-a(T-t)}}{a} \\ B(t, T) &= \exp \left\{ \int_t^T \left[ abe^{au} \left( \int_u^T e^{-ay} dy \right) - \frac{1}{2} \int_t^T e^{2au} \sigma^2 \left( \int_u^T e^{-ay} dy \right)^2 \right] du \right\} \\ &= \exp \left\{ \left( b - \frac{\sigma^2}{2a^2} \right) [A(t, T) - (T - t)] - \frac{\sigma^2 A(t, T)^2}{4a} \right\} \end{aligned}$$

### 3.4 Cox, Ingersoll and Ross (CIR) Model

Cox, Ingersoll and Ross[3] assumes that the instantaneous rate follows the SDE:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW_t \quad (3.21)$$

where again  $a$  and  $\sigma$  are non-negative constants while  $b \in \mathbb{R}$ , and  $W_t$  is a standard Brownian Motion. The main innovation CIR Model brings on top of Vasicek Model is that the standard deviation (or implied volatility) of interest rate process is not constant, but proportional to current level of instantaneous rate. In addition, the model avoids negative rates (if  $2ab > \sigma^2$ ) in contrast to Vasicek Model. Interest rates can attain negative values with a positive probability in Vasicek Model, which were considered as one of the drawbacks of the model. The CIR Model assumes the same drift term  $(a(b - r(t))dt)$  as in Vasicek Model. Hence the short-rates exhibit a mean-reversion property in CIR Model too. However, the interest rate process is not Gaussian in CIR Model, which makes computations harder. Nevertheless, it is possible to get mean and variance of the process and

the system has an analytical solution for the price of zero-coupon bond. Here we only need an analytical expression for the price of a discount bond and the remaining issues related to the model are beyond the scope of this thesis.

Given the short-rate  $r(t)$ , let as usual  $P(r(t), t, T)$  be the price of zero-coupon bond at time- $t$ , that matures at time- $T$  and pays \$1 at maturity. Applying Itô's Formula to the zero-coupon bond price, we obtain:

$$\begin{aligned} P(r(t), t, T) = & P(r(0), 0, T) + \int_0^t P_s(r(s), s, T) ds + \int_0^t P_r(r(s), s, T) dr(s) \\ & + \frac{1}{2} \int_0^t P_{rr}(r(s), s, T) d\langle r, r \rangle_s \end{aligned} \quad (3.22)$$

Note that in equation 3.22, by  $P_x$  we mean partial derivative of  $P$  with respect to  $x$ . Furthermore, we will be writing simply  $P$  (without its arguments) instead of  $P(r(t), t, T)$  from now on. Then the equation 3.22 can be written in differential form as:

$$\begin{aligned} dP = & P_t dt + P_r dr + \frac{1}{2} P_{rr} d\langle r, r \rangle_t \\ = & P_t dt + P_r \{a(b - r(t))dt + \sigma \sqrt{r(t)} dW_t\} + \frac{1}{2} P_{rr} \sigma^2 r(t) dt \end{aligned} \quad (3.23)$$

Then the rate of return on our discount bond is calculated as:

$$\frac{dP}{P} = \{a(b - r(t)) \frac{P_r}{P} + \frac{P_t}{P} + \frac{1}{2} \sigma^2 r(t) \frac{P_{rr}}{P}\} dt + \sigma \sqrt{r(t)} \frac{P_r}{P} dW_t \quad (3.24)$$

Let us express the rate of return on our discount bond in more general form as:

$$\frac{dP}{P} = \mu(r, t, T) dt + \nu(r, t, T) dW_t \quad (3.25)$$

where  $\mu(r, t, T)$  is mean return on the asset and  $\nu(r, t, T)$  is the volatility factor of the asset return. In perfect markets, under no-arbitrage condition, the return on any asset can be expressed as risk-free rate,  $r$ , plus some risk premium. This implies:

$$\mu(r, t, T) = r + \lambda^* \nu(r, t, T) \quad (3.26)$$

Here CIR assumes that the risk premium takes the form  $\lambda^* = \lambda \frac{\sqrt{r}}{\sigma}$ . Then combining equations 3.24, 3.25 and 3.26; and using the no arbitrage condition

$P(r, T, T) = 1$ , we arrive at the following second order partial differential equation (PDE) in  $r$  and  $t$ :

$$\begin{aligned} PDE : rP + \lambda rP_r &= P_r a(b - r) + P_t + \frac{1}{2} P_{rr} \sigma^2 r \\ BC : P(r, T, T) &= 1 \end{aligned} \quad (3.27)$$

The PDE in 3.27 has a solution of the form:

$$P(r(t), t, T) = B(t, T) \exp\{r(t)A(t, T)\} \quad (3.28)$$

where

$$A(t, T) \equiv \frac{e^{\phi_1 \tau} - 1}{\phi_1 + \phi_2 [e^{\phi_1 \tau} - 1]} \quad (3.29)$$

$$B(t, T) \equiv \left[ \frac{\phi_1 e^{\phi_2 \tau}}{\phi_1 + \phi_2 [e^{\phi_1 \tau} - 1]} \right]^{\phi_3} \quad (3.30)$$

and

$$\tau \equiv (T - t) \quad (3.31)$$

$$\phi_1 \equiv \sqrt{(a + \lambda)^2 + 2\sigma^2} \quad (3.32)$$

$$\phi_2 \equiv (a + \lambda + \phi_1)/2 \quad (3.33)$$

$$\phi_3 \equiv 2ab/\sigma^2 \quad (3.34)$$

Based on the prices of Turkish treasuries in the market in each day and assuming  $\lambda = 0$ , we will estimate  $a$ ,  $b$ ,  $\sigma$  and  $r$  numerically and get time series for those model parameters.

### 3.5 Nelson and Siegel (NS) Model

The Nelson-Siegel Model of term structure is one of the most common and hence most standard models of estimating a yield curve by central banks all around the world. In this study, the fitting performance of the Vasicek Model and the CIR Model is compared with that of this benchmark model. Since firstly the model is standard and can be found in any book on yield curve estimation and secondly it is not the main focus of this study, we only provide a summary of the basic

properties of the NS Model for the sake of completeness. Nelson and Siegel[12] proposes that the instantaneous forward rate,  $r(m)$ , for maturity  $m$  satisfies the following differential equation:

$$r(m) = \beta_0 + \beta_1 e^{-\frac{m}{\tau}} + \beta_2 \left[ \frac{m}{\tau} e^{-\frac{m}{\tau}} \right] \quad (3.35)$$

where  $\beta_0, \beta_1, \beta_2$  and  $\tau$  are the model parameters to be estimated. The parameter  $\tau$  is a time constant that indicates how fast the factors for  $\beta_1$  and  $\beta_2$  in equation-3.36 goes to zero when we do a least square regression. Note that a small (large) value for  $\tau$  indicates a rapid decay of these factors, which in turn causes the estimated yield curve fit to the short(long) maturity rates well while giving poor fit for long (short) maturities.

The yield on a bond of maturity  $m$  should be just an average of expected forward rates until time  $m$ . Hence the zero-coupon rate,  $R(m)$ , should be integral of  $r(m)$  from 0 to time  $m$ , divided by  $m$ :

$$R(m, \beta, \tau) = \beta_0 + \beta_1 \left[ \frac{1 - e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} \right] + \beta_2 \left[ \frac{1 - e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} - e^{-\frac{m}{\tau}} \right] \quad (3.36)$$

Using the zero rates implied by the equation-3.36, we can calculate the zero-coupon bond prices for each maturity in terms of model parameters in the form of the equation-4.2. Then as usual, the model parameters that minimize the difference between the market price and model implied price of bonds, that is the objective function in 4.3, are estimated.

