CANONICAL INDUCTION, GREEN FUNCTORS, LEFSCHETZ INVARIANT OF MONOMIAL G-POSETS

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Hatice Mutlu

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Laurence John Barker(Advisor)

Ali Sinan Sertöz

Fatma Altunbulak Aksu

Ergün Yalçın

Ayşe Çiğdem Özcan

Approved for the Graduate School of Engineering and Science:

Ezhan Karaşan
Director of the Graduate School

ABSTRACT

CANONICAL INDUCTION, GREEN FUNCTORS, LEFSCHETZ INVARIANT OF MONOMIAL G-POSETS

Hatice Mutlu
Ph.D. in Mathematics
Advisor: Laurence John Barker
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Green functors are a kind of group functor, rather like Mackey functors, but with a further multiplicative structure. They are defined on a category whose objects are finite groups and whose morphisms are generated by maps such as induction, restriction, inflation, deflation. The aim of this thesis is general formulation for canonical induction, suitable for Green functors, optionally equipped with inflations.

Let p be a prime number. In Section 3, we apply the Boltje's theory of canonical induction [1] to p-permutation modules and give a restriction-preserving $\mathbb{Z}[1/p]$-linear canonical induction formula from the inflations of projective modules.

In Section 4, we give a general formulation of canonical induction theory for Green biset functors equipped with induction, restriction, inflation maps.

Let G be a finite group and C be an abelian group. In Section 5, motivated in part by a search for connection with Peter Symonds' proof [2] of the integrality of a canonical induction formula, we introduce a Lefschetz invariant for the C-monomial Burnside ring. These invariants let us to construct generalize tensor induction functors associated to any C-monomial (G, H)-biset from the category of C-monomial G-posets to the category of C-monomial H-posets. We will show that these functors induce well-defined tensor induction maps from $B_C(G)$ to $B_C(H)$, which in turn gives a group homomorphism $B_C(G)^\times \to B_C(H)^\times$ between the unit groups of C-monomial Burnside rings.

Keywords: Green functors, p-permutation modules, canonical induction formula, Burnside ring, monomial Burnside ring, tensor induction, Lefschetz invariant.

ÖZET

KURALSAL İNDÜKSİYON, GREEN İZLEÇLERİ, TEK TERİMLİ G-KISMİ SIRALI KÜMELERİNİN LEFSCHETZ DEĞİŞMEZLERİ

Hatice Mutlu
Matematik, Doktora
Tez Danışmanı: Laurence John Barker
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Green izleçleri, Mackey izleçlerine benzeyen fakat çarpımsal yapısı da olan bir tür grup izlecidir. Objeleri sonlu gruplar olan morfizmleri indüksiyon, kısıtlama, şişirme, söndürme fonksiyonları ile üretilen bir kategoridir. Bu tezin amacı kuralsal indüksiyonun Green izleçleri için uygun olan, opsiyonel olarak şişirme fonksiyonları ile ilişkilendirilmiş genel formülünün elde edilmesidir.

p bir asal sayı olsun. 3. kısımda Boltje'nin kuralsal indüksiyon teorisi [1] p-permütasyon modüllerine uygulandı ve projektif modülerin şişirmelerinden kısıtlama koruyan bir $\mathbb{Z}[1/p]$-lineer kuralsal indüksiyon formülü verildi.

4. kısımda indüksiyon, kısıtlama, şişirme ile ilişkilendirilmiş Green ikili küme izleçleri için bir genel kuralsal indüksiyon teorisi verildi.

G bir sonlu grup ve C bir değişmeli grup olsun. 5. kısımda, kısmen Peter Symonds'ın tam sayı katsayılı kuralsal indüksiyon formülü ile bağlantı için yapılan araştırmadan motive olarak, C-tek terimli Burnside halkası için Lefschetz değişmezi tanımladık. Tanımladığımız değişmezler C-tek terimli G-kısmi sıralı kümelerinin kategorisinden C-tek terimli H-kısmi sıralı kümelerinin kategorisine herhangi bir C-tek terimli (G, H)-ikili kümesine ilişiklendirilmiş genel tensör indüksiyonu izleçleri inşa etmemize izin verir. Bu izleçler $B_C(G)$'den $B_C(H)$'e iyi tanımlı tensor indüksiyon fonksiyonlarını ortaya çıkarır. Bu fonksiyonlar da bize C-tek terimli Burnside halkalarının tersi olan elemanlarının grupları arasında $B_C(G)^\times \to B_C(H)^\times$ grup homomorfizması verir.

Anahtar sözcükler: Green izleçleri, p-permütasyon modülleri, kuralsal indüksiyon formülü, Burnside halkası, tek terimli Burnside halkası, tensör indüksiyonu, Lefschetz invaryantı.

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Contents

Chapter 1

Introduction

The Burnside ring, the monomial Burnside ring, the ordinary and p-modular character rings, the p-permutation ring arise frequently in representation theory of finite groups. These rings can be considered as functors associating a module $M(G)$ to a finite group G. Known examples include Mackey functors which associate subgroups of a fixed finite group to modules with induction, restriction and conjugation maps, when associated modules have extra multiplicative structure it is called a Green functor. There are biset theoretic approaches to Green functors; a Green biset functor has multiplicative structure and is equipped with induction, restriction, and, optionally, inflation and deflation maps, for details see [3], [4], and [5].

One significant application of the theory of Mackey functors is Boltje's canonical induction formula [6], [1] with a sufficient condition for integrality. The aim of this thesis is general formulation for canonical induction, suitable for Green functors, optionally equipped with inflations.

In Section 3, we apply the Boltje's theory of canonical induction [1] to p-permutation modules where p is a prime number. Let $\mathbb{F}$ an algebraically closed field of characteristic p and G be a finite group. We work on finite dimensional $\mathbb{F}G$-modules. In [7] Boltje gives a $\mathbb{Z}$-integral canonical induction formula which expresses any p-permutation $\mathbb{F}G$-module as a $\mathbb{Z}$-linear combination of modules induced from one dimensional p-permutation modules. In our work, we construct $\mathbb{Z}[1/p]$-linear canonical induction formula from the inflations of projective modules, which we call exprojective modules.

Let $T(G)$ be the Grothendieck ring for p-permutation modules. Lifting the induced modules of indecomposable exprojective modules, we introduce a ring $\mathcal{T}(G)$ which replaces the lower plus construction in [1]. We show how a linearization map $\mathcal{T}(G) \to T(G)$, extended to coefficients in $\mathbb{Z}[1/p]$, is split by a canonical induction map $\mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$. We also show that $\mathbb{K}\mathcal{T}(G)$ is $\mathbb{K}$-semisimple and commutative.

In Section 4, we give a general formulation of canonical induction theory for Green biset functors equipped with induction, restriction, inflation maps. Following Coşkun's construction [8] and [9], we replace the exprojective functor with an inflation-restriction Green functor A embedding in an inflation-restriction-induction Green functor M. Then we obtain an inflation-restriction-induction Green functor A_+. We construct the functor A_+ via adjunctions. Thus, T^{ex}, $T, \mathcal{T}$ are replaced by A, M, A_+. There is another way of constructing the functor A_+ in [10].

Let R be a commutative unital ring such that all group orders are invertible in R. Given an inflation-restriction R-homomorphism $\nu : A \to M$ and $\pi : M \to A$, we get inflation-restriction-induction R-homomorphism $\operatorname{lin}^{\nu} : A_+ \to M$ associated to ν called linearization map and $\operatorname{can}^{\pi} : M(G) \to A_+(G)$ associated to π. We extend to the coefficients $\operatorname{lin}^{\nu} : A_+ \to M$ and $\pi : M \to A$. Recovering [1, 6.4] we give a sufficient and necessary condition for $\operatorname{can}^{\pi} : M(G) \to A_+(G)$ to be a splitting of $\operatorname{lin}^{\nu} : A_+ \to M$.

In Section 5, we work on the monomial Burnside ring. Let G be a finite group. Thévenaz in [11] introduced the notion of the Lefschetz invariant of a G-poset as an element of the Burnside ring thus extending of the notion of the Euler-Poincaré characteristic of a poset. Now let C be an abelian group. The C-monomial Burnside ring is the Grothendieck ring of the category of C-fibred

G-sets. Motivated in part by a search for connection with Peter Symonds' proof [2] of the integrality of a canonical induction formula, we introduce a Lefschetz invariant for the C-monomial Burnside ring.

We introduce a new tool to work on the C-monomial Burnside ring, namely, a pair $(X, \mathfrak{l})$ consisting of a G-set and a functor $\mathfrak{l}$ from the transporter category $\widehat{X}$ of X to the category $\bullet_C$ with one object where morphisms are the elements of C and composition is multiplication in C. Extending this notion to G-posets, we introduce the category of C-monomial G-posets. Note that for a field k of characteristic p, the group of functors from the transporter category of the orbit category on the set of nontrivial p-subgroups to $\bullet_{k^\times}$ appears in connection with endotrivial modules. We define induction and restriction functors, and introduce the Lefschetz invariant of a C-monomial G-poset as an element of the C-monomial Burnside ring. We show that any element of the monomial Burnside ring can be expressed as the Lefschetz invariant of some (non unique) monomial G-poset. We show that restriction and induction functors are compatible with Lefschetz invariants. We also give a monomial analogue of the relation between the Euler-Poincaré characteristic of fixed points of a G-poset X and the Lefschetz invariant of X. This result lets us construct well-defined tensor induction maps $\mathcal{T}_{X,\mathfrak{l}} : B_C(G) \to B_C(H)$ for any finite C-monomial $(G \times H)$-set $(X, \mathfrak{l})$ and any finite groups G and H. In particular, we get induced group homomorphisms between the corresponding unit groups of monomial Burnside rings. These homomorphisms are similar to those obtained by Carman ([12]) for other familiar Green functors. These maps are not additive in general, but multiplicative and preserve identity elements. Moreover, under suitable conditions, these tensor induction functors and their associated tensor induction maps preserve composition. This gives rise to a (partial) fibred biset functor structure on the group of units of the monomial Burnside ring.

Chapter 2

Preliminaries

2.1 Restriction, induction and coinduction for rings

In this section, we recall the definition of restriction, induction and coinduction for rings. Then we state adjointness properties. All rings we consider are with identity. Modules will be left modules unless otherwise stated.

Let $\mathcal{E}$ and $\mathcal{S}$ be rings. Given a unital ring homomorphism $\alpha : \mathcal{S} \to \mathcal{E}$, any $\mathcal{E}$-module can be regarded as an $\mathcal{S}$-module via α. This induces a functor

$$_{\mathcal{S}}\operatorname{Res}^{\alpha}_{\mathcal{E}} - : \mathcal{E}\text{-mod} \to \mathcal{S}\text{-mod}$$

called the restriction.

We may regard $\mathcal{E}$ as a right $\mathcal{S}$-module by $es = e\alpha(s)$ for $s \in \mathcal{S}$ and $e \in \mathcal{E}$. Now let M be an (left) $\mathcal{S}$-module. Then $\mathcal{E} \otimes_{\mathcal{S}} M$ is an (left) $\mathcal{E}$-module by $e(e' \otimes m) = ee' \otimes m$ for any $e \in \mathcal{E}$ and $m \in M$. It's not hard to see that this action is well-defined as the natural action of M on itself commutes with the action of $\mathcal{S}$ on $\mathcal{E}$. The module $\mathcal{E} \otimes_{\mathcal{S}} M$ is called the induced module and denoted by $_{\mathcal{E}}\operatorname{Ind}^{\alpha}_{\mathcal{S}} M$. We obtain the induction functor

$$ {}_{\mathcal{E}}\operatorname{Ind}_{\mathcal{S}}^{\alpha} - := \mathcal{E} \otimes_{\mathcal{S}} - : \mathcal{E}\text{-mod} \to \mathcal{S}\text{-mod}. $$

Now we consider $\mathcal{E}$ as a left $\mathcal{S}$-module by $se = \alpha(s)e$ for any $e \in \mathcal{E}$ and $s \in \mathcal{S}$. Let M be an $\mathcal{S}$-module. Then, we can regard $\operatorname{Hom}_{\mathcal{S}}(\mathcal{E}, M)$ a (left) $\mathcal{E}$-module by $(e\phi)(e') = \phi(e'e)$ for any e, $e' \in \mathcal{E}$ $\phi \in \operatorname{Hom}_{\mathcal{S}}(\mathcal{E}, M)$. It's not hard to see that the natural action of $\mathcal{E}$ on itself commutes with the action of $\mathcal{S}$ on $\mathcal{E}$. The module $\operatorname{Hom}_{\mathcal{S}}(\mathcal{E}, M)$ is called the coinduced module and denoted by ${}_{\mathcal{E}}\operatorname{Coind}_{\mathcal{S}}^{\alpha} M$. Then we set the coinduction functor

$$ {}_{\mathcal{E}}\operatorname{Coind}_{\mathcal{S}}^{\alpha} := \operatorname{Hom}_{\mathcal{S}}(\mathcal{E}, -) : \mathcal{S}\text{-mod} \to \mathcal{E}\text{-mod}. $$

Now we will state an adjointness property, for the details and proof, see [13].

Theorem 2.1.1. *Let $\mathcal{E}$ and $\mathcal{S}$ be rings and $\alpha : \mathcal{S} \to \mathcal{E}$ be a unital ring homomorphism. Consider an $\mathcal{E}$-module M and an $\mathcal{S}$-module N. Then:*

(a) *(Adjointness of induction and restriction.) Naturally in M and N, there is an isomorphism*

$$ \operatorname{Hom}_{\mathcal{E}}\big({}_{\mathcal{E}}\operatorname{Ind}_{\mathcal{S}}^{\alpha}(N), M\big) \cong \operatorname{Hom}_{\mathcal{S}}\big(N, {}_{\mathcal{S}}\operatorname{Res}_{\mathcal{E}}^{\alpha}(M)\big) $$

whereby maps $\theta : {}_{\mathcal{E}}\operatorname{Ind}_{\mathcal{S}}^{\alpha}(N) \to M$ and $\phi : N \to {}_{\mathcal{S}}\operatorname{Res}_{\mathcal{E}}^{\alpha}(M)$ correspond if and only if, for all $e \in \mathcal{E}$ and $s \in \mathcal{S}$ and $n \in \mathcal{S}N$, we have

$$ \theta(e\alpha(s)n) = e\alpha(s)\phi(n). $$

(b) *(Adjointness of restriction and coinduction.) Naturally in M and N there is an isomorphism*

$$ \operatorname{Hom}_{\mathcal{S}}\big({}_{\mathcal{S}}\operatorname{Res}_{\mathcal{E}}^{\alpha}(M), N\big) \cong \operatorname{Hom}_{\mathcal{E}}\big(M, {}_{\mathcal{E}}\operatorname{Coind}_{\mathcal{S}}^{\alpha}(N)\big) $$

whereby maps $\phi : {}_{\mathcal{S}}\operatorname{Res}_{\mathcal{E}}^{\alpha}(M) \to N$ and $\psi : M \to {}_{\mathcal{E}}\operatorname{Coind}_{\mathcal{S}}^{\alpha}(N)$ correspond if and only if, for all $e \in \mathcal{E}$ and $s \in \mathcal{S}$ and $m \in \mathcal{E}M$, we have

$$ \phi\big(\alpha(s)em\big) = \psi(m)\big(\alpha(s)e\big). $$

Throughout this thesis, we will consider α as the inclusion $\mathcal{S} \hookrightarrow \mathcal{E}$ or the projection $\mathcal{S} \twoheadrightarrow \mathcal{E} = \mathcal{S}/\Delta$ where Δ is an ideal of $\mathcal{S}$. If α is an inclusion we will write ${}_{\mathcal{E}}\operatorname{Ind}_{\mathcal{S}}$, ${}_{\mathcal{S}}\operatorname{Res}_{\mathcal{E}}$ and ${}_{\mathcal{E}}\operatorname{Coind}_{\mathcal{S}}$ for the induction, restriction and coinduction functors, respectively. If α is a projection, we write ${}_{\mathcal{E}}\mathrm{Def}_{\mathcal{S}}$ and ${}_{\mathcal{S}}\operatorname{Inf}_{\mathcal{E}}$ for the induction and restriction, respectively.

2.2 The monomial Burnside ring

In this section, we define the monomial Burnside ring introduced by Dress ([14]). We follow the notation of Barker in [15].

Let C be an abelian group and G be a finite group. We define a *C-fibred G-set* to be a C-free $(C \times G)$-set with finitely many C-orbits. Let ${}_{C}FG\text{-}\mathsf{set}$ denote the category of C-fibred G-sets where morphisms are $(C \times G)$-equivariant maps. Let X and Y be C-fibred G-sets. Their coproduct $X \sqcup Y$ is defined to be the their coproduct as sets, with the obvious $(C \times G)$-action. The group C acts on $X \times Y$ by $c(x, y) = (cx, c^{-1}y)$ for any $c \in C$ and $(x, y) \in X \times Y$. We let $x \otimes y$ to denote the C-orbit of an element (x, y) of $X \times Y$ and $X \otimes Y$ denotes the set of C-orbits. The group $C \times G$ acts on $X \otimes Y$ by

$$(c, g)(x \otimes y) = cgx \otimes gy$$

for any $(c, g) \in C \times G$ and $x \otimes y \in X \otimes Y$. We call the C-fibred G-set $X \otimes Y$ the tensor product of X and Y.

The isomorphism class of a C-fibred G-set is denoted by $[X]$. We define the *C-monomial Burnside ring* $B_C(G)$ to be the Grothendieck group of the category of C-fibred G-sets, for relations given by $[X] + [Y] = [X \sqcup Y]$. The ring structure of $B_C(G)$ is induced by $[X] \cdot [Y] = [X \otimes Y]$. The set C with trivial G-action is the identity element and the empty set is the zero element. If C is trivial we recover the ordinary Burnside ring $B(G)$ of the group G.

Let X be a C-fibred G-set. We let $C\backslash X$ to denote the set of C-orbits on X. The set $C\backslash X$ is a G-set and $(C \times G)$-transitive if and only if $C\backslash X$ is G-transitive.

If $C\backslash X$ is transitive as a G-set it is isomorphic to G/U for some $U \leq G$. There exists a group homomorphism $\mu : U \to C$ such that if U is the stabilizer of the orbit Cx, then $ax = \mu(a)x$ for all $a \in U$. Since the stabilizer $(C \times G)_x$ of x in $C \times G$ is equal to

$$(C \times G)_x = \{\big(\mu(a)^{-1}, a\big) \mid a \in U\},$$

the C-fibred G-set X is determined up to isomorphism by the subgroup U and μ.

Conversely, let U be a subgroup of G, and $\mu : U \to C$ be a group homomorphism. Then we set $U_\mu = \{\big(\mu(a)^{-1}, a\big) \mid a \in U\}$, and denote by $[U, \mu]_G$ the C-fibred G-set $(C \times G)/U_\mu$. The pair (U, μ) is called a C-subcharacter of G. We denote the set of C-subcharacters by $\operatorname{ch}(G)$. The group G acts on $\operatorname{ch}(G)$ by conjugation. The G-set $\operatorname{ch}(G)$ is a poset with the relation $\leq$ defined by

$$(U, \mu) \leq (V, \nu) \Leftrightarrow U \leq V \text{ and } {}_U\operatorname{Res}_V \nu = \mu$$

for any (U, μ) and (V, ν) in $\operatorname{ch}(G)$.

As an abelian group we have

$$B_C(G) = \bigoplus_{(U,\mu) \in_G \operatorname{ch}(G)} \mathbb{Z}[U, \mu]_G$$

where (V, ν) runs over G-representatives of the C-subcharacters of G, details can be seen in [15].

The multiplication of two C-fibred G-sets $[U, \mu]_G$ and $[V, \nu]_G$ is given by

$$[U, \mu]_G \cdot [V, \nu]_G = \sum_{UgV \subseteq G} [U \cap {}^gV, \mu \cdot {}^g\nu]_G$$

where $\mu \cdot {}^g\nu$ denotes the restriction of the product of the characters μ and ${}^g\nu$.

2.3 p-permutation modules

In this section, we define the ring of p-permutation modules. We start by basic definitions which can be found in [16] and [13].

Let $\mathbb{F}$ be an algebraically closed field of characteristic p where p is prime.

Now we will set some notations. Let H be a subgroup of G. The induction functor from $\mathbb{F}H$-modules to $\mathbb{F}G$-modules will be denoted by ${}_G\operatorname{Ind}_H$. The restriction functor from $\mathbb{F}G$-modules to $\mathbb{F}H$-modules will be denoted by ${}_H\operatorname{Res}_G$. Now let K be a normal subgroup of G. The inflation functor from $\mathbb{F}G/K$-modules to $\mathbb{F}G$-modules will be denoted by ${}_G\operatorname{Inf}_{G/K}$. Given a group isomorphism $L \to G$, the isogation functor from $\mathbb{F}G$-modules to $\mathbb{F}L$-modules will be denoted by ${}_L\operatorname{Iso}_G$. Here, ${}_L\operatorname{Iso}_G(X)$ meant to be $\mathbb{F}L$-module obtained from an $\mathbb{F}G$-module X by transport of structure via the understood isomorphism.

Definition 2.3.1. *Given an indecomposable $\mathbb{F}G$-module M and a subgroup H of G, the module M is called H-projective provided M is a direct summand of an $\mathbb{F}G$-module induced from an $\mathbb{F}H$-module.*

Definition 2.3.2. *Let H be a subgroup of G. Let M and W be $\mathbb{F}G$-modules. Then the* trace map *or* transfer map

$$\operatorname{tr}_H^G : \operatorname{Hom}_{\mathbb{F}H}\big({}_H\operatorname{Res}_G(W), {}_H\operatorname{Res}_G(M)\big) \to \operatorname{Hom}_{\mathbb{F}G}(W, M)$$

is defined as follows. We choose a set $\{g_i \mid i \in I\}$ of left coset representatives of H in G and set

$$\operatorname{tr}_H^G(\phi)(m) = \sum_{i \in I} g_i \phi(g_i^{-1} m).$$

Note that this definition is independent of the choice of coset representatives of H in G because ϕ is an $\mathbb{F}H$-module homomorphism and this implies that $g_i\phi(g_i^{-1}m)$ only depends on the left coset g_iH where $g_i \in I$.

Proposition 2.3.3. *(Higman's Criterion) Let H be a subgroup of G and M be an $\mathbb{F}H$-module. Then M is H-projective if and only if the identity map on M belongs to the image of the relative trace map*

$$\operatorname{tr}_H^G : \operatorname{End}_{\mathbb{F}H}(M) \to \operatorname{End}_{\mathbb{F}G}(M).$$

If H is minimal subject to that condition, then H is called a *vertex* of M. It can be shown that the vertices of M are p-subgroups and any two vertices are G-conjugate.

Definition 2.3.4. *Given an indecomposable $\mathbb{F}G$-module V, a subgroup P of G is called vertex of V if V is a direct summand of ${}_G\operatorname{Ind}_P \operatorname{Res}_G(V)$ and P is minimal subject to this condition.*

Note that the vertices of an indecomposable $\mathbb{F}G$-module V are p-subgroups and any two vertices are G-conjugate.

Definition 2.3.5. *Let P be a vertex of V and let S be an indecomposable $\mathbb{F}P$-module such that W is a direct summand of ${}_P\operatorname{Ind}_G(S)$. We call S a* source *$\mathbb{F}P$-module of W.*

It can be shown that, if S and S_0 are source $\mathbb{F}P$-modules of W, then S is isomorphic to an $N_G(P)$-conjugate of S_0.

Definition 2.3.6. *Let M be an $\mathbb{F}G$-module. We call M a* permutation module *if there exists a G-set X with $M = \mathbb{F}X$, that is to say, M has a G-stable $\mathbb{F}$-basis.*

Theorem 2.3.7. *Let W be an $\mathbb{F}G$-module. The following two conditions are equivalent:*

(a) *For every indecomposable direct summand V of W, the source modules of V are trivial.*

(b) *For any p-subgroup P of G, there exists a P-stable basis for W .*

When the equivalent conditions of the theorem hold, the module W is called **p-permutation $\mathbb{F}G$-module.** Another popular name for it is **trivial source $\mathbb{F}G$-module**.

Note that p-permutation modules are preserved by direct sums, tensor products, induction, restriction and conjugation.

An $\mathbb{F}G$-module V is a p-permutation module if and only if every indecomposable direct summand of V is a p-permutation module. There are only finitely many isomorphism classes of indecomposable p-permutation modules.

2.4 Bisets and group categories

2.4.1 Bisets

Let G and H be finite groups. An (H, G)-biset X is both a left H-set and a right G-set, such that $h \cdot (x \cdot g) = (h \cdot x) \cdot g$ for all elements $h \in H$, $g \in G$ and $x \in X$.

We can regard an (H, G)-biset as a right $(H \times G)$-set equipped with the action given by

$$x \cdot (h, g) = h^{-1} \cdot x \cdot g$$

for all $x \in X$ and $h \in H$ and $g \in G$.

So all the properties of permutation sets can be applied to bisets. The biset Burnside group $B(H, G)$ is defined to be the Burnside group $B(H \times G)$.

An (H, G)-biset X is called transitive if for any elements $x, y \in X$ there exist an element $h \in H$ and an element $g \in G$ such that $h \cdot x \cdot g$ is equal to y. Thus, X is a transitive (H, G)-biset if and only if X is a transitive $(H \times G)$-set. There is a bijective correspondence between

(a) isomorphism classes $[X]$ of transitive (H, G)-bisets,

(b) conjugacy classes $[L]$ of subgroups of $H \times G$ where where the correspondence is given by $[X] \leftrightarrow [L]$ if and only if the stabilizer of a point $x \in X$ is in $[L]$.

We let $\left(\frac{H \times G}{L}\right)$ to denote a transitive biset with point stabilizer equal to L.

Let G be a finite group. Now we shall define the elementary bisets:

(1) If H is a subgroup of G, then the set G is an (H, G)-biset for the actions given by left and right multiplication in G. It is denoted by ${}_H\mathrm{res}_G$ (where res means restriction).

(2) In the same situation, the set G is a (G, H)-biset for the actions given by left and right multiplication in G. It is denoted by bf ${}_G\mathrm{ind}_H$ (where ind means induction).

(3) If N is a normal subgroup of G, and if $H = G/N$, the set H is a (G,H)-biset, for the right action of H by multiplication, and the left action of G by projection to H, and then left multiplication in H. It is denoted by ${}_G\mathrm{inf}_{G/N}$ (where inf means inflation).

(4) In the same situation, the set H is an (H,G)-biset, for the left action of H by multiplication, and the right action of G by projection to H, and then right multiplication in H. It is denoted by ${}_{G/H}\mathrm{def}_H$ (where def means deflation).

(5) If $f : G \to H$ is a group isomorphism, then the set H is an (H,G)-biset, for the left action of H by multiplication, and the right action of G given by taking image by f, and then multiplying on the right in H. It is denoted by $\mathrm{iso}(f)$, or ${}_H\mathrm{iso}_G$ if the isomorphism f is clear from the context.

Now we set some notations. Let G and H be finite groups. Let L be a subgroup of $H \times G$. We set

$$p_1(L) = \{h \in H \mid \exists g \in G,\ (h,g) \in L\}$$

$$p_2(L) = \{g \in G \mid \exists h \in H,\ (h,g) \in L\}$$

$$k_1(L) = \{h \in H : (h,1) \in L\}$$

$$k_2(L) = \{g \in G : (1,g) \in L\}.$$

Note that we have the subgroup inclusions $k_1(L) \subseteq p_1(M)$ and $k_2(L) \subseteq p_2(L)$. Now we will define the composition of bisets.

Definition 2.4.1. *Let G, H, and K be finite groups. If U is an (H,G)-biset, and V is a (K,H)-biset, the composition of V and U is the set of H-orbits on the cartesian product $V \times U$, where the right action of H is defined by*

$$\forall (v,u) \in V \times U, \forall h \in H,\ (v,u)\cdot h = (v\cdot h, h^{-1}\cdot u).$$

It is denoted by $V \times_H U$. The H-orbit of $(v,u) \in V \times U$ is denoted by $(v, {}_H u)$. The set $V \times_H U$ is a (K,G)-biset for the action defined by

$$\forall k \in K, \forall (v, {}_H u) \in V \times_H U, \forall g \in G,\ \ k\cdot(v, {}_H u)\cdot g = (k\cdot v, {}_H u\cdot g).$$

By linear extension, the Mackey product induces a bilinear map

$$B(K \times H) \times B(H \times G) \to B(K \times G).$$

For the transitive bisets $\left(\frac{K \times H}{L}\right)$ and $\left(\frac{H \times G}{N}\right)$, the Mackey product is explicitly given by

$$\left(\frac{K \times H}{L}\right) \times_H \left(\frac{H \times G}{N}\right) = \sum_{x \in p_2(L) \backslash K / p_1(N)} \left(\frac{H \times G}{L *^{(x,1)} N}\right)$$

where the subgroup $L * N$ of $K \times G$ is defined by

$$L * N = \{(k, g) \in K \times G : (k, h) \in L \, \text{and} \, (h, g) \in N \, \text{for some} \, h \in H\}.$$

Now we state Bouc's result which describes the decomposition of any transitive biset in terms of the transitive bisets.

Theorem 2.4.2. *Let H and K be finite groups. Let L be any subgroup of $H \times K$. Then*

$$\left(\frac{H \times K}{L}\right) = {}_H\mathrm{ind}_{p_1(L)} \mathrm{inf}_{p_1(L)/k_1(L)} \mathrm{iso}^{\phi}_{p_2(L)/k_2(L)} \mathrm{def}_{p_2(L)} \mathrm{res}_K.$$

The isomorphism

$$\phi : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$$

is the one given by associating $lk_2(L)$ to $mk_1(L)$ where for a given element $l \in p_2(L)$ we let m be the unique element in $p_1(L)$ such that $(m, l) \in L$.

The factorization in the above theorem is called the internal butterfly factorization. Now will state the Goursat's Lemma presented as in [17, 2.1].

Theorem 2.4.3. *(Goursat's Theorem) Let H and K be finite groups. There is a bijective correspondence between the subgroups $L \leq H \times K$ and the quintuples $(H_1, H_2, \phi, K_2, K_1)$ such that $H_2 \trianglelefteq H_1 \leq H$ and $K_2 \trianglelefteq K_1 \leq K$ and ϕ is an isomorphism $H_1/H_2 \to K_1/K_2$. The correspondence is such that $L \leftrightarrow \big(k_1(L), p_1(L), \phi, p_2(L), k_2(L)\big)$.*

This theorem tells us the internal butterfly factorization of $\frac{H \times K}{L}$ is unique in the sense that the possible quintuples $(H_1, H_2, \phi, K_2, K_1)$ comprise an orbit under the evident action of $H \times K$. The uniqueness comes from the fact that $[\frac{H\times K}{L}]$ where L is a subgroup of $H \times K$ and L is well-defined up to conjugation in $H \times K$. For each L, the associated subgroups H_1, H_2, ϕ, K_2, K_1 are such that the projection of L to H has image H_1 and kernel $K_1 = K_1 \times 1$.

Now we define the biset category. The biset category $\mathcal{C}$ is defined as the category whose objects are finite groups and morphisms are biset Burnside groups (see, [4]). Given an (F, G)-biset X, isomorphism class of X as an element of $B(F, G)$ is denoted by $[X]$. Given an (F, G)-biset X and a (G, H)-biset Y , then the composition of $[X]$ and $[Y]$ is $[X][Y] = [X \times_H Y]$ where $X \times_H Y$ denotes the set of H-orbits in $X \times Y$. Also, for a commutative unital ring R, the category $R\mathcal{C}$ is a category for which objects are finite groups and the morphisms are extensions of biset Burnside groups $RB(H, G) = R \otimes B(H, G)$.

Now we will define generalized restriction and induction. Given a group homomorphism $\alpha : G \to F$, we define transitive morphisms

$$_F\mathrm{ind}_G^{\alpha} = \Big[\frac{F \times G}{(\alpha(g), g) : g \in G}\Big], \quad {}_G\,\mathrm{res}_F^{\alpha} = \Big[\frac{G \times F}{(g, \alpha(g)) : g \in G}\Big]$$

called induction and restriction, respectively. It was shown in [3] that for a given diagram $F \overset{\alpha}{\leftarrow} I \overset{\theta}{\to} G \overset{\beta}{\leftarrow} J \overset{\phi}{\to} H$ in the category of finite groups, then

$$_F\mathrm{ind}_I^{\alpha}\,\mathrm{res}_G^{\theta}\mathrm{ind}_J^{\beta}\,\mathrm{res}_H^{\phi} = \sum_{\theta(I)g\beta(J)\subseteq G} {}_F\mathrm{ind}_{L_g}^{\alpha\gamma_g}\,\mathrm{res}_H^{\phi\psi_g}$$

where $I \overset{\gamma_g}{\leftarrow} L_g \overset{\psi_g}{\to} J$ is any pullback of $I \overset{\theta}{\to} L_g \overset{g\beta}{\leftarrow} J$.

Now following [18], we state basic results and set some notations. Let $\alpha : G \to F$ and $\beta : H \to G$ be group homomorphisms. Then

$$_F\mathrm{ind}_G^{\alpha}\mathrm{ind}_H^{\beta} = {}_F\,\mathrm{ind}_H^{\alpha\beta} \quad \text{and} \quad {}_H\mathrm{res}_G^{\beta}\mathrm{res}_F^{\alpha} = {}_H\,\mathrm{res}_F^{\alpha\beta}$$

If α is injective, $_F\mathrm{ind}_G^{\alpha}$ is called ordinary induction and $_G\mathrm{res}_F^{\alpha}$ is called ordinary restriction. If $\alpha : G \hookrightarrow F$ is an injection, we omit the symbol α from the notation, just writing $_F\mathrm{ind}_G$ and $_G\mathrm{res}_F$. If α is surjective, we have

$$_F\mathrm{def}_G = {}_F\mathrm{ind}_G^{\alpha} \quad \text{and} \quad {}_G\mathrm{inf}_F = {}_G\mathrm{res}_F^{\alpha}.$$

For arbitrary α we have factorizations,

$$ {}_F\mathrm{ind}_G^\alpha = {}_F\mathrm{ind}_{\alpha(G)}\mathrm{def}_G = \quad \text{and} \quad {}_G\mathrm{res}_F^\alpha =_G \mathrm{inf}_{\alpha(G)}\mathrm{res}_F. $$

If α is an isomorphism, we set

$$ {}_F\mathrm{iso}_G^\alpha = {}_F\mathrm{ind}_G^\alpha = {}_F\mathrm{res}_G^{\alpha^{-1}}. $$

The right-free transitive morphisms $F \leftarrow G$ are the morphisms that can be expressed in the form

$$ {}_F\mathrm{ind}_V\,\mathrm{res}_G^\beta = {}_F\mathrm{ind}_V\mathrm{inf}_{\beta(V)}^\beta\,\mathrm{res}_G = \left[\frac{F \times G}{\mathfrak{G}(V,\beta)}\right] $$

where $V \leq F$ and $\beta : V \to G$ and

$$ \mathfrak{G}(V,\beta) = \{(v,\beta(v)) : v \in V\} \subseteq F \times G. $$

Now we give the rightfree analogue of [18, Theorem 3.1].

Proposition 2.4.4. *(Mackey relation for right-free transitive morphisms.) Let $W \leq G$ and $V \leq F$ be finite groups. Let $\alpha : V \to G$ and $\beta : W \to H$ be group homomorphisms. Then*

$$ {}_F\mathrm{ind}_V\,\mathrm{res}_G^\alpha\mathrm{ind}_W\,\mathrm{res}_H^\beta = \sum_{\alpha(V)gW \subseteq G} {}_F\mathrm{ind}_{\alpha^{-1}({}^gW)}\mathrm{res}_H^{\beta_g c(g^{-1})\alpha_g} $$

where $\alpha_g :^g W \cap \alpha(V) \leftarrow \alpha^{-1}({}^gW)$ and $\beta_g : H \leftarrow W \cap {}^g\alpha(V)$ are restrictions of α and β.

2.4.2 Group categories

We shall start by stating some general definitions. We say that a category is R-linear (or R-preadditive) if the sets of morphisms are R-modules and the composition is R-bilinear. Functors between R-linear categories are required to act on morphisms as R-linear maps.

