Bloch Vector Formalism for Quantum Computation

by

Ahmet Tuna Bölükbaşı

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This is to certify that I have examined this copy of a master's thesis by

Ahmet Tuna Bölükbaşı

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

Prof. Tekin Dereli

Prof. Varga Kalantarov

Asst. Prof Alkan Kabakçıoğlu

Date:

Dedicated to the loving memory of my grandmother (TonTon):
Hayriye Güncü

ABSTRACT

The Bloch vector formalism provides a useful approach to both fundamental and applied problems of the finite dimensional, composite quantum systems, and therefore it is a very important tool for the research on quantum computation and information. This thesis deals with the systematic treatment of Bloch vector formalism for quantum mechanics of one node, two node and three node systems. Each chapter is divided into three sections dealing with states, measurements and time evolutions. Also, the geometric picture of the space of states is given in the section on one node systems. Classically correlated states i.e. separable states, for two and three node systems are investigated in the relevant chapters. I will also briefly mention the well-known EPR paradox, Bell's inequality, Mermin's proofs on the problems with the elements of reality by using GHZ state and the two different classes of the genuine tripartite entanglement to comprehend the unusual nature of entanglement.

ÖZETÇE

Bloch vektör formalizmi sonlu boyutlu bileşik kuantum sistemlerinin gerek temel gerekse uygulamalı problemleri için faydalı bir yaklaşım sunar, ve bu yüzden kuantum hesaplama ve enformasyonu için çok önemli bir araçtır. Bu tez bir, iki ve üç bileşenli sistemlerin Bloch vektör formalizmini sistematik olarak incelemektedir. Her bölüm kendi içinde durum vektörleri, ölçümler ve zaman evrimi olarak üçe ayrılmıştır. Ayrıca, durum uzaylarının geometrik resmi tek bileşenli sistemlerle ilgili olan bölümde verilmiştir. İki ve üç bileşenli sistemlerin klasik bağdaşık durumları yani ayrılabilir durumları, ilgili bölümlerde incelenmiştir. Ayrıca, kısaca EPR paradoksundan, Bell eşitsizliklerinden ve Mermin'in gerçeklik elemanı ile GHZ durumları arasındaki problemi çözümünden ve iki farklı gerçek üçlü dolanıklıktan, dolanıklığın alışılmadık doğasını kavramak için bahsedeceğim.

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INTRODUCTION

In the early 80's, Feynman started the discussion of whether computation can be done at the scale of quantum physics [1], because of the essential difficulties in simulating quantum mechanical systems on classical computers. Later, in 1985, Deutsch [2] published a paper showing that any physical process, in principle, could be modeled perfectly by a quantum computer. Thus, a quantum computer would have capabilities far beyond those of any traditional classical computer. After Deutsch published this paper, the search began to find interesting applications for such a machine. In 1994, Shor [3] set out a method for using quantum computers to solve an important problem in number theory, namely factorization. He showed how an ensemble of mathematical operations, designed specifically for a quantum computer, could be organized to enable such a machine to factor huge numbers extremely rapidly, much faster than is possible on conventional computers. With this breakthrough, quantum computing transformed from a mere academic curiosity directly into a subject of national and world interest.

To understand the power of quantum computation, let us compare the computation done classically and quantum mechanically [4],[6]. In classical computers, the elementary information unit is *bit* i.e. a value which is either 1 or 0. The corresponding quantum analog of bit is a qubit which is defined as a state in two dimensional Hilbert space $\mathcal{H}_2$. Two possible states for a qubit are the states $|1\rangle$ and $|0\rangle$ corresponding to the states 1 and 0 for a bit, is called the computational basis states, and form an orthonormal basis for this vector space. The general state of a qubit is a *superposition* of these two classical states,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

where $\alpha, \beta \in C$.

The computation is done on these qubits by unitary time evolution generated by the

Hamiltonian of the system, which are the analogs of the gates such as AND or NOT in classical computers. [4]

To build a quantum computer, we need to compose a large number N of these qubits. The Hilbert space of these qubits is the tensor product of the each of the Hilbert spaces $\mathcal{H}_{2^N} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_2$, and the power of quantum computation lies here. In classical a computer, when we consider N bits we know that each of them have the exact value of either 1 or 0. However, for quantum systems composed of N qubits it may not be possible to assign a quantum state $|\psi\rangle$ to each of them. For example, consider the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

which is a state of two qubits and it is not possible to assign a state vector to each subsystem. A state vector that can not be written as a tensor product of its subsystems is called *entangled*. It is the essence of the power of quantum computation and lets us to define new mathematical operations, designed specifically for a quantum computer. [4]

Erwin Schrödinger gave that name to the correlation of quantum nature 71 years ago.[5] Over the decades the meaning of the word "entanglement" has changed with the attitudes towards it. The various important persons in the fields of foundations of quantum physics and quantum information theory have described entanglement differently.[5]

A. Einstein/B. Podolsky/N. Rosen: An entangled wavefunction does not describe the physical reality in a complete way.

E. Schrödinger: For an entangled state the best possible knowledge of the whole does not include the best possible knowledge of its parts.

Entanglement is...

J. Bell: ...a correlation that is stronger than any classical correlation.

D. Mermin: ...a correlation that contradicts the theory of elements of reality.

A. Peres: ...a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians.

C. Bennett: ...a resource that enables quantum teleportation.

P. Shor: ...a global structure of the wavefunction that allows for faster algorithms.

A. Ekert: ...a tool for secure communication.

Because the entanglement makes it hard to deal with subsystems, it is better to use density matrix formalism of quantum mechanics [6],[7]. The density matrix was introduced by J. von Neumann in 1927 to describe statistical concepts in quantum mechanics. The representation of quantum mechanical states by density matrix enables the maximum information available on the system to be expressed in a compact manner by providing a uniform treatment of all states whether single pure states or mixed ensemble of states. Also, the state of a subsystem of an entangled state can be obtained from the density matrix of the system and it is called reduced density matrix. Therefore, density matrix formalism is essential for quantum computation and information which requires ultimate control over the quantum systems, and next chapter briefly deals with it.

A composite quantum systems can be thought in terms of the reduced density matrices of each subsystem and the correlation term between subsystems which can be obtained by comparing the resulting product of reduced density matrices with the density matrix of the total system. The correlation can have classical or quantum nature i.e. entanglement. To simplify all these calculations and make the results more transparent, it is better to expand the density matrix into a basis for the density matrix space. In particular, we can choose the basis constructed from identity matrix and traceless, Hermitian generators of $\mathsf{SU(N)}$. The vector consisting of the real coefficients of the generators of $\mathsf{SU(N)}$ is called the Bloch vector. The main advantage of using this basis is that the coefficients correspond to the expectation values of measurements by generators of $\mathsf{SU(N)}$ on density matrix of the system. Moreover, by reversing the argument, the state of an unknown quantum system (mixed or not) can be determined by expectation values of the generators of $\mathsf{SU(N)}$ and that is important for both experimentalists and theoreticians. Also, the representation of states by a Bloch vector allows to construct the geometry of the space of density matrix, for example, famous Bloch sphere is the space of the Bloch vectors of a qubit. The geometric picture of states is important to understand the types of correlation and the measure of entanglement.

We may also represent the Hermitian operators and the Hamiltonian of the system by using the same basis again. In particular, quantum mechanics as a whole can be formulated in the Bloch vector representation. The advantage of using this representation is that the nonlocal action of the operators can be analyzed transparently and this makes it easier to control the system's behavior for generating entanglement in the desired quantity. For example, one can study how a product state becomes correlated under a nonlocal Hamiltonian evolution, or for entangled states, after measuring one of the subsystems, what amount of correlations is materialized in the state of the other subsystem.

This thesis deals with the systematic treatment of Bloch vector formalism for quantum mechanics for one node, two node and three node systems. Each chapter divided into three sections dealing with states, measurements and time evolutions. In addition to that, the geometric picture of the space of states will be given in the section dealing with one node system. Classically correlated states i.e. separable states, for two and three node systems are investigated in the relevant chapters. I will also briefly mention the well-known EPR paradox, Bell's inequality, Mermin's proofs on the problems with the elements of reality by using GHZ states and the two different classes of the genuine tripartite entanglement to comprehend the unusual nature of entanglement.

Chapter 2

DENSITY MATRIX FORMALISM

2.1 Density Matrix Formalism for One Node Systems

Density Matrix Representation of Pure States

In a quantum system, the state vector specifies all the physical properties of the system. It provides the maximal information about the system and a system with a state vector description is said to be in a *pure state*. For a pure state $|\psi\rangle$, its density matrix is defined as [7]

$$\widehat{\rho} := |\psi\rangle\langle\psi|$$

Let $|i\rangle$ denote a complete orthonormal set of vectors spanning a Hilbert space of dimension N, then $|\psi\rangle$ can be represented by

$$|\psi\rangle = \sum_i a_i|i\rangle$$

and the density matrix of $|\psi\rangle$ is

$$\widehat{\rho} = \sum_{i,j} \rho_{ij}|i\rangle\langle j|, \qquad \rho_{ij} = a_i a_j^*$$

For an Hermitian operator $\widehat{A}$, the expectation value of the measurement is

$$\begin{aligned}
\langle\widehat{A}\rangle &= \langle\psi|\widehat{A}|\psi\rangle \\
&= \sum_{i,j} \rho_{ij}\langle j|\widehat{A}|i\rangle \\
&= \sum_{i,j} \langle i|\widehat{\rho}|j\rangle\langle j|\widehat{A}|i\rangle \\
&= \mathsf{Tr}\{\widehat{\rho}\widehat{A}\}. \qquad (2.1)
\end{aligned}$$

Let $|i\rangle$ $(i = 1, ..., N)$ be eigenvectors of $\widehat{A}$ with the eigenvalues A_i and the probability $\Pr(A_i)$ of getting A_i is

$$\Pr(A_i) = \langle\widehat{P}_i\rangle$$

where $\widehat{P}_i$ is projection operator defined as $\widehat{P}_i = |i\rangle\langle i|$. In terms of density matrix formalism the probability $\Pr(A_i)$ is equal to

$$\Pr(A_i) = \mathsf{Tr}\{\widehat{\rho}\widehat{P}_i\}$$

In wavefunction formalism, the state after the measurement $\widehat{A}$ for a particular outcome A_i is

$$|i\rangle = \frac{\widehat{P}_i|\psi\rangle}{a_i}$$

and its correspondence in density matrix formalism is

$$\begin{aligned} |i\rangle\langle i| &= \frac{\widehat{P}_i|\psi\rangle\langle\psi|\widehat{P}_i}{a_i a_i^*} \\ &= \frac{\widehat{P}_i\widehat{\rho}\widehat{P}_i}{\mathsf{Tr}\{\widehat{\rho}\widehat{P}_i\}} \\ &= \frac{\widehat{P}_i\widehat{\rho}\widehat{P}_i}{\Pr(A_i)} \end{aligned}$$

The time dependence of the state vector $|\psi(t)\rangle$ is determined by the Hamiltonian of the system according to *Schrödinger equation*

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \widehat{H}(t)|\psi(t)\rangle.$$

In density matrix formalism the time dependence of the density matrix $\widehat{\rho}(t)$ is defined by *Liouville equation* [7] ,

$$\begin{aligned} i\hbar\frac{d}{dt}\widehat{\rho}(t) &= i\hbar\frac{d|\psi(t)\rangle}{dt}\langle\psi(t)| + i\hbar|\psi(t)\rangle\frac{d\langle\psi(t)|}{dt} \\ &= \widehat{H}(t)|\psi(t)\rangle\langle\psi(t)| - |\psi(t)\rangle\langle\psi(t)|\widehat{H}(t) \\ &= [\widehat{H}(t), \widehat{\rho}(t)]. \end{aligned} \tag{2.2}$$

Density Matrix Representation of Ensemble of Pure States

An *ensemble* is a collection of N non-interacting systems of the same kind in the same or different quantum states [7]. One of the main advantage of density matrix formalism is achieved when we deal with a mixture of pure states $|\psi_i\rangle$ with weights p_i $(0 \le p_i \le 1,\ i = 1, ..., N)$. This ensemble can be represented by the density matrix

$$\begin{aligned} \widehat{\rho} &= \sum_i p_i|\psi_i\rangle\langle\psi_i| \\ &= \sum_i p_i\widehat{\rho}_i \end{aligned}$$

and a system having such a description is said to be in a *mixed state*. Mixed states do not admit state vector descriptions.

Let $|m\rangle$ denote a complete orthonormal set of vectors spanning a Hilbert space of dimension N then

$$\begin{aligned} |\psi_i\rangle &= \sum_m a_m^{(i)}|m\rangle \\ \widehat{\rho} &= \sum_{m,n} \rho_{mn}|m\rangle\langle n|, \qquad \rho_{mn} = \sum_i p_i a_m^{(i)} a_n^{(i)*} \end{aligned}$$

For an Hermitian operator $\widehat{A}$, the expectation value of the measurement for $|\psi_i\rangle$ is

$$\langle \widehat{A}\rangle_i = \mathsf{Tr}\{\widehat{\rho}_i\widehat{A}\}$$

and the expectation value of the measurement for mixture is

$$\begin{aligned} \langle \widehat{A}\rangle &= \sum_i p_i \mathsf{Tr}\{\widehat{\rho}_i\widehat{A}\ \} \\ &= \sum_{i,j} p_i\langle j|\widehat{\rho}_i\widehat{A}|j\rangle \\ &= \sum_j \langle j|\sum_i p_i\widehat{\rho}_i\widehat{A}|j\rangle \\ &= \mathsf{Tr}\{\widehat{\rho}\widehat{A}\} \end{aligned}$$

For a mixture, let $|m\rangle$ $(m = 1, ..., N)$ be the eigenvectors of $\widehat{A}$ with the eigenvalues A_m and the probability $\Pr(A_m)$ of getting A_m is

$$\begin{aligned} \Pr(A_m) &= \langle \widehat{P}_m\rangle \\ &= \mathsf{Tr}\{\widehat{\rho}\widehat{P}_m\} \\ &= \sum_j \langle j|\widehat{\rho}\widehat{P}_m|j\rangle \\ &= \langle m|\widehat{\rho}|m\rangle \\ &= \rho_{mm} \end{aligned}$$

The state after the measurement $\widehat{A}$ for a particular outcome A_m for $|\psi_i\rangle$ is

$$|m\rangle = \frac{\widehat{P}_m|\psi_i\rangle}{a_m^{(i)}}$$

and its correspondence in density matrix formalism for mixture is

$$
\begin{aligned}
|m\rangle\langle m| &= \frac{\widehat{P}_m|\psi_i\rangle\langle\psi_i|\widehat{P}_m}{a_m^{(i)}a_m^{(i)*}} \\
a_m^{(i)}a_m^{(i)*}|m\rangle\langle m| &= \widehat{P}_m|\psi_i\rangle\langle\psi_i|\widehat{P}_m \\
\sum_i p_i a_m^{(i)}a_m^{(i)*}|m\rangle\langle m| &= \sum_i p_i\widehat{P}_m\widehat{\rho}_i\widehat{P}_m \\
\rho_{mm}|m\rangle\langle m| &= \widehat{P}_m\widehat{\rho}\widehat{P}_m \\
|m\rangle\langle m| &= \frac{\widehat{P}_m\widehat{\rho}\widehat{P}_m}{\mathsf{Tr}\{\widehat{\rho}\widehat{P}_m\}}
\end{aligned}
$$

The time dependence of the mixture is

$$
\begin{aligned}
\widehat{\rho}(t) &= \sum_i p_i\widehat{\rho}_i(t) \\
i\hbar\frac{d}{dt}\widehat{\rho}(t) &= \sum_i p_i i\hbar\frac{d}{dt}\widehat{\rho}_i(t) \\
&= \sum_i p_i[\widehat{H}(t),\widehat{\rho}_i(t)] \\
&= [\widehat{H}(t),\sum_i p_i\widehat{\rho}_i(t)] \\
&= [\widehat{H}(t),\widehat{\rho}(t)]
\end{aligned}
$$

The fundamental equation of quantum mechanics remains the *same* in passing from pure states to ensemble of pure states.

Properties of the Density Matrix

Density matrices are Hermitian. For pure states this is obvious from the definition

$$\widehat{\rho} := |\psi\rangle\langle\psi|$$

For mixed states any real linear combination of Hermitian matrices are again Hermitian.

The trace of a density matrix is equal to 1.

$$\begin{aligned}
\mathsf{Tr}\{\widehat{\rho}\} &= \sum_j \langle j|\widehat{\rho}|j\rangle \\
&= \sum_j \rho_{jj} \\
&= \sum_{i,j} p_i a_j^{(i)} a_j^{(i)*} \\
&= \sum_i p_i \left(\sum_j a_j^{(i)} a_j^{(i)*} \right) \quad \text{since } |\psi_i\rangle \text{ is normalized} \\
&= \sum_i p_i \\
&= 1
\end{aligned}$$

Density matrices are nonnegative. This means that for any state $|\phi\rangle$, $\langle\phi|\widehat{\rho}|\phi\rangle \geq 0$

$$\begin{aligned}
\langle\phi|\widehat{\rho}|\phi\rangle &= \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle \\
&= \sum_i p_i |\langle\phi|\psi_i\rangle|^2 \\
&\geq 0.
\end{aligned}$$

The inverse is also true, i.e., any matrix that is Hermitian, has trace 1 and is nonnegative, is a density matrix for some ensemble of quantum states.

Note that the equality $\widehat{\rho}^2 = \widehat{\rho}$ satisfied any pure states is not valid for mixed states

$$\begin{aligned}
\widehat{\rho}^2 &= \left(\sum_i p_i |\psi_i\rangle\langle\psi_i| \right)^2 \\
&= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i| \\
&\neq \widehat{\rho}
\end{aligned}$$

furthermore

$$\mathsf{Tr}\{\widehat{\rho}^2\} \leq 1.$$

2.2 Density Matrix Formalism for Two Node Systems

Consider a quantum system with Hilbert spaces $\mathcal{H}(v)(\ v = 1, 2)$ whose dimension $\dim \mathcal{H}(v) = N_v$. The mathematical scheme appropriate to describe composite systems consisting of specific subsystems (nodes) $\mathcal{H}(1)$ and $\mathcal{H}(2)$ is the product Hilbert space

$$\mathcal{H} = \mathcal{H}(1) \otimes \mathcal{H}(2)$$

with dimension $N = \dim \mathcal{H} = N_1 N_2$. Let $|l(1)\rangle$ and $|m(2)\rangle$ denote complete orthonormal set of vectors spanning each Hilbert space $\mathcal{H}(1)$ and $\mathcal{H}(2)$, respectively. Then $|l(1)\rangle \otimes |m(2)\rangle$ denote a complete orthonormal set of vectors spanning the product Hilbert space $\mathcal{H}$ and a state $|\psi\rangle$ in that space is represented as

$$|\psi\rangle = \sum_{l,m} a_{lm} |l(1), m(2)\rangle$$

where $|l(1), m(2)\rangle = |l(1)\rangle \otimes |m(2)\rangle$. These state vectors obey the orthonormality relation

$$\langle l(1), m(2)|l'(1), m'(2)\rangle = \delta_{ll'}\delta_{mm'}$$

Let $|\psi_1\rangle$, $|\psi_2\rangle$ be state vectors in $\mathcal{H}(1)$ and $\mathcal{H}(2)$ respectively. The state vector of the composite system of the form

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

is called a product state vector. Similarly, let $\widehat{A}(1)$ and $\widehat{A}(2)$ be Hermitian operators acting on $\mathcal{H}(1)$ and $\mathcal{H}(2)$, respectively, Hermitian operator of the total system will be

$$\widehat{A} = \widehat{A}(1) \otimes \widehat{A}(2)$$

Let $\widehat{H}(1)$ and $\widehat{H}(2)$ be Hamiltonians of the system in $\mathcal{H}(1)$ and $\mathcal{H}(2)$, respectively, Hamiltonian of the composite system is

$$\widehat{H} = \widehat{H}(1) \otimes \widehat{H}(2).$$

Suppose we consider such a product state $|\psi_1\rangle \otimes |\psi_2\rangle$ subjected to a measurement of the product Hermitian operator $\widehat{A}(1) \otimes \widehat{A}(2)$. The outcome state $|i, j\rangle$ is determined by considering the measurements on subsystems separately and then taking a product of each outcome states $|i\rangle, |j\rangle$:

$$\begin{aligned} |\psi_1\rangle &\xrightarrow{\widehat{A}(1)} |i\rangle, \\ |\psi_2\rangle &\xrightarrow{\widehat{A}(2)} |j\rangle, \\ |\psi_1\rangle \otimes |\psi_2\rangle &\xrightarrow{\widehat{A}(1)\otimes\widehat{A}(2)} |i\rangle \otimes |j\rangle. \end{aligned}$$

Furthermore, the expectation value of the measurement $\widehat{A}(1) \otimes \widehat{A}(2)$ is the product of each expectation values:

$$\langle \widehat{A}(1) \otimes \widehat{A}(2)\rangle = \langle\psi_1|\widehat{A}(1)|\psi_1\rangle\langle\psi_2|\widehat{A}(2)|\psi_2\rangle.$$

Similarly, the time evolution of such a product state $|\psi_1\rangle \otimes |\psi_2\rangle$ with a product Hamiltonian $\widehat{H}(1) \otimes \widehat{H}(2)$ is determined by considering evolution of the subsystems separately and then taking a product of final states $|\psi_1(t)\rangle, |\psi_2(t)\rangle$:

$$\begin{aligned}
|\psi_1\rangle &\xrightarrow{\widehat{H}(1)} |\psi_1(t)\rangle, \\
|\psi_2\rangle &\xrightarrow{\widehat{H}(2)} |\psi_2(t)\rangle, \\
|\psi_1\rangle \otimes |\psi_2\rangle &\xrightarrow{\widehat{H}(1)\otimes\widehat{H}(2)} |\psi_1(t)\rangle \otimes |\psi_2(t)\rangle.
\end{aligned}$$

In such a totally product systems i.e. product states, product Hermitian operators and product Hamiltonians, composite systems are totally independent of each other.

On the other hand, consider the state $|\psi\rangle$ in a product Hilbert space $\mathcal{H}(1) \otimes \mathcal{H}(2)$ of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |2,2\rangle)$$

This state can not have separate state vectors for each node. Such a pure state that does not have separate state vector is called *entangled states* [6].

However, to simplify the expectation value equation of a measurement performed on a subsystem, we can ascribe a density matrix for subsystems taken out of the total density matrix of the system. Such a density matrix is called the *reduced density matrix* [6]. Let $|\psi\rangle$ be the state vector of the system such that

$$\begin{aligned}
|\psi\rangle &= \sum_{l,m} a_{lm}|l(1), m(2)\rangle, \\
\widehat{\rho} &= \sum_{l,m,l',m'} a_{lm}a^*_{l'm'}|l,m\rangle\langle l',m'|,
\end{aligned}$$

then the expectation value of the measurement is equal to

$$\begin{aligned}
\langle \widehat{A}(1) \otimes \widehat{1}(2)\rangle &= \mathsf{Tr}\{\widehat{\rho}\widehat{A}(1) \otimes \widehat{1}(2)\} \\
&= \sum_{\mathsf{i,j}} \langle \mathsf{i,j}| \left(\sum_{l,m,l',m'} a_{lm} a^*_{l'm'} |l,m\rangle\langle l',m'|\widehat{A}(1) \otimes \widehat{1}(2) \right) |i,j\rangle \\
&= \sum_{i,j,l,m,l',m'} a_{lm} a^*_{l'm'} \langle \mathsf{i,j}|l,m\rangle\langle l',m'|\widehat{A}(1) \otimes \widehat{1}(2)|i,j\rangle \\
&= \sum_{l,m,l',m'} a_{lm} a^*_{l'm'} \langle l',m'|\widehat{A}(1) \otimes \widehat{1}(2)|l,m\rangle \\
&= \sum_{l,m,l',m'} a_{lm} a^*_{l'm'} \langle l'|\widehat{A}(1)|l\rangle \otimes \langle m'|\widehat{1}(2)|m\rangle \\
&= \sum_{l,m,l'} a_{lm} a^*_{l'm} \langle l'|\widehat{A}(1)|l\rangle \\
&= \mathsf{Tr}\{\widehat{\rho}_1 \widehat{A}(1)\}
\end{aligned}$$

where

$$\widehat{\rho}_1 = \sum_{l,l'} \left(\sum_m a_{lm} a^*_{l'm} \right) |l\rangle\langle l'|.$$

Here $\widehat{\rho}_1$ is called *reduced density matrix* and it is the partial trace of $\widehat{\rho}$ with respect to second node

$$\begin{aligned}
\widehat{\rho}_1 = \mathsf{Tr}_2\{\widehat{\rho}\} = \quad & \mathsf{Tr}_2 \left\{ \sum_{l,m,l',m'} a_{lm} a^*_{l'm'} |l,m\rangle\langle l',m'| \right\} \\
&= \sum_{l,m,l',m'} a_{lm} a^*_{l'm} |l\rangle\langle l'|\langle m'|m\rangle \\
&= \sum_{l,l'} (\widehat{\rho}_1)_{ll'} |l\rangle\langle l'|, \qquad (\widehat{\rho}_1)_{ll'} = \sum_m a_{lm} a^*_{l'm}
\end{aligned}$$

For a two node system, the reduced density matrices of an entangled state vector correspond to mixed states, while the reduced density matrices of a product state vector correspond to pure states.

To justify calling $\widehat{\rho}$ a reduced density matrix, we now prove that it is a density matrix.

Reduced density matrices are Hermitian.

$$\begin{aligned}
(\widehat{\rho}_1)^\dagger_{ll'} = (\widehat{\rho}_1)^*_{l'l} &= \left(\sum_m a_{l'm} a^*_{lm} \right)^* \\
&= \sum_m (a_{l'm} a^*_{lm})^* \\
&= \sum_m a_{lm} a^*_{l'm} = (\widehat{\rho}_1)_{ll'}
\end{aligned}$$

The trace of the reduced density matrix is equal to 1.

$$\begin{aligned}\mathsf{Tr}\{\widehat{\rho}_1\} &= \sum_j \langle j|\widehat{\rho}_1|j\rangle \\ &= \sum_j (\rho_1)_{jj} \\ &= \sum_{j,m} a_{jm}a^*_{jm} = 1\end{aligned}$$

Reduced density matrices are nonnegative. This means that for any state $|\phi\rangle$, $\langle\phi|\widehat{\rho}_1|\phi\rangle \geq 0$

$$\begin{aligned}\langle\phi|\widehat{\rho}_1|\phi\rangle &= \sum_{l,m,l'} a_{lm}a^*_{l'm}\langle\phi|l\rangle\langle l'|\phi\rangle \\ &= \sum_m \left|\sum_l a_{lm}\langle\phi|l\rangle\right|^2 \\ &\geq 0.\end{aligned} \tag{2.3}$$

Therefore, reduced density matrix is a density matrix that represents a subsystem for measurements acted on that subsystem, i.e. subject to the type of Hermitian operators $\widehat{A}(1)\otimes\widehat{1}(2)$:

$$\langle\widehat{A}(1)\otimes\widehat{1}(2)\rangle = \mathsf{Tr}\{\widehat{\rho}\widehat{A}(1)\otimes\widehat{1}(2)\} = \mathsf{Tr}\{\widehat{\rho}_1\widehat{A}(1)\}.$$

However, reduced density matrix does not represent the subsystem behavior for the measurements of the type of product Hermitian operators $\widehat{A}(1)\otimes\widehat{A}(2)$ or non-product type Hermitian operators $\widehat{A}(1,2)$ acted on the total system:

$$\langle\widehat{A}(1)\otimes\widehat{A}(2)\rangle \neq \mathsf{Tr}\{\widehat{\rho}_1\widehat{A}(1)\}\mathsf{Tr}\{\widehat{\rho}_2\widehat{A}(2)\}. \tag{2.4}$$

Also note that two different state vectors of a two node system given by $|\psi_1\rangle$, $|\psi_2\rangle$ may correspond to the same reduced density matrix for each node. For example, consider the two state vectors

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |2,2\rangle), \quad \text{and} \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|1,2\rangle + |2,1\rangle)$$

then

$$\widehat{\rho}_1 = \frac{1}{2}\begin{pmatrix} |1,1\rangle\langle 1,1| + |1,1\rangle\langle 2,2| \\ +|2,2\rangle\langle 1,1| + |2,2\rangle\langle 2,2| \end{pmatrix}, \widehat{\rho}_2 = \frac{1}{2}\begin{pmatrix} |1,2\rangle\langle 1,2| + |1,2\rangle\langle 2,1| \\ +|2,1\rangle\langle 1,2| + |2,1\rangle\langle 2,1| \end{pmatrix}$$

so that

$$\begin{aligned} \mathsf{Tr}_2\{\widehat{\rho}_1\} &= \frac{1}{2}\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right), & \mathsf{Tr}_2\{\widehat{\rho}_2\} &= \frac{1}{2}\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) \\ \mathsf{Tr}_1\{\widehat{\rho}_1\} &= \frac{1}{2}\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right), & \mathsf{Tr}_1\{\widehat{\rho}_2\} &= \frac{1}{2}\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) \end{aligned} \tag{2.5}$$

These states are obviously different from each other but they have the same reduced density matrix for each node. Therefore, it is clear that the partial trace operation deletes some crucial information from density matrix of the total system and it is not possible to get back the density matrix of a total system from the reduced density matrices of the two subsystems.

To understand the nature of the inequality 2.4 and the information loss in partial trace operation in 2.5, *Bloch vector* [8],[9] will be defined to analyze the properties of density matrix deeper.

Chapter 3

BLOCH VECTOR FORMALISM FOR ONE NODE SYSTEMS

3.1 Bloch Vector Representation of States

The density matrix space $\mathcal{L}_{+,1}(H_N)$ for N level systems is defined by

$$\mathcal{L}_{+,1}(H_N) := \{\widehat{\rho} \in \mathcal{L}(H_N) : (i)\mathsf{Tr}\{\widehat{\rho}\} = 1, \quad (ii)\rho_i \geq 0 \quad (i = 1, ..., N)\}, \tag{3.1}$$

where $\mathcal{L}(H_N)$ denotes the space of $N \times N$ Hermitian matrices and ρ_i is the i^{th} eigenvalue of the $\widehat{\rho}$. As one of the properties of the density matrix,

$$\mathsf{Tr}\{\widehat{\rho}^2\} \leq 1$$

follows from the criteria (i) and (ii).