The values of the parameters in NS Model have certain implications for the shape, slope and location of the yield curve. For instance:

- i . Firstly as maturity tends to zero ( $m \rightarrow 0$ ),  $R(m, \beta, \tau) \rightarrow \beta_0 + \beta_1$ . Hence the estimated yield curve has an intercept of  $\beta_0 + \beta_1$  on  $y$ -axis.
- ii . Secondly as maturity tends to infinity ( $m \rightarrow \infty$ ),  $R(m, \beta, \tau) \rightarrow \beta_0$ . Hence the yield curve asymptotically converges to  $\beta_0$  on the long end.
- iii . Lastly the estimated yield curve makes a hump at maturity equal to  $\tau$ . Whether the hump will be a “u” or “n” depends on the sign of  $\beta_2$ . If  $\beta_2 < 0$

( $\beta_2 > 0$ ), then the yield curve makes a “u” (“n”) hump at  $m = \tau$  (Nelson and Siegel[12]).



## Chapter 4

# Data and Empirical Analysis

### 4.1 Data

As the data source, Borsa Istanbul (BIST) Daily Bulletin of Debt Securities Market is used to get Turkish treasury securities of varying maturities. The bulletin contains fixed-rate or flexible-rate government and private sector securities that are denominated in Turkish Lira. Among them, private sector securities are excluded since the firm characteristics affect the pricing of these bonds and create a huge amount of heterogeneity which makes comparison of different maturity cash-flows impossible. Furthermore, flexible-rate government securities are also excluded since they have different pricing dynamics, which is heavily affected from global uncertainty etc. As a result, there remain zero-coupon (discount) bonds and bonds with fixed frequency coupon payments for the estimation of market short-rates in Turkey. The inclusion of fixed-coupon paying bonds is important because it is hard to find zero-coupon discount bonds in every maturity. After inclusion of fixed-coupon bonds, the remaining set of bonds is rich enough to estimate yield curve in daily basis and contains bonds with legitimately comparable cash flows. The data set includes bond prices and certain bond characteristics on a daily basis since February 2005. A small portion of the employed data set together with the definitions of and explanations for the series is provided at

Appendix-A.

## 4.2 Estimation Methodology

The basic idea of yield curve estimation is to reveal the interest rate for each and every maturity as yield curve relates a zero rate (and implicitly forward-rates) with time-to-maturity. On an ordinary day, there are only a limited number of cash flows and the interest rate (or equivalently price) for these cash flows. Hence, in principle one tries to fill the remaining unobservable maturity-yield pairs to get a complete yield curve and reveal the entire spot or forward interest rates for each maturity. Normally, some sort of Bootstrap method can be employed to fill in the unobservable maturity-yield pairs. However this approach is far from being complete and reliable. Therefore, by assuming interest rate (or yield-to-maturity) takes a specific functional form that reflects the main properties of market behavior of interest rates, a complete yield curve is estimated by numerically solving the model parameters coming from the functional form assumed. This estimation is done by finding the parameter values that minimize an objective function, which is proportional to the difference between actual (or market) prices of bonds and fitted prices based on the theoretical model.

At time- $t$  (for our case on a typical day), the market price of a government bond- $i$  can be expressed as follows:

$$P_t^i = \sum_{j=1}^N C_j^i \times B(t, t_j) \quad (4.1)$$

where  $C_j^i$  denotes the specific coupon payment at time- $t_j$  of this bond (assuming there are remaining  $N$  coupon payment dates) and  $B(t, t_j)$  stands for the zero coupon bond price at time- $t$  of maturity- $t_j$  (or simply discount factor bringing the value of a time- $t_j$  cash flow to time- $t$ ). Indeed this theoretical discount factor is unobservable for any time- $t$ . However, one can model the zero coupon bond price  $B(t, t_j)$  in terms of unknown model parameters. As a result, the bond price

attained by the model is:

$$P_t^{i,fitted} = \sum_{j=1}^N C_j^i \times B^*(\beta, t, t_j) \quad (4.2)$$

where this time  $B^*(\beta, t, t_j)$  is the theoretical discount factor implied by the model and  $\beta$  is the parameter matrix. The set of parameters to be estimated in the Vasicek, CIR and NS models are respectively  $\beta_{Vasicek} = \{a, b, \sigma, r\}$ ,  $\beta_{CIR} = \{a, b, \sigma, r\}$  and  $\beta_{NS} = \{\beta_0, \beta_1, \beta_2, r\}$  where in all sets  $r$  stands for the short rate. The main objective in yield curve estimation is then to estimate the model parameters so that the difference between the market price  $P_t$  and the model fitted price  $P_t^{fitted}$  is minimal. Hence the objective function is expressed as:

$$\min_{\beta} \sum_{i=1}^M \frac{|P_t^i - P_t^{i,fitted}|}{W_t^i} \quad (4.3)$$

assuming there is M number of bonds in the given day. Here, as is the common practice among practitioners, the price difference is weighted by the inverse of Macaulay Duration ( $W_t^i$ ) to avoid the estimation error of very short term bonds (for instance bonds with just a few days of time-to-maturity) to cause substantial price differences.

### 4.3 Empirical Evidence

The estimation performance of the models will be evaluated by means of in-sample fitting power of the models. These are standard methods to evaluate how well the estimated model describes the data, here market behavior of bond prices or equivalently interest rates, compared to an alternative model. In-sample fitting is a measure of how well the model estimates the price of a bond that is included in the sample with which the theoretical model parameters are estimated. Although it is not used in this study, another common approach in the literature is to measure out-of-sample fitting performance, which measures how well the model estimates a bond price that is not included in the sample with which the parameters are estimated. Either way, the RMSE or MSE are the employed

statistics to compare the degree that alternative models fit to data:

$$RMSE = \sqrt{\sum_{i=1}^M \frac{(P_t^i - P_t^{i,fitted})^2}{M}} \quad (4.4)$$

$$MAE = \sum_{i=1}^M \frac{|P_t^i - P_t^{i,fitted}|}{M} \quad (4.5)$$

Among alternative models, the model with the lowest RMSE or MAE is said to be giving the best fit to data. In addition, the fitting power of these parsimonious short-term modeling strategies will be compared to the current benchmark term-structure modeling practice in the world i.e. the fitting power of our models will be compared to that of Nelson-Siegel (NS) framework due to Nelson and Siegel [12], which is currently most widely used model by central banks, also by Central Bank of the Republic of Turkey (CBRT).

The pioneering work on yield curve modeling in Turkey is Akıncı et al. [18] study, that estimates yield curve for Turkish treasury securities using Nelson-Siegel approach. Previously Alper et al.[19] estimated government securities yield curve for monthly data on period 1992-2004 using both McCulloch and Nelson-Siegel methods. But since the government securities market was not well-developed during that period, they can estimate yield curve at monthly frequency and only for very short maturities. Duran and Gülşen[20] recently estimated yield curve for Turkish inflation-indexed bonds using Extended Nelson-Siegel approach. They then calculated inflation compensation as the difference between nominal and inflation-indexed zero rates of each maturity. Another recent study on the yield curve modeling in Turkey is Çepni and Küçüksaraç[21] work that answers the question whether price minimization or yield minimization is a better approach in yield curve estimation and they conclude that estimating yield curve with the first one gives a better fit to data. That is why this study also uses the price minimization approach while estimating yield curve. The above studies were all on estimating a yield curve for government securities. In a novel study, Kanlı et al.[22] estimated a yield curve again using the NS method for corporate bonds, a market lately developed in Turkey.

### 4.3.1 In-Sample Fitting Performance

The estimation results for each model across time are reported in the Table 4.1. There the fitting performance of the models in different time periods in the sample is compared. For all alternative models, one striking observation emerges from estimation results: The fitting performance of all models is worst during the Global Financial Crisis (GFC) of 2008. This is true for both of the model checking statistics MAE and RMSE. This is considered to be a consequence of volatile financial environment during the crisis period. In addition, since in the period before the crisis there were only a limited number of bonds traded in the market and as a result Turkish capital markets were not deep enough, the models give poor fitting to data at pre-crisis period (2005-2007) too. On the other hand, in the relatively calm period after the crisis, all the three models achieve the best fitting performance.

