MATHEMATICAL ANALYSIS OF QUANTUM ENTANGLEMENT IN VARIOUS SYSTEMS

by

DUYGU İŞGÖREN YILMAZ

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Prof Dr. Nihat Çelebi

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Cenap Özel

Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Assoc. Prof. Dr Cenap ÖZEL

Supervisor

Examining Committee Members

Assistant Prof. Aliekber AKTAĞ

Assistant Prof. Erol YILMAZ

Associate Prof. Cenap ÖZEL

ABSTRACT

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İşgören Yılmaz, Duygu

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Supervisor: Associate Prof. Cenap ÖZEL

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In this thesis, we studied the entanglement properties of a system composed of one-to-three atoms, interacting stochastically with a cavity field. We also studied the entanglement properties of the systems under random phase telegraph noise. We obtained exact solutions for each systems. In order to examine atom-field, atom-atom and field-field entanglement in two level atom model we used the Jaynes-Cummings model. We will mainly focus on atom-atom entanglement. A strong atom-atom entanglement is observed in a stationary case. It has been observed that a strong noise helped in creating stationary atom-atom entanglement. We confined our research to random phase telegraph noise in order to understand the effects on atom-field interaction.

Keywords:Entanglement, Jaynes-Cummings model, random phase telegraph noise

ÖZET

KUVANTUM DOLAŞIKLIĞININ FARKLI SİSTEMLERDEKİ MATEMATİKSEL ANALİZİ

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Bu tezde, bir ila üç atomdan oluşan ve boşluk alanıyla stokastik olarak etkileşime giren bir sistemin çapraşıklık özelliklerini inceledik. Ayrıca, rasgele evreli telgraf gürültüsü altındaki sistemlerin özelliklerini de inceledik. İki seviyeli atom modelinde atom-alan, atom-atom ve alan-alan etkileşimini incelemek için Jaynes-Cummings modelini kullandık. Genel olarak atom-atom çapraşıklığına odaklanacağız. Güçlü bir atom-atom çapraşıklığı durağan bir durumda gözlemlenir. Güçlü bir gürültünün durağan atom-atom çapraşıklığının oluşmasında yardımcı olduğu gözlemlenmiştir. Atom-alan etkileşiminin etkilerini anlamak için araştırmamızı rasgele evreli telgraf gürültüsüyle sınırladık.

Anahtar kelimeler: Dolaşıklık, Jaynes-Cummings modeli, Rasgele Faz Telgraf Gürültüsü

To My Family

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CHAPTER 1
INTRODUCTION

In this chapter, we introduce a literature review of the thesis.

In quantum theory, discrete units are assigned to certain physical quantities, like the energy of an atom at rest. The discovery of waves that have discrete energy packets (quanta) behave in a manner similar to particles led a branch of physics that deals with atomic and subatomic systems is called quantum mechanics. It is the mathematical framework of many fields of physics and chemistry such as condensed matter physics, solid-state physics, atomic physics, molecular physics, computational chemistry, quantum chemistry, particle physics, and nuclear physics. The foundation of quantum mechanics was established during the first half of the twentieth century by Werner Heisenberg, Max Planck, Louis de Broglie, Albert Einstein, Niels Bohr, Erwin Schrödinger, Max Born, John von Neumann, Paul Dirac, Wolfgang Pauli, David Hilbert, and others. Some fundamental aspects of the theory are still actively studied [1]. Quantum mechanics is necessary to comprehend the behavior of systems at atomic length scales and smaller. For instance, if classical mechanics ruled the working cycle of an atom, electrons would travel towards and collide with the nucleus, which render the stable atoms impossible. However, electrons in reality remain in an uncertain, non-deterministic "smeared" (wave-particle wave function) orbital path around or even "through" the nucleus, denying classical electromagnetism[2]. Quantum mechanics was developed of to explain the atom is structure better, especially the spectra emission and absoption of light of different atomic species. The quantum theory is further developed to explain how electron is staying in its orbital. This could not be explained by Newton's laws and by Maxwell's classical electromagnetism laws [3]. While forming quantum mechanics, the state of a system at a specific time is defined by a complex wave function (sometimes known as orbitals in the case of atomic electrons), and more generally, elements of a complex vector space [4]. For example, electrons can be regarded to be somewhere within a region of space, but their exact positions are still unknown. Contours of constant probability, often referred to as "clouds" may be drawn

around the nucleus of an atom in order to conceptualize the most probable place of the electron. Heisenberg's uncertainty principle quantifies the inability to precisely find the particle given its conjugate[5]. The other exemplar that led to quantum mechanics was the study of electromagnetic waves such as light. By the time it was found in 1900 by Max Planck that the energy of waves could be defined as consisting of small packets or quanta, Albert Einstein used this idea to show that an electromagnetic wave such as light could be described by a particle called the photon with a discrete energy dependent on its frequency. This situation gave rise to a theory of unity between subatomic particles and electromagnetic waves which are called wave-particle duality where the particles and waves were neither one nor the other, but had specific properties of both. Quantum mechanics describes the world of the very small things and it is also needed to mention certain "macroscopic quantum systems" like superconductors and super fluids [6]. In general, quantum mechanics includes four classes of phenomena that classical physics cannot account for:

(I) the quantization (discretization) of certain physical quantities,

(II) wave-particle duality,

(III) the uncertainty principle,

(IV) quantum entanglement. Each of these phenomena is described in detail in subsequent sections [7].

Historically quantum mechanics [8]. began in 1838 with discovery of cathode rays by Michael Faraday. In 1859 Gustav Kirchhoff stated the black body radiation problem, in 1877 Ludwig Boltzmann suggested that the energy states of a physical system could be discrete, and in 1900 Max Planck stated that any energy is radiated and absorbed in quantities divisible by discrete 'energy elements', E, such that each of these energy elements is proportional to the frequency with which they each individually radiate energy, as defined by the following formula:

$$E = hv$$

where h is Planck's Action Constant. Planck insisted [9] that this was just an aspect of the processes of absorption and emission of radiation and did not have anything to do with the physical reality of the radiation itself. However, this situation did not explain

the photoelectric effect (1839), that shining light on certain materials can function to eject electrons from the material. In 1905 Albert Einstein [10] proposed that light itself consists of individual quanta, basing his work on Planck's quantum hypothesis. These later were called photons (1926). From Einstein's simple proposal many debates, theories and tests, and thus, the entire field of quantum physics was born.

The Jaynes-Cummings model proposed in 1963 by Edwin Jaynes and Fred Cummings in order to study the relationship between the quantum theory of radiation and the semi-classical theory in describing the phenomenon of spontaneous emission [11]. In the earlier semi-classical theory of field-atom interaction, only the atom is quantized and the field is treated as a definite function of time rather than as an operator. The JCM discovers how quantization of the radiation field affects the predictions for the evolution of the state of a two-level system in comparison with semi-classical theory of light-atom interaction. Later it was discovered that the revival of the atomic population inversion after its collapse is a direct consequence of discreteness of field states (photons)[12, 13].This is a pure quantum effect that can be described by the JCM but not with the semi-classical theory. Twenty four years later, Rempe, Walther, and Klein demonstrated a beautiful quantum collapse and revival in a one-atom maser[14]. Before that time research groups could build experimental setups capable of enhancing the coupling of an atom with a single field mode, at the same suppressing other modes. With the appearance of one-atom masers one could possible study the interaction of a single atom (usually a Rydberg atom) with a single resonant mode of the electromagnetic field in a cavity [15, 16] and study different aspects of the JCM. In order to see strong atom-field coupling in visible light frequencies hour-glass-type optical modes can help us because of their large mode volume which coincides with a strong field inside the cavity in the end[17]. In order to describe the interaction between an atom and a laser field more, the model is generalized in different methods. Some of the generalizations are applying initial conditions, considering dissipation and damping in the model [18, 19, 20], considering multilevel atoms and multiple atoms [21], and multimode description of the field [22]. It was also discovered that the atom and field exist in a macroscopic superposition state during the quiescent intervals of collapsed Rabi oscillations (a Schrödinger cat). This discovery presents the possibility to use the JCM in order to make the basic properties of quantum correlation (entanglement) clear[23].

In another work the JCM is used for modeling the transfer of quantum information [24].

"*Quantum entanglement*, which is also known as the quantum non-local connection, is a possible property of a quantum mechanical state of a system consisting of two or more objects in which the quantum states of these objects are linked together". The reason for this is that one object cannot be described sufficiently without full mention of its counterpart even if the separate objects are spatially separated in a space-like manner. This interconnection causes non-classical correlations between observable physical properties of remote systems which are often referred to as non-local correlations. Measuring any number of particles will result in an unpredictable series of measures that will tend more and more closely to half up and half down. However, if this experiment is done with entangled particles the results are quite different. When two members of an entangled pair are measured, one will always be spin-up and the other will be spin-down. The distance between the two particles is irrelevant. Theories involving 'hidden variables' have been proposed to explain this result; these hidden variables account for the spin of each particle, and are determined when the entangled pair is created. It may appear then that the hidden variables must be in communication no matter how far apart the particles are, that the hidden variable describing one particle must be able to change instantly when the other is measured. If the hidden variables stop interacting when they are far apart, the statistics of multiple measurements must obey an inequality (called Bell's inequality), which is, however, violated - both by quantum mechanical theory and in experiments.Bell inequalities concern measurements made by observers on pairs of particles that have interacted and then separated. According to quantum mechanics they are entangled while local realism limits the correlation of subsequent measurements of the particles. Different authors subsequently derived inequalities similar to Bell´s original inequality, collectively termed Bell inequalities. All Bell inequalities describe experiments in which the predicted result assuming entanglement differs from that following from local realism.

The assumption that measurement in effect "creates" the state of the measured quality goes back to the arguments of, among others: Schrödinger, and Einstein, Podolsky, and Rosen (see EPR paradox) concerning Heisenberg's uncertainty principle and its relation to observation. The analysis of entangled particles by means of Bell's the-

orem, can lead to an impression of non-locality that is, that there exists a connection between the members of such a pair that defies both classical and relativistic concepts of space and time. This is reasonable if it is assumed that each particle departs the scene of the pair's creation in an ambiguous state (as per a possible interpretation of Heisenberg). In such a case, for a given measurement either outcome remains a possibility; only measurement itself would precipitate a distinct value. On the other hand, if each particle departs the scene of its "entangled creation" with properties that would unambiguously determine the value of the quality to be subsequently measured, then a postulated instantaneous transmission of information across space and time would not be required to account for the result. The Bohm interpretation postulates that a guide wave exists connecting what are perceived as individual particles such that the supposed hidden variables are actually the particles themselves existing as functions of that wave. Observation of wave function collapse can lead to the impression that measurements performed on one system instantaneously influence other systems entangled with the measured system, even when far apart. Yet another interpretation of this phenomenon is that quantum entanglement does not necessarily enable the transmission of classical information faster than the speed of light because a classical information channel is required to complete the process. Entanglement is one of the properties of quantum mechanics that caused Einstein and others not to like the theory. In 1935, Einstein, Podolsky, and Rosen created the EPR paradox, a quantum-mechanical thought experiment with a highly counterintuitive and apparently non-local outcome, in response to Niels Bohr's advocacy of the belief that quantum mechanics as a theory was complete [25]. Einstein derided entanglement as Spukhafte Fernwirkung [26], or spooky action at a distance. He believed that future mathematicians would discover that quantum entanglement entailed nothing more or less than an error in their calculations. As he once wrote:" I find the idea quite intolerable that an electron exposed to radiation should choose of its own free will, not only its moment to jump off, but also its direction. In that case, I would rather be a cobbler, or even an employee in a gaming house, than a physicist [27]". On the other hand, quantum mechanics has succeeded in producing correct experimental predictions, and the strong correlations expected by the theory of quantum entanglement. One apparent way to explain found correlations in line with the predictions of quantum entanglement is an approach known as local

hidden variable theory. In this approach unknown, shared, local parameters cause the correlations. However, in 1964 John Stewart Bell derived an upper limit (known as Bell's inequality) on the strength of correlations for any theory obeying local realism. Quantum entanglement can cause stronger correlations which exceed this limit. As a result quantum entanglement can be experimentally distinguished from a broad class of local hidden-variable theories. Results of the later experiments are supported by quantum mechanics. However, there may be experimental problems, known as loopholes, which affect the validity of these experimental findings. Observations which are related to the entangled states seem to conflict with the property of relativity that information cannot be transferred faster than the speed of light. Despite the fact that two entangled systems appear to interact across large spatial separations, the current state of belief is that useful information cannot be transmitted in this way. This is the statement of the no-communication theorem. It is believed that it is possible to transmit information using a set of entangled states used in conjunction with a classical information channel even if information cannot be transmitted through entanglement alone. This process is called quantum teleportation. In spite of its name, quantum teleportation cannot allow information to be transmitted faster than light. The reason for this is that a classical information channel is required to complete the process. In addition experiments are underway to see if entanglement is the result of retro causality [28, 29].

In this thesis, we make use of these already cited entanglement works on the Jaynes-Cummings model, by taking the effects of noise as so called *random phase telegraph noise*. Jaynes-Cummings model studies the entanglement dynamics in a system of the interaction of atom its single-mode cavity field in the presence of random phase telegraph noise. Calculations, exacts solutions and discussion are given in later in details.

Burshtein introduced the influence of noise (jump-type) on the atom-field interactions in quantum optics was [30, 31, 32]. The simplest jump-like processes model is the two-state random telegraph. Eberly et al [33, 34] mentioned laser-atom interactions which are subjected to two-state random (phase and frequency) telegraph noise. Joshi et al [35, 36, 37, 38] introduced the incorporation of these random telegraph models into Jaynes-Cummings model as the stochastic fluctuations in the atom-field coupling parameter.

We also considered the JCM with stochastic fluctuations as it is applied to study the motion of an ion in a harmonic trap. This trap is in interaction with a stationary a traveling wave. The reason is that in some approximations [39] the equations governing the motion of the ion in the trap may be reduced to a form that is similar to the JCM. Here vibrational modes of quantized center-of-mass motion of the ion take the place of the field variables. At that point, the coupling coefficient includes the amplitude and the phase of the standing wave and the fluctuations in both amplitude and phase might well be considered.