Let $\mathcal{L}$ be a small R-linear category with objects finite groups. An $\mathcal{L}$-functor is defined to be a functor $\mathcal{L} \to R$-Mod. The $\mathcal{L}$-functors can be regarded as the modules as follows. Given F, $G \in \mathrm{Obj}(\mathcal{L})$, the R-module of morphisms $G \to F$ in $\mathcal{L}$ will be denoted by $\mathcal{L}(F, G)$. We let the quiver algebra of $\mathcal{L}$ to be the locally unital algebra

$$^{\oplus}\mathcal{L} = \bigoplus_{F,\, G \in Obj(\mathcal{L})} \mathcal{L}(F, G)$$

whose multiplication is given by composition and product of incompatible morphisms being zero. Equipped with the multiplication operation coming from composition of morphisms, products of incompatible morphisms being zero. Note that since $\mathcal{L}$ is a small category we can form the direct sum. Any element $x \in {}^{\oplus}\mathcal{L}$ can be written uniquely in the form

$$x = \sum_{F,\, G \in Obj(\mathcal{L})} {}_F x_G$$

where each ${}_F x_G \in \mathcal{L}(F,\, G)$ and only finitely many of the ${}_F x_G$ are non-zero. Using the notation ${}_F x_G y_H = {}_F x_G \cdot {}_G y_H$ for $x, y \in {}^{\oplus}\mathcal{L}$ and $F, G, H \in \mathrm{Obj}(\mathcal{L})$, the (F, H)-entry of the product xy is given by ${}_F(xy)_H = \sum_G {}_F x_G y_H$.

We will identify the $\mathcal{L}$-functors with the ${}^{\oplus}\mathcal{L}$-modules, each functor $G \to M(G)$ giving rise to a module $\bigoplus_G M(G)$, each module M giving rise to functor $G \mapsto \mathrm{id}_G M$.

A group category $\mathcal{X}$ is defined to be a linear subcategory of $\mathcal{C}$ such that every morphism in $\mathcal{X}$ is a linear combination of transitive morphisms in $\mathcal{C}$.

Now we set some notations. Let $\mathcal{X}$ be a group category. We let $\mathrm{tm}_{\mathcal{X}}$ to denote the set of transitive morphisms in $\mathcal{X}$. Given G, $G' \in \mathrm{Obj}(X)$, the set

$$\mathrm{tm}_{\mathcal{X}}(G', G) = \mathcal{X}(G', G) \cap \mathrm{tm}_{\mathcal{X}}$$

is the set of transitive morphisms from G' to G in $\mathcal{X}$. Note that we have a disjoint union

$$\mathrm{tm}_{\mathcal{X}} = \bigcup_{G',\, G \in \mathcal{X}} \mathrm{tm}_{\mathcal{X}}(G', G).$$

We have that $\mathrm{tm}_{\mathcal{X}}(G', G)$ is an R-basis for $R\mathcal{X}(G', G)$ and $\mathrm{tm}_{\mathcal{X}}$ is an R-basis for $R\mathcal{X}$. Therefore, as direct sums of R-modules we have

$$R\mathcal{X} = \bigoplus_{\alpha \in \mathrm{tm}_{\mathcal{X}}}, \qquad R\mathcal{X} = \bigoplus_{\alpha \in \mathrm{tm}_{\mathcal{X}}(G', G)} R\alpha.$$

Note that in the following chapters given a group category χ, the quiver algebra will be denoted by χ.

Chapter 3

A new canonical induction formula for p-permutation modules

Let p be a prime number and let $\mathbb{F}$ be an algebraically closed field of characteristic p. Let G be a finite group. A $\mathbb{Z}$-integral canonical induction formula for p-permutation module is given by Boltje in [7]. His formula expresses any p-permutation module, up to isomorphism, as a $\mathbb{Z}$-linear combination of modules induced from the 1-dimensional modules. In this chapter, we will introduce a special kind of p-permutation module which are inflations of projective modules called exprojective. We also give a $\mathbb{Z}[1/p]$-integral canonical induction formula, expressing any p-permutation $\mathbb{F}G$-module, up to isomorphism, as a $\mathbb{Z}[1/p]$-linear combination of modules induced from exprojective modules.

3.1 Exprojective modules

In this section, we work on the notion of exprojective module.

Definition 3.1.1. *Let M be an indecomposable $\mathbb{F}G$-module. We call M exprojective if the following equivalent conditions hold up to isomorphism: there exists a normal subgroup $K \trianglelefteq G$ such that M is inflated from a projective $\mathbb{F}G/K$-module; there exists $K \trianglelefteq G$ such that M is a direct summand of the permutation $\mathbb{F}G$-module $\mathbb{F}G/K$.*

The group G is called **p'-residue-free** if $G = O^{p'}(G)$, equivalently, G is generated by the Sylow p-subgroups of G. Let $\mathcal{Q}(G)$ denote the set of pairs (K, F), where K is a p'-residue-free normal subgroup of G and F is an indecomposable projective $\mathbb{F}G/K$-module, two such pairs (K, F) and (K', F') being deemed the same provided $K = K'$ and $F \cong F'$. We set an indecomposable exprojective $\mathbb{F}G$-module $M_G^{K,F} = {}_G\mathrm{Inf}_{G/K}(F)$.

Proposition 3.1.2. *Let M be an indecomposable $\mathbb{F}G$-module. Then M is exprojective if and only if the vertices of M act trivially on M.*

Proof. Suppose that M is exprojective. We let K and F to be as above. Then K is a direct summand of the regular $\mathbb{F}G/K$-module. But K acts trivially on F. Since the vertices of M are contained in K, they also act trivially on M.

Conversely, suppose that a vertex P of M acts trivially on M. Now let K be the smallest normal subgroup of G containing P. So K is the subgroup of G generated by the G-conjugates of P. Obviously, all of those conjugates act trivially on M. So K acts trivially on M. Thus, M is the inflation of an indecomposable $\mathbb{F}G/K$-module F. Note that F is projective by Higman's criterion. □

By considering vertices, we obtain the following result.

Proposition 3.1.3. *The condition $M \cong M_G^{K,F}$ characterizes a bijective correspondence between:*

(a) *the isomorphism classes of indecomposable exprojective $\mathbb{F}G$-modules M,*

(b) *the elements (K, F) of $\mathcal{Q}(G)$.*

Proof. The correspondence is given by $M \cong {}_G\mathrm{Inf}_{G/K}(F)$. Given M, we let $K = O^{p'}(\mathrm{Ker}(M))$. Then M is a direct summand of the $\mathbb{F}G$-module $\mathbb{F}G/K$. $\square$

Let P be a p-subgroup of G. Then the condition $E \cong {}_{N_G(P)}\mathrm{Inf}_{N_G(P)/P}(\overline{E})$ characterizes a bijective correspondence between, up to isomorphism, the indecomposable exprojective $\mathbb{F}N_G(P)$-modules E with vertex P and the indecomposable projective $\mathbb{F}N_G(P)/P$-modules $\overline{E}$. Thus, we can rewrite the well-known result of Bouc–Thévenaz [19, 2.9] on classification of the isomorphism classes of indecomposable p-permutation $\mathbb{F}G$-modules as follows. The set of pairs (P, E) where P is a p-subgroup of G and E is an exprojective $\mathbb{F}N_G(P)$-module with vertex P will be denoted by $\mathcal{P}(G)$. Two pairs (P, E) and (P', E') will be same if $P = P'$ and $E \cong E'$. Then the set $\mathcal{P}(G)$ is a G-set via the action on coordinates. The indecomposable p-permutation $\mathbb{F}G$-module with vertex P in Green correspondence with E will be denoted by $M^G_{P,E}$.

Theorem 3.1.4. *The condition $M \cong M^G_{P,E}$ characterizes a bijective correspondence between:*

(a) *the isomorphism classes of indecomposable p-permutation $\mathbb{F}G$-modules M,*

(b) *the G-conjugacy classes of elements $(P, E) \in \mathcal{P}(G)$.*

We let $T(G)$ to be the Grothendieck ring of the category of p-permutation $\mathbb{F}G$-modules, we mention that the split short exact sequences are the distinguished sequences determining the relations on $T(G)$. The multiplication on $T(G)$ is given by tensor product over $\mathbb{F}$. Given a p-permutation $\mathbb{F}G$-module X, we write $[X]$ to denote the isomorphism class of X. We understand that $[X] \in T(G)$. By Theorem 3.1.4, we have

$$T(G) = \bigoplus_{(P,E) \in_G \mathcal{P}(G)} \mathbb{Z}[M^G_{P,E}]$$

as a direct sum of regular $\mathbb{Z}$-modules, the notation indicating that the index runs over representatives of G-orbits.

Note that every exprojective module is a p-permutation module. So the indecomposable exprojective $\mathbb{F}G$-modules are the $\mathbb{F}G$-modules having the form $M^G_{P,E}$

where the element $(P, E) \in \mathcal{P}(G)$ satisfies some appropriate condition. Now we will give a necessary and sufficient condition for $M^G_{P,E}$ to be exprojective.

Proposition 3.1.5. *Let $(P, E) \in \mathcal{P}(G)$. Let K be the normal closure of P in G. Then $M^G_{P,E}$ is exprojective if and only if $N_K(P)$ acts trivially on E. In that case, K is p'-residue-free, P is a Sylow p-subgroup of K, we have $G = N_G(P)K$, the inclusion $N_G(P) \hookrightarrow G$ induces an isomorphism $N_G(P)/N_K(P) \cong G/K$, and $M^G_{P,E} \cong M^{K,F}_G$, where F is the indecomposable projective $\mathbb{F}G/K$-module determined, up to isomorphism, by the condition $E \cong {}_{N_G(P)} \operatorname{Inf}_{N_G(P)/N_K(P)} \operatorname{Iso}_{G/K}(F)$.*

Proof. We let $M = M^G_{P,E}$. Assume that M is an exprojective module. Proposition 3.1.2 implies that K acts trivially on M because K is generated by the vertices of M. Since $\mathbb{F}N_G(P)$-module ${}_{N_G(P)} \operatorname{Inf}_{N_G(P)/P} \overline{E}$ has vertex P and ${}_{N_G(P)} \operatorname{Inf}_{N_G(P)/P} \overline{E}$ is Green correspondent of M, module ${}_{N_G(P)} \operatorname{Inf}_{N_G(P)/P} \overline{E}$ is direct summand of ${}_{N_G(P)} \operatorname{Res}_G(M)$. So $N_K(P)$ acts trivially on M

Now assume that $N_K(P)$ acts trivially on E. Since P is a vertex of E, it is a Sylow p-subgroup of $N_K(P)$. Thus, P is a Sylow p-subgroup of K. Then Frattini argument implies $G = N_G(P)K$ and we have an isomorphism $N_G(P)/N_K(P) \cong G/K$ as specified. Now let $X = {}_G \operatorname{Ind}_{N_G(P)}(E)$. Since $N_K(P)$ acts trivially on M, the $\mathbb{F}G$=module X has well-defined $\mathbb{F}$-submodules

$$Y = \left\{ \sum\nolimits_k k \otimes_{N_G(P)} x : x \in E \right\} , \qquad Y' = \left\{ \sum\nolimits_k k \otimes_{N_G(P)} x_k : x_k \in E, \sum\nolimits_k x_k = 0 \right\}$$

summed over a left transversal $kN_K(P) \subseteq K$. Making use of the well-definedness, an easy manipulation shows that the action of $N_G(P)$ on X stabilizes Y and Y'. Similarly, K stabilizes Y and Y'. So Y and Y' are $\mathbb{F}G$-submodules of X. Since $|K : N_K(P)|$ is coprime to p, we have $Y \cap Y' = 0$. Since $|K : N_K(P)| = |G : N_G(P)|$, a consideration of dimensions yields $X = Y \oplus Y'$.

Fix a left transversal $\mathcal{L}$ for $N_K(P)$ in K. For $g \in N_G(P)$ and $\ell \in \mathcal{L}$, we can write ${}^g\ell = \ell_g h_g$ with $\ell_g \in \mathcal{L}$ and $h_g \in N_K(P)$. By the assumption on E again, $h_g x = x$ for all $x \in E$. So

$$g \sum\nolimits_\ell \ell \otimes x = \sum\nolimits_\ell {}^g\ell \otimes gx = \sum\nolimits_\ell \ell_g \otimes gx = \sum\nolimits_\ell \ell \otimes gx$$

summed over $\ell \in \mathcal{L}$. We have shown that ${}_{N_G(P)}\mathrm{Res}_G(Y) \cong E$. A similar argument involving a sum over $\mathcal{L}$ shows that K acts trivially on Y. Therefore, $Y \cong M_G^{K,F}$. On the other hand, Y is indecomposable with vertex P and, by the Green correspondence, $Y \cong M_{P,E}^G$. □

We let $T^{\mathrm{ex}}(G)$ to be the $\mathbb{Z}$-submodule of $T(G)$ spanned by the isomorphism classes of exprojective $\mathbb{F}G$-modules. The following proposition gives us that the $\mathbb{Z}$-submodule $T^{\mathrm{ex}}(G)$ of $T(G)$ is, in fact, a subring.

Proposition 3.1.6. *Given exprojective $\mathbb{F}G$-modules X and Y, then the $\mathbb{F}G$-module $X \otimes_{\mathbb{F}} Y$ is exprojective.*

Proof. We may assume that X and Y are indecomposable. Then X and Y are, respectively, direct summands of permutation $\mathbb{F}G$-modules having the form $\mathbb{F}G/K$ and $\mathbb{F}G/L$ where $K \trianglelefteq G \trianglerighteq L$. Then every indecomposable direct summand of $X \otimes Y$ is a direct summand of $\mathbb{F}G/(K \cap L)$ by Mackey decomposition and the Krull–Schmidt Theorem. □

We have

$$T^{\mathrm{ex}}(G) = \bigoplus_{(K,F) \in_G \mathcal{Q}(G)} \mathbb{Z}[M_G^{K,F}]$$

by Proposition 3.1.3.

3.2 A canonical induction formula

Let $\mathfrak{K}$ be a class of finite groups that is closed under taking subgroups. Let G be a finite group in $\mathfrak{K}$.

Applying the general theory in Boltje [1], we shall introduce a commutative ring $\mathcal{T}(G)$ and a ring epimorphism $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$. We aim to show that the $\mathbb{Z}[1/p]$-linear extension $\mathrm{lin}_G : \mathbb{Z}[1/p]\mathcal{T}(G) \to \mathbb{Z}[1/p]T(G)$ has a splitting

$\mathrm{can}_G : \mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$. Moreover, the map can_G is the unique splitting that commutes with restriction and isogation.

We adapt Boltje's construction in [1, 2.2] to the case of arbitrary $\mathfrak{K}$. Then we form the G-cofixed quotient $\mathbb{Z}$-module

$$\mathcal{T}(G) = \big(\bigoplus_{U \leq G} T^{\mathrm{ex}}(U)\big)_G$$

where G acts on the direct sum via the conjugation maps ${}_{gU}\mathrm{con}_U^g$. In fact, the functor $\mathcal{T}$ becomes a Green functor as in [1, 2.2], with the evident isogation maps by utilizing the Green functor structure of T, the restriction functor structure of T^{ex} and noting that $T^{\mathrm{ex}}(G)$ is a subring of $T(G)$. Moreover, since $T(G)$ is commutative, $\mathcal{T}(G)$ becomes a commutative ring. For any $x_U \in T^{\mathrm{ex}}(U)$, The image of x_U in $\mathcal{T}(G)$ will be denoted by $[U, x_U]_G$. Any $x \in \mathcal{T}(G)$ can be expressed in the form

$$x = \sum_{U \leq_G G} [U, x_U]_G$$

where the notation indicates that the index runs over representatives of the G-conjugacy classes of subgroups of G. Here x determines $[U, x_U]$ and x_G but not, in general, x_U. Let $\mathcal{R}(G)$ be the G-set of pairs (U, K, F) where $U \leq G$ and $(K, F) \in \mathcal{Q}(U)$. We have

$$\mathcal{T}(G) = \bigoplus_{U \leq_G G, (K,F) \in_{N_G(U)} \mathcal{Q}(U)} \mathbb{Z}[U, [M_U^{K,F}]] = \bigoplus_{(U,K,F) \in_G \mathcal{R}(G)} \mathbb{Z}[U, [M_U^{K,F}]] \ .$$

We set a $\mathbb{Z}$-linear map $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$ defined by $\mathrm{lin}_G[U, x_U] = {}_G\mathrm{ind}_U(x_U)$ for any $[U, x_U]$ in $\mathcal{T}(G)$. Then the family $(\mathrm{lin}_G : G \in \mathfrak{K})$ is a morphism of Green functors $\mathrm{lin} : \mathcal{T} \to T$ as in [1, 3.1]. In particular, the map $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$ is a ring homomorphism. We extend to coefficients in $\mathbb{Q}$ and obtain an algebra map

$$\mathrm{lin}_G : \mathbb{Q}\mathcal{T}(G) \to \mathbb{Q}T(G) \ .$$

We set a $\mathbb{Z}$-linear epimorphism $\pi_G : T(G) \to T^{\mathrm{ex}}(G)$ such that π_G acts as the identity on $T^{\mathrm{ex}}(G)$ and π_G annihilates the isomorphism class of every indecomposable non-exprojective p-permutation $\mathbb{Q}G$-module. We again extend to coefficients in $\mathbb{Q}$ and obtain a $\mathbb{Q}$-linear epimorphism $\pi_G : \mathbb{Q}T(G) \to \mathbb{Q}T^{\mathrm{ex}}(G)$. Following [1, 5.3a, 6.1a], we define a $\mathbb{Q}$-linear map

$$\operatorname{can}_G : \mathbb{Q}T(G) \to \mathbb{Q}\mathcal{T}(G),\ \xi \mapsto \frac{1}{|G|} \sum_{U,V \leq G} |U| \operatorname{möb}(U,V)[U, {}_U\mathrm{res}_V(\pi_V({}_V\mathrm{res}_G(\xi)))]_G$$

where möb() denotes the Möbius function on the poset of subgroups of G.

Theorem 3.2.1. *Consider the* $\mathbb{Q}$*-linear map* can_G*.*

(1) *We have* $\operatorname{lin}_G \circ \operatorname{can}_G = \mathrm{id}_{\mathbb{Q}T(G)}$*.*

(2) *For all* $H \leq G$*, we have* ${}_H\mathrm{res}_G \circ \operatorname{can}_G = \operatorname{can}_H \circ {}_H\mathrm{res}_G$*.*

(3) *For all* $L \in \mathfrak{K}$ *and isomorphisms* $\theta : L \leftarrow G$*, we have* ${}_L\mathrm{iso}^{\theta}_G \circ \operatorname{can}_G = \operatorname{can}_L \circ {}_L\mathrm{iso}^{\theta}_G$*.*

(4) $\operatorname{can}_G[X] = [X]$ *for all exprojective* $\mathbb{F}G$*-modules* X*.*

Those four properties, taken together for all $G \in \mathfrak{K}$*, determine the maps* can_G*.*

Proof. By [1, 6.4], part (1) will follow when we have checked that, for every indecomposable non-exprojective p-permutation $\mathbb{F}G$-module M, we have $[M] \in \sum_{K<G} {}_G\mathrm{ind}_K(\mathbb{Q}T(K))$. By [7, 2.1, 4.7], we may assume that G is p-hypoelementary. By [7, 1.3(b)], M is induced from $N_G(P)$ where P is a vertex of M. But M is non-exprojective, so P is not normal in G. The check is complete. Parts (2), (3), (4) follow from the proof of [1, 5.3a]. □

Obviously, parts (2) and (3) of the theorem implies that $\operatorname{can}_* : T \to \mathcal{T}$ is a morphism of restriction functors. Moreover, when $\mathfrak{K}$ is closed under the taking of quotient groups, the functors T, T^{ex}, $\mathcal{T}$ can be equipped with inflation maps, and the morphisms lin_* and can_* are compatible with inflation.

Corollary 3.2.2. *Given a p-permutation* $\mathbb{Q}G$*-module* X*, then*

$$[X] = \frac{1}{|G|} \sum_{U,V \leq G} |U| \operatorname{möb}(U,V)\, {}_G\mathrm{ind}_U \mathrm{res}_V(\pi_V({}_V\mathrm{res}_G[X]))\ .$$

Let M and X be p-permutation $\mathbb{F}G$-modules. We assume that M is indecomposable and denote the multiplicity of M as a direct summand of X by $m_G(M,X)$. We write $\pi_G(X)$ to denote the direct summand of X, well-defined up to isomorphism, such that $[\pi_G(X)] = \pi_G[X]$.

Lemma 3.2.3. *Let $\mathfrak{p}$ be a set of primes. Suppose that, for all $V \in \mathfrak{K}$, all p-permutation $\mathbb{F}V$-modules Y, all $U \lhd V$ such that V/U is a cyclic $\mathfrak{p}$-group, and all V-fixed elements $(K,F) \in \mathcal{Q}(U)$, we have*

$$m_U(M_U^{K,F}, \pi_U({}_U\operatorname{Res}_V(Y))) = \sum_{(J,E)\in\mathcal{Q}(V)} m_U(M_U^{K,F}, {}_U\operatorname{Res}_V(M_V^{J,E}))\, m_V(M_V^{J,E}, \pi_V(Y)) \,.$$

Then, for all $G \in \mathfrak{K}$, we have $|G|_{\mathfrak{p}'} \operatorname{can}_G[Y] \in \mathcal{T}(G)$, where $|G|_{\mathfrak{p}'}$ denotes the $\mathfrak{p}'$-part of $|G|$.

Proof. This is a special case of [1, 9.4]. □

Now we prove that can_G is $\mathbb{Z}[1/p]$-integral.

Theorem 3.2.4. *The $\mathbb{Q}$-linear map can_G restricts to a $\mathbb{Z}[1/p]$-linear map $\mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$.*

Proof. Let $\mathfrak{p}$ be the set of primes distinct from p. Let V, Y, U, K, F be as in the latest lemma. We will obtain the equality in the lemma. We may assume that Y is indecomposable. If Y is exprojective, then $\pi_U({}_U\operatorname{Res}_V(Y)) \cong {}_U\operatorname{Res}_V(Y)$ and $\pi_V(Y) \cong X$, whence the required equality is clear. So we may assume that Y is non-exprojective. Then $\pi_V(Y)$ is the zero module. Now it is enough to show that $M_U^{K,F}$ is not a direct summand of ${}_U\operatorname{Res}_V(Y)$. For a contradiction, suppose otherwise. The condition on $|V:U|$ implies that U contains the vertices of Y. So $Y \mid {}_V\operatorname{Ind}_U(X)$ for some indecomposable p-permutation $\mathbb{F}U$-module X. Since (K,F) is V-stable, a Mackey decomposition argument shows that $M_U^{K,F} \cong X$. We also have $K \lhd V$ by the V-stability of (K,F). Then

$$Y \mid {}_V\operatorname{Ind}_U \operatorname{Inf}_{U/K}(F) \cong {}_V\operatorname{Inf}_{V/K}\operatorname{Ind}_{U/K}(F) \,.$$

Thus, Y is exprojective. This is a contradiction, as required. □

Proposition 3.2.5. *The $\mathbb{Z}$-linear map $\operatorname{lin}_G : \mathcal{T}(G) \to T(G)$ is surjective. However, the $\mathbb{Z}[1/p]$-linear map $\operatorname{can}_G : \mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$ need not restrict to a $\mathbb{Z}$-linear map $T(G) \to \mathcal{T}(G)$. Indeed, putting $p = 3$ and $G = \mathrm{SL}_2(3)$, letting Y to be the isomorphically unique indecomposable non-simple non-projective p-permutation $\mathbb{F}G$-module and X to be the isomorphically unique 2-dimensional simple $\mathbb{F}Q_8$-module, then coefficient of the standard basis element $[Q_8, X]_G$ in $\operatorname{can}_G([Y])$ is equal to $2/3$.*

Proof. Since every 1-dimensional $\mathbb{F}G$-module is exprojective, the surjectivity of the $\mathbb{Z}$-linear map lin_G follows from Boltje [7, 4.7]. Routine techniques confirm the counter-example. $\square$

3.3 The $\mathbb{K}$-semisimplicity of the commutative algebra $\mathbb{K}\mathcal{T}(G)$

We write $\mathcal{I}(G)$ to denote the the G-set of pairs (P, s) where P is a p-subgroup of G and s is a p'-element of $N_G(P)/P$. We let $\mathbb{K}$ to be a field of characteristic zero such that $\mathbb{K}$ has roots of unity whose order is the p'-part of the exponent of G. We choose and fix an arbitrary isomorphism between a suitable torsion subgroup of $\mathbb{K} - \{0\}$ and a suitable torsion subgroup of $\mathbb{F} - \{0\}$. Note that Brauer characters of $\mathbb{F}G$-modules have values in $\mathbb{K}$. Let s be a p'-element of G. We set a species $\epsilon^G_{1,s}$ of $\mathbb{K}T(G)$, we mean, an algebra map $\mathbb{K}T(G) \to \mathbb{K}$, such that $\epsilon^G_{1,s}[M]$ is the value, at s, of the Brauer character of a p-permutation $\mathbb{F}G$-module M. Generally, for $(P, s) \in \mathcal{I}(G)$, we define a species $\epsilon^G_{P,s}$ of $\mathbb{K}T(G)$ such that $\epsilon^G_{P,s}[M] = \epsilon^{N_G(P)/P}_{1,s}[M(P)]$, where $M(P)$ denotes the P-relative Brauer quotient of M^P. The next result, well-known, can be found in Bouc–Thévenaz [19, 2.18, 2.19].

Theorem 3.3.1. *Given $(P, s), (P', s') \in \mathcal{I}(G)$, then $\epsilon^G_{P,s} = \epsilon^G_{P',s'}$ if and only if we have G-conjugacy $(P, s) =_G (P', s')$. The set $\{\epsilon^G_{P,s} : (P, s) \in_G \mathcal{I}(G)\}$ is the set of species of $\mathbb{K}T(G)$ and it is also a basis for the dual space of $\mathbb{K}T(G)$. The dual* *basis* $\{e^G_{P,s} : (P,s) \in_G \mathcal{I}(G)\}$ *is the set of primitive idempotents of* $\mathbb{K}T(G)$. *As a direct sum of trivial algebras over* $\mathbb{K}$, *we have*

$$\mathbb{K}T(G) = \bigoplus_{(P,s)\in_G \mathcal{I}(G)} \mathbb{K}e^G_{P,s} \ .$$

We let $\mathcal{J}(G)$ to denote the G-set of pairs (L,t) where L is a p'-residue-free normal subgroup of G and t is a p'-element of G/L. We define a species $\epsilon^{L,t}_G$ of $\mathbb{K}T^{\mathrm{ex}}(G)$ such that, given an indecomposable exprojective $\mathbb{F}G$-module M, then $\epsilon^{L,t}_G[M] = 0$ unless M is the inflation of an $\mathbb{F}G/L$-module $\overline{M}$, in which case, $\epsilon^{L,t}_G$ is the value, at t, of the Brauer character of $\overline{M}$.

Lemma 3.3.2. *Let* $(P,s) \in \mathcal{I}(G)$. *Let* L *be the normal closure of* P *in* G. *Let* t *be the image of* s *under the canonical homomorphism* $N_G(P)/P \to G/L$. *Then* $\epsilon^{L,t}_G[M] = \epsilon^G_{P,s}[M]$ *for all exprojective* $\mathbb{F}G$*-modules* M.

Proof. We may assume that M is indecomposable and that at least one of $\epsilon^{L,t}_G[M]$ and $\epsilon^G P, s[M]$ is nonzero. If $\epsilon^{L,t}_G[M] \neq 0$, then the required equality follows from the observation that L and perforce P act trivially on M. Now suppose $\epsilon^G P, s[M] \neq 0$. Then $M[P] \neq 0$. So P is contained in a vertex Q of M. The normal closure of Q in G contains L. Therefore, L acts trivially on M and, again, the required equality becomes clear. □

Now let $(L,t) \in \mathcal{J}(G)$ and let (P,s) run over mutually non-conjugate representatives of the elements of $\mathcal{I}(G)$ which satisfies the condition in the latest lemma. Then we set

$$e^{L,t}_G = \sum_{(P,s)} e^G_{P,s}.$$

Lemma 3.3.3. *Given* $(L,t),(L',t') \in \mathcal{J}(G)$, *then* $\epsilon^{L,t}_G = \epsilon^{L',t'}_G$ *if and only if* $L = L'$ *and* $t =_{G/L} t'$, *in other words,* $(L,t) =_G (L',t')$. *The set* $\{\epsilon^{L,t}_G : (L,t) \in_G \mathcal{J}(G)\}$ *is the set of species of* $\mathbb{K}T^{\mathrm{ex}}(G)$ *and it is also a basis for the dual space of* $\mathbb{K}T^{\mathrm{ex}}(G)$. *We have*

$$\mathbb{K}T^{\mathrm{ex}}(G) = \bigoplus_{(L,t)\in_G \mathcal{J}(G)} \mathbb{K}e^{L,t}_G$$

as a direct sum of trivial algebras over $\mathbb{K}$.

Proof. Since $\mathbb{K}T^{\mathrm{ex}}(G)$ is a subalgebra of $\mathbb{K}T(G)$, Theorem 3.3.1 implies that

$$\mathbb{K}T^{\mathrm{ex}}(G) = \bigoplus_{e \in \mathcal{M}} \mathbb{K}e$$

where $\mathcal{M}$ is the set of primitive idempotents of $\mathbb{K}T^{\mathrm{ex}}(G)$, furthermore, there is an equivalence relation $\equiv$ on the basis $\{e^G_{P,s}\}$ whereby $e^G_{P,s} \equiv e^G_{P',s'}$ provided $e^G_{P,s} \leq e \geq e^G_{P',s'}$ for some $e \in \mathcal{M}$. This is equivalent to the condition that $\epsilon(e^G_{P,s}) = \epsilon(e^G_{P',s'})$ for all species ϵ of $\mathbb{K}T^{\mathrm{ex}}(G)$. By the previous lemma, $\mathcal{M} = \{e^{L,t}_G\}$. The rest of the required conclusion is now clear. $\square$

We let $\mathcal{K}(G)$ to be the G-set of triples (V, L, t) where $V \leq G$ and $(L, t) \in \mathcal{J}(V)$. Now let $(L, t) \in \mathcal{J}(G)$. We set a species $\epsilon^G_{G,L,t}$ of $\mathbb{K}\mathcal{T}(G)$ such that, for $x \in \mathcal{T}(G)$ expressed as a sum as in Section 3.3,

$$\epsilon^G_{G,L,t}(x) = \epsilon^{L,t}_G(x_G)\ .$$

Now we define species for $\mathbb{K}\mathcal{T}(G)$. Given $(V, L, t) \in \mathcal{K}(G)$, we set a species $\epsilon^G_{V,L,t}$ such that

$$\epsilon^G_{V,L,t}(x) = \epsilon^V_{V,L,t}({}_V\mathrm{res}_G(x))\ .$$

Now we check that $\epsilon^G_{V,L,t}$ really is a species. Let x, $y \in \mathbb{K}\mathcal{T}(G)$. Then we have $(xy)_G = x_G y_G$ and $\mathrm{res}(xy) = \mathrm{res}(x)\mathrm{res}(y)$. Thus, $\big(\mathrm{res}(xy)\big)_V = \big(\mathrm{res}(x)\big)\big(\mathrm{res}(y)\big)$ and

$$\epsilon^G_{V,L,t}(xy) = \epsilon^G_{V,L,t}(x)\epsilon^G_{V,L,t}(y).$$

Using Lemma 3.3.3, a straightforward adaptation of the argument in [19, 2.18] gives the next result. This result also follows from Boltje—Raggi-Cárdenas—Valero-Elizondo [10, 7.5].

Theorem 3.3.4. *Given $(V, L, t), (V', L', t') \in \mathcal{K}(G)$, then $\epsilon^G_{V,L,t} = \epsilon^G_{V',L',t'}$ if and only if $(V, L, t) =_G (V', L', t')$. The set $\{\epsilon^G_{V,L,t} : (V, L, t) \in_G \mathcal{K}(G)\}$ is the set of species of $\mathbb{K}\mathcal{T}(G)$ and it is also a basis for the dual space of $\mathbb{K}\mathcal{T}(G)$. The dual basis $\{e^G_{V,L,t} : (V, L, t) \in_G \mathcal{K}(G)\}$ is the set of primitive idempotents of $\mathbb{K}\mathcal{T}(G)$. As a direct sum of trivial algebras over $\mathbb{K}$, we have*

$$\mathbb{K}\mathcal{T}(G) = \bigoplus_{(V,L,t) \in_G \mathcal{K}(G)} \mathbb{K}e^G_{V,L,t}\ .$$

Proof. Let $\epsilon = \epsilon^G_{V,L,t}$ and $\epsilon' = \epsilon^G_{V',L',t'}$. Plainly, if $(V, L, t) =_G (V', L', t')$, then $\epsilon = \epsilon'$. By considering the case where x has the form $x = [V, x_V]$, we deduce that $V' \leq_G V$. Similarly, $V \leq GV'$. So $V =_G V'$ and, replacing (V', L', t') with a suitable G-conjugate, we may assume that $V = V'$. Applying a formula for restriction in Boltje [1, Section 2], then Mackey decomposition,

$$1 = \epsilon'\big([V, e_V^{L',t'}]_G\big) = \epsilon\big([V, e_V^{L',t'}]_G\big)$$

$$= \epsilon^V_{V,L,t}]\Big({}_V\mathrm{res}_G\big([V, e_V^{L',t'}]_G\big)\Big) = \sum_{gV \subseteq N_G(V)} \epsilon_V^{L,t}\big(\mathrm{con}^g(e_V^{L',t'})\big)$$

Therefore, $(L, t) =_{N_G(V)} ({}^{L'}, t')$ and $(V, L, t) =_G (V', L', t')$.

The set of species $\mathcal{M}$ of $\mathbb{K}T(G)$ is $\mathbb{K}$-linearly independent. In particular, the subset $\{e^G_{V,L,t}\}$ of $\mathcal{M}$ is $\mathbb{K}$-linearly independent. Meanwhile, since

$$\{[V, e_V^{L,t}]_G \,:\, (V, L, t) \in \mathcal{K}(G)\}$$

is a $\mathbb{K}$-basis for $\mathbb{K}T(G)$, the dimension of $\mathbb{K}T(G)$ is equal to the number of G-orbits in $\mathcal{K}(G)$. Therefore, $\{e^G_{V,L,t}\}$ is a basis for the dual space of $\mathbb{K}T(G)$. Perforce, $\{e^G_{V,L,t}\} = \mathcal{M}$. The rest is clear. □

We have the following easy corollary on lifts of the primitive idempotents $e^G_{P,s}$.

Corollary 3.3.5. *Given $(P, s) \in \mathcal{I}(G)$, then $e^G_{\langle P,s\rangle,P,s}$ is the unique primitive idempotent e of $\mathbb{K}\mathcal{T}(G)$ such that $\mathrm{lin}_G(e) = e^G_{P,s}$.*

Chapter 4

A general theory of canonical induction for Green functors

4.1 Green category of a Mackey System

In this section, we recall definition of Mackey System and Green category from [3] and [18], but we also state a result describing a Green category associated with an arbitrary Mackey system.

We define a Mackey system on $\mathfrak{K}$ to be a subcategory $\mathcal{F}$ of the category of finite groups such that $\mathrm{obj}(\mathcal{F}) = \mathfrak{K}$ and, writing $\mathcal{F}(F, G)$ for the set of morphisms to a group $F \in \mathfrak{K}$ from a group $G \in \mathfrak{K}$, the following four axioms hold:

MS1 : For all $I \leq G \in \mathfrak{K}$, the inclusion $I \hookrightarrow G$ is in $\mathcal{F}(G, I)$.

MS2 : For all $I \leq G \in \mathfrak{K}$ and $g \in G$, the conjugation map $i \to^g i$ is in $\mathcal{F}({}^gI, I)$.

MS3 : For all $\theta \in \mathcal{F}(F, G)$, then the restriction $G \leftarrow \theta(G)$ is in $\mathcal{F}(\theta(G), G)$.