To find a basis of this space we will review an orthogonal basis of $\mathsf{su(N)}$ consisting of a set of elements $\widehat{\lambda}_i \in \mathcal{L}(H_N)$ $(i = 1, ..., N^2 - 1)$ which are defined as [8]

$$\{\widehat{\lambda}_i\}_{i=1}^{N^2-1} = \{\widehat{u}_{jk}, \widehat{v}_{jk}, \widehat{w}_l\} \quad 1 \leq j < k \leq N, \quad 1 \leq l \leq N-1$$

where

$$\begin{aligned}
\widehat{u}_{jk} &= |j\rangle\langle k| + |k\rangle\langle j| \\
\widehat{v}_{jk} &= i(|j\rangle\langle k| - |k\rangle\langle j|) \quad \text{where } i = \sqrt{-1} \\
\widehat{w}_l &= \sqrt{\frac{2}{l(l+1)}} \left(\sum_{j=1}^{l} |j\rangle\langle j| - l|l+1\rangle\langle l+1| \right)
\end{aligned}$$

and satisfy

$$(a)\ \widehat{\lambda}_i^\dagger = \widehat{\lambda}_i, \quad (b)\ \mathsf{Tr}\{\widehat{\lambda}_i\} = 0, \quad (c)\ \mathsf{Tr}\{\widehat{\lambda}_i\widehat{\lambda}_j\} = 2\delta_{ij}.$$

The commutator and anti-commutator of the basis elements [9] are

$$\begin{aligned}
[\widehat{\lambda}_i, \widehat{\lambda}_j] &= 2i\sum_k f_{ijk}\widehat{\lambda}_k \\
[\widehat{\lambda}_i, \widehat{\lambda}_j]_+ &= \frac{4}{N}\delta_{ij}\widehat{1} + 2\sum_k d_{ijk}\widehat{\lambda}_k
\end{aligned}$$

where f_{ijk}, d_{ijk} are the structure constants of the particular basis. By using them, two useful vector products are defined as

$$(a \wedge b)_i \quad : \quad = \sum_{j,k} f_{ijk} a_j b_k \tag{3.2}$$

$$(a \star b)_i \quad : \quad = \sum_{j,k} d_{ijk} a_j b_k \tag{3.3}$$

In particular, these basis matrices $\widehat{\lambda}_i$ are equal to the Pauli matrices for $N = 2$ and Gell-Mann matrices for $N = 3$. However, for Gell-Mann matrices the labeling is not the standard one.

Note that this orthogonal basis of $\mathsf{su(N)}$ and identity matrix $\widehat{1}$ help to form the complete orthonormal basis $\{\widehat{Q}_i\}_{i=0}^{N^2-1}$ for the space of $N \times N$ Hermitian matrices $\mathcal{L}(H_N)$

$$\widehat{Q}_0 = \frac{1}{\sqrt{N}}\widehat{1}, \widehat{Q}_j = \frac{1}{\sqrt{2}}\widehat{\lambda}_j \quad (j = 1, ..., N^2 - 1) \tag{3.4}$$

in the sense of the Hilbert–Schmidt product

$$\mathsf{Tr}\{\widehat{Q}_i, \widehat{Q}_j\} = \delta_{ij}.$$

In this basis a general density matrix $\widehat{\rho}$ can be decomposed as

$$\begin{aligned} \widehat{\rho} &= \sum_{i=0}^{N^2-1} \mathsf{Tr}\{\widehat{\rho}\widehat{Q}_i\}\widehat{Q}_i \\ &= \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i \widehat{\lambda}_i \end{aligned} \tag{3.5}$$

where $\mathsf{Tr}\{\widehat{\rho}\widehat{\lambda}_i\} = \langle\widehat{\lambda}_i\rangle = \lambda_i \in \mathbf{R}$ and the density matrix can be represented by the real vector $\lambda = \{\lambda_i, i = 1, ..., N^2 - 1\}$ which is called *Bloch vector* . Note that λ_i is the expectation value of $\widehat{\lambda}_i$.

Since the complete orthonormal basis $\{\widehat{Q}_i\}_{i=0}^{N^2-1}$ is for the space of $N \times N$ Hermitian matrices $\mathcal{L}(H_N)$ and the space of density matrices $\mathcal{L}_{+,1}(H_N)$ is a subspace of it, not all $\lambda \in \mathbf{R}^{N^2-1}$ corresponds to a density matrix. The density matrix criteria (i) and (ii) in 3.1 bring some restrictions on λ to represent a density matrix.

3.1.1 Characterization of the Bloch vectors

It is clear that the matrix $\widehat{\rho}'$

$$\widehat{\rho}' = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i \widehat{\lambda}_i \text{ where } \lambda_i \in \mathbf{R} \tag{3.6}$$

is Hermitian and its trace is 1 since $\widehat{\lambda}_i$'s are Hermitian and traceless. Therefore, if its all eigenvalues are nonnegative $\rho'_i \geq 0$ $(i = 1, ..., N)$, it is a density matrix . For what value of λ, are all of its eigenvalues nonnegative?

Although a nonnegativity criterion was given in my previous studies [19] for three and four levels systems, it is better to follow Kimura's formulation [10] because of its applicability to N level systems.

Consider the characteristic equation of $\widehat{\rho}'$

$$\begin{aligned}
\det(\widehat{\rho}' - x\widehat{1}) &= 0 \\
&= \prod_{i=1}^{N}(x - \rho'_i) = 0 \text{ where } \rho'_i \in \mathbf{R} \ (i = 1, ..., N) \text{ since } \widehat{\rho}' \text{ is Hermitian} \\
&= \sum_{j=0}^{N}(-1)^j a_j(\lambda) x^{N-j} = 0 \quad (a_0(\lambda) = 1) \text{ where } a_j(\lambda) \in \mathbf{R} \ (i = 0, ..., N)
\end{aligned}$$

the following theorem [10] reads the necessary and sufficient condition for $\rho'_i \geq 0 (i = 1, ..., N)$.

Theorem 1 *Consider an algebraic equation of degree $N \geq 1$:*

$$\sum_{j=0}^{N}(-1)^j a_j x^{N-j} = \prod_{i=1}^{N}(x - x_i) = 0, \qquad (a_0 = 1)$$

which has only real roots $x_i \in \mathbf{R}$ $(i = 1, ..., N)$. The necessary and sufficient condition that all the roots x_i's to be non-negative is that all the coefficients a_i's are non-negative

$$x_i \geq 0 \ (i = 1, ..., N) \Leftrightarrow a_i \geq 0 \ (i = 1, ..., N)$$

Therefore for $a_j(\lambda) \geq 0$ $(i = 1, ..., N)$, $\widehat{\rho}'$ is a density matrix. That is the space of Bloch vectors for N level systems $\mathcal{B}_N(\lambda)$ is given by

$$\mathcal{B}_N(\lambda) := \{\lambda \in \mathbf{R}^{N^2-1} : a_j(\lambda) \geq 0 \ (j = 1, ..., N)\}. \tag{3.7}$$

Note also that the coefficients $a_j(\lambda)$ are given by Newton's formula [10]

$$k a_k(\lambda) = \sum_{q=1}^{k}(-1)^{q-1} C(N, q) a_{k-N} \quad (1 \leq k \leq N) \tag{3.8}$$

where $C(N,q)=\sum_i^N(\rho_i')^q=\mathsf{Tr}\{(\widehat{\rho}')^q\}$. Explicitly,

$$\begin{aligned}
a_1(\lambda) &= 1\\
a_2(\lambda) &= \frac{1}{2}(1-\mathsf{Tr}\{\widehat{\rho}^2\})\\
a_3(\lambda) &= \frac{1}{6}(1-3\mathsf{Tr}\{\widehat{\rho}^2\}+2\mathsf{Tr}\{\widehat{\rho}^3\})\\
a_4(\lambda) &= \frac{1}{24}(1-6\mathsf{Tr}\{\widehat{\rho}^2\}+8\mathsf{Tr}\{\widehat{\rho}^3\}+3(\mathsf{Tr}\{\widehat{\rho}^2\})^2-6\mathsf{Tr}\{\widehat{\rho}^4\})\\
a_5(\lambda) &= \dots
\end{aligned}$$

In appendix A.1, $C(N,q)$'s are explicitly calculated for the Bloch vector representation of $\widehat{\rho}'$. By using these results, the concrete expressions of coefficients $a_i(\lambda)$'s in 3.8 in terms of Bloch vector and the structure constants are given as

$$\begin{aligned}
a_1(\lambda) &= 1\\
2!a_2(\lambda) &= \frac{N-1}{N}-\frac{1}{2}|\lambda|^2\\
3!a_3(\lambda) &= \frac{(N-1)(N-2)}{N^2}-\frac{3(N-2)}{2N}|\lambda|^2+\frac{1}{2}\sum_{i,j,k}d_{ijk}\lambda_i\lambda_j\lambda_k\\
4!a_4(\lambda) &= \frac{(N-1)(N-2)(N-3)}{N^3}-\frac{3(N-2)(N-3)}{N^2}|\lambda|^2+\frac{3(N-2)}{4N}|\lambda|^4\\
&\quad +\frac{2(N-2)}{N}\sum_{i,j,k}d_{ijk}\lambda_i\lambda_j\lambda_k-\frac{3}{4}\sum_{i,j,k,m,l}d_{ijm}d_{mkl}\lambda_i\lambda_j\lambda_k\lambda_l\\
5!a_5(\lambda) &= \dots
\end{aligned}$$

Note that $a_i(\lambda)$'s are meaningful only for $i\leq N$.

Pure States and the Length of a Bloch Vector

Consider a density matrix $\widehat{\rho}$ in its Bloch vector representation

$$\widehat{\rho}=\frac{1}{N}\widehat{1}+\frac{1}{2}\sum_{i=1}^{N^2-1}\lambda_i\widehat{\lambda}_i\ .$$

Then the state is pure if $\widehat{\rho}^2=\widehat{\rho}$ i.e. $\mathsf{Tr}\{\widehat{\rho}^2\}=1$:

$$\begin{aligned}
\mathsf{Tr}\{\widehat{\rho}^2\} &= \frac{1}{N}+\frac{1}{2}|\lambda|^2=1,\\
|\lambda|^2 &= \frac{2(N-1)}{N}.
\end{aligned}$$

Note that for N level systems the *length* of the Bloch vector $|\lambda|$ is restricted by $a_2(\lambda)$ such that

$$\begin{aligned} a_2(\lambda) &\geq 0 \\ \frac{1}{2}\left(\frac{N-1}{N} - \frac{1}{2}|\lambda|^2\right) &\geq 0 \\ |\lambda|^2 &\leq \frac{2(N-1)}{N} \end{aligned}$$

Therefore the space $\mathcal{B}_N(\lambda)$ of Bloch vectors for N level systems is the subset of the ball $\mathbf{B}^{N^2-1}\left(\frac{2(N-1)}{N}\right)$ in the $\mathbf{R}^{N^2-1}$:

$$\mathcal{B}_N(\lambda) \subseteq \mathbf{B}^{N^2-1}\left(\frac{2(N-1)}{N}\right)$$

and the common points of $\mathcal{B}_N(\lambda)$ and $\mathbf{B}^{N^2-1}\left(\frac{2(N-1)}{N}\right)$ correspond to pure states.

Two Level Systems

For two level systems, the necessary and sufficient condition for $\widehat{\rho}'$ in 3.6 is

$$\begin{aligned} a_2(\lambda) &\geq 0 \\ \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}|\lambda|^2\right) &\geq 0 \\ |\lambda|^2 &\leq 1 \end{aligned}$$

and the space of Bloch vectors in 3.7 is given by

$$\mathcal{B}_2(\lambda) := \{\lambda \in \mathbf{R}^3 : |\lambda|^2 \leq 1\}.$$

Note that this is the definition of the famous *Bloch sphere* [6]. Since the whole surface consists of common points of the ball $\mathbf{B}^3(1)$, the surface of the sphere correspond to pure states while inside is mixed.

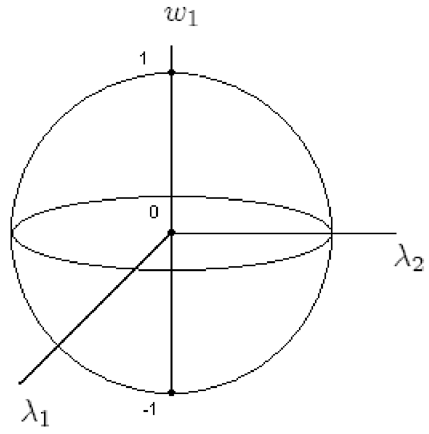

Figure 3.1: Bloch Sphere

Three and Four Level Systems

For three level systems, the necessary and sufficient conditions for $\widehat{\rho}'$ in 3.6 are

$$\begin{aligned} a_2(\lambda) &\geq 0, \quad |\lambda|^2 \leq \frac{4}{3} \\ a_3(\lambda) &\geq 0, \quad \frac{2}{9} - \frac{1}{2}|\lambda|^2 + \frac{1}{2}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k \geq 0 \end{aligned}$$

and the space of Bloch vectors in 3.7 is given by

$$\mathcal{B}_3(\lambda) := \{\lambda \in \mathbf{R}^8 : |\lambda|^2 \leq \frac{4}{3}, \quad \frac{2}{9} - \frac{1}{2}|\lambda|^2 + \frac{1}{2}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k \geq 0\}$$

For four level systems, the necessary and sufficient conditions for $\widehat{\rho}'$ given in 3.6 are

$$\begin{aligned} a_2(\lambda) &\geq 0, \quad |\lambda|^2 \leq \frac{3}{2} \\ a_3(\lambda) &\geq 0, \quad \frac{3}{8} - \frac{3}{4}|\lambda|^2 + \frac{1}{2}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k \geq 0 \\ a_4(\lambda) &\geq 0, \quad \frac{3}{32} - \frac{3}{8}|\lambda|^2 + \frac{3}{4}|\lambda|^4 + \sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k - \frac{3}{4}\sum_{i,j,k,m,l} d_{ijm}d_{mkl}\lambda_i\lambda_j\lambda_k\lambda_l \geq 0 \end{aligned}$$

and the space of Bloch vectors in 3.7 is given by

$$\begin{aligned} \mathcal{B}_4(\lambda) := \{\lambda \in \mathbf{R}^{15} : |\lambda|^2 \leq \frac{3}{2}, \frac{3}{8} - \frac{3}{4}|\lambda|^2 + \frac{1}{2}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k \geq 0, \\ \frac{3}{32} - \frac{3}{8}|\lambda|^2 + \frac{3}{4}|\lambda|^4 + \sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k - \frac{3}{4}\sum_{i,j,k,m,l} d_{ijm}d_{mkl}\lambda_i\lambda_j\lambda_k\lambda_l \geq 0\} \end{aligned}$$

3.1.2 Geometry of the Bloch Vectors

It is apparent that the geometry of the Bloch vector spaces for three and higher level systems is not trivial as that of two level systems. However, the geometry of the diagonalized states are relatively easy and the orbit of the diagonalized state space under the action of $\mathsf{SU(N)}$ will give the total Bloch vector space. For example, consider the N level case. The diagonalized state

$$\widehat{\rho}_D = \frac{1}{2}\widehat{1} + \frac{1}{2}\sum_{l=1}^{N-1} w_l \widehat{w}_l, \text{ with } \sum_{l=1}^{N-1} w_l^2 \leq \frac{2(N-1)}{N}$$

Then the space $\mathcal{L}_{+,1}(H_N)$ of density matrices is given by

$$\mathcal{L}_{+,1}(H_N) = \{\mathsf{U}\widehat{\rho}_D\mathsf{U}^\dagger : \mathsf{U} \in \mathsf{SU(N)}\}.$$

The corresponding effect of the unitary transformation U on a density matrix $\widehat{\rho}$ for the Bloch vector λ is a rotation around origin, since there is a homomorphism (*see* Appendix B) between $\mathsf{SU(N)}$ and $\mathsf{SO(N^2-1)}$. That is, the unitary transformation U of density matrix $\widehat{\rho}$

$$\mathsf{U}\widehat{\rho}\mathsf{U}^\dagger = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i \mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger$$

is equal to an application of $\mathsf{O} \in \mathsf{SO(N^2{-}1)}$ to λ and following equality holds:

$$\begin{aligned}
\sum_{i=1}^{N^2-1} \lambda_i \mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger &= \sum_{j=1}^{N^2-1}\Big(\sum_{i=1}^{N^2-1} \mathsf{O}_{ji}\lambda_i\Big)\widehat{\lambda}_j, \\
\sum_{i=1}^{N^2-1} \lambda_i \mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger\} &= \sum_{j=1}^{N^2-1}\Big(\sum_{i=1}^{N^2-1} \mathsf{O}_{ji}\lambda_i\Big)\mathsf{Tr}\{\widehat{\lambda}_m\widehat{\lambda}_j\}, \\
\sum_{i=1}^{N^2-1} \lambda_i \mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger\} &= 2\sum_{i=1}^{N^2-1} \mathsf{O}_{mi}\lambda_i.
\end{aligned}$$

The corresponding element O of $\mathsf{SO(N^2{-}1)}$ for the unitary transformation U is

$$\begin{aligned}
\frac{1}{2}\mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger\} &= \mathsf{O}_{mi} \\
\mathsf{Ad(U)} &= \mathsf{O}
\end{aligned} \tag{3.9}$$

For two level systems the diagonalized density matrix $\widehat{\rho}_D$ is of the form

$$\widehat{\rho}_D = \frac{1}{2}\widehat{1} + \frac{1}{2}w_1\widehat{w}_1, \text{ with } w_1^2 \leq 1$$

and the Bloch vector space corresponding to $\widehat{\rho}_D$ is

$$\mathcal{B}_2(w_1) := \{w_1 \in \mathbf{R} : w_1^2 \leq 1\}$$

where pure states are $w_1^2 = 1$ and mixed states are $w_1^2 < 1$. Since the homomorphism between $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ is onto, all elements of $\mathsf{SO}(3)$ can be applied to $\mathcal{B}_2(w_1)$ and the orbit of it is the Bloch sphere.

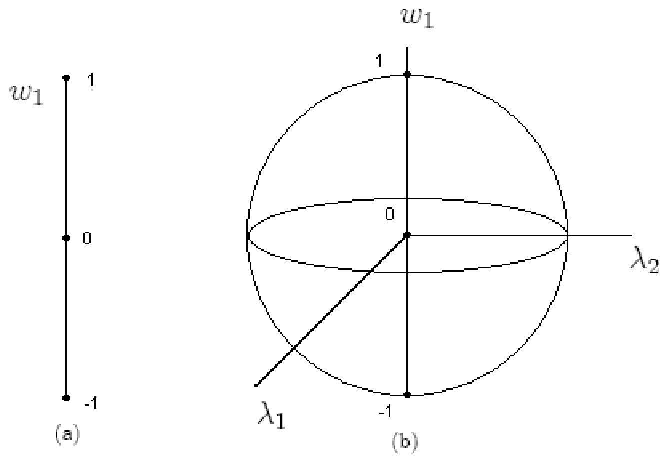

Figure 3.2: Interval of w_1 and Bloch Sphere

(a) For a particular basis in which density matrix is diagonal, the values $w_1 = \pm 1$ correspond to pure states, and the interval $-1 < w_1 < 1$ corresponds to mixed states.

(b) For any basis, pure states lie on the surface of the sphere $\lambda^2 = 1$ and mixed states lie inside the sphere $0 \leq \lambda^2 < 1$.

In the same manner, it is also possible to generate the three level and four level analogs of the Bloch sphere.

For three level system, when we diagonalize the density matrix $\widehat{\rho}_D$, it is written as

$$\widehat{\rho}_D = \frac{1}{3}\widehat{1} + \frac{1}{2}(w_1\widehat{w}_1 + w_2\widehat{w}_2)$$

where $\widehat{\lambda}_3 = \widehat{w}_1$ and $\widehat{\lambda}_8 = \widehat{w}_2$. In this case, the density matrix conditions

$$|\lambda|^2 \leq \frac{4}{3} \;\text{ and }\; \frac{2}{9} - \frac{1}{2}|\lambda|^2 + \frac{1}{2}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k \geq 0$$

reduce to

$$w_1^2 + w_2^2 \leq \frac{4}{3} \quad \text{and} \quad \frac{4}{9} - w_1^2 - w_2^2 - \frac{\sqrt{3}}{3} w_2^3 + \sqrt{3} w_1^2 w_2 \geq 0.$$

Their common area defines an equilateral triangle with vertices at $(w_1, w_2) = (1, \frac{1}{\sqrt{3}})$, $(-1, \frac{1}{\sqrt{3}})$ and $(0, -\frac{2}{\sqrt{3}})$.

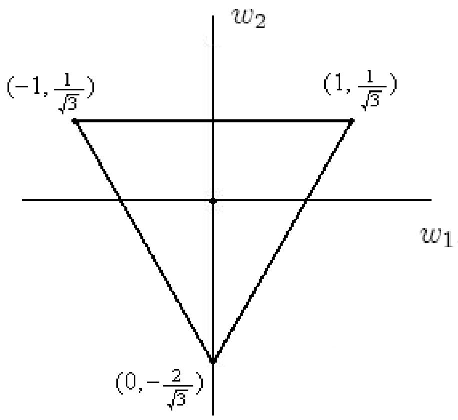

Figure 3.3: Geometry of Diagonalized Three Level Density Matrix

These vertices correspond to pure states and the points inside the triangle correspond to mixed states. Then the orbit of this triangle under $\mathsf{SU}(3)$ (i.e. adjoint representation of $\mathsf{SU}(3)$ given in 3.9) gives the generalization of Bloch sphere for three level systems. Although for all $\mathsf{O} \in \mathsf{SO}(3)$, there exists a $\mathsf{U} \in \mathsf{SU}(2)$ such that $\mathsf{O}=\mathsf{Ad}(\mathsf{U})$, $\mathsf{Ad}(\mathsf{SU}(3))$ is only an eight-parameter subgroup of twenty-eight-parameter group $\mathsf{SO}(8)$. Therefore not all the orthogonal rotations on real eight-dimensional space is applicable.

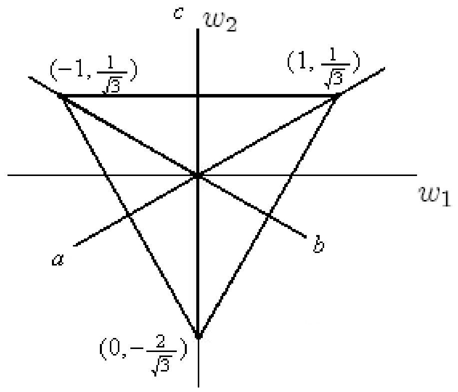

Figure 3.4: The Axes of Symmetry

This restriction is apparent when we consider the possible action of $\mathsf{SU}(3)$ on $w_1w_2-plane$. All possible rotation are $\frac{2\pi}{3}$ around origin of $w_1w_2-plane$ and π rotations around $a,b,c-axis$.

For four level systems, when we diagonalize the density matrix $\widehat{\rho}_D$, it is written as

$$\widehat{\rho}_D=\frac{1}{4}\widehat{1}+\frac{1}{2}(w_1\widehat{w}_1+w_2\widehat{w}_2+w_3\widehat{w}_3)$$

where $\widehat{\lambda}_3=\widehat{w}_1$, $\widehat{\lambda}_8=\widehat{w}_2$ and $\widehat{\lambda}_{15}=\widehat{w}_3$. In this case, density matrix conditions

$$|\lambda|^2\leq\frac{3}{2},\ \frac{3}{8}-\frac{3}{4}|\lambda|^2+\frac{1}{2}\sum_{i,j,k}d_{ijk}\lambda_i\lambda_j\lambda_k\geq\ 0$$

and

$$\frac{3}{32}-\frac{3}{8}|\lambda|^2+\frac{3}{4}|\lambda|^4+\sum_{i,j,k}d_{ijk}\lambda_i\lambda_j\lambda_k-\frac{3}{4}\sum_{i,j,k,m,l}d_{ijm}d_{mkl}\lambda_i\lambda_j\lambda_k\lambda_l\geq\ 0$$

reduce to

$$w_1^2+w_2^2+w_3^2\leq\frac{3}{2}$$

$$\frac{3}{4}-\frac{3}{2}(w_1^2+w_2^2+w_3^2)-\frac{\sqrt{3}}{3}w_2^3-\sqrt{3}w_1^2w_2+\frac{\sqrt{6}}{2}w_3(w_1^2+w_2^2-\frac{2}{3}w_3^2)\geq 0$$

and

$$\begin{aligned}
& \frac{3}{32} - \frac{3}{8}(w_1^2 + w_2^2 + w_3^2) + \frac{3}{4}(w_1^2 + w_2^2 + w_3^2)^2 \\
- \quad & \frac{\sqrt{3}}{3} w_2^3 - \sqrt{3} w_1^2 w_2 + \frac{\sqrt{6}}{2} w_3 (w_1^2 + w_2^2 - \frac{2}{3} w_3^2) \\
- \quad & \frac{3}{4}(\frac{1}{2} w_1^4 + \frac{1}{2} w_2^4 + \frac{2}{3} w_3^4 + 2\sqrt{2}(w_1^2 + \frac{1}{3} w_2^2) w_2 w_3 + w_1^2 w_2^2) \geq 0
\end{aligned}$$

Their common area defines a tetrahedron with vertices at $(w_1, w_2, w_3) = (1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}})$, $(-1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}})$, $(0, -\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}})$, and $(0, 0, -\frac{3}{\sqrt{6}})$.

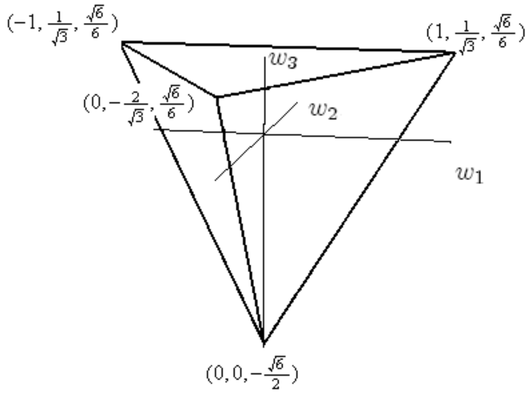

Figure 3.5: Geometry of Diagonalized Four Level Density Matrix

These vertices correspond to pure states and the points inside the tetrahedron correspond to mixed states . Then the orbit of this tetrahedron under $\mathsf{SU}(4)$ (i.e. adjoint representation of $\mathsf{SU}(4)$ given in 3.9) gives the generalization of Bloch sphere to four level systems.

General Case: $N \geq 2$ **Level Systems**

The coordinates of the pure states $(\rho_D)_{ii}$ for diagonalized density matrix representation $\widehat{\rho}_D$

for N level systems in $w_1 w_2 ... w_{N-1}-$coordinate system are

$$\begin{aligned}
(\rho_D)_{ii} &= \mathbf{w}^i = (w_1^i, w_2^i, w_3^i, ..., w_{N-1}^i) \\
(\rho_D)_{11} &= \mathbf{w}^1 = \left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, ..., \sqrt{\frac{2}{(N-1)N}}\right) \\
(\rho_D)_{22} &= \mathbf{w}^2 = \left(-1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, ..., \sqrt{\frac{2}{(N-1)N}}\right) \\
(\rho_D)_{33} &= \mathbf{w}^3 = \left(0, -\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}}, ..., \sqrt{\frac{2}{(N-1)N}}\right) \\
(\rho_D)_{44} &= \mathbf{w}^4 = \left(0, 0, -\frac{3}{\sqrt{6}}, ..., \sqrt{\frac{2}{(N-1)N}}\right) \\
& \quad\; .... \\
& \quad\; .... \\
(\rho_D)_{ii} &= \mathbf{w}^i = \left(0, 0, 0, ..., -\sqrt{\frac{2(i-1)}{i}}, ..., \sqrt{\frac{2}{(N-1)N}}\right) \\
& \quad\; .... \\
& \quad\; .... \\
(\rho_D)_{NN} &= \mathbf{w}^N = \left(0, 0, 0, ..., -\sqrt{\frac{2(N-1)}{N}}\right)
\end{aligned}$$

these points are the vertices of a *regular* $(N-1)$-*simplex* [11], i.e. generalized equilateral triangle to $(N-1)$ dimensional space or *hypertetrahedron* and it is denoted by $\triangle^{N-1}$.In one dimension, the simplex is the line segment $[-1, 1]$. In two dimensions, the 2-simplex is the convex hull of the equilateral triangle. In three dimensions, the 3-simplex is the convex hull of the tetrahedron. The following figures show the graphs for the n-simplexes from n=2 to 7. Note that the graph of an $n-$simplex is the *complete graph* of $n+1$ vertices.

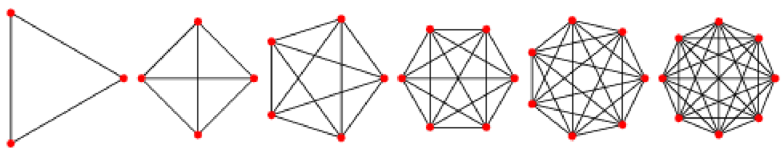

Figure 3.6: n-simplexes with n=2 to 7

Definition 2 *If $a_1, a_2, ..., a_{n+1}$ are $n+1$ points in $\mathbf{R}^n$ such that $a_2 - a_1, a_3 - a_1, ..., a_{n+1} - a_1$ are linearly independent, then the convex hull of these points*

$$\left\{ x| \quad x = \sum_{i=1}^{n+1} t_i a_i \text{ where } \sum_{i=1}^{n+1} t_i = 1 \text{ and } t_i \geq 0(\ i = 1, ..., n+1) \right\}$$

is an $\mathbf{n}$***-simplex*** *and it is* ***regular*** *if the distance between any two vertices is same. The points $a_1, a_2, ..., a_{n+1}$ are called the* ***vertices*** *of the n-simplex.*

Note that, $\mathbf{w}^i$'s are N points in $\mathbf{R}^{N-1}$ such that $\mathbf{w}^2 - \mathbf{w}^1, \mathbf{w}^3 - \mathbf{w}^1, ..., \mathbf{w}^N - \mathbf{w}^1$are linearly independent, and a general element $\overline{\mathbf{w}}$ from its convex hull

$$\begin{aligned} \overline{\mathbf{w}} &= \sum_{i=1}^{N} t_i \mathbf{w}^i \text{ where } \sum_{i=1}^{n} t_i = 1 \\ \overline{w}_l &= \sum_{i=1}^{N} t_i w_l^i \end{aligned}$$

represents a diagonalized density matrix $\widehat{\rho}_D$ since

$$\begin{aligned} \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{l=1}^{N-1} \overline{w}_l \ \widehat{w}_l &= \frac{1}{N}\widehat{1}\sum_{i=1}^{N} t_i + \frac{1}{2}\sum_{l=1}^{N-1}\left(\sum_{i=1}^{N} t_i w_l^i\right)\widehat{w}_l \\ &= \sum_{i=1}^{N} t_i \left(\frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{l=1}^{N-1} w_l^i \widehat{w}_l\right) \\ &= \sum_{i=1}^{N} t_i \widehat{\rho}_{ii} = \widehat{\rho}_D. \end{aligned}$$

By reversing the above equation every diagolized matrix is an element of the convex hull of $\mathbf{w}^i$'s, since every $\widehat{\rho}_D$ is written as

$$\widehat{\rho}_D = \sum_{i=1}^{N} t_i \widehat{\rho}_{ii} \text{ where } \sum_{i=1}^{n} t_i = 1.$$

Therefore, $\mathbf{w}^i$'s are the vertices of a n-simplex, and this n-simplex is also *regular* by the following argument. It is clear that the length of each $\mathbf{w}^i$ is the same since they are pure states and equal to

$$|\mathbf{w}^i| = \sqrt{\frac{2(N-1)}{N}}(\ i = 1, ..., N).$$

If we show that the angle between any two $\mathbf{w}^i$ is same then the distance between any two vertices is same.