The noise in this study is uncontrollable. The reason is the stochastic behavior of the system itself, not any environment effect. There are some works with environmental noise which we did not consider in this study.

The plan of this thesis is as follows; In section 2 a theoritical background of the thesis is presented. There is an overview regarding quantum entanglement and some crucial entanglement measures. Apart from this, we mention the mathematical theory of the stochastic analysis to deal with noisy atom-field interactions. In section 3, by using the Jaynes Cummings model, we approach to the system of an atom coupled to a single mode cavity field where random phase telegraph noise present analytically. In section 4, by using the two atom generalization of the Jaynes-Cummings model, we analyze the interaction of two atoms which are coupled to a single mode cavity field when random phase telegraph noise is present and we calculate system's concurrence because there is two-partite entanglement here. In section 5, by using the three atom generalization of the Jaynes-Cummings model, we analyze the interaction of three atoms which are coupled to a single mode cavity field with random phase telegraph noise. Then we assess the fidelity of the system. Because there is three atom one field and we calculate fidelity of the subsystem to find if the system is entangled or not. In these sections (3-5), we examine the dynamics of the entanglement between the atoms and the fields and reveal the effects of this noise on the entanglement dynamics.

CHAPTER 2
BASIC CONCEPT

In this chapter we mention quantum entanglement theory, some entanglement measures and stochastic analysis. This chapter is based on the books of Stroock [40], Rao [41], Nelson [42], Çınlar [43].

2.1 Stochastic Analysis

Stochastic analysis is generally a study which comprises the stochastic processes' structural and inferential properties. Models of dynamic system which are open to uncertainty are called *stochastic processes*. A stochastic process (sometimes called random process), is the counterpart of a deterministic process (or deterministic system) in probability theory. It is just a random variable sequence and these random variables are ordered by an index set. Probability distributions define lack of determinacy in the future of a stochastic or random process. Rather than dealing with only one possible reality of how the process might evolve under time, one should take this fact into consideration. We can infer from this situation that even if the initial condition (or starting point) is known, there are many possibilities that the process might evolve to. However, some paths are more probable than others. The object can be described as an indexed family of random variables $X_t, t \in T$ on a probability space. This expression includes some hidden conditions regarding the family. In order to be more clear, we use the axiomatic theory of probability. In this theory we show how the basic probability space may be created, with the available initial information, in order that a stochastic process may be defined on it.

2.1.1 Random Variables

A random variable is a function and it relates a unique numerical value with each result of an experiment. The result of an experiment does not need to be a number. For example, the result when a coin is tossed can be 'heads' or 'tails'. However, we generally represent the results as numbers. The random variable's value will vary from

trial to trial as the experiment is repeated. There are two types of random variable. These are discrete and continuous random variables

Discrete random variable:A random variable has an related probability distribution.

Continuous random variable:A random variable has an related probability density function.

Some examples about discrete and continuous random variable are:

1.A coin is tossed ten times. The random variable X is the number of tails which are shown where X is discrete random variable.

2.A light bulb is burned to the time it burns out. The random variable Y is its lifetime in hours where Y is a continuous random variable.

Definition 2.1 *(Expected Value) A random variable's expected value shows its average over all variables. It is a useful summary value (a number) of the variable's distribution.*

The expected value gives a general idea of the behavior of a random variable without giving full details of its probability distribution (if it is discrete) or its probability density function(if it is continuous). Two random variables which have the same expected value may have very different distributions. Variance is the other useful descriptive measures affecting the shape of the distribution, for instance variance exist. The variance of a random variable is a not a negative number. It shows how widely spread the values of the random variable can be. In case the variance is larger, observations are scattered on average. The variance shows how closely concentrated round the expected value the distribution is; it is a measure of the 'spread' of a distribution about its average value.

2.1.2 Probability Distribution

A probability distribution define either the probability of each value of an unidentified discreate random variable or probability of the value falling within a particular continuous interval in probability theory and statistics. The probability distribution describes the range of possible values that a random variable can attain and the probability that the value of the random variable is within any (measurable) subset of that range. The probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values. It is also called the probability function. The

probability distribution of a discrete random variable X is a function which gives the probability $p(x_i)$ that the random variable is equal to x_i, for each value x_i:

$$p(x_i) = P(X = x_i)$$

2.1.3 Stochastic Process

A stochastic process is a sequence of random variables. As a result, it is characterized by the joint distribution of its component random variables. However, characterizing a stochastic process in this way causes some technical problems. The reason is that there are an infinite number of random variables to characterize. In order to solve this issue, a stochastic process should be defined by using an algorithm to generate its sample paths rather than specifying the joint distribution of its component random variables directly.

For instance, an n-dimensional stochastic process is $\{S_n; n = 0, 1, 2, ...\}$ where

$$S_n \equiv \begin{pmatrix} S_{1,n} \\ S_{2,n} \\ . \\ . \\ S_{m,n} \end{pmatrix}$$

and each element $S_{i,n}$ as a scalar random variable. We can give some examples about stochastic process

(i) $\{S_n; n = 0, 1, 2, ...\}$, where the state space of S_n is $\{1, 2, 3, 4\}$ represents which of four types of transactions a person sends to an on-line data-base and n is the number of transactions.

(ii) $\{S_n; n = 0, 1, 2, ...\}$, where the state space of S_n is $\{1, 2\}$ represents if an electronic component is acceptable or defective, and n is the number of components.

(iii) $\{X_t; t \geq 0\}$, where the state space of X_t is $\{0, 1, 2, ...\}$ represents the number of accidents at an intersection and time t corresponds to weeks.

(iv) $\{X_t; t \geq 0\}$, where the state space of X_t is $\{0, 1, 2, ...s\}$ represents the number of copies of a software and time t corresponds to weeks.

(v) $\{X_t; t \geq 0\}$, where the state space of X_t is $\{0, 1, 2, ...\}$ represents the number

of cars parked in garage, and time t corresponds to weeks.

2.2 Poisson Process.

A Poisson process, named after the French mathematician Siméon - Denis Poisson, is the stochastic process in which events occur continuously and independently of one another (the word event used here is not an instance of the concept of event frequently used in probability theory) [44].

Some examples of Poisson processes are telephone calls arriving at a switchboard, the radioactive decay of atoms, page view requests to a website, and rainfall.

The Poisson process is a continuous-time process and the simplest example of a birth-death process. It is also a collection $N(t) : t \geq 0$ of random variables, where $N(t)$ is the number of events that have occurred up to time t (starting from time 0). Each realization of the process $N(t)$ is a non-negative integer-valued step function that is non-decreasing, but for intuitive purposes it is usually easier to think of it as a point pattern on $[0, \infty)$ (the points in time where the step function jumps, i.e. the points in time where an event occurs). In this section we will look at the counting process connected to parameter λ. In a Poisson process, the arrival rate stays constant. These processes are occasionally used as stochastic processes that model arrival processes to service systems. In some cases, Poisson processes are representations of real-world systems. We start with a set of assumptions which underlie Poisson process and obtain the probability distribution for $N(t)$. Those assumptions are open to intuitive interpretations. We will then study some properties and several variants of the Poisson process.

The counting process $N = N(t), t \geq 0$ is a Poisson process with rate $\lambda > 0$, if it shows the following properties:

(i)$N(0) = 0$

(ii)it satisfies the stationary and independent increment properties

(iii)$PN(h) = 1 = \lambda h + o(h)$

(iv)$PN(h) \geq 2 = o(h)$.

Property (i) simply starts the clock of counting from time 0. According to property (ii), in case we change the time origin of the counting process (if it will not alter the probabilistic behaviors of the process) the *length* of the interval at stake is important. In line with the independent-increment property, the number of arrivals in two disjoint

intervals have to be independent. Property (iii) implies that in a small interval the probability of one arrival is approximately equal to the arrival rate λ times the length of the interval, and Property (iv) says that in a small interval the probability of more than one arrival is negligible. Hence if arrivals occur in batches of more than one, the number of arrivals in an interval will not follow a Poisson process. However, the number of *arrival occurrences* in the same interval could follow a Poisson process. By using some of the previous properties, Poisson processes can used to model some arrival processes. Many times, however, the Poisson processes are useful approximations for arrival processes in practice. Poisson processes are also examples of continuous-time Markov processes.

There are two types of Poisson process.These are homogeneous Poisson process and non- homogeneous Poisson process

1.Homogeneous Poisson process The homogeneous Poisson process is one of the most well-known Lévy processes. This process is characterized by a rate parameter λ, also known as intensity, such that the number of events in time interval $(t, t+\tau)$ follows a Poisson distribution with associated parameter λ_τ. This relation is given as

$$P[(N(t+\tau) - N(t)) = k] = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} \qquad k = 0, 1, ...,$$

where $N(t+\tau) - N(t)$ is the number of events in time interval $(t, t+\tau)$.

Just as a Poisson random variable is characterized by its scalar parameter λ, a homogeneous Poisson process is characterized by its rate parameter λ, which is the expected number of *events* or *arrivals* that occur per unit time. N(t) is a sample homogeneous Poisson process, not to be confused with a density or distribution function.

2.Non-Homogeneous Poisson process

In general, the rate parameter may change over time; such a process is called a non-homogeneous Poisson process or inhomogeneous Poisson process. In this case, the generalized rate function is given as $\lambda(t)$. Now the expected number of events between time a and time b is

$$\lambda_{a,b} = \int_a^b \lambda(t)dt$$

Thus, the number of arrivals in the time interval (a, b), given as N(b) - N(a), follows

a Poisson distribution with associated parameter $\lambda_{a,b}$

$$P[(N(b) - N(a)) = k] = \frac{e^{-\lambda_a}(\lambda_{a,b})^k}{k!, b} \qquad k = 0, 1, ...,$$

A homogeneous Poisson process may be viewed as a special case when $\lambda(t) = \lambda$, a constant rate.

2.3 Markov Process.

Important stochastic process classes are Markov processes and Markov chains. To make a more specific distinction, let us define Markov Processes and Markov Chains in a more detailed way. Here we will compare these two concepts.

2.3.1 Markov Processes.

A Markov process, named after the Russian mathematician Andrey Markov, is a mathematical model for the random evolution of a memoryless system [45].

Markov process is a discrete time process where the future behavior is related to the present behavior and is not related to the past behavior. Markov process is the continuous time version of the Markov chain and many sequence models are in fact Markov processes. A Markov process looks at a sequence of events, and analyzes the tendency of one event to be followed by another. Using this analysis, you can generate a new sequence of random but related events, which will look similar to the original.

A Markov process is useful for analyzing dependent random events, that is, events whose likelihood depends on what happened last. It would not be a good way to model a coin flip, for example, since every time you toss the coin, it has no memory of what happened before. The sequence of heads and tails are not inter-related. They are independent events. But many random events are affected by what happened before. For example, if a truck driver does not sleep at night, he falls asleep the next day while driving and in a moment he crashes into a car, which is followed by a chain accident. These are not independent events.

We have discrete S states, also in a Markov process. However, the transition behavior is different from the Markov chain. In each states there are various possible events which lead to transition. The event that causes a transition from state i to j,

where $j = i$, takes place after an exponential amount of time, say with parameter q_{ij}. As a result, the transitions occur in random time points in this model. According to the properties of exponential random variables we have the following situations:

(i) In state i a transition takes place after an exponential amount of time with parameter

$$\sum_{j \neq i} q_{ij}$$

(ii) The system makes a transition to the state j with the following probability.

$$p_{ij} = q_{ij} / \sum_{k \neq i} q_{ik}$$

$$q_{ij} = -\sum_{j \neq i} q_{ij}, \qquad i \in S$$

The Q matrix which has the q_{ij} elements is called as the generator of the Markov processes. Note that the definition of q_{ii} states that the prom sums of Q are equal to 0.

(iii) All Markov states communicate with each other

(iv) Markov process does not drift to infinity

Different from the discrete time state in Markov chain, you do not have to worry about periodicity. The randomness of time spent by the system for each state quarantess that the probability $p_i(t)$ will go to the p limit. Limit probabilities or balance probabilities can again be calculated by balance equations. The balance equations balance the flow-out and flow-into of one situation. The flow is the mean transition number per time unit. If the system is in state i, then events that cause the system to make a transition to state j occur with a frequency or rate q_{ij}. Hence, the mean number of transitions per time limit from i to j is equal to $p_i q_{ij}$. This situation leads to the following balance equations:

$$p_i = \sum_{j \neq i} q_{ij} = q_{ij} = \sum_{j \neq i} p_j q_{ji}, \qquad i \in S$$

$$0 = \sum_{j \in S} p_j q_{ji}$$

In vector-matrix notation this becomes, with p the row vector with elements p_i

$$0 = pQ$$

Together with the normalization equation

$$\sum_{i \in S} p_i = 1$$

The solution of the set of equations is unique.

The stochastic process $\{Y = Y_t; t \in R_+\}$ is said to be a*Markov process* with state space E provided that for any $t, s \geq 0$ and $j \in E$, $P\{Y_{t+s} = j | Y_u; u \leq t\} = P\{Y_{t+s} = j | Y_t\}$ the conditional probability appearing above may, in general, depend on both t and s. When $P\{Y_{t+s} = j | Y_t = i\} = P_s = (i, j)$ is independent of $t \geq 0$ for all i, $j \in E$ and $s \geq 0$, the process Y is a *time-homogeneous* Markov process

In probability theory, a continuous-time Markov process is a stochastic process $X(t) : t \geq 0$ that satisfies the Markov property and takes values from a set called the state space. The Markov property states that at any times $s > t > 0$, the conditional probability distribution of the process at time s given the whole history of the process up to and including time t, depends only on the state of the process at time t. In effect, the state of the process at time s is conditionally independent of the history of the process before time t, given the state of the process at time t.