MS4 : For all $\theta \in \mathcal{F}(F, G)$ such that θ is a group isomorphism, then $\theta^{-1} \in \mathcal{F}(G, F)$.

Consider a pair of Mackey systems $(\mathcal{I}, \mathcal{R})$ such that the following three conditions hold:

GC1 : The category $\mathcal{I}$ is a subcategory of $\mathcal{R}$.

GC2 : Let $F \overset{\alpha}{\leftarrow} I \overset{\theta}{\rightarrow} G$ be a diagram with $\alpha \in \mathrm{mor}(\mathcal{I})$ and $\theta \in \mathrm{mor}(\mathcal{R})$. If θ is surjective and there exists a group homomorphism β such that $\alpha = \beta\theta$, then $\beta \in \mathrm{mor}(\mathcal{I})$. If α is surjective and there exists a group homomorphism ϕ such that $\theta = \phi\alpha$, then $\phi \in \mathrm{mor}(\mathcal{R})$.

GC3 : For every diagram $I \overset{\theta}{\rightarrow} G \overset{\beta}{\leftarrow} J$ with $\theta \in \mathrm{mor}(\mathcal{R})$ and $\beta \in \mathrm{mor}(\mathcal{I})$ there is a pullback $I \overset{\gamma}{\leftarrow} L \overset{\psi}{\rightarrow} J$ with $\gamma \in \mathrm{mor}(\mathcal{I})$ and $\psi \in \mathrm{mor}(\mathcal{R})$. Let $\mathcal{G}$ be the linear subcategory of $\mathcal{C}$ such that $\mathrm{obj}(\mathcal{G}) = \mathfrak{K}$ and the morphisms in $\mathcal{G}$ are generated by the morphisms ind^{α} and res^{θ} where $\alpha \in \mathrm{mor}(\mathcal{I})$ and $\beta \in \mathrm{mor}(\mathcal{R})$. We call $\mathcal{G}$ a Green category on $\mathfrak{K}$ (see, [3]). The homomorphisms in $\mathcal{I}$ are called the induction homomorphisms for $\mathcal{G}$ and the homomorphisms in $\mathcal{R}$ are called the restriction homomorphisms for $\mathcal{G}$.

Theorem 4.1.1. *Given a Mackey system $\mathcal{R}$ on $\mathfrak{K}$ let $\mathcal{I}$ be the subcategory of $\mathcal{R}$ whose morphisms are the injective morphisms in $\mathcal{R}$. Let $\mathcal{G}$ be the linear subcategory of $\mathcal{C}$ such that obj$(\mathcal{G}) = \mathfrak{K}$ and the morphisms in $\mathcal{G}$ are generated by the morphisms* ind^{α} *and* res^{θ} *where $\alpha \in$ mor$(\mathcal{I})$ and $\beta \in$ mor$(\mathcal{R})$. Then $\mathcal{G}$ is a Green category.*

Proof. Let $\mathcal{R}$ be a Mackey system on $\mathfrak{K}$. $\mathcal{I}$ be the subcategory of $\mathcal{R}$ whose morphisms are the injective homomorphisms in $\mathcal{R}$. Let $\mathcal{G}$ be the linear subcategory of $\mathcal{C}$ such that $\mathrm{obj}(\mathcal{G}) = \mathfrak{K}$ and the morphisms in $\mathcal{G}$ are generated by the morphisms ind^{α} and res^{θ} where $\alpha \in \mathrm{mor}(\mathcal{I})$ and $\beta \in \mathrm{mor}(\mathcal{R})$. Condition GC1 is trivial. We show that $(\mathcal{I}, \mathcal{R})$, satisfies the conditions GC2 and GC3.

Let $F \overset{\alpha}{\leftarrow} I \overset{\theta}{\rightarrow} G$ be a diagram with $\alpha \in \mathrm{mor}(\mathcal{I})$ and $\theta \in \mathrm{mor}(\mathcal{R})$. Suppose that θ is surjective and there exists a group homomorphism β such that $\alpha = \beta\theta$. Since $\alpha = \beta\theta$ and α is injective, θ is injective. So $\beta = \alpha\theta^{-1}$, thus $\beta \in \mathrm{mor}(\mathcal{I})$. Now suppose that α is surjective and there exists a group homomorphism ϕ such that $\theta = \phi\alpha$. Since α is surjective and injective, $\phi = \theta\alpha^{-1}$ so $\phi \in \mathrm{mor}(\mathcal{R})$ as required. Hence $(\mathcal{I}, \mathcal{R})$ satisfies the condition GC2.

Let $I \overset{\theta}{\to} G \overset{\beta}{\leftarrow} J$ be a diagram with $\theta \in \mathrm{mor}(\mathcal{R})$ and $\beta \in \mathrm{mor}(\mathcal{I})$. We set $L = \{i \in I \mid \theta(i) \in \beta(J)\}$. Take γ as an inclusion map, then $\gamma \in \mathrm{mor}(\mathcal{I})$. Given $l \in L$, then $\theta(l) = \beta(j)$ for some $j \in J$. We set $\psi(l) = j$. Since β is injective and ψ is well-defined, we get $\theta\gamma = \beta\psi$. Consider the following diagram:

$$\psi = \beta^{-1}\theta\gamma : L \to I \to Im\beta \cap Im\theta \overset{\beta^{-1}}{\to} J.$$

Thus, $\psi \in \mathrm{mor}(\mathcal{R})$. $\square$

4.2 A quadruple of group categories

In this section, following Coşkun's construction in [8], we introduce a quadruple of group categories $(\mathcal{S}, \mathcal{W}, \mathcal{E}, \mathcal{N})$ and state some properties. We also give definition of Green functor as in [3].

Consider a quadruple of group categories $(\mathcal{S}, \mathcal{W}, \mathcal{E}, \mathcal{N})$ such that the following conditions are hold:

- $\mathcal{N}$ is generated by ${}_G\mathrm{ind}_H^{\alpha}\,\mathrm{res}_F^{\theta}$ for all group homomorphism $\theta : H \to F$ and injective group homomorphisms $\alpha : H \to G$.
- $\mathcal{W}$ is generated by ${}_G\mathrm{ind}_H^{\alpha}\mathrm{inf}_{H/K}$ for all injective group homomorphisms $\alpha : H \to G$.
- $\mathcal{E}$ is generated by ${}_H\,\mathrm{res}_F^{\theta}$ for all injective group homomorphisms $\theta : H \to F$.
- $\mathcal{S}$ is generated by ${}_H\mathrm{inf}_{H/K}$.

Following is an immediatte corollary of the Theorem 4.1.1:

Corollary 4.2.1. *$\mathcal{N}$ is a Green category.*

A crucial theorem in Coskun's approach was [8, Theorem 3.2], we have the following adaptation of that result.

Proposition 4.2.2. *There is a $(\mathcal{W}, \mathcal{E})$ - bimodule isomorphism $\Phi : \mathcal{W} \otimes_{\mathcal{S}} \mathcal{E} \to \mathcal{N}$ given by ${}_F\mathrm{ind}_I^{\alpha}\mathrm{inf} \otimes_{\mathcal{S}} \mathrm{res}^{\theta} \to {}_F\mathrm{ind}_I^{\alpha}\mathrm{infres}^{\theta}$.*

Now we set some notations. As direct sums of regular $\mathbb{Z}$-modules, we define

$$\Delta_{\mathcal{W}} = \bigoplus_{\alpha \in \mathrm{tm}_{\mathcal{W}} - \mathrm{tm}_{\mathcal{S}}} \mathbb{Z}\alpha, \qquad \Delta_{\mathcal{W}}(G', G) = \bigoplus_{\alpha \in \mathrm{tm}_{\mathcal{W}}(G',G) - \mathrm{tm}_{\mathcal{S}}(G',G)}$$

where $G, G' \in \mathfrak{K}$. Clearly, we have

$$\Delta_{\mathcal{W}}(G', G) = \mathcal{W}(G', G) \cap \Delta_{\mathcal{W}}$$

and

$$\Delta_{\mathcal{W}} = \bigcup_{G', G \in \mathfrak{K}} \Delta_{\mathcal{W}}(G', G).$$

Therefore, as a direct sum of $\mathbb{Z}$-modules,

$$\mathcal{W} = \mathcal{S} \oplus \Delta_{\mathcal{W}}, \qquad \mathcal{W}(G', G) = \mathcal{S}(G', G) \oplus \Delta_{\mathcal{W}}.$$

Defining $\Delta_{\mathcal{E}}$ and $\Delta_{\mathcal{E}}(G', G)$ similarly, the observations that we have just made still hold with $\mathcal{E}$ in place of $\mathcal{W}$. Now we state an obvious observation.

Proposition 4.2.3. *The $\mathbb{Z}$-submodule $\Delta_{\mathcal{W}}$ is an ideal of $\mathcal{W}$.*

We let $\pi_{\mathcal{S}}^{\mathcal{W}} : \mathcal{W} \to \mathcal{S}$ to be the $\mathbb{Z}$-module epimorphism with kernel $\Delta_{\mathcal{W}}$. The latest proposition says that $\pi_{\mathcal{S}}^{\mathcal{W}}$ is a ring homomorphisms. Thus, we have a diagram of algebra maps as shown, where the four arrows bearing upwards denote inclusions and the arrow bearing downwards denotes the projection.

$$\begin{tikzcd}
 & & \mathcal{N} & \\
 & \mathcal{W} \arrow[ur] \arrow[dl] & & \mathcal{E} \arrow[ul] \\
\mathcal{S} \arrow[rr, dash] & & \mathcal{S} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Given a $R\mathcal{W}$-module N, we make the evident identification

$$_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}(N) = N$$

as $R\mathcal{S}$-modules. Given an $R\mathcal{S}$-module M, we make the identification

$$_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}(M) = M$$

as $\mathcal{S}$-modules, the extension to $\mathcal{W}$ being such that the module is annihilated by $\Delta_G\mathcal{W}$. The following two remarks are obvious.

Remark 4.2.4. *Given an $R\mathcal{S}$-module M, then the canonical identification of $R\mathcal{S}$-modules*

$$_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}(M) = {}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}(M) = M$$

yields a natural equivalence $_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}} \cong \operatorname{id}_{R\mathcal{S}}$.

Remark 4.2.5. *Let A be an $R\mathcal{E}$-module. Then using the canonical identification of $R\mathcal{W}$-modules $_{R\mathcal{W}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Ind}_{\mathcal{E}} = {}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}}(A)$ and the canonical identification of $\mathcal{S}$-modules $_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A) = A$ we obtain an identification of $R\mathcal{W}$-modules*

$$_{R\mathcal{W}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}}(A) = {}_{R\mathcal{W}}\operatorname{Ind}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A)$$

whereby $w \otimes_{\mathcal{E}} a = w \otimes_{\mathcal{S}} a$ for $w \in \mathcal{W}$ and $a \in A$. In particular, we have natural equivalence

$$_{R\mathcal{W}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}} \cong {}_{R\mathcal{W}}\operatorname{Ind}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}.$$

Now we give an analogue of the latest remark for coinduction. Both the proposition and the remark are adapted from Coskun [8, 3.2].

Proposition 4.2.6. *Let M be an $R\mathcal{W}$-module. Then there is a natural $R\mathcal{E}$-isomorphism*

$$ {}_{R\mathcal{E}}\operatorname{Coind}{}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}(M) \cong {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Coind}{}_{R\mathcal{W}}(M)$$

such that, for elements

$$\epsilon \in \operatorname{Coind}\operatorname{Res}(M) = \operatorname{Hom}{}_{R\mathcal{E}}\big({}_{R\mathcal{S}}R\mathcal{E}_{R\mathcal{E}}, {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(M)\big),$$

$$\nu \in \operatorname{Res}\operatorname{Coind}(M) = \operatorname{Hom}{}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{N}_{R\mathcal{E}}, M)$$

we have $\epsilon \leftrightarrow \nu$ provided ϵ is a restriction of ν. In particular, we have a natural equivalence

$$ {}_{R\mathcal{E}}\operatorname{Coind}{}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}} = {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Coind}{}_{R\mathcal{W}}.$$

Proof. We apply Proposition 4.2.2 and adjunctions, we obtain $R\mathcal{E}$-isomorphisms

$$\operatorname{Hom}{}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{N}_{R\mathcal{E}}, M) \cong \operatorname{Hom}{}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{W} \otimes_{R\mathcal{S}} R\mathcal{E}_{R\mathcal{E}}, M)$$

$$\operatorname{Hom}{}_{R\mathcal{S}}\big({}_{R\mathcal{S}}R\mathcal{E}_{R\mathcal{E}}, \operatorname{Hom}{}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{W}_{R\mathcal{S}}, M)\big) \cong \operatorname{Hom}{}_{R\mathcal{S}}\big({}_{R\mathcal{S}}R\mathcal{E}_{R\mathcal{E}}, {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}(M)\big).$$

Let $\nu' \in \operatorname{Hom}(R\mathcal{W} \otimes_{R\mathcal{S}} R\mathcal{E}, M)$ and $\epsilon' \in \operatorname{Hom}\big(R\mathcal{E}, \operatorname{Hom}(R\mathcal{W}, M)\big)$. Now suppose that the elements are in correspondence $\nu \leftrightarrow \nu' \leftrightarrow \epsilon' \leftrightarrow \epsilon$. Then

$$\nu(we) = \nu'(w \otimes e) = \epsilon'(e)(w) = w\epsilon(e)$$

$w \in R\mathcal{W}$ and $e \in R\mathcal{E}$. Since ν is an $R\mathcal{W}$-homomorphism, we see that ν and ϵ determine each other via the equality $\nu(e) = \epsilon(e)$. $\square$

We define a **Green** $R\mathcal{E}$-functor to be an $R\mathcal{E}$-functor A such that:

- **GF1** : $A(G)$ is a unital ring for each $G \in \mathfrak{K}$,
- **GF2** : ${}_I\operatorname{res}_G^{\theta} : A(G) \to A(I)$ is a unital ring homomorphism for each $\theta : I \to G$ in $\mathcal{F}$.

Generalizing a definition in Romero [5, Section 2], Barker in [3] defines a Green $R\mathcal{N}$-functor to be an $R\mathcal{N}$-functor $\mathcal{A}$ such that:

- Each $\mathcal{A}(G)$ is unital ring.
- Each ${}_I\operatorname{Res}^{\theta}_G : \mathcal{A}(G) \to \mathcal{A}(I)$ is a unital ring homomorphism for each ring homomorphism $\theta : I \to G$.
- Given an injective $\alpha : I \to F$ in $\mathcal{F}$ and elements $a \in \mathcal{A}(F)$ and $b \in \mathcal{A}(I)$, then

$$ {}_F\mathrm{ind}^{\alpha}_I({}_I\operatorname{res}^{\alpha}_F(a) \cdot b) = a \cdot {}_F\mathrm{ind}^{\alpha}_I(b), \quad {}_F\mathrm{ind}^{\alpha}_I(b \cdot {}_I\operatorname{res}^{\alpha}_F(a)) = {}_F\mathrm{ind}^{\alpha}_I(b) \cdot a. $$

4.3 An embedding of the Burnside ring into the biset endomorphism ring

Let $G \in \mathfrak{K}$. Let $B(G)$ be the Burnside ring of G. Now will describe $RB(G)$ as an algebra, in fact, as a subalgebra $R\mathcal{N}(G)$, here $R\mathcal{N}(G) = R\operatorname{End}_{R\mathcal{N}}(G)$.

The next result describes RB as an algebra, in fact, as a subalgebra of $R\mathcal{N}$. Note that that Proposition 4.3.1 and Lemmas 4.3.2 and 4.3.3 are all based on [1, 1.5] which, in turn, has a citation acknowleding a work of Dress.

Proposition 4.3.1. *There is a counital injective algebra map*

$$ \Delta_G : RB(G) \to R\mathcal{N}(G) $$

given by the condition that, for $U \leq G$, we have

$$ \Delta_G(d^G_U) = {}_G\mathrm{ind}_U\mathrm{res}_G. $$

Proof. Proof is by the Mackey Product Formula. □

Now we state some properties of the algebra map Δ_G.

Lemma 4.3.2. *Let $H \leq G$ and $\phi : G \to G'$ be group isomorphism. Let $x \in RB(G)$ and $y \in RB(H)$. Then:*

(1) ${}_{G'}\mathrm{iso}_G^{\phi} \cdot \Delta_G(x) = \Delta_{G'}\big({}_{G'}\mathrm{iso}_G^{\phi}(x)\big) \cdot {}_{G'}\mathrm{iso}_G^{\phi}$.

(2) ${}_H\mathrm{res}_G \cdot \Delta_G(x) = \Delta_H\big({}_H\mathrm{res}_G(x)\big) \cdot {}_H\mathrm{res}_G$.

(3) $\Delta_G\big({}_G\mathrm{ind}_H(y)\big) = {}_G\mathrm{ind}_H \cdot \Delta_H(y) \cdot {}_H\mathrm{res}_G$.

(4) $\Delta_G(x) \cdot {}_G\mathrm{ind}_H = {}_G\mathrm{ind}_H \cdot \Delta_H\big({}_H\mathrm{res}_G(x)\big)$.

Lemma 4.3.3. *Let M be an $R\mathcal{N}$-module.*

(1) *Given $x \in RB(G)$ and isogation ${}_{G'}\mathrm{iso}_G^{\phi}$ in $\mathcal{N}$ and $m \in M(G)$, then*

$$
{}_{G'}\mathrm{iso}_G^{\phi}\big(\Delta_G(x) \cdot m\big) = \Delta_{G'}\big({}_{G'}\mathrm{iso}_G^{\phi}(x)\big) \cdot {}_{G'}\mathrm{iso}_G^{\phi}(m).
$$

(2) *Given $x \in RB(G)$ and $H \leq G$ and $m \in M(G)$, then*

$$
{}_H\mathrm{res}_G\big(\Delta_G(x) \cdot m\big) = \Delta_H\big({}_H\mathrm{res}_G(x)\big) \cdot {}_H\mathrm{res}_G(m)
$$

(3) *Given $y \in RB(H)$ and $H \leq G$ and $m \in M(G)$, then*

$$
\Delta_G\big({}_G\mathrm{ind}_H(y)\big) \cdot m = {}_H\mathrm{ind}_G\big(\Delta_H(y) \cdot {}_H\mathrm{res}_G(m)\big).
$$

(4) *Given $x \in RB(G)$ and $H \leq G$ and $n \in M(H)$, then*

$$
\Delta_G(x) \cdot {}_G\mathrm{ind}_H(n) = {}_G\mathrm{ind}_H\big(\Delta_H({}_H\mathrm{res}_G(x)) \cdot n\big).
$$

Proof. This is immediate from the previous lemma. □

Now we set some notations. Let M be an $R\mathcal{N}$-module. We set

$$
\mathcal{I}(M)(G) = \sum_{H<G} {}_G\mathrm{ind}_H\big(M(H)\big) = \sum_{H<G} \mathrm{Im}\big({}_G\mathrm{ind}_H : M(H) \to M(G)\big),
$$

$$
\mathcal{K}(M)(G) = \bigcap_{H<G} \mathrm{Ker}\big({}_H\mathrm{res}_G : M(G) \to M(H)\big).
$$

Now using Proposition 4.3.1, we can regard $M(G)$ as an $RB(G)$-module via Δ_G. Thus, an element $x \in RB(G)$ sends an element $m \in M(G)$ to the element $\Delta_G(x)m \in M(G)$. The next result is an adaptation of [7, 6.2].

Proposition 4.3.4. *Suppose that $|G|$ is invertible in R. Let M be an $R\mathcal{N}$-module. Then*

$$\mathcal{K}(M)(G) = \Delta_G(e_G^G)M(G) \qquad \mathcal{I}(M)(G) = \Delta_G(1 - e_G^G)M(G).$$

In particular, as a direct sum of $RB(G)$-modules, $M(G) = \mathcal{K}(M)(G) \oplus \mathcal{I}(M)(G)$.

The following result and its proof generalize [1, 6.3].

Corollary 4.3.5. *Let M be an $R\mathcal{N}$-module. Given $m \in M(G)$, then $m = 0$ if and only if $\Delta_G(e_G^G)m = 0$ and ${}_H\mathrm{res}_G(m) = 0$ for all strict subgroups $H < G$.*

Proof. Proposition 4.3.4 says that $\Delta_G(e_G^G)m = 0$ if and only if $m \in \mathcal{I}(M)(G)$. Obviously, the restriction condition holds if and only if $m \in \mathcal{K}(M)(G)$. The proposition also tells us that zero is the unique element of $M(G)$ satisfying both of those conditions. $\square$

4.4 Lower plus functor

In this section, we will replace T^{ex} with a Green $R\mathcal{E}$-functor A and $\mathcal{T}$ with $\mathrm{Lplus}\,(A)$ functor. We show that $\mathrm{Lplus}\,(A)$ covers the classical cofixed module construction $A_+(G)$ of Boltje [1]. We will also mention a result which states that given a Green $R\mathcal{E}$-functor A, we make $A_+(G)$ a ring and show that A_+ is a Green $R\mathcal{N}$-functor.

Let A be a Green $R\mathcal{E}$- functor. We make the usual identification

$$A = \bigoplus_G A(G).$$

As an $R\mathcal{N}$-functor, we define the lower plus functor

$${}_{R\mathcal{N}}\,\mathrm{Lplus}_{\,R\mathcal{E}} : R\mathcal{E}\text{-Mod} \to R\mathcal{N}\text{-Mod}$$

such that

$${}_{R\mathcal{N}}\,\mathrm{Lplus}_{\,R\mathcal{E}}(A) = R\mathcal{N} \otimes_{R\mathcal{E}} A.$$

Now we incorporate some definitions and notations from [1]. Fix a group $G \in \mathfrak{K}$ and let A be an $R\mathcal{E}$-module. We make $\bigoplus\limits_{U \leq G} A(U)$ an RG-module such that action of an element $g \in G$ restricts to the conjugation map ${}_{{}^gU}\operatorname{con}^g_U : A(U) \to A({}^gU)$ for each U. Consider the G-cofixed quotient R-module

$$A_+(G) = \big(\bigoplus_{U \leq G} A(U) \big)_G.$$

We can make the identification

$$A_+(G) = \bigoplus_{U \leq_G G} A(U)_{N_G(U)}.$$

Thus, any element $x \in A_+(G)$ can be written in the form

$$x = \sum_{U \leq_G G} [U, x_U]_G$$

where $x_U \in A(U)$ with image $[U, x_U] \in A(U)_{N_G(U)} \leq A_+(G)$.

If we write $x' = \sum\limits_{U \leq_G G} [U, x'_U]_G$ where U running over the same representatives of conjugacy classes of subgroups, then $x = x'$ if and only if each $x_U - x'_U$ belongs to the $RN_G(U)$-submodule of $A(U)$ spanned by the elements having the form $(n - n')a$ with $n, n' \in N_G(U)$ and $a \in A(U)$. Note that the expression $x = \sum_{U \leq_G G} [U, x_U]_G$ is unique up to this condition. Note that the action of U on $A(U)$ is trivial and $A(U)$ is actually an $RN_G(U)/U$-module.

Proposition 4.4.1. *Let A be an $R\mathcal{E}$-module and let $G \in \mathfrak{K}$. Then there is an R-linear isomorphism*

$$s_G : {}_{R\mathcal{N}}\operatorname{Lplus}_{R\mathcal{E}}(A)(G) \to A_+(G)$$

given by $s_G({}_G\mathrm{ind}_H \otimes_{R\mathcal{E}} x_U) = [H, a_H]_G$ for any $H \leq G$ and any $a_H \in A(H)$.

Proof. Given $U \leq G$, let ${}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} A(U)$ denote the R-submodule consisting of those elements that can be written in the form ${}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} x_U$ with $x_U \in A(U)$. We aim to show that as a direct sum of R-modules we have

$$\operatorname{Lplus}(A)(G) = \bigoplus_{U \leq_G G} {}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} A(U).$$

The R-module

$$\mathrm{Lplus}\,(A)(G) = (R\mathcal{N} \otimes_{R\mathcal{E}} A)(G) = \mathrm{iso}_G \cdot R\mathcal{N} \otimes_{R\mathcal{E}} A(G)$$

is spanned by the elements having the form $\alpha \otimes_{R\mathcal{E}} x_W$ where $W \in \mathfrak{K}$ and $\alpha : W \to G$ is a transitive morphism in $\mathcal{N}$. We set $\alpha = {}_G\mathrm{ind}_U \cdot \mu$, where μ is a transitive morphism in $\mathcal{W}$, then $\alpha \in {}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} A(U)$. Thus,

$$\mathrm{Lplus}\,(A)(G) = \bigoplus_{U \leq_G G} {}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} A(U).$$

Now we will show that the sum is direct. It's enough to show that

$$\mathrm{iso}_G \cdot R\mathcal{N} = \bigoplus_{U \leq_G G} {}_G\mathrm{ind}_U \cdot R\mathcal{E}.$$

We set $\alpha = {}_G\mathrm{ind}_{U'} \cdot \mu'$ where μ' is a transitive morphism in $\mathcal{W}$. Then, $\alpha \in {}_U\mathrm{ind}_G \otimes_{R\mathcal{E}} A(U)$. Therefore,

$$\mathrm{Lplus}\,(A)(G) = \sum_{U \leq_G G} {}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} A(U).$$

Now to show that this sum is direct, it suffices to show that

$$\mathrm{iso}_G \cdot R\mathcal{N} = \sum_{U \leq_G G} {}_G\mathrm{ind}_U \cdot R\mathcal{E}.$$

Let us also $\alpha = {}_G\mathrm{ind}_{u'} \cdot \mu'$, where μ' is a transitive morphism in $\mathcal{W}$. Choosing internal butterfly factorizations of μ and μ' in $\mathcal{C}$, we obtain two internal butterfly factorizations of in $\mathcal{C}$. By the uniqueness of the internal butter y factorization up to conjugation, there exists an element $g \in G$ such that ${}^gU = u'$ and the following diagram in $\mathcal{C}$ commutes.

$$\begin{tikzcd}
& H \arrow[dl, "{}_G\mathrm{ind}_H"'] \arrow[dr, "\mu"] \arrow[dd, "{}_{H'}\mathrm{con}^g_H"] & \\
G \arrow[dr, "{}_G\mathrm{ind}_{H'}"'] & & W \arrow[dl, "\mu'"] \\
& H' &
\end{tikzcd}$$

Now we show that there is a well-defined R-linear map $s'_G : A_+(G) \to \mathrm{Lplus}\,(A)(G)$ defined by $s'_G[U, x_U]_G = {}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} x_U$. Indeed, it will then be clear that s_G and s'_G are mutual inverses. Fixing U, and elements x_U and x'_U of $A(U)$ such that $[U, x_U]_G = [U, x'_U]_G$, we must show that ${}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} x_U = {}_G\mathrm{ind}_U \otimes_{R\mathcal{E}} x'_U$. We may assume that $x'_U = {}^n x_U$ for some $n \in N_G(U)$, in other words, $x'_U = {}_U\mathrm{con}^n_U(x_U)$. The required equality follows, because ${}_G\mathrm{ind}_U\mathrm{con}^n_U = {}_G\mathrm{ind}_U$. $\square$

The R-linear isomorphism s_G defined in the previous proposition combine to form an R-linear isomorphism

$$s : \mathrm{Lplus}\,(A) \to A_+$$

where $A_+ = \bigoplus_{G \in \mathfrak{K}} A_+(G)$. We make A_+ an $R\mathcal{N}$-module via s. Thus, s becomes an $R\mathcal{N}$-module isomorphism.

Now we consider the following five kinds of transitive morphism: the isogations ${}_{G'}\mathrm{iso}^{\phi}_G$ where $\phi : G \to G'$ is a group isomorphism; the internal induction ${}_G\mathrm{ind}_H$ and internal restriction ${}_H\mathrm{res}_G$ where $H \leq G$; the internal inducation ${}_G\mathrm{inf}_{G/N}$ where $N \trianglelefteq G$. Let us introduce variables

$$U \leq G, \qquad V \leq H, \qquad N \leq W \leq G$$

taking on roles such that, when considering elements

$$x \in A_+(G), \qquad y \in A_+(H), \qquad z \in A_+(G/N)$$

we may conventionally write

$$x = \sum_{U \leq_G G} [U, x_U]_G, \qquad y = \sum_{V \leq_H H} [V, y_V]_H, \qquad z = \sum_{W/V \leq_{G/N} G/N} [W/N, x_{W/N}]_{G/N}$$

where $x_U \in A(U)$ and similarly for y_V and $z_{W/N}$. The next result explicitly describes the actions of those four kinds of transitive morphism on the $R\mathcal{N}$-module $\mathrm{Lplus}\,(A) \cong A_+$.

Proposition 4.4.2. *With the notation above, we have*

(1) ${}_G\mathrm{ind}_H[V, y_V]_H = [V, {}_G\mathrm{ind}_H(y_V)]_G$.

(2) ${}_H\mathrm{res}_G[U, x_U]_G = \sum_{HgU \subseteq G} [H \cap {}^gU, {}_{H \cap {}^gU}\mathrm{con}^g_{H^g \cap U}\mathrm{res}_U(x_U)]_H$

(3) *If* ${}_{G'}\mathrm{iso}^{\phi}_{G} \in \mathcal{N}$ *then* ${}_{G'}\mathrm{iso}^{\phi}_{G}[U', {}_{G'}\mathrm{iso}^{\phi}_{G}(x_U)]_{G'}$ *where* $U' = \phi(U)$.

(4) *If* ${}_{G}\mathrm{inf}_{G/N} \in \mathcal{N}$*, then* ${}_{G}\mathrm{inf}_{G/N}[W/N, z_{W/N}]_{G/N} = [W, {}_{G}\mathrm{inf}_{G/N}(z_{W/N})]_{G}$.

Proof. As an abuse of notation, we identify Lplus (A) with A_+ via s, writing

$$ {}_{G}\mathrm{ind}_{U} \otimes_{R\mathcal{E}} x_U = [U, x_U]_G. $$

Part (1) is now clear. For the rest, we apply the commutation relations for induction recorded in the previous subsection. Part (2) holds because

$$ {}_{H}\mathrm{res}_{G}\mathrm{ind}_{U} x_U = \sum_{HgU \subseteq G} {}_{H}\mathrm{ind}_{H \cap {}^{g}U}\mathrm{con}^{g}_{H^{g} \cap U}\mathrm{res}_{U} \otimes x_U $$

$$ = \sum_{HgU \subseteq G} {}_{H}\mathrm{ind}_{H \cap {}^{g}U} \otimes {}_{H \cap {}^{g}U}\mathrm{con}^{g}_{H^{g} \cap U}\mathrm{res}_{U}(x_U). $$

Part (3) is very easy. Part (4) holds because

$$ {}_{G}\mathrm{inf}_{G/N}\mathrm{ind}_{W/N} \otimes x_U = {}_{G}\mathrm{ind}_{W}\mathrm{inf}_{W/N} \otimes x_U = {}_{G}\mathrm{ind}_{W} \otimes {}_{W}\mathrm{inf}_{W/N}(x_U). $$

A similar argument yields Part (5). □

Note that the local case involving induction and restriction between subgroups of a fixed group G, parts (1), (2), (3) of the latest proposition agree with the defining formulas in [1, Section 2], for the actions of induction, restriction and conjugation on A_+.

Let $G \in \mathfrak{K}$. Now we will give a ring structure to $A_+(G)$. We want to get an injective R-map $\theta : A_+ \to E$ such that $\theta(A_+)$ is a locally unital R-subalgebra of E. Then A_+ becomes a locally unital R-algebra with multiplication such that θ becomes an R-algebra isomorphism.

Now consider $\sigma : R\mathcal{N} \to E$ be the representation of A_+ as an $R\mathcal{N}$-functor. For any ${}_{F}x_{G} \in R\mathcal{N}(F, G)$ and $U \leq G$ and $s \in A(U)$, we get

$$ \sigma({}_{F}x_{G})({}_{G}\mathrm{ind}_{U} \otimes s) = ({}_{F}x_{G}\mathrm{ind}_{U}) \otimes s. $$

We can embed $A(G)$ into the $A_+(G)$ via the identification $a = \mathrm{iso}_G \otimes a$ for $a \in A(G)$. Now we can write

$$_G\mathrm{ind}_U \otimes s = \sigma(_G\mathrm{ind}_U)s.$$

So given any element $a \in A_+(G)$ can be written in the form

$$a = \sum_{U \leq_G G} \sigma(_G\mathrm{ind}_U)s_U.$$

Now we define an R-map $\nu : A \to E$ such that

$$\nu(b)\sigma(_G\mathrm{ind}_U)s = \sigma(_G\mathrm{ind}_U)(_U\,\mathrm{res}_G(b) \cdot s)$$

for any $b \in A(G)$. We need to show that ν is well-defined. Suppose that $\sigma(_G\mathrm{ind}_U)s = \sigma(_G\mathrm{ind}_{U'})s'$. Then $U' = {}^gU$ and $s' = \mathrm{con}^g(s)$ for some $g \in G$. We get

$$\sigma(_G\mathrm{ind}_U)(_U\mathrm{res}_G(b) \cdot s) = \sigma(_G\mathrm{ind}_{U'})(_{U'}\mathrm{res}_G(b) \cdot s').$$

Let $b_1, b_2 \in A(G)$ then by using the restriction axiom for A, we obtain

$$\nu(b_1)(\nu(b_2)\sigma(_G\mathrm{ind}_U s)).$$

Since $A_+(G)$ is spanned by the elements having the form $\sigma(_G\mathrm{ind}_U)s$, we get $\nu(b_1)\nu(b_2) = \nu(b_1 b_2)$. Plainly, $\nu(1_{A(G)}) = 1_{A_+(G)}$. We have shown that $\nu : A \to E$ is locally unital algebra map.

Lemma 4.4.3. *Given $b \in A(G)$ and $t \in A(J)$ and an injective homomorphism $\beta : J \to G$ in $\mathcal{F}$, then*

$$\nu(b)\sigma(_G\mathrm{ind}_J^\beta)t = \sigma(_G\mathrm{ind}_J^\beta)(_J\,\mathrm{res}_G^\beta(b) \cdot t).$$

Proof. This follows by an easy manipulation involving the equalities $_G\mathrm{ind}_J^\beta = {}_G\mathrm{ind}_{\beta(J)}\mathrm{iso}_J^\beta$ and $_J\mathrm{res}_G^\beta = {}_J\mathrm{iso}_{\beta(J)}^{\beta^{-1}}\mathrm{res}_G$. □

Now we change the notation again. Let $U \leq G$ and $s \in A(U)$. We write

$$_G\mathrm{ind}_U(s) = \sigma(_G\mathrm{ind}_U)s.$$

So the R-module A_+ is spanned by the elements having the form ${}_G\mathrm{ind}_U(s)$.

Let $\theta : A_+ \to E$ be the R-linear map defined by

$$\theta({}_G\mathrm{ind}_U(s)) = \sigma({}_G\mathrm{ind}_U)\nu(s)\sigma({}_U\mathrm{res}_G).$$

The map θ is injective because $\theta({}_G\mathrm{ind}_U(s))1_{A(G)} = {}_G\mathrm{ind}_U(s)$. So we can regard $\theta(A_+)$ as a copy of A_+. The following lemma is an analogue of the [3, Lemma 3.2].