When the alternative models are compared, the CIR Model gives the poorest fit to the data while the Vasicek Model achieves a similar performance with the benchmark model of the Central Bank of Turkey (CBRT), i.e. the Nelson-Siegel (NS) Model. Estimation of bond prices with CIR methodology of term structure yields a better fit to data only in the Post-Tapering period (June 2013 onwards). This conclusion does not change if MAE is used instead of RMSE as the indicator of fitting performance. Although the CIR Model is theoretically a more complex model compared to the Nelson-Siegel and Vasicek Models, increased theoretical complexity does not pay off in the form of better fit to data.

Although the results for the Vasicek Model and Nelson-Siegel Model are very similar, when sub-periods are considered, the Vasicek Model results in more reliable yield curve estimation compared to the Nelson-Siegel Model only during the GFC period, while for the rest of the time the latter is more reliable. However it should be noted that the difference in fitting performance of two models is not substantial. Although it is theoretically the simplest model among alternatives,

the Vasicek Model describes the market behavior of interest rates to a good extent. Hence, it is the most economic model among alternatives in the sense that it achieves the same or better fitting to data in theoretically the simplest way.

Table 4.1: In-Sample Fitting Performance Across Time

|                                                                               |                | Before GFC<br>(02.05-12.07) | During GFC<br>(01.08-12.09) | After GFC<br>(01.10-05.13) | Post-Tapering<br>(06.13-08.17) | The Whole Sample<br>(02.05-08.17) |
|-------------------------------------------------------------------------------|----------------|-----------------------------|-----------------------------|----------------------------|--------------------------------|-----------------------------------|
| <b>NS</b>                                                                     | <b>RMSE</b>    | 0.2452<br>(0.1024)          | 0.5328<br>(0.5590)          | 0.2338<br>(0.1259)         | 0.2669<br>(0.1122)             | 0.2953<br>(0.2678)                |
|                                                                               | <b>MAE</b>     | 0.1707<br>(0.0814)          | 0.3198<br>(0.3327)          | 0.1440<br>(0.0638)         | 0.1924<br>(0.0764)             | 0.1945<br>(0.1600)                |
|                                                                               | <b>Av. Er.</b> | -0.0036<br>(0.0206)         | -0.0249<br>(0.0319)         | 0.0083<br>(0.0200)         | 0.0169<br>(0.0303)             | 0.0038<br>(0.0298)                |
| <b>Vasicek</b>                                                                | <b>RMSE</b>    | 0.2511<br>(0.1529)          | 0.5224<br>(0.5518)          | 0.2379<br>(0.1471)         | 0.2676<br>(0.1390)             | 0.2963<br>(0.2760)                |
|                                                                               | <b>MAE</b>     | 0.1699<br>(0.0974)          | 0.3077<br>(0.3138)          | 0.1433<br>(0.0713)         | 0.1879<br>(0.0864)             | 0.1907<br>(0.1571)                |
|                                                                               | <b>Av. Er.</b> | -0.0039<br>(0.0244)         | -0.0308<br>(0.0433)         | 0.0091<br>(0.0285)         | 0.0184<br>(0.0514)             | 0.0029<br>(0.0428)                |
| <b>CIR</b>                                                                    | <b>RMSE</b>    | 0.4293<br>(0.2781)          | 0.5401<br>(0.5509)          | 0.2322<br>(0.1515)         | 0.2506<br>(0.1325)             | 0.3329<br>(0.3040)                |
|                                                                               | <b>MAE</b>     | 0.2263<br>(0.1241)          | 0.3153<br>(0.3122)          | 0.1376<br>(0.0714)         | 0.1764<br>(0.0814)             | 0.1994<br>(0.1620)                |
|                                                                               | <b>Av. Er.</b> | -0.0194<br>(0.0438)         | -0.0314<br>(0.0436)         | 0.0073<br>(0.0291)         | 0.0152<br>(0.0511)             | -0.0024<br>(0.0468)               |
| *Standard deviations are in the paranthesis. Av. Er. stands for average error |                |                             |                             |                            |                                |                                   |

Lastly, for all three alternative models the average difference between market prices and model estimated prices of bonds are quite small. Average error in the whole sample for the Nelson-Siegel, the Vasicek and the CIR models are respectively 0.0038, 0.0029 and -0.0024. The fact that average errors are so close to zero reflects that models on average depict the market prices to a good extent. This is a natural result of the chosen objective function in which the market and modeled bond price difference is minimized. But the models may systematically underestimate or overestimate bond prices in the given sub-periods. For instance,

all the three models overestimate the bond prices before and during the GFC, but underestimate them after 2010. Nevertheless, the average errors of the models for sub-periods are very close to 0 too.

Besides checking model fitting performance across time, one can also compare how successful are the models in capturing prices of bonds of varying maturities. Table 4.2 reports the estimated model fitting statistics of the models across maturity. For all the three alternative models, as the maturity increases, the degree to which models describes the market data deteriorates. Models give poor fitting to data for longer maturities. This phenomenon is a natural consequence of the structure of government bonds market in Turkey where for a long time the maturities at which the government can make borrowing from the market were quite low. After the financial crisis of 2001 in Turkey, The Undersecretariat of Treasury can only borrow at very short maturities. This continued to be the case even for a long time after the crisis due to worries about government may default on its debt on the one hand and relatively underdevelopment of financial markets on the other. That is why all the alternative models give a better fit to short-end of yield curve. There have always been enough bonds with short-maturities so that the models can properly estimate a yield curve on the short-end.

As it is the case above, the CIR Model yields the poorest fitting performance in most of the term-to-maturities, while the Vasicek and the Nelson-Siegel models achieve a similar fitting performance across maturities. While the Vasicek Model describes the market data at very short maturities (up to 1 year) and at maturities between 2 to 5 years, the Nelson-Siegel Model gives a better fit to data at the rest. Nevertheless again the difference in fitting performance of the two models is not substantial.

Table 4.2: In-Sample Fitting Performance Across Maturity

|                                                                               |                | Up to 1-Year       | 1 to 2 Years        | 2 to 5 Years        | 5 to 10 Years      | The Whole Sample    |
|-------------------------------------------------------------------------------|----------------|--------------------|---------------------|---------------------|--------------------|---------------------|
| <b>NS</b>                                                                     | <b>RMSE</b>    | 0.0953<br>(0.0727) | 0.2623<br>(0.4126)  | 0.4237<br>(0.4354)  | 0.4613<br>(0.3886) | 0.2953<br>(0.2678)  |
|                                                                               | <b>MAE</b>     | 0.0739<br>(0.0564) | 0.2199<br>(0.3430)  | 0.3750<br>(0.3970)  | 0.4217<br>(0.3883) | 0.1945<br>(0.1600)  |
|                                                                               | <b>Av. Er.</b> | 0.0080<br>(0.0401) | -0.0538<br>(0.1550) | -0.0874<br>(0.0568) | 0.2894<br>(0.3910) | 0.0032<br>(0.0298)  |
| <b>Vasicek</b>                                                                | <b>RMSE</b>    | 0.0920<br>(0.0715) | 0.2733<br>(0.4123)  | 0.4134<br>(0.4973)  | 0.4630<br>(0.4491) | 0.2963<br>(0.2760)  |
|                                                                               | <b>MAE</b>     | 0.0692<br>(0.0537) | 0.2267<br>(0.3299)  | 0.3699<br>(0.4612)  | 0.4243<br>(0.4428) | 0.1907<br>(0.1571)  |
|                                                                               | <b>Av. Er.</b> | 0.0044<br>(0.0412) | -0.0356<br>(0.1759) | -0.1086<br>(0.2971) | 0.2786<br>(0.5183) | 0.0029<br>(0.0428)  |
| <b>CIR</b>                                                                    | <b>RMSE</b>    | 0.0923<br>(0.0739) | 0.2689<br>(0.4176)  | 0.5995<br>(0.7802)  | 0.4604<br>(0.4675) | 0.3329<br>(0.3041)  |
|                                                                               | <b>MAE</b>     | 0.0697<br>(0.0556) | 0.2258<br>(0.3367)  | 0.5280<br>(0.6923)  | 0.4246<br>(0.4640) | 0.1995<br>(0.1620)  |
|                                                                               | <b>Av. Er.</b> | 0.0028<br>(0.0412) | 0.0175<br>(0.1928)  | -0.2467<br>(0.5811) | 0.1864<br>(0.5721) | -0.0024<br>(0.0468) |
| *Standard deviations are in the paranthesis. Av. Er. stands for average error |                |                    |                     |                     |                    |                     |