2.3.2 The Chapman-Kolmogorov equation

In mathematics, specifically in probability theory and in particular the theory of Markovian stochastic processes, the *Chapman-Kolmogorov equation* is an identity relating the joint probability distributions of different sets of coordinates on a stochastic process. The equation was arrived at independently by both the British mathematician Sydney Chapman and the Russian mathematician Andrey Kolmogorov.[46]

Suppose that $\{f_i\}$ is an indexed collection of random variables, that is, a stochastic process. Let$p_{i_1,...,i_n}(f_1, ..., f_n)$ be the joint probability density function of the values of the random variables $\{f_i\}$ to $\{f_n\}$. Then, the Chapman-Kolmogorov equation is

$$p_{i_1,...,i_{n-1}}(f_1,...,f_{n-1}) = \int_{-\infty}^{\infty} p_{i_1,...,i_n}(f_1,...,f_n)df_n \tag{2.1}$$

2.3.3 Markov Chains

A Markov chain, which is studied in discrete $0, 1, 2, ...$ time intervals is characterized by a S state set and p_{ij} transition probabilities. Here $p_{ij}$is the probability that the Markov chain is in the present time point in state i and is is in next time point in state j. The P matrix which has the p_{ij} elements, is called the transmission probability of the Markov process. Note that the p_{ij} definition shows that the row sums of P are equal to 1. Under the following conditions:

(i) All states of Markov chain communicate with each other (i.e. it is possible to go from each state to another, provided that it is more than one step),

(ii) Markov chain is not periodic (a periodic Markov chain is a chain in which, e.g., you can only return to a state in an even number of steps),

(iii) Markov chain does not drift to infinity

As n drifts to infinity, the probability $\pi(n)$ that the system is in state i at time point n converges to a limit π_i. These limit or equilibrium probabilities can be calculated from a set called balance equations. The balance equations balance the probability of leaving and entering a state in equilibrium. This situation leads to the following equations:

$$\pi_i \sum_{j \neq i} p_{ij} = \sum_{j \neq i} \pi_j p_{ji} \qquad i \in S$$

$$\pi_i = \sum_{j \in S} \pi_j p_{ji} \qquad i \in S$$

In vector-matrix notation this becomes, with the row vector with elements,

$$\pi_i = \pi P$$

With the following normalization equation

$$\sum_{i \in S} \pi_i = 1$$

The solution of the set of equations is unique.

2.4 Random Phase Telegraph Noise.

The effect of the noise on the atom-field was first expressed by Burshtein [47]. Random phase telegraph noise is a noise in interaction and a solution might be obtained for the model under noise. This solution is used to study the atom-field interaction's entanglement dynamics. The degree of the entanglement first decreases to a minimum value and then increases to an adequately high intensity

Strong atom interactions that are subjected to jump-type (random telegraph) random-phase noise are discussed. Physically, the jumps may arise from laser fluctuations, from collisions of various kinds, or from external forces. The discussion is carried out in two stages. First, direct and partially heuristic calculations determine the laser spectrum and also give a third-order differential equation for the average inversion of a two-level atom on resonance. At this stage a number of general features of the interaction are able to be studied easily. Second, it is shown that the theory of generalized Poisson processes allows laser-atom interactions in the presence of random telegraph noise of all kinds (not only phase noise) to be treated systematically, by means of a master equation first used in the context of quantum optics by Burshtein. The Burshtein equation is used to obtain an exact expression for the two-level atom's steady-state resonance fluorescence spectrum, when the exciting laser exhibits phase telegraph noise [48].

We extend our study of the effects of jump-type noise on laser-atom interactions to frequency-telegraph noise. Such noise can be used as a model of collisional effects, in which the atomic transition frequency randomly jumps, or as a model of finite laser bandwidth effects, in which the laser frequency exhibits random jumps. We show that these two types of frequency noise can be distinguished in light-scattering spectra. We also discuss examples which demonstrate both temporal and spectral motional narrowing, non exponential correlations, and non-Lorentzian spectra. Its exact solubility in finite terms makes the frequency-telegraph noise model an attractive alternative to the white-noise Ornstein-Uhlenbeck frequency noise model which has been previously

applied to laser-atom interactions.

2.5 The Qubit.

The complex quantum mechanical system is a two-state system which is defined by a vector in two-dimensional complex Hilbert space and this system is called a *qubit*. The favorite qubit of many physicists is, a spinning particle like electron. We will use spins of such particles to show entanglement. If we measure the z-component of the spin, the result is either up or down. This measurement corresponds to the observable operator which we call Z. The state of a particle is an eigenstate of the observable. We demonstrate these states by $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, for spin parallel to and anti parallel to the z-axis. They are orthogonal and we take them as the unit basis vectors of the spin Hilbert space. In addition to the eigenstates, the particle may be in any superposition of the eigenstates

$$|\psi\rangle = \alpha|\uparrow_z\rangle + \beta|\downarrow_z\rangle \tag{2.2}$$

such that $|\alpha|^2 + |\beta|^2 = 1$. When the spin along the z-axises measured, we obtain +1 with a probability of $|\alpha|^2$ and -1 with probability $|\beta|^2$. After the measurement the state collapses into the corresponding eigenstate. As the overall phase of a quantum state does not have any physical significance, only the phase difference between α and $\beta$is important, and for convenience we often choose α to be real. To include the property $|\alpha|^2 + |\beta|^2 = 1$ we write the spin state (2.2) of the particle as

$$|\psi\rangle = \cos(\frac{\theta}{2})|\uparrow_z\rangle + e^{i\phi}\sin(\frac{\theta}{2})|\downarrow_z\rangle \tag{2.3}$$

where $\alpha = \cos(\frac{\theta}{2})$ and $\beta = e^{i\phi}\sin(\frac{\theta}{2})$

Thus the state demonstrates the state of the spin $\frac{1}{2}$ particle, and thereby any of qubit. All possible states are covered when the parameters are restricted to $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$. Hence, any state corresponds to a point on the unit sphere, with the azimuthal angle ϕ and the polar angle θ. Such a sphere is called a *Bloch sphere* and is a very useful tool for "visualizing" states of a qubit.

We may choose another set of orthonormal basis vectors. For instance

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle), \qquad |\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle - |\downarrow_z\rangle) \tag{2.4}$$

These are the eigenvectors of the X-operator, and they are the resulting states if we measure the spin along the x-axis. Likewise, we may measure the spin along any axis, and find a corresponding set of eigenvectors. If we denote the basis vectors

$$|\uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.5}$$

then the operators for the spin along the x, y and z axes are nothing more than the Pauli matrices

$$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

In physics, the Pauli matrices are a set of 2×2 complex Hermitian and unitary matrices [49]. The name refers to Wolfgang Pauli.

For the spin along any other unit vector, say $a\overrightarrow{x} + b\overrightarrow{y} + c\overrightarrow{z}$ where $a^2 + b^2 + c^2 = 1$, the spin operator is $aX + bY + cZ$. All operators of this form have eigenvalues ± 1 corresponding to spin parallel and anti parallel to the vector.

As stated before, other systems are good qubits in terms of being a spin $\frac{1}{2}$ particle. Polarization of a photon and the energy levels of a two-level system are more popular qubits. They can be represented by a Bloch sphere just like the spin of an electron. The good thing about a spin $\frac{1}{2}$ particle is the fact that the vectors on the Bloch sphere corresponds to actual spin directions in real space. As a result a qubit in a state, which is close to the y-axis on the Bloch sphere has a spin pointing in a direction close to the y-axis.

When a spin $\frac{1}{2}$ particle is present, the basis vectors $|0\rangle$ and $|1\rangle$ are denoted with $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$. According to the sign in the superposition, the superpositions of these states related to the spin along the x-axis often denote $|+\rangle$ and $|-\rangle$.

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2.7}$$

The unitary Pauli operators stated in (2.6) can be written as the following in the context of general qubits

$$X \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \tag{2.8}$$

$$Y \equiv |1\rangle\langle 0| - i|0\rangle\langle 1| \tag{2.9}$$

$$Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1| \tag{2.10}$$

X is a *bit flip operator* which turns a $|0\rangle$ into a $|1\rangle$ or $|1\rangle$ into $|0\rangle$. Z is *the phase flip* operator which transits between the $\alpha|0\rangle + \beta|1\rangle$ and $\alpha|0\rangle - \beta|1\rangle$ states.

The name *qubit* comes from *quantum bit* and it plays the same role in quantum information theory as the bit does in classical information theory. The bit is a system which can be in one of two states, usually denoted 0 and 1. As a result, each bit is able to store enough information to reply one yes/no question and the qubit has a richer structure. It is capable of being in the $|0\rangle$ and $|1\rangle$ states and just like a bit, it can be a continuum of superpositions between those states, which are given by two parameters.

2.6 Entangled State.

We now will look at the *composite* quantum systems. A composite system is a system which comprises of two or more parts, and the simplest system consists of two qubits and these two systems are called A and B. Any state of each of the two systems may be written as

$$|\psi\rangle_A = \alpha|0\rangle_A + |1\rangle_A \qquad |\psi\rangle_B = \gamma|0\rangle_B + \delta|1\rangle_B \tag{2.11}$$

with $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$ The composite state of the two systems is their tensor product.

$$|\psi_{prod}\rangle = |\psi\rangle_A \otimes |\phi\rangle_B \tag{2.12}$$

This state is referred to as *product state*. However, product states are not the physically realizable states. Unless the two systems interact with each other, the superposition of the product states is observed. As a result, a general composite state can be

$$|\psi\rangle = \sum_{ij} \alpha_{ij} |\psi_i\rangle_A \otimes |\phi_j\rangle_B \tag{2.13}$$

where $\sum |\alpha_{ij}|^2 = 1$ and the sets $\{|\psi_i\rangle\}$ and $\{|\phi_j\rangle\}$ are orthonormal bases for the two subsystems. Any composite state that is not a product state is called an *entangled state*.

A composite quantum state which consists of only two parts is called a *bipartite state*, but a composite quantum state, which consists of more than two parts is called *multipartite state*. For bipartite qubit states, four entangled states play a major role, namely the singlet state

$$|\psi^-\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.14}$$

and the three triplet states

$$|\psi^+\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.15}$$

$$|\phi^-\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.16}$$

$$|\phi^+\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.17}$$

where we have used $|ij\rangle$ as a shorthand notation for $|i\rangle \otimes |j\rangle$. They are called *Bell*

states or EPR pairs. Together they from an orthonormal basis for the state space of two qubits, called the *Bell basis*. The Bell states are maximally entangled and one can be converted into another by applying a unitary transform locally on any one of the subsystems. Note that if we measure the state of one qubit in a Bell state (that is, measure the Z operator which has eigenvalues ± 1), we immediately know the state of the other particle. In the singlet Bell state, a measurement of qubit A will yield one of the eigenstates $|0\rangle$ and $|1\rangle$, each with a probability of $\frac{1}{2}$. These results leave qubit B in state $|1\rangle$ or $|0\rangle$, respectively.

For a single qubit we could always change to another basis where the outcome of a Z measurement would be given. For a spin $\frac{-1}{2}$ particle the spin is always pointing in some direction. However, the state is not one of the basis states, but it is shown as a superposition in one of the basis states. In case the particle is entangled with another particle, the direction of the spin of that particle cannot be defined well.

Actually, for particles in one of the Bell states, the probability for measuring the spin of the particle to "up" (while ignoring the other particle)is $\frac{1}{2}$ for any direction.

2.7 Pure and Mixed States.

In quantum physics, a quantum state is a mathematical object that fully describes a quantum system. One typically imagines some experimental apparatus and procedure which "prepares" is quantum state; the mathematical object then reflects the set up of the apparatus. Quantum states can be statistically mixed, corresponding to an experiment involving a random change of the parameters. States obtained in this way are called **mixed states**, as opposite to **pure states**, which cannot be described as a mixture of others. When performing a certain measurement on a quantum state, the result generally described by a probability distribution, and the form that this distribution takes is completely determined by the quantum state and the observable describing the measurement. However, unlike in classical mechanics, the result of a measurement on even a pure quantum state is only determined probabilistically. This reflects a core difference between classical and quantum physics.

2.7.1 Pure States.

Mathematically, a pure quantum state is typically represented by a vector in a Hilbert space. In physics, bra-ket notation is often used to denote such vectors. A pure quantum state is a state which can be described by a single ket vector [50]. A simple criterion for describing a pure state is that the trace of ρ is equal to 1 if the state is pure. When we come up with a bipartite state we first think whether it is entangled or not. A pure state can only be separated if it is a product state.

$$|\psi_{prod}\rangle = |\psi\rangle_A \otimes |\phi\rangle_B \tag{2.18}$$

A pure state which can be written as $|\psi\rangle_A$ and $|\phi\rangle_B$ might be a product state. If the state is not a product state, it an entangled state.

In order to write the product state in(2.18) right basis vectors should exist. For this, the Schmidt decomposition is useful [51]. The number of terms in the Schmidt decomposition is called *Schmidt number* or *Schmidt rank*.

$$|\psi\rangle = \sum_{k=1} n\lambda_k |a_k^{'}\rangle \otimes |b_k^{'}\rangle \tag{2.19}$$

The Schmidt coefficient is the square root of the eigenvalues of any of the reduced density matrices. Hence it is the number of nonzero eigenvalues of ρ_A or ρ_B.

What the Schmidt number tells physically how many degrees of freedom are entangled and as the product state can be written as (2.18) it is equal to 1. For pure states, the amount of mixture is a good measurement method.

The only classification we make for bipartite pure states is in separable and entangled states.

2.7.2 Mixed States.

A mixed quantum state is a statistical ensemble of pure states and it cannot be described as a ket vector. Instead of this, it is described by density matrix (or density operator) which is usually shown as ρ [52]. A simple criterion for describing a mixed state is that the trace of ρ_2 is less than 1 if the state is mixed. The structure of mixed state bipartite entanglement is richer than the pure state. For example there are entangled

states that can not be distilled to Bell states, and cannot used in any application where entanglement is an important resource. Apart from this, as the task of knowing whether a mixed state is separable or not gets more difficult, it gives rise to classes of states which can be identified by other criteria. A state can be converted to another state which has less or equal entanglement. However, in mixed states, there are some states which cannot be converted to any other. This makes the mixed state bipartite entanglement's structure richer than the pure state equivalent.

2.8 Density Operators and Mixed States.

Let us describe the state of a subsystem. This subsystem is entangled with another subsystem and we do not have access to the latter. We cannot use state vectors for this purpose, and we need another representation of quantum states, that is *density operators*. In the laboratory it is not possible to isolate the quantum systems completely, because these systems become entangled with the environment due to unwanted, but inevitable interactions. However, we need to describe the system without considering the environment.