Note that there is a 1-to-1 correspondence between $(\rho_D)_{ii}$ and $\mathbf{w}^i$ for ($i = 1, ..., N$). Now, consider the unitary matrix $\mathsf{U} \in \mathsf{SU(N)}$ in the block form

$$\mathsf{U} = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \widehat{1} \end{pmatrix}$$

when we apply U to all $(\rho_D)_{ii}$, only $(\rho_D)_{11}$ and $(\rho_D)_{22}$ are transformed into each other, while nothing happens to the other $(\rho_D)_{ii}$'s. Note that, the corresponding action of U on Bloch vector $\mathbf{w}^i$, is similar i.e. only $\mathbf{w}^1$ and $\mathbf{w}^2$ are transformed into each other, while nothing happens to the other $\mathbf{w}^i$'s. Since this action is the adjoint representation O of U as given in 3.9and it is an orthogonal transformation, the length of each $\mathbf{w}^i$ and the inner product between them remains the same:

$$\langle \mathbf{w}^i | \mathbf{w}^j \rangle = \langle \mathsf{O}\mathbf{w}^i | \mathsf{O}\mathbf{w}^j \rangle \quad (i, j = 1, ..., N).$$

Therefore, the angle $\angle(\mathbf{w}^1, \mathbf{w}^i)$ between $\mathbf{w}^1$ and any $\mathbf{w}^i$ are the same as the angle $\angle(\mathbf{w}^2, \mathbf{w}^i)$ between $\mathbf{w}^2$ and $\mathbf{w}^i$:

$$\angle(\mathbf{w}^1, \mathbf{w}^i) = \angle(\mathbf{w}^2, \mathbf{w}^i) \quad (i = 1, ..., N)$$

Without loss of generality, it is possible to find a unitary matrix $\mathsf{U} \in \mathsf{SU(N)}$ such that any $\mathbf{w}^m$ and $\mathbf{w}^n$ transforms into each other, while nothing happens to the other $\mathbf{w}^i$'s. Then

$$\angle(\mathbf{w}^m, \mathbf{w}^i) = \angle(\mathbf{w}^n, \mathbf{w}^i) \quad (i, n, m = 1, ..., N)$$

so the angles between any $\mathbf{w}^i$ is the same. Therefore, n-simplex is *regular.*

3.2 Bloch Vector Formalism of Measurements

In the complete orthonormal basis $\{\widehat{Q}_i\}_{i=0}^{N^2-1}$, introduced in 3.4, any $N \times N$ Hermitian matrix $\widehat{A}$ can be written as [9]

$$\begin{aligned} \widehat{A} &= \sum_{i=0}^{N^2-1} \mathsf{Tr}\{\widehat{A}\widehat{Q}_i\}\widehat{Q}_i \\ &= \frac{1}{N}\mathsf{Tr}\{\widehat{A}\}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} A_i \widehat{\lambda}_i \end{aligned}$$

where $A_i = \mathsf{Tr}\{\widehat{A}\widehat{\lambda}_i\}$. Therefore, for a density matrix $\widehat{\rho}$ of an N-level system

$$\widehat{\rho} = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1}\lambda_i\widehat{\lambda}_i$$

the expectation value of the Hermitian operator $\widehat{A}$ is given by 2.1

$$\langle\widehat{A}\rangle = \mathsf{Tr}\{\widehat{\rho}\widehat{A}\}$$

By combining the Bloch vector representations of $\widehat{\rho}$ and $\widehat{A}$, one obtains the expression

$$\langle\widehat{A}\rangle = \frac{1}{N}\mathsf{Tr}\{\widehat{A}\} + \frac{1}{2}\sum_{i=1}^{N^2-1} A_i\lambda_i \tag{3.10}$$

Let $|a_i\rangle$ $(i = 1, ..., N)$ be eigenvectors of $\widehat{A}$ with the eigenvalues a_i and the probability $\Pr(a_i)$ of getting a_i from the measurement of $\widehat{A}$ for a density matrix $\widehat{\rho}$ is

$$\Pr(a_i) = \langle\widehat{P}_{a_i}\rangle \tag{3.11}$$

where $\widehat{P}_{a_i}$ is projection operator for the eigenvectors of $\widehat{A}$ defined as $\widehat{P}_{a_i} = |a_i\rangle\langle a_i|$. Its Bloch vector representation is

$$\widehat{P}_{a_i} = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{j=1}^{N^2-1}\lambda_j^{a_i}\ \widehat{\lambda}_j \text{ where } \lambda_j^{a_i} = \mathsf{Tr}\{\widehat{P}_{a_i}\widehat{\lambda}_j\} \tag{3.12}$$

since $\mathsf{Tr}\{\widehat{P}_{a_i}\} = 1$. Combining 3.11 and 3.12, the probability $\Pr(a_i)$ can be expressed as

$$\Pr(a_i) = \frac{1}{N} + \frac{1}{2}\sum_{j=1}^{N^2-1}\lambda_j^{a_i}\lambda_j.$$

Restricting ourselves to the projections $\widehat{P}_i = |i\rangle\langle i|$, i.e. the projections onto the basis vectors of the representation of the generators $\widehat{\lambda}_j$ of $\mathsf{SU(N)}$, the Bloch vector representation of $\widehat{P}_i$ is

$$\widehat{P}_i = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{c=1}^{N-1} w_c^i\ \widehat{w}_c \text{ where } w_c^i = \mathsf{Tr}\{\widehat{P}_i\widehat{w}_c\}$$

and the probability of getting state $|i\rangle$ after measurement

$$\Pr(|i\rangle) = \langle\widehat{P}_i\rangle = \frac{1}{N} + \frac{1}{2}\sum_{c=1}^{N-1} w_c^i\ \widehat{w}_c$$

3.3 Dynamics of Bloch Vector

Consider the Liouville equation

$$i\hbar \frac{d}{dt}\widehat{\rho}(t) = [\widehat{H}(t), \widehat{\rho}(t)].$$

When $\widehat{\rho}(t)$ is replaced by its Bloch vector representation

$$\widehat{\rho}(t) = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i(t)\widehat{\lambda}_i,$$

the following equation is obtained

$$i\hbar \frac{1}{2}\sum_{i=1}^{N^2-1} \frac{d\lambda_i(t)}{dt}\widehat{\lambda}_i = \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i(t)\ [\widehat{H}(t), \widehat{\lambda}_i]$$

Multiplying both sides by $\widehat{\lambda}_j$ and taking trace of both sides, we get

$$i\hbar \frac{1}{2}\sum_{i=1}^{N^2-1} \frac{d\lambda_i(t)}{dt}\mathsf{Tr}\{\widehat{\lambda}_i\widehat{\lambda}_j\} = \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i(t)\mathsf{Tr}\{[\widehat{H}(t), \widehat{\lambda}_i]\widehat{\lambda}_j\}.$$

By using the trace relation $\mathsf{Tr}\{\widehat{\mathsf{A}}[\widehat{\mathsf{B}}, \widehat{\mathsf{C}}]\} = \mathsf{Tr}\{\widehat{\mathsf{B}}[\widehat{\mathsf{C}}, \widehat{\mathsf{A}}]\} = \mathsf{Tr}\{\widehat{\mathsf{C}}[\widehat{\mathsf{A}}, \widehat{\mathsf{B}}]\}$ in cyclic permutation, we obtain

$$\begin{aligned} i\hbar \frac{d\lambda_j(t)}{dt} &= \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i(t)\mathsf{Tr}\{\widehat{H}(t)[\widehat{\lambda}_i, \widehat{\lambda}_j]\} \\ i\hbar \frac{d\lambda_j(t)}{dt} &= i\sum_{i,k=1}^{N^2-1} \lambda_i(t) f_{ijk}\mathsf{Tr}\{\widehat{H}(t)\widehat{\lambda}_k\} \\ \hbar \frac{d\lambda_j(t)}{dt} &= \sum_{i,k=1}^{N^2-1} \lambda_i(t) f_{ijk} H_k(t) \\ \frac{d\lambda_j(t)}{dt} &= \frac{1}{\hbar}\sum_{i,k=1}^{N^2-1} \lambda_i(t) f_{ijk} H_k(t) \end{aligned}$$

Changing the dummy variables we get the product defined in 3.2:

$$\begin{aligned} \frac{d\lambda_i(t)}{dt} &= \frac{1}{\hbar}\sum_{i,k=1}^{N^2-1} f_{ijk} H_j(t)\lambda_k(t) \\ \frac{d\lambda_i(t)}{dt} &= \frac{1}{\hbar}(\mathbf{H}(t) \wedge \lambda(t))_i. \end{aligned} \tag{3.13}$$

For two level systems, the structure constants f_{ijk} are identical with the elements of the totally antisymmetric ϵ tensor, and the 3.13 reduces to the well known *Bloch equation*

$$\frac{d}{dt}\lambda(t) = \frac{1}{\hbar}(\mathbf{H}(t) \times \lambda(t))_i.$$

Chapter 4

BLOCH VECTOR FORMALISM FOR TWO NODE SYSTEMS

4.1 Bloch Vector Representation of States

The density matrix space of the composite systems $\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2})$ is the tensor product of each density matrix space $\mathcal{L}^v_{+,1}(H_{Nv})$ $(v = 1, 2)$ for N_v level subsystems and it is also equal to the density matrix space of N_1N_2 levels system which is given by

$$\begin{aligned}\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2}) &:= \mathcal{L}^1_{+,1}(H_{N_1}) \otimes \mathcal{L}^2_{+,1}(H_{N_2}) = \mathcal{L}_{+,1}(H_{N_1N_2}) \\ &= \{\widehat{\rho} \in \mathcal{L}(H_{N_1N_2}) : (i)\ \mathsf{Tr}\{\widehat{\rho}\}= 1, \\ &\qquad (ii)\ \rho_i \geq 0 \quad (i = 1, ..., N_1N_2)\}\end{aligned}$$

where $\mathcal{L}(H_{N_1N_2})$ is the space of the $N_1N_2 \times N_1N_2$ Hermitian matrices. The orthonormal basis $\left\{\widehat{Q}_{jk}\right\}_{j,k=0}^{s_1,s_2}$ $(s_1 = N_1^2 - 1, s_2 = N_2^2 - 1)$ for $\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2})$ is the tensor product of the orthonormal bases $\left\{\frac{1}{\sqrt{N_v}}\widehat{1}(v), \frac{1}{\sqrt{2}}\widehat{\lambda}_j(v)\right\}$ of each subsystem $\mathcal{L}^v_{+,1}(H_{Nv})$ $(v = 1, 2)$

$$\begin{aligned}\widehat{Q}_{00} &= \frac{1}{\sqrt{N_1N_2}}\widehat{1}, \\ \widehat{Q}_{j0} &= \frac{1}{\sqrt{2}}\widehat{\lambda}_j(1) \otimes \frac{1}{\sqrt{N_2}}\widehat{1}(2), \\ \widehat{Q}_{0k} &= \frac{1}{\sqrt{N_1}}\widehat{1}(1) \otimes \frac{1}{\sqrt{2}}\widehat{\lambda}_k(2), \\ \widehat{Q}_{jk} &= \frac{1}{2}\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2).\end{aligned} \tag{4.1}$$

It is orthonormal in the sense of Hilbert-Schmidt product

$$\mathsf{Tr}\left\{\widehat{Q}_{jk}\widehat{Q}_{j'k'}\right\} = \delta_{jj'}\delta_{kk'}.$$

Therefore, a density matrix $\widehat{\rho} \in \mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2})$ can be represented by [9]

$$\begin{aligned}
\widehat{\rho} &= \sum_{j,k=0}^{s_1,s_2} \mathsf{Tr}\left\{\widehat{\rho}\widehat{Q}_{jk}\right\}\widehat{Q}_{jk} \\
&= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\
&\quad + \frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]
\end{aligned} \tag{4.2}$$

where $\lambda_j(1), \lambda_k(2)$ and $\lambda_{jk}(1,2)$ are

$$\begin{aligned}
\lambda_j(1) &= \mathsf{Tr}\left\{\widehat{\rho}\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\right\} = \langle\widehat{\lambda}_j(1)\rangle, \\
\lambda_k(2) &= \mathsf{Tr}\left\{\widehat{\rho}\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\right\} = \langle\widehat{\lambda}_k(2)\rangle, \\
\lambda_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{\rho}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\right\} = \langle\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\rangle.
\end{aligned}$$

The corresponding Bloch vectors can be represented by $\lambda(v)= \{\lambda_1(v), \lambda_2(v), ..., \lambda_{s_v}(v)\}$ $(v = 1,2)$. The tensor $\lambda_{jk}(1,2)$ which describes the relative phases between nodes is called *coherence tensor*. For a product state, coherence tensor is equal to

$$\lambda_{jk}(1,2) = \lambda_j(1)\lambda_k(2)$$

and it is convenient to introduce the *correlation* tensor $C_{jk}(1,2)$ by

$$C_{jk}(1,2) = \lambda_{jk}(1,2) - \lambda_j(1)\lambda_k(2). \tag{4.3}$$

It is named that way because it gives rise to correlations in particular measurements, which will be discussed in the measurement section in detail.

Note that each Bloch vector $\lambda(v)$ represents the corresponding reduced density matrix $\widehat{\rho}_v$ as

$$\begin{aligned}
\lambda(1) &\longleftrightarrow \widehat{\rho}_1 = \mathsf{Tr}_2\{\widehat{\rho}\} \\
\lambda(2) &\longleftrightarrow \widehat{\rho}_2 = \mathsf{Tr}_1\{\widehat{\rho}\}
\end{aligned}$$

Also note that, it is mentioned in 2.5 that two different state vectors of a two node system may give the same reduced density matrices for each node and the partial trace operation deletes some crucial information from the density matrix of the total system so that it is not possible to regain the density matrix of the total system from reduced density matrices of the two subsystems. Now it is clear that the deleted information is the coherence tensor $\lambda_{jk}(1,2)$, which characterizes the relative phases between the reduced density matrices of each node, and it does not affect the result of measurement performed only on a subsystem i.e. the measurement of the type $\widehat{A}(1)\otimes\widehat{1}(2)$ or $\widehat{1}(1)\otimes\widehat{A}(2)$, as it will be analyzed in detail in Section 4.2.

4.1.1 Characterization of Bloch Vectors and Coherence Tensor

Since the complete orthonormal basis $\left\{\widehat{Q}_{jk}\right\}_{j,k=0}^{s_1,s_2}$ is for the space of $N_1N_2\times N_1N_2$ Hermitian matrices $\mathcal{L}(H_{N_1N_2})$ and the density matrix space of the composite systems $\mathcal{L}_{+,1}(H_{N_1}\otimes H_{N_2})$ is a subspace of it, not all $\lambda(1)$, $\lambda(2)$ and $\lambda_{jk}(1,2)$ correspond to a density matrix just as the case for one node systems. The nonnegativity requirement of being a density matrix can be again satisfied by the treatment followed for one node systems, since the density matrix space of a two node system $\mathcal{L}_{+,1}(H_{N_1}\otimes H_{N_2})$ is equal to density matrix space of a one node system $\mathcal{L}_{+,1}(H_{N_1N_2})$. However, this time we use the basis of two node system$\left\{\widehat{Q}_{jk}\right\}_{j,k=0}^{s_1,s_2}$ $(s_1=N_1^2-1, s_2=N_2^2-1)$ instead of the basis of one node system $\left\{\widehat{Q}_i\right\}_{i=0}^{(N_1N_2)^2-1}$.

Recall that

$$
\begin{aligned}
\widehat{\rho}' &= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2) \\
&+\frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\
&+\frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] \\
&+\frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]
\end{aligned}
$$

is a nonnegative matrix if and only if all a_i's given by 3.8 are nonnegative:

$$
ka_k(\lambda)=\sum_{q=1}^{k}(-1)^{q-1}C(N,q)a_{k-N_1N_2} \quad (1\le k\le N_1N_2).
$$

Explicitly, we have

$$
\begin{aligned}
a_1(\lambda) &= 1 \\
a_2(\lambda) &= \frac{1}{2}(1 - \mathsf{Tr}\{\widehat{\rho}^2\}) \\
a_3(\lambda) &= \frac{1}{6}(1 - 3\mathsf{Tr}\{\widehat{\rho}^2\} + 2\mathsf{Tr}\{\widehat{\rho}^3\}) \\
a_4(\lambda) &= \frac{1}{24}(1 - 6\mathsf{Tr}\{\widehat{\rho}^2\} + 8\mathsf{Tr}\{\widehat{\rho}^3\} + 3(\mathsf{Tr}\{\widehat{\rho}^2\})^2 - 6\mathsf{Tr}\{\widehat{\rho}^4\}) \\
a_5(\lambda) &= ...
\end{aligned}
$$

In appendix A.2, $C(N,q)$'s are explicitly calculated from the Bloch vector representation 4.2, and the concrete expressions of coefficients $a_i(\lambda)$'s are given as

$$
\begin{aligned}
a_1(\lambda) &= 1 \qquad (4.4) \\
2a_2(\lambda) &= \frac{N_1N_2 - 1}{N_1N_2} - \frac{1}{2N_2}|\lambda(1)|^2 - \frac{1}{2N_1}|\lambda(2)|^2 - \frac{1}{4}|\lambda(1,2)|^2 \\
3!a_3(\lambda) &= \frac{(N_1N_2)^2 - 2N_1N_2 + 3}{(N_1N_2)^2} \\
&\quad + \frac{3}{2N_1N_2^2}|\lambda(1)|^2 + \frac{3}{2N_1^2N_2}|\lambda(2)|^2 \\
&\quad + \frac{3}{4N_1N_2}|\lambda(1,2)|^2 \\
&\quad + \frac{1}{2N_2^2}\sum_{i,j,k} d_{ijk}(1)\lambda_i(1)\lambda_j(1)\lambda_k(1) \\
&\quad + \frac{1}{2N_1^2}\sum_{i,j,k} d_{ijk}(2)\lambda_i(2)\lambda_j(2)\lambda_k(2) \\
&\quad + \frac{3}{N_1N_2}\sum_{j,k=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_k(2)\lambda_j(1) \\
&\quad + \frac{3}{4N_2}\sum_{j,k,i,m=1}^{s_1,s_2} d_{ijm}(1)\lambda_{jk}(1,2)\lambda_i(1)\lambda_{mk}(1,2) \\
&\quad + \frac{3}{4N_1}\sum_{j,k,i,m=1}^{s_1,s_2} d_{kim}(2)\lambda_{jk}(1,2)\lambda_i(2)\lambda_{jm}(1,2) \\
&\quad + \frac{1}{8}\sum_{j,k,m,N,p,r=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_{mN}(1,2)\lambda_{pr}(1,2)d_{jmp}(1)d_{kNr}(2) \\
4!a_4(\lambda) &= ....
\end{aligned}
$$

note that $a_i(\lambda)$'s are meaningful only for $i \leq N_1N_2$.

It is clear that these inequalities ($a_i(\lambda) \geq 0$, $1 \leq i \leq N_1N_2$) are not easy to bring a concrete characterization of the Bloch vectors $\lambda(v)(v = 1,2)$ and the coherence tensor

$\lambda_{jk}(1,2)(j = 1,...,s_1, k = 1,...,s_2)$. However, each reduced density matrix $\widehat{\rho}'_v$, obtained from $\widehat{\rho}'$, have to be nonnegative, so its Bloch vector $\lambda(v)$ have to satisfy the Bloch vector characterization of one node systems.

$$\begin{aligned}
a_1(\lambda(v)) &= 1 \qquad (4.5)\\
2!a_2(\lambda(v)) &= \frac{N-1}{N} - \frac{1}{2}|\lambda(v)|^2\\
3!a_3(\lambda(v)) &= \frac{(N-1)(N-2)}{N^2} - \frac{3(N-2)}{2N}|\lambda(v)|^2 + \frac{1}{2}\sum_{i,j,k} d_{ijk}(v)\lambda_i(v)\lambda_j(v)\lambda_k(v)\\
4!a_4(\lambda(v)) &= \frac{(N-1)(N-2)(N-3)}{N^3} - \frac{3(N-2)(N-3)}{N^2}|\lambda(v)|^2 + \frac{3(N-2)}{4N}|\lambda(v)|^4\\
&+\frac{2(N-2)}{N}\sum_{i,j,k} d_{ijk}(v)\lambda_i(v)\lambda_j(v)\lambda_k(v)\\
&-\frac{3}{4}\sum_{i,j,k,m} d_{ijm}(v)d_{mkl}(v)\lambda_i(v)\lambda_j(v)\lambda_k(v)\lambda_l(v)
\end{aligned}$$

note that here $a_i(\lambda(v))$'s are meaningful only for $i \leq N_v$. However, it is not necessary to consider the nonnegativity conditions of 4.5 in addition to the nonnegativity conditions of 4.4, since a reduced density matrix $\widehat{\rho}'_v$ obtained from a density matrix $\widehat{\rho}'$ have to be nonnegative as proved in 2.3, and for every density matrix $\widehat{\rho}'_v$ satisfying nonnegativity conditions of 4.5, there exist a density matrix of $\widehat{\rho}' = \widehat{1}\otimes\widehat{\rho}_v$ which satisfies the nonnegativity conditions of 4.4. Therefore, the nonnegativity conditions of 4.5 are already embedded in the nonnegativity conditions of 4.4.

4.1.2 Classification of States

Pure states vs. Mixed States

Recall that a state $\widehat{\rho}$ is pure if $\widehat{\rho}^2 = \widehat{\rho}$ i.e. $\mathsf{Tr}\{\widehat{\rho}^2\} = 1$. In appendix A.2, it has been shown that

$$\mathsf{Tr}\{\widehat{\rho}^2\} = \frac{1}{N_1N_2} + \frac{1}{2N_2}|\lambda(1)|^2 + \frac{1}{2N_1}|\lambda(2)|^2 + \frac{1}{4}|\lambda(1,2)|^2$$

therefore a state is pure if and only if

$$\frac{1}{N_2}|\lambda(1)|^2 + \frac{1}{N_1}|\lambda(2)|^2 + \frac{1}{2}|\lambda(1,2)|^2 = \frac{2(N_1N_2-1)}{N_1N_2}.$$

A state which is not pure is a mixed state.

Product States

A density matrix of a composite system $\widehat{\rho}$ is called product if

$$\widehat{\rho} = \widehat{\rho}_1 \otimes \widehat{\rho}_2$$

where $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are reduced density matrices of each subsystem. Note that, if $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are pure then $\widehat{\rho}$ is pure. Otherwise $\widehat{\rho}$ is mixed since $\mathsf{Tr}\{\widehat{\rho}^2\} = \mathsf{Tr}\{\widehat{\rho}_1\}\mathsf{Tr}\{\widehat{\rho}_2\} = 1$ if and only if $\mathsf{Tr}\{\widehat{\rho}_1\} = \mathsf{Tr}\{\widehat{\rho}_2\} = 1$ and that is the case when both $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are pure.

The product states are completely independent states even if the total system is mixed which corresponds to the case at least one of the subsystems has mixed density matrix, there is neither classical nor quantum correlation. For this reason, they are also called *uncorrelated states.*

Nonproduct States: Separable and Entangled

There are two types of non-product states: separable and non-separable. Separable quantum states citewerner are states without quantum correlation (i.e. entanglement) and given as

$$\widehat{\rho}_{sep} = \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i \qquad \sum_i p_i = 1$$

where $\widehat{\rho}(1)_i$ and $\widehat{\rho}(2)_i$ can be pure or mixed density matrices of each subsystem, and p_i's are the weights of $\widehat{\rho}(1)_i$ and $\widehat{\rho}(2)_i$ in the mixture. The subsystems of a separable state can be produced separately by the different parties whereas this is not possible for non-separable states. *The non-separable states are called entangled states.*

Consider two parties who can communicate with each other while preparing $\widehat{\rho}_{sep}$. First they prepare $\widehat{\rho}(1)_1 \otimes \widehat{\rho}(2)_1$ with probability p_1 then keep it in the box 1, then prepare $\widehat{\rho}(1)_2 \otimes \widehat{\rho}(2)_2$ with probability p_2 then keep it in the box 2, and so on... It is not necessary to keep them in boxes but $\widehat{\rho}(1)_i$ and $\widehat{\rho}(1)_j$ should not mix each other, i.e. they should be remain distinguishable. Then the mathematical form of the final state will be

$$\widehat{\rho}_{sep} = \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i \ . \tag{4.6}$$

Consider a measurement $\widehat{P}_{\widehat{\rho}(1)_1}(1) \otimes \widehat{1}(2)$ on the system which selects only $\widehat{\rho}(1)_1$ in the first node and nothing happens in the second node. The state $\widehat{\rho}'$ after the measurement is

$$\widehat{\rho}' = \widehat{\rho}(1)_1 \otimes \widehat{\rho}(2)_1$$

which means measuring process on node 1 attains a value on node 2. However, note that this value replacement is only possible if the observer of the second party removes all of the states except $\widehat{\rho}(2)_1$.

The correlations of separable states are called classical correlations, Werner [12] comments on this states as follows. "...there are usually very different ways of preparing the same state $\widehat{\rho}_{sep}$, *classical correlation* does not mean that the state has actually been prepared in the manner described, but only that its *statistical properties can be reproduced by a classical mechanism.*" For example, they can be obtained by partial trace operation performed on higher level systems. Suppose we consider a GHZ state

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle) \\ \widehat{\rho} &= |GHZ\rangle\langle GHZ| \end{aligned}$$

The partial trace operation performed on the third node gives a separable density matrix

$$\begin{aligned} \mathsf{Tr}_3\{\widehat{\rho}\} &= \frac{1}{2}|0,0\rangle\langle 0,0| + \frac{1}{2}|1,1\rangle\langle 1,1| \\ &= \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|. \end{aligned}$$

Bruss [5] comments on separability problem as "Finding a decomposition as in 4.6 for a given $\widehat{\rho}$, or proving that it does not exist, is a non-trivial task which has been solved explicitly only for a few cases. Therefore this simple-looking definition of separability is by no means "user-friendly", and we are in demand of criteria that are easier to test." One of the tests used in determining whether a give mixed state is separable or not is the "Peres-Horodecki separability criterion" [13],[14].

The Peres-Horodecki Separability Criterion

Consider a separable density matrix $\widehat{\rho}$

$$\begin{aligned} \widehat{\rho} &= \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i \\ \rho_{m\mu,n\nu} &= \sum_i p_i (\widehat{\rho}(1)_i)_{mn} (\widehat{\rho}(2)_i)_{\mu\nu} \end{aligned}$$

where Latin indices refer to the first subsystem, Greek indices to the second one. Now, let us define a new matrix

$$\sigma_{n\mu,m\nu} = \rho_{m\mu,n\nu}$$

the Latin indices of ρ have been transposed, but not the Greek ones. This operation is called *partial transposition* and it is not a unitary transformation but, nevertheless, the $\widehat{\sigma}$ matrix is Hermitian and, we have

$$\widehat{\sigma} = \sum_i p_i (\widehat{\rho}(1)_i)^T \otimes \widehat{\rho}(2)_i$$

since the transposed matrices $(\widehat{\rho}(1)_i)^T = (\widehat{\rho}(1)_i)^*$ are nonnegative matrices with unit trace, $\widehat{\sigma}$ is also a legitimate density matrix.

Also, Peres [13] noted that when applied to entangled states, the partial transposition need not result in a physical state. This is related to the fact that entangled states do not admit a description in which each subsystem's state is specified without reference to other subsystems, as is the case with separable states. The partial transposition of entangled states can result in negative operators, i.e. operators with negative eigenvalues. Therefore, nonnegativity under partial transposition (PPT) is a necessary condition for being separable.

On the other hand, the question of whether the condition is also sufficient to guarantee separability is answered by Horodecki [14] by proving that it is sufficient only for systems of dimensionality 2×2 and 2×3.

The Peres-Horodecki Separability Criterion in Bloch Vector Representation

Consider the density matrices,

$$\begin{aligned}\widehat{\rho}(1)_i &= \frac{1}{N_1}\widehat{1}(1) + \frac{1}{2}\sum_{j=1}^{s_1} \lambda_j(1)_i \widehat{\lambda}_j(1),\\ \widehat{\rho}(2)_i &= \frac{1}{N_2}\widehat{1}(2) + \frac{1}{2}\sum_{k=1}^{s_2} \lambda_k(2)_i \widehat{\lambda}_k(2).\end{aligned}$$

Their tensor product is

$$\begin{aligned}\widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i &= \frac{1}{N_1 N_2}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)_i[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)]\\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)_i[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)]\\ &+\frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_j(1)_i\lambda_k(2)_i[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned}$$

and their mixture is

$$\begin{aligned}\sum_i p_i\widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i &= \frac{1}{N_1 N_2}\widehat{1} + \frac{1}{2N_2}\sum_{j=1}^{s_1}\left(\sum_i p_i\lambda_j(1)_i\right)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)]\\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2}\left(\sum_i p_i\lambda_k(2)_i\right)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)]\\ &+\frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\left(\sum_i p_i\lambda_j(1)_i\lambda_k(2)_i\right)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)].\end{aligned}$$

A density matrix $\widehat{\rho}$ of an N_1N_2-levels system

$$\begin{aligned}\widehat{\rho} &= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2)+\frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)]\\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)]+\frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)].\end{aligned}$$

is separable if it satisfies the following equations

$$\begin{aligned}\lambda_j(1) &= \sum_i p_i\lambda_j(1)_i\\ \lambda_k(2) &= \sum_i p_i\lambda_k(2)_i\\ \lambda_{jk}(1,2) &= \sum_i p_i\lambda_j(1)_i\lambda_k(2)_i.\end{aligned}$$

Note that while there is no restriction on the number of partitions i, $\lambda_j(1),\lambda_k(2)$ and $\lambda_{jk}(1,2)$ and also $\lambda_j(1)_i,\lambda_k(2)_i$ and $\lambda_{jk}(1,2)_i$ have to satisfy inequalities 4.4.