While estimating the zero-coupon yield curve, the values for model parameters that minimize the objective function in 4.3 are also obtained. These model parameters in each model have specific economic meaning and each reveals important facts about market behavior. Therefore the first thing to look at is how these parameters change over time. In Figure 4.1 and Figure 4.2, the time series of the estimated model parameters respectively for the Vasicek and the CIR Models are graphed. An important observation about the time series is that, while the estimated model parameters show a high degree of volatility, the estimated short-rate series are relatively stable. The analyzed one-factor models compromise parameter stability in exchange for stability of the short-rate process. The average in the whole sample period for estimated Vasicek Model parameters  $a$  (mean reversion or speed of adjustment parameter),  $b$  (long-run mean parameter),  $\sigma$  (volatility parameter) and  $r$  (short-rate) are respectively 1.21, 0.31, 0.29

and 0.10. On the other hand, average values for the same parameters ( $a, b, \sigma$  and  $r$ ) in the CIR Model are respectively 0.90, 0.41, 0.56 and 0.10.

Figure 4.1: Vasicek Coefficients

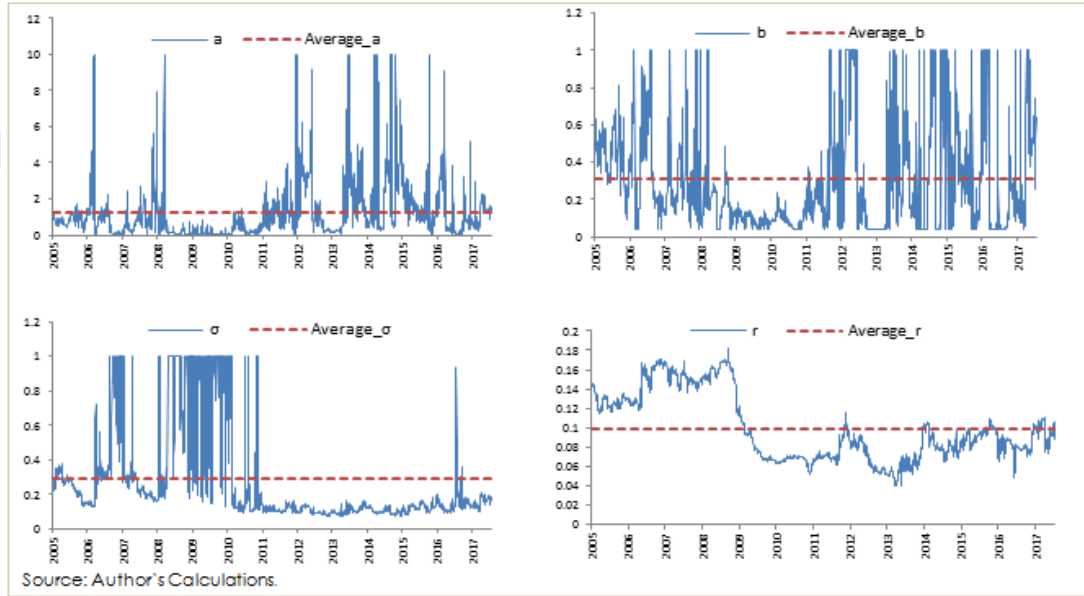
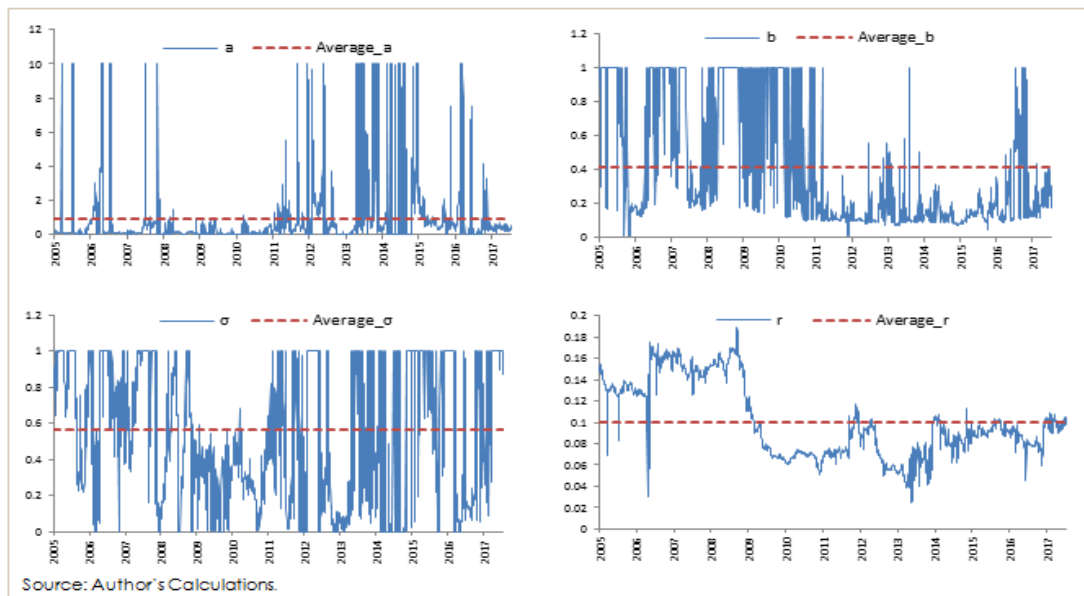


Figure 4.2: CIR Coefficients



### 4.3.2 Estimated Yield Curves

Figure 4.3, Figure 4.4 and Figure 4.5 provide the estimated yield curves for the specific dates that were chosen in Chapter-2 to depict a yield curve with Bloomberg data. Compare yield curves estimated by alternative modeling strategies with the yield curve given by Bloomberg, which is one of the reliable sources of data by practitioners. In terms of shape, the estimated yield curves are rather similar to Bloomberg yield curves. One advantage of the modeling strategies discussed in this thesis is that one can obtain zero rates (and forward rates) of any maturity (on a continuum) whereas Bloomberg provides zero rates only for a few term-to-maturities.

Figure 4.3: Upward Sloping Yield Curve (02.01.2013)

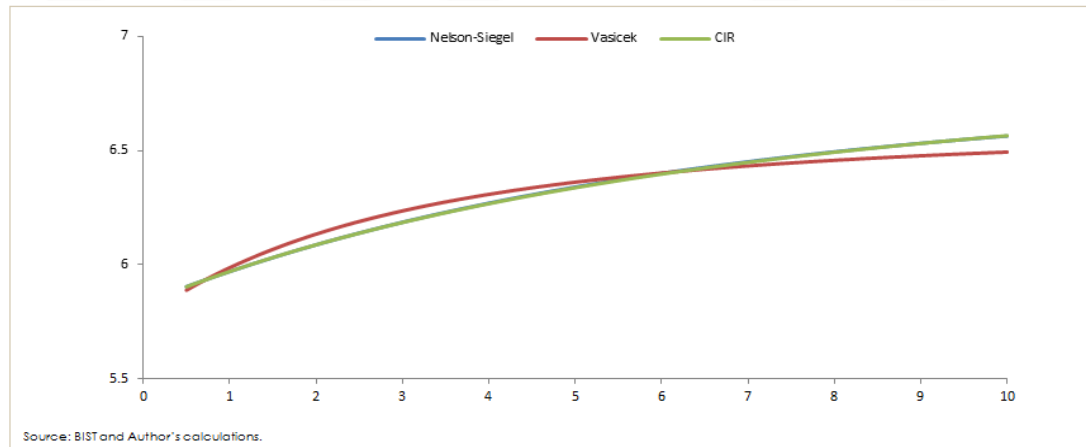


Figure 4.4: Humped Yield Curve (27.01.2014)