In the density operators, quantum states are defined by operators on the system's Hilbert space rather than unit vectors on it. The corresponding density operator is the projection operator $|\psi\rangle\langle\psi|$ for any quantum state vector$|\psi\rangle$. Density operators are also called *density matrices* because linear operators may be represented by matrices. Mixed quantum states are described by density matrices.

Up until now we used density operators instead of vectors. Suppose that there is a machine in our laboratory which outputs particles in state $|\psi\rangle$. But sometimes events happen without our control and knowledge (a needle in the electric grid, opening an electromagnet in the lab near, etc.) cause the machine to lose its function. As a result, states which are close to, but not exactly $|\psi\rangle$ are produced. The state $|\psi^{'}\rangle$ is always created when the machine shows dysfunction, and the possibility for this to happen to any particle is p. With this uncertain situation, we cannot describe the state by a state vector. We have to use density matrices for describing it as

$$\rho = (1-p)|\psi\rangle\langle\psi| + p|\psi^{'}\rangle\langle\psi^{'}| \tag{2.20}$$

A state which we can represent with a unit vector is a pure state. In a pure state we *know maximum* about a state. The rest of the states are *convex combinations* of pure states, and these are called *mixed states*. If $\rho = \rho^2$, the state is a pure state and if $\rho \neq \rho^2$ the state is a mixed state.

The density operator does not express which pure states the state was prepared from. The states which come from different ensembles and which have the same density operator are indistinguishable. As a result, the density operator describes the mixed state as much as possible. This in distinguish ability causes the density operator to be more intuitive representation of a state than the state vector. For instance, two vectors which differ by a phase factor represent the same state, because the state vector has an arbitrary phase factor. However, the phase factor vanishes because of the complex conjugation when the vector is put together with its dual vector in order to form a pure state density operator. As a result, all state vectors, which represent the same state are represented by the same density operator [53]. Some of the properties of density operators are

(i) *positive*. Positive definite operators satisfy vector $|\upsilon\rangle$, $\langle\upsilon|\rho|\upsilon\rangle \geq 0$, for any $|\upsilon\rangle$ and have only Hermitian with positive eigenvalues.

(ii) the trace of a density operator is unity, $tr(\rho) = 1$.

It turns out that any operator satisfying these two criteria is a pure state.

First, we will see that the density operators are used to describe individual parts of an entangled system. After that, we will study that when the composite system is in a pure state, the individual subsystems are in mixed states (if the subsystem are entangled).

Formally, ignoring some degrees of freedom in a system (like the ones corresponding to a particle out of reach)is done by *tracing out* the relevant degrees of freedom from the density operator. This is also called taking the partial trace with respect to the degrees of freedom that are to be traced out. Any bipartite state can be written as

$$\rho = \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l|$$

and the partial trace with respect to system B of a bipartite system is defined as

$$\begin{aligned} tr_B(\rho) &\equiv \sum_{ijkl} c_{ijkl}|a_i\rangle\langle a_j| \otimes tr(|b_k\rangle\langle b_l|) \\ &= \sum_{ijkl} c_{ijkl}\langle b_l|b_k\rangle|a_i\rangle\langle a_j| \\ &= \sum_{ij} C_{ij}|a_i\rangle\langle a_j| \end{aligned} \tag{2.21}$$

with $C_{ij} = \sum_{kl} c_{ijkl}\langle b_l|b_k\rangle$. The density operator obtained by tracing out one part of the system is called the *reduced density operator*.

The density matrix is

$$\rho = |\alpha|^2|00\rangle\langle 00| + \alpha\beta^*|00\rangle\langle 11| + \alpha_*\beta|11\rangle\langle 00| + |\beta|^2|11\rangle\langle 11| \tag{2.22}$$

so the reduced density matrix becomes

$$\begin{aligned} \rho_A &= tr_B(\rho) \\ &= |\alpha|^2\langle 0|0\rangle|0\rangle\langle 0| + \alpha\beta^*\langle 0|1\rangle|0\rangle\langle 1| + \alpha_*\beta\langle 1|0\rangle|1\rangle\langle 0| + |\beta|^2\langle 1|1\rangle|1\rangle\langle 1| \\ &= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| \end{aligned} \tag{2.23}$$

which is just what we expected.

In section 2.6 we saw what we meant by an entangled state as long as the states were pure. But this leaves the question of when a mixed state is entangled. Consider for instance the mixture of two Bell states.

$$\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-| \tag{2.24}$$

Written out in an orthonormal product basis, this is

$$\begin{aligned} \rho &= \frac{1}{4}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\ &\quad + \frac{1}{4}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|) \\ &= \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| \end{aligned} \tag{2.25}$$

which is a mixture of the product states $|00\rangle$ and $|11\rangle$ .So the mixed state can be realized by both an ensemble of maximally entangled states and an ensemble of product states. We say that a mixed state is *separable* if and only if it can be realized as a mixture (convex combination) of locally prepared states. That is, it can be written as

$$\sum_i p_i \rho_i^{(A)} \otimes \rho_i^B \tag{2.26}$$

where $\{p_i\}$ forms a probability distribution; $p_i \geq 0$ and $\sum_i p_i = 1$. This also means that the state can be written as a mixture of pure product states,

$$\sum_{kl} p_{kl} |\psi_k^{(A)}\rangle\langle\psi_k^{(A)}| \otimes |\psi_l^{(B)}\rangle\langle\psi_l^{(B)}| \tag{2.27}$$

Hence a mixture of entangled states need not be entangled, but a mixture of separable states is always separable. Mixture is a process which destroys entanglement. This is because by discarding information about which of a number of entangled states the system is in, it can no longer be distinguished from a mixture of separable states.

2.9 Separable States.

In quantum mechanics, separable quantum states are states which do not have quantum entanglement. As mentioned in section (2.7), a bipartite state is called separable if it can be realized as a mixture of product states. Take two noninteracting systems A and B into account, with respective Hilbert spaces H_A and H_B. The Hilbert space of the composite system is the tensor product $H_A \otimes H_B$. In case the first system is in $|\psi\rangle_A$ state and the second system is in the $|\phi\rangle_B$ state, the state of the composite system is $|\psi\rangle_A \otimes |\phi\rangle_B$. We can call states of the composite system which can be represented in this form *separable states* or *product states*. If a state is inseparable we call it an entangled state. First of all, we take separability for pure states and then we consider separability regarding mixed states.

2.9.1 Separable Pure States

If a pure state $|\psi\rangle \in H_1 \otimes H_2$ can be written as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ where $|\psi_i\rangle$ is a pure state of the i-th subsystem, it is separable. Otherwise it is entangled. The embedding of a product of states into the product space is presented by the Segre embedding. In other words, a quantum-mechanical pure state is separable if it is in the image of the Segre embedding.

2.9.2 Separable Mixed States

Let us think about the mixed state case. The composite system's mixed state is defined by a density matrix ρ acting on $H_1 \otimes H_2$. ρ is separable if there exist $p_k \geq 0$, $\left\{\rho_1^k\right\}$ and $\left\{\rho_2^k\right\}$ which are mixed states of the respective subsystem such that

$$\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k$$

where

$$\sum_k p_k = 1$$

Otherwise ρ is an entangled state and we can understand from the definition that the family of the separable states is a convex set.

2.9.3 Separability criterion

Usually there is no easily computable, necessary and sufficient criterion, but an easy to compute necessary condition exists. This is known as the *Positive Partial Transpose criterion* or *Peres-Horodecki criterion*. In case the state is separable, the partial transpose of the density operator regarding one subsystem becomes positive. A transpose of an operator has to be considered in the light of a basis and the resulting operator depends on this basis. However, because the different bases are connected by a unitary transformation, the eigenvalues do not depend of the basis. Consequently, any orthonormal basis will do for taking the transposition.

In the systems which are in 2 x 2 and 2 x 3 dimension, the PPT criterion is sufficient for the state to be separable[54]. However, for larger systems, there are entangled PPT states [55]. The PPT-criterion is useful and as a result the class of PPT states is a

crucial state class. It is also larger than the separable state classes . Any factor that is valid for all PPT states, is also true for separable states.

2.10 von Neumann Entropy.

In this section we briefly discuss entropy of a mixed state and how it can be viewed as a measure of entanglement [56] von Neumann entropy refers to the extension of classical entropy concepts to the field of quantum mechanics in quantum statistical mechanics. John von Neumann established the correct mathematical framework for quantum mechanics (Mathematische Grundlagen der Quantenmechanik). In this work, he mentioned a theory of measurement. In this measurement, the usual notion of wave collapse is defined as an irreversible process [57]. von Neumann entropy gives the entropy of an ensemble associated to a mixed state with density matrix ρ as follows:

$$S(\rho) \equiv -tr(\rho log \rho) \tag{2.28}$$

where λ_i are the eigenvalues of ρ. It is most easily calculated from the non-zero eigenvalues λ_i of ρ as

$$S(\rho) = -\sum_i \lambda_i log \lambda_i \tag{2.29}$$

Note that zero eigenvalues do not make any contribution.

von Neumann entropy is often regarded as a generalization of the Shannon entropy. Although Shannon entropy causes a special case in quantum information theory in case of orthogonal states, the historical information is different. von Neumann entropy was introduced in 1932 by von Neumann himself and Shannon published his mathematical theory about communication in 1948. von Neumann proposed that Shannon should call his uncertainty function "entropy". This definition was used before in statistical physics. It was derived by using notions such as the ideal gas law and other laws of thermodynamics. According to the criterion is the von Neumann entropy is 0 for a pure state, and positive for a mixed state. In order to calculate, we have to find a basis

in which ρ possesses a diagonal representation. We note that the entropy$S(\rho)$ times the Boltzmann constant k_B is equal to the thermodynamical or physical entropy. If the system is finite (finite dimensional matrix representation) the entropy describes the departure of our system from a pure state [58]. In other words, it measures the degree of mixture of our state describing a given finite system

1.Some properties of the von Neumann entropy

(i) $S(\rho)$ is only zero for pure states.

(ii) $S(\rho)$ is maximum and equal to lnN for a maximally mixed state and N is the dimension of the Hilbert space.

(iii)$S(\rho)$ is invariant under changes in the basis of ρ, that is $S(\rho) = S(U\rho U^{\dagger})$, with U a unitary transformation.

(iv) $S(\rho)$ is concave, in other words, given a collection of positive numbers λ_i which sum to unity $(\sum_i \lambda_{i=1})$ and density operators ρ_i, we have $(\sum_{i=1}^{k} \lambda_i \rho_i) \geq \sum_{i=1}^{k} \lambda_i S(\rho_i)$

(v) $S(\rho)$ is additive. In the case of two density matrices ρ_A, ρ_B which define independent systems A and B, then $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$ Instead, if ρ_A, ρ_B are the reduced density matrices of the general stateρ_{AB}, then $|S(\rho_A) - S(\rho_B)| \leq \rho_{AB} \leq S(\rho_A) + S(\rho_B)$

This right hand inequality is known as sub additivity. The two inequalities together are sometimes known as the triangle inequality. They were proved in 1970 by Huzihiro Araki and Elliott H. Lieb [59]. In Shannon's theory, the entropy of a composite system cannot be lower than the entropy of any its parts. However, in quantum theory this is not the case, i.e., it is possible that $S(\rho_{AB}) = 0$ while $S(\rho_A) > 0$ and $S(\rho_B) > 0$. Actually, this can be regarded as an indicator of an entangled state ρ_{AB}.

(vi) The von Neumann entropy is also strongly sub additive. Given three Hilbert spaces, A,B,C, $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$

This is a much more difficult theorem and was proved in 1973 by Elliott H. Lieb and Mary Beth Ruskai,[60] using a matrix inequality of Elliott H. Lieb[56] proved in 1973.

2.Uses of the von Neumann entropy

The von Neumann entropy is used widely in different forms (relative entropies, conditional entropies etc.) within the scope of quantum information theory. Entanglement measures are based on some quantity which is directly related to the von Neumann

entropy. However, in the literature there have been several papers which are about the possible inadequacy of the Shannon information measure, and as a result of the von Neumann entropy as an appropriate quantum generalization of Shannon entropy. The main idea here is that in classical measurement the Shannon information measure is a natural measure of our missing knowledge about the properties of a system. The existence of these properties are independent of measurement. Conversely, it cannot be claimed that quantum measurement show the properties of a system that existed before the measurement was performed. This controversy has caused some authors to show the non-additivity property of Tsallis' entropy (a generalization of the standard Boltzmann-Gibbs entropy) as the main reason for recovering a true quanta information measure in the quantum context.

2.11 Entanglement of Formation

The*Entanglement of Formation* was historically the first entanglement measure to appear. In the literature it is generally called *entanglement of creation*. It was meant to be the asymptotic entanglement cost, but this interpretation rests on the weak additivity which is strongly conjectured, but still not proved. However, the regularization of it has been shown to be equal to the asymptotic entanglement cost [61].

The entanglement of formation is a direct generalization of the entropy of entanglement in mixed states. The entanglement cost in Bell states of preparing a pure state is obtained by the entropy of entanglement, in the asymptotic limit. As a result, it was normal to define the entanglement of formation for a pure state as the entropy of entanglement. For a given ensemble of pure states $\varepsilon = p_i, |\psi_i\rangle$ the entanglement of formation is the average of the entropy of entanglement for the states in the ensemble

$$E_f(\varepsilon) = \sum_i p_i E_E(|\psi_i\rangle) \tag{2.30}$$

.

A mixed state can be described by a multiple pure state ensembles which have different entanglement of formation. As any of those ensembles describes the mixed state, the natural definition for the entanglement of formation regarding a mixed state is the entanglement of formation for the "most economic" ensemble. In other words,

the entanglement of formation for a mixed state is defined as

$$E_f(\rho) = inf_\varepsilon \sum_i p_i E_E(|\psi_i\rangle) \tag{2.31}$$

Here, infimum is taken over all ensembles $\varepsilon = p_i, |\psi_i\rangle$ that describes the state ρ.

How is entanglement of formation not necessarily equal to the entanglement cost. The asymptotic entanglement cost cannot be larger than the entanglement of formation. Consider that you have to create a large number $n_{out}(n_{out} \to \infty)$ of some bipartite state ρ_{out}. You will find the pure state realization $p_i, |\psi_i\rangle$ of ρ_{out} that has the lowest average entropy of entanglement (equal to the entanglement of formation).