In Bloch vector representation

$$(\widehat{\rho}(1)_i)^T = \frac{1}{N_1}\widehat{1}+\frac{1}{2}\sum_{j=1}^{s_1}\lambda_j(1)_i(\widehat{\lambda}_j(1))^T$$

and since

$$\begin{aligned}(\widehat{u}_{jk})^T &= \widehat{u}_{jk}\\ (\widehat{v}_{jk})^T &= -\widehat{v}_{jk}\\ (\widehat{w}_l)^T &= \widehat{w}_l\end{aligned}$$

where $1\leq j<k\leq N$, $1\leq l\leq N-1$, so $(\widehat{\rho}(1)_i)^T$ is represented by the same Bloch vector as $\widehat{\rho}(1)_i$ except v components. They are negative of that of $\widehat{\rho}(1)_i$:

$$\begin{aligned}\widehat{\rho}(1)_i &\longleftrightarrow (u_{jk},v_{jk},w_l)=\lambda(1),\\ (\widehat{\rho}(1)_i)^T &\longleftrightarrow (u_{jk},-v_{jk},w_l)=\lambda_T(1).\end{aligned}$$

Therefore, the partial transposition of a state with respect to first subsystem is

$$\begin{aligned}\mathsf{T}_1(\widehat{\rho}) &= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2)+\frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_{j_T}(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)]\\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)]+\frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{j_Tk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned}$$

where $\lambda_{j_T}(1)$ is the Bloch vector components of $\lambda_T(1)$ and $\lambda_{j_Tk}(1,2)$ is the coherence tensor components of $\mathsf{T}_1(\widehat{\rho})$.

4.2 Bloch Vector Formalism of Measurements

In the complete orthonormal basis $\left\{\widehat{Q}_{jk}\right\}_{j,k=0}^{s_1,s_2}$ ($s_1 = N_1^2 - 1, s_2 = N_2^2 - 1$) for the space of $N_1N_2 \times N_1N_2$ Hermitian matrices $\mathcal{L}(H_{N_1N_2})$ any Hermitian matrix $\widehat{A} \in \mathcal{L}(H_{N_1N_2})$ can be written as [9]

$$\begin{aligned}\widehat{A} &= \frac{1}{N_1N_2}\mathsf{Tr}\{\widehat{A}\}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1} A_{j0}\ [\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2} A_{0k}\ [\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} A_{jk}\ [\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned}$$

where A_{j0}, A_{0k} and A_{jk} are

$$\begin{aligned}A_{j0} &= \mathsf{Tr}\left\{\widehat{A}\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\right\}, \\ A_{0k} &= \mathsf{Tr}\left\{\widehat{A}\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\right\}, \\ A_{jk} &= \mathsf{Tr}\left\{\widehat{A}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\right\}.\end{aligned}$$

For a general density matrix $\widehat{\rho}$ of an N_1N_2-level system

$$\begin{aligned}\widehat{\rho} &= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned} \tag{4.7}$$

the expectation value of $\widehat{A}$ is again written as

$$\langle\widehat{A}\rangle = \mathsf{Tr}\{\widehat{\rho}\widehat{A}\}.$$

As the correlation tensor defined for density matrices, it is useful to define the *correlation tensor of measurement* K_{jk} as

$$K_{jk} := A_{jk} - A_{j0}A_{0k} \tag{4.8}$$

4.2.1 Measurements on a Subsystem

Consider the measurement performed only on the first subsystem of the general density matrix $\widehat{\rho}$ in 4.7. The decomposition of the Hermitian operator $\widehat{A}(1)\otimes\widehat{1}(2)$ in the orthonormal basis is

$$\widehat{A}(1)\otimes\widehat{1}(2) = \frac{1}{N_1}\mathsf{Tr}\{\widehat{A}(1)\}\widehat{1} + \frac{1}{2}\sum_{j=1}^{s_1} A_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)]$$

where

$$A_j(1) = \mathsf{Tr}\{\widehat{A}(1)\widehat{\lambda}_j(1)\}$$

and the expectation value of the measurement is

$$\langle \widehat{A}(1) \otimes \widehat{1}(2) \rangle = \frac{1}{N_1}\mathsf{Tr}\{\widehat{A}(1)\} + \frac{1}{2}\sum_{j=1}^{s_1} A_j(1)\lambda_j(1)$$

just as in the 3.10. The expectation value of the measurement is determined by the reduced density matrix of the first node

$$\widehat{\rho}_1 = \mathsf{Tr}_2\{\widehat{\rho}\} = \frac{1}{N_1}\widehat{1} + \frac{1}{2}\sum_{j=1}^{s_1} \lambda_j(1)\widehat{\lambda}_j(1)$$

and the expectation value equation is of the form

$$\langle \widehat{A}(1) \otimes \widehat{1}(2) \rangle = \mathsf{Tr}\{\widehat{\rho}_1 \widehat{A}(1)\}.$$

Let the eigenvectors $|i(1)\rangle$ of $\widehat{A}(1)$ be the basis for the representation of the generators $\widehat{\lambda}_j(1)$,. The state $\widehat{\rho}'$ after the measurement is determined by the projection operator $\widehat{P}_i(1) \otimes \widehat{1}(2)$ as

$$\widehat{\rho}' = \frac{1}{p_i(1)}\widehat{P}_i(1) \otimes \widehat{1}(2)\ \widehat{\rho}\ \widehat{P}_i(1) \otimes \widehat{1}(2) \tag{4.9}$$

where the projection operator $\widehat{P}_i(1)$ represented by

$$\widehat{P}_i(1) = \frac{1}{N_1}\widehat{1}(1) + \frac{1}{2}\sum_{c=1}^{N_1-1} w_c^i(1)\widehat{w}_c(1)$$

and $p_i(1)$ is the probability of finding state $|i(1)\rangle$ is

$$p_i(1) = \mathsf{Tr}\{\widehat{\rho}_1 \widehat{P}_i(1)\} = \frac{1}{N_1} + \frac{1}{2}\sum_{c=1}^{N_1-1} w_c(1)w_c^i(1). \tag{4.10}$$

By using the facts

$$\begin{aligned} \widehat{P}_i(1)\widehat{v}_{mn}(1)\widehat{P}_i(1) &= \widehat{P}_i(1)\widehat{u}_{mn}(1)\widehat{P}_i(1) = 0, \\ \widehat{P}_i(1)\widehat{w}_c(1)\widehat{P}_i(1) &= w_c^i(1)\widehat{P}_i(1), \\ \widehat{P}_i(1)\widehat{1}(1)\widehat{P}_i(1) &= \widehat{P}_i(1). \end{aligned} \tag{4.11}$$

the equation 4.10 becomes

$$\begin{aligned} p_i(1)\widehat{\rho}' &= \frac{1}{N_1 N_2}\widehat{P}_i(1) \otimes \widehat{1}(2) + \frac{1}{2N_2}\sum_{c=1}^{N_1-1} w_c(1)w_c^i(1)[\widehat{P}_i(1) \otimes \widehat{1}(2)] \\ &+ \frac{1}{2N_1}\sum_{k=1}^{s_2} \lambda_k(2)[\widehat{P}_i(1) \otimes \widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1} \lambda_{s'+c\ k}(1,2)w_c^i(1)[\widehat{P}_i(1) \otimes \widehat{\lambda}_k(2)] \end{aligned}$$

where $s' = N_1^2 - N_1$. It is clear that the state of the whole system after the measurement will be the tensor product of the individual states of each node:

$$\widehat{\rho}' = \widehat{\rho}_1' \otimes \widehat{\rho}_2'.$$

Therefore, the state of the first node is

$$\begin{aligned}\widehat{\rho}_1' &= \mathsf{Tr}_2\{\widehat{\rho}'\} \\ &= \widehat{P}_i(1)\end{aligned}$$

as expected, and the state of the second node is

$$\begin{aligned}\widehat{\rho}_2' &= \mathsf{Tr}_1\{\widehat{\rho}'\} \\ &= \frac{1}{p_i(1)}\left(\frac{1}{N_1} + \frac{1}{2}\sum_{c=1}^{N_1-1} w_c(1)w_c^i(1)\right)\frac{1}{N_2}\widehat{1}(2) \\ &\quad + \frac{1}{2N_1p_i(1)}\sum_{k=1}^{s_2}\lambda_k(2)\widehat{\lambda}_k(2) \\ &\quad + \frac{1}{4p_i(1)}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}\lambda_{s'+c\ k}(1,2)w_c^i(1)\ \widehat{\lambda}_k(2)\end{aligned} \tag{4.12}$$

by expressing the coherence tensor $\lambda_{jk}(1,2)$ in terms of the correlation defined in 4.3 as

$$\lambda_{jk}(1,2) = C_{jk}(1,2) + \lambda_j(1)\lambda_k(2).$$

Then 4.12 becomes

$$\begin{aligned}\widehat{\rho}_2' &= \frac{1}{N_2}\widehat{1}(2) + \frac{1}{2N_1p_i(1)}\sum_{k=1}^{s_2}\lambda_k(2)\widehat{\lambda}_k(2) \\ &\quad + \frac{1}{4p_i(1)}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}\lambda_{s'+c}(1)\lambda_k(2)w_c^i(1)\widehat{\lambda}_k(2) \\ &\quad + \frac{1}{4p_i(1)}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}C_{s'+c\ k}(1,2)w_c^i(1)\widehat{\lambda}_k(2) \\ &= \frac{1}{N_2}\widehat{1}(2) + \frac{1}{2p_i(1)}\sum_{k=1}^{s_2}\lambda_k(2)\left(\frac{1}{2}\sum_{c=1}^{N_1-1}\lambda_{s'+c}(1)w_c^i(1) + \frac{1}{N_1}\right)\widehat{\lambda}_k(2) \\ &\quad + \frac{1}{4p_i(1)}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}C_{s'+c\ k}(1,2)w_c^i(1)\widehat{\lambda}_k(2) \\ &= \widehat{\rho}_2 + \frac{1}{4p_i(1)}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}C_{s'+c\ k}(1,2)w_c^i(1)\widehat{\lambda}_k(2).\end{aligned} \tag{4.13}$$

Note that the correlation tensor determines the state of the second subsystem after the measurement, while it has no importance in the measurement process of the first node. Although our interest is to measure the first subsystem, the second subsystem is also measured *indirectly* and for such indirect measurements correlation tensor plays an important role.

Note also that for pure composite systems, the states of both nodes after the measurement are definitely pure, although for mixed composite systems the state of the first subsystem is pure but second one could be pure or mixed depending on how much mixed the composite system is. For example, consider the mixed density matrix

$$\widehat{\rho} = \frac{1}{3}(|00\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|)$$

Performing a projection $\widehat{P}_0(1) \otimes \widehat{1}(2)$ on $\widehat{\rho}$ results in

$$\begin{aligned} \widehat{\rho}' &= \frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 01|) \\ &= |0\rangle\langle 0| \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \end{aligned}$$

where the state of the second subsystem is a mixed state. However, if we performed the projection $\widehat{P}_1(1) \otimes \widehat{1}(2)$, the state after the measurement would become

$$\begin{aligned} \widehat{\rho}' &= |10\rangle\langle 10| \\ &= |1\rangle\langle 1| \otimes |0\rangle\langle 0|, \end{aligned}$$

where now both subsystems are in pure states.

4.2.2 Product Measurements

For a two node density matrix $\widehat{\rho}$ in 4.7, consider the measurement operator

$$\begin{aligned} \widehat{A}(1) \otimes \widehat{A}(2) &= \frac{1}{N_1 N_2}\mathsf{Tr}\{\widehat{A}(1)\}\mathsf{Tr}\{\widehat{A}(2)\}\widehat{1}(1) \otimes \widehat{1}(2) \\ &+ \frac{1}{2N_2}\mathsf{Tr}\{\widehat{A}(2)\}\sum_{j=1}^{s_1} A_j(1)[\widehat{\lambda}_j(1) \otimes \widehat{1}(2)] \\ &+ \frac{1}{2N_1}\mathsf{Tr}\{\widehat{A}(1)\}\sum_{k=1}^{s_2} A_k(2)[\widehat{1}(1) \otimes \widehat{\lambda}_k(2)] \\ &+ \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} A_j(1)A_k(2)[\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)] \end{aligned}$$

where

$$\begin{aligned} A_j(1) &= \mathsf{Tr}\left\{\widehat{A}(1)\widehat{\lambda}_j(1)\right\} \\ A_k(2) &= \mathsf{Tr}\left\{\widehat{A}(2)\widehat{\lambda}_k(2)\right\} \end{aligned}$$

performed on both subsystems simultaneously. If the state of the composite system is product i.e. $\widehat{\rho} = \widehat{\rho}(1) \otimes \widehat{\rho}(2)$, then the measurements are totally independent from each other and its properties investigated in section 3.2.

If the state of the composite system $\widehat{\rho}$ in 4.7 is nonproduct i.e.

$$\lambda_{jk}(1,2) \neq \lambda_j(1)\lambda_k(2),$$

then the expectation value of the total measurement will be

$$\begin{aligned} \mathsf{Tr}\{\widehat{\rho}\widehat{A}(1) \otimes \widehat{A}(2)\} &= \frac{1}{N_1 N_2}\mathsf{Tr}\{\widehat{A}(1)\}\mathsf{Tr}\{\widehat{A}(2)\} + \frac{1}{2N_2}\mathsf{Tr}\{\widehat{A}(2)\}\sum_{j=1}^{s_1}\lambda_j(1)A_j(1) \\ &+ \frac{1}{2N_1}\mathsf{Tr}\{\widehat{A}(1)\}\sum_{k=1}^{s_2}\lambda_k(2)A_k(2) + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)A_j(1)A_k(2) \end{aligned}$$

and since all the eigenvectors of the product Hermitian matrices correspond to product states, the state of the system $\widehat{\rho}'$ after the measurement will be the tensor product of density matrices of each subsystem:

$$\widehat{\rho}' = \widehat{\rho}'_1 \otimes \widehat{\rho}'_2.$$

Note that, the expectation value of the total measurement is not the same as the product of the expectation values of the measurements performed on each subsystem, i.e.

$$\begin{aligned} \langle \widehat{A}(1) \otimes \widehat{A}(2)\rangle_{\widehat{\rho}} &\neq \langle \widehat{A}(1) \otimes \widehat{1}(2)\rangle_{\widehat{\rho}}\ \langle \widehat{1}(1) \otimes \widehat{A}(2)\rangle_{\widehat{\rho}} \\ &\neq \langle \widehat{A}(1)\rangle_{\widehat{\rho}_1}\ \langle \widehat{A}(2)\rangle_{\widehat{\rho}_2}. \end{aligned}$$

The expectation value of independent measurements performed on subsystems are

$$\begin{aligned} \mathsf{Tr}\left\{\widehat{\rho}\widehat{A}(1) \otimes \widehat{1}(2)\right\} &= \frac{1}{N_1}\mathsf{Tr}\{\widehat{A}(1)\} + \frac{1}{2}\sum_{j=1}^{s_1}\lambda_j(1)A_j(1), \\ \mathsf{Tr}\left\{\widehat{\rho}\widehat{1}(1) \otimes \widehat{A}(2)\right\} &= \frac{1}{N_2}\mathsf{Tr}\{\widehat{A}(2)\} + \frac{1}{2}\sum_{k=2}^{s_2}\lambda_k(2)A_k(2). \end{aligned}$$

Therefore, the relation between the expectation value of the total measurement and the product of the expectation values of independent measurements is

$$\langle \widehat{A}(1) \otimes \widehat{A}(2) \rangle_{\widehat{\rho}} = \langle \widehat{A}(1) \rangle_{\widehat{\rho}_1} \; \langle \widehat{A}(2) \rangle_{\widehat{\rho}_2} + \frac{1}{4} \sum_{j,k=1}^{s_1,s_2} C_{jk}(1,2) A_j(1) A_k(2) \tag{4.14}$$

where the correlation tensor makes the difference. Note that for uncorrelated systems, this term vanishes. Therefore, the states that have non zero correlation tensor are called *correlated states.* Note also that the correlation can be of the classical nature or quantum nature i.e. entangled. One of the tests that whether correlated states are classically correlated or not is understood through Bell's inequalities. All classically correlated states satisfy Bell's inequalities, however, a state satisfying Bell's inequality may be separable or entangled. Before going on to Bell's inequality, it is worthwhile to mention briefly the EPR paradox .

EPR Paradox and Bell's Inequality

EPR paradox is based on two assumptions [15] :

(i) The reality criterion: If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

(ii) The locality criterion: The measurement results of one system cannot affect the measurement results performed on a second system, if at the time of measurement the two systems are well separated.

These assumptions are shortly called *local realism.* Consider the spin state $|\psi\rangle$ of a composite system

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

which is written in the basis of $\widehat{\sigma}_z$ and shared by two party, traditionally named Alice and Bob. Suppose that Alice performs a measurement along the $z-$axis on her qubit and finds the result $|0\rangle$. Then she perfectly predicts the state of Bob's particle, which is definitely $|0\rangle$. Therefore, Bob's particle's spin state has *elements of reality* by the reality criterion. Moreover, Bob's particle's spin state along the $z-$axis was $|0\rangle$ even before the measurement by the locality criterion.

On the other hand she could measure along the $y-$axis and would get either $|0\rangle_y$ or $|1\rangle_y$ and again she could predict the state of Bob's particle to be either $|0\rangle_y$ or $|1\rangle_y$. Therefore, it is possible to assign a value for both y and z components of Bob's particle's spin at the same time by using the locality criterion. This is in contradiction with uncertainty principle. Therefore, one may conclude that quantum mechanics is an incomplete theory i.e. the representation of the physical system by $|\psi\rangle$ is incomplete. There should be some other hidden variables which determine the spin state of Bob's particle.

However, J. S. Bell, by assuming that outcomes of the measurements performed on spatially separated, but entangled, subsystems are determined by local hidden variables, derived his famous inequality. A generalization of Bell's original inequality known as the CHSH inequality was experimentally verified by Aspect and his collaborators [15] and as explained in the following.

Consider a two node system whose subsystems are both two levels and the measurements for the first subsystem $\widehat{A}(1), \widehat{B}(1)$ and for the second subsystem $\widehat{A}(2), \widehat{B}(2)$ are defined in the Bloch vector representation as

$$\begin{aligned}
\widehat{A}(1) &= 2[\sin a(1), 0, \cos a(1)] \\
\widehat{B}(1) &= 2[\sin b(1), 0, \cos b(1)] \\
\widehat{A}(2) &= 2[\sin a(2), 0, \cos a(2)] \\
\widehat{B}(2) &= 2[\sin b(2), 0, \cos b(2)]
\end{aligned}$$

Furthermore, the results of all measurements are either 1 or -1. Therefore, the results of the all product measurements are

$$\begin{aligned}
\widehat{A}(1) \otimes \widehat{A}(2) &= \pm 1 \\
\widehat{A}(1) \otimes \widehat{B}(2) &= \pm 1 \\
\widehat{B}(1) \otimes \widehat{A}(2) &= \pm 1 \\
\widehat{B}(1) \otimes \widehat{B}(2) &= \pm 1
\end{aligned}$$

Of course, for any actual measurement only one out of those four possible angle settings can be realized. Local realism, however, assumes that all these data "exist" nevertheless i.e. without any actually performed measurement. Under these assumptions, consider the variable

$$S = \widehat{A}(1) \otimes \widehat{A}(2) + \widehat{A}(1) \otimes \widehat{B}(2) + \widehat{B}(1) \otimes \widehat{A}(2) - \widehat{B}(1) \otimes \widehat{B}(2), \tag{4.15}$$

thus it takes values either $+2$ or -2. For any state $\widehat{\rho}$, since the expectation value of each measurement is between -1 and $+1$,

$$|\langle S\rangle| = |\langle \widehat{A}(1)\otimes\widehat{A}(2)\rangle + \langle \widehat{A}(1)\otimes\widehat{B}(2)\rangle + \langle \widehat{B}(1)\otimes\widehat{A}(2)\rangle - \langle \widehat{B}(1)\otimes\widehat{B}(2)\rangle| \leq 2 \qquad (4.16)$$

This inequality represents a specific statement of the *Bell's inequality* and called CHSH inequality [16].

Now consider the singlet state $|\psi\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1,-1\rangle - |-1,1\rangle).$$

Its Bloch vector representation is equal to

$$\lambda_i(v) = 0 \text{ and } C_{jk}(1,2) = -\delta_{jk} \ (v = 1,2).$$

By using 4.14 for a measurement $\widehat{A}(1)\otimes\widehat{A}(2)$, the expectation value of the measurement turns out to be

$$\begin{aligned}
\langle \widehat{A}(1)\otimes\widehat{A}(2)\rangle &= \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} C_{jk}(1,2)A_j(1)A_k(2) \\
&= -\sin a(1)\sin a(2) - \cos a(1)\cos a(2) \\
&= -\cos[a(2)-a(1)]
\end{aligned}$$

When we apply this results in 4.15 by choosing the angles as

$$a(2)-a(1) = b(1)-a(2) = b(2)-b(1) = \theta$$

and

$$b(2)-a(1) = 3\theta$$

we get

$$S = 3\cos\theta - \cos 3\theta$$

which violates the limit postulated by 4.16 by taking value $|\langle S\rangle| = 2\sqrt{2}$ when $\theta = \pi/4$.

On the other hand, for a product state $\widehat{\rho} = \widehat{\rho}(1)\otimes\widehat{\rho}(2)$, since

$$\langle \widehat{A}(1)\otimes\widehat{A}(2)\rangle = \langle \widehat{A}(1)\rangle\langle \widehat{A}(2)\rangle,$$

the CHSH inequality reduces to a form

$$\left|\langle \widehat{A}(1)\rangle \left(\langle \widehat{A}(2)\rangle + \langle \widehat{B}(2)\rangle\right) + \langle \widehat{B}(1)\rangle \left(\langle \widehat{A}(2)\rangle - \langle \widehat{B}(2)\rangle\right)\right| \leq 2.$$

In order to attain the maximum values inside the absolute value $|\langle\widehat{A}(1)\rangle| = |\langle\widehat{B}(1)\rangle| = 1$ i.e. there exist four possibilities for the $\langle\widehat{A}(1)\rangle$ and $\langle\widehat{B}(1)\rangle$ as a pair and it is clear that for all of these the inequality is satisfied.

Also for a separable states $\widehat{\rho} = \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i$, since

$$\langle\widehat{A}(1) \otimes \widehat{A}(2)\rangle = \sum_i p_i \langle\widehat{A}(1)\rangle_i \langle\widehat{A}(2)\rangle_i \ ,$$

the CHSH inequality reduces to the form

$$\begin{aligned}\left|\sum_i p_i \left[\langle\widehat{A}(1)\rangle_i\langle\widehat{A}(2)\rangle_i + \langle\widehat{A}(1)\rangle_i\langle\widehat{B}(2)\rangle_i + \langle\widehat{B}(1)\rangle_i\langle\widehat{A}(2)\rangle_i - \langle\widehat{B}(1)\rangle_i\langle\widehat{B}(2)\rangle_i\right]\right| &\leq 2, \\ \left|\sum_i p_i \left[\langle\widehat{A}(1)\rangle_i \left(\langle\widehat{A}(2)\rangle_i + \langle\langle\widehat{B}(2)\rangle_i\right) + \langle\widehat{B}(1)\rangle_i \left(\langle\widehat{A}(2)\rangle_i - \langle\widehat{B}(2)\rangle_i\right)\right]\right| &\leq 2.\end{aligned}$$

But we have already proven that

$$\left|\langle\widehat{A}(1)\rangle_i \left(\langle\widehat{A}(2)\rangle_i + \langle\langle\widehat{B}(2)\rangle_i\right) + \langle\widehat{B}(1)\rangle_i \left(\langle\widehat{A}(2)\rangle_i - \langle\widehat{B}(2)\rangle_i\right)\right| \leq 2 \ ,$$

for all i, therefore

$$\left|\sum_i p_i \left[\langle\widehat{A}(1)\rangle_i \left(\langle\widehat{A}(2)\rangle_i + \langle\langle\widehat{B}(2)\rangle_i\right) + \langle\widehat{B}(1)\rangle_i \left(\langle\widehat{A}(2)\rangle_i - \langle\widehat{B}(2)\rangle_i\right)\right]\right| \leq 2$$

and the inequality is satisfied. Therefore, the Bell's inequality is a test of entanglement: any state violating it is entangled, although the reverse of this statement is not true.

4.2.3 Nonproduct measurements

When we consider a two node system as a whole, we can measure it by just one measurement apparatus. Such a measurement $\widehat{A}(1,2)$ is represented by

$$\begin{aligned}\widehat{A}(1,2) &= \frac{1}{N_1 N_2}\mathsf{Tr}\{\widehat{A}(1,2)\}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1} A_{j0}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2} A_{0k}(1,2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned}$$

where

$$\begin{aligned}A_{j0}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{\lambda}_j(1)\otimes\widehat{1}(2)\right\} \\ A_{0k}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{1}(1)\otimes\widehat{\lambda}_k(2)\right\} \\ A_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\right\}\end{aligned}$$

which satisfy the following inequality:

$$A_{jk}(1,2) \neq A_{j0}(1,2)A_{0k}(1,2).$$

Consequently, correlation tensor of measurement $K_{jk}(1,2)$ is non zero.

When such a measurement $\widehat{A}(1,2)$ is applied to the state $\widehat{\rho}$ given by 4.7, the expectation value of the measurement will be

$$\begin{aligned}\mathsf{Tr}\{\widehat{\rho}\widehat{A}(1,2)\} &= \frac{1}{N_1N_2}\mathsf{Tr}\{\widehat{A}(1,2)\} + \frac{1}{2N_2}\sum_{j=1}^{s_1} A_{0j}(1,2)\lambda_j(1) \\ &+ \frac{1}{2N_1}\sum_{k=1}^{s_2} A_{k0}(1,2)\lambda_k(2) + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2)\lambda_{jk}(1,2). \qquad (4.17)\end{aligned}$$

Note that, since at least one of the eigenvectors of the nonproduct Hermitian matrices have to be nonproduct, i.e. entangled (since it is also pure), the state of the total system after a nonproduct measurement can be product or nonproduct. Therefore, the Hilbert space of states $\mathcal{H}(1,2)$ after measurement is of the form

$$\mathcal{H}(1,2) = [\mathcal{H}'(1) \otimes \mathcal{H}'(2)] \oplus \mathcal{H}_E(1,2)$$

where $\mathcal{H}_E(1,2)$ is the space of the entangled states.

The fact is that at least $N_1 + N_2 - 1$ eigenvectors of the nonproduct Hermitian matrices is nonproduct. Because in the case there is only one nonproduct state, the product subspace $\mathcal{H}'(1) \otimes \mathcal{H}'(2)$ has dimension, $N_1N_2 - 1$ and it can not be a product space, since $N_1N_2 - 1$ is not a product of two integer a and b such that $a < N_1$ and $b < N_2$. The product subspace $\mathcal{H}'(1) \otimes \mathcal{H}'(2)$ can be barely accomplished when the dimension of the first node is $(N_1 - 1)$ and that of second node is $(N_2 - 1)$, and the remaining dimensions belong to the space of the entangled states $\mathcal{H}_E(1,2)$ which is $N_1 + N_2 - 1$.