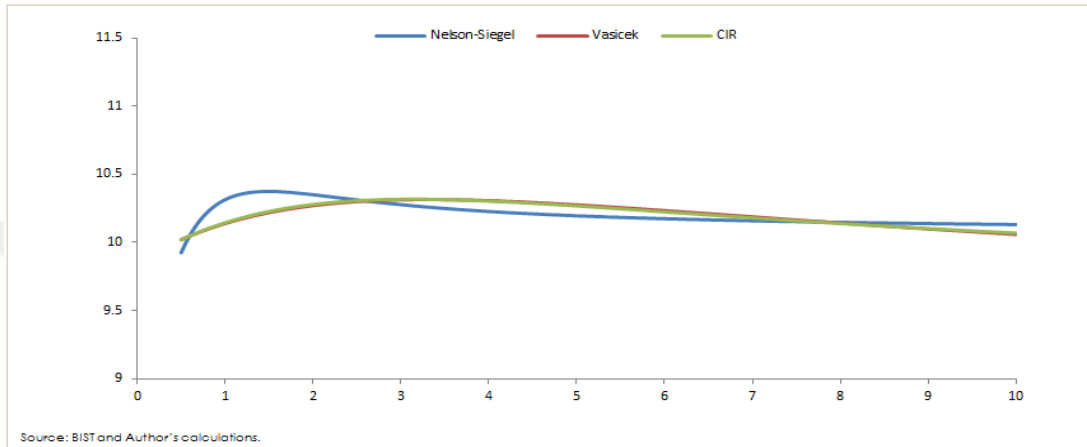
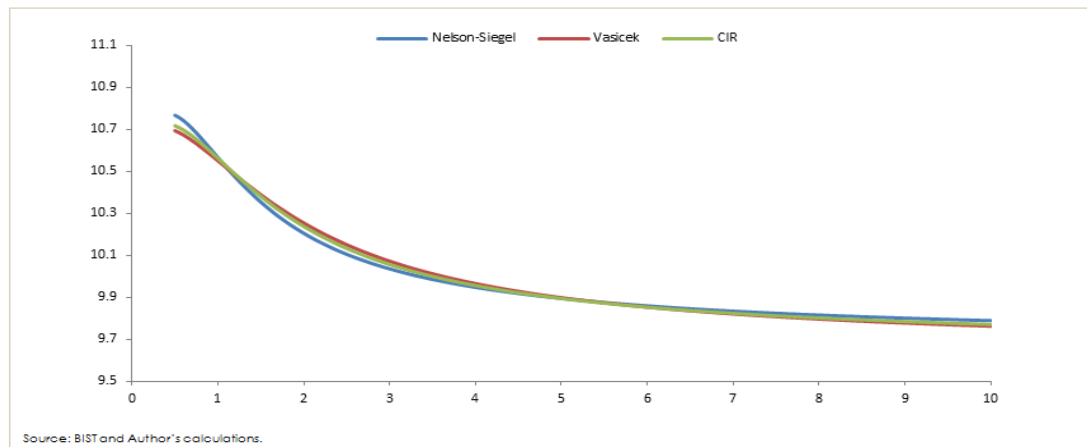


Figure 4.5: Downward Sloping Yield Curve (31.03.2014)



### 4.3.3 Time Series of Estimated Zero Rates

Figure 4.6, Figure 4.7, Figure 4.8 and Figure 4.9 depict the estimated zero-rates for various maturities. Beside the rate estimated through the three of the modeling strategies, those that are reported by Bloomberg are also provided for comparison. As discussed above, the models estimate the prices of bonds of shorter maturities better. That is why the estimated zero rates of various models are quite close to each other for 6-month and 2-year of term-to-maturities (Figure 4.6 and Figure 4.7). The disagreement among zero rates estimated through various models increases as the maturity rises. For the long maturities on the yield curve, the alternative models result in rates away from each other (Figure 4.8 and Figure 4.9). Hence the estimated zero-rates are less reliable on the longer maturities. This phenomenon occurs since there are a very limited number of bonds with long-maturities. For instance a bond whose maturity is equal to 10 years is first issued on January 27th, 2010 while there is almost no bond whose maturity is longer than 10 years during the whole sample period. Indeed, the dispersion among zero rates of the long maturities estimated through different models is higher before 2010, when the maturities in the bond market were very short. Therefore it is rather expected to observe the models describing the data poorly for the long-maturities and yielding zero rates far away from each other.