From the definition of E_f it is easy to see that it satisfies (E0a) and (E1a). (E0a) is a mixture of pure states. (E1a) reduces to the entropy of entanglement on pure states. Monotonicity (E2a), continuity (E3a) and sub additivity (E5a) are shown in [61, 53]. The way of the entropy of entanglement was extended to mixed states in the case of the entanglement of formation, can also be used for other measures on pure states. Measures in this way are called *convex roof measures* [62, 63].

There are two methods for calculate entanglement. One of them is ***concurrence***. Concurrence is defined for a two qubit system. For a general state ρ, let $\tilde{\rho}$ be the spin-flipped state

$$\tilde{\rho} \equiv (Y \otimes Y)\rho^*(Y \otimes Y) \tag{2.32}$$

where the Y is the Pauli Y operator and ρ^* is the complex conjugate of ρ, both taken in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Let the Hermitian matrix R be defined as

$$r \equiv \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \tag{2.33}$$

with eigenvalues in decreasing order $\{\lambda_i\}$. The concurrence is defined as

$$C(\rho) \equiv max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{2.34}$$

There have been various attempts to generalize the concept of concurrence. Uhlmann [63] considered general conjugations instead of the special (2.32). This was further generalized to concurrence vectors by Auderaert *et al.* [64]. The previously mentioned convex roof extended negativity coincides with the concurrence in the case of two qubits [65], and thus provides another generalization. Some of those generalizations-along with other aspects of concurrence and entanglement of formation-are reviewed in [66].

The other method is ***fidelity***. Fidelity is a popular measure of distance between density operators and it is not a metric system, but has some useful properties. Fidelity as a distance measure between pure states was called *transition probability.* For two states given by unit vectors φ, ψ it is $|\langle\phi, \psi\rangle|^2$. For a pure state (vector ψ) and a mixed state (density matrix ρ) this generalizes to $\langle\psi|\rho|\psi\rangle$. For two density matrices ρ, σ it is generalized as the largest fidelity between any two purifications of the given states [67]. According to a theorem, this situation leads to the expression

$$F(\rho, \sigma) = \left(tr\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2 \tag{2.35}$$

This is exactly the expression used by Richard Jozsa [70],where the term fidelity appears to have been used first.

However, one can also start from $|\phi\rangle, \langle\psi|$, leading to the alternative

$$F(\rho, \sigma) = tr\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \tag{2.36}$$

used in [71]. This second quantity is sometimes denoted as $\sqrt{F}$ and called *square root fidelity.* It has no interpretation as a probability, but appears in some estimates in a simpler way.

Basic Properties

If $\rho = |\psi\rangle\langle\psi|$ is pure, then $F(\rho, \sigma) = \langle\psi|\sigma|\psi\rangle$ and if both states are pure i.e. $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, then $F(\rho, \sigma) = ||\psi\rangle\langle\phi||^2$.

Other properties:

1. $0 \leq F(\rho, \sigma) \leq 1$

2. $F(\rho,\sigma) = F(\sigma,\rho)$

3. $F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1 \otimes \sigma_1)F(\rho_2 \otimes \sigma_2)$

4. $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho,\sigma)$

5. $F(\rho, \alpha\sigma_1 + (1-\alpha)\sigma_2) \geq \alpha F(\rho,\sigma_1) + (1-\alpha)F(\rho,\sigma_2), \alpha \in [0,1]$

Classical Fidelity

Fidelity is also defined for classical probability distributions. Let $\{p_i\}$ and $\{q_i\}$ where $i = 1, 2, ..., n$ be probability distributions [68] The fidelity between p and q is defined as

$$F(\{p_i\},\{q_i\}) = \sum_{i=1} n\sqrt{p_i, q_i} \tag{2.37}$$

Superfidelity

Superfidelity is a similarity measure between density operators.It is defined as

$$G(\rho,\sigma) = tr\rho\sigma + \sqrt{1-tr(\rho^2)}\sqrt{1-tr(\sigma^2)} \tag{2.38}$$

Superfidelity was introduced in[69] as an upper bound for fidelity.

Fidelity Of Quantum States

Fidelity is a measure of the *closeness* of two quantum states in quantum information theory. It is not a metric on the space of density matrices. In probability theory, in case two random variables $p = (p_1), ..., p_n$ and $q = q_1, ..., q_n$ are given on the probability space, we have $X = \{1, 2, ..., n\}$. The fidelity of p and q is defined to be the quantity

$$F(p,q) = \sum_i \sqrt{p_i q_i} \tag{2.39}$$

That is, the fidelity $F(p,q)$ is the inner product of $(\sqrt{p_1}, ..., \sqrt{p_n})$ and $(\sqrt{q_1}, ..., \sqrt{q_n})$ and it is viewed as vectors in Euclidean space. Please note that when $p = q$, $F(p,q) = 1$. In general, $0 \leq F(p,q) \leq 1$.

Making the appropriate modification for the matricidal notion of square root and mimicking the above definition gives the fidelity of two quantum state.

If two density matrices like ρ and σ are given, the fidelity is defined by

$$F(p,q) = Tr\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right] \tag{2.40}$$

Hilbert-Schmidt inner product takes the place of the Euclidean inner product from the classical definition. When the states are classical,(ρ and σ commute), the definition coincides with that for probability distributions.

CHAPTER 3
INTERACTION ONE ATOM-ONE FIELD WITH RANDOM PHASE TELEGRAPH NOISE

In this chapter the interaction of a two-level atom with a single-mode field is review in the environment or the random phase telegraph noise by the JCM. We include the effects of stochastic phase fluctuations are modeled by random telegraph process and an equation for the density operator averaged over the distribution of fluctuations has been obtained.The solution of this equation is used to study the resulting decoherence effects in the dynamical behavior of the atom and the statistical properties of the field. We solve the problem by two different techniques in this chapter. This chapter is based on the article of Lawande, Joshi *et al.* [72, 73]

3.1 Generalized Jaynes-Cumming Model with Random Phase Telegraph Noise

3.1.1 The Formulation of the Problem

The interaction of a single two-level atom with a single mode of radiation field is considered. While the atom is characterized by spin $\frac{-1}{2}$ angular-momentum operators $S_{\pm}$, S_z the field is described by the annihilation and the creation operators a and $a^{\dagger}$. We assume that the field is in resonance with the atomic transition frequency ω_0. In the general rotating-wave approximation, the Hamiltonian of the system has the following form

$$H = \omega_0 S_z + \omega_0 a^{\dagger} a + [g^*(t) S_+ a + g(t) S_- a^{\dagger}] \tag{3.1}$$

where the coupling g(t) between the atom and the field is time dependent.In this section an alternative model that represents noise by means of discrete jump process will be studied. A simple example of such a jump process is two state random telegraph. In order to show such a model we assume that

$$g(t) = g_0 e^{-i\phi(t)} \tag{3.2}$$

where the $\phi(t)$ is treated as a stochastic variable while the non-stochastic amplitude g_0 is a positive reel quantity. In the following we have telegraph noise where $\phi(t)$ randomly jumps between two possible states a and $-a$, the correlation function

$$\begin{aligned} \langle g(t)g(t)^{'}\rangle &= g_0^2\langle e^{\{i[\phi(t)\phi(t)^{'}]\}}\rangle \\ &= g_0^2[cos^2 a + sin^2 a e^{(-2)|t-t^{'}|/T}] \end{aligned} \tag{3.3}$$

where the τ_0 is the mean dwell time of the telegraph. The jumps are separated by an average time interval called the mean dwell time.

3.1.2 Solution

We will examine the dynamics in the telegrapher noise. For this, we produce equation of motion from the Liouville-von Neumann [56] equation for the density operator and the interaction representation can be used.

$$i\partial\rho/\partial t = [H_1, \rho] = H_1\rho - \rho H_1 \tag{3.4}$$

where H_1 shows the Hamiltonian interaction

$$H_1 = g_0[e^{i\phi(t)}S_+ a + e^{-i\phi(t)}S_- a^\dagger] \tag{3.5}$$

We now introduce the states equation

$$\begin{aligned} |\Psi_m\rangle &= |m, e\rangle \\ |\Psi_m\rangle &= |m+1, g\rangle \end{aligned} \tag{3.6}$$

and

$$\rho^{\pm}_{mn}(t) = \langle \Psi^{\pm}_{m} | \rho(t) | \psi^{\mp}_{n} \rangle \tag{3.7}$$

By using the (3.7) equation we can write the equations below;

$$\rho^{++}_{mn}(t) = \langle \Psi^{+}_{m} | \rho(t) \Psi^{+}_{n} \rangle \tag{3.8}$$

$$\rho^{+-}_{mn}(t) = \langle \Psi^{+}_{m} | \rho(t) | \Psi^{-}_{n} \rangle \tag{3.9}$$

$$\rho^{-+}_{mn}(t) = \langle \Psi^{-}_{m} | \rho(t) | \Psi^{+}_{n} \rangle \tag{3.10}$$

$$\rho^{--}_{mn}(t) = \langle \Psi^{-}_{m} | \rho(t) | \Psi^{-}_{n} \rangle \tag{3.11}$$

From the equation above we can write(m=n)

$$\rho^{++}_{mn}(t) = \langle ne | \rho(t) | ne \rangle \tag{3.12}$$

$$\rho^{+-}_{mn}(t) = \langle ne | \rho(t) | n+1g \rangle \tag{3.13}$$

$$\rho^{-+}_{mn}(t) = \langle n+1g | \rho(t) | ne \rangle \tag{3.14}$$

$$\rho^{--}_{mn}(t) = \langle n+1g | \rho(t) | n+1g \rangle \tag{3.15}$$

By using the equation (3.4) we get

$$i\partial\rho/\partial t = H_1 \rho^{++}_{mn}(t) - \rho^{++}_{mn}(t) H_1 \tag{3.16}$$

$$i\partial\rho/\partial t = H_1\rho_{mn}^{+-}(t) - \rho_{mn}^{+-}(t)H_1 \tag{3.17}$$

$$i\partial\rho/\partial t = H_1\rho_{mn}^{-+}(t) - \rho_{mn}^{-+}(t)H_1 \tag{3.18}$$

$$i\partial\rho/\partial t = H_1\rho_{mn}^{--}(t) - \rho_{mn}^{--}(t)H_1 \tag{3.19}$$

From equations (3.16), (3.17), (3.18), (3.19), we obtain the equations below.

$$\frac{d}{dt}\rho_{mn}^{\pm,\mp}(t) = ie^{\pm i\phi(t)}[\alpha_n\rho_{mn}^{\pm,\pm}(t) - \alpha_m\rho_{mn}^{\mp,\mp}(t)] \tag{3.20}$$

$$\frac{d}{dt}\rho_{mn}{}^{\pm,\pm}(t) = i[\alpha_n e^{\mp i\phi(t)}\rho_{mn}^{\pm,\mp}(t) - \alpha_m e^{\pm i\phi(t)}\rho_{mn}^{\mp,\pm}(t)] \tag{3.21}$$

If we use the equations (3.20) and (3.21) we can write the equations below.

$$\frac{d}{dt}\rho_{mn}^{++}(t) = i[\alpha_n e^{-i\phi(t)}\rho_{mn}^{+-}(t) - \alpha_m e^{+i\phi(t)}\rho_{mn}^{-+}(t)] \tag{3.22}$$

$$\frac{d}{dt}\rho_{mn}^{+-}(t) = ie^{+i\phi(t)}[\alpha_n\rho_{mn}^{++}(t) - \alpha_m\rho_{mn}^{--}(t)] \tag{3.23}$$

$$\frac{d}{dt}\rho_{mn}^{-+}(t) = ie^{-i\phi(t)}[\alpha_n\rho_{mn}^{--}(t) - \alpha_m\rho_{mn}^{++}(t)] \tag{3.24}$$

$$\frac{d}{dt}\rho_{mn}^{--}(t) = i[\alpha_n e^{+i\phi(t)}\rho_{mn}^{-+}(t) - \alpha_m e^{-i\phi(t)}\rho_{mn}^{+-}(t)] \tag{3.25}$$

When we write 1 for + and 2 for - in the equations above, we find the equations below.

$$\frac{d}{dt}\rho_{11}(t) = i[\alpha_n e^{-i\phi(t)}\rho_{12}(t) - \alpha_m e^{+i\phi(t)}\rho_{21}(t)] \tag{3.26}$$

$$\frac{d}{dt}\rho_{12}(t) = ie^{+i\phi(t)}[\alpha_n\rho_{11}(t) - \alpha_m\rho_{22}(t)] \tag{3.27}$$

$$\frac{d}{dt}\rho_{21}(t) = ie^{-i\phi(t)}[\alpha_n\rho_{22}(t) - \alpha_m\rho_{11}(t)] \tag{3.28}$$

$$\frac{d}{dt}\rho_{22}(t) = i[\alpha_n e^{+i\phi(t)}\rho_{21}(t) - \alpha_m e^{-i\phi(t)}\rho_{12}(t)] \tag{3.29}$$

Here $\alpha_n = g_0\sqrt{n+1}$

These equations should be solved by taking the stochastic nature of the phase variable $\phi(t)$ into account. We can first solve equations(3.23) and (3.24) and insert to equations (3.22) and (3.25). Then we have the suitable solution.