Lemma 4.4.4. *Let A be a Green $R\mathcal{E}$-functor, and let $F, G, I \in \mathfrak{K}$.*

1. *Given $b \in A(G)$ and a restriction homomorphism $\theta : I \to G$ in $\mathcal{F}$, then*

$$\sigma({}_I\,\mathrm{res}^{\theta}_G)\nu(b) = \nu({}_I\,\mathrm{res}^{\theta}_G(b)) \cdot \sigma({}_I\,\mathrm{res}^{\theta}_G).$$

2. *Given $a \in A(F)$ and an injective homomorphism $\alpha : I \to F$ in $\mathcal{F}$, then*

$$\nu(a)\sigma({}_F\mathrm{ind}^{\alpha}_I) = \sigma({}_F\mathrm{ind}^{\alpha}_I)\nu({}_I\,\mathrm{res}^{\alpha}_F(a)).$$

Proof. 1. Let $K \le G$ and $t \in A(K)$. Using Lemma 4.4.3, definition of ν and Mackey relation we obtain

$$\sigma({}_I\,\mathrm{res}_G)\nu(b)({}_I\mathrm{ind}_K(t)) = \sigma({}_I\,\mathrm{res}_G)\sigma({}_G\mathrm{ind}_K)({}_K\,\mathrm{res}_G(b)\cdot t)$$

$$= \sum_{IgK\subseteq G} \sigma({}_I\mathrm{ind}_{I\cap {}^gK} con^{g}_{I^g\cap K}\,\mathrm{res}_K)({}_K\,\mathrm{res}_G(b)\cdot t)$$
$$= \sum_{IgK\subseteq G} \sigma({}_I\mathrm{ind}_{I\cap {}^gK})({}_{I\cap {}^gK}\,\mathrm{res}_G(b)\cdot {}_{I\cap {}^gK}\,\mathrm{con}\,^{g}_{I^g\cap K}\,\mathrm{res}_K(t))$$
$$= \sum_{IgK\subseteq G} \nu(b)\sigma({}_I\mathrm{ind}_{I\cap {}^gK})({}_{I\cap {}^gK}\,\mathrm{con}\,^{g}_{I^g\cap K}\,\mathrm{res}_K(t))$$
$$= \nu(b)\sum_{IgK\subseteq G} \sigma({}_I\mathrm{ind}_{I\cap {}^gK}\,\mathrm{con}\,^{g}_{I^g\cap K}\,\mathrm{res}_K(t))$$
$$= \nu(b)\sigma({}_I\,\mathrm{res}_G)\sigma({}_G\mathrm{ind}_K)(t)$$
$$= \nu({}_I\,\mathrm{res}_G(b))\cdot\sigma({}_I\,\mathrm{res}_G).$$

2. Let $K \leq I$ and $t \in A(K)$. Using Lemma 4.4.3 and the definition of ν,

$$\nu(a)\sigma({}_F\mathrm{ind}_I^\alpha)({}_I\mathrm{ind}_K(t)) = \nu(a)\sigma({}_F\mathrm{ind}_K^\alpha)t = \sigma({}_F\mathrm{ind}_K^\alpha)({}_K\operatorname{res}_F^\alpha(a)\cdot t)$$

$$= \sigma({}_F\mathrm{ind}_I^\alpha)\sigma({}_I\mathrm{ind}_K)({}_K\operatorname{res}_I\operatorname{res}_F^\alpha(a)\cdot t) = \sigma({}_F\mathrm{ind}_I^\alpha)\nu({}_I\operatorname{res}_F^\alpha(a))({}_I\mathrm{ind}_K(t)).$$

□

Now using Mackey relation and above lemma, we obtain

$$\sigma({}_G\mathrm{ind}_U)\nu(s)\sigma({}_U\operatorname{res}_G)\sigma({}_G\mathrm{ind}_V)\nu(t)\sigma({}_V\operatorname{res}_G)$$

$$= \sum_{UgV\subseteq G}\sigma({}_G\mathrm{ind}_U)\nu(s)\sigma({}_U\mathrm{ind}_{U\cap{}^gV}\operatorname{con}{}^g_{U^g\cap V}\operatorname{res}_V)\nu(t)\sigma({}_V\operatorname{res}_G)$$

$$= \sum_{UgV\subseteq G}\sigma({}_G\mathrm{ind}_{U\cap{}^gV})\nu({}_{U\cap{}^gV}\operatorname{res}_U(s)\cdot{}_{U\cap{}^gV}\operatorname{con}{}^g_{{}^gU\cap V}\operatorname{res}_V)\sigma({}_{U\cap{}^gV}\operatorname{res}_G).$$

Then by using the injectivity of θ we get the formula

$${}_G\mathrm{ind}_U(s)\cdot_G\mathrm{ind}_V(t) = \sum_{UgV\subseteq G}{}_G\mathrm{ind}_{U\cap{}^gV}({}_{U\cap{}^gV}\operatorname{res}_U(s)\cdot{}_{U\cap{}^gV}\operatorname{res}_{{}^gV}\operatorname{con}{}^g_V(t)).$$

Theorem 4.4.5. *Given a Green $R\mathcal{E}$-functor A, then A_+ is a Green $R\mathcal{N}$-functor.*

Proof. Given a homomorphism $\alpha : H \to G$ we show that

$${}_H\operatorname{res}_G^\alpha : A_+(H) \to A_+(G)$$

is a ring homomorphism. Let ${}_G\mathrm{ind}_U(s) \in A_+(G)$, ${}_G\mathrm{ind}_V(t) \in A_+(G)$. Let

$$\mathcal{B} = {}_H\operatorname{res}_G^\alpha({}_G\mathrm{ind}_U(s))\cdot{}_H\operatorname{res}_G^\alpha({}_G\mathrm{ind}_V(t))$$

and

$$\mathcal{C} = {}_H\operatorname{res}_G^\alpha({}_G\mathrm{ind}_U(s)\cdot{}_G\mathrm{ind}_V(t))$$

We must show that $\mathcal{B} = \mathcal{C}$. Then

$$\mathcal{B} = \sum_{\alpha(H)gU\subseteq G}{}_H\mathrm{ind}_{\alpha^{-1}({}^gU)}\operatorname{res}_U^{c(g^{-1})\alpha_g}(s). \sum_{\alpha(H)eV\subseteq G}{}_H\mathrm{ind}_{\alpha^{-1}({}^eV)}\operatorname{res}_V^{c(e^{-1})\alpha_e}(t)$$

where $\alpha_g : \alpha^{-1}({}^gU) \to {}^gU \cap \alpha(H)$ and $\alpha_e : \alpha^{-1}({}^eU) \to {}^eU \cap \alpha(H)$

$$= \sum_{\substack{\alpha(H)gU\subseteq G\\ \alpha(H)eV\subseteq G\\ \alpha^{-1}({}^gU)h\alpha^{-1}({}^eV)\subseteq H}} {}_H\mathrm{ind}_{\alpha^{g,h,e}_{H,U,V}}\left({}_{\alpha^{g,h,e}_{H,U,V}}\mathrm{res}_U^{c(g^{-1})\alpha_g}(s)\cdot{}_{\alpha^{g,h,e}_{H,U,V}}\mathrm{res}_V^{c(e^{-1})\alpha_e c(h^{-1})}(t)\right)$$

where $\alpha^{g,h,e}_{H,U,V} = \alpha^{-1}({}^gU)\cap {}^h\alpha^{-1}({}^eV)$

$$= \sum_{\substack{g,e\in G\\ h\in H}} \frac{1}{|\alpha(H)gU||\alpha(H)eV||\alpha^{-1}({}^gU)h\alpha^{-1}({}^eV)|}$$

$${}_H\mathrm{ind}_{\alpha^{g,h,e}_{H,U,V}}\left({}_{\alpha^{g,h,e}_{H,U,V}}\mathrm{res}_U^{c(g^{-1})\alpha_g}(s)\cdot{}_{\alpha^{g,h,e}_{H,U,V}}\mathrm{res}_V^{c(e^{-1})\alpha_e c(h^{-1})}(t)\right)$$

$$= \sum_{\substack{g,e\in G\\ h\in H}} \frac{|\alpha^{-1}({}^gU)\cap {}^h\alpha^{-1}({}^eV)|}{|\alpha(H)|^2|U||V|}$$

$${}_H\mathrm{ind}_{\alpha^{g,h,e}_{H,U,V}}\left({}_{\alpha^{g,h,e}_{H,U,V}}\mathrm{res}_U^{c(g^{-1})\alpha_g}(s)\cdot{}_{\alpha^{g,h,e}_{H,U,V}}\mathrm{res}_V^{c(e^{-1})\alpha_e c(h^{-1})}(t)\right)$$

$$= \sum_{g,e\in G} \frac{|\alpha^{-1}({}^gU)\cap \alpha^{-1}({}^eV)|}{|\alpha(H)||U||V|}{}_H\mathrm{ind}_{\alpha^{g,e}_{U,V}}\left({}_{\alpha^{g,e}_{U,V}}\mathrm{res}_U^{c(g^{-1})\alpha_g}(s)\cdot{}_{\alpha^{g,e}_{U,V}}\mathrm{res}_V^{c(e^{-1})\alpha_e c(-1)}(t)\right)$$

where $\alpha^{g,e}_{U,V} = \alpha^{-1}({}^gU)\cap\alpha^{-1}({}^eV)$. On the other hand,

$$\mathcal{C} = \sum_{\substack{UfV\subseteq G\\ \alpha(H)kU\cap^f V\subseteq G}} {}_H\mathrm{ind}_{\alpha^{-1}({}^k(U\cap^fV))}\mathrm{res}_{U\cap^fV}^{c(k^{-1})\alpha_k}\left({}_{U\cap^fV}\mathrm{res}_U(s)\cdot{}_{U\cap^fV}\mathrm{res}_V^{c(f^{-1})}(t)\right)$$

$$= \sum_{f,k\in G} \frac{1}{|UfV|}\frac{1}{|\alpha(H)k(U\cap{}^fV)|}$$

$${}_H\mathrm{ind}_{\alpha^{k,f}_{U,V}}\left({}_{\alpha^{k,f}_{U,V}}\mathrm{res}_U^{c(k^{-1})\alpha_k}(s)\cdot{}_{\alpha^{k,f}_{U,V}}\mathrm{res}_V^{c(f^{-1})c(k^{-1})\alpha_k}(t)\right)$$

where $\alpha^{k,f}_{U,V} = \alpha^{-1}({}^k(U\cap^f V))$

$$= \sum_{f,k\in G}\frac{|\alpha(H)\cap{}^kU\cap{}^{kf}V|}{|\alpha(H)||U||V|}{}_H\mathrm{ind}_{\alpha^{k,f}_{U,V}}\left({}_{\alpha^{k,f}_{U,V}}\mathrm{res}_U^{c(k^{-1})\alpha_k}(s)\cdot{}_{\alpha^{k,f}_{U,V}}\mathrm{res}_V^{c(f^{-1})c(k^{-1})\alpha_k}(t)\right)$$

Thus, $\mathcal{B}=\mathcal{C}$. Now we show that given $H\leq G$,

$${}_G\mathrm{ind}_H({}_H\mathrm{res}_G(x)\cdot y) = x\cdot{}_G\mathrm{ind}_H(y),$$

for all $x\in A_+(G)$ and $y\in A_+(H)$. Without loss of generality we may assume that $x={}_G\mathrm{ind}_U(u)$ for some $U\leq G$ and $y={}_H\mathrm{ind}_V(v)$ for some $V\leq H$. Now let $\mathcal{F}={}_G\mathrm{ind}_H({}_H\mathrm{res}_G(x)\cdot y)$ and $\mathcal{K}=x\cdot{}_G\mathrm{ind}_H$. Then

$$\mathcal{F} = \sum_{\substack{HgU\subseteq G\\ (H\cap^gU)hV\subseteq H}} {}_G\mathrm{ind}_{U\cap^{g^{-1}h}V}\left({}_{U\cap^{g^{-1}h}V}\mathrm{res}_U(u)\cdot{}_{U\cap^{g^{-1}h}V}\mathrm{res}_{g^{-1}hV}con_U^{g^{-1}h}\right)$$

$$= \sum_{\substack{g \in G \\ h \in H}} \frac{1}{|HgU|} \frac{1}{|(H \cap {}^gU)hV|} {}_G\mathrm{ind}_{U \cap {}^{g^{-1}h}V} ({}_{U \cap {}^{g^{-1}h}V} \mathrm{res}_U(u) \cdot {}_{U \cap {}^{g^{-1}h}V} \mathrm{res}_{{}^{g^{-1}h}V} con_U^{g^{-1}h})$$

$$= \sum_{\substack{g \in G \\ h \in H}} \frac{|{}^{hg}U \cap V|}{|H||U||V|} {}_G\mathrm{ind}_{U \cap {}^{g^{-1}h}V} ({}_{U \cap {}^{g^{-1}h}V} \mathrm{res}_U(u) \cdot {}_{U \cap {}^{g^{-1}h}V} \mathrm{res}_{{}^{g^{-1}h}V} con_U^{g^{-1}h})$$

$$= \sum_{g \in G} \frac{|{}^gU \cap V|}{|U||V|} {}_G\mathrm{ind}_{U \cap {}^{g^{-1}h}V} ({}_{U \cap {}^{g^{-1}h}V} \mathrm{res}_U(u) \cdot {}_{U \cap {}^{g^{-1}h}V} \mathrm{res}_{{}^{g^{-1}h}V} con_U^{g^{-1}h})$$

On the other hand we have,

$$\begin{aligned} \mathcal{K} &= \sum_{UfV \subseteq G} {}_G\mathrm{ind}_{U \cap {}^fV} ({}_{U \cap {}^fV} \mathrm{res}_U(u) \cdot_{U \cap {}^fV} \mathrm{res}_{{}^fV} \mathrm{con}\,{}^f_V(v)) \\ &= \sum_{f \in G} \frac{1}{|UfV|} {}_G\mathrm{ind}_{U \cap {}^fV} ({}_{U \cap {}^fV} \mathrm{res}_U(u) \cdot_{U \cap {}^fV} \mathrm{res}_{{}^fV} \mathrm{con}\,{}^f_V(v)) \\ &= \sum_{f \in G} \frac{|{}^fU \cap V|}{|U||V|} {}_G\mathrm{ind}_{U \cap {}^fV} ({}_{U \cap {}^fV} \mathrm{res}_U(u) \cdot_{U \cap {}^fV} \mathrm{res}_{{}^fV} \mathrm{con}\,{}^f_V(v)). \end{aligned}$$

Thus, $\mathcal{F} = \mathcal{K}$. □

4.5 Upper plus functor

In this section, we will introduce the lower plus functor $\mathrm{Lplus}\,(A)$. We show that $\mathrm{Lplus}\,(A)$ coincides with the classical fixed module construction $A^+(G)$ of Bolje [1].

The upper plus functor is defined as

$$\mathrm{Uplus} = {}_{R\mathcal{N}}\,\mathrm{Uplus}\,{}_{R\mathcal{E}} = {}_{R\mathcal{N}}\,\mathrm{Coind}\,{}_{R\mathcal{W}}\,\mathrm{Inf}_{R\mathcal{S}}\,\mathrm{Res}_{R\mathcal{E}} : R\mathcal{E}\text{-Mod} \to R\mathcal{N}\text{-Mod}.$$

Let $G \in \mathfrak{K}$. Given $R\mathcal{E}$-module A, we have

$$\mathrm{Uplus}\,(A) = {}_{R\mathcal{N}}\,\mathrm{Coind}\,{}_{R\mathcal{W}}\,\mathrm{Inf}_{R\mathcal{S}}\,\mathrm{Res}_{R\mathcal{E}}(A) = \mathrm{Hom}\,{}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{N}_{R\mathcal{N}}, {}_{R\mathcal{W}}A).$$

As before, $\bigoplus_{U \leq G} A(U)$ is an RG-module. Now consider the G-fixed R-submodule

$$A^+(G) = \big(\bigoplus_{U \leq G} A(U)\big)^G.$$

We can make the identification

$$A^+(G) = \bigoplus_{U \leq_G G} A(U)^{N_G(U)}.$$

Thus, any element $\xi \in A_+(G)$ can be written in the form

$$\xi = \sum_{U \leq_G G} [U, \xi^U]^G$$

where $\xi^U \in A(U)$ with image $[U, \xi^U] \in A(U)^{N_G(U)} \leq A^+(G)$.

Proposition 4.5.1. *Let C be an $R\mathcal{S}$-module. Then there is an R-linear isomorphism $t_G : {}_{R\mathcal{N}} \operatorname{Uplus}_{R\mathcal{S}}(A)(G) \to A^+(G)$ such that, given $x \in \operatorname{Uplus}(A)(G)$, then*

$$t_G(x) = \sum_{U \leq_G G} [U, x({}_U\mathrm{res}_G)]^G.$$

Proof. We start by showing that $[U, x({}_U\mathrm{res}_G)]^G$ makes sense. Let $g \in N_G(U)$. Then

$${}_U\mathrm{con}^g_U\big(\xi({}_U\mathrm{res}_G)\big) = \xi({}_U\mathrm{con}^g_U\mathrm{res}_G) = \xi({}_U\mathrm{res}_G).$$

Thus, $\xi({}_U\mathrm{res}_G) \in C(U)^{N_G(U)}$ so the specified expression makes sense. Then

$$\operatorname{Uplus}(C)(G) = \mathrm{iso}_G \cdot \operatorname{Hom}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{N}_{R\mathcal{N}}, {}_{R\mathcal{W}}C) = \operatorname{Hom}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{N} \cdot \mathrm{iso}_G, {}_{R\mathcal{W}}C).$$

Dualizing an equality demonstrated in the proof of Proposition 4.4.1, we have

$$R\mathcal{N} \cdot \mathrm{iso}_G = \bigoplus_{U \leq_G G} R\mathcal{W} \cdot {}_U\mathrm{res}_G.$$

The bijectivity of the R-linear map t_G is now clear. □

In analogy with the treatment of $\operatorname{Lplus}(A)$ and A_+ in Section 4.3, we combine the isomorphisms t_G to obtain an R-linear isomorphism

$$t : \operatorname{Uplus}(C) \to C^+$$

where $C^+ = \bigoplus_{G \in \mathfrak{K}} C^+(G)$. We make C^+ become an $R\mathcal{N}$-module in such a way that t is an $R\mathcal{N}$-module isomorphism. Let us introduce elements

$$\xi \in C^+(G), \qquad \eta \in C^+(H), \qquad \zeta \in C^+(G/N)$$

to be conventionally written as

$$\xi = \sum_{U \leq G} [U, \xi^G]^G, \qquad \eta = \sum_{V \leq_H H} [V, \eta^V]^H, \qquad \zeta = \sum_{W/N \leq_{G/N} G/N} [W/N, \zeta^{W/N}]^{G/N}$$

where, noting that $N_G(W/N)/N = N_{G/N}(W/N)$, we have

$$\xi^U \in C(U)^{N_G(U)}, \qquad \eta^V \in C(V)^{N_H(U)}, \qquad \zeta_{W/N} \in C(W/N)^{N_G(W)/N}.$$

Proposition 4.5.2. *Let C be an $R\mathcal{S}$-module. Consider elements $\xi \in C^+(G)$ and $\eta \in C^+(G)$ as above. Then;*

(1) *If $\xi \in {}_G\mathrm{ind}_H(\eta)$, then $\eta^U = \sum_{gH \subseteq G \; U \leq {}^gH} {}_U\mathrm{con}^g_{U^g}(\eta^{U^g})$.*

(2) *If $\eta = {}_H\mathrm{res}_G(\xi)$, then each $\eta^V = \xi^V$.*

(3) *Given a group isomorphism $\phi : G \to G'$ such that ${}_{G'}\mathrm{iso}^\phi_G$ is in $\mathcal{N}$, and writing $U' = \phi(U)$ and $\zeta' = {}_{G'}\mathrm{iso}^\phi_G(\zeta)$, then $\zeta'^{U'} = \zeta^U$. In particular, given $g \in G$, writing $H' = {}^gH$ and $V' = {}^gV$ and $\eta^p rime = {}_{H'}\mathrm{con}^g_H$, then $\eta'^{V'} = \eta^V$.*

(4) *Consider an element $\zeta \in C^+(G/N)$ as above. If $\xi = {}_G\mathrm{inf}_{G/N}(\zeta)$ then*

$$\xi^U = {}_U\mathrm{inf}_{U/(U \cap N)}\mathrm{iso}_{UN/N}(\xi^{UN/N}).$$

Proof. Via t, we make an identification $\mathrm{Uplus}\,(C) = C^+$. Thus $\zeta^U = \zeta({}_U\mathrm{res}_G)$. The assumption $\zeta = {}_G\mathrm{ind}_H(\eta)$ implies that

$$\zeta^U = \big({}_G\mathrm{ind}_H(\eta)\big)({}_U\mathrm{res}_G) = \eta({}_U\mathrm{res}_G\mathrm{ind}_H) = \sum_{UgH \subseteq G} \eta({}_U\mathrm{ind}_{U \cap {}^gH}\mathrm{con}^g_{U^g \cap H}\mathrm{res}_H)$$

$$= \sum_{UgH \subseteq G} {}_U\mathrm{ind}_{U \cap {}^gH}\big(\eta({}_{U \cap {}^gH}\mathrm{con}^g_{U^g \cap H}\mathrm{res}_H)\big).$$

The term $\eta({}_{U \cap {}^gH}\mathrm{con}^g_{U^g \cap H}\mathrm{res}_H)$ belongs to the $R\mathcal{W}$-module C, which is annihilated by all proper inductions. So ${}_U\mathrm{ind}_{U \cap {}^gH}\big(\eta({}_{U \cap {}^gH}\mathrm{con}^g_{U^g \cap H}\mathrm{res}_H)\big) = 0$ unless $U \leq {}^gH$. In that case, $UgH = g \cdot g^{-1}UgH = gH$. Part (1) holds because

$$\zeta^U = \sum_{gH \subseteq G : U \leq {}^gH} \eta({}_U\mathrm{con}^g_{U^g}\mathrm{res}_H) = \sum_{gH \subseteq G : U \leq {}^gH} {}_U\mathrm{con}^g_{U^g}\big(\eta({}_{{}^gU}\mathrm{res}_H)\big).$$

Part (2) holds because, if $\eta = {}_H\mathrm{res}_G(\xi)$, then

$$\eta^V = \big({}_H\mathrm{res}_G(\xi)\big)({}_V\mathrm{res}_H) = \xi({}_V\mathrm{res}_H\mathrm{res}_G) = \xi^V.$$

Consider a group epimorphism $\pi : G \to F$, and suppose that the inflation ${}_G\mathrm{inf}_F^\pi$ is in $\mathcal{N}$. For the moment, assume that $\zeta \in C^+(F)$. If $\xi = {}_G\mathrm{inf}_F^\pi(\eta)$, then

$$\xi^U = \big({}_G\mathrm{inf}_F^\pi(\zeta)\big)({}_U\mathrm{res}_G) = \zeta({}_U\mathrm{res}_G\mathrm{inf}_F^\pi).$$

Replacing π with ϕ, part (3) follows easily. Now replace $\pi : G \to F$ with the canonical epimorphism $G \to G/N$. Using the commutation relation for inflation and restriction,

$$\xi^U = \zeta({}_U\mathrm{inf}_{U/(U\cap N)}\mathrm{iso}_{UN/N}\mathrm{res}_{G/N}).$$

Noting that ζ commutes with inflation and isogation, part (4) now follows. $\square$

4.6 Canonical induction formula

In this section, we will define linearization map, mark map, tom Dieck map and canonical induction map.

4.6.1 Linearization map

Given an $R\mathcal{N}$-module M and $R\mathcal{E}$-module A, by Theorem 2.1.1 we have the following adjunction :

$$\mathrm{Hom}_{R\mathcal{N}}({}_{R\mathcal{N}}\,\mathrm{Lplus}_{R\mathcal{E}}(A), M) \cong \mathrm{Hom}_{R\mathcal{E}}(A, {}_{R\mathcal{E}}\,\mathrm{Res}_{R\mathcal{N}}(M)).$$

We define the **linearization map** associated with $\nu \in \mathrm{Hom}_{R\mathcal{E}}(A, {}_{R\mathcal{E}}\,\mathrm{Res}_{R\mathcal{N}}(M))$ to be the $\mathcal{RN}$-homomorphism

$$\mathrm{lin}^\nu \in \mathrm{Hom}_{R\mathcal{N}}({}_{R\mathcal{N}}\,\mathrm{Lplus}_{R\mathcal{E}}(A), M).$$

Proposition 4.6.1. *Let A be an $R\mathcal{E}$-module, M be an $R\mathcal{N}$-module, $\nu : A \to {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M)$ be an $R\mathcal{N}$-homomorphism and $l : {}_{R\mathcal{N}}\operatorname{Lplus}_{R\mathcal{E}}(A) \to M$. Then the following are equivalent:*

(a) *We have $l = lin^{\nu}$,*

(b) *We have $l(n \otimes a) = n\nu(a)$ for all $n \in R\mathcal{N}$ and $a \in A$.*

Proof. Proof is obvious by the definition of linearization map. □

We obtain the following corollary.

Corollary 4.6.2. *Let A be an $R\mathcal{E}$-module A and M be an $R\mathcal{N}$-module. Given an $R\mathcal{E}$-homomorphism $\nu : A \to {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M)$, then the associated morphism $\operatorname{lin}^n u : A_+ \to M$ is given by the equality*

$$\operatorname{lin}_G(x) = \sum_{U \in_G S(G)} {}_G\operatorname{ind}_U\big(\nu_U(x_U)\big)$$

where $x \in A_+(G)$ and $x = \sum_{U \in_G S(G)} [U, x_U]_G$

If we take $(\mathcal{N}, \mathcal{E}, \mathcal{W}, \mathcal{S})$ to be the classical quadruple, and take $\mathfrak{K}$ as the set of subgroups of a fixed finite group G such that S is the group category generated by the conjugation maps ${}_{H^g}\operatorname{con}^g_H$ with $g \in G \geq H$. Suppose also that ν is an embedding $A \hookrightarrow \operatorname{Res}(M)$. This is the case considered in [20]. The linearization homomorphism lin is denoted by $b^{M,A}$ in [[20], 3.1].

4.6.2 Mark map

Let A and C be $R\mathcal{E}$-modules. Applying Remark 4.2.4, we make the identification

$${}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A) = {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A).$$

Then the adjunction between induction and restriction in Theorem 2.1.1 becomes a $\mathbb{Z}$-linear isomorphism

$$\Theta^{C,A} : \operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C), {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A)\big)$$

$$\to \operatorname{Hom}_{R\mathcal{W}}\big({}_{R\mathcal{W}}\operatorname{Ind}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C), {}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A)\big).$$

Now by applying the Remark 4.2.5 we make the identification

$${}_{R\mathcal{W}}\operatorname{Ind}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C) = {}_{R\mathcal{W}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}}(C).$$

Then the adjunction between restriction and coinduction in Theorem 2.1.1 becomes a $\mathbb{Z}$-linear isomorphism

$$\Psi^{C,A} : \operatorname{Hom}_{R\mathcal{W}}\big({}_{R\mathcal{W}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}}(C), {}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A)\big)$$

$$\to \operatorname{Hom}_{R\mathcal{N}}\big({}_{R\mathcal{N}}\operatorname{Lplus}_{R\mathcal{E}}(C), {}_{R\mathcal{N}}\operatorname{Uplus}_{R\mathcal{E}}(A)\big).$$

We define the mark map for A with respect to $(\mathcal{S}, \mathcal{W}, \mathcal{E}, \mathcal{N})$ to be the $R\mathcal{N}$-module map

$$\operatorname{mk}^{A} = \Psi^{A,A}\Theta^{A,A}(\operatorname{id}_A) : \operatorname{Lplus}(A) \to \operatorname{Uplus}(A)$$

where id_A to denotes the identity endomorphism of ${}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A)$. Now to express the mark map more explictly, we consider five mutually corresponding maps $\phi \leftrightarrow \phi' \leftrightarrow \theta' \leftrightarrow \theta \leftrightarrow \psi$ where

$$\begin{aligned}
\phi &\in \operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A), {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C)\big),\\
\phi' &\in \operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A), {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C)\big),\\
\theta' &\in \operatorname{Hom}_{R\mathcal{W}}\big({}_{R\mathcal{W}}\operatorname{Ind}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A), {}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C)\big),\\
\theta &\in \operatorname{Hom}_{R\mathcal{W}}\big({}_{R\mathcal{W}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}}(A), {}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C)\big),\\
\psi &\in \operatorname{Hom}_{R\mathcal{N}}\big({}_{R\mathcal{N}}\operatorname{Ind}_{R\mathcal{E}}(A), {}_{R\mathcal{N}}\operatorname{Coind}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(C)\big)\\
&= \operatorname{Hom}_{R\mathcal{N}}\big(\operatorname{Lplus}(A), \operatorname{Uplus}(C)\big).
\end{aligned}$$

Now we examine the relationship between the map ϕ and $\psi = \Psi(\Theta(\phi))$ by using the latest two remarks and the formulas for the two adjunctions in Theorem 2.1.1.The correspondences $\phi \leftrightarrow \phi' \leftrightarrow \theta' \leftrightarrow \theta$ are characterized by the equalities

$$w\phi(a) = w\phi'(a) = \theta'(w \otimes_{\mathcal{S}} a) = \theta(w \otimes_{\mathcal{E}} a)$$

for any $w \in \mathcal{W}$ and $a \in C$. The correspondence $\theta \leftrightarrow \psi$ is characterized by

$$w\phi(nw \otimes_{R\mathcal{E}} a) = \psi(w \otimes_{R\mathcal{E}} a)(n)$$

where $n \in \mathcal{N}$. Thus, we obtain the following result.

Proposition 4.6.3. *Let A and C be $R\mathcal{E}$-modules. Then the $\mathbb{Z}$-module isomorphism*

$$\Theta^{C,A} : \mathrm{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{S}}\mathrm{Res}_{R\mathcal{E}}(C), {}_{R\mathcal{S}}\mathrm{Res}_{R\mathcal{E}}(A)\big) \to \mathrm{Hom}_{R\mathcal{N}}\big(\mathrm{Lplus}\,(C), u'(A)\big)$$

is such that an $R\mathcal{S}$-module map $\phi : C \to A$ and an $R\mathcal{N}$-module map ψ : $\mathrm{Lplus}\,(C) \to \mathrm{Uplus}\,(A)$, satisfy $\psi = \Theta^{C,A}(\varphi)$ if and only if

$$w\phi(a) = \theta(w \otimes_{R\mathcal{E}} a), \quad \theta(nw \otimes_{R\mathcal{E}} c) = \psi(w \otimes_{R\mathcal{E}} c)(n)$$

for some $R\mathcal{W}$-module map $\theta : \mathrm{Lplus}\,(C) \to {}_{R\mathcal{W}}\mathrm{Inf}_{R\mathcal{S}}\mathrm{Res}_{R\mathcal{E}}(A)$, where $c \in C$ and $w \in \mathcal{W}$ and $n \in \mathcal{N}$. In particular, now with $a \in A$, we have

$$mk^A(w \otimes_{R\mathcal{E}} a)(n) = \theta(nw \otimes_{R\mathcal{E}} a)$$

where $\theta(w \otimes_{R\mathcal{E}} a) = wa$. Note that $wa = 0$ when $w \in \Delta_{\mathcal{W}}$.

Now we will describe the mark map more explicitly. Let $G \in \mathfrak{K}$. We define an R-linear map

$$\mathrm{br}_G^A : A_+(G) \to A(G)$$

called the **Brauer map** on $A_+(G)$, given by

$$\mathrm{br}_G^A\Big(\sum_{U \leq_G G} [U, x_U]\Big) = x_G.$$

Note that the definition of the map br_G^A makes sense because G acts trivially on the R-submodule $A(G)$ of $\bigoplus_{U \leq G} A(U)$ and $[G, x_G]_G$ can be interpreted as an element of the R-submodule $A(G) = A(G)_G$ of the R-module $A_+(G) = \big(\bigoplus_{U \leq G} A(U)\big)_G$.

Lemma 4.6.4. *the R-linear maps $br_G^A : A_+(G) \to A(G)$ combine to form an $R\mathcal{S}$-module map*

$$br^A : {}_{R\mathcal{S}}\mathrm{Res}_{R\mathcal{N}}(A_+) \to {}_{R\mathcal{S}}\mathrm{Res}_{R\mathcal{E}}(A).$$

Proof. Note that Proposition 4.4.2 says that br^A commutes with isogations and internal inflations. With the aid of a suitable commutation formula for an internal induction and an arbitrary inflation, the proof of that proposition can easily be adapted to show that br^A commutes with arbitrary inflations. $\square$

Proposition 4.6.5. *Let $x \in A_+(G)$. Put $\xi = mk_G^A(x)$. Write $x = \sum_{U' \leq_G G} [U, x_U]_G$ and $\xi = \sum_{U' \leq_G G} [U', \xi^{U'}]^G$. Then*

$$\xi^{U'} = br_{U'}^A\big({}_U\mathrm{res}_G(x)\big) = \sum_{U \leq_G G,\, gU \subseteq G : U' \leq {}^gU} {}_{U'}\,\mathrm{res}_{{}^gU}\,\mathrm{con}_U^g(x_U) = \sum_{U : U' \leq U \leq G} \frac{|N_G(U)|}{|U|}{}_{U'}\,\mathrm{res}_U(x_U).$$

Proof. We may assume that $x = [U, x_U]_G$. Then

$$\xi^{U'} = \mathrm{mk}_G^A(x)({}_{U'}\mathrm{res}_G) = \mathrm{mk}_G^A({}_G\mathrm{ind}_H \otimes_{R\mathcal{E}} x_U)({}_{U'}\mathrm{res}_G).$$

We apply the correspondance $\phi \leftrightarrow \theta \leftrightarrow \psi$ in Proposition 4.6.3. We set $\phi = \mathrm{id}_A$ and $\psi = \mathrm{mk}^A$. Then we obtain

$$\xi^{U'} = \sum_{U'gU \subseteq G} {}_{U'}\mathrm{ind}_{U' \cap {}^gU}\Big(\mathrm{con}^g\big({}_{U'^g \cap U}\mathrm{res}_U(x_U)\big)\Big).$$

Now we fix the element g. Then $\mathrm{con}^g\big({}_{U'^g \cap U}\mathrm{res}_U(x_U)\big)$ can be interpreted as an element of $({}_{R\mathcal{W}}A)(U)$. Then since the $R\mathcal{W}$-module ${}_{R\mathcal{W}}A = {}_{\mathcal{W}}\mathrm{Inf}_{\mathcal{S}}\,\mathrm{Res}_{\mathcal{E}}(A)$ is annihilated by inductions, the term of the sum indexed by g vanishes unless $U' \leq {}^gU$. Thus,

$$\xi^{U'} = \sum_{U'gU \subseteq G \,:\, U' \leq {}^gU} \mathrm{con}^g\big({}_{U'^g \cap U}\mathrm{res}_U(x_U)\big).$$

Since $U' \leq {}^gU$ we have $U'gU = gU$. We let ξ to be arbitrary. Using the commutation relation for conjugation and restriction, we get

$$\xi^{U'} = \sum_{U \leq_G G,\, gU \subseteq G \,:\, U' \leq {}^gU} {}_{U'^g \cap U}\mathrm{res}_U\big(\mathrm{con}^g(x_U)\big).$$

Now we change the indexing so that g runs over coset representatives $gN_H(U) \subseteq G$ instead of $gU \subseteq G$, then ${}^gU$ runs over all the subgroups of G without repetitions. So

$$\xi^{U'} = \sum_{U \leq G \,:\, U' \leq U} \frac{|N_G(U)|}{|U|}{}_{U'}\mathrm{res}_U(x_U).$$

Now by Proposition 4.4.2 we obtain

$$\xi^{U'} = \mathrm{br}^A_{U'}\big(\sum_{U'gU\subseteq G}[U'\cap {}^gU, \mathrm{con}^g\big({}_{U'^g\cap U}\mathrm{res}_U(x_U)\big)]_{U'}\big) = \mathrm{br}^A_{U'}\big({}_U\mathrm{res}_G(x)\big).$$

□

In the previous proposition we show that if we take $\mathfrak{K}$ as the set of subgroups of a fixed finite group G, the mark map and Brauer map br^A_H for $H \leq G$ coincides with with the maps ρ^G_H and π^G_H in [[20], Section 2.3]. The next proposition reduces to [[20], Proposition 2.4].