Figure 4.6: Estimated 6-Month Zero Rates

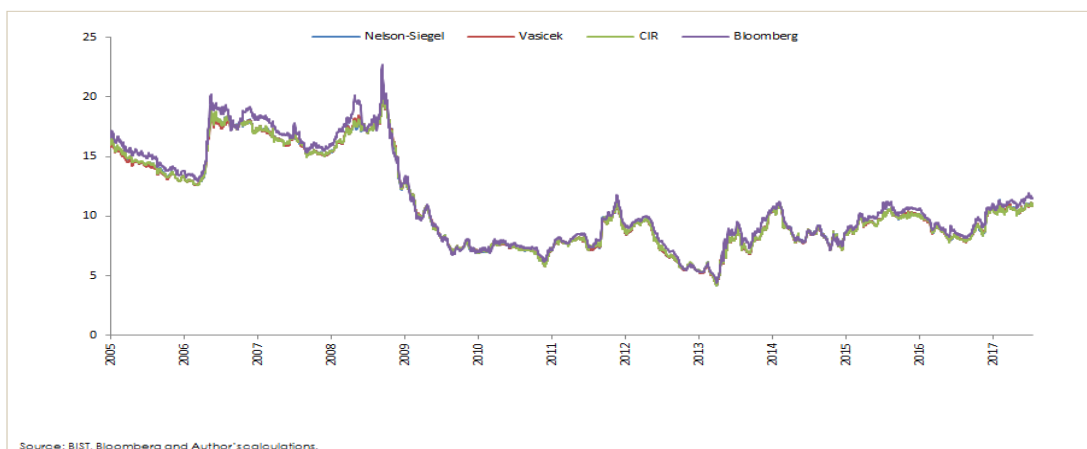


Figure 4.7: Estimated 2-Year Zero Rates

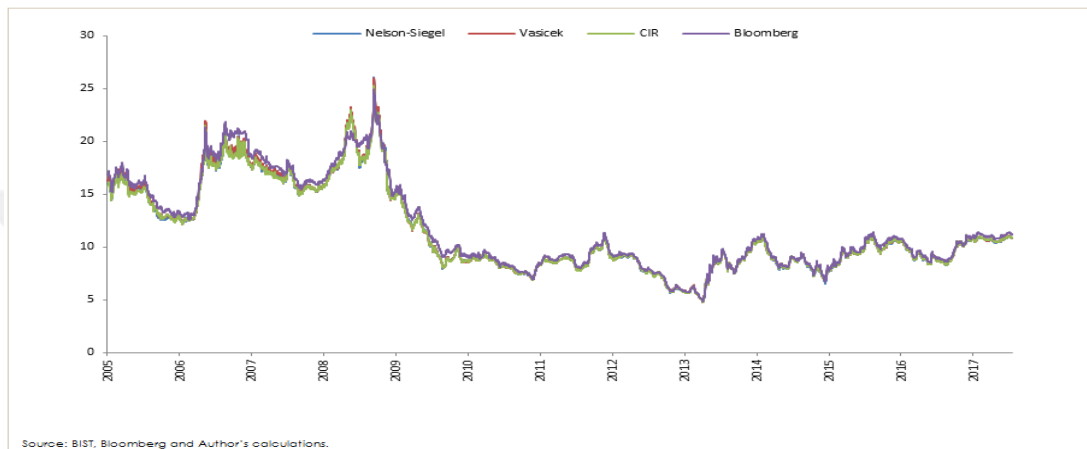


Figure 4.8: Estimated 5-Year Zero Rates

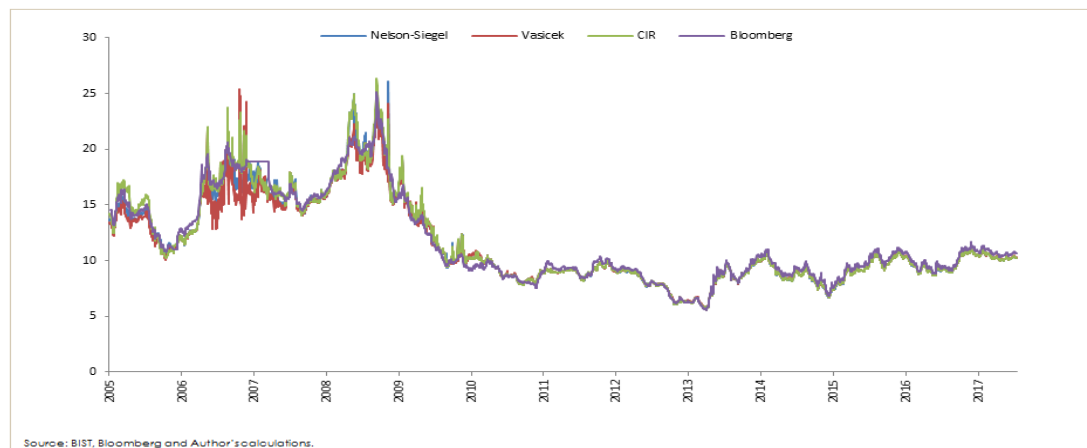
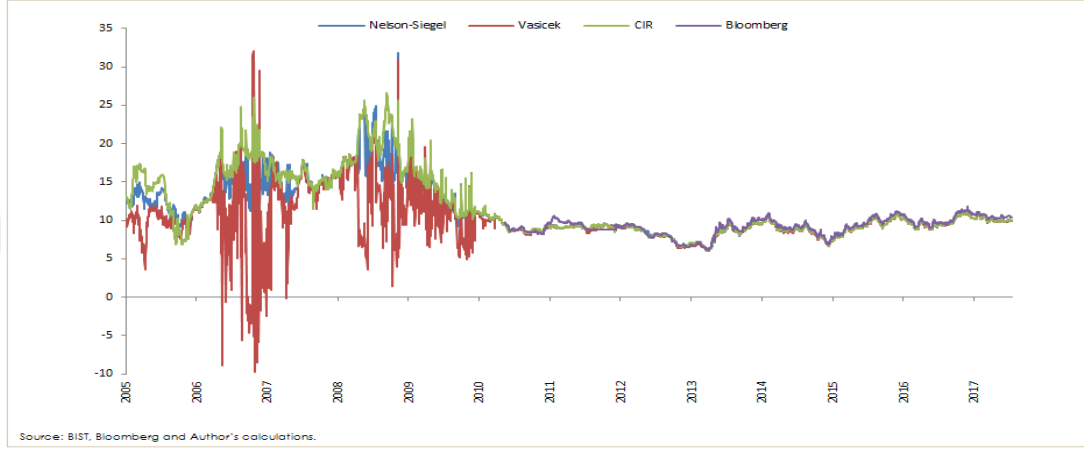


Figure 4.9: Estimated 10-Year Zero Rates



Beside the observation that models disagree on zero-rates of long maturities, another striking regularity emerges from estimated zero-rates: Interest rates (of all maturities) enter into a new regime after the GFC period. While before 2010 the rates were high (in level) and volatile, after 2010 zero rates follow a regime in which they attain low levels and display low volatility. The reduced levels of interest rates are mostly due to fall in inflation levels thank to credible inflation targeting by the Central Bank (CBRT) and fall in country risk premiums due to benign global and domestic financial environment. On the other hand, thank to improvements in financial structure after completing the reforms and to relatively more predictable global financial environment, interest rates follow a less volatile path in the post-crisis period.

Furthermore, for zero rates of 10-year maturity the Vasicek Model gives negative rates in the period before 2010. As it is shown in the Table 4.1 and Table 4.2, the Vasicek Model still yields good fitting to the price of existing bonds. But while trying to model long-term rates where there are a limited number of bonds whose maturity is close to these terms, the Vasicek Model may result in unreliable estimation. As it is discussed in Chapter-3, it is theoretically possible to obtain

negative rates with the Vasicek Model and this is considered as a major drawback of the model by practitioners. This disadvantage of the model manifests itself in Figure 4.9 when one try to model term-to-maturities for which there is limited or no data.



## Chapter 5

### Conclusion

This thesis investigates the extent to which the two of the most common one-factor short-rate models are able to describe the market behavior of risk free Turkish treasuries for the post-2005 period. The investigated models are those widely used ones in the literature, which has analytical solutions, namely the Vasicek Model and the Cox-Ingersoll-Ross (CIR) Model among equilibrium models. The unexpected movements in interest rates have the potential to cause huge capital losses and cash flow problems for market players. Hence every market participant desires to have a good model that sufficiently describes the market behavior of interest rates. Those one-factor short-rate models are simple theoretical tools that are used to reveal interest rate dynamics and estimate term-structure of interest rates to a good extent.

The thesis firstly provides the mathematical background that is needed to solve the models. Then the terminology of bond markets is discussed for readers who are not familiar with the finance literature. After building the necessary mathematical and financial structure, the stochastic mechanics of interest rate modeling is provided and in light of it the zero-coupon bond prices in the two models are solved, which are needed to numerically estimate model coefficients and then implied bond prices. The success of any model depends on how close it

estimates a bond price to the market price. In the empirical part, the fitting performance of these two models is compared via standard model checking statistics (RMSE and MAE) together with the current benchmark term structure model, i.e. Nelson-Siegel Model. While so doing, the model fitting statistics for each model is compared across different time periods and across maturity.

The empirical analysis yields two striking facts: Firstly, all of the alternative models exhibit their worst fit to data during the GFC, which is inferred to be a consequence of volatile global financial environment during that time. Besides, the models also yield a poor fit to data before the GFC period, when Turkish bond market was relatively underdeveloped. Secondly, the model fitting performance deteriorates as maturity rises. This observation is a natural consequence of the fact that during the whole sample period, there has been very limited number of bonds whose maturity is long enough. Of course, it is an open question that how would the model fitting be across maturity if the bonds are evenly distributed across maturities.

When the alternative models are compared, the CIR Model gives the worst fitting to data despite its theoretical complexity. On the other hand, the Vasicek Model achieves a high fitting performance very close to the Nelson-Siegel Model. Despite it is theoretically the simplest (and one of the first) interest rate model, the Vasicek Model describes the market behavior of interest rates to a good extent. But one of the main drawbacks of the model exhibits itself as well: Namely the estimated 10-year zero rates get negative values, which is considered as a main disadvantage of the model.

The main contribution of this thesis is that it investigates how well two of the main interest rate models describe the market dynamics of bond prices compared to the Nelson-Siegel Model, which is the benchmark model used by the Central Bank (CBRT). Estimating interest rates with one of these three alternatives has substantial benefits for practitioners. For instance, one can get interest rate of

any maturity for all days when markets are open, whereas the main financial data sources (Bloomberg or Reuters) provide interest rates only for a few terms with discontinuity from time to time. In this thesis, the models are evaluated based on their in-sample fitting performance. Comparing the out-of-sample fitting performance is left to future research. Moreover, only two of the most common models are analyzed in this thesis. A complete and exhaustive analysis of other alternative approaches in the literature is beyond the scope of this study.