We considered that $\rho_{mn}^{\pm,\mp} = 0$ and $t^{'} < t$. Now we add the suitable equations to the equations (3.22) and (3.25)

$$\begin{aligned}
\frac{d}{dt'}\rho_{mn}^{++}(t) &= i\left[\alpha_n e^{-i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{+-}(t')dt' - \alpha_m e^{+i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{-+}(t')dt'\right] \\
&= i\alpha_n e^{-i\phi(t)}\int_0^t dt' i\left[\alpha_n e^{+i\phi(t')}\rho_{mn}^{++}(t') - \alpha_m e^{+i\phi(t')}\rho_{mn}^{--}(t')\right] \\
&\quad - \alpha_m e^{+i\phi(t)}\int_0^t dt' i\left[[\alpha_n e^{-i\phi(t')}\rho_{mn}^{--}(t') - \alpha_m e^{-i\phi(t')}\rho_{mn}^{++}(t')\right] \\
&= -\int_0^t [dt'\{\alpha_n e^{-i[\phi(t)-\phi(t')]}][\alpha_n\rho_{mn}^{++}(t') - \alpha_m\rho_{mn}^{--}(t')] \\
&\quad + \alpha_m e^{i[\phi(t)-\phi(t')]}[\alpha_m\rho_{mn}^{++}(t') - \alpha_n\rho_{mn}^{--}(t)]\}
\end{aligned} \tag{3.30}$$

Likewise

$$\begin{aligned}
\frac{d}{dt'}\rho_{mn}^{--}(t) &= i\left[\alpha_n e^{+i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{-+}(t')dt' - \alpha_m e^{-i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{+-}(t')dt'\right] \\
&= i\alpha_n e^{+i\phi(t)}\int_0^t dt' i\left[\alpha_n e^{-i\phi(t')}\rho_{mn}^{--}(t') - \alpha_m e^{-i\phi(t')}\rho_{mn}^{++}(t')\right] \\
&\quad - \alpha_m e^{-i\phi(t)}\int_0^t dt' i\left[[\alpha_n e^{+i\phi(t')}\rho_{mn}^{++}(t') - \alpha_m e^{+i\phi(t')}\rho_{mn}^{--}(t')\right] \\
&= \int_0^t [dt'\{\alpha_n e^{-i[\phi(t)-\phi(t')]}][\alpha_n\rho_{mn}^{++}(t') - \alpha_m\rho_{mn}^{--}(t')] \\
&\quad + \alpha_m e^{i[\phi(t)-\phi(t')]}[\alpha_m\rho_{mn}^{++}(t') - \alpha_n\rho_{mn}^{--}(t)]\}
\end{aligned} \tag{3.31}$$

By describing these equations, we have assumed that $\rho_{mn}^{\pm,\mp} = (0)$ and the reason will be explained later. Similar equations for the evolution of $\rho_{mn}^{\pm,\mp} = (t)$ can be produced, but we do not need them in the present paper. While describing equations (3.22) and (3.25) in the above form the idea that lies behind reproducing the equation is to make use of the Markovian property of the telegrapher noise. According to this, since $t^{'} < t$, the part which involves the exponential factor in equations (3.30)and (3.31) gets decoupled from the other part and this depends on $t^{'}$. This factor lets us take the average over the stochastic distribution of the telegrapher noise. For this, we must use $\langle\rangle$ symbol in order to show such an average and write

$$c(t-t^{'}) = \langle e^{\bar{+}i[\phi(t)-\phi(t^{'})]}\rangle \tag{3.32}$$

Here we take average of equation (3.30) and equation (3.31) then we use equation (3.32).

$$\begin{aligned}
\frac{d}{dt}\langle\rho_{mn}^{++}(t)\rangle &= -\int_0^t dt^{'}\{\alpha_n\langle e^{-i[\phi(t)-\phi(t^{'})]}\rangle[\alpha_n\langle\rho_{mn}^{++}(t^{'})\rangle - \alpha_m\langle\rho_{mn}^{--}(t^{'})\rangle] \\
&\quad + \alpha_m\langle e^{i[\phi(t)-\phi(t^{'})]}\rangle[\alpha_m\langle\rho_{mn}^{++}(t^{'})\rangle - \alpha_n\langle\rho_{mn}^{--}(t^{'})\rangle]\} \\
&= -\int_0^t dt^{'}c(t-t^{'})\{(\alpha_n^2+\alpha_m^2)\langle\rho_{mn}^{++}(t^{'})\rangle - 2\alpha_n\alpha_m\langle\rho_{mn}^{--}(t^{'})\rangle\}
\end{aligned} \tag{3.33}$$

The average of equation (3.31)

$$\begin{aligned}
\frac{d}{dt}\langle\rho_{mn}^{--}(t)\rangle &= \int_0^t dt^{'}\{\alpha_m\langle e^{-i[\phi(t)-\phi(t^{'})]}\rangle[\alpha_n\langle\rho_{mn}^{++}(t^{'})\rangle - \alpha_m\langle\rho_{mn}^{--}(t^{'})\rangle] \\
&\quad + \alpha_n\langle e^{i[\phi(t)-\phi(t^{'})]}\rangle[\alpha_m\langle\rho_{mn}^{++}(t^{'})\rangle - \alpha_n\langle\rho_{mn}^{--}(t^{'})\rangle]\} \\
&= \int_0^t dt^{'}c(t-t^{'})\{2\alpha_n\alpha_m\langle\rho_{mn}^{++}(t^{'})\rangle - (\alpha_n^2+\alpha_m^2)\langle\rho_{mn}^{--}(t^{'})\rangle\}
\end{aligned} \tag{3.34}$$

for it to be easy we may introduce two quantities:

$$\mathbb{F}_{mn}^{\pm}(t) = \frac{1}{2}\{\langle\rho_{mn}^{++}(t)\rangle \pm \langle\rho_{mn}^{--}(t)\rangle\} \tag{3.35}$$

We add between the equations (3.33) and (3.34)

$$\begin{aligned}
\frac{d}{dt}\{\langle\rho_{mn}^{++}(t)\rangle + \langle\rho_{mn}^{--}(t)\rangle\} &= \int_0^t [dt^{'}c(t-t^{'})\{-(\alpha_m-\alpha_n)^2\langle\rho_{mn}^{++}(t^{'})\rangle - (\alpha_m-\alpha_n)^2\langle\rho_{mn}^{--}(t^{'})\rangle\}] \\
2\frac{d}{dt}F_{mn}^{+}(t) &= \int_0^t [dt^{'}c(t-t^{'})[-(\alpha_m-\alpha_n)^2]\{\langle\rho_{mn}^{++}(t^{'})\rangle + \langle\rho_{mn}^{--}(t^{'})\rangle\}] \\
\frac{d}{dt}F_{mn}^{+}(t) &= \int_0^t [dt^{'}c(t-t^{'})[-(\alpha_m-\alpha_n)^2]\frac{1}{2}\{\langle\rho_{mn}^{++}(t^{'})\rangle + \langle\rho_{mn}^{--}(t^{'})\rangle\}] \\
\frac{d}{dt}F_{mn}^{+}(t) &= -(\Omega_{mn}^{-})^2\int_0^t [dt^{'}c(t-t^{'})F_{mn}^{+}(t^{'})] \qquad (3.36)
\end{aligned}$$

Then we make subtraction between the equations (3.33) and(3.34)

$$\begin{aligned}
\frac{d}{dt}\{\langle\rho_{mn}^{++}(t)\rangle - \langle\rho_{mn}^{--}(t)\rangle\} &= \int_0^t [dt^{'}c(t-t^{'})\{(\alpha_m+\alpha_n)^2\langle\rho_{mn}^{++}(t^{'})\rangle - (\alpha_m+\alpha_n)^2\langle\rho_{mn}^{--}(t^{'})\rangle\}] \\
2\frac{d}{dt}F_{mn}^{+}(t) &= \int_0^t [dt^{'}c(t-t^{'})[(\alpha_m+\alpha_n)^2]\{\langle\rho_{mn}^{++}(t^{'})\rangle - \langle\rho_{mn}^{--}(t^{'})\rangle\}] \\
\frac{d}{dt}F_{mn}^{+}(t) &= \int_0^t [dt^{'}c(t-t^{'})[(\alpha_m+\alpha_n)^2]\frac{1}{2}\{\langle\rho_{mn}^{++}(t^{'})\rangle - \langle\rho_{mn}^{--}(t^{'})\rangle\}] \\
\frac{d}{dt}F_{mn}^{+}(t) &= -(\Omega_{mn}^{+})^2\int_0^t [dt^{'}c(t-t^{'})F_{mn}^{-}(t^{'})] \qquad (3.37)
\end{aligned}$$

where $(\Omega_{mn}^{\pm})^2 = (\alpha_m \pm -\alpha_n)^2 = g_0^2(\sqrt{m+1} \pm -\sqrt{n+1})^2$

Equations (3.36) and (3.37) can be solved by using Laplace transform techniques. The form of the correlation function $c(t)$ depends on the telegrapher process. We get solution of (3.36) and (3.37)in the two cases of telegraph noise and use it in order to mention the behavior of atomic observables and the field statistics.The correlation function $c(t-t^{'})$ for the phase telegraph is presented by equation (3.3)

$$C(t-t^{'}) = \cos^2 a + \sin^2 a e^{\frac{-2|t-t^{'}|}{T}} \qquad (3.38)$$

When we put equation (3.38)in equation (3.36)and (3.37) then we take the Laplace transform with respect to t, we get $\widetilde{F}_{mn}^{\pm}(z)$. First of all, we find $\widetilde{F}_{mn}^{+}(z)$;

$$
\begin{aligned}
\frac{d}{dt}F^{+}_{mn}(t) &= -(\Omega^{-}_{mn})^2 \int_0^t dt^{'} c(t-t^{'}) F^{+}_{mn}(t^{'}) \\
\mathcal{L}\{\frac{d}{dt}F^{+}_{mn}(t)\} &= -\Omega^{-}_{mn})^2 L\Big\{ \int_0^t dt^{'} c(t-t^{'}) F^{+}_{mn}(t^{'}) \Big\} \\
z\widetilde{F}^{+}_{mn}(z) - F^{+}_{mn}(0) &= -(\Omega^{-}_{mn})^2 L\{c(t)\}\widetilde{F}^{+}_{mn}(z) \\
\widetilde{F}^{+}_{mn}(z) &= \frac{F^{+}_{mn}(0)}{z + (\Omega^{-}_{mn})^2 L\{c(t)\}}
\end{aligned} \tag{3.39}
$$

Since $c(t) = \cos^2 a + \sin^2 a e^{\frac{-2|t|}{T}}$

$$
\mathcal{L}\{c(t)\} = \frac{\cos^2 a}{z} + \frac{\sin^2 a}{\frac{2}{T} + z} \tag{3.40}
$$

After adding equation (3.40) in equation (3.39) and then simplifying it, we find

$$
\widetilde{F}^{+}_{mn}(z) = \frac{z(z + 2/T)F^{+}_{mn}(0)}{z^3 + 2z^2/T + (\Omega^{-}_{mn})^2 2\cos^2 a/T + z(\Omega^{-}_{mn})^2} \tag{3.41}
$$

By applying the same procedure in the equation above, we have

$$
\widetilde{F}^{-}_{mn}(z) = \frac{z(z + 2/T)F^{-}_{mn}(0)}{z^3 + 2z^2/T + (\Omega^{+}_{mn})^2 2\cos^2 a/T + z(\Omega^{+}_{mn})^2} \tag{3.42}
$$

by equations (3.41)and(3.42) we get

$$
\widetilde{F}^{\pm}_{mn}(z) = \frac{z(z + 2/T)F^{\pm}_{mn}(0)}{z^3 + 2z^2/T + (\Omega^{\pm}_{mn})^2 \frac{2\cos^2 a}{T} + z(\Omega^{\pm}_{mn})^2} \tag{3.43}
$$

Here $\widetilde{F}^{\pm}_{mn}(z)$ is the Laplace transform of $F^{\pm}_{mn}(t)$.

If n=m

$$
(\Omega^{+}_{mn})^2 = (\alpha_m - \alpha_n)^2 = (\alpha_n - \alpha_n)^2 = 0 \tag{3.44}
$$

and we have from equation (3.36),we get

$$\begin{aligned}
\frac{d}{dt}F^{+}_{mn}(t) &= 0 \\
\mathcal{L}\{\frac{d}{dt}F^{+}_{mn}(t)\} &= \mathcal{L}\{0\} \\
z\widetilde{F}^{+}_{mn}(z) - F^{+}_{mn}(0) &= 0 \\
z\widetilde{F}^{+}_{mn}(z) &= F^{+}_{mn}(0) \\
\widetilde{F}^{+}_{mn}(z) &= \frac{F^{+}_{mn}(0)}{z}
\end{aligned} \tag{3.45}$$

When we take inverse Laplace transform of the equation (3.45), we have

$$F^{+}_{mn}(t) = F^{+}_{mn}(0) \tag{3.46}$$

If $n \neq m$,

we take the inverse Laplace transform of (3.41) and (3.42) we get the general solution.

$$F^{\pm}_{mn}(t) = F^{\pm}_{mn}(0)\{\sum_{j=1}^{3} \frac{\lambda_j(\lambda_j + 2/T)}{\prod_{k\neq j}(\lambda_j - \lambda_k)} e^{\lambda_j t}\} \tag{3.47}$$

where λ_j are roots of the cubic equation in(3.41) and (3.42)

$$\lambda^3 + (2/T)\lambda^2 + (\Omega^{\pm}_{mn})^2\lambda + (2/T)(\Omega^{\pm}_{mn})^2 cos^2 a = 0 \tag{3.48}$$

This equation has one negative real and a pair of complex-conjugate roots.