4.3 Dynamics of Bloch Vectors and Coherence Tensor

For a general state of a two node system $\widehat{\rho}$ given in 4.2, consider a general Hamiltonian $\widehat{H}$ of the system

$$\begin{aligned}\widehat{H} &= \frac{1}{N_1N_2}\mathsf{Tr}\left\{\widehat{H}\right\}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1} H_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\ &+ \frac{1}{2N_1}\sum_{k=1}^{s_2} H_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} H_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned}$$

where

$$\begin{aligned} H_j(1) &= \mathsf{Tr}\left\{\widehat{H}\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\right\}, \\ H_k(2) &= \mathsf{Tr}\left\{\widehat{H}\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\right\}, \\ H_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{H}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\right\}. \end{aligned}$$

To get the dynamical equation for the Bloch vectors and the coherence tensor, we now insert the Hamiltonian and the density matrix into the Liouville equation

$$i\hbar\frac{d}{dt}\widehat{\rho} = [\widehat{H}, \widehat{\rho}].$$

Note that, the commutator of product operators satisfies the following property [9]

$$\begin{aligned} \left[\widehat{A}(1)\otimes\widehat{B}(2), \widehat{C}(1)\otimes\widehat{D}(2)\right] &= \frac{1}{2}\left[\widehat{A}(1), \widehat{C}(1)\right]\otimes\left[\widehat{B}(2), \widehat{D}(2)\right]_+ \\ &\quad +\frac{1}{2}\left[\widehat{A}(1), \widehat{C}(1)\right]_+\otimes\left[\widehat{B}(2), \widehat{D}(2)\right] \end{aligned} \tag{4.18}$$

so, the time evolution of the components of the Bloch vectors are determined by

$$\begin{aligned} \frac{d\lambda_m(1)}{dt} &= \frac{1}{N_2\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'}(1)\lambda_j(1) \\ &\quad +\frac{1}{2\hbar}\sum_{j',j,k} f_{j'jm}(1)H_{j'k}(1,2)\lambda_{jk}(1,2), \end{aligned} \tag{4.19}$$

$$\begin{aligned} \frac{d\lambda_n(2)}{dt} &= \frac{1}{N_1\hbar}\sum_{k,k'} f_{k'kn}(2)H_{k'}(2)\lambda_k(2) \\ &\quad +\frac{1}{2\hbar}\sum_{k',k,j} f_{k'kn}(2)H_{jk'}(1,2)\lambda_{jk}(1,2), \end{aligned} \tag{4.20}$$

and that of the coherence tensor reads

$$\begin{aligned} \frac{d\lambda_{mn}(1,2)}{dt} &= \frac{1}{N_2\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'n}(1,2)\lambda_j(1) \\ &\quad +\frac{1}{N_1\hbar}\sum_{k',k} f_{k'kn}(2)H_{mk'}(1,2)\lambda_k(2) \\ &\quad +\frac{1}{N_2\hbar}\sum_{j,j'} f_{j'jm}(1)H_{j'}(1)\lambda_{jn}(1,2) \\ &\quad +\frac{1}{N_1\hbar}\sum_{k,k'} f_{k'kn}(2)H_{k'}(2)\lambda_{mk}(1,2) \\ &\quad +\frac{1}{2\hbar}\sum_{j,k,j',k'} d_{k'kn}(2)f_{j'jm}(1)H_{j'k'}(1,2)\lambda_{jk}(1,2) \\ &\quad +\frac{1}{2\hbar}\sum_{j,k,j',k'} d_{j'jm}(1)f_{k'kn}(2)H_{j'k'}(1,2)\lambda_{jk}(1,2). \end{aligned} \tag{4.21}$$

By defining

$$
\begin{aligned}
\Omega_{mj}(1) &:= \frac{1}{N_2\hbar}\sum_{j'} f_{j'jm}(1)H_{j'}(1) \\
\Omega_{nk}(2) &:= \frac{1}{N_1\hbar}\sum_{k'} f_{k'kn}(2)H_{k'}(2) \\
Q_{mjk}(1) &:= \frac{1}{\hbar}\sum_{j'} f_{j'jm}(1)H_{j'k}(1,2) \\
Q_{jnk}(2) &:= \frac{1}{\hbar}\sum_{k'} f_{k'kn}(2)H_{jk'}(1,2) \\
D_{nkmj} &:= \frac{1}{\hbar}\sum_{j',k'}[d_{k'kn}(2)f_{j'jm}(1) + d_{j'jm}(1)f_{k'kn}(2)]H_{j'k'}(1,2)
\end{aligned}
$$

time derivative of the bloch vectors and coherence tensor become

$$
\frac{d\lambda_m(1)}{dt} = \sum_j \Omega_{mj}(1)\lambda_j(1) + \frac{1}{2}\sum_{j,k} Q_{mjk}(1)\lambda_{jk}(1,2), \tag{4.22}
$$

$$
\frac{d\lambda_n(2)}{dt} = \sum_k \Omega_{nk}(2)\lambda_k(2) + \frac{1}{2}\sum_{k,j} Q_{jnk}(2)\lambda_{jk}(1,2), \tag{4.23}
$$

$$
\begin{aligned}
\frac{d\lambda_{mn}(1,2)}{dt} = & \sum_j \Omega_{mj}(1)\lambda_{jn}(1,2) + \sum_k \Omega_{nk}(2)\lambda_{mk}(1,2) \\
& + \frac{1}{N_2}\sum_j Q_{mjn}(1)\lambda_j(1) + \frac{1}{N_1}\sum_k Q_{mnk}(2)\lambda_k(2) \\
& + \frac{1}{2}\sum_{j,k} D_{nkmj}\lambda_{jk}(1,2).
\end{aligned} \tag{4.24}
$$

4.3.1 Product Hamiltonian

A general state of a two node system $\widehat{\rho}$ having product Hamiltonian $\widehat{H}(1) \otimes \widehat{H}(2)$, the dynamical evolution of the system will be governed by the tensor products of the unitary operators $\exp(-\frac{i}{h}\widehat{H}(1)\ t)$ and $\exp(-\frac{i}{h}\widehat{H}(2)\ t)$:

$$
\begin{aligned}
\mathsf{U}(t) &= \exp(-\frac{i}{h}\widehat{H}(1)\ t) \otimes \exp(-\frac{i}{h}\widehat{H}(2)\ t) \\
&= \mathsf{U}(1,t) \otimes \mathsf{U}(2,t).
\end{aligned}
$$

If $\widehat{\rho}$ is a separable state given by

$$
\widehat{\rho} = \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i,
$$

then it will stay as separable since

$$\begin{aligned}\widehat{\rho}(t) &= \mathsf{U}(t)\widehat{\rho}\mathsf{U}^\dagger(t) \\ &= \sum_i p_i \mathsf{U}(1,t)\widehat{\rho}(1)_i\mathsf{U}^\dagger(1,t) \otimes \mathsf{U}(2,t)\widehat{\rho}(2)_i\mathsf{U}^\dagger(2,t) \\ &= \sum_i p_i \widehat{\rho}(1;t)_i \otimes \widehat{\rho}(2;t)_i.\end{aligned}$$

Also note that, any separable state at a later time

$$\begin{aligned}\widehat{\rho}(t) &= \mathsf{U}(t)\widehat{\rho}\mathsf{U}^\dagger(t) \\ &= \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i\end{aligned}$$

should have been separable in the past since

$$\begin{aligned}\widehat{\rho} &= \mathsf{U}(t)^\dagger\widehat{\rho}(t)\mathsf{U}(t) \\ &= \sum_i p_i \mathsf{U}(1,t)^\dagger\widehat{\rho}(1)_i\mathsf{U}(1,t) \otimes \mathsf{U}(2,t)^\dagger\widehat{\rho}(2)_i\mathsf{U}(2,t) \\ &= \sum_i p_i \widehat{\rho}(1;-t)_i \otimes \widehat{\rho}(2;-t)_i\end{aligned}$$

Therefore any *entangled state stays entangled* under time evolution governed by the product Hamiltonians.

Note also that for a general density matrix in 4.2, the action of tensor product of unitary operators is

$$\begin{aligned}\mathsf{U}(t)\widehat{\rho}\mathsf{U}(t)^\dagger &= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2) + \frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)[\mathsf{U}(1,t)\widehat{\lambda}_j(1)\mathsf{U}(1,t)^\dagger\otimes\widehat{1}(2)] \\ &+\frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\mathsf{U}(2,t)\widehat{\lambda}_k(2)\mathsf{U}(2,t)^\dagger] \\ &+\frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\mathsf{U}(1,t)\widehat{\lambda}_j(1)\mathsf{U}(1,t)^\dagger\otimes\mathsf{U}(2,t)\widehat{\lambda}_k(2)\mathsf{U}(2,t)^\dagger].\end{aligned}$$

Using the adjoint representations of $\mathsf{SU(N_1)}$ and $\mathsf{SU(N_2)}$

$$\frac{1}{2}\mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}(v,t)\widehat{\lambda}_i\mathsf{U}^\dagger(v,t)\} = \mathsf{O}_{mi}(v,t),$$

the action on the Bloch vectors and the coherence tensor can be seen as

$$\begin{aligned}\mathsf{O}_{jm}(1,t)\lambda_m(1) &= \lambda_j(1;t), \\ \mathsf{O}_{km}(2,t)\lambda_m(2) &= \lambda_k(2;t), \\ \mathsf{O}_{mi}(1,t)\mathsf{O}_{kl}(2,t)\lambda_{ml}(1,2) &= \lambda_{jk}(1,2;t),\end{aligned}$$

so the correlation tensor $C_{jk}(1,2)$ will evolve according to

$$\mathsf{O}_{mi}(1,t)\mathsf{O}_{kl}(2,t)C_{ml}(1,2) = C_{jk}(1,2;t) \tag{4.25}$$

4.3.2 Nonproduct Hamiltonian

It is the nonproduct Hamiltonians that generate the correlation tensor. It was already shown that coherence tensor satisfies the differential equation 4.21. The time derivative of the correlation tensor $C_{mn}(1,2)$ is given by

$$\frac{dC_{mn}(1,2)}{dt} = \frac{d\lambda_{mn}(1,2)}{dt} - \frac{d\lambda_m(1)}{dt}\lambda_n(2) - \lambda_m(1)\frac{d\lambda_n(2)}{dt}.$$

Substituting in the equations 4.22, 4.23 and 4.24, we get

$$\begin{aligned}
\frac{dC_{mn}(1,2)}{dt} &= \sum_j \Omega_{mj}(1)C_{jn}(1,2) + \sum_k \Omega_{nk}(2)C_{mk}(1,2) \\
&+\frac{1}{N_2}\sum_j Q_{mjn}(1)\lambda_j(1) + \frac{1}{N_1}\sum_k Q_{mnk}(2)\lambda_k(2) \\
&-\frac{1}{2}\sum_{j,k} Q_{mjk}(1)C_{jk}(1,2)\lambda_n(2) - \frac{1}{2}\sum_{k,j} Q_{jnk}(2)C_{jk}(1,2)\lambda_m(1) \\
&-\frac{1}{2}\sum_{j,k} Q_{mjk}(1)\lambda_j(1)\lambda_k(2)\lambda_n(2) - \frac{1}{2}\sum_{k,j} Q_{jnk}(2)\lambda_j(1)\lambda_k(2)\lambda_m(1) \\
&+\frac{1}{2}\sum_{j,k} D_{mjnk}\ C_{jk}(1,2) + \frac{1}{2}\sum_{j,k} D_{mjnk}\ \lambda_j(1)\lambda_k(2)
\end{aligned} \tag{4.26}$$

When the Hamiltonian is product and the correlation tensor is initially zero ($C_{jk}(1,2) = 0$), then the correlation tensor stays zero, i.e. $\frac{d}{dt}C_{jk}(1,2) = 0$ by 4.25. So,

$$\begin{aligned}
0 &= \frac{1}{N_2}\sum_j Q_{mjn}(1)\lambda_j(1) + \frac{1}{N_1}\sum_k Q_{mnk}(2)\lambda_k(2) \\
&-\frac{1}{2}\sum_{j,k} Q_{mjk}(1)\lambda_j(1)\lambda_k(2)\lambda_n(2) - \frac{1}{2}\sum_{k,j} Q_{jnk}(2)\lambda_j(1)\lambda_k(2)\lambda_m(1) \\
&+\frac{1}{2}\sum_{j,k} D_{mjnk}\ \lambda_j(1)\lambda_k(2).
\end{aligned} \tag{4.27}$$

When we implement this result into 4.26, the time derivative of $C_{ij}(1,2)$, which is initially zero, becomes

$$\begin{aligned}\frac{dC_{mn}(1,2)}{dt} &= \frac{1}{N_2}\sum_j Q^K_{mjn}(1)\lambda_j(1) + \frac{1}{N_1}\sum_k Q^K_{mnk}(2)\lambda_k(2)\\ &-\frac{1}{2}\sum_{j,k} Q^K_{mjk}(1)\lambda_j(1)\lambda_k(2)\lambda_n(2) - \frac{1}{2}\sum_{k,j} Q^K_{jnk}(2)\lambda_j(1)\lambda_k(2)\lambda_m(1)\\ &+\frac{1}{2}\sum_{j,k} D^K_{mjnk}\ \lambda_j(1)\lambda_k(2)\end{aligned}$$

where

$$\begin{aligned}Q^K_{mjk}(1) &:= \frac{1}{\hbar}\sum_{j'} f_{j'jm}(1)K_{j'k}(1,2)\\ Q^K_{jmk}(2) &:= \frac{1}{\hbar}\sum_{k'} f_{k'km}(2)K_{jk'}(1,2)\\ D^K_{nkmj} &:= \frac{1}{\hbar}\sum_{j',k'}[d_{k'kn}(2)f_{j'jm}(1) + d_{j'jm}(1)f_{k'kn}(2)]K_{j'k'}(1,2)\end{aligned}$$

where the correlation tensor $K_{jk}(1,2)$ for Hermitian operators given in 4.8 as

$$K_{jk}(1,2) := H_{jk}(1,2) - H_j(1)H_k(2)$$

After grouping the common terms, it reduces to

$$\begin{aligned}\frac{dC_{mn}(1,2)}{dt} &= \frac{1}{N_2}\sum_j Q^K_{mjn}(1)\lambda_j(1) + \frac{1}{N_1}\sum_k Q^K_{mnk}(2)\lambda_k(2)\\ &-\frac{1}{2}\sum_{j,k}\left[Q^K_{mjk}(1)\lambda_n(2) - Q^K_{jnk}(2)\lambda_m(1) + D^K_{mjnk}\right]\lambda_j(1)\lambda_k(2)\end{aligned}$$

This is the equation that shows how an uncorrelated states become correlated under non-product Hamiltonian evolution of the system.

Chapter 5

BLOCH VECTOR FORMALISM FOR THREE NODE SYSTEMS

5.1 Bloch Vector Representation of States

The density matrix space of the three node systems $\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2} \otimes H_{N_3})$ is the tensor products of each density matrix space $\mathcal{L}^v_{+,1}(H_{Nv})$ $(v = 1, 2, 3)$ for N_v level systems and it is also equal to the density matrix space of $N_1N_2N_3$ levels systems which is given by

$$\begin{aligned}
\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2} \otimes H_{N_3}) : \; &= \; \mathcal{L}^1_{+,1}(H_{N_1}) \otimes \mathcal{L}^2_{+,1}(H_{N_2}) \otimes \mathcal{L}^3_{+,1}(H_{N_3}) = \mathcal{L}_{+,1}(H_{N_1N_2N_3}) \\
&= \; \{\widehat{\rho} \in \mathcal{L}(H_{N_1N_2N_3}) : (i) \ \mathsf{Tr}\{\widehat{\rho}\}= 1, \\
&\qquad (ii) \ \rho_i \geq 0 \quad (i = 1, ..., N_1N_2N_3)\} \quad (5.1)
\end{aligned}$$

where $\mathcal{L}(H_{N_1N_2N_3})$ is the space of $N_1N_2N_3 \times N_1N_2N_3$ Hermitian matrices and the orthonormal basis $\left\{\widehat{Q}_{jk}\right\}_{j,k,l=0}^{s_1,s_2,s_3}$ $(s_1 = N_1^2 - 1, s_2 = N_2^2 - 1, s_3 = N_3^2 - 1)$ for $\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2} \otimes H_{N_3})$ is the tensor product of the orthonormal bases $\left\{\frac{1}{\sqrt{n_v}}\widehat{1}(v), \frac{1}{\sqrt{2}}\widehat{\lambda}_j(v)\right\}$ of each subsystem $\mathcal{L}^v_{+,1}(H_{Nv})$ $(v = 1, 2, 3)$

$$\begin{aligned}
\widehat{Q}_{000} &= \frac{1}{\sqrt{N_1N_2N_3}}\widehat{1}, \\
\widehat{Q}_{j00} &= \frac{1}{\sqrt{2}}\widehat{\lambda}_j(1) \otimes \frac{1}{\sqrt{N_2}}\widehat{1}(2) \otimes \frac{1}{\sqrt{N_3}}\widehat{1}(3), \text{ etc.} \\
\widehat{Q}_{jk0} &= \frac{1}{\sqrt{N_1}}\widehat{1}(1) \otimes \frac{1}{\sqrt{2}}\widehat{\lambda}_k(2) \otimes \frac{1}{\sqrt{N_3}}\widehat{1}(3), \text{ etc.} \\
\widehat{Q}_{jkl} &= \frac{1}{2\sqrt{2}}\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{\lambda}_l(3). \qquad (5.2)
\end{aligned}$$

It is orthonormal in the sense of Hilbert-Schmidt product

$$\mathsf{Tr}\left\{\widehat{Q}_{jkl}\widehat{Q}_{j'k'l'}\right\} = \delta_{jj'}\delta_{kk'}\delta_{ll'}$$

therefore the density matrix $\widehat{\rho} \in \mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2} \otimes H_{N_3})$ can be represented by [9]

$$\begin{aligned}
\widehat{\rho} &= \sum_{j,k,l=0}^{s_1,s_2,s_3} \mathsf{Tr}\left\{\widehat{\rho}\widehat{Q}_{jkl}\right\}\widehat{Q}_{jkl} \\
&= \frac{1}{N_1N_2N_3}\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3) + \frac{1}{2N_2N_3}\sum_{j=1}^{s_1}\lambda_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)] \\
&\quad +\frac{1}{2N_1N_3}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] + \frac{1}{2N_1N_2}\sum_{l=1}^{s_3}\lambda_l(3)[\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&\quad +\frac{1}{4N_3}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] + \frac{1}{4N_2}\sum_{j,l=1}^{s_1,s_3}\lambda_{jl}(1,3)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&\quad +\frac{1}{4N_1}\sum_{k,l=1,}^{s_2,s_3}\lambda_{kl}(2,3)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)] + \frac{1}{8}\sum_{j,k,l=1}^{s_1,s_2,s_3}\lambda_{jkl}(1,2,3)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned} \tag{5.3}$$

where $\lambda_j(1)...,\lambda_{jk}(1,2)...$ and $\lambda_{jkl}(1,2,3)$ are

$$\begin{aligned}
\lambda_j(1) &= \mathsf{Tr}\left\{\widehat{\rho}\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)\right\} = \langle\widehat{\lambda}_j(1)\rangle, \text{ ... etc.} \\
\lambda_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{\rho}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)\right\} = \langle\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\rangle, \text{ ... etc.} \\
\lambda_{jk}(1,2,3) &= \mathsf{Tr}\left\{\widehat{\rho}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)\right\} = \langle\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)\rangle
\end{aligned}$$

and the corresponding Bloch vectors are $\lambda(v)= \{\lambda_1(v),\lambda_2(v),...,\lambda_{s_v}(v)\}$ $(v = 1,2,3)$. For three node systems, there are three 2-node correlation tensors defined by

$$\begin{aligned}
C_{jk}(1,2) &:= \lambda_{jk}(1,2) - \lambda_j(1)\lambda_k(2) \\
C_{jl}(1,3) &:= \lambda_{jl}(1,3) - \lambda_j(1)\lambda_l(3) \\
C_{kl}(2,3) &:= \lambda_{kl}(2,3) - \lambda_k(2)\lambda_l(3)
\end{aligned} \tag{5.4}$$

and in addition to that there exist a 3-node correlation tensor defined by

$$\begin{aligned}
C_{jkl}(1,2,3) &:= \lambda_{jkl}(1,2,3) - \lambda_j(1)\lambda_k(2)\lambda_l(3) \\
&\quad -\lambda_j(1)C_{kl}(2,3) - \lambda_k(2)C_{jl}(1,3) - \lambda_l(3)C_{jk}(1,2)
\end{aligned} \tag{5.5}$$

Note that the reduced density matrices $\widehat{\rho}_i$ for each node and the reduced density matrices $\widehat{\rho}_{ij}$ for each two node subsystem can be obtained by the following trace operations

$$\begin{aligned}
\widehat{\rho}_i &= \mathsf{Tr}_{jk}\{\widehat{\rho}\} \\
\widehat{\rho}_{ij} &= \mathsf{Tr}_k\{\widehat{\rho}\}
\end{aligned}$$

where $i,j,k = 1,2,3$.

5.1.1 Characterization of Bloch Vectors and Coherence Tensors

The characterization of Bloch vectors and coherence tensors can be accomplished by a similar treatment followed in two node systems, that is, by using the equality of $\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2} \otimes H_{N_3})$ and $\mathcal{L}_{+,1}(H_{N_1N_2N_3})$, it is possible to use the theorem 1 and its result which says nonnegativity condition is satisfied if and only if all a_i's given by 3.8 are nonnegative. However, this time the states are represented in the basis of $\mathcal{L}_{+,1}(H_{N_1} \otimes H_{N_2} \otimes H_{N_3})$ given by 5.2 instead of the basis of $\mathcal{L}_{+,1}(H_{N_1N_2N_3})$ given by 3.4. Unfortunately, because the amount and the complexity of the inequalities to be satisfied is impracticable, the conditions will not be written explicitly but it is clear that they exist and have similar structures with one node and two node cases.

5.1.2 Classification of States

Pure states

Recall that a general density matrix $\widehat{\rho}$ is pure if $\mathsf{Tr}\{\widehat{\rho}^2\} = 1$, and the trace of $\widehat{\rho}^2$ is

$$\begin{aligned}\mathsf{Tr}\{\widehat{\rho}^2\} &= \frac{1}{N_1N_2N_3} + \frac{1}{2N_2N_3}|\lambda(1)|^2 + \frac{1}{2N_1N_3}|\lambda(2)|^2 + \frac{1}{2N_1N_2}|\lambda(3)|^2 \\ &+ \frac{1}{4N_3}|\lambda(1,2)|^2 + \frac{1}{4N_2}|\lambda(1,3)|^2 + \frac{1}{4N_1}|\lambda(2,3)|^2 + \frac{1}{8}|\lambda(1,2,3)|^2\end{aligned}$$

and if $\mathsf{Tr}\{\widehat{\rho}^2\} < 1$, $\widehat{\rho}$ is a mixed state.

Product States

A density matrix of a three node system $\widehat{\rho}$ is called product if

$$\widehat{\rho} = \widehat{\rho}_1 \otimes \widehat{\rho}_2 \otimes \widehat{\rho}_3$$

where $\widehat{\rho}_1, \widehat{\rho}_2$ and $\widehat{\rho}_3$ are the reduced density matrices of each subsystem. Note that, if the $\widehat{\rho}_1$, $\widehat{\rho}_2$ and $\widehat{\rho}_3$ are pure than $\widehat{\rho}$ is pure, otherwise $\widehat{\rho}$ is mixed since

$$\mathsf{Tr}\{\widehat{\rho}^2\} = \mathsf{Tr}\{\widehat{\rho}_1\}\mathsf{Tr}\{\widehat{\rho}_2\}\mathsf{Tr}\{\widehat{\rho}_3\} = 1$$

if and only if

$$\mathsf{Tr}\{\widehat{\rho}_1\} = \mathsf{Tr}\{\widehat{\rho}_2\} = \mathsf{Tr}\{\widehat{\rho}_3\} = 1$$

and that is the case $\widehat{\rho}_1$, $\widehat{\rho}_2$ and $\widehat{\rho}_3$ are pure.

Entanglement of Pure States

In three node systems, a node can be entangled with one of the other nodes or both of the two nodes and such entangled states are called bipartite and tripartite entangled states, respectively. The state of a bipartite entangled systems is of the form

$$\widehat{\rho} = \widehat{\rho}_{ab} \otimes \widehat{\rho}_c$$

where the entanglement is between node a and node b, and its properties is not different from that of two node systems. On the other hand, the tripartite entanglement has stronger than bipartite entanglement in the sense of the violation of the generalized Bell's inequalities of three node systems, as it will be discussed in subsection 5.2.4. Note that, a pure state is called tripartite entangled if its 3-node correlation tensor $C_{jkl}(1, 2, 3)$ is non zero.

One of the important properties of the tripartite systems is two inequivalent tripartite entanglement classes that they have [17]. The meaning of the inequality is that a member of one class can not be transformed into a member of other class by means of stochastic local operations and classical communications (SLOCC). If the probability is one then the performed local operations corresponds to tensor products of unitary transformations and of course transformed states have to be equivalent. However, tensor product of unitary transformations is not the only way of obtaining one state from another, the projection operations may also help to achieve this and this is the reason of talking about probability. Mathematically, two pure states of a multipartite system are equivalent under SLOCC if and only if they are related by a local invertible operator i.e. operators of the form

$$\widehat{O} = \widehat{O}_1 \otimes \widehat{O}_2 \otimes ... \otimes \widehat{O}_N$$

where $\widehat{O}_i$'s are invertible operators [17].

Consider a tripartite entangled state which can written as a linear combination of two terms such as

$$|\phi_2\rangle = r_0|a_0, b_0, c_0\rangle + r_1 e^{i\theta}|a_1, b_1, c_1\rangle$$

Note the fact that, invertible local operators do not increase the number of terms in a decomposition because they are local and they do not decrease the number of terms because they are invertible.

By using this fact, $|\phi_2\rangle$ is SLOCC equivalent to maximally entangled GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$$

Also note that every pure tripartite entangled state can be written as a linear combination of at most three terms

$$|\phi_3\rangle = r_0|a_0, b_0, c_0\rangle + r_1 e^{i\theta_1}|a_1, b_1, c_1\rangle + r_2 e^{i\theta_2}|a_2, b_2, c_2\rangle$$

and a tripartite entangled state which can not be written as linear combination of two terms is SLOCC equivalent to W state

$$|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$$

by using the above fact again.

Separable States

There are two different types of separable states: those that can be expressed as a convex sum of product states;

$$\widehat{\rho} = \sum_i p_i \widehat{\rho}(1)_i \otimes \widehat{\rho}(2)_i \otimes \widehat{\rho}(3)_i \qquad \sum_i p_i = 1$$

and those that can be expressed as a convex sum of product of a state with a bipartite entangled states such as

$$\widehat{\rho} = \sum_i p_i \widehat{\rho}(1,2)_i \otimes \widehat{\rho}(3)_i \qquad \sum_i p_i = 1$$

where $\widehat{\rho}(1,2)_i$ is an entangled state. Such a separable states are also called *biseparable.*

5.2 Bloch Vector Formalism of Measurements

In the complete orthonormal basis $\left\{\widehat{Q}_{jk}\right\}_{j,k,l=0}^{s_1,s_2,s_3}$ $(s_1 = N_1^2 - 1, s_2 = N_2^2 - 1, s_3 = N_3^2 - 1)$ for the space of $N_1N_2N_3 \times N_1N_2N_3$ Hermitian matrices $\mathcal{L}(H_{N_1N_2N_3})$, any Hermitian matrix

$\widehat{A} \in \mathcal{L}(H_{N_1N_2N_3})$ can be written as [9]

$$\begin{aligned}
\widehat{A} \;&=\; \frac{1}{N_1N_2N_3}\mathsf{Tr}\{\widehat{A}\}\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3) + \frac{1}{2N_2N_3}\sum_{j=1}^{s_1} A_{j00}\;[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_3}\sum_{k=1}^{s_2} A_{0k0}\;[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] + \frac{1}{2N_1N_2}\sum_{l=1}^{s_3} A_{00l}\;[\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_3}\sum_{j,k=1}^{s_1,s_2} A_{jk0}\;[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] + \frac{1}{4N_2}\sum_{j,l=1}^{s_1,s_3} A_{j0l}\;[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_1}\sum_{k,l=1,}^{s_2,s_3} A_{0kl}\;[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)] + \frac{1}{8}\sum_{j,k,l=1}^{s_1,s_2,s_3} A_{jkl}\;[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}$$

where $A_{j00}, ..., A_{jk0}, ..., A_{jkl}$ are

$$\begin{aligned}
A_{j00} &= \mathsf{Tr}\left\{\widehat{A}\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)\right\}, \text{ ... etc.} \\
A_{jk0} &= \mathsf{Tr}\left\{\widehat{A}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)\right\}, \text{ ... etc.} \\
A_{jkl} &= \mathsf{Tr}\left\{\widehat{A}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)\right\}
\end{aligned}$$

Note that there are three 2-node correlation tensors of measurement given as

$$\begin{aligned}
K_{jk0} &:= A_{jk0} - A_{j00}A_{0k0} \\
K_{j0l} &:= A_{j0l} - A_{j00}A_{00l} \\
K_{0kl} &:= A_{0kl} - A_{0k0}A_{00l} \qquad (5.6)
\end{aligned}$$

and in addition to that there exist a 3-node correlation tensor of measurement defined by

$$\begin{aligned}
K_{jkl} &:= A_{jkl} - A_{j00}A_{0k0}A_{00l} \qquad (5.7) \\
&\quad - A_{j00}K_{0kl} - A_{0k0}K_{j0l} - A_{00l}K_{jk0}
\end{aligned}$$

5.2.1 Measurements on One Subsystem

Consider the measurement performed only on the first subsystem of the state $\widehat{\rho}$ given in 5.3, the representation of such a measurement $\widehat{A}(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)$ is

$$\widehat{A}(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3) = \frac{1}{N_1}\mathsf{Tr}\left\{\widehat{A}(1)\right\}\widehat{1} + \frac{1}{2}\sum_{j=1}^{s_1} A_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)]$$

where

$$A_j(1) = \mathsf{Tr}\{\widehat{A}(1)\widehat{\lambda}_j(1)\}$$

the expectation value of this measurement is

$$\langle \widehat{A}(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \rangle = \frac{1}{N_1} \mathsf{Tr} \left\{ \widehat{A}(1) \right\} + \frac{1}{2} \sum_{j=1}^{s_1} A_j(1) \lambda_j(1).$$

It is determined by the reduced density matrix of the first node $\widehat{\rho}_1$

$$\widehat{\rho}_1 = \mathsf{Tr}_{23}\{\widehat{\rho}\} = \frac{1}{N_1} \widehat{1}(1) + \frac{1}{2} \sum_{j=1}^{s_1} \lambda_j(1) \widehat{\lambda}_j(1)$$

and the expectation value equation is of the form

$$\langle \widehat{A}(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \rangle = \mathsf{Tr}\{\widehat{\rho}_1 \widehat{A}(1)\}$$

Let the eigenvectors $|f(1)\rangle$ of $\widehat{A}(1)$ be the basis for the representation of the generators $\widehat{\lambda}_j(1)$, and the state $\widehat{\rho}'$ after the measurement is determined by the projection operator $\widehat{P}_f(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3)$ as

$$\widehat{\rho}' = \frac{1}{p_f(1)} \widehat{P}_f(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \ \widehat{\rho} \ \widehat{P}_f(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \tag{5.8}$$

where the projection operator $\widehat{P}_f(1)$ is represented by

$$\widehat{P}_f(1) = \frac{1}{N_1} \widehat{1}(1) + \frac{1}{2} \sum_{c=1}^{N_1 - 1} w_c^f(1) \widehat{w}_c(1) \tag{5.9}$$

and $p_f(1)$, the probability of finding state $|f(1)\rangle$, is

$$p_f(1) = \mathsf{Tr}\{\widehat{\rho}_1 \widehat{P}_f(1)\} = \frac{1}{N_1} + \frac{1}{2} \sum_{c=1}^{N_1 - 1} w_c(1) w_c^f(1) \tag{5.10}$$

By using the facts

$$\begin{aligned} \widehat{P}_f(1) \widehat{v}_{mn}(1) \widehat{P}_f(1) &= \widehat{P}_f(1) \widehat{u}_{mn}(1) \widehat{P}_f(1) = 0 \\ \widehat{P}_f(1) \widehat{w}_c(1) \widehat{P}_f(1) &= w_c^f(1) \widehat{P}_f(1) \\ \widehat{P}_f(1) \widehat{1}(1) \widehat{P}_f(1) &= \widehat{P}_f(1) \end{aligned} \tag{5.11}$$

the equation 5.8 becomes

$$\begin{aligned}
p_f(1)\widehat{\rho}' &= \frac{1}{N_1N_2N_3}\widehat{P}_f(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3) \\
&+\frac{1}{2N_2N_3}\sum_{c=1}^{N_1-1} w_c(1)w_c^f(1)[\widehat{P}_f(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_3}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{P}_f(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_2}\sum_{l=1}^{s_3}\lambda_l(3)[\widehat{P}_f(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_3}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}\lambda_{s'+c\ k}(1,2)w_c^f(1)[\widehat{P}_f(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{4N_2}\sum_{l=1}^{s_3}\sum_{c=1}^{N_1-1}\lambda_{s'+c\ l}(1,3)w_c^f(1)[\widehat{P}_f(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_1}\sum_{k,l=1}^{s_2,s_3}\lambda_{kl}(2,3)[\widehat{P}_f(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)] \\
&\frac{1}{8}\sum_{k,l=1}^{s_2,s_3}\sum_{c=1}^{N_1-1}\lambda_{s'+ckl}(1,2,3)w_c^f(1)[\widehat{P}_f(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}$$

where $s' = N_1^2 - N_1$. It is clear that the state $\widehat{\rho}'$ of the whole system after the measurement will be the tensor product of the state $\widehat{\rho}'_1$ of first node and the state $\widehat{\rho}'_{23}$ of the second and third node

$$\widehat{\rho}' = \widehat{\rho}'_1 \otimes \widehat{\rho}'_{23}$$

where the state of the first node is

$$\begin{aligned}
\widehat{\rho}'_1 &= \mathsf{Tr}_{23}\{\widehat{\rho}'\} \\
&= \widehat{P}_f(1)
\end{aligned}$$

as expected and the state $\widehat{\rho}'_{23}$ of the second and third node is

$$\begin{aligned}
\widehat{\rho}'_{23} &= \mathsf{Tr}_1\{\widehat{\rho}'\} \\
&= \frac{1}{N_2N_3}\widehat{1}(2)\otimes\widehat{1}(3) \\
&\quad+\frac{1}{2N_1N_3p_f(1)}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&\quad+\frac{1}{2N_1N_2p_f(1)}\sum_{l=1}^{s_3}\lambda_l(3)[\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&\quad+\frac{1}{4N_3p_f(1)}\sum_{k=1}^{s_2}\sum_{c=1}^{N_1-1}\lambda_{s'+c\ k}(1,2)w_c^f(1)[\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&\quad+\frac{1}{4N_2p_f(1)}\sum_{l=1}^{s_3}\sum_{c=1}^{N_1-1}\lambda_{s'+c\ l}(1,3)w_c^f(1)[\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&\quad+\frac{1}{4N_1p_f(1)}\sum_{k,l=1}^{s_2,s_3}\lambda_{kl}(2,3)[\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)] \\
&\quad\frac{1}{8p_f(1)}\sum_{k,l=1}^{s_2,s_3}\sum_{c=1}^{N_1-1}\lambda_{s'+ckl}(1,2,3)w_c^f(1)[\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}$$

by using of the relations between the coherence tensors and correlation tensors given in 5.4 and 5.5, $\widehat{\rho}'_{23}$ reduces to the following form

$$\begin{aligned}
\widehat{\rho}'_{23} &= \widehat{\rho}_{23}+\left[\frac{1}{4p_f(1)}\sum_{c=1}^{N_1-1}\sum_{k=1}^{s_2}C_{s'+ck}(1,2)w_c^f(1)\widehat{\lambda}_k(2)\right]\otimes\widehat{\rho}_3 \\
&+\ \widehat{\rho}_2\otimes\left[\frac{1}{4p_f(1)}\sum_{c=1}^{N_1-1}\sum_{l=1}^{s_3}C_{s'+cl}(1,3)w_c^f(1)\widehat{\lambda}_l(3)\right] \\
&+\ \frac{1}{8p_f(1)}\sum_{c=1}^{N_1-1}\sum_{k,l=1}^{s_2,s_3}C_{s'+ckl}(1,2,3)w_c^f(1)\ [\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}\tag{5.12}$$

It is worth to compare this result with its analog for two node systems given in 4.13 to comprehend the meaning of common nonlocal-effect terms.