# Bibliography

- [1] O. Vasicek, “An equilibrium characterization of the term structure,” *Journal of Financial Econometrics*, vol. 5, pp. 177–188, 1977.
- [2] R. Rendleman and B. Bartter, “The pricing of options on debt securities,” *Journal of Financial and Quantitative Analysis*, vol. 15, pp. 11–24, 1980.
- [3] J. C. Cox, J. E. Ingersoll and A. S. Ross, “A theory of the term structure of the interest rates,” *Econometrica*, vol. 55, no. 2, pp. 385–407, 1985.
- [4] M. J. Brennan and E.S. Schwartz, “An equilibrium model of bond pricing and a test of market efficiency,” *Journal of Financial and Quantitative Analysis*, vol. 17, no. 3, pp. 301–329, 1982.
- [5] F. A. Longstaff and E. S. Schwartz, “Time-dependent variance and the pricing of options,” *Journal of Finance*, vol. 47, no. 4, pp. 1259–1282, 1992.
- [6] J. Hull, “Options, futures and other derivatives,” *8th ed.*, Pearson Education, 2011.
- [7] T. S. Y. Ho and S. B. Lee, “Term structure movements and pricing interest rate contingent claims,” *Journal of Finance*, vol. 41, pp. 1011–1029, 1986.
- [8] F. Black and P. Karasinski, “Bond and option pricing when short rates are lognormal,” *Financial Analysts Journal*, pp. 52–59, 1991.
- [9] J. Hull and A. White, “Pricing interest rate derivative securities,” *The Review of Financial Studies*, vol. 3, no. 4, pp. 573–592, 1990.

- [10] D. Heath, R. Jarrow and A. Morton, “Bond pricing and the term structure of the interest rates: A new methodology,” *Econometrica*, vol. 60, no. 1, pp. 77–105, 1992.
- [11] A. Brace, D. Gatarek and M. Musiela, “The market model of interest rate dynamics,” *Mathematical Finance*, vol. 1, no. 2, pp. 127–155, 1997.
- [12] C. R. Nelson and A. F. Siegel, “Parsimonious modeling of yield curves,” *The Journal of Business*, vol. 60, no. 4, pp. 473–489, 1987.
- [13] D. L. Cohn, “Measure theory,” *2nd ed.*, *Birkhuser Basel*, 2013.
- [14] I. Karatzas and S. E. Shreve, “Brownian motion and stochastic calculus,” *Springer-Verlag*, 1988.
- [15] B. Øksendal, “Stochastic differential equations: An introduction with applications,” *4th ed.*, *Springer-Verlag*, 1998.
- [16] D. Lamberton and B. Lapeyre, “Introduction to stochastic calculus applied to finance,” *2nd ed.*, *Chapman and Hall/CRC*, 2008.
- [17] M. J. Harrison and D. M. Kreps, “Martingales and arbitrage in multiperiod securities market,” *Journal of Economic Theory*, vol. 29, pp. 381–408, 1979.
- [18] Ö. Akıncı, B. Gürcihan, R. Gürkaynak and Ö. Özel, “Devlet ic borçlanma senetleri için getiri eğrisi tahmini (in turkish),” *Central Bank of the Republic of Turkey Working Paper Series*, no. 06/08, 2006.
- [19] C. E. Alper, A. Akdemir and K. Kazimov, “Estimating the term structure of government securities in turkey,” *Boğaziçi University Economics Working Paper*, no. ISS/EC-2004-03, 2004.
- [20] M. Duran and E. Gülşen, “Estimating inflation compensation for turkey using yield curves,” *Economic Modeling*, vol. 32, no. C, pp. 592–601, 2013.
- [21] O. Çepni and D. Küçükşaraç, “Optimal mix of the extended nelson siegel model for turkish sovereign yield curve,” *Central Bank of the Republic of Turkey Research Notes in Economics*, no. 17/02, 2017.

- [22] İ. B. Kanlı, D. Küçüksaraç and Ö. Özel, “Yield curve estimation for corporate bonds in turkey,” *Central Bank of the Republic of Turkey Working Paper Series*, no. 13/26, 2013.



# Appendix A

## Data

Below is data for 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup> of January,2016. I provide data of only these three days for illustration purpose as the whole data set is too big to be put in the appendix. The definitions of and explanation for given series:

-**Quotation Date**: The date at which quotation made, i.e. the selling or purchase deal is agreed upon and contract signed.

-**Price**: The market price of bond in the given day.

-**Date of Retirement**: The date at which the given bond will be retired. The difference between the date of retirement and the quotation date gives the maturity of the bond.

-**With Coupon**: Dummy variable that takes value 1 if the given bond has intermediary coupon payments and 0 if it is a zero-coupon bond.

-**Benchmark**: Dummy variable that takes value 1 if the given bond is the benchmark and 0 otherwise. The Undersecretariat of Treasury periodically announces the benchmark bond whose maturity is closest to 2 years among its outstanding bond portfolio.

- **Coupon Payment**: The amount paid in Turkish Liras in each coupon payment.

- **Coupon Frequency**: The variable that shows how many times the given bond makes coupon payments per year, generally being 2 corresponding to semiannual coupon payments.

- Stopage:** Dummy variable for stoppage (or capital tax) imposed on the given bond, taking value 1 if there is tax imposed and 0 otherwise.
- Clear Price:** Bond price free of taxes and accrued interest.
- Accrued Interest:** The amount of interest earned from the emission date of the bond until the day of the exchange.