We can get the approximate solution in the two limits of large and small T. In case T is rather large ($g_0 T \gg 1$), the real approximate roots of the cubic equation (3.48) are

$$\lambda_1 = -(\frac{2}{T})\cos^2 a \tag{3.49}$$

$$\lambda_{2,3} = \pm i(\Omega_{mn}^{\mp}) - \frac{\sin^2 a}{T} \tag{3.50}$$

In this part, we rewrite the solution by the Laplace transform techniques. First we will find the elements of matrix.

$$\begin{aligned} H_{int}(t)|ne\rangle &= g^*(t)S_+a + g(t)S_-a^\dagger|ne\rangle \\ &= g(t)\sqrt{n+1}|n+1g\rangle \\ H_{int}(t)|n+1g\rangle &= g^*(t)S_+a + g(t)S_-a^\dagger|n+1g\rangle \\ &= g^*(t)\sqrt{n+1}|ne\rangle \end{aligned} \tag{3.51}$$

$$\begin{aligned} \langle ne|H_{int}(t)|ne\rangle &= \langle ne|g(t)\sqrt{n+1}|n+1g\rangle \\ &= g(t)\sqrt{n+1}\langle ne|n+1g\rangle \\ &= g(t)\sqrt{n+1}0 \\ &= 0 \end{aligned} \tag{3.52}$$

$$\begin{aligned} \langle ne|H_{int}(t)|n+1g\rangle &= \langle ne|g^*(t)\sqrt{n+1}|ne\rangle \\ &= g^*(t)\sqrt{n+1}\langle ne|ne\rangle \\ &= g^*(t)\sqrt{n+1} \end{aligned} \tag{3.53}$$

$$\begin{aligned}\langle n+1g|H_{int}(t)|ne\rangle &= \langle n+1g|g(t)\sqrt{n+1}|n+1g\rangle \\ &= g(t)\sqrt{n+1}\langle n+1g|n+1g\rangle \\ &= g(t)\sqrt{n+1} \end{aligned} \tag{3.54}$$

$$\begin{aligned}\langle n+1g|H_{int}(t)|n+1g\rangle &= \langle n+1g|g^*(t)\sqrt{n+1}|ne\rangle \\ &= g(t)^*\sqrt{n+1}\langle n+1g|ne\rangle \\ &= g(t)^*\sqrt{n+1}0 \\ &= 0 \end{aligned} \tag{3.55}$$

Now we can write the matrix

$$H_{int}(t) = \begin{pmatrix} 0 & g^*(t)\sqrt{n+1} \\ g(t)\sqrt{n+1} & 0 \end{pmatrix} \tag{3.56}$$

In order to illustrate our model we assume that

$$g(t) = g_0 e^{[+i\phi(t)]} \tag{3.57}$$

where the non-stochastic amplitude g_0 is a positive real quantity and the phase $\phi(t)$ is regarded as a stochastic variable.

The correlation functions $(g(t), g(t^{'}))$ are different from each other in the two cases. However, in the present work we only mention the random phase telegraph noise; as a result, the interaction Hamiltonian $H_{int}(t)$ depends parametrically on ϕ. That is, the randomness element is introduced in the time dependence of the Hamiltonian $H = H_0 + H_{int}(t)$ only by a change n $\phi(t)$.

$$dQ(t) = exp(\frac{-\tau}{\tau_0})\frac{d\tau}{\tau_0} \tag{3.58}$$

The distribution above mentions the probability of duration of each such jump interval.

To specify the dynamical evolution of system we have to know the unitary transformation $U(t,t^{'})$ such that

$$\rho(t) = U(\phi(t),t,t^{'})\rho(t^{'})U^{-1}(\phi(t),t,t^{'}) \tag{3.59}$$

If $U = e^{(-iH_{int})}$, we have the following equation

$$U = \begin{pmatrix} \frac{1}{2}e^{-i\sqrt{1+n}+g_0}(1+e^{2i\sqrt{1+n}+g_0}) & -\frac{1}{2}e^{-i\phi(t)}e^{-i\sqrt{1+n}+g_0}(-1+e^{2i\sqrt{1+n}+g_0}) \\ -\frac{1}{2}e^{i\phi(t)}e^{-i\sqrt{1+n}+g_0}(-1+e^{2i\sqrt{1+n}+g_0}) & \frac{1}{2}e^{-i\sqrt{1+n}+g_0}(1+e^{2i\sqrt{1+n}+g_0}) \end{pmatrix} \tag{3.60}$$

We make the equation(3.60) simpler and get the unitary transformation

$$U = \begin{pmatrix} \cos[i\sqrt{1+n}+g_0] & -e^{-i\phi(t)}\sin[i\sqrt{1+n}+g_0] \\ -e^{i\phi(t)}\sin[i\sqrt{1+n}+g_0] & \cos[i\sqrt{1+n}+g_0]) \end{pmatrix} \tag{3.61}$$

If $\cos(\theta_n(t)) = \cos[i\sqrt{1+n}+g_0]$,then $\theta_n(t) = \sqrt{1+n}+g_0$ and $\theta_n = g_0\sqrt{1+n}$.

The elements which we will average in the calculations of G are the ones which contain the $exp(\pm i\phi(t))$ factors. The phases are equally probable as $dQ = \frac{d\phi}{2\pi}$ and as a result most of the terms disappear after averaging process. The remaining non-vanishing elements of G are

$G_{11}^{11} = \int U_{11}U_{11}^{-1}d\phi$ $\quad$ $G_{11}^{21} = \int U_{21}U_{11}^{-1}d\phi$

$G_{12}^{11} = \int U_{12}U_{11}^{-1}d\phi$ $\quad$ $G_{12}^{21} = \int U_{22}U_{11}^{-1}d\phi$

$G_{21}^{11} = \int U_{11}U_{21}^{-1}d\phi$ $\quad$ $G_{21}^{21} = \int U_{21}U_{21}^{-1}d\phi$

$G_{22}^{11} = \int U_{12}U_{21}^{-1}d\phi$ $\quad$ $G_{22}^{21} = \int U_{22}U_{12}^{-1}d\phi$

$G_{11}^{12} = \int U_{11}U_{12}^{-1}d\phi$ $\quad$ $G_{11}^{22} = \int U_{21}U_{12}^{-1}d\phi$

$G_{12}^{12} = \int U_{12}U_{12}^{-1}d\phi$ $\quad$ $G_{12}^{22} = \int U_{22}U_{12}^{-1}d\phi$

$G_{21}^{12} = \int U_{11}U_{22}^{-1}d\phi$ $\quad$ $G_{21}^{22} = \int U_{21}U_{22}^{-1}d\phi$

$G_{22}^{12} = \int U_{12}U_{22}^{-1}d\phi$ $\quad$ $G_{22}^{22} = \int U_{22}U_{22}^{-1}d\phi$

First we find inverse of U.

$$U^{-1} = \begin{pmatrix} \cos[g(t)i\sqrt{1+n}t] & e^{-i\phi(t)}\sin[g(t)i\sqrt{1+n}t] \\ e^{i\phi(t)}\sin[g(t)i\sqrt{1+n}t] & \cos[g(t)i\sqrt{1+n}t]) \end{pmatrix} \tag{3.62}$$

Next we define

$$G^{im}_{lk} = \int U_{ik}(\phi;\tau,t)U^{-1}_{lm}(\phi;\tau,t)dQ(\phi)$$

In case of

$$G^{im}_{lk} = G^{mi}_{kl} \tag{3.63}$$

we get

$$\rho_{im}(\tau) = e^{(-\tau/\tau_0)}Tr[G^{im}(\tau,0)\rho(0)] + \frac{1}{\tau_0}\int_0^\tau dte^{(-\tau-t)/\tau_0}Tr[G^{im}(\tau,t)\rho(t)] \tag{3.64}$$

Here we will find $G^{11}_{11}, G^{22}_{22}, G^{11}_{22}, G^{11}_{12}, G^{11}_{21}$

$$\begin{aligned} G^{11}_{11} &= \int_0^{2\pi} U_{11}(\phi;\tau,t)U^{-1}_{11}(\phi;\tau,t)dQ(\phi) \\ &= \int_0^{2\pi} \cos[\theta_n(\tau-t)]\cos[\theta_n(\tau-t)]\frac{d\phi}{2\pi} \\ &= \cos^2\theta_n(\tau-t) \end{aligned} \tag{3.65}$$

$$\begin{aligned} G^{11}_{12} &= \int_0^{2\pi} U_{12}(\phi;\tau,t)U^{-1}_{21}(\phi;\tau,t)dQ(\phi) \\ &= \int_0^{2\pi} -ie^{-i\phi(t)}\sin[\theta_n(\tau-t)]\cos[\theta_n(\tau-t)]\frac{d\phi}{2\pi} \\ &= 0 \end{aligned} \tag{3.66}$$

$G^{11}_{21} = 0$ because $G^{11}_{12} = G^{11}_{21}$

$$
\begin{aligned}
G_{22}^{11} &= \int_0^{2\pi} U_{12}(\phi;\tau,t)U_{21}^{-1}(\phi;\tau,t)dQ(\phi) \\
&= \int_0^{2\pi} (-i)e^{-i\phi(t)}\sin[\theta_n(\tau-t)](+i)e^{i\phi(t)\sin[\theta_n(\tau-t)]\frac{d\phi}{2\pi}} \\
&= \sin^2[\theta_n(\tau-t)] \qquad (3.67)
\end{aligned}
$$

We can writeG^{11} by the equations above

$$
G^{11} = \begin{pmatrix} G_{11}^{11} & G_{12}^{11} \\ G_{21}^{11} & G_{22}^{11} \end{pmatrix} = \begin{pmatrix} \cos^2\theta_n(\tau) & 0 \\ 0 & \sin^2\theta_n(\tau) \end{pmatrix} \qquad (3.68)
$$

$G_{11}^{22} = \sin^2\theta_n(\tau-t)$ because $G_{11}^{22} = G_{22}^{11}$

$$
\begin{aligned}
G_{12}^{22} &= \int_0^{2\pi} U_{22}(\phi;\tau,t)U_{12}^{-1}(\phi;\tau,t)dQ(\phi) \\
&= \int_0^{2\pi} \cos\theta_n(t)e^{-i\phi(t)}\sin\theta_n(t)d\phi \\
&= 0 \qquad (3.69)
\end{aligned}
$$

$G_{21}^{11} = 0$ because $G_{12}^{11} = G_{21}^{11}$

$G_{22}^{22} = \cos^2\theta_n(\tau-t)$ because $G_{22}^{22} = G_{11}^{11}$

We can writeG^{22} by the equations above

$$
G^{22} = \begin{pmatrix} G_{11}^{22} & G_{12}^{22} \\ G_{21}^{22} & G_{22}^{22} \end{pmatrix} = \begin{pmatrix} \sin^2\theta_n(\tau-t) & 0 \\ 0 & \cos^2\theta_n(\tau-t) \end{pmatrix} \qquad (3.70)
$$

We don't have to calculate all the elements of the density matrix. For two-level atom we have $\rho_{11} = 1-\rho_{22}$ and $\rho_{12} = \rho_{21}^*$; so it is enough to know only ρ_{11} and ρ_{12} in order to determine the system's complete dynamics.

Firstly, we find $\rho_{11}(\tau)$ and then $\rho_{22}(\tau)$ by using equation (3.64)

$$\rho_{11}(\tau) = e^{(-\tau/\tau_0)} Tr[G^{11}(\tau,0)\rho(0)] + \frac{1}{\tau_0}\int_0^\tau dt e^{(-\tau-t)/\tau_0} Tr[G^{11}(\tau,t)\rho(t)] \quad (3.71)$$

First we write

$$\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \quad (3.72)$$

then we multiply $\rho(0)$ and equation (3.68)

$$G^{11}(\tau,0)\rho(0) = \begin{pmatrix} \cos^2[\theta_n(\tau)]\rho_{11}(0) & \cos^2[\theta_n(\tau)]\rho_{12}(0) \\ \sin^2[\theta_n(\tau)]\rho_{21}(0) & \sin^2[\theta_n(\tau)]\rho_{22}(0) \end{pmatrix} \quad (3.73)$$

Now we track the equation (3.73) and find

$$\begin{aligned} Tr[G^{11}(\tau,0)\rho(0)] &= \rho_{11}(0)\cos^2[\theta_n(\tau)] + \rho_{22}(0)\sin^2[\theta_n(\tau)] \\ &= \rho_{11}(0) + [\rho_{22}(0) - \rho_{11}(0)]\sin^2[\theta_n(\tau)] \end{aligned} \quad (3.74)$$

We write $\rho(t)$ and below.

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} \quad (3.75)$$

then we multiply $\rho(t)$ and equation (3.68)

$$G^{11}(\tau,t)\rho(t) = \begin{pmatrix} \cos^2[\theta_n(\tau-t)]\rho_{11}(t) & \cos^2[\theta_n(\tau-t)]\rho_{12}(t) \\ \sin^2[\theta_n(\tau)-t]\rho_{21}(t) & \sin^2[\theta_n(\tau-t)]\rho_{22}(t) \end{pmatrix} \quad (3.76)$$

Now we track the equation (3.76) and find

$$\begin{aligned} Tr[G^{11}(\tau,t)\rho(t)] &= \rho_{11}(t)\cos^2[\theta_n(\tau-t)] + \rho_{22}(t)\sin^2[\theta_n(\tau-t)] \\ &= \rho_{11}(t) + [\rho_{22}(t)-\rho_{11}(t)]\sin^2[\theta_n(\tau-t)] \end{aligned} \tag{3.77}$$

Hence we can write $\rho_{11}exp\Big(+ \frac{\tau}{\tau_0}\Big)$ like this

$$\begin{aligned} \rho_{11}exp\Big(+ \frac{\tau}{\tau_0}\Big) &= (\rho_{11}(0) + [\rho_{22}(0)-\rho_{11}(0)]\sin^2[\theta_n\tau] \\ &\quad + \frac{1}{\tau_0}\int_0^\tau exp\Big(+ \frac{t}{\tau_0}\Big)\{\rho_{11}(t) + [\rho_{22}(t)-\rho_{11}(t)]\sin^2[\theta_n(\tau-t)]\} \end{aligned} \tag{3.78}$$

In a similar way $\rho_{22}exp\Big(+ \frac{\tau}{\tau_0}\Big)$ can write like this

$$\begin{aligned} \rho_{22}exp\Big(+ \frac{\tau}{\tau_0}\Big) &= (\rho_{22}(0) + [\rho_{11}(0)-\rho_{22}(0)]\sin^2[\theta_n\tau] \\ &\quad + \frac{1}{\tau_0}\int_0^\tau exp\Big(+ \frac{t}{\tau_0}\Big)\{\rho_{22}(t) + [\rho_{11}(t)-\rho_{22}(t)]\sin^2[\theta_n(\tau-t)]\} \end{aligned} \tag{3.79}$$

The quantity of experimental interest is called inversion and it is described as $W_n = \rho_{11} - \rho_{22}$ and can be evaluated with the use of equations(3.78) and (3.79)

$$\begin{aligned} &(\rho_{11}-\rho_{22})exp\Big(+ \frac{\tau}{\tau_0}\Big) = \Big\{[\rho_{11}(0)-\rho_{22}] + 2[\rho_{22}(0)-\rho_{11}(0)]\sin^2[\theta_n\tau] \\ &+\frac{1}{\tau_0}\int_0^\tau exp\Big(+ \frac{t}{\tau_0}\Big)\Big\{[\rho_{11}(0)-\rho_{22}] + 2[\rho_{22}(t)-\rho_{11}(t)]\sin^2[\theta_n(\tau-t)]\} \end{aligned} \tag{3.80}$$