Proposition 4.6.6. *Let $G \in \mathfrak{K}$ and suppose that $|G|$ is invertible in R. Then the $R\mathcal{N}$-module map mk^A_G is an isomorphism. Given $\xi \in A_+(G)$ and writing $\xi = \sum_{U\leq_{U'}G}[U', \xi^{U'}]^G$ then, summing over pairs (U'', U') of subgroups $U'' \leq G \geq U'$, we have*

$$(\mathrm{mk}^A_G)^{-1}(\xi) = \frac{1}{|G|}\sum_{U'',U'\leq G}|U''|\mathrm{m\ddot{o}b}(U'', U')[U'', {}_{U''}\mathrm{res}_{U'}(\xi^{U'})]^G.$$

Putting $x = (mk^A_G)^{-1}(\xi)$ and writing $x = \sum_{U\leq_G G}[U, x_U]G$, then

$$x_U = \frac{1}{|G|}\sum_{U\leq_G G,\, N_G(U)g\subseteq G}|U|\mathrm{m\ddot{o}b}(U^g, U')_{U^g}\,\mathrm{con}^g_U\,\mathrm{res}_{U'}(\xi^{U'}).$$

Proof. We will define an R-linear map $\mathrm{km}^A_G : A^+(G) \to A_+(G)$ by

$$\mathrm{km}^A_G(\xi) = \sum_{U'', U'\leq G}|U''|\mathrm{m\ddot{o}b}(U'', U')[U'', {}_{U''}\mathrm{res}_{U'}(\xi^{U'})]^G.$$

We set $\widetilde{x} = \mathrm{km}^A_G(\xi)$ and write $\widetilde{x} = \sum_{\widetilde{U'}\leq_G G}[\widetilde{U'}, \widetilde{x}_{\widetilde{U'}}]^G$. Then

$$\widetilde{x}_{\widetilde{U}} = \sum_{U'\leq G,\, N_G(\widetilde{U})g\subseteq G}|\widetilde{U}|\mathrm{m\ddot{o}b}(\widetilde{U}^g, U')_{\widetilde{U}}\mathrm{con}^g{}_{\widetilde{U}^g}\mathrm{res}_{U'}(\xi^{U'}).$$

Assume that $\xi = \mathrm{mk}^A_G(x)$. Since $\mathrm{m\ddot{o}b}(\widetilde{U}^g, U') = 0$ unless $\widetilde{U}^g \leq U'$, we have

$$\widetilde{x}_{\widetilde{U}} = \sum_{U',U\leq G,\, N_G(U)g\subseteq G,:\widetilde{U}^g\leq U'\leq U}|\widetilde{U}|\mathrm{m\ddot{o}b}(\widetilde{U}^g, U')\frac{|N_G(U)|}{|U|}{}_{\widetilde{U}}\mathrm{con}^g{}_{\widetilde{U}^g}\mathrm{res}_U(x_U).$$

But $\sum_{U'} \text{möb}(\widetilde{U}^g, U') = \lfloor \widetilde{U}^g = U' \rfloor$, summed over $\widetilde{U}^g \le U' \le U$, hence

$$\widetilde{x}_{\widetilde{U}} = |N_G(\widetilde{U})| \sum_{N_G(U)g \subseteq G} {}_{\widetilde{U}}\text{con}^g{}_{\widetilde{U}^g}(x_{\widetilde{U}^g}) = |G| x_{\widetilde{U}}.$$

Thus, $\text{km}_G^A\big(\text{mk}_G^A(x)\big) = \widetilde{x} = |G| x_{\widetilde{U}}$.

For arbitrary ξ, and letting $\widetilde{x} = \text{km}_G^A(\xi)$, put $\widetilde{\xi} = \text{mk}_G^A(\widetilde{x})$ and write

$$\widetilde{\xi} = \sum_{U' \le_G G} [\widetilde{U}^g, \widetilde{\xi}^{\widetilde{U}'}]^G.$$

Using Proposition 4.6.5 and formula above for $\widetilde{x}_{\widetilde{U}}$, we have

$$\begin{aligned}
\widetilde{\xi}^{\widetilde{U}'} &= \sum_{\widetilde{U} \,:\, \widetilde{U}' \le \widetilde{U} \le G} \frac{|N_G(\widetilde{U})|}{|\widetilde{U}|} {}_{\widetilde{U}'}\text{res}_{\widetilde{U}}(\widetilde{x}_{\widetilde{U}}) \\
&= \sum_{\widetilde{U}, U' \le G, N_G(\widetilde{U})g \subseteq G \,:\, \widetilde{U}' \le \widetilde{U}} |N_G(\widetilde{U})| \text{möb}(\widetilde{U}^g, U')_{\widetilde{U}^g}\text{res}_{\widetilde{U}}\text{con}^g_{\widetilde{U}^g}\text{res}_{U'}(\xi^{U'}) \\
&= \sum_{\widetilde{U}, U' \le G, N_G(\widetilde{U})g \subseteq G \,:\, \widetilde{U}'^g \le \widetilde{U}^g \le U'} |N_G(\widetilde{U})| \text{möb}(\widetilde{U}^g, U')_{\widetilde{U}'}\text{con}^g_{\widetilde{U}'^g}\text{res}_{U'}(\xi^{U'}).
\end{aligned}$$

Fixing g and U', then the sum over $\widetilde{U}$ vanishes except when $\widetilde{U}'^g = U'$. So

$$\widetilde{\xi}^{\widetilde{U}'} = |N_G(\widetilde{U}')| \sum_{N_G(\widetilde{U})g \subseteq G} {}_{\widetilde{U}'}\text{con}^g_{\widetilde{U}'^g}(\xi^{\widetilde{U}'^g}) = |G| \xi^{\widetilde{U}'}.$$

Therefore, $\text{mk}_G^A\big(\text{km}_G^A(\xi)\big) = \widetilde{\xi} = |G|\xi$. We have shown that mk_G^A and $\text{km}_G^A/|G|$ are mutual inverses. $\square$

Remark 4.6.7. *For each $U \le G$, let $a_U \in A(U)_{N_G(U)}$ and write $x = \sum_{U \le G} [U, a_U]_G$. Then, as an element of $a_U \in A(U)_{N_G(U)}$, we have*

$$x_U = \sum_{N_G(U)g \subseteq G} {}_U\text{con}^g_{U^g}(a_{U^g}).$$

In particular, $x = \sum_{U \le G} \frac{|N_G(U)|}{|G|}[U, x_U]_G$.

Using the latest remark and Proposition 4.6.6, we obtain the following formula for mk^{-1}.

Proposition 4.6.8. *Given $\xi \in A^+(G)$, letting $x = \operatorname{mk}_G^{-1}(\xi)$, then*

$$x_U = \frac{|U|}{|N_G(U)|} \sum_{U' \le G} \text{möb}(U, U')_U \text{res}_{U'}(\xi^{U'})$$

as an element of $A(U)_{N_G(U)}$, for all $U \le G$. Here, we understand the term in the sum to be zero when $U \not\le U'$.

4.6.3 Tom Dieck map

We consider an arbitrary $R\mathcal{S}$-homomorphism

$$\pi \in \operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M), A\big).$$

We want to define an $R\mathcal{E}$-homomorphism

$$\operatorname{die} = \operatorname{die}^{\pi} \in \operatorname{Hom}_{R\mathcal{E}}\big({}_{R\mathcal{E}}M, {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}u'(A)\big).$$

Let M be an $R\mathcal{E}$-module and A be an $R\mathcal{S}$-module. Applying Theorem 2.1.1, Remark 4.2.4 and Proposition 4.2.6 we obtain R-isomorphism

$$\operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(M), A\big) \cong \operatorname{Hom}_{R\mathcal{E}}\big(M, {}_{R\mathcal{E}}\operatorname{Coind}_{R\mathcal{S}}(A)\big)$$

$$\cong \operatorname{Hom}_{R\mathcal{E}}\big(M, {}_{R\mathcal{E}}\operatorname{Coind}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}(A)\big) \cong \operatorname{Hom}_{R\mathcal{E}}\big(M, {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Uplus}_{R\mathcal{S}}(A)\big).$$

Consider corresponding elements $\pi \leftrightarrow \operatorname{die}'' \leftrightarrow \operatorname{die}' \leftrightarrow \operatorname{die}$ where

$$\begin{aligned}
\operatorname{die}'' &\in \operatorname{Hom}\big(M, \operatorname{Coind}(A)\big) = \operatorname{Hom}_{R\mathcal{S}}\big(M, \operatorname{Hom}_{R\mathcal{S}}({}_{R\mathcal{S}}R\mathcal{E}_{R\mathcal{E}}, A)\big)\\
\operatorname{die}' &\in \operatorname{Hom}\big(M, \operatorname{Coind}\operatorname{Res}\operatorname{Inf}(A)\big)\\
&\qquad = \operatorname{Hom}_{R\mathcal{E}}\big(M, \operatorname{Hom}_{R\mathcal{S}}({}_{R\mathcal{S}}R\mathcal{E}_{R\mathcal{E}}, {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}(A))\big)\\
\operatorname{die} &\in \operatorname{Hom}\big(M, \operatorname{Res}\operatorname{Uplus}(A)\big) = \operatorname{Hom}_{R\mathcal{E}}\big(M, \operatorname{Hom}_{R\mathcal{W}}({}_{R\mathcal{W}}R\mathcal{N}_{R\mathcal{E}}, {}_{R\mathcal{W}}\operatorname{Inf}_{R\mathcal{S}}(A)\big).
\end{aligned}$$

Thus, given $\pi \in \operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M), A\big)$ there exists a map $\operatorname{die}^{\pi} \in \operatorname{Hom}_{R\mathcal{E}}\big({}_{R\mathcal{E}}M, {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Uplus}(A)\big)$ associated to π. The map die^{π} is called **the tom Dieck map** associated to π.

Proposition 4.6.9. *Let M be an $R\mathcal{E}$-module, A be an $R\mathcal{S}$-module, $\pi : {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M) \to A$ be an $R\mathcal{S}$-module map and $d : M \to {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}\operatorname{Uplus}_{R\mathcal{S}}(A)$ an $R\mathcal{E}$-module map. Then $d = die^{\pi}$ if and only if $\pi(em) = d(m)(e)$ for all $m \in RM$ and $e \in R\mathcal{E}$.*

Proof. With the notation above, the correspondences

$$\pi \leftrightarrow \operatorname{die}'' \leftrightarrow \operatorname{die}' \leftrightarrow \operatorname{die}^{\pi}$$

are characterized by the equalities

$$\pi(em) = \operatorname{die}''(m)(e) = \operatorname{die}'(m)(e) = \operatorname{die}(m)(e).$$

□

Applying Proposition 4.5.1, we can make an identification

$$\operatorname{Hom}_{R\mathcal{W}}(R\mathcal{N}, {}_{R\mathcal{W}}A) = {}_{R\mathcal{N}}\operatorname{Uplus}_{R\mathcal{S}}(A) = A^{+}$$

by writing $\xi = \sum_{U \leq_G G} \big[U, \xi({}_U\mathrm{res}_G)\big]^G$, where ξ is an element of the coordinate module

$$\operatorname{Hom}_{R\mathcal{W}}(R\mathcal{N} \cdot \mathrm{iso}_G, {}_{R\mathcal{W}}A) = A^{+}(G) = \bigoplus_{U \leq_G G} A(U)^{N_G(U)}.$$

Corollary 4.6.10. *Given an $R\mathcal{E}$-module A, an $R\mathcal{S}$-module M and an $R\mathcal{S}$-homomorphism $\pi : {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M) \to A$, then the tom Dieck homomorphism $\operatorname{die}^{\pi} : M \to {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(A^{+})$ associated with π is given by the condition that, for $m \in M(G)$, we have*

$$\operatorname{die}^{\pi}(m) = \sum_{U \in_G G} [U, \pi_U({}_U\,\mathrm{res}_G(m))]^G.$$

Proof. By the latest proposition, $\operatorname{die}_G(m)({}_U\mathrm{res}_G) = \pi_U\big({}_U\mathrm{res}_G(m)\big)$. □

4.6.4 Canonical induction formula

Let M be an $R\mathcal{E}$-module and A be an $R\mathcal{E}$-module. Given

$$\nu \in \operatorname{Hom}_{R\mathcal{E}}\big(A, {}_{R\mathcal{E}}\operatorname{Res}_{R\mathcal{N}}(M)\big), \quad \pi \in \operatorname{Hom}_{R\mathcal{S}}\big({}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{N}}(M), {}_{R\mathcal{S}}\operatorname{Res}_{R\mathcal{E}}(A)\big),$$

We define an $R\mathcal{E}$-homomorphism $\operatorname{can}^{\pi} : M \to A_{+}$ called **the canonical induction homomorphism** associated with π, such that

$$\operatorname{can}^{\pi} = \operatorname{mk}^{-1} \circ die^{\pi}.$$

Here obviously the homomorphims $\mathrm{mk} : A_+ \to A^+$ and $\mathrm{mk}^{-1} : A^+ \to A_+$ are $R\mathcal{N}$-isomorphisms. We obtain the following commutative triangle of $R\mathcal{N}$-homomorphism.

$$\begin{tikzcd} A_+ \arrow[rr, "\mathrm{mk}"] & & A^+ \\ & M \arrow[ul, "\mathrm{can}"] \arrow[ur, "\mathrm{die}"'] & \end{tikzcd}$$

Now, we shall give an explicit formula for $\mathrm{can} : M \to A_+$ as a generalization of [20, 6.1a].

Proposition 4.6.11. *Given $m \in M(G)$, then*

$$\mathrm{can}_G(m) = \sum_{U,U' \leq G} |U| \mathrm{m\ddot{o}b}(U,U') \Big[U, {}_U \mathrm{res}_{U'}\big(\pi_{U'}({}_{U'}\mathrm{res}_G(m))\big)\Big]_G .$$

Proof. We set $\xi = \mathrm{die}_G(m)$ and $x = \mathrm{mk}_G^{-1}(\xi)$. By Corollary 4.6.10, $\xi^{U'} = \pi_{U'}\big({}_{U'}\mathrm{res}_G(m)\big)$. By Proposition 4.6.8,

$$x_U = \frac{|U|}{|N_G(U)|} \sum_{U' \leq G} \mathrm{m\ddot{o}b}(U,U')_U \mathrm{res}_{U'}\Big(\pi_{U'}\big({}_{U'}\mathrm{res}_G(m)\big)\Big).$$

The required equality now follows from Remark 4.6.7. □

The next result is an adaptation of [1, 6.4]. When ν is a monomorphism, the result follows immediately from the statement of [1, 6.4]. In general, the argument in [1, 6.4] carries though without any essential change.

Theorem 4.6.12. *Still assuming that the group orders are invertible in R, let M be an $R\mathcal{N}$-module. Then the following two conditions are equivalent:*

(a) *We have* $\mathrm{lin}^{\nu} \circ \mathrm{can}^{\pi} = \mathrm{id}_M$.

(b) *For all $G \in \mathfrak{K}$ and $m \in M(G)$, we have* $\nu_G(\pi_G(m)) - m \in \sum_{H \leq G} {}_G\mathrm{ind}_H\big(M(H)\big)$.

Proof. We set $x = \mathrm{can}_G(m)$, then by Corollary 4.6.2 we have

$$\mathrm{lin}_G\big(\mathrm{can}_G(m)\big) = \mathrm{lin}_G(x) = \sum_{U \leq_G G} {}_G\mathrm{ind}_U\big(\nu_U(x_U)\big).$$

Now by using the formula for x_U in the proof of the latest result we have

$$\operatorname{lin}_G\big(\operatorname{can}_G(m)\big)$$

$$= \sum_{U \leq_G G, U' \leq G} \frac{|U|}{|N_G(U)|} \text{möb}(U, U')_G \text{ind}_U(\nu_U({}_U\text{res}_{U'}(\pi_{U'}({}_{U'}\text{res}_G(m))))).$$

Since ν_U is an $R\mathcal{E}$-homomorphism,

$$\nu_U({}_U\text{res}_{U'}(\pi_{U'}({}_{U'}\text{res}_G(m)))) = {}_U\text{res}_{U'}(\nu_{U'}\pi_{U'}({}_{U'}\text{res}_G(m))).$$

Now by Proposition 4.3.4, $\Delta(e_G^G)\nu^G(\pi_G(m))$ annihilates $\mathcal{I}(M)(G)$, thus,

$$\Delta_G(e_G^G) \operatorname{lin}_G\big(\operatorname{can}_G(m)\big) = \Delta_G(e_G^G)\nu_G(\pi_G(m)).$$

Now suppose that (a) holds. Then the latest equality becomes

$$\Delta_G(e_G^G)m = \Delta_G(e_G^G)\nu_G(\pi_G(m)),$$

which is equivalent to condition (b).

Conversely, suppose that condition (b) holds. Assume that inductively the condition (a) holds for all strict subgroups $H < G$. Since lin_G and can_G both commute with restriction we have

$${}_H\text{res}_G\Big(\operatorname{lin}_G\big(\operatorname{can}_G(m)\big)\Big) = \operatorname{lin}_H\Big(\operatorname{can}_H\big({}_H\text{res}_G(m)\big)\Big) = {}_H\text{res}_G(m).$$

$\square$

When the equivalent conditions (1) and (2) of the theorem hold, we call $\operatorname{can}^{\pi}()$ a **canonical induction formula** for $\operatorname{lin}^{\nu}()$.

Remark 4.6.13. *With the notation of the latest theorem, if* $\operatorname{can}()$ *is a canonical induction formula for* $\operatorname{lin}()$*, then*

$$m = \sum_{U \leq_G G, U' \leq G} \frac{|U|}{|N_G(U)|} \text{möb}(U, U')_G \text{ind}_U({}_U\,\text{res}_{U'}(\nu_{U'}\pi_{U'}({}_{U'}\,\text{res}_G(m)))).$$

Chapter 5

Lefschetz invariant attached to monomial posets

Let G be a finite group and C be an abelian group. In this Chapter, we introduce the category of C-monomial G-posets which gives a new description of the C-monomial Burnside ring $B_C(G)$. We also introduce the category of C-monomial G-posets and define Lefschetz invariant attached to a C-monomial G-poset. Then we construct the generalized tensor induction.

5.1 Category of C-monomial G-sets

Given a G-set X, we will denote its transporter category by $\widehat{X}$. Objects of $\widehat{X}$ are the elements of X and given x, y in X the set of morphisms from x to y is

$$\mathrm{Hom}_{\widehat{X}}(x, y) = \{g \in G \mid gx = y\}.$$

Let $\bullet_C$ denote the category with one object where morphisms are the elements of C and composition is multiplication in C. Now we define C-monomial G-sets as follows.

Definition 5.1.1. *A* C-monomial G-set *is a pair* $(X, \mathfrak{l})$ *consisting of a finite* G*-set* X *and a functor* $\mathfrak{l} : \widehat{X} \to \bullet_C$*.*

Given $x, y \in X$ and $g \in G$ such that $gx = y$, we get an element $\mathfrak{l}(g, x, y)$ of C, such that $\mathfrak{l}(h, y, z)\mathfrak{l}(g, x, y) = \mathfrak{l}(hg, x, z)$ if $h \in G$ and $hy = z$, and $\mathfrak{l}(1, x, x) = 1$ for any $x \in X$.

Let $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$ be C-monomial G-sets. The pair $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$ is a map of C-monomial G-sets where $f : X \to Y$ is a map of G-sets and $\lambda : \mathfrak{l} \to \mathfrak{m} \circ f$ is a natural transformation. The category whose objects are C-monomial G-sets and morphisms are the maps of C-monomial G-sets will be denoted by ${}_C MG\textsf{-set}$.

Let $(X, \mathfrak{l})$ and $(X', \mathfrak{l}')$ be C-monomial G-sets. The disjoint union of C-monomial G-sets is defined as $(X, \mathfrak{l}) \sqcup (X', \mathfrak{l}') = (X \sqcup X', \mathfrak{l} \sqcup \mathfrak{l}')$ where $X \sqcup X'$ is the disjoint union of G-sets and

$$\mathfrak{l} \sqcup \mathfrak{l}' : \widehat{X \sqcup X'} \to \bullet_C$$

is the functor such that

$$(\mathfrak{l} \sqcup \mathfrak{l}')(g, z_1, z_2) = \begin{cases} \mathfrak{l}(g, z_1, z_2) & z_1,\ z_2 \in X \\ \mathfrak{l}'(g, z_1, z_2) & z_1, z_2 \in X' \end{cases}$$

for any $z_1,\ z_2 \in X \sqcup X'$ such that $gz_1 = z_2$ for some $g \in G$.

We define the product of C-monomial G-sets as $(X \times X', \mathfrak{l} \times \mathfrak{l}')$ where $X \times X'$ is the product of G-sets and $\mathfrak{l} \times \mathfrak{l}' : \widehat{X \times Y} \to \bullet_C$ is the functor defined by

$$(\mathfrak{l} \times \mathfrak{l}')\big(g, (x, x'), (y, y')\big) = \mathfrak{l}(g, x, y)\mathfrak{l}'(g, x', y')$$

for $g \in G$ and $(x, x'), (y, y') \in X \times X'$ such that $g(x, x') = (y, y')$.

We want to show that the categories ${}_C MG\textsf{-set}$ and ${}_C FG\textsf{-set}$ are equivalent. We define a functor $F : {}_C MG\textsf{-set} \to {}_C FG\textsf{-set}$ such that given a C-monomial G-set $(X, \mathfrak{l})$

$$F(X,\mathfrak{l}) = C \times_{\mathfrak{l}} X,$$

where $C \times X$ is the direct product endowed with the $(C \times G)$-action defined by $(k,g)(c,x) = \big(kc\mathfrak{l}(g,x,gx), gx\big)$ for any $(k,g) \in C \times G$ and $(c,x) \in C \times X$.

Let $(f,\lambda) : (X,\mathfrak{l}) \to (Y,\mathfrak{m})$ be a map of C-monomial G-sets. We define

$$F(f,\lambda) : C \times_{\mathfrak{l}} X \to C \times_{\mathfrak{m}} Y$$

by $F(f,\lambda)(c,x) = (c\lambda_x, f(x))$ for any $(c,x) \in C \times_{\mathfrak{l}} X$. Then $F(f,\lambda)$ is a $(C \times G)$-map: indeed, given $(k,g) \in C \times G$ and $(c,x) \in C \times X$, we have

$$\begin{aligned}(k,g)F(f,\lambda)(c,x) &= (k,g)\big(c\lambda_x, f(x)\big) = \big(kc\lambda_x\mathfrak{m}(g,f(x),f(gx)), f(gx)\big)\\ &= \big(kc\lambda_{gx}\mathfrak{l}(g,x,gx), f(gx)\big) = F(f,\lambda)\big(kc\mathfrak{l}(g,x,gx), gx\big)\\ &= F(f,\lambda)\big((k,g)(c,x)\big).\end{aligned}$$

Given a C-monomial G-set $(X,\mathfrak{l})$, we consider the map of C-monomial G-sets $(\mathrm{id}_X, 1_X) : (X,\mathfrak{l}) \to (X,\mathfrak{l})$ where $\mathrm{id}_X : X \to X$ is the identity map of G-sets and $1_X : \mathfrak{l} \to \mathfrak{l} \circ \mathrm{id}_X$ is the identity natural transformation. Then we obviously $F(\mathrm{id}_X, 1_X) : C \times_{\mathfrak{l}} X \to C \times_{\mathfrak{l}} X$ is the identity map of C-fibred G-posets. Now let $(f_1,\lambda_1) : (X_1,\mathfrak{l}_1) \to (X_2,\mathfrak{l}_2)$ and $(f_2,\lambda_2) : (X_2,\mathfrak{l}_2) \to (X_3,\mathfrak{l}_3)$ be maps of C-monomial G-sets. Then given $(c,x) \in C \times_{\mathfrak{l}_1} X_1$ we have

$$F\big((f_2,\mathfrak{l}_2) \circ (f_1,\mathfrak{l}_1)\big)(c,x) = F(f_2 \circ f_1, \mathfrak{l}_2\mathfrak{l}_1)(c,x) = (c(\lambda_2\lambda_1)_x, f_2(f_1(x)))$$

$$= (c\lambda_{2x}\lambda_{1x}, f_2(f_1(x))) = F(f_2,\lambda_2)(c\lambda_{1x}, f_1(x)) = F(f_2,\lambda_2) \circ F(f_1,\lambda_1)(c,x).$$

Thus, $F : {}_C MG\text{-}\mathsf{set} \to {}_C FG\text{-}\mathsf{set}$ is a functor.

Lemma 5.1.2. *Let C be an abelian group and G be a finite group. Then the above functor $F : {}_C MG\text{-}\mathsf{set} \to {}_C FG\text{-}\mathsf{set}$ is an equivalence of categories.*

Proof. We need to show that F is fully faithful and essentially surjective. Now we prove that F is essentially surjective. Let X be a C-fibred G-set and $C\backslash X$ denote the set of C-orbits. Obviuosly $C\backslash X$ is a G-set. Now let Cx, Cy be elements of $C\backslash X$ such that $Cgx = Cy$ for some $g \in G$. Then there exists a unique $c \in C$ such that $gx = cy$. We define a functor $\mathfrak{l} : \widehat{C\backslash X} \to \bullet_C$ by $\mathfrak{l}(g, Cx, Cy) = c$. Now we have $F(C\backslash X, \mathfrak{l}) = C \times_{\mathfrak{l}} (C\backslash X)$. We choose a set $[C\backslash X]$ of G-representatives of the G-action on $C\backslash X$. Given $x \in X$, then there exits a unique $C\sigma_x \in [C\backslash X]$ such that $x \in C\sigma_x$. Since X is C-free, there exists a unique $c_x \in C$ such that $x = c_x\sigma_x$. We define a $(C \times G)$-map $f : X \to C \times_{\mathfrak{l}} (C\backslash X)$ such that $f(x) = (c_x, C\sigma_x)$. Then

$$(c, g)f(x) = (c, g)(c_x, C\sigma_x) = (c_x c\mathfrak{l}(g, C\sigma_x, Cg\sigma_x), Cg\sigma_x) = (c_x c, Cg\sigma_x)$$

$$= (c_{cgx}, Cg\sigma_x) = f\big((c, g)x\big).$$

So f is a $(C \times G)$-map and clearly an isomorphism. Thus, F is essentially surjective. Given C-monomial G-sets $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$ be we need to show that the map

$$\overline{F} : \operatorname{Hom}\big((X, \mathfrak{l}), (Y, \mathfrak{m})\big) \to \operatorname{Hom}\big(F(X, \mathfrak{l}), F(Y, \mathfrak{m})\big)$$

induced by F is surjective and injective. Let $\varphi : C \times_{\mathfrak{l}} X \to C \times_{\mathfrak{m}} Y$ be a $(C \times G)$-map. Given $(1, x) \in C \times_{\mathfrak{l}} X$, let $\varphi(1, x) = (c_x, z_x)$ for $(c_x, z_x) \in C \times Y$. Since φ is a $(C \times G)$-map, we get

$$\varphi(c, x) = (cc_x, z_x)$$

and

$$\varphi(1, gx) = \big(c_x\mathfrak{m}(g, z_x, gz_x)\mathfrak{l}^{-1}(g, x, gx), gz_x\big)$$

for any $c \in C$ and $g \in G$. We define a map

$$(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$$

such that $f : X \to Y$ is defined by $f(x) = z_x$ and $\lambda : \mathfrak{l} \to \mathfrak{m} \circ f$ is defined by $\lambda_x = c_x$ for any $x \in X$. Clearly, f is a G-set map. Let $x \in X$ and $g \in G$. Then

$$\mathfrak{m}\big(g, f(x), f(gx)\big)\lambda_x = \mathfrak{m}\big(g, f(x), f(gx)\big)c_x$$

$$= c_x\mathfrak{m}\big(g, f(x), f(gx)\big)\mathfrak{l}^{-1}(g, x, gx)\mathfrak{l}(g, x, gx) = \mathfrak{l}(g, x, gx)c_{gx} = \mathfrak{l}(g, x, gx)\lambda_{gx}.$$

So $\lambda : \mathfrak{l} \to \mathfrak{m} \circ f$ is a natural transformation and (f, λ) is a map of C-monomial G-sets. Thus, $\overline{F}(f, \lambda) = \varphi$ and $\overline{F}$ is surjective. The injectivity is clear, so F is fully faithful. $\square$

Proposition 5.1.3. *Let G be a finite group. Then $B_C(G)$ is isomorphic to the Grothendieck ring of the category ${}_CMG$-set, for relations given by decomposition into disjoint unions of C-monomial G-sets and multiplication induced by product of C-monomial G-sets.*

Proof. We let $B^1_C(G)$ denote the Grothendieck ring of the category ${}_CMG$-set. The equivalence

$$F : {}_CMG\text{-}\mathsf{set} \to {}_CFG\text{-}\mathsf{set}$$

induces a bijection

$$\widehat{F} : B^1_C(G) \to B_C(G)$$

such that

$$\widehat{F}\Big(\big[(X, \mathfrak{l})\big]\Big) = [C \times_{\mathfrak{l}} X]$$

for any C-monomial G-set $(X, \mathfrak{l})$. Now we show that $\widehat{F}$ is a ring homomorphism. Let $(X_1, \mathfrak{l}_1)$ and $(X_2, \mathfrak{l}_2)$ be C-monomial G-sets. Then

$$\begin{aligned}\widehat{F}\Big(\big[(X_1, \mathfrak{l}_1)\big] + \big[(X_1, \mathfrak{l}_1)\big]\Big) &= \widehat{F}\Big(\big[(X_1, \mathfrak{l}_1) \sqcup (X_1, \mathfrak{l}_1)\big]\Big) = \widehat{F}\Big(\big[(X_1 \sqcup X_2, \mathfrak{l}_1 \sqcup \mathfrak{l}_2)\big]\Big)\\ &= \big[C \times_{\mathfrak{l}_1 \sqcup \mathfrak{l}_2} (X_1 \sqcup X_2)\big] = \big[(X_1, \mathfrak{l}_1) \sqcup (X_2, \mathfrak{l}_2)\big] = [C \times_{\mathfrak{l}_1} X_1] + [C \times_{\mathfrak{l}_2} X_2].\end{aligned}$$

For multiplicativity of $\widehat{F}$ we define a map

$$f : C \times_{\mathfrak{l}_1 \times \mathfrak{l}_2} (X_1 \times X_2) \to (C \times_{\mathfrak{l}_1} X_1) \times_C (C \times_{\mathfrak{l}_2} X_2)$$

such that $f\big(c, (x_1, x_2)\big) = (c, x_1) \times_C (1, x_2)$. Let $(k, g) \in C \times G$ and $\big(c, (x_1, x_2)\big) \in C \times_{\mathfrak{l}_1 \times \mathfrak{l}_2} (X_1 \times X_2)$. Then

$$\begin{aligned}(k, g)f\big(c, (x_1, x_2)\big) &= (k, g)\big((c, x_1) \times_C (1, x_2)\big) = \big((k, g)(c, x_1) \times_C (1, g)(1, x_2)\big)\\ &= \big(kc\mathfrak{l}_1(g, x_1, gx_1), gx_1\big) \times_C \big(\mathfrak{l}_2(g, x_2, gx_2), gx_2\big)\\ &= (kc\mathfrak{l}_1(g, x_1, gx_1)\mathfrak{l}_2(g, x_2, gx_2), gx_1) \times_C (1, gx_2)\\ &= f\big(kc\mathfrak{l}_1(g, x_1, gx_1)\mathfrak{l}_2(g, x_2, gx_2), g(x_1, x_2)\big)\\ &= f\big((k, g)(c, (x_1, x_2))\big).\end{aligned}$$

So f is a $(C \times G)$-map and obviously, f is a $(C \times G)$-isomorphism. Using f we get

$$\begin{aligned} &\widehat{F}\big([X_1, \mathfrak{l}_1] \cdot [X_2, \mathfrak{l}_2]\big) = \widehat{F}\big([X_1 \times X_2, \mathfrak{l}_1 \times \mathfrak{l}_2]\big) \\ = &\big[C \times_{\mathfrak{l}_2 \times \mathfrak{l}_2} (X_1 \times X_2)\big] = \big[(C \times_{\mathfrak{l}_1} X_1) \times_C (C \times_{\mathfrak{l}_2} X_2)\big]. \end{aligned}$$

Thus, the desired result follows.

□

Remark 5.1.4. *Let $(X, \mathfrak{l})$ be a C-monomial G-set. For all $x \in X$, we get a character $\mathfrak{l}_x : G_x \to C$ defined by $\mathfrak{l}_x(g) = \mathfrak{l}(g, x, x)$ for $g \in G_x$. On the other hand given a subgroup U of G and a group homomorphism $\mu : U \to C$ we get a C-monomial G-set $(G/U, \widehat{\mu})$ where and $\widehat{\mu} : \widehat{G/U} \to \bullet_C$ is the functor such that given gU, $kU \in G/U$ if $hgU = kU$ for some $g \in G$ then $\widehat{\mu}(h, gU, kU) = \mu(k^{-1}hg)$. Moreover, $[U, \mu]_G$ and $[G/U, \widehat{\mu}]$ represents the same element in $B_C(G)$.*

5.2 Lefschetz invariant of monomial posets

5.2.1 The category of C-monomial G-posets

Let X be a G-poset. We work on the category $\widehat{X}$ whose objects are the elements of X and given x, $y \in X$ the set of morphisms from x to y is

$$\mathrm{Hom}_{\widehat{X}}(x, y) = \{g \in G \mid gx \leq y\}.$$

Now we define a C-monomial G-poset as follows.

Definition 5.2.1. *A C-*monomial *G-*poset *is a pair $(X, \mathfrak{l})$ consisting of a G-poset X and a functor $\mathfrak{l} : \widehat{X} \to \bullet_C$.*

Given $x, y \in X$ and $g \in G$ such that $gx \leq y$, we get an element $\mathfrak{l}(g, x, y)$ of C, such that $\mathfrak{l}(h, y, z)\mathfrak{l}(g, x, y) = \mathfrak{l}(hg, x, z)$ if $h \in G$ and $hy \leq z$, and $\mathfrak{l}(1, x, x) = 1$ for any $x \in X$.

Let $(X,\mathfrak{l})$ and $(Y,\mathfrak{m})$ be C-monomial G-posets. The pair $(f,\lambda) : (X,\mathfrak{l}) \to (Y,\mathfrak{m})$ is a map of C-monomial G-posets where $f : X \to Y$ is a map of G-posets and $\lambda : \mathfrak{l} \to \mathfrak{m} \circ f$ is a natural transformation. The category whose objects are C-monomial G-posets and morphisms are the maps of C-monomial G-posets will be denoted by ${}_CMG\text{-}\mathsf{poset}$. We can define product and disjoint union of C-monomial G-posets as for C-monomial G-sets. We identify the category ${}_CMG\text{-}\mathsf{poset}$ with G-poset when C is the trivial group.

Remark 5.2.2. *Given a C-monomial G-poset $(X,\mathfrak{l})$, we can define a character $\mathfrak{l}_x : G_x \to C$ by $\mathfrak{l}_x(g) = \mathfrak{l}(g,x,x)$ for any $x \in X$. If $x \le y$, since the following diagram is commutative*

$$\begin{tikzcd}
\mathfrak{l}(x) \arrow[r, "\mathfrak{l}(1,x,y)"] \arrow[d, "\mathfrak{l}(g,x,x)"'] & \mathfrak{l}(y) \arrow[d, "\mathfrak{l}(g,y,y)"] \\
\mathfrak{l}(x) \arrow[r, "\mathfrak{l}(1,x,y)"'] & \mathfrak{l}(y),
\end{tikzcd}$$

then

$$\mathrm{res}^{G_x}_{G_x\cap G_y}\mathfrak{l}_x = \mathrm{res}^{G_y}_{G_x\cap G_y}\mathfrak{l}_y.$$

Given a subgroup H of G and a C-monomial H-poset $(X,\mathfrak{l})$, let $G\times_H X$ denote the quotient of $G \times X$ by the action of H. We make the set $G \times_H X$ a G-set by the action $g(u,_H x) = (gu,_H x)$, for any $g \in G$, and $(u,_H x) \in G \times_H X$. There is an order relation $\le$ on $G \times_H X$ given by

$$\forall (u,_H x),\, (v,_H y) \in G \times_H X,\ (u,_H x) \le (v,_H y) \Leftrightarrow \exists h \in H,\, u = vh,\, x \le h^{-1}y.$$

Note that

$$G \times_H X = \bigsqcup_{g\in G/H} g \times_H X,$$

so it's enough to consider the chains of type $(u,_H x_0) < ... < (u,_H x_n)$ in $G \times_H X$ for some $u \in G$ and a chain $x_0 < ... < x_n$ in X for some $n \in \mathbb{N}$.