5.2.2 Product Measurements on Two Subsystem

Consider a product measurement performed on the first and the second nodes of the state $\widehat{\rho}$ given in 5.3, the representation of such a measurement $\widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{1}(3)$ is

$$\begin{aligned}
\widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{1}(3) &= \frac{1}{N_1 N_2} \mathsf{Tr}\left\{\widehat{A}(1)\widehat{A}(2)\right\} \widehat{1}(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \\
&+ \frac{1}{2N_2} \sum_{j=1}^{s_1} A_j(1)[\widehat{\lambda}_j(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3)] \\
&+ \frac{1}{2N_1} \sum_{k=1}^{s_2} A_k(2)[\widehat{1}(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)] \\
&+ \frac{1}{4} \sum_{j,k=1}^{s_1,s_2} A_j(1) A_k(2)[\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)]
\end{aligned}$$

and the expectation value of this measurement is

$$\begin{aligned}
\langle \widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{1}(3) \rangle &= \frac{1}{N_1 N_2} \mathsf{Tr}\left\{\widehat{A}(1)\widehat{A}(2)\right\} + \frac{1}{2N_2} \sum_{j=1}^{s_1} A_j(1)\lambda_j(1) \\
&+ \frac{1}{2N_1} \sum_{k=1}^{s_2} A_k(2)\lambda_k(2) + \frac{1}{4} \sum_{j,k=1}^{s_1,s_2} A_j(1) A_k(2) \lambda_{jk}(1,2)
\end{aligned}$$

note that the expectation value of the measurement is determined by the reduced density matrix for the first two node $\widehat{\rho}_{12}$

$$\begin{aligned}
\widehat{\rho}_{12} = \mathsf{Tr}_3\{\widehat{\rho}\} &= \frac{1}{N_1 N_2} \widehat{1}(1) \otimes \widehat{1}(2) + \frac{1}{2N_2} \sum_{j=1}^{s_1} \lambda_j(1)[\widehat{\lambda}_j(1) \otimes \widehat{1}(2)] \\
&+ \frac{1}{2N_1} \sum_{k=1}^{s_2} \lambda_k(2)[\widehat{1}(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)] \\
&+ \frac{1}{4} \sum_{j,k=1}^{s_1,s_2} \lambda_{jk}(1,2)[\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)]
\end{aligned}$$

so the expectation value equation is of the form

$$\langle \widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{1}(3) \rangle = \mathsf{Tr}\{\widehat{\rho}_{12} \widehat{A}(1) \otimes \widehat{A}(2)\}$$

Let the eigenvectors $|f(1)\rangle$ and $|g(2)\rangle$ of $\widehat{A}(1)$ and $\widehat{A}(2)$ be the basis for the representation of the generators $\widehat{\lambda}_j(1)$ and $\widehat{\lambda}_k(2)$. The state $\widehat{\rho}'$ of the whole system after the measurement is determined by the projection operator $\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3)$ as

$$\widehat{\rho}' = \frac{1}{p_{f,g}(1,2)} \widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3) \; \widehat{\rho} \; \widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3) \tag{5.13}$$

where the projection operators $\widehat{P}_f(1)$ and $\widehat{P}_g(2)$ are

$$\begin{aligned}
\widehat{P}_f(1) &= \frac{1}{N_1}\widehat{1}(1) + \frac{1}{2}\sum_{c=1}^{N_1-1} w_c^f(1)\widehat{w}_c(1) \\
\widehat{P}_g(2) &= \frac{1}{N_2}\widehat{1}(2) + \frac{1}{2}\sum_{d=1}^{N_2-1} w_d^g(2)\widehat{w}_d(2)
\end{aligned}$$

and $p_{f,g}(1,2)$, the probability of finding state $|f(1)\rangle \otimes |g(2)\rangle$, is

$$\begin{aligned}
p_{f,g}(1,2) &= \mathsf{Tr}\{\widehat{\rho}\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3)\} \\
&= \frac{1}{N_1 N_2} + \frac{1}{2N_2}\sum_{c=1}^{N_1-1} w_c(1)w_c^f(1) + \frac{1}{2N_1}\sum_{d=1}^{N_2-1} w_d(2)w_d^g(2) \\
&\quad + \frac{1}{4}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} \lambda_{s_1'+c\ s_2'+d}(1,2)w_c^f(1)w_d^g(2) \qquad (5.14)
\end{aligned}$$

where $s_q' = N_q^2 - N_q$ $(q = 1, 2)$. By using the facts given in 5.11, the equation 5.13 becomes

$$\begin{aligned}
p_{f,g}(1,2)\widehat{\rho}' &= \frac{1}{N_1 N_2 N_3}\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3) \\
&\quad + \frac{1}{2N_2 N_3}\sum_{c=1}^{N_1-1} w_c(1)w_c^f(1)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3)] \\
&\quad + \frac{1}{2N_1 N_3}\sum_{d=1}^{N_2-1} w_d(2)w_d^g(2)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3)] \\
&\quad + \frac{1}{2N_1 N_2}\sum_{l=1}^{s_3} \lambda_l(3)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{\lambda}_l(3)] \\
&\quad + \frac{1}{4N_3}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} \lambda_{s_1'+c\ s_2'+d}(1,2)w_c^f(1)w_d^g(2)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3)] \\
&\quad + \frac{1}{4N_2}\sum_{l=1}^{s_3}\sum_{c=1}^{N_1-1} \lambda_{s_1'+c\ l}(1,3)w_c^f(1)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{\lambda}_l(3)] \\
&\quad + \frac{1}{4N_1}\sum_{d=1}^{N_2-1}\sum_{l=1}^{s_3} \lambda_{s_2'+d\ l}(2,3)w_d^g(2)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{\lambda}_l(3)] \\
&\quad + \frac{1}{8}\sum_{l=1}^{s_3}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} \lambda_{s_1'+c\ s_2'+d\ l}(1,2,3)w_c^f(1)w_d^g(2)[\widehat{P}_f(1) \otimes \widehat{P}_g(2) \otimes \widehat{\lambda}_l(3)]
\end{aligned}$$

It is clear that the state of the whole system after the measurement will be the tensor product of the individual state of each node,

$$\widehat{\rho}' = \widehat{\rho}_1' \otimes \widehat{\rho}_2' \otimes \widehat{\rho}_3'$$

so the state of the first node is,

$$\begin{aligned}
\widehat{\rho}'_1 &= \mathsf{Tr}_{23}\{\widehat{\rho}'\} \\
&= \frac{1}{p_{f,g}(1,2)}[\frac{1}{N_1N_2}\widehat{P}_f(1) + \frac{1}{2N_2}\sum_{c=1}^{N_1-1} w_c(1)w_c^f(1)\widehat{P}_f(1) \\
&\quad + \frac{1}{2N_1}\sum_{d=1}^{N_2-1} w_d(2)w_d^g(2)\widehat{P}_f(1) \\
&\quad + \frac{1}{4}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} \lambda_{s'_1+c\ s'_2+d}(1,2)w_c^f(1)w_d^g(2)\ \widehat{P}_f(1)] \\
&= \widehat{P}_f(1)
\end{aligned}$$

as expected; and similarly the state of the second node is

$$\widehat{\rho}'_2 = \widehat{P}_g(2) = \frac{1}{N_2}\widehat{1}(2) + \frac{1}{2}\sum_{d=1}^{N_2-1} w_d^g(2)\widehat{w}_d(2)$$

and finally the state of third node is

$$\begin{aligned}
\widehat{\rho}'_3 &= \frac{1}{N_3}\widehat{1}(3) + \frac{1}{2p_{f,g}(1,2)}\sum_{l=1}^{s_3}[\ \frac{1}{N_1N_2}\lambda_l(3) \\
&\quad + \frac{1}{2N_2}\sum_{c=1}^{N_1-1} \lambda_{s'_1+c\ l}(1,3)w_c^f(1) \\
&\quad + \frac{1}{2N_1}\sum_{d=1}^{N_2-1} \lambda_{s'_2+d\ l}(2,3)w_d^g(2) \\
&\quad + \frac{1}{4}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} \lambda_{s'_1+c\ s'_2+d\ l}(1,2,3)w_c^f(1)w_d^g(2)\ ]\ \widehat{\lambda}_l(3)
\end{aligned}$$

by using the relations between the coherence tensor and correlation tensor given in 5.4 and eq. 5.5, the final form of $\widehat{\rho}'_3$ becomes

$$\begin{aligned}
\widehat{\rho}'_3 &= \widehat{\rho}_3 + p_g(2)\ \frac{1}{4p_{f,g}(1,2)}\sum_{l=1}^{s_3}\sum_{c=1}^{N_1-1} C_{s'_1+c\ l}(1,3)w_c^f(1)\widehat{\lambda}_l(3) \\
&\quad + p_f(1)\ \frac{1}{4p_{f,g}(1,2)}\sum_{l=1}^{s_3}\sum_{d=1}^{N_2-1} C_{s'_2+d\ l}(2,3)w_d^g(2)\widehat{\lambda}_l(3) \\
&\quad + \frac{1}{8p_{f,g}(1,2)}\sum_{l=1}^{s_3}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} C_{s'_1+c\ s'_2+d\ l}(1,2,3)w_c^f(1)w_d^g(2)\widehat{\lambda}_l(3) \qquad (5.15)
\end{aligned}$$

where $p_f(1)$ and $p_g(2)$ are

$$\begin{aligned} p_f(1) &= \frac{1}{N_1} + \frac{1}{2}\sum_{c=1}^{N_1-1} w_c(1) w_c^f(1) \\ p_g(2) &= \frac{1}{N_2} + \frac{1}{2}\sum_{d=1}^{N_2-1} w_d(2) w_d^g(2) \end{aligned}$$

and observe that they are also the probability of getting $|f(1)\rangle$ and $|g(2)\rangle$ in the independent projections $\widehat{P}_f(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3)$ and $\widehat{1}(1) \otimes \widehat{P}_g(2) \otimes \widehat{1}(3)$.

It is worth to compare this result with 4.13 and 5.12 to comprehend the meaning of the common nonlocal-effect terms.

5.2.3 Nonproduct Measurements on Two Subsystem

By considering the first two node of a three node system as a whole, we can measure it by just one measurement apparatus. Such a measurement $\widehat{A}(1,2) \otimes \widehat{1}(3)$ is represented by

$$\begin{aligned} \widehat{A}(1,2) \otimes \widehat{1}(3) &= \frac{1}{N_1 N_2} \mathsf{Tr}\{\widehat{A}(1,2)\} \widehat{1}(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \\ &+ \frac{1}{2N_2} \sum_{j=1}^{s_1} A_{j0}(1,2) [\widehat{\lambda}_j(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3)] \\ &+ \frac{1}{2N_1} \sum_{k=1}^{s_2} A_{0k}(1,2) [\widehat{1}(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)] \\ &+ \frac{1}{4} \sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2) [\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)] \end{aligned}$$

where

$$\begin{aligned} A_{j0}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{\lambda}_j(1) \otimes \widehat{1}(2)\right\} \\ A_{0k}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{1}(1) \otimes \widehat{\lambda}_k(2)\right\} \\ A_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)\right\} \end{aligned}$$

and they satisfy the following inequality

$$A_{jk}(1,2) \neq A_{j0}(1,2) A_{0k}(1,2)$$

When such a measurement is performed on the state $\widehat{\rho}$ given in 5.3, the expectation value of the measurement is

$$\begin{aligned}\mathsf{Tr}\{\widehat{\rho}\widehat{A}(1,2)\otimes\widehat{1}(3)\} &= \frac{1}{N_1N_2}\mathsf{Tr}\{\widehat{A}(1,2)\} + \frac{1}{2N_2}\sum_{j=1}^{s_1} A_{j0}(1,2)\lambda_j(1) \\ &\quad + \frac{1}{2N_1}\sum_{k=1}^{s_2} A_{0k}(1,2)\lambda_k(2) + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2)\lambda_{jk}(1,2)\end{aligned}$$

Note that, the expectation value of the measurement is again determined by the reduced density matrix for the first two node $\widehat{\rho}_{12}$

$$\begin{aligned}\widehat{\rho}_{12} = \mathsf{Tr}_3\{\widehat{\rho}\} &= \frac{1}{N_1N_2}\widehat{1}(1)\otimes\widehat{1}(2) \\ &\quad + \frac{1}{2N_2}\sum_{j=1}^{s_1}\lambda_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)] \\ &\quad + \frac{1}{2N_1}\sum_{k=1}^{s_2}\lambda_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)] \\ &\quad + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2}\lambda_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)]\end{aligned}$$

Recall that in two node systems, the state of the system after a nonproduct measurement could be product or nonproduct. Similarly, the state $\widehat{\rho}'_{12}$ of the first two part of the total systems after the measurement can be product or nonproduct, but the state of the total system after measurement is definitely the tensor product of $\widehat{\rho}'_{12}$ and $\widehat{\rho}'_3$

$$\widehat{\rho}' = \widehat{\rho}'_{12}\otimes\widehat{\rho}'_3$$

Let the eigenvectors $|i(1,2)\rangle$ of $\widehat{A}(1,2)$ be the basis for the representation of $\left\{\widehat{Q}_{jk}\right\}_{j,k=0}^{s_1,s_2}$. The state $\widehat{\rho}'$ after the measurement is determined by the projection operator $\widehat{P}_i(1,2)\otimes\widehat{1}(3)$ as

$$\widehat{\rho}' = \frac{1}{p_i(1,2)}\widehat{P}_i(1,2)\otimes\widehat{1}(3)\ \widehat{\rho}\ \widehat{P}_i(1,2)\otimes\widehat{1}(3) \tag{5.16}$$

where the projection operator $\widehat{P}_i(1,2)$ is

$$\begin{aligned}
\widehat{P}_i(1,2) \otimes \widehat{1}(3) &= \frac{1}{N_1 N_2}\widehat{1}(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \\
&+ \frac{1}{2N_2}\sum_{c=1}^{N_1-1} w_c^i(1)\widehat{w}_c(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \\
&+ \frac{1}{2N_1}\sum_{d=1}^{N_2-1} w_d^i(2)\widehat{1}(1) \otimes \widehat{w}_d(2) \otimes \widehat{1}(3) \\
&+ \frac{1}{4}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} w_{cd}^i(1,2)\ \widehat{w}_c(1) \otimes \widehat{w}_d(2) \otimes \widehat{1}(3)
\end{aligned}$$

and $p_i(1,2)$, the probability of finding state $|i(1,2)\rangle$, is

$$\begin{aligned}
p_i(1,2) &= \mathsf{Tr}\{\widehat{\rho}\widehat{P}_i(1,2)\} \\
&= \frac{1}{N_1 N_2} + \frac{1}{2N_2}\sum_{c=1}^{N_1-1} w_c(1) w_c^i(1) + \frac{1}{2N_1}\sum_{d=1}^{N_2-1} w_d(2) w_d^i(2) \\
&+ \frac{1}{4}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1} \lambda_{s_1'+c\ s_2'+d}(1,2) w_{cd}^i(1,2)
\end{aligned}$$

where $s_q' = N_q^2 - N_q$. By using the following facts

$$\begin{aligned}
\widehat{P}_i(1,2)\ \widehat{v}_{mn}(1) \otimes \widehat{1}(2)\widehat{P}_i(1,2) &= \widehat{P}_i(1,2)\ \widehat{1}(1) \otimes \widehat{v}_{mn}(2)\widehat{P}_i(1,2)\ = 0 \\
\widehat{P}_i(1,2)\ \widehat{u}_{mn}(1) \otimes \widehat{1}(2)\widehat{P}_i(1,2) &= \widehat{P}_i(1,2)\ \widehat{1}(1) \otimes \widehat{u}_{mn}(2)\widehat{P}_i(1,2)\ = 0 \\
\widehat{P}_i(1,2)\ \ \widehat{w}_c(1) \otimes \widehat{1}(2)\ \ \widehat{P}_i(1,2) &= w_c^i(1)\widehat{P}_i(1,2) \\
\widehat{P}_i(1,2)\ \widehat{1}(1) \otimes \widehat{w}_d(2)\widehat{P}_i(1,2) &= w_d^i(1)\widehat{P}_i(1,2) \\
\widehat{P}_i(1,2)\ \widehat{w}_c(1) \otimes \widehat{w}_d(2)\widehat{P}_i(1,2) &= w_{cd}^i(1,2)\widehat{P}_i(1,2) \\
\widehat{P}_i(1,2)\ \widehat{1}(1) \otimes \widehat{1}(2)\ \widehat{P}_i(1,2) &= \widehat{P}_i(1,2)
\end{aligned}$$

the equation 5.16 becomes

$$\begin{aligned}
p_i(1,2)\widehat{\rho}' &= \frac{1}{N_1N_2N_3}\widehat{P}_i(1,2)\otimes\widehat{1}(3) \\
&+\frac{1}{2N_2N_3}\sum_{c=1}^{N_1-1} w_c(1)w_c^i(1)\widehat{P}_i(1,2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_3}\sum_{d=1}^{N_2-1} w_d(1)w_d^i(1)[\widehat{P}_i(1,2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_2}\sum_{l=1}^{s_3}\lambda_l(3)[\widehat{P}_i(1,2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_3}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1}\lambda_{s_1'+c\ s_2'+d}(1,2)w_{cd}^i(1,2)[\widehat{P}_i(1,2)\otimes\widehat{1}(3)] \\
&+\frac{1}{4N_2}\sum_{l=1}^{s_3}\sum_{c=1}^{N_1-1}\lambda_{s_1'+c\ l}(1,3)w_c^i(1)[\widehat{P}_i(1,2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_1}\sum_{d=1}^{N_2-1}\sum_{l=1}^{s_3}\lambda_{s_2'+d\ l}(2,3)w_d^i(2)[\widehat{P}_i(1,2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{8}\sum_{l=1}^{s_3}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1}\lambda_{s_1'+c\ s_2'+d\ l}(1,2,3)w_{cd}^i(1,2)[\widehat{P}_i(1,2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}$$

The state $\widehat{\rho}'_{12}$ of the first two nodes after the measurement is

$$\widehat{\rho}'_{12} = \widehat{P}_i(1,2) \tag{5.17}$$

and by tracing out the first and second nodes, the state of the third node is

$$\begin{aligned}
\widehat{\rho}'_3 &= \frac{1}{N_3}\widehat{1}(3) + \frac{1}{2p_i(1,2)}\sum_{l=1}^{s_3}[\frac{1}{N_1N_2}\lambda_l(3) + \frac{1}{2N_2}\sum_{c=1}^{N_1-1}\lambda_{s_1'+c\ l}(1,3)w_c^i(1) \\
&\qquad + \frac{1}{2N_1}\sum_{d=1}^{N_2-1}\lambda_{s_2'+d\ l}(2,3)w_d^i(2) \\
&\qquad + \frac{1}{4}\sum_{d=1}^{N_2-1}\sum_{c=1}^{N_1-1}\lambda_{s_1'+c\ s_2'+d\ l}(1,2,3)w_{cd}^i(1,2)]\ \widehat{\lambda}_l(3)
\end{aligned}$$

by using the relation between the coherence tensor and correlation tensor given in 5.4 and 5.5, the final form of $\widehat{\rho}'_3$ becomes

$$
\begin{aligned}
\widehat{\rho}_3' = \widehat{\rho}_3 &+ \frac{1}{N_1}\ \frac{1}{4p_i(1,2)} \sum_{l=1}^{s_3} \sum_{d=1}^{N_2-1} C_{s_2'+d\ l}(2,3) w_d^i(2) \widehat{\lambda}_l(3) \\
&+ \frac{1}{N_2}\ \frac{1}{4p_i(1,2)} \sum_{l=1}^{s_3} \sum_{c=1}^{N_1-1} C_{s_1'+c\ l}(1,3) w_c^i(1) \widehat{\lambda}_l(3) \\
&+ \frac{1}{8p_i(1,2)} \sum_{l=1}^{s_3} \sum_{d=1}^{N_2-1} \sum_{c=1}^{N_1-1} [C_{s_1'+c\ s_2'+d\ l}(1,2,3) + C_{s_2'+d\ l}(2,3) w_c(1) \\
&\qquad + C_{s_1'+c\ l}(1,3) w_d(2)]\ w_{cd}^i(1,2)\ \widehat{\lambda}_l(3)
\end{aligned}
$$

It is worth to compare this result with 4.13, 5.12 and 5.15 to comprehend the meaning of the common nonlocal-effect terms.

5.2.4 Product Measurements I

For the three node density matrix $\widehat{\rho}$ given in 5.3, consider the measurement $\widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{A}(3)$,

$$
\begin{aligned}
\widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{A}(3) \quad = \quad & \frac{1}{N_1 N_2 N_3} \mathsf{Tr}\{\widehat{A}(1)\} \mathsf{Tr}\{\widehat{A}(2)\} \mathsf{Tr}\{\widehat{A}(3)\} \widehat{1}(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3) \\
& + \frac{1}{2N_2N_3} \mathsf{Tr}\{\widehat{A}(2)\} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{j=1}^{s_1} A_j(1) [\widehat{\lambda}_j(1) \otimes \widehat{1}(2) \otimes \widehat{1}(3)] \\
& + \frac{1}{2N_1N_3} \mathsf{Tr}\{\widehat{A}(1)\} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{k=1}^{s_2} A_k(2) [\widehat{1}(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)] \\
& + \frac{1}{2N_1N_2} \mathsf{Tr}\{\widehat{A}(1)\} \mathsf{Tr}\{\widehat{A}(2)\} \sum_{l=1}^{s_3} A_l(3) [\widehat{1}(1) \otimes \widehat{1}(2) \otimes \widehat{\lambda}_l(3)] \\
& + \frac{1}{4N_3} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{j,k=1}^{s_1,s_2} A_j(1) A_k(2) [\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{1}(3)] \\
& + \frac{1}{4N_2} \mathsf{Tr}\{\widehat{A}(2)\} \sum_{j,l=1}^{s_1,s_3} A_j(1) A_l(3)\ [\widehat{\lambda}_j(1) \otimes \widehat{1}(2) \otimes \widehat{\lambda}_l(3)] \\
& + \frac{1}{4N_1} \mathsf{Tr}\{\widehat{A}(1)\} \sum_{k,l=1}^{s_2,s_3} A_k(2) A_l(3)\ [\widehat{1}(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{\lambda}_l(3)] \\
& + \frac{1}{8} \sum_{j,k,l=1}^{s_1,s_2,s_3} A_j(1) A_k(2) A_l(3)\ [\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2) \otimes \widehat{\lambda}_l(3)]
\end{aligned}
$$

where

$$
\begin{aligned}
A_j(1) &= \mathsf{Tr}\left\{\widehat{A}(1)\widehat{\lambda}_j(1)\right\} \\
A_k(2) &= \mathsf{Tr}\left\{\widehat{A}(2)\widehat{\lambda}_k(2)\right\} \\
A_l(3) &= \mathsf{Tr}\left\{\widehat{A}(3)\widehat{\lambda}_l(3)\right\}
\end{aligned}
$$

the expectation value of this measurement is

$$
\begin{aligned}
\langle \widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{A}(3) \rangle &= \frac{1}{N_1 N_2 N_3} \mathsf{Tr}\{\widehat{A}(1)\}\mathsf{Tr}\{\widehat{A}(2)\}\mathsf{Tr}\{\widehat{A}(3)\} \\
&+ \frac{1}{2N_2N_3} \mathsf{Tr}\{\widehat{A}(2)\}\mathsf{Tr}\{\widehat{A}(3)\} \sum_{j=1}^{s_1} A_j(1)\lambda_j(1) \\
&+ \frac{1}{2N_1N_3} \mathsf{Tr}\{\widehat{A}(1)\}\mathsf{Tr}\{\widehat{A}(3)\} \sum_{k=1}^{s_2} A_k(2)\lambda_k(2) \\
&+ \frac{1}{2N_1N_2} \mathsf{Tr}\{\widehat{A}(1)\}\mathsf{Tr}\{\widehat{A}(2)\} \sum_{l=1}^{s_3} A_l(3)\lambda_l(3) \\
&+ \frac{1}{4N_3} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{j,k=1}^{s_1,s_2} A_j(1)\ A_k(2)\lambda_{jk}(1,2) \\
&+ \frac{1}{4N_2} \mathsf{Tr}\{\widehat{A}(2)\} \sum_{j,l=1}^{s_1,s_3} A_j(1)A_l(3)\lambda_{jl}(1,3) \\
&+ \frac{1}{4N_1} \mathsf{Tr}\{\widehat{A}(1)\} \sum_{k,l=1,}^{s_2,s_3} A_k(2)A_l(3)\lambda_{kl}(2,3) \\
&+ \frac{1}{8} \sum_{j,k,l=1}^{s_1,s_2,s_3} A_j(1)A_k(2)A_l(3)\lambda_{jkl}(1,2,3)
\end{aligned}
$$

and since all eigenvectors of the product Hermitian matrices are product states, the state $\widehat{\rho}'$ of the system after the measurement is the product of the density matrices of each subsystem.

$$
\widehat{\rho}' = \widehat{\rho}_1' \otimes \widehat{\rho}_2' \otimes \widehat{\rho}_3'
$$

Note that, the expectation value of the total measurement is not equal to the product of the expectation values of the independent measurements, i.e.

$$
\langle \widehat{A}(1) \otimes \widehat{A}(2) \otimes \widehat{A}(3) \rangle_{\widehat{\rho}} \neq \langle \widehat{A}(1) \rangle_{\widehat{\rho}_1}\ \langle \widehat{A}(2) \rangle_{\widehat{\rho}_2} \langle A(3) \rangle_{\widehat{\rho}_3}
$$

The expectation value of the independent measurements are

$$\begin{aligned}
\mathsf{Tr}\{\widehat{\rho}_1\widehat{A}(1)\} &= \frac{1}{N_1}\mathsf{Tr}\{\widehat{A}(1)\} + \frac{1}{2}\sum_{j=1}^{s_1}\lambda_j(1)A_j(1), \\
\mathsf{Tr}\{\widehat{\rho}_2\widehat{A}(2)\} &= \frac{1}{N_2}\mathsf{Tr}\{\widehat{A}(2)\} + \frac{1}{2}\sum_{k=1}^{s_2}\lambda_k(2)A_k(2), \\
\mathsf{Tr}\{\widehat{\rho}_3\widehat{A}(3)\} &= \frac{1}{N_3}\mathsf{Tr}\{\widehat{A}(3)\} + \frac{1}{2}\sum_{l=1}^{s_3}\lambda_l(3)A_l(3).
\end{aligned}$$

Therefore, the relation between the expectation value of the whole measurement and the product of the expectation values of the independent measurements is

$$\begin{aligned}
\langle\widehat{A}(1)\otimes\widehat{A}(2)\otimes\widehat{A}(3)\rangle_{\widehat{\rho}} &= \langle\widehat{A}(1)\rangle_{\widehat{\rho}_1}\ \langle\widehat{A}(2)\rangle_{\widehat{\rho}_2}\langle A(3)\rangle_{\widehat{\rho}_3} \\
&+\frac{1}{4N_3}\mathsf{Tr}\{\widehat{A}(3)\}\sum_{j,k=1}^{s_1,s_2}C_{jk}(1,2)A_j(1)A_k(2) \\
&+\frac{1}{4N_2}\mathsf{Tr}\{\widehat{A}(2)\}\sum_{j,l=1}^{s_1,s_3}C_{jl}(1,3)A_j(1)A_l(3) \\
&+\frac{1}{4N_1}\mathsf{Tr}\{\widehat{A}(1)\}\sum_{k,l=1}^{s_2,s_3}C_{kl}(2,3)A_k(2)A_l(3) \\
&+\frac{1}{8}\sum_{j,k,l=1}^{s_1,s_2,s_3}[C_{jkl}(1,2,3)+\lambda_j(1)C_{kl}(2,3) \\
&\qquad + \lambda_k(2)C_{jl}(1,3)+\lambda_l(3)C_{jk}(1,2)]A_j(1)A_k(2)A_l(3)
\end{aligned}$$

where the 2-node and 3-node correlation tensors, which are zero for an uncorrelated system, make the difference as expected.