Table A.1: Daily Bond Market Data

| Quot.<br>Date | Price     | Date of<br>Retire. | With<br>Coupon | Bench. | Coupon<br>Pay. | Coupon<br>Freq. | Stoppage | Clear<br>Price | Accrued<br>Interest |
|---------------|-----------|--------------------|----------------|--------|----------------|-----------------|----------|----------------|---------------------|
| 04.01.2016    | 91919.91  | 05.02.2020         | 1              | 0      | 3700           | 2               | 1        | 88972.11       | 2947.80             |
| 04.01.2016    | 100926.56 | 08.03.2017         | 1              | 0      | 4500           | 2               | 1        | 98033.71       | 2892.86             |
| 04.01.2016    | 84069.18  | 08.03.2023         | 1              | 0      | 3550           | 2               | 1        | 81923.58       | 2145.60             |
| 04.01.2016    | 95882     | 08.06.2016         | 0              | 0      | 0              | 0               | 1        | 95882          | 0                   |
| 04.01.2016    | 99739.55  | 08.07.2020         | 1              | 0      | 4700           | 2               | 1        | 95271.97       | 4467.58             |
| 04.01.2016    | 97454.25  | 10.07.2019         | 1              | 0      | 4250           | 2               | 1        | 93414.42       | 4039.84             |
| 04.01.2016    | 99528     | 12.01.2022         | 1              | 0      | 4750           | 2               | 1        | 95195.59       | 4332.42             |
| 04.01.2016    | 87118.22  | 12.03.2025         | 1              | 0      | 4000           | 2               | 1        | 84854.48       | 2263.74             |
| 04.01.2016    | 97346     | 13.04.2016         | 0              | 0      | 0              | 0               | 1        | 97346          | 0                   |
| 04.01.2016    | 102649.96 | 13.07.2016         | 1              | 0      | 4100           | 2               | 1        | 98752.71       | 3897.25             |
| 04.01.2016    | 94128.84  | 14.02.2018         | 1              | 0      | 3150           | 2               | 1        | 91740.38       | 2388.46             |
| 04.01.2016    | 99052.87  | 14.06.2017         | 1              | 0      | 4800           | 2               | 1        | 98551.77       | 501.10              |
| 04.01.2016    | 91854.25  | 14.09.2022         | 1              | 0      | 4250           | 2               | 1        | 89449.03       | 2405.22             |
| 04.01.2016    | 96341.92  | 14.11.2018         | 1              | 0      | 4400           | 2               | 1        | 95205.66       | 1136.26             |
| 04.01.2016    | 90708.83  | 14.12.2016         | 0              | 0      | 0              | 0               | 1        | 90708.83       | 0                   |
| 04.01.2016    | 104100.99 | 15.01.2020         | 1              | 0      | 5250           | 2               | 1        | 99312.52       | 4788.46             |
| 04.01.2016    | 98983.26  | 16.11.2016         | 1              | 0      | 4100           | 2               | 1        | 97924.47       | 1058.79             |
| 04.01.2016    | 94081.6   | 17.08.2016         | 0              | 0      | 0              | 0               | 1        | 94081.60       | 0                   |
| 04.01.2016    | 100802.75 | 20.03.2024         | 1              | 0      | 5200           | 2               | 1        | 98059.9        | 2742.86             |
| 04.01.2016    | 95073.62  | 20.06.2018         | 1              | 0      | 4150           | 2               | 1        | 94800          | 273.63              |
| 04.01.2016    | 103832.96 | 24.02.2016         | 1              | 0      | 5350           | 2               | 1        | 99982.14       | 3850.82             |
| 04.01.2016    | 94533.84  | 24.07.2024         | 1              | 0      | 4500           | 2               | 1        | 90775.6        | 3758.24             |
| 04.01.2016    | 103818.5  | 27.01.2016         | 1              | 0      | 4500           | 2               | 1        | 99887.19       | 3931.32             |
| 04.01.2016    | 101640.93 | 27.03.2019         | 1              | 0      | 5200           | 2               | 1        | 98898.07       | 2742.86             |
| 04.01.2016    | 91923.76  | 27.09.2023         | 1              | 0      | 4400           | 2               | 1        | 89772.11       | 2151.65             |
| 05.01.2016    | 91542.26  | 05.02.2020         | 1              | 0      | 3700           | 2               | 1        | 88574.13       | 2968.13             |
| 05.01.2016    | 100818.93 | 08.03.2017         | 1              | 0      | 4500           | 2               | 1        | 97901.35       | 2917.58             |
| 05.01.2016    | 83515.1   | 08.03.2023         | 1              | 0      | 3550           | 2               | 1        | 81350          | 2165.11             |
| 05.01.2016    | 98912.26  | 08.07.2020         | 1              | 0      | 4700           | 2               | 1        | 94418.85       | 4493.41             |
| 05.01.2016    | 96778.81  | 10.07.2019         | 1              | 0      | 4250           | 2               | 1        | 92715.62       | 4063.19             |
| 05.01.2016    | 98058.51  | 12.01.2022         | 1              | 0      | 4750           | 2               | 1        | 93700          | 4358.52             |
| 05.01.2016    | 86265.05  | 12.03.2025         | 1              | 0      | 4000           | 2               | 1        | 83979.34       | 2285.71             |
| 05.01.2016    | 97385     | 13.04.2016         | 0              | 0      | 0              | 0               | 1        | 97385          | 0                   |
| 05.01.2016    | 102669.91 | 13.07.2016         | 1              | 0      | 4100           | 2               | 1        | 98750.13       | 3919.78             |
| 05.01.2016    | 93960.14  | 14.02.2018         | 1              | 0      | 3150           | 2               | 1        | 91554.37       | 2405.77             |
| 05.01.2016    | 98827.47  | 14.06.2017         | 1              | 0      | 4800           | 2               | 1        | 98300          | 527.47              |
| 05.01.2016    | 90979.31  | 14.09.2022         | 1              | 0      | 4250           | 2               | 1        | 88550.74       | 2428.57             |
| 05.01.2016    | 95960.43  | 14.11.2018         | 1              | 0      | 4400           | 2               | 1        | 94800          | 1160.44             |
| 05.01.2016    | 90694.63  | 14.12.2016         | 0              | 0      | 0              | 0               | 1        | 90694.63       | 0                   |
| 05.01.2016    | 103752.11 | 15.01.2020         | 1              | 0      | 5250           | 2               | 1        | 98934.8        | 4817.31             |
| 05.01.2016    | 94771.42  | 20.06.2018         | 1              | 0      | 4150           | 2               | 1        | 94475          | 296.43              |
| 05.01.2016    | 103861.84 | 24.02.2016         | 1              | 0      | 5350           | 2               | 1        | 99981.62       | 3880.22             |
| 05.01.2016    | 93820.3   | 24.07.2024         | 1              | 0      | 4500           | 2               | 1        | 90037.34       | 3782.97             |
| 05.01.2016    | 103846.35 | 27.01.2016         | 1              | 0      | 4500           | 2               | 1        | 99890.31       | 3956.04             |
| 05.01.2016    | 101196.42 | 27.03.2019         | 1              | 0      | 5200           | 2               | 1        | 98425          | 2771.43             |
| 06.01.2016    | 91165.73  | 05.02.2020         | 1              | 0      | 3700           | 2               | 1        | 88177.27       | 2988.46             |
| 06.01.2016    | 100794.3  | 08.03.2017         | 1              | 0      | 4500           | 2               | 1        | 97851.99       | 2942.31             |

|            |           |            |   |   |      |   |   |          |         |
|------------|-----------|------------|---|---|------|---|---|----------|---------|
| 06.01.2016 | 82984.61  | 08.03.2023 | 1 | 0 | 3550 | 2 | 1 | 80800    | 2184.62 |
| 06.01.2016 | 95932     | 08.06.2016 | 0 | 0 | 0    | 0 | 1 | 95932    | 0       |
| 06.01.2016 | 98550.67  | 08.07.2020 | 1 | 0 | 4700 | 2 | 1 | 94031.44 | 4519.23 |
| 06.01.2016 | 96843.68  | 10.07.2019 | 1 | 0 | 4250 | 2 | 1 | 92757.14 | 4086.54 |
| 06.01.2016 | 97602.32  | 12.01.2022 | 1 | 0 | 4750 | 2 | 1 | 93217.7  | 4384.62 |
| 06.01.2016 | 85290.83  | 12.03.2025 | 1 | 0 | 4000 | 2 | 1 | 82983.14 | 2307.69 |
| 06.01.2016 | 97385     | 13.04.2016 | 0 | 0 | 0    | 0 | 1 | 97385    | 0       |
| 06.01.2016 | 102740.24 | 13.07.2016 | 1 | 0 | 4100 | 2 | 1 | 98797.94 | 3942.31 |
| 06.01.2016 | 93848.07  | 14.02.2018 | 1 | 0 | 3150 | 2 | 1 | 91425    | 2423.08 |
| 06.01.2016 | 99024.64  | 14.06.2017 | 1 | 0 | 4800 | 2 | 1 | 98470.8  | 553.85  |
| 06.01.2016 | 90676.92  | 14.09.2022 | 1 | 0 | 4250 | 2 | 1 | 88225    | 2451.92 |
| 06.01.2016 | 95734.61  | 14.11.2018 | 1 | 0 | 4400 | 2 | 1 | 94550    | 1184.62 |
| 06.01.2016 | 90728.64  | 14.12.2016 | 0 | 0 | 0    | 0 | 1 | 90728.64 | 0       |
| 06.01.2016 | 103292.9  | 15.01.2020 | 1 | 0 | 5250 | 2 | 1 | 98446.75 | 4846.15 |
| 06.01.2016 | 98960.23  | 16.11.2016 | 1 | 0 | 4100 | 2 | 1 | 97856.38 | 1103.85 |
| 06.01.2016 | 99251.23  | 20.03.2024 | 1 | 0 | 5200 | 2 | 1 | 96451.23 | 2800    |
| 06.01.2016 | 94469.23  | 20.06.2018 | 1 | 0 | 4150 | 2 | 1 | 94150    | 319.23  |
| 06.01.2016 | 103900.32 | 24.02.2016 | 1 | 0 | 5350 | 2 | 1 | 99990.7  | 3909.62 |
| 06.01.2016 | 93117.84  | 24.07.2024 | 1 | 0 | 4500 | 2 | 1 | 89310.15 | 3807.69 |
| 06.01.2016 | 103874.04 | 27.01.2016 | 1 | 0 | 4500 | 2 | 1 | 99893.27 | 3980.77 |
| 06.01.2016 | 101048    | 27.03.2019 | 1 | 0 | 5200 | 2 | 1 | 98248    | 2800    |
| 06.01.2016 | 90725     | 27.09.2023 | 1 | 0 | 4400 | 2 | 1 | 88525    | 2200    |