Given $(u,_H x), (u,_H y) \in G \times_H X$ and $g \in G$ such that $g(u,_H x) \le (u,_H y)$, there exists $h \in H$ such that $gu = uh$ and $hx \le y$. *The induced C-monomial*

G-poset $\mathrm{Ind}_H^G(X,\mathfrak{l})$ of $(X,\mathfrak{l})$ is defined to be the pair $(G\times_H X, G\times_H \mathfrak{l})$ where $G\times_H \mathfrak{l} : \widehat{G\times_H X} \to \bullet_C$ is given by

$$(G\times_H \mathfrak{l})\big(g,(u,_H x),(u,_H y)\big) = \mathfrak{l}(h,x,y).$$

Now we need to show that $(G\times_H X, G\times_H \mathfrak{l})$ is a C-monomial G-poset.

Given $(u,_H x)$, $(u,_H y)$, $(u,_H z) \in G\times_H X$ and g, $g' \in G$ such that

$$g(u,_H x) \leq (u,_H y)$$

and

$$g'(u,_H y) \leq (u,_H z),$$

there exist some h, $h' \in H$ such that

$$gu = uh,\ \ g'u = uh',\ \ hx \leq y,\ \ h'y \leq z.$$

Let $t = h'h \in H$. Then $g'gu = uh'h = ut$ and $tx = h'hx \leq z$. We obtain

$$\begin{aligned}(G\times_H \mathfrak{l})\big(g'g,(u,_H x),(u,_H z)\big) &= \mathfrak{l}(t,x,z) = \mathfrak{l}(h'h,x,z) = \mathfrak{l}(h',y,z)\mathfrak{l}(h,x,y)\\ &= (G\times_H \mathfrak{l})\big(g',(u,_H x),(u,_H y)\big)(G\times_H \mathfrak{l})\big(g,(u,_H y),(u,_H z)\big).\end{aligned}$$

Obviously, given any $(u,_H x) \in G\times_H X$ we have $(G\times_H \mathfrak{l})\big(1,(u,_H x),(u,_H x)\big) = 1$. So $G\times_H \mathfrak{l}$ is a functor. Thus, $\mathrm{Ind}_H^G(X,\mathfrak{l})$ is a C-monomial G-poset.

Let $(Y,\mathfrak{m})$ be a C-monomial G-poset. *The restriction* $\mathrm{Res}_H^G(Y,\mathfrak{m})$ of $(Y,\mathfrak{m})$ is defined to be the pair $(\mathrm{Res}_H^G Y,\ \mathrm{res}_H^G \mathfrak{m})$ where $\mathrm{Res}_H^G Y$ is the restriction of the G-poset Y to H-poset and $\mathrm{res}_H^G \mathfrak{m}$ is the restriction of the functor $\mathfrak{m}$ from $\widehat{Y}$ to $\widehat{\mathrm{Res}_H^G Y}$.

Proposition 5.2.3. *Let G be a finite group.*

1. *If Y is a finite G-poset, denote by $1_Y : \widehat{Y} \to \bullet_C$ the trivial functor defined by $1_Y(g,x,y) = 1$ for any $g \in G$ and $x,y \in Y$ such that $gx \leq y$. Then the assignment $Y \mapsto (Y,1_Y)$ is a functor τ_G from G-poset to ${}_CMG$-poset.*

2. *Let H be a subgroup of G. The assignment $(X,\mathfrak{l}) \mapsto \mathrm{Ind}_H^G(X,\mathfrak{l})$ is a functor $\mathrm{Ind}_H^G : {}_CMH$-$\mathsf{poset} \to {}_CMG$-$\mathsf{poset}$, and the assignment $(Y,\mathfrak{m}) \mapsto \mathrm{Res}_H^G(Y,\mathfrak{m})$ is a functor $\mathrm{Res}_H^G : {}_CMG$-$\mathsf{poset} \to {}_CMH$-$\mathsf{poset}$.*

3. *Moreover the diagrams*

$$\begin{tikzcd} H\text{-poset} \arrow[r, "\mathrm{Ind}_H^G"] \arrow[d, "\tau_H"'] & G\text{-poset} \arrow[d, "\tau_G"'] \\ {}_CMH\text{-poset} \arrow[r, "\mathrm{Ind}_H^G"] & {}_CMG\text{-poset} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G\text{-poset} \arrow[r, "\mathrm{Res}_H^G"] \arrow[d, "\tau_G"'] & H\text{-poset} \arrow[d, "\tau_H"'] \\ {}_CMG\text{-poset} \arrow[r, "\mathrm{Res}_H^G"] & {}_CMH\text{-poset} \end{tikzcd}$$

of categories and functors are commutative.

Proof. 1. Given a map of G-posets $f : X \to Y$, we define

$$\tau_G(f) = (f, 1_f) : (X, 1_X) \to (Y, 1_Y),$$

where $1_f : 1_X \to 1_Y \circ f$ is given by $1_{f_x} = 1$ for any $x \in X$. Clearly, $(f, 1_f)$ is a map of C-monomial G-posets and τ_G is a functor.

2. Given a map of C-monomial H-posets $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$, we define

$$\mathrm{Ind}_H^G(f, \lambda) = (G \times_H f, G \times_H \lambda) : (G \times_H X, G \times_H \mathfrak{l}) \to (G \times_H Y, G \times_H \mathfrak{m})$$

where

$$G \times_H f : G \times_H X \to G \times_H Y$$

is given by $(G \times_H f)(u,_H x) = \big(u,_H f(x)\big)$ and

$$G \times_H \lambda : G \times_H \mathfrak{l} \to (G \times_H \mathfrak{m}) \circ (G \times_H f)$$

is given by $(G \times_H \lambda)_{(u,_H x)} = \lambda_x$ for any $(u,_H x) \in G \times_H X$. Obviously, $G \times_H f$ is a map of C-monomial G-posets. Now we prove that $G \times_H \lambda$ is a natural transformation. Given $(u,_H x)$, $(u,_H y) \in G \times_H X$ and $g \in G$ such that $g(u,_H x) \leq (u,_H y)$ there exists $h \in H$ such that $gu = uh$ and $hx \leq y$. Then we obtain

$$(G \times_H \mathfrak{m})\Big(g, \big(u,_H f(x)\big), \big(u,_H f(y)\big)\Big)(G \times_H \lambda)_{(u,_H x)} = \mathfrak{m}\big(h, f(x), f(y)\big)\lambda_x$$

$$= \lambda_y \mathfrak{l}(h, x, y) = (G \times_H \lambda)_{(u,_H y)}(G \times_H \mathfrak{l})\big(g, (u,_H x), (u,_H y)\big)$$

because $\lambda : \mathfrak{l} \to \mathfrak{m} \circ f$ is a natural transformation. Let $(\,\mathrm{id}_X, \mathrm{id}_{\mathfrak{l}}) : (X, \mathfrak{l}) \to (X, \mathfrak{l})$ be a map of C-monomial G-posets where $\mathrm{id}_X : X \to X$ is the identity map on the H-set X and $\mathrm{id}_{\mathfrak{l}} : \mathfrak{l} \to \mathfrak{l} \circ \mathrm{id}_X$ is the identity transformation. $\mathrm{id}_X : X \to X$ is the identity map on the H-set X and $\mathrm{id}_{\mathfrak{l}} : \mathfrak{l} \to \mathfrak{l} \circ \mathrm{id}_X$ is the identity transformation. We obtain

$$\mathrm{Ind}_H^G(\mathrm{id}_X, \mathrm{id}_{\mathfrak{l}}) = (\mathrm{id}_{G\times_H X}, \mathrm{id}_{G\times_H \mathfrak{l}}).$$

Given maps of C-monomial H-posets $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$ and $(t, \beta) : (Y, \mathfrak{m}) \to (Z, \mathfrak{r})$, we have

$$\mathrm{Ind}_H^G(t, \beta) \circ \mathrm{Ind}_H^G(f, \lambda) = (G \times_H t, G \times_H \beta) \circ (G \times_H f, G \times_H \lambda)$$

$$= \big(G \times_H (t \circ f), G \times_H (\beta \circ \lambda)\big) = G \times (t \circ f, \beta \circ \lambda)\, \mathrm{Ind}_H^G\big((t, \beta) \circ (f, \lambda)\big).$$

Thus, $\mathrm{Ind}_H^G : {}_CMH\textsf{-poset} \to {}_CMG\textsf{-poset}$ is a functor.

Given a map of C-monomial G-posets $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$, we consider the pair

$$\mathrm{Res}_H^G(f, \lambda) = (f|_H, \lambda|_H) : (\mathrm{Res}_H^G X, \mathrm{res}_H^G \mathfrak{l}) \to (\mathrm{Res}_H^G Y, \mathrm{res}_H^G \mathfrak{m})$$

where $f|_H : \mathrm{Res}_H^G X \to \mathrm{Res}_H^G Y$ is defined as the restriction of map of G-posets f to map of H-posets and $\lambda|_H : \mathrm{res}_H^G \mathfrak{l} \to \mathrm{res}_H^G \mathfrak{m} \circ f|_H$ is defined as the restriction of λ. Obviously, $\mathrm{Res}_H^G : {}_CMG\textsf{-poset} \to {}_CMH\textsf{-poset}$ is a functor.

3. Given an H-poset X, we have

$$\tau_G \circ \mathrm{Ind}_H^G(X) = \tau_G(G \times_H X) = (G \times_H X, 1_{G\times_H X}) = \mathrm{Ind}_H^G(X, 1_X) = \mathrm{Ind}_H^G \circ \tau_H(X).$$

So first diagram is commutative.

Given a G-poset Y, we have

$$\tau_H \circ \mathrm{Res}_H^G(Y) = \tau_H(\mathrm{Res}_H^G Y) = (\mathrm{Res}_H^G Y, 1_{\mathrm{Res}_H^G Y})$$

$$= (\mathrm{Res}_H^G Y, \mathrm{res}_H^G 1_Y) = \mathrm{Res}_H^G(Y, 1_Y) = \mathrm{Res}_H^G \circ \tau_H(Y).$$

So the second diagram is commutative. □

Proposition 5.2.4. *Let G be a finite group and H be subgroup of G. Then the functor $\mathrm{Ind}_H^G : {}_CMH\textsf{-poset} \to {}_CMG\textsf{-poset}$ is left adjoint to the functor $(Y, \mathfrak{m}) \mapsto \mathrm{Res}_H^G(Y, \mathfrak{m})$.*

Proof. Given any C-monomial H-poset $(X, \mathfrak{l})$ and any C-monomial G-poset $(Y, \mathfrak{m})$, we will show that there is a bijection

$$\mathrm{Hom}_{CMG}\big(\mathrm{Ind}_H^G(X,\mathfrak{l}),(Y,\mathfrak{m})\big) \cong \mathrm{Hom}_{CMH}\big((X,\mathfrak{l}),\, \mathrm{Res}_H^G(Y,\mathfrak{m})\big)$$

natural in $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$. We set

$$\varphi:\ \mathrm{Hom}_{CMG}\big(\mathrm{Ind}_H^G(X,\mathfrak{l}),(Y,\mathfrak{m})\big) \to \mathrm{Hom}_{CMH}\big((X,\mathfrak{l}),\, \mathrm{Res}_H^G(Y,\mathfrak{m})\big)$$

where

$$\varphi : (f, \lambda) \mapsto \big(\varphi(f), \varphi(\lambda)\big)$$

such that

$$\varphi(f) : X \to \mathrm{Res}_H^G(Y)$$

given by $\varphi(f)(x) = f(1,_H x)$ and

$$\varphi(\lambda) : \mathfrak{l} \to \mathrm{res}\mathfrak{m} \circ \varphi(f)$$

given by $\varphi(\lambda)_x = \lambda_{(1,_H x)}$ for any $x \in X$. Clearly, $\varphi(f)$ is a map of H-posets. Now we show that

$$\varphi(\lambda) : \mathfrak{l} \to \mathrm{res}\mathfrak{m} \circ \varphi(f)$$

is a natural transformation.

Given any x, $y \in X$ and $g \in G$ such that $gx \le y$, we get

$$\begin{aligned}\mathfrak{m}\big(h,&\varphi(f)(x), \varphi(f)(y)\big)\varphi(\lambda)_x = \mathfrak{m}\big(h, f(1,_H x), f(1,_H y)\big)\lambda_{(1,_H x)}\\ &= \lambda_{(1,_H y)}\mathfrak{l}(h,x,y) = \varphi(\lambda)_y(G\times_H \mathfrak{l})\big(h,(1,_H x),(1,_H y)\big).\end{aligned}$$

We set an inverse map to φ as

$$\theta:\ \mathrm{Hom}_{CMH}\big((X,\mathfrak{l}),\, \mathrm{Res}_H^G(Y,\mathfrak{m})\big) \to \mathrm{Hom}_{CMG}\big(\mathrm{Ind}_H^G(X,\mathfrak{l}),(Y,\mathfrak{m})\big)$$

where

$$\theta : (\psi, \beta) \mapsto \big(\theta(\psi), \theta(\beta)\big)$$

such that

$$\theta(\psi) : G \times_H X \to Y$$

given by $\theta(\psi)(u,_H x) = u\psi(x)$ and

$$\theta(\beta) : G \times_H \mathfrak{l} \to \mathfrak{m} \circ \theta(\psi)$$

given by

$$\theta(\beta)_{(u,_H x)} = \mathfrak{m}\big(u, \psi(x), u\psi(x)\big)\beta_x$$

for any $(u,_H x) \in G \times_H X$. Clearly, the map $\theta(\psi)$ is a map of G-posets. Now we prove that $\theta(\beta)$ is a natural transformation. Given any $(u,_H x)$, $(u,_H y) \in G \times_H X$ and any $g \in G$ such that $g(u,_H x) \leq (u,_H y)$ there exists some $h \in H$ such that $gu = uh$ and $hx \leq y$. Then we get

$$\begin{aligned}
&\mathfrak{m}\big(g, \theta(\psi)(u,_H x), \theta(\psi)(u,_H y)\big)\theta(\beta)_{(u,_H x)} = \mathfrak{m}\big(g, u\psi(x), u\psi(y)\big)\mathfrak{m}\big(u, \psi(x), u\psi(x)\big)\beta_x \\
&\quad = \mathfrak{m}\big(u, \psi(y), u\psi(y)\big)\mathfrak{m}\big(h, \psi(x), \psi(y)\big)\beta_x = \mathfrak{m}\big(u, \psi(y), u\psi(y)\big)\beta_y \mathfrak{l}(h, x, y) \\
&\quad = \theta(\beta)_{(u,_H y)}(G \times_H \mathfrak{l})\big(h, (u_H x), (u,_H y)\big).
\end{aligned}$$

Obviuously, φ and θ are mutual inverse maps, and natural in $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$. $\square$

5.2.2 The Lefschetz invariant attached to a C-monomial G-poset

Given a C-monomial G-poset $(X, \mathfrak{l})$, we define the *Lefschetz invariant* $\Lambda_{(X,\mathfrak{l})}$ of $(X, \mathfrak{l})$ as

$$\Lambda_{(X,\mathfrak{l})} = \sum_{x_0 < \ldots < x_n \in_G X} (-1)^n \big[G_{x_0,\ldots,x_n}, \operatorname{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}(\mathfrak{l}_{x_0})\big]_G$$

where $x_0 < \ldots < x_n$ runs over G-representatives of the chains in X. Note that $\Lambda_{(X,\mathfrak{l})}$ is an element of $B_C(G)$. Here, the group $G_{x_0,\ldots,x_n}$ denotes the stabilizer of the set $\{x_0, \ldots, x_n\}$, that is $G_{x_0,\ldots,x_n} = \cap_{i=0}^n G_{x_i}$ and $\operatorname{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}(\mathfrak{l}_{x_0})$ denotes the restriction of the character $\mathfrak{l}_{x_0}$ introduced in Remark 5.1.4. If for some $n \in \mathbb{N}$ we have a chain $x_0 < \ldots < x_n$ in X then by Remark 5.2.2 we have

$$\operatorname{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}} \mathfrak{l}_{x_0} = \operatorname{Res}^{G_{x_i}}_{G_{x_0,\ldots,x_n}} \mathfrak{l}_{x_i}$$

for any $0 \leq i \leq n$.

Given a C-monomial G-poset $(X, \mathfrak{l})$ and $n \in \mathbb{N}$ we denote the set of chains in X with $n+1$ by $\mathrm{Sd}_n(X)$. Clearly, the set $\mathrm{Sd}_n(X)$ is a G-set. We set a pair $(\mathrm{Sd}_n(X), \mathfrak{l}_n)$ where $\mathfrak{l}_n : \widehat{\mathrm{Sd}_n(X)} \to \bullet_C$ is defined by

$$\mathfrak{l}_n(g, x_0 < \ldots < x_n, y_0 < \ldots < y_n) = \mathfrak{l}(g, x_0, y_0)$$

for any $x_0 < \ldots < x_n$, and $y_0 < \ldots < y_n$ in $\mathrm{Sd}_n(X)$ such that

$$g(x_0 < \ldots < x_n) = y_0 < \ldots < y_n$$

for some $g \in G$. Now we show that $\mathfrak{l}_n : \widehat{\mathrm{Sd}_n(X)} \to \bullet_C$ is a functor. Obviously, $\mathfrak{l}_n(1, x_0 < \ldots < x_n, x_0 < \ldots < x_n) = 1$ for any element $x_0 < \ldots < x_n$ of $\mathrm{Sd}_n(X)$. Given any elements $x_0 < \ldots < x_n$, $y_0 < \ldots < y_n$, $z_0 < \ldots < z_n$ of $\mathrm{Sd}_n(X)$ and for any g, $h \in G$ such that

$$g(x_0 < \ldots < x_n) = y_0 < \ldots < y_n$$

and

$$h(y_0 < \ldots < y_n) = z_0 < \ldots < z_n$$

we have

$$\begin{aligned}
&\mathfrak{l}_n(h, y_0 < \ldots < y_n, z_0 < \ldots < z_n)\mathfrak{l}_n(g, x_0 < \ldots < x_n, y_0 < \ldots < y_n) \\
&= \mathfrak{l}(h, y_0, z_0)\mathfrak{l}(g, x_0, y_0) = \mathfrak{l}(hg, x_0, z_0) \\
&= \mathfrak{l}_n(hg, x_0 < \ldots < x_n, z_0 < \ldots < z_n).
\end{aligned}$$

So $\mathfrak{l}_n : \widehat{\mathrm{Sd}_n(X)} \to \bullet_C$ is a functor which means $(\mathrm{Sd}_n(X), \mathfrak{l}_n)$ is a C-monomial G-set.

Remark 5.2.5. *Given a C-monomial G-poset $(X, \mathfrak{l})$, we have the following isomorphism of monomial G-sets:*

$$\big(\mathrm{Sd}_n(X), \mathfrak{l}_n\big) \cong \bigsqcup_{x_0 < \ldots < x_n \in_G \mathrm{Sd}_n(X)} \big(G/G_{x_0,\ldots,x_n}, \widehat{\mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}}(\mathfrak{l}_{x_0})\big)$$

for any $n \in \mathbb{N}$.

Proof. We will denote the set of representatives of the G-action on $\operatorname{Sd}_n(X)$ by $[G/\operatorname{Sd}_n(X)]$. We consider a chain $x = x_0 < ... < x_n$ in $\operatorname{Sd}_n(X)$. Then there exist some $g_x \in G$ and a unique $\sigma_x \in [G/\operatorname{Sd}_n(X)]$ such that $x = g_x\sigma_x$ where $\sigma_x = \sigma_{x_0} < ... < \sigma_{x_n}$. We set

$$(f,\lambda) : \big(\operatorname{Sd}_n(X), \mathfrak{l}_n\big) \to \bigsqcup_{x_0<...<x_n \in_G \operatorname{Sd}_n(X)} \big(G/G_{x_0,\dots,x_n}, \widehat{\operatorname{Res}^{G_{x_0}}_{G_{x_0,\dots,x_n}}}(\mathfrak{l}_{x_0})\big)$$

where $f(x) = g_x G_{\sigma_x} \in G/G_{\sigma_x}$ and $\lambda_x = \mathfrak{l}(g_x^{-1}, g_x\sigma_{x_0}, \sigma_{x_0})$. It's clear that

$$f : \operatorname{Sd}_n(X) \to \bigsqcup_{x_0<...<x_n \in_G \operatorname{Sd}_n(X)} G/G_{x_0,\dots,x_n}$$

is an isomorphism of G-sets. Now we will prove that

$$\lambda : \mathfrak{l}_n \to \bigsqcup_{x_0<...<x_n \in_G Sd_n(X)} \widehat{\operatorname{Res}^{G_{x_0}}_{G_{x_0,\dots,x_n}}}(\mathfrak{l}_{x_0}) \circ f$$

is a natural transformation. Given chains $x = x_0 < ... < x_n$, and $y = y_0 < ... < y_n$ in $\operatorname{Sd}_n(X)$ and $g \in G$ such that $gx = y$, there exist a unique $\sigma_x, \sigma_y \in [G/\operatorname{Sd}_n(X)]$ such that $x = g_x\sigma_x$ and $y = g_y\sigma_y$ for some g_x and g_y in G. We get $x_0 = g_x\sigma_{x_0}$ and $y_0 = g_y\sigma_{y_0}$ so $y_0 = gx_0 = gg_x\sigma_{x_0}$. Then uniqueness implies $\sigma_{x_0} = \sigma_{y_0}$ and so $g_y^{-1}gg_x \in G_{\sigma_{x_0}}$. We set $r = \widehat{\operatorname{Res}^{G_{x_0}}_{G_x}}(\mathfrak{l}_{x_0})\big(g, f(x), f(y)\big)\lambda_x$, then

$$\begin{aligned} r &= \widehat{\mathfrak{l}_{x_0}}(g, g_xG_{\sigma_x}, g_yG_{\sigma_y})\mathfrak{l}(g_x^{-1}, g_x\sigma_{x_0}, \sigma_{x_0}) = \mathfrak{l}_{x_0}(g_y^{-1}gg_x)\mathfrak{l}(g_x^{-1}, g_x\sigma_{x_0}, \sigma_{x_0}) \\ &= \mathfrak{l}(g_y^{-1}gg_x, x_0, x_0)\mathfrak{l}(g_x^{-1}, g_x\sigma_{x_0}, \sigma_{x_0}) \\ &= \mathfrak{l}(g_x, \sigma_{x_0}, g_x\sigma_{x_0})\mathfrak{l}(g, g_x\sigma_{x_0}, gg_x\sigma_{x_0})\mathfrak{l}(g_y^{-1}, gg_x\sigma_{x_0}, \sigma_{x_0})\mathfrak{l}(g_x^{-1}, g_x\sigma_{x_0}, \sigma_{x_0}) \\ &= \mathfrak{l}(g, x_0, y_0)\mathfrak{l}(g_y^{-1}, g_y\sigma_{y_0}, \sigma_{y_0}) \\ &= \mathfrak{l}_n(g, x, y)\lambda_y. \end{aligned}$$

□

Now we can write the Lefschetz invariant of a C-monomial G-set $(X, \mathfrak{l})$ as

$$\Lambda_{(X,\mathfrak{l})} = \sum_{x_0<...<x_n \in_G X} (-1)^n \big[G_{x_0,\dots x_n}, \operatorname{Res}^{G_{x_0}}_{G_{x_0,\dots,x_n}}(\mathfrak{l}_{x_0})\big]_G = \sum_{n\in\mathbb{N}} (-1)^n \big(\operatorname{Sd}_n(X), \mathfrak{l}_n\big)$$

by Remark 5.2.5. It follows that $\Lambda_X = \Lambda_{\tau_G(X)}$, where Λ_X is the Lefschetz invariant of the G-poset X as in [4].

We can similarly define the *reduced Lefschetz invariant* of $(X, \mathfrak{l})$

$$\widetilde{\Lambda}_{(X,\mathfrak{l})} = \Lambda_{(X,\mathfrak{l})} - [G, 1_G]_G$$

where 1_G is the trivial character of G.

Lemma 5.2.6. *Let G be a finite group and C be an abelian group.*

1. *Let $(X, \mathfrak{l})$ be a C-monomial G-set, viewed a a C-monomial G-poset ordered by the equality relation on X. Then $\Lambda_{(X,\mathfrak{l})} = [C \times_{\mathfrak{l}} X]$ in $B_C(G)$.*

2. *Let $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$ be C-monomial G-posets. Then $\Lambda_{(X \sqcup Y, \mathfrak{r})} = \Lambda_{(X,\mathfrak{l})} + \Lambda_{(Y,\mathfrak{m})}$ in $B_C(G)$.*

3. *Given C-monomial G-posets $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$, we have $\Lambda_{(X\times Y, \mathfrak{l}\times\mathfrak{m})} = \Lambda_{(X,\mathfrak{l})}\Lambda_{(Y,\mathfrak{m})}$ in $B_C(G)$.*

Proof. 1. and 2. are clear.

3. Considering the inclusion

$$B_C(G) \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} B_C(G)$$

we identify the elements of $B_C(G)$ with their image in $\mathbb{Q} \otimes_{\mathbb{Z}} B_C(G)$.

We rearrange the chains in $X \times Y$ as in the proof of Lemma 11.2.9 in [4]. Let $n \in \mathbb{N}$ and $z = z_0 < ... < z_n$ be a chain in $X \times Y$. We will denote the projection of z on X by z_X and on Y by z_Y. Then z_X is a chain in X with order $i+1$ for some $i \leq n$ and z_Y is a chain in Y with order $j+1$ for some $j \leq n$ such that $i + j = n$. We will denote the chain $s_0 < ... < s_i$ by $\underline{s}_i$ and the chain $t_0 < ... < t_j$ by $\underline{t}_j$. Then

$$\Lambda_{(X\times Y, \mathfrak{l}\times\mathfrak{m})} = \sum_{\substack{n\in\mathbb{N},\\ z\in_G \operatorname{Sd}_n(X\times Y)}} (-1)^n \big[G_z, \operatorname{Res}^{G_{z_0}}_{G_z}(\mathfrak{l}_{z_0})\big]_G$$

$$= \sum_{\substack{n\in\mathbb{N},\\ z\in \mathrm{Sd}_n(X\times Y)}} (-1)^n \frac{|G_z|}{|G|}[G_z,\, \mathrm{Res}^{G_{z_0}}_{G_z}(\mathfrak{l}_{z_0})]_G = \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y}} \Gamma_{\underline{s}_i,\underline{t}_j}$$

where

$$\begin{aligned}
\Gamma_{\underline{s}_i,\underline{t}_j} &= \sum_{\substack{n\in\mathbb{N}\\ z\in \mathrm{Sd}_n(X\times Y):z_X=\underline{s}_i,\, z_Y=\underline{t}_j}} (-1)^n \frac{|G_{\underline{s}_i}\cap G_{\underline{t}_j}|}{|G|}\big[G_{\underline{s}_i}\cap G_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{G_{t_0}}_{G_{\underline{t}_j}}(\mathfrak{m}_{t_0})\big]_G \\
&= \frac{|G_{\underline{s}_i}\cap G_{\underline{t}_j}|}{|G|}\big[G_{\underline{s}_i}\cap G_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{G_{t_0}}_{G_{\underline{t}_j}}(\mathfrak{m}_{t_0})\big]_G \sum_{\substack{n\in\mathbb{N}\\ z\in \mathrm{Sd}_n(X\times Y):z_X=\underline{s}_i,\, z_Y=\underline{t}_j}} (-1)^n \\
&= \frac{|G_{\underline{s}_i}\cap G_{\underline{t}_j}|}{|G|}\big[G_{\underline{s}_i}\cap G_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{G_{t_0}}_{G_{\underline{t}_j}}(\mathfrak{m}_{t_0})\big]_G (-1)^{i+j}.
\end{aligned}$$

We get,

$$\Lambda_{(X\times Y,\mathfrak{l}\times\mathfrak{m})} = \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y}} (-1)^{i+j}\frac{|G_{\underline{s}_i}\cap G_{\underline{t}_j}|}{|G|}\big[G_{\underline{s}_i}\cap G_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{G_{t_0}}_{G_{\underline{t}_j}}(\mathfrak{m}_{t_0})\big]_G.$$

On the other hand

$$\begin{aligned}
\Lambda_{(X,\mathfrak{l})}\Lambda_{(Y,\mathfrak{m})} &= \sum_{\substack{i\in\mathbb{N}\\ \underline{s}_i\in_G X}} (-1)\big[G_{\underline{s}_i},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\big] \sum_{\substack{j\in\mathbb{N}\\ \underline{t}_j\in_G Y}} (-1)^j\big[G_{\underline{t}_j},\, \mathrm{Res}^{G_{t_0}}_{G_{\underline{t}_j}}(\mathfrak{m}_{t_0})\big]_G \\
&= \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y\\ G_i gG_{\underline{t}_j}\subseteq G}} (-1)^{i+j}\frac{|G_{\underline{s}_i}||G_{\underline{t}_j}|}{|G|^2}\big[G_{\underline{s}_i}\cap {}^gG_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{{}^gG_{t_0}}_{{}^gG_{\underline{t}_j}}({}^g\mathfrak{m}_{t_0})\big]_G \\
&= \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y\\ g\in G}} (-1)^{i+j}\frac{|G_{\underline{s}_i}||G_{\underline{t}_j}|}{|G|^2|G_{\underline{s}_i}gG_{\underline{t}_j}|}\big[G_{\underline{s}_i}\cap {}^gG_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{{}^gG_{t_0}}_{{}^gG_{\underline{t}_j}}({}^g\mathfrak{m}_{t_0})\big]_G \\
&= \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y\\ g\in G}} (-1)^{i+j}\frac{|G_{\underline{s}_i}\cap {}^gG_{\underline{t}_j}|}{|G|^2}\big[G_{\underline{s}_i}\cap {}^gG_{\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{{}^gG_{t_0}}_{{}^gG_{\underline{t}_j}}({}^g\mathfrak{m}_{t_0})\big]_G \\
&= \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y\\ g\in G}} (-1)^{i+j}\frac{|G_{\underline{s}_i}\cap G_{g\underline{t}_j}|}{|G|^2}\big[G_{\underline{s}_i}\cap G_{g\underline{t}_j},\, \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\,\mathrm{Res}^{G_{gt_0}}_{G_{g\underline{t}_j}}(\mathfrak{m}_{gt_0})\big]_G
\end{aligned}$$

$$= \sum_{\substack{i,j\in\mathbb{N}\\ \underline{s}_i\in X\\ \underline{t}_j\in Y}} (-1)^{i+j} \frac{|G_{\underline{s}_i} \cap G_{\underline{t}_j}|}{|G|} \big[G_{\underline{s}_i} \cap G_{\underline{t}_j},\ \mathrm{Res}^{G_{s_0}}_{G_{\underline{s}_i}}(\mathfrak{l}_{s_0})\, \mathrm{Res}^{G_{t_0}}_{G_{\underline{t}_j}}(\mathfrak{m}_{t_0})\big]_G.$$

Thus, $\Lambda_{(X\times Y,\mathfrak{l}\times\mathfrak{m})} = \Lambda_{(X,\mathfrak{l})}\Lambda_{(Y,\mathfrak{m})}$. □

By the first assertion of of Lemma 5.2.6 every positive element of $B_C(G)$ is in of the form $\Lambda_{(X,\mathfrak{l})}$ for some C-monomial G-poset $(X,\mathfrak{l})$. Let the set $X = \{a,b,c,d,e\}$ be the G-poset with the ordering $\{a \le c, a \le d, a \le e, b \le c, b \le d, b \le e\}$ and the trivial G-action. Then $\Lambda_{\tau_G(X)} = -1_{B_C(G)}$. So as a consequence of Lemma 5.2.6 we get the following corollary.

Corollary 5.2.7. *Any element of the monomial Burnside ring can be expressed as the Lefschetz invariant of some (non unique) monomial G-poset.*

Proposition 5.2.8. *Let H be a subgroup of G. Given a C-monomial H-poset $(X,\mathfrak{l})$, we have*

$$\mathrm{Ind}_H^G(\Lambda_{(X,\mathfrak{l})}) = \Lambda_{\mathrm{Ind}_H^G(X,\mathfrak{l})}.$$

Proof. If we show that there exists a C-monomial G-set isomorphism between

$$\big(G \times_H \mathrm{Sd}_n(X), G \times_H \mathfrak{l}_n\big)$$

and

$$\big(\mathrm{Sd}_n(G \times_H X), (G \times_H \mathfrak{l})_n\big)$$

for any $n \in \mathbb{N}$ then the proof is done.

Given any chain $(u,_H x_0 < \ldots < x_n)$ in $G \times_H \mathrm{Sd}_n(X)$, we set

$$(f_n, \mathrm{id}) : \big(G \times_H \mathrm{Sd}_n(X), G \times_H \mathfrak{l}_n\big) \to \big(\mathrm{Sd}_n(G \times_H X), (G \times_H \mathfrak{l})_n\big)$$

where

$$f_n : G \times_H \mathrm{Sd}_n(X) \to \mathrm{Sd}_n(G \times_H X)$$

such that

$$f_n(u,_H x_0 < \ldots < x_n) = \big((u,_H x_0) < \ldots < (u,_H x_n)\big).$$

Given any chain $(u_0,_H x_0) < \ldots < (u_n,_H x_n)$ in $\mathrm{Sd}_n(G \times_H X)$, there exist some $h_i \in H$ such that $u_i h_i = u_{i+1}$ and $h_i^{-1} x_i < x_{i+1}$ for all $0 \leq i \leq n-1$. Then

$$f_n\big(u_0,_H x_0 < h_0 x_1 < \ldots < h_0 \ldots h_{n-1} x_n\big) = (u_0,_H x_0) < \ldots < (u_n,_H x_n).$$

Clearly, f_n is a map of G-sets and injective.

Now we need to show that $G \times_H \mathfrak{l}_n = (G \times_H \mathfrak{l})_n \circ f_n$. Given any element k of G and chain $(u,_H x_0 < \ldots < x_n)$ in in $G \times_H \mathrm{Sd}_n(X)$ such that

$$k(u,_H x_0 < \ldots < x_n) = (v,_H y_0 < \ldots < y_n),$$

there exists some $h \in H$ such that $ku = vh$ and $hx_i = y_i$ for all $0 \leq i \leq n$. We get

$$\begin{aligned}
&(G \times_H \mathfrak{l})_n\big(k, f_n(u,_H x_0 < \ldots < x_n), f_n(v,_H y_0 < \ldots < y_n)\big)\\
&= (G \times_H \mathfrak{l})_n\big(k, (u,_H x_0) < \ldots < (u,_H x_n), (v,_H y_0) < \ldots < (v,_H y_n)\big)\\
&= \mathfrak{l}_n(h, x_0 < \ldots < x_n, y_0 < \ldots < y_n)\\
&= (G \times_H \mathfrak{l}_n)\big(k, (u,_H x_0 < \ldots < x_n), (v,_H y_0 < \ldots < y_n)\big).
\end{aligned}$$

$\square$

Now we set some notations. Consider a G-poset and an element x in X. The pairs $(]x, \cdot[_X, \mathfrak{l}_{>x})$ and $(]\cdot, x[_X, \mathfrak{l}^{<x})$ will denote the C-monomial G_x-posets where

$$]x, \cdot[_X = \{y \in X \mid x < y\}, \qquad]\cdot, x[_X = \{y \in X \mid y < x\}$$

which are G_x-posets and $\mathfrak{l}_{>x} : \widehat{]x, \cdot[_X} \to \bullet_C$ and $\mathfrak{l}^{<x} : \widehat{]\cdot, x[_X} \to \bullet_C$ are the restrictions of the functor $\mathfrak{l}$.