As J. S. Bell proved that hidden variable theory would not fit with the predictions of quantum mechanics by using the bipartite entangled sates, D. Mermin [18] demonstrated the incompatibility of the reality criterion with quantum theory by using the tripartite entangled states.

GHZ State and the Elements of Reality

Consider the GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$$

which is written in the basis generated from the eigenvectors of $\widehat{\sigma}_z$. Note that the Hermitian operators $\widehat{\sigma}_x(j)$ and $\widehat{\sigma}_y(j)$ satisfy the following equalities for j^{th} subsystem,

$$\begin{aligned}
\widehat{\sigma}_x(j)|0\rangle_j &= |1\rangle_j \\
\widehat{\sigma}_x(j)|1\rangle_j &= |0\rangle_j \\
\widehat{\sigma}_y(j)|0\rangle_j &= i|1\rangle_j \\
\widehat{\sigma}_y(j)|1\rangle_j &= -i|0\rangle_j
\end{aligned} \tag{5.18}$$

and GHZ is an eigenstate of the following operators

$$\begin{aligned}
\widehat{\sigma}_x(1)\otimes\widehat{\sigma}_y(2)\otimes\widehat{\sigma}_y(3)|GHZ\rangle &= |GHZ\rangle \\
\widehat{\sigma}_y(1)\otimes\widehat{\sigma}_x(2)\otimes\widehat{\sigma}_y(3)|GHZ\rangle &= |GHZ\rangle \\
\widehat{\sigma}_y(1)\otimes\widehat{\sigma}_y(2)\otimes\widehat{\sigma}_x(3)|GHZ\rangle &= |GHZ\rangle \\
\widehat{\sigma}_x(1)\otimes\widehat{\sigma}_x(2)\otimes\widehat{\sigma}_x(3)|GHZ\rangle &= -|GHZ\rangle
\end{aligned}$$

Since the spin along x and y directions of a given particle anticommute i.e.

$$[\widehat{\sigma}_x(1),\widehat{\sigma}_y(1)]_+ = 0,$$

and

$$(\widehat{\sigma}_x(j))^2 = (\widehat{\sigma}_y(j))^2 = \widehat{1}(j),$$

these operators satisfy

$$[\widehat{\sigma}_x(1)\widehat{\sigma}_y(2)\widehat{\sigma}_y(3)]\,[\widehat{\sigma}_y(1)\widehat{\sigma}_x(2)\widehat{\sigma}_y(3)]\,[\widehat{\sigma}_y(1)\widehat{\sigma}_y(2)\widehat{\sigma}_x(3)] = -\widehat{\sigma}_x(1)\widehat{\sigma}_x(2)\widehat{\sigma}_x(3).$$

For the GHZ state, the operators $[\widehat{\sigma}_x(1)\widehat{\sigma}_y(2)\widehat{\sigma}_y(3)]$, $[\widehat{\sigma}_y(1)\widehat{\sigma}_x(2)\widehat{\sigma}_y(3)]$, and $[\widehat{\sigma}_y(1)\widehat{\sigma}_y(2)\widehat{\sigma}_x(3)]$ all have eigenvalue $+1$, which implies according to quantum theory that one can predict with certainty the measurement outcome of the spin along the x direction of any one of the three particles by measuring the spin along the y direction of the other two particles. If the outcomes along y are the same, then the subsequent measurement outcome along x would be $+1$, and if the y outcomes are different, then the subsequent outcome along x would be -1. Similarly, we can also predict the measurement outcome of the spin along the y direction of any particle with certainty, by measuring the spin of the other two particles, along x for one and y for the other.

Now, if we assume that a measurement performed on any one particle doesn't affect the other two particles (the locality criterion of EPR), then the reality criterion of EPR implies that there exist six elements of physical reality, denoted by $\sigma_x(1)$, $\sigma_x(2)$, $\sigma_x(3)$, $\sigma_y(1)$, $\sigma_y(2)$, and $\sigma_y(3)$. Although we can not know the values for all six elements of reality, from 5.18, their values must satisfy

$$\begin{aligned}
\sigma_x(1)\sigma_y(2)\sigma_y(3) &= 1 \\
\sigma_y(1)\sigma_x(2)\sigma_y(3) &= 1 \\
\sigma_y(1)\sigma_y(2)\sigma_x(3) &= 1 \\
\sigma_x(1)\sigma_x(2)\sigma_x(3) &= -1
\end{aligned} \tag{5.19}$$

On the other hand, since each $\sigma_y(j)$ is either $+1$ or -1 and each occurs twice, we have

$$\sigma_x(1)\sigma_y(2)\sigma_y(3)\ \sigma_y(1)\sigma_x(2)\sigma_y(3)\ \sigma_y(1)\sigma_y(2)\sigma_x(3) = \sigma_x(1)\sigma_x(2)\sigma_x(3) = 1$$

and this is a contradiction with 5.19. The essence of the contradiction is considering $\sigma_x(i)$'s appeared in all measurements as same.

Note also that, for the generalized Bell's inequalities of the three node systems [9], the violation of the tripartite entanglement is much greater then that of bipartite entanglement. Consider the following inequality

$$|\sigma_x(1)\sigma_y(2)\sigma_y(3)\ +\sigma_y(1)\sigma_x(2)\sigma_y(3)+\ \sigma_y(1)\sigma_y(2)\sigma_x(3)-\sigma_x(1)\sigma_x(2)\sigma_x(3)| \leq 2$$

but it is violated for GHZ state, by taking value 4.

5.2.5 Product Measurements II

For the three node density matrix $\widehat{\rho}$ given in 5.3, consider the Hermitian operator $\widehat{A}(1,2)\otimes \widehat{A}(3)$,

$$
\begin{aligned}
\widehat{A}(1,2)\otimes\widehat{A}(3) &= \frac{1}{N_1N_2N_3}\mathsf{Tr}\{\widehat{A}(1,2)\}\mathsf{Tr}\{\widehat{A}(3)\}\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3) \\
&+\frac{1}{2N_2N_3}\mathsf{Tr}\{\widehat{A}(3)\}\sum_{j=1}^{s_1} A_{j0}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_3}\mathsf{Tr}\{\widehat{A}(3)\}\sum_{k=1}^{s_2} A_{0k}(1,2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_2}\mathsf{Tr}\{\widehat{A}(1,2)\}\sum_{l=1}^{s_3} A_{l}(3)[\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_3}\mathsf{Tr}\{\widehat{A}(3)\}\sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{4N_2}\sum_{j,l=1}^{s_1,s_3} A_{j0}(1,2)A_l(3)\ [\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_1}\sum_{k,l=1,}^{s_2,s_3} A_{0k}(1,2)A_l(3)\ [\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{8}\sum_{j,k,l=1}^{s_1,s_2,s_3} A_{jk}(1,2)A_l(3)\ [\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}
$$

where

$$
\begin{aligned}
A_l(3) &= \mathsf{Tr}\left\{\widehat{A}(3)\widehat{\lambda}_l(3)\right\} \\
A_{j0}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{\lambda}_j(1)\otimes\widehat{1}(2)\right\} \\
A_{0k}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{1}(1)\otimes\widehat{\lambda}_k(2)\right\} \\
A_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{A}(1,2)\ \widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\right\}
\end{aligned}
$$

and they satisfy the following inequality

$$A_{jk}(1,2) \neq A_{j0}(1,2)A_{0k}(1,2)$$

so the 2-node correlation tensor $K_{jk}(1,2)$ for the measurement is nonzero while the 3-node correlation tensor of measurement defined given in 5.7 is zero.

$$K_{jk}(1,2)\neq 0 \qquad K_{jkl}(1,2,3)=0$$

When such a measurement is performed on the state $\widehat{\rho}$ given in 5.3, the expectation value of the measurement is

$$
\begin{aligned}
\langle \widehat{A}(1,2) \otimes \widehat{A}(3) \rangle \quad = \quad & \frac{1}{N_1 N_2 N_3} \mathsf{Tr}\{\widehat{A}(1,2)\}\mathsf{Tr}\{\widehat{A}(3)\} \\
& + \frac{1}{2N_2 N_3} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{j=1}^{s_1} A_{j0}(1,2)\lambda_j(1) \\
& + \frac{1}{2N_1 N_3} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{k=1}^{s_2} A_{0k}(1,2)\lambda_k(2) \\
& + \frac{1}{2N_1 N_2} \mathsf{Tr}\{\widehat{A}(1,2)\} \sum_{l=1}^{s_3} A_l(3)\lambda_l(3) \\
& + \frac{1}{4N_3} \mathsf{Tr}\{\widehat{A}(3)\} \sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2)\lambda_{jk}(1,2) \\
& + \frac{1}{4N_2} \sum_{j,l=1}^{s_1,s_3} A_{j0}(1,2)A_l(3)\lambda_{jl}(1,3) \\
& + \frac{1}{4N_1} \sum_{k,l=1}^{s_2,s_3} A_{0k}(1,2)A_l(3)\lambda_{kl}(2,3) \\
& + \frac{1}{8} \sum_{j,k,l=1}^{s_1,s_2,s_3} A_{jk}(1,2)A_l(3)\ \lambda_{jkl}(1,2,3)
\end{aligned}
$$

Recall that for two node systems, the state after the measurement could be product or nonproduct, so the state $\widehat{\rho}'_{12}$ of the first two part of the whole systems after the measurement can be product or nonproduct, but the state of the total system after measurement is definitely tensor product of $\widehat{\rho}'_{12}$ and $\widehat{\rho}'_3$

$$\widehat{\rho}' = \widehat{\rho}'_{12} \otimes \widehat{\rho}'_3$$

Note that, the expectation value of the total measurement is not equal to the product of the expectation values of the measurements performed separately, i.e.

$$\langle \widehat{A}(1) \otimes \widehat{A}(2) \rangle_{\widehat{\rho}} \neq \langle \widehat{A}(1,2) \rangle_{\widehat{\rho}_{12}} \langle A(3) \rangle_{\widehat{\rho}_3}$$

The expectation values of the independent measurements performed on subsystems are

$$\begin{aligned}
\mathsf{Tr}\{\widehat{\rho}_{12}\widehat{A}(1,2)\} &= \frac{1}{N_1N_2}\mathsf{Tr}\{\widehat{A}(1,2)\} + \frac{1}{2N_2}\sum_{j=1}^{s_1} A_{j0}(1,2)\lambda_j(1) \\
&+\frac{1}{2N_1}\sum_{k=1}^{s_2} A_{0k}(1,2)\lambda_k(2) + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} A_{jk}(1,2)\lambda_{jk}(1,2) \\
\mathsf{Tr}\left\{\widehat{\rho}_3\widehat{A}(3)\right\} &= \frac{1}{N_3}\mathsf{Tr}\{\widehat{A}(3)\} + \frac{1}{2}\sum_{l=1}^{s_3}\lambda_l(3)A_l(3)
\end{aligned}$$

Therefore, the relation between the expectation value of the total measurement and the product of the expectation values of the independent measurements is

$$\begin{aligned}
\langle\widehat{A}(1)\otimes\widehat{A}(2)\rangle_{\widehat{\rho}} &= \langle\widehat{A}(1)\rangle_{\widehat{\rho}_1}\ \langle\widehat{A}(2)\rangle_{\widehat{\rho}_2}\langle A(3)\rangle_{\widehat{\rho}_3} \\
&+\frac{1}{4N_2}\sum_{j,l=1}^{s_1,s_3} A_{j0}(1,2)A_l(3)C_{jl}(1,3) + \frac{1}{4N_1}\sum_{k,l=1}^{s_2,s_3} A_{0k}(1,2)A_l(3)\lambda_{kl}(2,3) \\
&+\frac{1}{8}\sum_{j,k,l=1}^{s_1,s_2,s_3} A_{jk}(1,2)A_l(3)\left[\ C_{jkl}(1,2,3) + \lambda_j(1)C_{kl}(2,3) + \lambda_k(2)C_{jl}(1,3)\right]
\end{aligned}$$

5.2.6 Nonproduct Measurements

When we consider a three node system as a whole, we can measure it by just one measurement apparatus. Such a measurement $\widehat{A}(1,2,3)$ is represented by

$$\begin{aligned}
\widehat{A}(1,2,3) &= \frac{1}{N_1N_2N_3}\mathsf{Tr}\{\widehat{A}(1,2,3)\}\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3) \\
&+\frac{1}{2N_2N_3}\sum_{j=1}^{s_1} A_{j00}(1,2,3)\ [\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_3}\sum_{k=1}^{s_2} A_{0k0}(1,2,3)\ [\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{2N_1N_2}\sum_{l=1}^{s_3} A_{00l}(1,2,3)\ [\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_3}\sum_{j,k=1}^{s_1,s_2} A_{jk0}(1,2,3)\ [\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)] \\
&+\frac{1}{4N_2}\sum_{j,l=1}^{s_1,s_3} A_{j0l}(1,2,3)\ [\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{4N_1}\sum_{k,l=1,}^{s_2,s_3} A_{0kl}(1,2,3)\ [\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)] \\
&+\frac{1}{8}\sum_{j,k,l=1}^{s_1,s_2,s_3} A_{jkl}(1,2,3)\ [\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}$$

where $A_{j00}, ..., A_{jk0}, ..., A_{jkl}$ are

$$\begin{aligned}
A_{j00}(1,2,3) &= \mathsf{Tr}\left\{\widehat{A}(1,2,3)\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)\right\},\ \text{... etc.}\\
A_{jk0}(1,2,3) &= \mathsf{Tr}\left\{\widehat{A}(1,2,3)\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)\right\},\ \text{... etc.}\\
A_{jkl}(1,2,3) &= \mathsf{Tr}\left\{\widehat{A}(1,2,3)\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)\right\}
\end{aligned}$$

and the 3-node correlation tensor of measurement is nonzero

$$K_{jkl}(1,2,3) \neq 0.$$

Therefore, this is the measurement that generates the 3-node correlation. The states after the measurement can be tripartite entangled, bipartite entangled or product. Therefore, the Hilbert space of the system after the measurement is of the form

$$\begin{aligned}
\mathcal{H}(1,2,3) &= [\mathcal{H}'(1)\otimes\mathcal{H}'(2)\otimes\mathcal{H}'(3)]\\
&\quad\oplus\left[\mathcal{H}'_E(1,2)\otimes\mathcal{H}'(3)\right]\oplus\left[\mathcal{H}'_E(1,3)\otimes\mathcal{H}'(2)\right]\oplus\left[\mathcal{H}'_E(2,3)\otimes\mathcal{H}'(1)\right]\\
&\quad\oplus\mathcal{H}'_E(1,2,3)
\end{aligned}$$

where $\mathcal{H}'_E(1,2,3)$ is the space of tripartite entangled states, and $\mathcal{H}'_E(i,j)$'s are the spaces of bipartite entangled states. Since the space of product states $\mathcal{H}'(1)\otimes\mathcal{H}'(2)\otimes\mathcal{H}'(3)$ has at most $(N_1-1)(N_2-1)(N_3-1)$ dimensions, the remaining $N_1N_2+N_1N_3+N_2N_3-N_1-N_2-N_3+1$ dimensional space consists of tripartite or bipartite entangled states.

5.3 Dynamics of Bloch Vectors and Coherence Tensors

For a general state of a three node system $\widehat{\rho}$ given in eq. 5.3, consider a general Hamiltonian $\widehat{H}$ of the system

$$\begin{aligned}
\widehat{H} &= \frac{1}{N_1N_2N_3}\mathsf{Tr}\{\widehat{H}\}\widehat{1}+\frac{1}{2N_2N_3}\sum_{j=1}^{s_1}H_j(1)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)]\\
&\quad+\frac{1}{2N_1N_3}\sum_{k=1}^{s_2}H_k(2)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)]+\frac{1}{2N_1N_2}\sum_{l=1}^{s_3}H_l(3)[\widehat{1}(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)]\\
&\quad+\frac{1}{4N_3}\sum_{j,k=1}^{s_1,s_2}H_{jk}(1,2)[\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)]+\frac{1}{4N_2}\sum_{j,l=1}^{s_1,s_3}H_{jl}(1,3)[\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{\lambda}_l(3)]\\
&\quad+\frac{1}{4N_1}\sum_{k,l=1,}^{s_2,s_3}H_{kl}(2,3)[\widehat{1}(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)]
\end{aligned}$$

where

$$\begin{aligned} H_j(1) &= \mathsf{Tr}\left\{\widehat{H}\widehat{\lambda}_j(1)\otimes\widehat{1}(2)\otimes\widehat{1}(3)\right\}, \ldots \text{ etc.} \\ H_{jk}(1,2) &= \mathsf{Tr}\left\{\widehat{H}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{1}(3)\right\}, \ldots \text{ etc.} \end{aligned}$$

Because three node interaction has no physical meaning

$$H_{jkl}(1,2,3) = \mathsf{Tr}\left\{\widehat{H}\widehat{\lambda}_j(1)\otimes\widehat{\lambda}_k(2)\otimes\widehat{\lambda}_l(3)\right\} = 0$$

To get the dynamical equation of the Bloch vectors, 2-node coherence tensors and 3-node coherence tensor, we now insert the Hamiltonian and the density matrix into the Liouville equation

$$i\hbar\frac{d}{dt}\widehat{\rho} = [\widehat{H},\widehat{\rho}]$$

by using the commutator relation given in 4.18, the time derivative of the components of the Bloch vectors are

$$\begin{aligned} \frac{d\lambda_m(1)}{dt} &= \frac{1}{N_2N_3\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'}(1)\lambda_j(1) + \frac{1}{2N_3\hbar}\sum_{j',j,k} f_{j'jm}(1)H_{j'k}(1,2)\lambda_{jk}(1,2) \\ &+\frac{1}{2N_2\hbar}\sum_{j',j,l} f_{j'jm}(1)H_{j'l}(1,3)\lambda_{jl}(1,3) \\ \frac{d\lambda_n(2)}{dt} &= \frac{1}{N_1N_3\hbar}\sum_{k',k} f_{k'kn}(2)H_{k'}(2)\lambda_k(2) + \frac{1}{2N_3\hbar}\sum_{k',k,j} f_{k'kn}(2)H_{jk'}(1,2)\lambda_{jk}(1,2) \\ &+\frac{1}{2N_1\hbar}\sum_{k',k,l} f_{k'kn}(2)H_{k'l}(2,3)\lambda_{kl}(2,3) \\ \frac{d\lambda_o(3)}{dt} &= \frac{1}{N_1N_2\hbar}\sum_{l',l} f_{l'lo}(3)H_{l'}(3)\lambda_l(3) + \frac{1}{2N_2\hbar}\sum_{l',l,j} f_{l'lo}(3)H_{jl'}(1,3)\lambda_{jl}(1,3) \\ &+\frac{1}{2N_1\hbar}\sum_{l',l,k} f_{l'lo}(3)H_{kl'}(2,3)\lambda_{kl}(2,3) \end{aligned}$$

and those of the 2-node coherence tensors read

$$
\begin{aligned}
\frac{d\lambda_{mn}(1,2)}{dt} \quad = \quad & \frac{1}{N_2N_3\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'n}(1,2)\lambda_j(1) \\
& +\frac{1}{N_1N_3\hbar}\sum_{k',k} f_{k'kn}(2)H_{mk'}(1,2)\lambda_k(2) \\
& +\frac{1}{N_2N_3\hbar}\sum_{j,j'} f_{j'jm}(1)H_{j'}(1)\lambda_{jn}(1,2) \\
& +\frac{1}{N_1N_3\hbar}\sum_{k',k} f_{k'kn}(2)H_{k'}(2)\lambda_{mk}(1,2) \\
& +\frac{1}{2N_3\hbar}\sum_{j',k',j,k} d_{k'kn}(2)f_{j'jm}(1)H_{j'k'}(1,2)\lambda_{jk}(1,2) \\
& +\frac{1}{2N_3\hbar}\sum_{j',k',j,k} d_{j'jm}(1)f_{k'kn}(2)H_{j'k'}(1,2)\lambda_{jk}(1,2) \\
& +\frac{1}{4\hbar}\sum_{j',j,l} f_{j'jm}(1)H_{j'l}(1,3)\lambda_{jnl}(1,2,3) \\
& +\frac{1}{4\hbar}\sum_{k',k,l} f_{k'kn}(2)H_{k'l}(2,3)\lambda_{mkl}(1,2,3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_{mo}(1,3)}{dt} \quad = \quad & \frac{1}{N_3N_2\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'o}(1,3)\lambda_j(1) \\
& +\frac{1}{N_1N_2\hbar}\sum_{l',l} f_{l'lo}(3)H_{ml'}(1,3)\lambda_l(3) \\
& +\frac{1}{N_3N_2\hbar}\sum_{j,j'} f_{j'jm}(1)H_{j'}(1)\lambda_{jo}(1,3) \\
& +\frac{1}{N_1N_2\hbar}\sum_{l',l} f_{l'l\ o}(3)H_{l'}(3)\lambda_{ml}(1,3) \\
& +\frac{1}{2N_2\hbar}\sum_{j',l,j,l'} d_{l'lo}(3)f_{j'jm}(1)H_{j'l'}(1,3)\lambda_{jl}(1,3) \\
& +\frac{1}{2N_2\hbar}\sum_{j',l,j,l'} d_{j'jm}(1)f_{l'lo}(3)H_{j'l'}(1,3)\lambda_{jl}(1,3) \\
& +\frac{1}{4\hbar}\sum_{j',j,k} f_{j'jm}(1)H_{j'k}(1,2)\lambda_{jko}(1,2,3) \\
& +\frac{1}{4\hbar}\sum_{l',k,l} f_{l'lo}(3)H_{kl'}(2,3)\lambda_{mkl}(1,2,3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_{no}(2,3)}{dt} \quad = \quad & \frac{1}{N_3 N_1 \hbar} \sum_{k,k'} f_{k'kn}(2) H_{k'o}(2,3) \lambda_k(2) \\
& + \frac{1}{N_2 N_1 \hbar} \sum_{l',l} f_{l'lo}(3) H_{nl'}(2,3) \lambda_l(3) \\
& + \frac{1}{N_3 N_1 \hbar} \sum_{k,k'} f_{k'kn}(2) H_{k'}(2) \lambda_{ko}(2,3) \\
& + \frac{1}{N_2 N_1 \hbar} \sum_{l',l} f_{l'lo}(3) H_{l'}(3) \lambda_{nl}(2,3) \\
& + \frac{1}{2N_1 \hbar} \sum_{k,l,k',l'} d_{l'lo}(3) f_{k'kn}(2) H_{k'l'}(2,3) \lambda_{kl}(2,3) \\
& + \frac{1}{2N_1 \hbar} \sum_{k,l,k',l'} d_{k'kn}(2) f_{l'lo}(3) H_{k'l'}(2,3) \lambda_{kl}(2,3) \\
& + \frac{1}{4\hbar} \sum_{k',k,j} f_{k'kn}(2) H_{jk'}(1,2) \lambda_{jko}(1,2,3) \\
& + \frac{1}{4\hbar} \sum_{l',l,j} f_{l'lo}(3) H_{jl'}(1,3) \lambda_{jnl}(1,2,3)
\end{aligned}
$$

finally that of 3-node coherence tensor reads

$$
\begin{aligned}
\frac{d}{dt}\lambda_{mno}(1,2,3) \quad = \quad & \frac{1}{N_2N_3\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'o}(1,3)\lambda_{jn}(1,2) \\
& +\frac{1}{N_1N_3\hbar}\sum_{k',k} f_{k'kn}(2)H_{k'o}(2,3)\lambda_{mk}(1,2) \\
& +\frac{1}{N_2N_3\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'n}(1,2)\lambda_{jo}(1,3) \\
& +\frac{1}{N_1N_2\hbar}\sum_{l',l} f_{l'lo}(3)H_{nl'}(2,3)\lambda_{ml}(1,3) \\
& +\frac{1}{N_1N_3\hbar}\sum_{k',k} f_{k'kn}(2)H_{mk'}(1,2)\lambda_{ko}(2,3) \\
& +\frac{1}{N_1N_2\hbar}\sum_{l',l} f_{l'lo}(3)H_{ml'}(1,3)\lambda_{nl}(2,3) \\
& +\frac{1}{N_2N_3\hbar}\sum_{j',j} f_{j'jm}(1)H_{j'}(1)\lambda_{jno}(1,2,3) \\
& +\frac{1}{N_1N_3\hbar}\sum_{k',k} f_{k'kn}(2)H_{k'}(2)\lambda_{mko}(1,2,3) \\
& +\frac{1}{N_1N_2\hbar}\sum_{l',l} f_{l'lo}(3)H_{l'}(3)\lambda_{mnl}(1,2,3) \\
& +\frac{1}{2N_3\hbar}\sum_{j',k',j,k} d_{k'kn}(2)f_{j'jm}(1)H_{j'k'}(1,2)\lambda_{jko}(1,2,3) \\
& +\frac{1}{2N_3\hbar}\sum_{j',k',j,k} d_{j'jm}(1)f_{k'kn}(2)H_{j'k'}(1,2)\lambda_{jko}(1,2,3) \\
& +\frac{1}{2N_2\hbar}\sum_{j',l',j,l} d_{l'lo}(3)f_{j'jm}(1)H_{j'l'}(1,3)\lambda_{jnl}(1,2,3) \\
& +\frac{1}{2N_2\hbar}\sum_{j',l',j,l} d_{j'jm}(1)f_{l'lo}(3)H_{j'l'}(1,3)\lambda_{jnl}(1,2,3) \\
& +\frac{1}{2N_1\hbar}\sum_{k',l',k,l} d_{l'lo}(3)f_{k'kn}(2)H_{k'l'}(2,3)\lambda_{mkl}(1,2,3) \\
& +\frac{1}{2N_1\hbar}\sum_{k',l',k,l} d_{k'kn}(2)f_{l'lo}(3)H_{k'l'}(2,3)\lambda_{mkl}(1,2,3)
\end{aligned}
$$

by defining the terms

$$
\begin{aligned}
\Omega_{mj}(1) &: = \frac{1}{N_2N_3\hbar}\sum_{j'} f_{j'jm}(1)H_{j'}(1) \\
\Omega_{nk}(2) &: = \frac{1}{N_1N_3\hbar}\sum_{k'} f_{k'kn}(2)H_{k'}(2) \\
\Omega_{ol}(3) &: = \frac{1}{N_1N_2\hbar}\sum_{l'} f_{l'lo}(3)H_{l'}(3) \\
Q_{mjk}(1;1,2) &: = \frac{1}{\hbar}\sum_{j'} f_{j'jm}(1)H_{j'k}(1,2) \\
Q_{jnk}(2;1,2) &: = \frac{1}{\hbar}\sum_{k'} f_{k'kn}(2)H_{jk'}(1,2) \\
Q_{mjl}(1;1,3) &: = \frac{1}{\hbar}\sum_{j'} f_{j'jm}(1)H_{j'l}(1,3) \\
Q_{jol}(3;1,3) &: = \frac{1}{\hbar}\sum_{l'} f_{l'lo}(3)H_{jl'}(1,3) \\
Q_{nkl}(2;2,3) &: = \frac{1}{\hbar}\sum_{k'} f_{k'kn}(2)H_{k'l}(2,3) \\
Q_{kol}(3;2,3) &: = \frac{1}{\hbar}\sum_{l'} f_{l'lo}(3)H_{kl'}(2,3) \\
D_{nkmj}(1,2) &: = \frac{1}{N_3\hbar}\sum_{j',k'}[d_{k'kn}(2)f_{j'jm}(1) + d_{j'jm}(1)f_{k'kn}(2)]H_{j'k'}(1,2) \\
D_{olmj}(1,3) &: = \frac{1}{N_2\hbar}\sum_{j',l'}[d_{l'lo}(3)f_{j'jm}(1) + d_{j'jm}(1)f_{l'lo}(3)]H_{j'l'}(1,3) \\
D_{olnk}(2,3) &: = \frac{1}{N_1\hbar}\sum_{k',l'}[d_{l'lo}(3)f_{k'kn}(2) + d_{k'kn}(2)f_{l'lo}(3)]H_{k'l'}(2,3)
\end{aligned}
$$

the time derivative equations take their final form as

$$
\begin{aligned}
\frac{d\lambda_m(1)}{dt} &= \sum_j \Omega_{mj}(1)\lambda_j(1) + \frac{1}{2}\sum_{j,k} Q_{mjk}(1;1,2)\lambda_{jk}(1,2) + \frac{1}{2}\sum_{j,l} Q_{mjl}(1;1,3)\lambda_{jl}(1,3) \\
\frac{d\lambda_n(2)}{dt} &= \sum_k \Omega_{nk}(2)\lambda_k(2) + \frac{1}{2}\sum_{k,j} Q_{jnk}(2;1,2)\lambda_{jk}(1,2) + \frac{1}{2}\sum_{k,l} Q_{nkl}(2;2,3)\lambda_{kl}(2,3) \\
\frac{d\lambda_o(3)}{dt} &= \sum_l \Omega_{ol}(3)\lambda_l(3) + \frac{1}{2}\sum_{l,j} Q_{jol}(3;1,3)\lambda_{jl}(1,3) + \frac{1}{2}\sum_{l,k} Q_{kol}(3;2,3)\lambda_{kl}(2,3)
\end{aligned}
$$

$$\begin{aligned}
\frac{d\lambda_{mn}(1,2)}{dt} &= \sum_j \Omega_{mj}(1)\lambda_{jn}(1,2) + \sum_k \Omega_{nk}(2)\lambda_{mk}(1,2) \\
&+ \frac{1}{N_2 N_3}\sum_j Q_{mjn}(1;1,2)\lambda_j(1) + \frac{1}{N_1 N_3}\sum_k Q_{mnk}(2;1,2)\lambda_k(2) \\
&+ \frac{1}{4}\sum_{j,l} Q_{mjl}(1;1,3)\lambda_{jnl}(1,2,3) + \frac{1}{4}\sum_{k,l} Q_{mkl}(2;2,3)\lambda_{mkl}(1,2,3) \\
&+ \frac{1}{2}\sum_{,j,k} D_{nkmj}(1,2)\lambda_{jk}(1,2)
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_{mo}(1,3)}{dt} &= \sum_j \Omega_{mj}(1)\lambda_{jo}(1,3) + \sum_l \Omega_{ol}(3)\lambda_{ml}(1,3) \\
&+ \frac{1}{N_3 N_2}\sum_j Q_{mjo}(1;1,3)\lambda_j(1) + \frac{1}{N_1 N_2 \hbar}\sum_l Q_{mol}(3;1,3)\lambda_l(3) \\
&+ \frac{1}{4}\sum_{j,k} Q_{mjk}(1;1,2)\lambda_{jko}(1,2,3) + \frac{1}{4}\sum_{k,l} Q_{kol}(3;2,3)\lambda_{mkl}(1,2,3) \\
&+ \frac{1}{2}\sum_{l,j} D_{olmj}(1,3)\lambda_{jl}(1,3)
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_{no}(2,3)}{dt} &= \sum_k \Omega_{nk}(2)\lambda_{ko}(2,3) + \sum_l \Omega_{ol}(3)\lambda_{nl}(2,3) \\
&+ \frac{1}{N_3 N_1}\sum_k Q_{nko}(2;2,3)\lambda_k(2) + \frac{1}{N_2 N_1}\sum_l Q_{nol}(3;2,3)\lambda_l(3) \\
&+ \frac{1}{4}\sum_{k,j} Q_{jnk}(2;1,2)\lambda_{jko}(1,2,3) + \frac{1}{4}\sum_{l,j} Q_{jol}(3;1,3)\lambda_{jnl}(1,2,3) \\
&+ \frac{1}{2}\sum_{k,l} D_{olnk}(2,3)\lambda_{kl}(2,3)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\lambda_{mno}(1,2,3) &= \sum_j \Omega_{mj}(1)\lambda_{jno}(1,2,3) + \sum_k \Omega_{nk}(2)\lambda_{mko}(1,2,3) + \sum_l \Omega_{ol}(3)\lambda_{mnl}(1,2,3) \\
&+ \frac{1}{N_2 N_3}\sum_j Q_{mjo}(1;1,3)\lambda_{jn}(1,2) + \frac{1}{N_1 N_3}\sum_k Q_{nko}(2;2,3)\lambda_{mk}(1,2) \\
&+ \frac{1}{N_2 N_3}\sum_j Q_{mjn}(1;1,2)\lambda_{jo}(1,3) + \frac{1}{N_1 N_2}\sum_l Q_{nol}(3;2,3)\lambda_{ml}(1,3) \\
&+ \frac{1}{N_1 N_3}\sum_k Q_{mnk}(2;1,2)\lambda_{ko}(2,3) + \frac{1}{N_1 N_2}\sum_l Q_{mol}(3;1,3)\lambda_{nl}(2,3) \\
&+ \frac{1}{2}\sum_{j,k} D_{nkmj}(1,2)\lambda_{jko}(1,2,3) + \frac{1}{2}\sum_{j,l} D_{olmj}(1,3)\lambda_{jnl}(1,2,3) \\
&+ \frac{1}{2}\sum_{k,l} D_{olnk}(2,3)\lambda_{mkl}(1,2,3)
\end{aligned}$$

the time derivative of the 2-node correlation tensors are as following

$$\begin{aligned}
\frac{dC_{mn}(1,2)}{dt} &= \sum_j \Omega_{mj}(1)C_{jn}(1,2) + \sum_k \Omega_{nk}(2)C_{mk}(1,2) \\
&-\frac{1}{2}\sum_{j,k} Q_{mjk}(1;1,2)\lambda_n(2)\lambda_{jk}(1,2) + \frac{1}{N_2N_3}\sum_j Q_{mjn}(1;1,2)\lambda_j(1) \\
&-\frac{1}{2}\sum_{j,l} Q_{mjl}(1;1,3)\lambda_n(2)\lambda_{jl}(1,3) + \frac{1}{4}\sum_{j,l} Q_{mjl}(1;1,3)\lambda_{jnl}(1,2,3) \\
&-\frac{1}{2}\sum_{k,j} Q_{jnk}(2;1,2)\lambda_m(1)\lambda_{jk}(1,2) + \frac{1}{N_1N_3}\sum_k Q_{mnk}(2;1,2)\lambda_k(2) \\
&-\frac{1}{2}\sum_{k,l} Q_{nkl}(2;2,3)\lambda_m(1)\lambda_{kl}(2,3) + \frac{1}{4}\sum_{k,l} Q_{mkl}(2;2,3)\lambda_{mkl}(1,2,3) \\
&+\frac{1}{2}\sum_{j,k} D_{nkmj}(1,2)\lambda_{jk}(1,2)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}C_{mo}(1,3) &= \sum_j \Omega_{mj}(1)C_{jo}(1,3) + \sum_l \Omega_{ol}(3)C_{ml}(1,3) \\
&-\frac{1}{2}\sum_{j,k} Q_{mjk}(1;1,2)\lambda_o(3)\lambda_{jk}(1,2) + \frac{1}{4}\sum_{j,k} Q_{mjk}(1;1,2)\lambda_{jko}(1,2,3) \\
&-\frac{1}{2}\sum_{j,l} Q_{mjl}(1;1,3)\lambda_o(3)\lambda_{jl}(1,3) + \frac{1}{N_3N_2}\sum_j Q_{mjo}(1;1,3)\lambda_j(1) \\
&-\frac{1}{2}\sum_{l,j} Q_{jol}(3;1,3)\lambda_m(1)\lambda_{jl}(1,3) + \frac{1}{N_1N_2}\sum_l Q_{mol}(3;1,3)\lambda_l(3) \\
&-\frac{1}{2}\sum_{l,k} Q_{kol}(3;2,3)\lambda_m(1)\lambda_{kl}(2,3) + \frac{1}{4}\sum_{k,l} Q_{kol}(3;2,3)\lambda_{mkl}(1,2,3) \\
&+\frac{1}{2}\sum_{l,j} D_{olmj}(1,3)\lambda_{jl}(1,3)
\end{aligned}$$

$$
\begin{aligned}
\frac{dC_{no}(2,3)}{dt} \quad = \quad & \sum_{k} \Omega_{nk}(2)C_{ko}(2,3) + \sum_{l} \Omega_{ol}(3)C_{nl}(2,3) \\
& -\frac{1}{2}\sum_{k,j} Q_{jnk}(2;1,2)\lambda_o(3)\lambda_{jk}(1,2) + \frac{1}{4}\sum_{k,j} Q_{jnk}(2;1,2)\lambda_{jko}(1,2,3) \\
& -\frac{1}{2}\sum_{l,j} Q_{jol}(3;1,3)\lambda_n(2)\lambda_{jl}(1,3) + \frac{1}{4}\sum_{l,j} Q_{jol}(3;1,3)\lambda_{jnl}(1,2,3) \\
& -\frac{1}{2}\sum_{k,l} Q_{nkl}(2;2,3)\lambda_o(3)\lambda_{kl}(2,3) + \frac{1}{N_3N_1}\sum_{k} Q_{nko}(2;2,3)\lambda_k(2) \\
& -\frac{1}{2}\sum_{l,k} Q_{kol}(3;2,3)\lambda_n(2)\lambda_{kl}(2,3) + \frac{1}{N_2N_1}\sum_{l} Q_{nol}(3;2,3)\lambda_l(3) \\
& +\frac{1}{2}\sum_{k,l} D_{olnk}(2,3)\lambda_{kl}(2,3)
\end{aligned}
$$

and the time derivative of the 3-node correlation tensor is equal to

$$
\begin{aligned}
&\frac{dC_{mno}(1,2,3)}{dt} = \\
&\sum_j \Omega_{mj}(1)C_{jno}(1,2,3) + \sum_k \Omega_{nk}(2)C_{mko}(1,2,3) + \sum_l \Omega_{ol}(3)C_{mnl}(1,2,3) \\
&+\frac{1}{N_2N_3}\sum_j Q_{mjn}(1;1,2)C_{jo}(1,3) + \frac{1}{N_2N_3}\sum_j Q_{mjo}(1;1,3)C_{jn}(1,2) \\
&+\frac{1}{N_1N_3}\sum_k Q_{mnk}(2;1,2)C_{ko}(2,3) + \frac{1}{N_1N_3}\sum_k Q_{nko}(2;2,3)C_{mk}(1,2) \\
&+\frac{1}{N_1N_2}\sum_l Q_{mol}(3;1,3)C_{nl}(2,3) + \frac{1}{N_1N_2}\sum_l Q_{nol}(3;2,3)C_{ml}(1,3) \\
&-\frac{1}{4}\sum_{j,k} Q_{mjk}(1;1,2)\left[\lambda_n(2)\lambda_{jko}(1,2,3) + 2\lambda_{jk}(1,2)C_{no}(2,3) - 2\lambda_n(2)\lambda_o(3)\lambda_{jk}(1,2)\right] \\
&-\frac{1}{4}\sum_{j,l} Q_{mjl}(1;1,3)\left[\lambda_o(3)\lambda_{jnl}(1,2,3) + 2\lambda_{jl}(1,3)C_{no}(2,3) - 2\lambda_n(2)\lambda_o(3)\lambda_{jl}(1,3)\right] \\
&-\frac{1}{4}\sum_{k,j} Q_{jnk}(2;1,2)\left[\lambda_m(1)\lambda_{jko}(1,2,3) + 2\lambda_{jk}(1,2)C_{mo}(1,3) - 2\lambda_m(1)\lambda_o(3)\lambda_{jk}(1,2)\right] \\
&-\frac{1}{4}\sum_{k,l} Q_{mkl}(2;2,3)\left[\lambda_o(3)\lambda_{mkl}(1,2,3) + 2\lambda_{kl}(2,3)C_{mo}(1,3) - 2\lambda_m(1)\lambda_o(3)\lambda_{kl}(2,3)\right] \\
&-\frac{1}{4}\sum_{l,j} Q_{jol}(3;1,3)\left[\lambda_m(1)\lambda_{jnl}(1,2,3) + 2\lambda_{jl}(1,3)C_{mn}(1,2) - 2\lambda_m(1)\lambda_n(2)\lambda_{jl}(1,3)\right] \\
&-\frac{1}{4}\sum_{k,l} Q_{kol}(3;2,3)\left[\lambda_n(2)\lambda_{mkl}(1,2,3) + 2\lambda_{kl}(2,3)C_{mn}(1,2) - 2\lambda_m(1)\lambda_n(2)\lambda_{kl}(2,3)\right] \\
&+\frac{1}{2}\sum_{j,k} D_{nkmj}(1,2)C_{jk\_o}(1,2;3) + \frac{1}{2}\sum_{j,l} D_{olmj}(1,3)C_{j\_n\_l}(1;2;3) \\
&+\frac{1}{2}\sum_{k,l} D_{olnk}(2,3)C_{m\_kl}(1;2,3)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{m\_no}(1;2,3) &: = \lambda_{mno}(1,2,3) - \lambda_m(1)\lambda_{no}(2,3) \\
C_{m\_n\_o}(1;2;3) &: = \lambda_{mno}(1,2,3) - \lambda_n(2)\lambda_{mo}(1,3) \\
C_{mn\_o}(1,2;3) &: = \lambda_{mno}(1,2,3) - \lambda_o(3)\lambda_{mn}(1,2)
\end{aligned}
$$

As in the two node systems, the correlation tensors of the states experience orthogonal transformations under products Hamiltonians of the type $\widehat{H}_1 \otimes \widehat{H}_2 \otimes \widehat{H}_3$ so product states stay product. A product state only gets correlated under a nonproduct Hamiltonian.

Consider the evolution of a product states i.e.

$$C_{jk}(1,2) = C_{jl}(1,3) = C_{kl}(2,3) = C_{jkl}(1,2,3) = 0$$

under a nonproduct Hamiltonian. The time derivative of the tripartite correlation can be found by similar treatment that we followed in two node case, and it is equal to

$$\begin{aligned}
\frac{d}{dt}C_{mno}(1,2,3) &= \frac{1}{4}\sum_{j,k}\left[Q^K_{mjk}(1;1,2)\lambda_n(2) + Q^K_{jnk}(2;1,2)\lambda_m(1)\right]\lambda_j(1)\lambda_k(2)\lambda_o(3) \\
&+\frac{1}{4}\sum_{j,l}\left[Q^K_{mjl}(1;1,3)\lambda_o(3) + Q^K_{jol}(3;1,3)\lambda_m(1)\right]\lambda_j(1)\lambda_n(2)\lambda_l(3) \\
&+\frac{1}{4}\sum_{k,l}\left[Q^K_{mkl}(2;2,3)\lambda_o(3) + Q^K_{kol}(3;2,3)\lambda_n(2)\right]\lambda_m(1)\lambda_k(2)\lambda_l(3)
\end{aligned}$$

where

$$\begin{aligned}
Q^K_{mjk}(1;1,2) &: = \frac{1}{\hbar}\sum_{j'} f_{j'jm}(1)K_{j'k}(1,2) \\
Q^K_{jnk}(2;1,2) &: = \frac{1}{\hbar}\sum_{k'} f_{k'kn}(2)K_{jk'}(1,2) \\
Q^K_{mjl}(1;1,3) &: = \frac{1}{\hbar}\sum_{j'} f_{j'jm}(1)K_{j'l}(1,3) \\
Q^K_{jol}(3;1,3) &: = \frac{1}{\hbar}\sum_{l'} f_{l'lo}(3)K_{jl'}(1,3) \\
Q^K_{nkl}(2;2,3) &: = \frac{1}{\hbar}\sum_{k'} f_{k'kn}(2)K_{k'l}(2,3) \\
Q^K_{kol}(3;2,3) &: = \frac{1}{\hbar}\sum_{l'} f_{l'lo}(3)K_{kl'}(2,3)
\end{aligned}$$

Chapter 6

CONCLUSION

This thesis is mainly concerned with the Bloch vector formalism for finite dimensional, composite quantum systems. Through this formalism, it is possible to represent pure and mixed states by the expectation values of the generators of $\mathsf{SU(N)}$ and that make it convenient to deal with quantum systems both theoretically and experimentally.

Before giving the detail of this representation, the fundamental properties of the density matrix formalism is briefly explained in the Chapter 2. Then, the Bloch vector representation is introduced and its characterization is given in Chapter 3. In addition to that, the Chapter 3 includes original results about the geometry of the Bloch vector space, which is helpful to understand the nature of the entangled states. Chapter 3 is concluded with the introduction of the Bloch vector formalism for the measurement and the time evolution of the systems.

In the Chapter 4 and Chapter 5, the Bloch vector formalism for bipartite and tripartite systems is deeply analyzed by considering all types of the Hermitian matrices used in measurements and Hamiltonians responsible for the time evolution of the systems. Since the state of the subsystems and the correlation between them are expressed clearly, their interactions with nonlocal Hermitian matrices that take a role both in the measurement and in the evolution of the system, became more meaningful and this is very important for controlling the behavior of the quantum systems. Although this thesis deals only with fundamental properties of quantum mechanical systems, it will have straightforward applications to physical systems.

Appendix A

A.1

The first few power of $\widehat{\rho}$ represented in the basis given eq. 3.4 is as followings

$$\begin{aligned}
\widehat{\rho} &= \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i \widehat{\lambda}_i \\
\widehat{\rho}^2 &= \frac{1}{N^2}\widehat{1} + \frac{1}{N}\sum_{i=1}^{N^2-1} \lambda_i \widehat{\lambda}_i + \frac{1}{4}\sum_{i,j=1}^{N^2-1} \lambda_i \lambda_j \widehat{\lambda}_i \widehat{\lambda}_j \\
\widehat{\rho}^3 &= \frac{1}{N^3}\widehat{1} + \frac{3}{2N^2}\sum_{i=1}^{N^2-1} \lambda_i \widehat{\lambda}_i + \frac{3}{4N}\sum_{i,j=1}^{N^2-1} \lambda_i \lambda_j \widehat{\lambda}_i \widehat{\lambda}_j + \frac{1}{8}\sum_{i,j,k=1}^{N^2-1} \lambda_i \lambda_j \lambda_k \widehat{\lambda}_i \widehat{\lambda}_j \widehat{\lambda}_k \\
\widehat{\rho}^4 &= ....
\end{aligned}$$

for these terms $C(N,q) = \mathsf{Tr}\{\widehat{\rho}^q\}$ equals to

$$\begin{aligned}
C(N,1) &= 1 \\
C(N,2) &= \frac{1}{N} + \frac{1}{2}|\lambda|^2 \\
C(N,3) &= \frac{1}{N^2} + \frac{3}{2N}|\lambda|^2 + \frac{1}{4}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k \\
C(N,4) &= \frac{1}{N^3} + \frac{3}{N^2}|\lambda|^2 + \frac{1}{N}\sum_{i,j,k} d_{ijk}\lambda_i\lambda_j\lambda_k + \frac{1}{4N}|\lambda|^4 + \frac{1}{8}\sum_{i,j,k,m} d_{ijm}d_{mkl}\lambda_i\lambda_j\lambda_k\lambda_l \\
C(N,5) &= ....
\end{aligned}$$

A.2

The first few power of $\widehat{\rho}$ represented in the basis given eq. 4.1 is as followings

$$
\begin{aligned}
\widehat{\rho} &= \frac{1}{n_1 n_2}\widehat{1} + \frac{1}{2n_2}\sum_{j=1}^{s_1} \lambda_j(1)[\widehat{\lambda}_j(1) \otimes \widehat{1}(2)] \\
&+ \frac{1}{2n_1}\sum_{k=1}^{s_2} \lambda_k(2)[\widehat{1}(1) \otimes \widehat{\lambda}_k(2)] + \frac{1}{4}\sum_{j,k=1}^{s_1,s_2} \lambda_{jk}(1,2)[\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)] \\
\widehat{\rho}^2 &= \left(\frac{1}{n_1 n_2}\right)^2 \widehat{1} + \frac{1}{n_1 n_2^2}\sum_{j=1}^{s_1} \lambda_j(1)[\widehat{\lambda}_j(1) \otimes \widehat{1}(2)] \\
&+ \frac{1}{n_1^2 n_2}\sum_{k=1}^{s_2} \lambda_k(2)[\widehat{1}(1) \otimes \widehat{\lambda}_k(2)] \\
&+ \frac{1}{2n_1 n_2}\sum_{j,k=1}^{s_1,s_2} (\lambda_{jk}(1,2) + \lambda_j(1)\lambda_k(2))[\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)] \\
&+ \frac{1}{4n_2^2}\sum_{j,i=1}^{s_1} \lambda_j(1)\lambda_i(1)[\widehat{\lambda}_j(1)\widehat{\lambda}_i(1) \otimes \widehat{1}(2)] \\
&+ \frac{1}{4n_1^2}\sum_{j,i=1}^{s_1} \lambda_j(2)\lambda_i(2)[\widehat{1}(1) \otimes \widehat{\lambda}_j(2)\widehat{\lambda}_i(2)] \\
&+ \frac{1}{4n_2}\sum_{j,k,i=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_i(1)[\widehat{\lambda}_i(1)\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)] \\
&+ \frac{1}{4n_1}\sum_{j,k,i=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_i(2)[\widehat{\lambda}_j(1) \otimes \widehat{\lambda}_k(2)\widehat{\lambda}_i(2)] \\
&+ \frac{1}{16}\sum_{j,k,m,n=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_{mn}(1,2)[\widehat{\lambda}_j(1)\widehat{\lambda}_m(1) \otimes \widehat{\lambda}_k(1)\widehat{\lambda}_n(2)]
\end{aligned}
$$

for these terms $C(N,q) = \mathsf{Tr}\{\widehat{\rho}^q\}$ equals to

$$
\begin{aligned}
C(N,2) &= \frac{1}{n_1 n_2} + \frac{1}{2n_2}|\lambda(1)|^2 + \frac{1}{2n_1}|\lambda(2)|^2 + \frac{1}{4}|\lambda(1,2)|^2 \\
C(N,3) &= \left(\frac{1}{n_1 n_2}\right)^2 + \frac{3}{2n_1 n_2^2}|\lambda(1)|^2 + \frac{3}{2n_1^2 n_2}|\lambda(2)|^2 + \frac{3}{4n_1 n_2}|\lambda(1,2)|^2 \\
&\quad + \frac{1}{4n_2^2}\sum_{i,j,k} d_{ijk}(1)\lambda_i(1)\lambda_j(1)\lambda_k(1) \\
&\quad + \frac{1}{4n_1^2}\sum_{i,j,k} d_{ijk}(2)\lambda_i(2)\lambda_j(2)\lambda_k(2) \\
&\quad + \frac{3}{2n_1 n_2}\sum_{j,k=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_k(2)\lambda_j(1) \\
&\quad + \frac{3}{8n_2}\sum_{j,k,im=1}^{s_1,s_2} d_{ijm}(1)\lambda_{jk}(1,2)\lambda_i(1)\lambda_{mk}(1,2) \\
&\quad + \frac{3}{8n_1}\sum_{j,k,i,m=1}^{s_1,s_2} d_{kim}(2)\lambda_{jk}(1,2)\lambda_i(2)\lambda_{jm}(1,2) \\
&\quad + \frac{1}{16}\sum_{j,k,m,n,d,r=1}^{s_1,s_2} \lambda_{jk}(1,2)\lambda_{mn}(1,2)\lambda_{dr}(1,2)d_{jmd}(1)d_{knr}(2) \\
C(N,3) &= \ldots
\end{aligned}
$$

Appendix B

The orthonormal basis of $\mathsf{su(N)}$ consisting of a set of elements $\widehat{\lambda}_i$ $(i = 1, ..., N^2 - 1)$ satisfying the relations

$$(a)\widehat{\lambda}_i^\dagger = \widehat{\lambda}_i,\ (b)\mathsf{Tr}\{\widehat{\lambda}_i\} = 0,\ (c)\mathsf{Tr}\{\widehat{\lambda}_i\widehat{\lambda}_j\} = 2\delta_{ij} \tag{B.1}$$

and it is an $N^2 - 1$ dimensional real vector space in the sense of Hilbert–Schmidt product

$$\langle A, B\rangle = \frac{1}{2}\mathsf{Tr}\{A^\dagger B\} = \frac{1}{2}\mathsf{Tr}\{AB\}$$

Note that, $\mathsf{su(N)}$ is isomorphic to $\mathbf{R}^{N^2-1}$.

Now suppose that U is an element of $\mathsf{SU(N)}$ and A is an element of $\mathsf{su(N)}$, and consider $\mathsf{U}A\mathsf{U}^\dagger$, since $\mathsf{Tr}\{AB\} = \mathsf{Tr}\{BA\}$,

$$\mathsf{Tr}\{\mathsf{U}A\mathsf{U}^\dagger\} = \mathsf{Tr}\{A\} = 0$$

and

$$(\mathsf{U}A\mathsf{U}^\dagger)^\dagger = (\mathsf{U}^\dagger)^\dagger A^\dagger \mathsf{U}^\dagger = \mathsf{U}A\mathsf{U}^\dagger$$

and so $\mathsf{U}A\mathsf{U}^{-1}$ is again in $\mathsf{su(N)}$. Furthermore, for a fixed U, the map Φ_U from $\mathsf{su(N)}$ to itself defined as

$$\Phi_\mathsf{U}(A) := \mathsf{U}A\mathsf{U}^{-1} \text{ where } \Phi : \mathsf{SU(N)} \to \mathsf{su(N)}$$

is linear in $\mathsf{su(N)}$. Moreover, given $\mathsf{U} \in \mathsf{SU(N)}$ and $A, B \in \mathsf{su(N)}$, we have

$$\langle \Phi_\mathsf{U}(A), \Phi_\mathsf{U}(B)\rangle = \frac{1}{2}\mathsf{Tr}\{\mathsf{U}A\mathsf{U}^{-1}\mathsf{U}B\mathsf{U}^{-1}\} = \frac{1}{2}\mathsf{Tr}\{AB\} = \langle A, B\rangle$$

Thus, Φ_U is an orthogonal transformation in $\mathsf{su(N)}$ and since we identified $\mathsf{su(N)}$ with $\mathbf{R}^{N^2-1}$, Φ_U is an element of $\mathsf{O}(N^2 - 1)$. So, Φ is a map from $\mathsf{su(N)}$ to $\mathsf{O}(N^2 - 1)$.

Note that

$$\Phi_{\mathsf{U}_1\mathsf{U}_2} = \mathsf{U}_1\mathsf{U}_2 A \mathsf{U}_2^{-1}\mathsf{U}_1^{-1} = (\mathsf{U}_1\mathsf{U}_2)A(\mathsf{U}_1\mathsf{U}_2)^{-1} = \Phi_{\mathsf{U}_1}\Phi_{\mathsf{U}_2}$$

Since $\Phi_{\mathsf{U}_1\mathsf{U}_2} = \Phi_{\mathsf{U}_1}\Phi_{\mathsf{U}_2}$, we see that Φ is continuous and, thus, a Lie group homomorphism between $\mathsf{su(N)}$ to $\mathsf{O(N^2{-}1)}$.

Recall that every element of $\mathsf{O}(\mathsf{N}^2{-}1)$ has determinant ±1. Now, $\mathsf{su}(\mathsf{N})$ is connected, Φ is continuous, and Φ_I is equal to I, which has determinant one. It follows that Φ must actually map $\mathsf{su}(\mathsf{N})$ into the $\mathsf{SO}(\mathsf{N}^2{-}1)$.

The map Φ is two-to-one, since $\mathsf{U} \in \mathsf{su}(\mathsf{N})$, $\Phi_{\mathsf{U}} = \Phi_{-\mathsf{U}}$. Also, it is onto for $N = 2$ and not onto for $N \geq 3$ since dimension of $\mathsf{SO}(\mathsf{N})$ is $N^2 - 1$ while dimension of $\mathsf{SO}(\mathsf{N}^2{-}1)$ is $\frac{(N^2-1)(N^2-2)}{2}$.

Unitary Transformation of the Density Matrix

The unitary transformation U of density matrix $\widehat{\rho}$

$$\mathsf{U}\widehat{\rho}\mathsf{U}^\dagger = \frac{1}{N}\widehat{1} + \frac{1}{2}\sum_{i=1}^{N^2-1} \lambda_i \mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger$$

is equal to an application of $\mathsf{O} \in \mathsf{SO}(\mathsf{N}^2{-}1)$ to λ, therefore following equality holds,

$$\begin{aligned}
\sum_{i=1}^{N^2-1} \lambda_i \mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger &= \sum_{j=1}^{N^2-1}(\sum_{i=1}^{N^2-1} \mathsf{O}_{ji}\lambda_i)\widehat{\lambda}_j \\
\sum_{i=1}^{N^2-1} \lambda_i \mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger\} &= \sum_{j=1}^{N^2-1}(\sum_{i=1}^{N^2-1} \mathsf{O}_{ji}\lambda_i)\mathsf{Tr}\{\widehat{\lambda}_m\widehat{\lambda}_j\} \\
\sum_{i=1}^{N^2-1} \lambda_i \mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger\} &= 2\sum_{i=1}^{N^2-1} \mathsf{O}_{mi}\lambda_i
\end{aligned}$$

and the corresponding element O of $\mathsf{SO}(\mathsf{N}^2{-}1)$ for the unitary transformation U is

$$\frac{1}{2}\mathsf{Tr}\{\widehat{\lambda}_m\mathsf{U}\widehat{\lambda}_i\mathsf{U}^\dagger\} = \mathsf{O}_{mi}$$

BIBLIOGRAPHY

[1] R.P. Feynman. International Journal of Theoretical Physics, 21, 467 (1982).

[2] D. Deutsch. Proceedings of the Royal Society of London A, 400, 97, (1985).

[3] P. Shor, Proc. 35nd Annual Symposium on Foundations of Computer Science (Shafi Goldwasser, ed.), IEEE Computer Society Press 124-134,(1994).

[4] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information, Cambridge University Press, Cambridge, U.K. (2000)

[5] D. Bruss, J. Math. Phys. 43, 4237 (2002)

[6] J. Preskill, Physics 229: Advanced mathematical methods of physics— Quantum computation and information, California Institute of Technology (1998) URL:http//www.theory.caltech.edu/people/preskill/ph229/

[7] K. Blum, Density Matrix Theory and Applications, Plenum Press, New York (1981)

[8] F.T. Hioe and J.H. Eberly, Phys. Rev. Lett. 47, 838 (1981)

[9] G. Mahler, V.A. Weberruss, Quantum Networks: Dynamics of Open Nanostructures, Springer, Berlin, (1995)

[10] G. Kimura, Phys. Lett. A 314, 339 (2003)

[11] H. S. M. Coxeter, Regular Polytopes, 3rd. ed., Dover Publications, (1973)

[12] R. F. Werner, Phys. Rev. A 40, 4277 (1989)

[13] A. Peres, Phys. Rev. Lett. 77, 1413 (1996)

[14] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).

[15] A. Peres, Quantum Theory: Concepts and Methods, Kluwer Academic, Dordrecht, (1993)

[16] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys. Rev. Lett. 23, 880 (1969).

[17] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000)

[18] N. D. Mermin, Physics Today 43, 9 (1990)

[19] A. T. Bolukbasi and T. Dereli, E-print quant-ph/0511111; J. Phys. Conf. Ser., 36, 28 (2006).