Lemma 5.2.9. *Let $(X, \mathfrak{l})$ be a monomial G-poset. We have*

$$\Lambda_{(X,\mathfrak{l})} = -\sum_{x \in [G/X]} \mathrm{Ind}_{G_x}^G \big([G_x, \mathfrak{l}_x]_{G_x} \cdot \widetilde{\Lambda}_{]x,\cdot[_X}\big).$$

Proof.

$$\Lambda_{(X,\mathfrak{l})} = \sum_{x_0 < \ldots < x_n \in_G X} (-1)^n \big[G_{x_0,\ldots,x_n}, \mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}(\mathfrak{l}_{x_0})\big]_G$$

$$
\begin{aligned}
&= \sum_{x_0 \in_G X} \sum_{x_1 < \ldots < x_n \in_G X : x_0 < x_1} (-1)^n \big[G_{x_0,\ldots,x_n}, \operatorname{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}(\mathfrak{l}_{x_0}) \big]_G \\
&= \sum_{x_0 \in_G X} \operatorname{Ind}^G_{G_{x_0}} \sum_{x_1 < \ldots < x_n \in_{G_{x_0}}]x_0, \cdot[_X} (-1)^n \big[G_{x_0,\ldots,x_n}, \operatorname{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}(\mathfrak{l}_{x_0}) \big]_{G_{x_0}} \\
&= \sum_{x_0 \in_G X} \operatorname{Ind}^G_{G_{x_0}} \big[G_{x_0}, \mathfrak{l}_{x_0} \big]_{G_{x_0}} \sum_{x_1 < \ldots < x_n \in_{G_{x_0}}]x_0, \cdot[_X} (-1)^n \big[G_{x_0,\ldots,x_n}, 1_{G_{x_0,\ldots,x_n}} \big]_{G_{x_0}} \\
&= -\sum_{x \in_G X} \operatorname{Ind}^G_{G_x} \big([G_x, \mathfrak{l}_x]_{G_x} \cdot \widetilde{\Lambda}_{]x,\cdot[_X} \big).
\end{aligned}
$$

$\square$

Remark 5.2.10. *We will define the* opposite *of a C-monomial G-poset. Let $(X, \mathfrak{l})$ be a C-monommial G-poset. $(X^{\mathrm{op}}, \mathfrak{l}^{\mathrm{op}})$ be the pair where X^{op} is the opposite G-poset with the order $\leq^{\mathrm{op}}$ given by*

$$\forall x, y \in X, g \in G, \; gx \leq^{\mathrm{op}} y \Leftrightarrow y \leq gx$$

and $\mathfrak{l}^{\mathrm{op}} : \widehat{X^{\mathrm{op}}} \to \bullet_C$ is given by

$$\mathfrak{l}^{\mathrm{op}}(g, x, y) = \mathfrak{l}^{-1}(g^{-1}, y, x)$$

for any $x, y \in X^{\mathrm{op}}$ and $g \in G$ such that $gx \leq^{\mathrm{op}} y$. Clearly, the pair $(X^{\mathrm{op}}, \mathfrak{l}^{\mathrm{op}})$ is a C-monomial G-poset. Given a map of C-monomial G-posets $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$ we obtain the following commmutative diagram

$$
\begin{tikzcd}
\mathfrak{l}(x) \arrow[r, "\lambda_x"] \arrow[d, "\mathfrak{l}^{\mathrm{op}}(g,x,x')"'] & \mathfrak{m} \circ f(x) \arrow[d, "\mathfrak{m}^{\mathrm{op}}(g, f(x), f(x'))"] \\
\mathfrak{l}(x') \arrow[r, "\lambda_{x'}"'] & \mathfrak{m} \circ f(x').
\end{tikzcd}
$$

*for any $gx \leq^{\mathrm{op}} x'$. It's clear that $(\mathfrak{l}^{\mathrm{op}})_x(g) = \mathfrak{l}^{-1}(g^{-1}, x, x) = \mathfrak{l}(g, x, x) = \mathfrak{l}_x(g)$, for any $x \in X$ and $g \in G_x$. Thus we obtain a functor from ${}_CMG$-*poset *to ${}_CMG$-*poset *by mapping $(X, \mathfrak{l})$ to $(X^{\mathrm{op}}, \mathfrak{l}^{\mathrm{op}})$. Then we get $\Lambda_{(X,\mathfrak{l})} = \Lambda_{(X^{\mathrm{op}}, \mathfrak{l}^{\mathrm{op}})}$.*

Now, consider a map of C-monomial G-posets $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$ and an element y of Y. Then folowing the notation in [21], we set

$$f^y = \{x \in X \mid f(x) \leq y\}, \qquad f_y = \{x \in X \mid f(x) \geq y\}$$

which are both G_y-posets. The pairs $(f^y, \mathfrak{l}_{|f^y})$ and $(f_y, \mathfrak{l}_{|f_y})$ will denote the C-monomial G_y-posets where $\mathfrak{l}_{|f^y} : \widehat{f^y} \to \bullet_C$ and $\mathfrak{l}_{|f_y} : \widehat{f_y} \to \bullet_C$ are restrictions of the functor $\mathfrak{l}$ to $\widehat{f^y}$ and $\widehat{f_y}$, respectively.

Example 5.2.11. *Given a map of C-monomial G-posets $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$, we define a G-poset $X *_{f,\lambda} Y$ with underlying G-set $X \sqcup Y$ as follows: for $z, z' \in X \sqcup Y$, we define*

$$z \leq z' \Leftrightarrow \begin{cases} z, z' \in X & \text{and} \quad z \leq z' \in X \\ z, z' \in Y & \text{and} \quad z \leq z' \in Y \\ z \in X, z' \in Y & \text{and} \quad f(z) \leq z' \in Y \end{cases}.$$

We set a functor $\mathfrak{l} *_{f,\lambda} \mathfrak{m} : \widehat{X \sqcup Y} \to \bullet_C$ given by

$$(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g, z, z') = \begin{cases} \mathfrak{l}(g, z, z') & \text{if } z,\ z' \in X \\ \mathfrak{m}(g, z, z') & \text{if } z,\ z' \in Y \\ \mathfrak{m}(g, f(z), z')\lambda_z & \text{if } z \in X,\ z' \in Y. \end{cases}.$$

for any $z,\ z' \in X *_{f,\lambda} Y$ and $g \in G$ such that $gz \leq z'$.

Given $z_1,\ z_2,\ z_3 \in X *_{f,\lambda} Y$ and $g,\ g' \in G$ such that $gz_1 \leq z_2$ and $g'z_2 \leq z_3$ we want to show that

$$(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g'g, z_1, z_3) = (\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g', z_2, z_3)(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g, z_1, z_2).$$

There are four cases to consider:

- $z_1,\ z_2,\ z_3 \in X$
- $z_1,\ z_2 \in X$ *and* $z_3 \in Y$
- $z_1 \in X$ *and* $z_2,\ z_3 \in Y$
- $z_1,\ z_2,\ z_3 \in Y$.

In the first case we have

$$(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g'g, z_1, z_3) = \mathfrak{l}(g'g, z_1, z_3) = \mathfrak{l}(g', z_2, z_3)\mathfrak{l}(g, z_1, z_2)$$

$$= (\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g', z_2, z_3)(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g, z_1, z_2).$$

In the second case, we use the naturality of λ *and obtain*

$$(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g'g, z_1, z_3) = \mathfrak{m}\big(g'g, f(z_1), z_3\big)\lambda_{z_1} = \mathfrak{m}\big(g', f(z_2), z_3\big)\mathfrak{m}\big(g, f(z_1), f(z_2)\big)\lambda_{z_1}$$

$$= \mathfrak{l}(g, z_1, z_2)\mathfrak{m}\big(g', f(z_2), z_3\big)\lambda_{z_2} = (\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g', z_2, z_3)(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g, z_1, z_2).$$

In the third case, we have

$$(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g'g, z_1, z_3) = \mathfrak{m}\big(g'g, f(z_1), z_3\big)\lambda_{z_1} = \mathfrak{m}\big(g', f(z_2), z_3\big)\mathfrak{m}\big(g, f(z_1), f(z_2)\big)\lambda_{z_1}$$

$$= (\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g, z_1, z_2)(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g', z_2, z_3).$$

In the fourth case since

$$(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g'g, z_1, z_3) = \mathfrak{m}\big(g'g, z_1, z_3\big) = \mathfrak{m}(g', z_2, z_3)\mathfrak{m}(g, z_1, z_2)$$

$$= (\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g', z_2, z_3)(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(g, z_1, z_2).$$

Let $z \in X *_{f,\lambda} Y$*, we have* $(\mathfrak{l} *_{f,\lambda} \mathfrak{m})(1, z, z) = 1$*. Thus,* $(X *_{f,\lambda} Y, \mathfrak{l} *_{f,\lambda} \mathfrak{m})$ *is a* C*-monomial* G*-poset.*

Lemma 5.2.12. *Let* $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$ *be a map of* C*-monomial* G*-posets. Then* $\Lambda_{(X *_{f,\lambda} Y, \mathfrak{l} *_{f,\lambda} \mathfrak{m})} = \Lambda_{(Y,\mathfrak{m})}$.

Proof. Given $z \in Z = X *_{f,\lambda} Y$, if $z \in X$ we define $g :]z, \cdot[_Z \to [f(z), \cdot[_Y$ such that

$$g(t) = \begin{cases} f(t) & \text{if } t \in X \\ t & \text{if } t \in Y \end{cases}$$

and $g' : \big[f(z), \cdot\big[\to \big]z, \cdot\big[$ such that $g'(s) = s$. Then g and g' are maps of G_z-posets such that $g \circ g' = \mathrm{Id}$ and $\mathrm{Id} \le g' \circ g$. Then by [21, Lemma 4.2.4 and Proposition 4.2.5], we obtain $\widetilde{\Lambda}_{]z,\cdot[} = \widetilde{\Lambda}_{[f(z),\cdot[} = 0$ for any $z \in X$. Thus,

$$\Lambda_{(X *_{f,\lambda} Y, \mathfrak{l} *_{f,\lambda} \mathfrak{m})} = -\sum_{z \in [G \backslash X *_{f,\lambda} Y]} \mathrm{Ind}_{G_z}^G([G_z, \mathfrak{l}_z]_{G_z} \cdot \widetilde{\Lambda}_{]z,\cdot[})$$

$$= -\sum_{y \in [G \backslash Y]} \mathrm{Ind}_{G_y}^G([G_y, \mathfrak{l}_y]_{G_y} \cdot \widetilde{\Lambda}_{]y,\cdot[}) = \Lambda_{(Y,\mathfrak{m})}.$$

□

Now we will give an analogue of Proposition 4.2.7. in [21], which in turn was inspired by a much deeper theorem of Quillen in [22].

Proposition 5.2.13. *Let $(f, \lambda) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$ be a map of C-monomial G-posets. Then in $B_C(G)$*

$$\widetilde{\Lambda}_{(Y,\mathfrak{m})} = \widetilde{\Lambda}_{(X,\mathfrak{l})} + \sum_{y \in G \backslash Y} \mathrm{Ind}_{G_y}^{G} \big(\widetilde{\Lambda}_{f^y} \widetilde{\Lambda}_{(]y,\cdot[_Y, \mathfrak{m}^{>y})}\big).$$

$$\widetilde{\Lambda}_{(Y,\mathfrak{m})} = \widetilde{\Lambda}_{(X,\mathfrak{l})} + \sum_{y \in G \backslash Y} \mathrm{Ind}_{G_y}^{G} \big(\widetilde{\Lambda}_{f_y} \widetilde{\Lambda}_{(]\cdot,y[_Y, \mathfrak{m}_{<y})}\big).$$

Proof. We will follow the proof of Proposition 4.2.7 in [21]. Let $n \in \mathbb{N}$ and consider a chain $z = z_0 < ... < z_n \in \mathrm{Sd}_n(X *_{f,\lambda} Y)$. The chain z can be of two types, depending on $z_n \in X$ or $z_n \in Y$. If z is of the first type we obtain

$$\mathrm{Res}_{G_{z_0,\ldots,z_n}}^{G_{z_n}} (\mathfrak{l} *_{f,\lambda} \mathfrak{m})_{n_{z_n}} = \mathrm{Res}_{G_{z_0,\ldots,z_n}}^{G_{z_n}} \mathfrak{l}_n.$$

If z is of the second type then it has a smallest element $y = z_i$ in Y. Then the sequence can be written as

$$x_0 < ... < x_{i-1} < y < y_0 < ... < y_{n-i-1}$$

such that $x_0 < ... < x_{i-1}$ is in $\mathrm{Sd}_{i-1}(f^y)$, and $y_0 < ... < y_{n-i-1}$ is in $\mathrm{Sd}_{n-i-1}(]y, \cdot[_Y)$. We obtain

$$\mathrm{Res}_{G_{z_0,\ldots,z_n}}^{G_{z_n}} (\mathfrak{l} *_{f,\lambda} \mathfrak{m})_{n_{z_n}} = \mathrm{Res}_{G_{z_0,\ldots,z_n}}^{G_y} (\mathfrak{m}).$$

Now the chain $x_0 < ... < x_{i-1}$ will be denoted by $\underline{x}_{i-1}$ and the chain $y_0 < ... < y_{n-i-1}$ will be denoted by $\underline{y}_{n-i-1}$. Applying Lemma 5.2.6 and Lemma 5.2.12 we obtain

$$\begin{aligned}
\Lambda_{(Y,\mathfrak{m})} = \Lambda_{(X *_{f,\lambda} Y, \mathfrak{l} *_{f,\lambda} \mathfrak{m})} &= \sum_{n \in \mathbb{N}} (-1)^n \big(\mathrm{Sd}_n(X *_{f,\lambda} Y), (\mathfrak{l} *_{f,\lambda} \mathfrak{m})_n\big) \\
&= \sum_{\substack{n \in \mathbb{N} \\ z_0 < \ldots < z_n \in \mathrm{Sd}_n(\mathfrak{l} *_{f,\lambda} \mathfrak{m})}} (-1)^n \big[G_{z_0,\ldots,z_n}, \mathrm{Res}_{G_{z_0,\ldots,z_n}}^{G_{z_n}} (\mathfrak{l} *_{f,\lambda} \mathfrak{m})_n\big]_G \\
&= \sum_{n \in \mathbb{N}} (-1)^n \big(\mathrm{Sd}_n(X), \mathfrak{l}_n\big)
\end{aligned}$$

$$+\sum_{y\in[G\backslash Y]}\mathrm{Ind}_{G_y}^{G}\sum_{i=0}^{n}\sum_{\substack{\underline{x}_{i-1}\in\mathrm{Sd}_{i-1}(f^y)\\ \underline{y}_{n-i-1}\in\mathrm{Sd}_{n-i-1}(]y,\cdot[_{G_y})}}\big[G_{\underline{x}_{i-1},y,\underline{y}_{n-i-1}},\,\mathrm{Res}^{G_y}_{G_{\underline{x}_{i-1},y,\underline{y}_{n-i-1}}}\mathfrak{m}_y\big]_{G_y}$$

$$=\Lambda_{(X,\mathfrak{l})}+\sum_{y\in[G\backslash Y]}\mathrm{Ind}_{G_y}^{G}\big(\widetilde{\Lambda}_{(f^y,1_{f^y})}\widetilde{\Lambda}_{(]y,\cdot[_Y,\mathfrak{m}_{>y})}\big).$$

The second assertion follows from Remark 5.2.10. □

Corollary 5.2.14. *Let $(f,\lambda):(X,\mathfrak{l})\to(Y,\mathfrak{m})$ be a map of C-monomial G-posets. If $\Lambda_{f^y}=0$ for all $y\in Y$ (resp. if $\Lambda_{f_y}=0$ for all $y\in Y$), then $\Lambda_{X,\mathfrak{l}}=\Lambda_{Y,\mathfrak{m}}$.*

Remark 5.2.15. *The assumption of this corollary is fulfilled in particular if $f:\widehat{X}\to\widehat{Y}$ admits a right adjoint g, in other words if there exists a map of posets $g:Y\to X$ such that $f(x)\le y\Leftrightarrow x\le g(y)$ for any $x\in X$ and $y\in Y$, i.e. equivalently if $f\circ g(y)\le y$ and $g\circ f(x)\le x$ for any $x\in X$ and any $y\in Y$.*

Let $(X,\mathfrak{l})$ be a C-monomial G-set. Then we can rewrite the Lefschetz invariant of $(X,\mathfrak{l})$ as

$$\Lambda_{(X,\mathfrak{l})}=\sum_{x_0<\ldots<x_n\in_G X}(-1)^n\big[G_{x_0,\ldots x_n},\,\mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}(\mathfrak{l}_{x_0})\big]_G$$

$$=\sum_{(V,\nu)\in_G\mathrm{ch}(G)}\gamma^{X,\mathfrak{l}}_{V,\nu}[V,\nu]_G$$

where

$$\gamma^{X,\mathfrak{l}}_{V,\nu}=\sum_{\substack{x_0<\ldots<x_n\in_G X\\ (G_{x_0,\ldots,x_n},\,\mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}\mathfrak{l}_{x_0})=_G(V,\nu)}}(-1)^n=\frac{1}{|}N_G(V,\nu):V|\sum_{\substack{x_0<\ldots<x_n\in X\\ (G_{x_0,\ldots,x_n},\,\mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}\mathfrak{l}_{x_0})=(V,\nu)}}(-1)^n.$$

Let (U,μ) be a subcharacter of G. The set $(X,\mathfrak{l})^{U,\mu}$ will be

$$(X,\mathfrak{l})^{U,\mu}=\{x\in X^U\mid \mathrm{Res}^{G_x}_U\mathfrak{l}_x=\mu\}$$

Then we obtain

$$\chi\big((X,\mathfrak{l})^{U,\mu}\big)=\sum_{\substack{n\in\mathbb{N}\\ x_0<\ldots<x_n\in X^U\\ \mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}\mathfrak{l}_{x_0}=\mu}}(-1)^n=\sum_{\substack{(V,\nu)\in\mathrm{ch}(G)\\ U\subseteq V\\ \mathrm{Res}^V_U\nu=\mu}}m^{X,\mathfrak{l}}_{V,\nu}$$

where

$$m_{V,\nu}^{X,\mathfrak{l}} = \sum_{\substack{n\in\mathbb{N} \\ x_0<\ldots<x_n\in X \\ (G_{x_0,\ldots,x_n},\, \mathrm{Res}^{G_{x_0}}_{G_{x_0,\ldots,x_n}}\mathfrak{l}_{x_0})=(V,\nu)}} (-1)^n.$$

So $|N_G(V,\nu):V|m_{V,\nu}^{X,\mathfrak{l}} = \gamma_{V,\nu}^{X,\mathfrak{l}}$. We will use this fact to prove the following lemma.

Lemma 5.2.16. *Let $(X,\mathfrak{l})$ and $(Y,\mathfrak{m})$ be C-monomial G-posets then $\Lambda_{(X,\mathfrak{l})} = \Lambda_{(Y,\mathfrak{m})}$ if and only if $\chi\big((X,\mathfrak{l})^{U,\mu}\big) = \chi\big((Y,\mathfrak{m})^{U,\mu}\big)$ for every C-subcharacter (U,μ) of G.*

Proof. Assume $\Lambda_{(X,\mathfrak{l})} = \Lambda_{(Y,\mathfrak{m})}$. We get

$$\sum_{(V,\nu)\in_G \mathrm{ch}(G)} \gamma_{V,\nu}^{X,\mathfrak{l}}[V,\nu]_G = \sum_{(V,\nu)\in_G \mathrm{ch}(G)} \gamma_{V,\nu}^{Y,\mathfrak{m}}[V,\nu]_G$$

$$\sum_{(V,\nu)\in_G \mathrm{ch}(G)} (\gamma_{V,\nu}^{X,\mathfrak{l}} - \gamma_{V,\nu}^{Y,\mathfrak{m}})[V,\nu]_G = 0.$$

So $\gamma_{V,\nu}^{X,\mathfrak{l}} = \gamma_{V,\nu}^{Y,\mathfrak{m}}$ and then $m_{V,nu}^{X,\mathfrak{l}} = m_{V,nu}^{Y,\mathfrak{m}}$ for every C-subcharacter (V,ν) of G. Then we obtain

$$\sum_{(U,\mu)\leq(V,\nu)\in_G \mathrm{ch}(G)} m_{V,\nu}^{X,\mathfrak{l}} = \sum_{(U,\mu)\leq(V,\nu)\in_G \mathrm{ch}(G)} m_{V,\nu}^{Y,\mathfrak{m}}.$$

Then for any C-subcharacter (U,μ) of G, we have $\chi\big((X,\mathfrak{l})^{U,\mu}\big) = \chi\big((Y,\mathfrak{m})^{U,\mu}\big)$.

Now assume that given any C-subcharacter (U,μ) of G we have $\chi\big((X,\mathfrak{l})^{U,\mu}\big) = \chi\big((Y,\mathfrak{m})^{U,\mu}\big)$. Then we obtain

$$\sum_{(U,\mu)\leq(V,\nu)\in \mathrm{ch}(G)} m_{V,\nu}^{X,\mathfrak{l}} = \sum_{(U,\mu)\leq(V,\nu)\in \mathrm{ch}(G)} m_{V,\nu}^{Y,\mathfrak{m}},$$

$$\sum_{(U,\mu)\leq(V,\nu)\in \mathrm{ch}(G)} (m_{V,\nu}^{X,\mathfrak{l}} - m_{V,\nu}^{Y,\mathfrak{m}}) = 0.$$

Let z denote the matrix with coefficients

$$z(U,\mu;V,\nu) = \lfloor (U,\mu)\leq(V,\nu)\rfloor = \begin{cases} 1 & \text{if } (U,\mu)\leq(V,\nu) \\ 0 & \text{otherwise} \end{cases}$$

for any C-subcharacters (U,μ), (V,ν). If we list the C-subcharacters in non-decreasing order of size of the subgroups, the matrix z is upper triangular with nonzero diagonal coefficients. Thus, z is nonsingular and so $m^{X,\mathfrak{l}}_{V,\nu} = m^{Y,\mathfrak{m}}_{V,\nu}$. This implies $\gamma^{X,\mathfrak{l}}_{V,\nu} = \gamma^{Y,\mathfrak{m}}_{V,\nu}$. We get

$$\Lambda_{(X,\mathfrak{l})} = \sum_{(V,\nu)\in_G \mathrm{ch}(G)} \gamma^{X,\mathfrak{l}}_{V,\nu}[V,\nu]_G = \sum_{(V,\nu)\in_G \mathrm{ch}(G)} \gamma^{Y,\mathfrak{m}}_{V,\nu}[V,\nu]_G = \Lambda_{(Y,\mathfrak{m})}.$$

This proves the lemma. □

5.3 Generalized tensor induction

Let G and H be finite groups. A C-monomial $(G\times H)$-set (U,λ) will be called a C-monomial (G,H)-biset, and usually denoted by U_λ for simplicity.

Let U_λ be a C-monomial $(G\times H)$-set. Given elements u and u' in U, the set of morphisms from u to u' in $\widehat{U}$ is

$$\mathrm{Hom}_{\widehat{U}}(u,u') = \{(g,h)\in G\times H \mid gu = u'h\}.$$

Given $(g,h) \in \mathrm{Hom}_{\widehat{U}}(u,u')$, the image of (g,h) under λ will be denoted by $\lambda(g,h,u,u')$.

Let U_λ be a C-monomial (G,H)-biset and V_ρ be a C-monomial (H,K)-biset. Consider the set

$$U_\lambda \circ V_\rho = \{(u,v)\in U\times V \mid \forall h\in H_u\cap H_v,\ \lambda(1,h,u,u)\rho(h,1,v,v) = 1\}.$$

Then we obtain an H-set $U_\lambda \circ V_\rho$ with the action

$$\forall (u,v)\in U_\lambda\circ V_\rho,\ \forall h\in H,\ h(u,v) = (uh^{-1},hv).$$

Indeed, the condition that we impose on $U_\lambda\circ V_\rho$ amounts to saying that given $(u,v)\in U_\lambda\circ V_\rho$, the linear character $\xi_{u,v}: h\mapsto \lambda(1,h,u,u)\rho(h,1,v,v)$ of $H_u\cap H_v$ is trivial. Moreover we have $\xi_{ux,x^{-1}v}(h) = \xi_{u,v}(xhx^{-1}) = 1$ for $x\in H$ and $h\in H_{ux}\cap H_{x^{-1}v}$, i.e. $xhx^{-1}\in H_u\cap H_v$.

Now we set some notations. The set of H-orbits on $U_\lambda\circ V_\rho$ will be denoted by $U_\lambda \circ_H V_\rho$ and the H-orbit containing (u,v) will be denoted by $(u,_H v)$. The set

$U_\lambda \circ_H V_\rho$ is (G,K)-biset with the action

$$(u,_H v) \in U_\lambda \circ_H V_\rho,\ (g,k) \in G \times K,\ g(u,_H v)k = (gu,_H vk).$$

We consider the pair $(U_\lambda \circ_H V_\rho, \lambda \times \rho)$, where $\lambda \times \rho$ is defined as follows: if $(u,_H v)$ and $(u',_H v') \in U_\lambda \circ_H V_\rho$ and $(g,k) \in G \times K$ are such that $g(u,_H v) = (u',_H v')k$, then there exists $h \in H$ such that $gu = u'h$ and $hv = v'k$. This element h need not be unique, but it is well defined up to multiplication on the right by an element of $H_u \cap H_v$. We set

$$(\lambda \times \rho)\big(g, k, (u,_H v), (u',_H v'\big) = \lambda(g,h,u,u')\rho(h,k,v,v'),$$

which does not depend on the choice of h, by the defining property of $U_\lambda \circ V_\rho$. It's not hard to show that $(U_\lambda \circ_H V_\rho, \lambda \times \rho)$ is a C-monomial (G,K)-biset. Note that if V is left free or λ and ρ are both equal to the trivial functor we have $U_\lambda \circ_H V_\rho = U \times_H V$.

Let G and H be finite groups. Let (U,λ) be a C-monomial (G,H)-biset. Let $(X,\mathfrak{l})$ be a C-monomial G-poset. The set of G-equivariant maps $f : U \to X$ such that

$$\mathfrak{l}\big(g, f(u), f(u)\big) = \lambda(g,1,u,u)$$

for all $u \in U$ and $g \in G_u$ will be denoted by $t_{U,\lambda}(X,\mathfrak{l})$. The set $t_{U,\lambda}(X,\mathfrak{l})$ is an H-poset with the action $(hf)(u) = f(uh)$, for any $h \in H$, for any $f \in t_{U,\lambda}(X,\mathfrak{l})$, for any $u \in U$. We define an order $\le$ on $t_{U,\lambda}(X,\mathfrak{l})$ as follows:

$$\forall f, f' \in t_{U,\lambda}(X,\mathfrak{l}),\ f \le f' \Leftrightarrow \forall u \in U, f(u) \le f'(u) \,\text{in}\, X.$$

We want to define a functor $\mathfrak{L}_{U,\lambda} : \widehat{t_{U,\lambda}(X,\mathfrak{l})} \to \bullet_C$. Given any elements f, and f' in $t_{U,\lambda}(X,\mathfrak{l})$ and $h \in H$ such that $hf \le f'$, we choose a set $[G\backslash U]$ of representatives of G-orbits of U. Let u be an element of U. Then there exist some $g_{h,u} \in G$ and a unique $\sigma_h(u) \in [G\backslash U]$ such that

$$uh = g_{h,u}\sigma_h(u).$$

We obtain $g_{h,u} f\big(\sigma_h(u)\big) \le f'(u)$ because $hf \le f'$. Then we set

$$\mathfrak{L}_{U,\lambda}(h,f,f') = \prod_{u \in [G\backslash U]} \mathfrak{l}\Big(g_{h,u}, f\big(\sigma_h(u)\big), f'(u)\Big)\lambda^{-1}\big(g_{h,u}, h, \sigma_h(u), u\big).$$

Now we need to show that this definition does not depend on the choice of $g_{h,u}$. Assume that there exist $g_{h,u}$, $g'_{h,u} \in G$ such that

$$uh = g_{h,u}\sigma_h(u) = g'_{h,u}\sigma_h(u).$$

So there exists $w \in G_{\sigma_h(u)}$ such that $g_{h,u} = g'_{h,u}w$. We get

$$\mathfrak{l}\Big(w, f\big(\sigma_h(u)\big), f\big(\sigma_h(u)\big)\Big) = \lambda\big(w, 1, \sigma_h(u), \sigma_h(u)\big).$$

Furthermore, we get the following commutative diagram:

$$\begin{tikzcd}
\sigma_h(u) \arrow[rr, "w"] \arrow[dr, "g_{h,u}"'] & & \sigma_h(u) \arrow[dl, "g'_{h,u}"] \\
& u &
\end{tikzcd}$$

Thus,

$$\begin{aligned}
\mathfrak{L}_{U,\lambda}(h, f, f') &= \prod_{u\in[G\backslash U]} \mathfrak{l}\Big(g_{h,u}, f\big(\sigma_h(u)\big), f'\big(u\big)\Big)\lambda^{-1}\big(g_{h,u}, h, \sigma_h(u), u\big) \\
&= \prod_{u\in[G\backslash U]} \mathfrak{l}\Big(g'_{h,u}w, f\big(\sigma_h(u)\big), f'\big(u\big)\Big)\lambda^{-1}\big(g'_{h,u}w, h, \sigma_h(u), u\big) \\
&= \prod_{u\in[G\backslash U]} \mathfrak{l}\Big(g'_{h,u}, f\big(\sigma_h(u)\big), f'\big(u\big)\Big)\lambda^{-1}\big(g'_{h,u}, h, \sigma_h(u), u\big).
\end{aligned}$$

Definition 5.3.1. *The above construction* $T_{U,\lambda} : (X, \mathfrak{l}) \mapsto \big(t_{U,\lambda}(X, \mathfrak{l}), \mathfrak{L}_{U,\lambda}\big)$ *is called the* generalized tensor induction for C-monomial G-posets*, associated to* (U, λ).

Lemma 5.3.2. *Let* G *and* K *be finite groups and* U *be a* (G, K)*-biset. Then there exists a bijection between the sets* $\{(u, t) \mid u \in [G\backslash U/K],\ t \in [(K \cap G^u)\backslash K]\}$ *and* $[G\backslash U]$.

Proof. Given $u \in [G\backslash U/K]$ and $t \in [(K \cap G^u)\backslash K]$, there exist some $g_{t,u} \in G$ and a unique $\sigma_t(u) \in [G\backslash U]$ such that

$$ut = g_{t,u}\sigma_t(u).$$

We define a bijection $\psi : \{(u, t) \mid u \in [G\backslash U/K],\ t \in [(K \cap G^u)\backslash K]\} \to [G\backslash U]$ such that $\psi(u, t) = \sigma_t(u)$. $\square$

Lemma 5.3.3. *Let G and H be finite groups, (U,λ) be a monomial (G,H)-biset and $(X,\mathfrak{l})$ be a C-monomial G-poset.*

1. *$\big(t_{U,\lambda}(X,\mathfrak{l}),\mathfrak{L}_{U,\lambda}\big)$ is a C-monomial H-poset.*

2. *$\big(t_{U,\lambda}(X,\mathfrak{l}),\mathfrak{L}_{U,\lambda}\big)$ does not depend on the choice of representative set $[G\backslash U]$, up to isomorphism.*

Proof. 1. We need to show that $\mathfrak{L}_{U,\lambda} : \widehat{t_{U,\lambda}(X,\mathfrak{l})} \to \bullet_C$ is a functor. Let $u \in [G\backslash U]$. Given $h,\, h' \in H$ and $f,\, f',\, f'' \in t_{U,\lambda}(X,\mathfrak{l})$ such that $hf \le f'$ and $h'f' \le f''$, there exist some $g_{h,u}$, $g_{h',u}$, $g_{h'h,u}$ in G and unique elements $\sigma_h(u)$, $\sigma_{h'}(u)$, $\sigma_{h'h}(u)$ in $[G\backslash U]$ such that

$$uh = g_{h,u}\sigma_h(u),\ uh' = g_{h',u}\sigma_{h'}(u),\ uh'h = g_{h'h,u}\sigma_{h'h}(u).$$

Also there exist some $g_{h,\sigma_{h'}(u)} \in G$ and a unique $\sigma_h\big(\sigma_{h'}(u)\big) \in [G\backslash U]$ such that

$$\sigma_{h'}(u)h = g_{h,\sigma_{h'}(u)}\sigma_h\big(\sigma_{h'}(u)\big).$$

Then we obtain

$$uh'h = g_{h',u}g_{h,\sigma_{h'}(u)}\sigma_h\big(\sigma_{h'}(u)\big)$$

and

$$\sigma_{h'h}(u) = \sigma_h\big(\sigma_{h'}(u)\big).$$

Then there exists $w \in G_{\sigma_{h'h}(u)}$ such that

$$g_{h'h,u} = g_{h',u}g_{h,\sigma_{h'}(u)}w.$$

We obtain the following commutative diagram:

$$\begin{tikzcd}
\sigma_{h'h}(u) \arrow[rr, "w"] \arrow[dr, "g_{h'h,u}"'] & & \sigma_{h'h}(u) \arrow[dl, "g_{h',u}g_{h,\sigma_{h'}(u)}"] \\
& uh'h &
\end{tikzcd}$$

Then we get

$$\mathfrak{l}\Big(w, f\big(\sigma_{h'h}(u)\big), f\big(\sigma_{h'h}(u)\big)\Big) = \lambda\big(w, 1, \sigma_{h'h}(u), \sigma_{h'h}(u)\big)$$

because $w \in G_{\sigma_{h'h}(u)}$. We set $L = \mathfrak{L}_{U,\lambda}(h'h, f, f'')$ and obtain

$$L = \prod_{u \in [G \backslash U]} \mathfrak{l}\Big(g_{h'h,u}, f\big(\sigma_{h'h}(u)\big), f''\big(u\big)\Big)\lambda^{-1}\big(g_{h'h,u}, h'h, \sigma_{h'h}(u), u\big)$$

$$= \prod_{u \in [G \backslash U]} \mathfrak{l}\Big(g_{h',u}g_{h,\sigma_{h'}(u)}w, f\big(\sigma_{h'h}(u)\big), f''\big(u\big)\Big)\lambda^{-1}\big(g_{h',u}g_{h,\sigma_{h'}(u)}w, h'h, \sigma_{h'h}(u), u\big)$$

$$= \prod_{u \in [G \backslash U]} \mathfrak{l}\Big(g_{h',u}g_{h,\sigma_{h'}(u)}, f\big(\sigma_{h'h}(u)\big), f''\big(u\big)\Big)\lambda^{-1}\big(g_{h',u}g_{h,\sigma_{h'}(u)}, h'h, \sigma_{h'h}(u), u\big)$$

$$= \mathfrak{L}(h', f', f'')\mathfrak{L}(h, f, f').$$

Now let $f \in T_{U,\lambda}(X, \mathfrak{l})$. Then we get

$$\mathfrak{L}(1, f, f) = \prod_{u \in [G \backslash U]} \mathfrak{l}\big(1, f(u), f(u)\big)\lambda^{-1}(1, 1, u, u) = 1.$$

Thus, $\mathfrak{L}_{U,\lambda} : \widehat{t_{U,\lambda}(X, \mathfrak{l})} \to \bullet_C$ is a functor.

2. Let $h \in H$ and $f,\ f' \in t_{U,\lambda}(X, \mathfrak{l})$ such that $hf \leq f'$. Let $S = [G \backslash U]$ and let S' be the another choice of representatives. For any $u' \in S'$, there exist some $a_u \in G$, and a unique $u \in S$ such that $u' = a_u u$. Then there exist some $g_{h,a_u u}$, $g_{h,u} \in G$, a unique $\sigma'_h(a_u u) \in S'$, and a unique $\sigma_h(u) \in S$ such that

$$a_u uh = g_{h,a_u u}\sigma'_h(a_u u)$$

and

$$uh = g_{h,u}\sigma_h(u).$$

So

$$a_u uh = a_u g_{h,u}\sigma_h(u) = a_u g_{h,u} a^{-1}_{\sigma_h(u)} a_{\sigma_h(u)}\sigma_h(u).$$

Then $\sigma'_h(a_u u) = a_{\sigma_h(u)}\sigma_h(u)$. Note that $a_{\sigma_h(u)}\sigma_h(u) \in S'$. We obtain the following commutative diagram:

$$\begin{tikzcd}
a_{\sigma_h(u)} f\big(\sigma_h(u)\big) \arrow[r, "a_u g_{h,u} a^{-1}_{\sigma_h(u)}"] \arrow[d, "a^{-1}_{\sigma_h(u)}"'] & a_u f'\big(u\big) \\
f\big(\sigma_h(u)\big) \arrow[r, "g_{h,u}"'] & f(u). \arrow[u, "a_u"']
\end{tikzcd}$$

We set $L = \mathfrak{L}'_{U,\lambda}(h, f, f')$ and get

$$L= \prod_{a_u u \in S'} \mathfrak{l}\Big(a_u g_{h,u} a^{-1}_{\sigma_h(u)}, f\big(a_{\sigma_h(u)}\sigma_h(u)\big), f'\big(a_u u\big)\Big)\lambda^{-1}\big(a_u g_{h,u} a^{-1}_{\sigma_h(u)}, h, a_{\sigma_h(u)}\sigma_h(u), a_u u\big)$$
$$= \mathfrak{L}'_{U,\lambda}(h, f, f') = \mathfrak{L}_{U,\lambda}(h, f, f')\alpha_{f'}\alpha_f^{-1}$$

where

$$\alpha_{f'} = \prod_{u\in S} \mathfrak{l}\big(a_u, f'(u), a_u f'(u)\big)\lambda^{-1}\big(a_u, 1, u, a_u u\big)$$

and

$$\alpha_f^{-1} = \prod_{u\in S} \mathfrak{l}\big(a_u^{-1}, a_u f(u), f(u)\big)\lambda^{-1}\big(a_u^{-1}, 1, a_u u, u\big).$$

□

Proposition 5.3.4. *Let G and H be finite groups and (U, λ) be a C-monomial (G, H)-biset.*

1. *Let $(X, \mathfrak{l})$, $(X', \mathfrak{l}')$ be C-monomial G-posets then*

$$T_{U,\lambda}\big((X, \mathfrak{l}) \times (X', \mathfrak{l}')\big) \cong T_{U,\lambda}(X, \mathfrak{l}) \times T_{U,\lambda}(X', \mathfrak{l}').$$

2. $T_{U,\lambda} : {}_C MG\text{-}\mathsf{poset} \to {}_C MH\text{-}\mathsf{poset}$ *is a functor.*

Proof. 1. is clear.

2. Given a map of C-monomial G-posets $(\varphi, \beta) : (X, \mathfrak{l}) \to (Y, \mathfrak{m})$, we define a map of C-monomial G-posets

$$\big(T_{U,\lambda}(\varphi), T_{U,\lambda}(\beta)\big) : \big(t_{U,\lambda}(X, \mathfrak{l}), \mathfrak{L}\big) \to \big(t_{U,\lambda}(Y, \mathfrak{m}), \mathfrak{M}\big)$$

where

$$T_{U,\lambda}(\varphi) : t_{U,\lambda}(X, \mathfrak{l}) \to t_{U,\lambda}(Y, \mathfrak{m})$$

such that $T_{U,\lambda}(\varphi)(f) = \varphi \circ f$ and

$$T_{U,\lambda}(\beta) : \mathfrak{L}(f) \to \mathfrak{M} \circ T_{U,\lambda}(\varphi)(f)$$

such that

$$T_{U,\lambda}(\beta) = \prod_{u\in[G\backslash U]} \beta_{f(u)}$$

for any $f \in t_{U,\lambda}(X,\mathfrak{l})$. Obviously, $\varphi \circ f : U \to X \to Y$ is a map of G-posets. Let $g \in G_u$ and $u \in U$. Then the naturality of $\beta : \mathfrak{l} \to \mathfrak{m} \circ \varphi$ gives the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{l}\big(f(u)\big) \arrow[r, "\beta_{f(u)}"] \arrow[d, "\mathfrak{l}\left(g,f(u),f(u)\right)"'] & \mathfrak{m}\circ\varphi\big(f(u)\big) \arrow[d, "\mathfrak{m}\left(g,\varphi\circ f(u),\varphi\circ f(u)\right)"] \\
\mathfrak{l}\big(f(u)\big) \arrow[r, "\beta_{f(u)}"'] & \mathfrak{m}\circ\varphi\big(f(u)\big).
\end{tikzcd}$$

Then

$$\beta_{f(u)}\mathfrak{l}\Big(g, f(u), f(u)\Big) = \mathfrak{m}\Big(g, \varphi\big(f(u)\big), \varphi\big(f(u)\big)\Big)\beta_{f(u)}.$$

Note that $g \in G_{f(u)}$ so

$$\mathfrak{l}\Big(g, f(u), f(u)\Big) = \lambda(g, 1, u, u).$$

So we obtain

$$\mathfrak{m}\Big(g, \varphi\big(f(u)\big), \varphi\big(f(u)\big)\Big) = \lambda(g, 1, u, u).$$

Thus, $\varphi \circ f \in t_{U,\lambda}(Y, \mathfrak{m})$. Now we will show that

$$T_{U,\lambda}(\beta) : \mathfrak{L} \to \mathfrak{M} \circ T_{U,\lambda}(\varphi)$$

is a natural transformation. Given $f,\ f' \in t_{U,\lambda}(X,\mathfrak{l})$ and $h \in H$ such that $hf \leq f'$, we need to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{L}\big(f\big) \arrow[r, "T_{U,\lambda}(\beta)_f"] \arrow[d, "\mathfrak{L}(h,f,f')"'] & \mathfrak{M}\circ T_{U,\lambda}(\varphi)(f) \arrow[d, "\mathfrak{M}(h,\varphi\circ f,\varphi\circ f')"] \\
\mathfrak{L}\big(f'\big) \arrow[r, "T_{U,\lambda}(\beta)_{f'}"'] & \mathfrak{M}\circ T_{U,\lambda}(\varphi)(f').
\end{tikzcd}$$

Given $u \in [G\backslash U]$, then there exist some $g_{h,u} \in G$ and a unique $\sigma_h(u) \in [G\backslash U]$ such that

$$uh = g_{h,u}\sigma_h(u).$$

But $\beta : \mathfrak{l} \to \mathfrak{m} \circ \varphi$ is a natural transformation, so we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{l}\Big(f\big(\sigma_h(u)\big)\Big) \arrow[r, "\beta_{f(\sigma_h(u))}"] \arrow[d, "\mathfrak{l}\left(g_{h,u},\, f(\sigma_h(u)),\, f'(u)\right)"'] & \mathfrak{m} \circ \varphi\Big(f\big(\sigma_h(u)\big)\Big) \arrow[d, "\mathfrak{m}\left(g_{h,u},\, \varphi\left(f\left(\sigma_h(u)\right)\right),\, \varphi\left(f'\left(u\right)\right)\right)"] \\
\mathfrak{l}\big(f'(u)\big) \arrow[r, "\beta_{f'(u)}"'] & \mathfrak{m} \circ \varphi\big(f'(u)\big).
\end{tikzcd}$$

We set $T = T_{U,\lambda}(\beta)_{f'} \circ \mathfrak{L}(h, f, f')$, and by the commutativity of the above diagram we obtain

$$\begin{aligned}
T &= T_{U,\lambda}(\beta)_{f'}\Bigg(\prod_{u\in[G\backslash U]} \mathfrak{l}\Big(g_{h,u}, f\big(\sigma_h(u)\big), f'\big(u\big)\Big)\lambda^{-1}\big(g_{h,u}, h, \sigma_h(u), u\big)\Bigg) \\
&= \prod_{u\in[G\backslash U]} \beta_{f'(u)}\mathfrak{l}\Big(g_{h,u}, f\big(\sigma_h(u)\big), f'(u)\Big)\lambda^{-1}\big(g_{h,u}, h, \sigma_h(u), u\big) \\
&= \prod_{u\in[G\backslash U]} \mathfrak{m}\bigg(g_{h,u}, \varphi\Big(f\big(\sigma_h(u)\big)\Big), \varphi\Big(f'\big(u\big)\Big)\bigg)\lambda^{-1}\big(g_{h,u}, h, \sigma_h(u), u\big)\beta_{f\left(\sigma_h(u)\right)} \\
&= \mathfrak{M}(h, \varphi \circ f, \varphi \circ f')\beta_f.
\end{aligned}$$

So $T_{U,\lambda}(\beta) : \mathfrak{L} \to \mathfrak{M} \circ T_{U,\lambda}(\varphi)$ is a natural transformation. Thus,

$$\big(T_{U,\lambda}(\varphi), T_{U,\lambda}(\beta)\big) : \big(t_{U,\lambda}(X, \mathfrak{l}), \mathfrak{L}\big) \to \big(t_{U,\lambda}(Y, \mathfrak{m}), \mathfrak{M}\big)$$

is a map of C-monomial G-posets. $\square$

Lemma 5.3.5. *Let G, H and K be finite groups. If U is a (G, H)-biset and V is a left free (H, K)-biset, then the map $(u, v) \in U \times V \mapsto (u,_H v) \in U \times_H V$ restricts to a bijection $\pi : [G\backslash U] \times [H\backslash V] \to [G\backslash(U \times_H V)]$, where brackets denote sets of representatives of orbits.*

Proof. Given $(u, v) \in U \times V$, there exists $v_0 \in [G\backslash V]$ and $h \in H$ such that $v = hv_0$. Then there exists $u_0 \in [G\backslash U]$ and $g \in G$ such that $uh = gu_0$. We get $(u,_H v) = g(u_0,_H v_0)$ and so π is surjective. Let (u_0, v_0) and (u_1, v_1) are pairs in $[G\backslash U] \times [H\backslash V]$ lying in the same G=orbit. Then there exists $g \in G$ and $h \in H$ such that $(gu_0, v_0) = (u_1h^{-1}, hv_1)$. We get $hv_1 = v_0$, so $v_0 = v_1 = hv_1$, and $h = 1$ since H act freely on V. Now $gu_0 = u_1$, so $u_0 = u_1$, and π is injective. $\square$

Proposition 5.3.6. *Let G, H and K be finite groups.*

1. *Let $(\bullet, 1)$ be the C-monomial G-poset where $\bullet$ is G-poset with one element and $1 : \bullet \to \bullet_C$ is the functor such that $1(g, \bullet, \bullet) = 1$. Then $T_{U,\lambda}(\bullet, 1) = (\bullet, 1)$.*

2. *Let $(\emptyset, z)$ be the empty C-monomial (G, H)-poset. Then $T_{\emptyset,z}$ is the constant functor with value $(\bullet, 1)$.*

3. *Let (U, λ) and (U', λ') be C-monomial (G, H)-bisets and let $(X, \mathfrak{l})$ be a C-monomial G-poset then*

$$T_{U\sqcup U',\lambda\sqcup\lambda'}(X, \mathfrak{l}) = T_{U,\lambda}(X, \mathfrak{l})T_{U',\lambda'}(X, \mathfrak{l}).$$

4. *Let id_G stand for the identity (G, G)-biset. Then $T_{\mathrm{id}_G,1}(X, \mathfrak{l}) = (X, \mathfrak{l})$ for any C-monomial G-poset $(X, \mathfrak{l})$.*

5. *Let (V, ρ) be a C-monomial left free (H, K)-biset, and (U, λ) be a C-monomial (H, G)-biset. Then*

$$T_{V,\rho} \circ T_{U,\lambda} = T_{U\times_H V,\lambda\times\rho}.$$

Proof. 1., 2., 3. and 4. are clear.

5. We have $U_\lambda \circ_H V_\rho \cong {}_GU \times_H V_K$ because V is left free. For any C-monomial G-poset $(X, \mathfrak{l})$ we need to show that

$$\Big(t_{V,\rho}\big(t_{U,\lambda}(X, \mathfrak{l}), \mathfrak{L}_{U,\lambda}\big), \mathfrak{L}_{V,\rho} \circ \mathfrak{L}_{U,\lambda}\Big) = \big(t_{U\times_H V,\lambda\times\rho}(X, \mathfrak{l}), \mathfrak{L}_{U\times_H V,\lambda\times\rho}\big).$$

We set $\varphi : t_{V,\rho}\big(t_{U,\lambda}(X, \mathfrak{l}), \mathfrak{L}_{U,\lambda}\big) \to t_{U\times_H V,\lambda\times\rho}(X, \mathfrak{l})$ such that

$$\varphi(f)(u, {}_Hv) = f(v)(u)$$

for any $f \in t_{V,\rho}\big(t_{U,\lambda}(X, \mathfrak{l}), \mathfrak{L}_{U,\lambda}\big)$ and $(u, {}_Hv) \in U \times_H V$. Obviously, the map, $\varphi(f)$ is a map of G-posets and φ is a map of K-posets. Now consider an element g of $G_{(u,{}_Hv)}$ then we get $g \in G_u$ because V is H-free. We obtain

$$\mathfrak{l}\Big(g, \varphi\big(f\big)(u,{}_H v), \varphi\big(f\big)(u,{}_H v)\Big) = \mathfrak{l}\Big(g, f(v)(u), f(v)(u)\Big)$$

$$= \lambda(g,1,u,u)\rho(1,1,v,v) = (\lambda\times\rho)\big(g,1,(u,{}_Hv),(u,{}_Hv)\big).$$

and so $\varphi(f)\in t_{U\times_H V,\lambda\times\rho}(X,\mathfrak{l})$. We set

$$\theta : t_{U\times_H V,\lambda\times\rho}(X,\mathfrak{l}) \to t_{V,\rho}\big(t_{U,\lambda}(X,\mathfrak{l}),\mathfrak{L}_{U,\lambda}\big)$$

such that $\theta(t)(v)(u) = t(u,{}_Hv)$ for any $t\in t_{U\times_H V,\lambda\times\rho}(X,\mathfrak{l})$, $u\in U$ and $v\in V$. We need to show that $\theta(t)\in t_{V,\rho}\big(t_{U,\lambda}(X,\mathfrak{l}),\mathfrak{L}_{U,\lambda}\big)$. Obviously, the map $\theta(t)$ is a map of H-sets. Moreover, for any $v\in V$ we have $H_v=1$ because V is H-free. Then

$$\mathfrak{L}_{U,\lambda}\big(1,\theta(t)(v),\theta(t)(v)\big) = 1 = \rho(1,1,v,v).$$

Obviously, the map $\theta(t)(v)$ is a map of G-sets. Then given $g\in G_u$ we get $g\in G_{(u,{}_Hv)}$ and so

$$\mathfrak{l}\Big(g,\theta\big(t\big)(v)(u),\theta\big(t\big)(v)(u)\Big) = \mathfrak{l}\Big(g,t(u,{}_Hv),t(u,{}_Hv)\Big)$$

$$= \lambda(g,1,u,u)\rho(1,1,v,v) = \lambda(g,1,u,u).$$

Thus, $\theta(t)\in t_{V,\rho}\big(t_{U,\lambda}(X,\mathfrak{l}),\mathfrak{L}_{U,\lambda}\big)$. Now we will prove that $\mathfrak{L}_{V,\rho}\circ\mathfrak{L}_{U,\lambda} = \mathfrak{L}_{U\times_H V,\lambda\times\rho}$. Let $k\in K$ and $f,f'\in t_{V,\rho}\big(t_{U,\lambda}(X,\mathfrak{l}),\mathfrak{L}_{U,\lambda}\big)$ such that $kf\le f'$. Then given $v\in[H\backslash V]$, there exist a unique $\sigma_k(v)\in[H\backslash V]$ and some $h_{k,v}\in H$ such that

$$vk = h_{k,v}\sigma_k(v).$$

Now consider an element u of $[G\backslash U]$, then there exist a unique $\sigma_{h_{k,v}}(u)\in[G\backslash U]$ and some $g_{h_{k,v},u}\in G$ such that

$$uh_{k,v} = g_{h_{k,v},u}\sigma_{h_{k,v}}(u).$$

We get

$$(u,{}_Hv) = (uh_{k,v}{h_{k,v}}^{-1},{}_H v) = (uh_{k,v},{}_H {h_{k,v}}^{-1}v)$$

$$= \big(g_{h_{k,v},u}\sigma_{h_{k,v}}(u),{}_H\,\sigma_k(v)k^{-1}\big) = g_{h_{k,v},u}\big(\sigma_{h_{k,v}}(u),{}_H\,\sigma_k(v)\big)k^{-1}.$$

We obtain

$$(u,{}_Hv)k = g_{h_{k,v},u}\big(\sigma_{h_{k,v}}(u),{}_H\,\sigma_k(v)\big).$$

Then

$$\big(\sigma_{h_{k,v}}(u),{}_H\,\sigma_k(v)\big) = \sigma_k(u,{}_Hv)$$

and

$$g_{h_{k,v},u} = g_{k,(u,_H v)} w$$

for some $w \in G_{\sigma_k(u,_H v)}$. We get the following commutative diagram:

$$\begin{tikzcd}
\sigma_k(u,{}_H v) \arrow[rr, "w"] \arrow[dr, "g_{h_{k,v},u}"'] & & \sigma_k(u,{}_H v) \arrow[dl, "g_{k,(u,_H v)}"] \\
& (u,{}_H v)k &
\end{tikzcd}$$

Using the commutativity of the above diagram and Lemma 5.3.5 we get

$$\mathfrak{L}_{V,\rho} \circ \mathfrak{L}_{U,\lambda}(k, f, f') = \prod_{v \in [H \backslash V]} \mathfrak{L}_{U,\lambda}\Big(h_{k,v}, f\big(\sigma_k(v)\big), f'\big(v\big)\Big)\rho^{-1}\big(h_{k,v}, k, \sigma_k(v), v\big)$$

$$= \prod_{\substack{u \in [G\backslash U] \\ v \in [H \backslash V]}} \mathfrak{l}\Big(g_{h_{k,v},u}, f\big(\sigma_k(v)\big)(\sigma_{h_{k,v}}(u)), f'(v)(u)\Big)$$
$$\lambda^{-1}\big(g_{h_{k,v},u}, h_{k,v}, \sigma_{h_{k,v}}(u), u\big)\rho^{-1}\big(h_{k,v}, k, \sigma_k(v), v\big)$$

$$= \prod_{(u,_H v) \in [G \backslash (U \times_H V)]} \mathfrak{l}\Big(g_{h_{k,v},u}, f\big(\sigma_{h_{k,v}}(u),_H \sigma_k(v)\big), f'\big(u,{}_H v\big)\Big)$$
$$(\lambda \times \rho)^{-1}\Big(g_{h_{k,v},u}, k, \big(\sigma_{h_{k,v}}(u),{}_H \sigma_k(v)\big), (u,{}_H v)\Big)$$

$$= \prod_{(u,_H v) \in [G \backslash (U \times_H V)]} \mathfrak{l}\Big(g_{k,(u,_H v)} w, f\big(\sigma_k(u,{}_H v)\big), f'\big(u,{}_H v\big)\Big)$$
$$(\lambda \times \rho)^{-1}\Big(g_{k,(u,_H v)} w, k, \big(\sigma_{h_{k,v}}(u),{}_H \sigma_k(v)\big), (u,{}_H v)\Big)$$

$$= \prod_{(u,_H v) \in [G \backslash (U \times_H V)]} \mathfrak{l}\Big(\, g_{k,(u,_H v)}, f\big(\sigma_k(u,{}_H v)\big), f'\big(u,{}_H v\big)\Big)$$
$$(\lambda \times \rho)^{-1}\Big(g_{k,(u,_H v)}, k, \big(\sigma_{h_{k,v}}(u),{}_H \sigma_k(v)\big), (u,{}_H v)\Big)$$

$$= \mathfrak{L}_{U \times_H V}(X, \mathfrak{l}).$$

$\square$

Remark 5.3.7. *The following example shows that the assumption that V is left free seems to be necessary for Assertion 5. Suppose that $H = N \rtimes K$ is a semidirect product of a normal subgroup N with K. Let G be the group K, viewed as a subgroup of H. Let moreover U be the set H, viewed as a (G, H)-biset by left* *and right multiplication, and let V be the set K, acted on by K on the right by multiplication, and by H on the left by projection to $K = H/N$, followed by multiplication in K. Let moreover λ and ρ be equal to the trivial functor on $\widehat{U}$ and $\widehat{V}$, respectively.*

Then $U_\lambda \circ_H V_\rho = U \times_H V$, as λ and ρ are both trivial. Moreover $U \times_H V = G \times_H K$ is equal to the identity (K, K)-biset (this makes sense since $G = K$), so $T_{U\times_H V, \lambda\times\rho} = T_{\mathrm{id}_K,1}$ is the identity functor, by Assertion 4.

On the other hand $G\backslash U = K\backslash(NK) \cong N$, and $H\backslash V$ has cardinality 1. So in the computation of the functor $\mathcal{L}_{U,1}$ appearing in $T_{U,1}(X, \mathfrak{l})$, we have a product of values of $\mathfrak{l}$, indexed by N. So the composition $T_{V,1} \circ T_{U,1}$ cannot act in general as the identity on $(X, \mathfrak{l})$, if N is non trivial. Hence $T_{V,\lambda} \circ T_{U,\rho} \neq T_{U\times_H V,\lambda\times\rho}$ in this situation.

Remark 5.3.8. *Let G and H be finite groups, and U be a (finite) (G, H)-biset. Then one can check that the diagram*

$$\begin{tikzcd} G\text{-poset} \arrow[r, "\tau_G"] \arrow[d, "T_U"] & {}_C MG\text{-poset} \arrow[d, "T_{U,1_U}"] \\ H\text{-poset} \arrow[r, "\tau_H"] & {}_C MH\text{-poset} \end{tikzcd}$$

of categories and functors is commutative, up to isomorphism, where the functor T_U on the left is the usual generalized tensor induction functor for G-posets.

Lemma 5.3.9. *Let G and H be finite groups, and let (U, λ) be a C-monomial (G, H)-biset. Then there exists a unique map*

$$\mathcal{T}_{U,\lambda} : B_C(G) \to B_C(H)$$

such that $\mathcal{T}_{U,\lambda}(\Lambda_{(X,\mathfrak{l})}) = \Lambda_{T_{U,\lambda}(X,\mathfrak{l})}$ for any finite C-monomial G-poset $(X, \mathfrak{l})$.

Proof. Let $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$ be finite C-monomial G-posets. We show that if $\Lambda_{(X,\mathfrak{l})} = \Lambda_{(Y,\mathfrak{m})}$ in $B_C(G)$, then $\Lambda_{T_{U,\lambda}(X,\mathfrak{l})} = \Lambda_{T_{U,\lambda}(Y,\mathfrak{m})}$ in $B_C(H)$. By Lemma 5.2.16, it's enough to show that for any (K, θ) of $\mathrm{ch}(G)$ we have $\chi\Big(T_{U,\lambda}\big((X, \mathfrak{l})\big)^{K,\theta}\Big) = \chi\Big(T_{U,\lambda}\big((Y, \mathfrak{m})\big)^{K,\theta}\Big)$. Now consider elements $u \in [G\backslash U/K]$, $k \in K$ and $t \in [K \cap$

$G^u\backslash K]$ then there exist a unique $\sigma_k(ut) \in [G\backslash U]$ and some $g_{k,ut} \in G$ such that

$$utk = g_{k,ut}\sigma_k(ut).$$

Also there exist some $c_{k,t} \in K \cap G^u$ and a unique $\tau_k(t) \in [K \cap G^u\backslash K]$ such that

$$tk = c_{k,t}\tau_k(t).$$

Since $c_{k,t} \in K \cap G^u$, there exists $\gamma_{k,t,u} \in G$ such that

$$uc_{k,t} = \gamma_{k,t,u}u.$$

Then

$$utk = uc_{k,t}\tau_k(t) = \gamma_{k,t,u}u\tau_k(t) = g_{k,ut}\sigma_k(ut).$$

So $\sigma_k(ut) = u\tau_k(t)$ and there exists $w \in G_{\sigma_k(ut)}$ such that $g_{k,ut} = \gamma_{k,t,u}w$.Then we obtain the following diagram:

$$\begin{tikzcd}
\sigma_k(ut) \arrow[rr, "w"] \arrow[dr, "g_{k,ut}"'] & & \sigma_k(ut) \arrow[dl, "\gamma_{k,t,u}"] \\
& utk &
\end{tikzcd}$$

Given $f \in t_{U,\lambda}(X,\mathfrak{l})^{K,\theta}$ and $k \in K$ such that $kf = f$, we obtain

$$f(utk) = f(ut) = f(u) = \gamma_{k,t,u}f\big(u\tau_k(t)\big) = \gamma_{k,t,u}f(u),$$

because f is K-fixed. This implies $\gamma_{k,t,u} \in G_{f(u)}$. Hence,

$$f\big(\sigma_k(ut)\big) = f\big(u\tau_k(t)\big) = \gamma_{k,t,u}f(utk) = \gamma_{k,t,u}f(u) = f(u).$$

Setting, $\gamma_{k,u} = \prod\limits_{t\in[K\cap G^u\backslash K]} \gamma_{k,t,u}$ and $\phi_u(k) = \prod\limits_{t\in[K\cap G^u\backslash K]} \lambda^{-1}(\gamma_{k,t,u}, k, \sigma_k(ut), ut)$ we obtain

$$\begin{aligned}
\mathfrak{L}(k,f,f) &= \prod_{u\in[G\backslash U]} \mathfrak{l}\Big(g_{k,u}, f\big(\sigma_k(u)\big), f\big(u\big)\Big)\lambda^{-1}\big(g_{k,u}, k, \sigma_k(u), u\big) \\
&= \prod_{\substack{u\in[G\backslash U/K] \\ t\in[K\cap G^u\backslash K]}} \mathfrak{l}\Big(g_{k,ut}, f\big(\sigma_k(ut)\big), f\big(ut\big)\Big)\lambda^{-1}\big(g_{k,ut}, k, \sigma_k(ut), ut\big) \\
&= \prod_{\substack{u\in[G\backslash U/K] \\ t\in[K\cap G^u\backslash K]}} \mathfrak{l}\Big(\gamma_{k,t,u}w, f\big(\sigma_k(ut)\big), f\big(ut\big)\Big)\lambda^{-1}\big(\gamma_{k,t,u}w, k, \sigma_k(ut), ut\big)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{\substack{u\in[G\backslash U/K]\\ t\in[K\cap G^u\backslash K]}} \mathfrak{l}\Big(\gamma_{k,t,u}, f\big(\sigma_k(ut)\big), f\big(ut\big)\Big)\lambda^{-1}\big(\gamma_{k,t,u}, k, \sigma_k(ut), ut\big)\\
&= \prod_{u\in[G\backslash U/K]} \mathfrak{l}_{f(u)}(\gamma_{k,u})\phi_u(k)\\
&= \theta(k).
\end{aligned}
$$

Let Ξ be the family of the sets $\xi = \{\xi_u\}_{u\in[G\backslash U/K]}$ where $\xi_u : {}^uK \to C$ is a character such that $\operatorname{res}^{G_{f(u)}}_{{}^uK}(\mathfrak{l}_{f(u)}) = \xi_u$ and

$$\theta(k) = \prod_{u\in[G\backslash U/K]} \xi_u(\gamma_{k,u})\phi_u(k)$$

for all $k \in K$ and $u \in [G\backslash U/K]$. Now we show that

$$T_{U,\lambda}(X,\mathfrak{l})^{K,\theta} = \bigsqcup_{\xi\in\Xi} \prod_{u\in[G\backslash U/K]} (X,\mathfrak{l})^{{}^uK,\xi_u}.$$

Let $f \in T_{U,\lambda}(X,\mathfrak{l})^{K,\theta}$, then $f(guk) = gf(u)$ for all $g \in G$, $u \in U$, and $k \in K$. So to determine f, it's enough to know $f(u)$ for $u \in [G\backslash U/K]$. Let $f(u) = x_u$. Then $\operatorname{res}^{G_{x_u}}_{{}^uK}\mathfrak{l}_{x_u} \in \{\xi_u\}_{u\in[G\backslash U/K]}$ for some $\{\xi_u\}_{u\in[G\backslash U/K]} \in \xi$. For the converse, we choose $x_u \in X$ for any $u \in [G\backslash U/K]$. Let $v \in V$ then $v = guk$ for some $g \in G$, for some $k \in K$ and a unique $u \in [G\backslash U/K]$. We set $f(v) = gx_u$. Now f is well defined if and only if $gx_u = x_u$ whenever $g \in {}^uK$ or equivalently $x_u \in X^{{}^uK}$. We want that $f \in T_{U,\lambda}(X,\mathfrak{l})^{K,\theta}$. If $x_u \in (X,\mathfrak{l})^{{}^uK,\xi_u}$ then $\operatorname{res}^{G_{x_u}}_{{}^uK}(\mathfrak{l}_{x_u}) = \xi_u$ and

$$\theta(k) = \prod_{u\in[G\backslash U/K]} \xi_u(\gamma_{k,u})\phi_u(k).$$

So $f \in T_{U,\lambda}(X,\mathfrak{l})^{K,\theta}$. Now using [4, Lemma 11.2.9], we get

$$\chi\Big(T_{U,\lambda}(X,\mathfrak{l})^{K,\theta}\Big) = \sum_{\xi\in\Xi} \prod_{u\in[G\backslash U/K]} \chi\big((X,\mathfrak{l})^{{}^uK,\xi_u}\big).$$

Thus, if $\Lambda_{(X,\mathfrak{l})} = \Lambda_{(Y,\mathfrak{m})}$ then $\Lambda_{T_{U,\lambda}(X,\mathfrak{l})} = \Lambda_{T_{U,\lambda}(Y,\mathfrak{m})}$. So we can define a map

$$\mathcal{T}_{U,\lambda} : B_C(G) \to B_C(H)$$

such that $\mathcal{T}_{U,\lambda}(a) = \Lambda_{T_{U,\lambda}(X,\mathfrak{l})}$ where $(X,\mathfrak{l})$ is a C-monomial G-poset such that $a = \Lambda_{(X,\mathfrak{l})}$, as in Corollary 5.2.7. □

Proposition 5.3.10. *Let G and H be finite groups, and let (U, λ) be a C-monomial (G, H)-biset.*

1. $\mathcal{T}_{U,\lambda}([G, 1_G]_G) = [H, 1_H]_H$.

2. $\mathcal{T}_{U,\lambda}(ab) = \mathcal{T}_{U,\lambda}(a)\mathcal{T}_{U,\lambda}(b)$, *for any* a, $b \in B_C(G)$.

In particular, the restriction of $\mathcal{T}_{U,\lambda}$ *to* $B_C(G)^\times$ *is a group homomorphism*

$$\mathcal{T}^\times_{U,\lambda} : B_C(G)^\times \to B_C(H)^\times.$$

Proof. 1. Since $\Lambda_{(\bullet,1)} = [H, 1_H]_H$, by the first assertion of Proposition 5.3.6 we get

$$\mathcal{T}_{U,\lambda}([G, 1]_G) = \Lambda_{T_{U,\lambda}(\bullet,1)} = \Lambda_{(\bullet,1)} = [H, 1_H]_H.$$

2. Given a, $b \in B_C(G)$, by Corollary 5.2.7 there exist C-monomial G-posets $(X, \mathfrak{l})$ and $(Y, \mathfrak{m})$ such that $\Lambda_{(X,\mathfrak{l})} = a$ and $\Lambda_{(Y,\mathfrak{m})} = b$. We get

$$\mathcal{T}_{U,\lambda}(ab) = \mathcal{T}_{U,\lambda}(\Lambda_{(X,\mathfrak{l})}\Lambda_{(Y,\mathfrak{m})}) = \mathcal{T}_{U,\lambda}(\Lambda_{X\times Y,\mathfrak{l}\times\mathfrak{m}}) = \Lambda_{T_{U,\lambda}(X\times Y,\mathfrak{l}\times\mathfrak{m})}$$

$$= \Lambda_{T_{U,\lambda}(X,\mathfrak{l})}\Lambda_{T_{U,\lambda}(Y,\mathfrak{m})} = \mathcal{T}_{U,\lambda}(a)\mathcal{T}_{U,\lambda}(b).$$

□

Proposition 5.3.11. *Let G, H, and K be finite groups.*

1. *Let* id_G *stand for the identity (G, G)-biset. Then* $\mathcal{T}_{\mathrm{id}_G,1_G}$ *is the identity map of* $B_C(G)$.

2. *Let (U, λ) and (U', λ') be C-monomial (G, H)-bisets. Then for any* $a \in B_C(G)$

$$\mathcal{T}_{U\sqcup U',\lambda\sqcup\lambda'}(a) = \mathcal{T}_{U,\lambda}(a)\mathcal{T}_{U',\lambda'}(a).$$

3. *Let (U, λ) be a C-monomial (G, H)-biset and let (V, ρ) be a monomial left free (H, K)-biset then*

$$\mathcal{T}_{V,\rho} \circ \mathcal{T}_{U,\lambda} = \mathcal{T}_{U\times_H V,\lambda\times\rho}.$$

Proof. Given $a \in B_C(G)$, we have $a = \Lambda_{(X,\mathfrak{l})}$ for some C-monomial G-poset $(X, \mathfrak{l})$ by Corollary 5.2.7.

1. We obtain

$$\mathcal{T}_{\mathrm{id}_G,1_G}(a) = \mathcal{T}_{\mathrm{id}_G,1_G}(\Lambda_{(X,\mathfrak{l})}) = \Lambda_{T_{\mathrm{id}_G,1_G}(X,\mathfrak{l})} = \Lambda_{(X,\mathfrak{l})} = a$$

by the third assertion of Proposition 5.3.6.

2. We obtain

$$\begin{aligned}\mathcal{T}_{U\sqcup U',\lambda\sqcup\lambda'}(a) = \mathcal{T}_{U\sqcup U',\lambda\sqcup\lambda'}(\Lambda_{(X,\mathfrak{l})}) &= \Lambda_{T_{U\sqcup U',\lambda\sqcup\lambda'}(X,\mathfrak{l})}\\ &= \Lambda_{T_{U,\lambda}(X,\mathfrak{l})\times T_{U',\lambda'}(X,\mathfrak{l})} = \Lambda_{T_{U,\lambda}(X,\mathfrak{l})}\Lambda_{T_{U',\lambda'}(X,\mathfrak{l})}\\ &= \mathcal{T}_{U,\lambda}(\Lambda_{(X,\mathfrak{l})})\mathcal{T}_{U',\lambda'}(\Lambda_{(X,\mathfrak{l})}) = \mathcal{T}_{U,\lambda}(a)\mathcal{T}_{U',\lambda'}(a).\end{aligned}$$

by the second assertion of Proposition 5.3.6.

3. We obtain

$$\mathcal{T}_{V,\rho} \circ \mathcal{T}_{U,\lambda}(a) = \mathcal{T}_{V,\rho} \circ \mathcal{T}_{U,\lambda}(\Lambda_{(X,\mathfrak{l})}) = \Lambda_{T_{V,\rho}\circ T_{U,\lambda}(X,\mathfrak{l})}$$

$$= \Lambda_{T_{U\times_H V,\lambda\times\rho}(X,\mathfrak{l})} = \mathcal{T}_{U\times_H V,\lambda\times\rho}(\Lambda_{(X,\mathfrak{l})}) = \mathcal{T}_{U\times_H V,\lambda\times\rho}(a)$$

by the fourth assertion of Proposition 5.3.6. □

Corollary 5.3.12. *Let G and H be finite groups. The map $(U, \lambda) \mapsto \mathcal{T}^{\times}_{U,\lambda}$ of Proposition 5.3.10 extends to a bilinear map*

$$B_C(G, H) \times B_C(G)^{\times} \to B_C(H)^{\times}.$$

Proof. This follows from Assertion 2 of Proposition 5.3.6 and Assertion 2 of Proposition 5.3.11, and from the fact that the map $\mathcal{T}^{\times}_{(U,\lambda)}$ depends only on the isomorphism class of (U, λ). □

Remark 5.3.13. *It follows from Remark 5.3.8 that if U is a finite (G, H)-biset, the square*

$$\begin{tikzcd} B(G) \arrow[r, "t_G"] \arrow[d, "\mathcal{T}_U"] & B_C(G) \arrow[d, "\mathcal{T}_{U,1_U}"] \\ B(H) \arrow[r, "t_H"] & B_C(H) \end{tikzcd}$$

of groups and multiplicative maps, is commutative, where $\mathcal{T}_U$ on the left is the usual generalized tensor induction map for Burnside rings, and the horizontal maps t_G and t_H are the ring homomorphisms induced by the functors τ_G and τ_H.

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