Since

$$W_n = \rho_{11} - \rho_{22}, W_n(0) = \rho_{11}(0) - \rho_{22}(0) \tag{3.81}$$

We replace (3.83) in equation (3.82) we find the equation below.

$$W_n exp\left(+ \frac{\tau}{\tau_0}\right) = W_n(0) + [1 - 2\sin^2(\theta_n \tau)] + \frac{1}{\tau_0}\int_0^\tau exp\left(+ \frac{t}{\tau_0}\right)\{1 - 2\sin^2[\theta_n(\tau - t)]\}W_n(t)dt\} \quad (3.82)$$

This equation describes the relaxation of the population and can be solved easily by the procedure of differentiation. We will differentiate (3.82) three times and after carrying out necessary simplification we get

$$\frac{dW_n}{d\tau}e^{\frac{\tau}{\tau_0}} + \frac{1}{\tau_0}W_n e^{\frac{\tau}{\tau_0}} = W_n(0)[-(2\theta_n)\sin(2\theta_n\tau)] + \frac{1}{\tau_0}\int_0^\tau e^{\frac{t}{\tau_0}}[-(2\theta_n)\sin(2\theta_n(t-\tau))]W_n(t)dt + e^{\frac{\tau}{\tau_0}}W_n(\tau)\frac{1}{\tau_0} \quad (3.83)$$

$$\begin{aligned}\frac{d^2W_n}{d\tau^2}e^{\frac{\tau}{\tau_0}} + \frac{dW_n}{d\tau}e^{(\frac{\tau}{\tau_0})\frac{2}{\tau_0}} + \frac{1}{\tau_0^2}W_n e^{\frac{\tau}{\tau_0}} &= W_n(0)[-4\theta_n^2]\cos(2\theta_n\tau) \\ &\quad + \frac{1}{\tau_0}\int_0^\tau e^{(\frac{t}{\tau_0})}[-(4\theta_n^2)\cos(2\theta_n(\tau-t))]W_n(t)dt \\ &\quad + e^{\frac{\tau}{\tau_0}}W_n(\tau)\frac{1}{\tau_0^2} + e^{\frac{\tau}{\tau_0}}\frac{dW_n}{d\tau}\frac{1}{\tau_0} \\ \left(\frac{d^2W_n}{d\tau^2} + \frac{dW_n}{d\tau}\frac{1}{\tau_0}\right)e^{\frac{\tau}{\tau_0}} &= (-(4\theta_n^2))W_n e^{\frac{\tau}{\tau_0}} \end{aligned} \quad (3.84)$$

$$\left(\frac{d^3W_n}{d\tau^3} + \frac{d^2W_n}{d\tau^2}\frac{1}{\tau_0}\right)e^{\frac{\tau}{\tau_0}} + \left(\frac{d^2W_n}{d\tau^2}\frac{1}{\tau_0} + \frac{dW_n}{d\tau}\frac{1}{\tau_0^2}\right)e^{\frac{\tau}{\tau_0}} = \frac{dW_n}{d\tau}(-4\theta_n^2)e^{(\frac{\tau}{\tau_0})} + (-4\theta_n^2)\frac{1}{\tau_0}W_n e^{(\frac{\tau}{\tau_0})} \quad (3.85)$$

If we simplify equations (3.83), (3.84), (3.85) we find the equation below;

$$\frac{d^3W_n}{d\tau^3} + \frac{2}{\tau_0}\frac{d^2W_n}{d\tau^2} + \left(\frac{1}{\tau_0^2} + (2\theta_n)^2\right)\frac{dW_n}{d\tau} + \frac{(2\theta_n)^2}{\tau_0}W_n = 0 \quad (3.86)$$

where the required solution should satisfy the following initial conditions:

$$\frac{dW_n}{d\tau} = 0 \qquad \frac{d^2W_n}{d\tau^2} = -(2\theta_n)^2 W_n(0) \tag{3.87}$$

We make use of the Laplace transform to solve (3.86) together with the initial conditions mentioned in equation (3.87) and obtain the following expression for the time evolution of inversion $W_n(t)$:

$$\mathcal{L}\left\{ \frac{d^3W_n}{d\tau^3} + \frac{2}{\tau_0}\frac{d^2W_n}{d\tau^2} + \left(\frac{1}{\tau_0^2} + (2\theta_n)^2\right)\frac{dW_n}{d\tau} + \frac{(2\theta_n)^2}{\tau_0} W_{n,\tau,s} \right\} = 0 \tag{3.88}$$

We find

$$\tilde{W_n}(s) = \frac{s^2 + \frac{2s}{\tau_0} + \frac{1}{\tau_0^2}}{s^3 + \frac{2s^2}{\tau_0} + \frac{s}{\tau_0^2} + (2\theta_n)^2 s + \frac{(2\theta_n)^2}{\tau_0}} W_n(0) \tag{3.89}$$

For $W_n(0) = 1$ we take the inverse Laplace transform of $W_n(s)$.

We get;

$$\begin{aligned}
W_n(t) &= e^{-t/\tau_0}\left\{ \cos[\frac{t\sqrt{1-16\tau_0^2\theta_n^2}}{2\tau_0}] + \sin[\frac{\frac{t\sqrt{1-16\tau_0^2\theta_n^2}}{2\tau_0}}{\sqrt{1-16\tau_0^2\theta_n^2}}] \right\} \\
&= e^{-t/2\tau_0}\left\{ \cos[\frac{t\sqrt{1-16\tau_0^2 g_0^2\sqrt{n+1}}}{2\tau_0}] + \sin[\frac{\frac{1-16\tau_0^2 g_0^2\sqrt{n+1}}{\sqrt{1-16\tau_0^2 g_0^2\sqrt{n+1}}}}{1-16\tau_0^2 g_0^2\sqrt{n+1}}] \right\} \\
&= e^{-t/2\tau_0}\left\{ \cos[\frac{4tg_0(n+1)^{1/2}\tau_0\sqrt{\frac{t}{4g_0^2\tau_0^2(n+1)} - 1}}{2\tau_0}] \right\} \\
&= (cos[2g_0\sqrt{n+1}Z_n t]\frac{1}{2g_0\tau_0\sqrt{n+1}Z_n} sin[2g_0\sqrt{n+1}Z_n t])e^{-\frac{t}{\tau_0}} W_n(0)
\end{aligned} \tag{3.90}$$

in which

$$Z_n = (\frac{1}{16g_0^2\tau_0^2(n+1)} - 1)^{\frac{1}{2}} \tag{3.91}$$

To find the elements of density matrices we use the master equation (3.64). We

use Laplace transform techniques during the process. Then our master equation (3.64) becomes equal to;

$$\rho_{im}(\tau) \quad = \quad e^{(-\tau/\tau_0)}Tr[G^{im}(\tau,0)\rho(0)] + \frac{1}{\tau_0}\int_0^\tau dt e^{(-\tau-t)/\tau_0}Tr[G^{im}(\tau,t)\rho(t)]$$

First we write

$$\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}, \rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix}, \rho(\tau) = \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) \end{pmatrix}$$

By using the equations above we can write the matrices $U(\phi;\tau,0)$ and $U^{-1}(\phi;\tau,0)$.

$$U(\phi;\tau,0) = \begin{pmatrix} \cos(\theta_n\tau) & -i\sin(\theta_n\tau)e^{-i\phi(t)} \\ -i\sin(\theta_n\tau)e^{i\phi(t)} & \cos(\theta_n\tau) \end{pmatrix}$$

$$U^{-1}(\phi;\tau,0) = \begin{pmatrix} \cos(\theta_n\tau) & i\sin(\theta_n\tau)e^{-i\phi(t)} \\ i\sin(\theta_n\tau)e^{i\phi(t)} & \cos(\theta_n\tau)) \end{pmatrix}$$

Similarly

$$U(\phi;\tau,t) = \begin{pmatrix} \cos(\theta_n(\tau-t)) & -i\sin(\theta_n(\tau-t))e^{-i\phi(t)} \\ -i\sin(\theta_n(\tau-t))e^{i\phi(t)} & \cos(\theta_n(\tau-t))) \end{pmatrix}$$

$$U^{-1}(\phi;\tau,t) = \begin{pmatrix} \cos(\theta_n(\tau-t)) & i\sin(\theta_n(\tau-t))e^{-i\phi(t)} \\ i\sin(\theta_n(\tau-t))e^{i\phi(t)} & \cos(\theta_n(\tau-t))) \end{pmatrix}$$

We find matrix A by multiplying $U(\phi;\tau,0)$, $\rho(0)$ and $U^{-1}(\phi;\tau,0)$

$$A = U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)$$

The elements of matrix A are;

$$
\begin{aligned}
A_{11} &= \cos(\theta_n\tau)^2\rho_{11}(0) + 1/2ie^{-i\phi(t)}\sin(2\theta_n\tau)(e^{2i\phi(t)}\rho_{12}(0) - \rho_{21}(0))\sin(\theta_n(\tau))^2\rho_{22}(0) \\
A_{12} &= \cos(\theta_n\tau)^2\rho_{12}(0) + e^{-2i\phi}\sin(\theta_n\tau)^2\rho_{21}(0) + \frac{1}{2}\sin(2\theta\tau)(i\cos\phi(t) + \sin\phi(t)) \\
&\quad (\rho_{11}(0) - \rho_{22}(0)) \\
A_{21} &= e^{2i\phi}\sin(\theta_n\tau)^2\rho_{12}(0) + \cos(\theta_n\tau)^2\rho_{21}(0) - \frac{1}{2}ie^{i\phi}\sin(2\theta_n\tau)(\rho_{11}(0) - \rho_{22}(0)) \\
A_{22} &= \sin(\theta_n\tau)^2\rho_{11}(0) - \frac{1}{2}ie^{-i\phi}\sin(2\theta_n\tau)(e^{2i\phi}\rho_{12}(0) - \rho_{21}(0)) + \cos(\theta_n\tau)^2\rho_{22}(0)
\end{aligned} \tag{3.92}
$$

Now we find integral of matrix A;

$$
A_{int} = \frac{1}{2\pi}\int_0^{2\pi} U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0) = \begin{pmatrix} \cos^2(\theta_n\tau)\rho_{11}(0) & \cos^2(\theta_n\tau)\rho_{12}(0) \\ +\sin^2(\theta_n\tau)\rho_{22}(0) & \\ \cos^2(\theta_n\tau)\rho_{21}(0) & \sin^2(\theta_n\tau)\rho_{11}(0) \\ & +\cos^2(\theta_n\tau)\rho_{22}(0) \end{pmatrix} \tag{3.93}
$$

We find matrix B by multiplying $U(\phi;\tau,t)$, $\rho(t)$ and $U^{-1}(\phi;\tau,t)$

$$
\begin{aligned}
B_{11} &= \cos(\theta_n(\tau))^2\rho_{11}(t) + 1/2ie^{-i\phi(t)}\sin(2\theta_n(\tau))(e^{2i\phi(t)}\rho_{12}(t) - \rho_{21}(t))\sin(\theta_n(\tau))^2\rho_{22}(t) \\
B_{12} &= \cos(\theta_n\tau)^2\rho_{12}(t) + e^{-2i\phi}\sin(\theta_n\tau)^2\rho_{21}(t) + \frac{1}{2}\sin(2\theta\tau)(i\cos\phi(t) + \sin\phi(t)) \\
&\quad (\rho_{11}(t) - \rho_{22}(t)) \\
B_{21} &= e^{2i\phi}\sin(\theta_n\tau)^2\rho_{12}(t) + \cos(\theta_n\tau)^2\rho_{21}(t) - \frac{1}{2}ie^{i\phi}\sin(2\theta_n\tau)(\rho_{11}(t) - \rho_{22}(t)) \\
B_{22} &= \sin(\theta_n\tau)^2\rho_{11}(t) - \frac{1}{2}ie^{-i\phi}\sin(2\theta_n\tau)(e^{2i\phi}\rho_{12}(t) - \rho_{21}(t)) + \cos(\theta_n\tau)^2\rho_{22}(t)
\end{aligned} \tag{3.94}
$$

Now we find integral of matrix B;

$$B_{int} = \frac{1}{2\pi}\int_0^{2\pi} U(\phi;\tau,0)\rho(t)U^{-1}(\phi;\tau,0) = \begin{pmatrix} \cos^2(\theta_n\tau)\rho_{11}(t) & \cos^2(\theta_n\tau)\rho_{12}(t) \\ +\sin^2(\theta_n\tau)\rho_{22}(t) & \\ \cos^2(\theta_n\tau)\rho_{21}(t) & \sin^2(\theta_n\tau)\rho_{11}(t) \\ & +\cos^2(\theta_n\tau)\rho_{22}(t) \end{pmatrix} \quad (3.95)$$

We multiply A_{int} by $e^{\frac{-\tau}{\tau_0}}$ and multiply B_{int} with $e^{-\frac{\tau-t}{\tau_0}}$.We sum equations. Then we take Laplace transform with respect to τ

$$\mathcal{L}\left\{ \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) \end{pmatrix} \right\} = \mathcal{L}\left\{ e^{-\frac{\tau}{\tau_0}} \begin{pmatrix} \cos^2(\theta_n\tau)\rho_{11}(0) & \cos^2(\theta_n\tau)\rho_{12}(0) \\ +\sin^2(\theta_n\tau)\rho_{22}(0) & \\ \cos^2(\theta_n\tau)\rho_{21}(0) & \sin^2(\theta_n\tau)\rho_{11}(0) \\ & +\cos^2(\theta_n\tau)\rho_{22}(0) \end{pmatrix} \right\}$$

$$+\mathcal{L}\left\{ \frac{1}{\tau_0}\int_0^{2\pi} dt e^{-\frac{\tau-t}{\tau_0}} \begin{pmatrix} \cos^2(\theta_n(\tau-t))\rho_{11}(t) & \cos^2(\theta_n(\tau-t))\rho_{12}(t) \\ +\sin^2(\theta_n(\tau-t))\rho_{22}(t) & \\ \cos^2(\theta_n(\tau-t))\rho_{21}(t) & \sin^2(\theta_n(\tau-t))\rho_{11}(t) \\ & +\cos^2(\theta_n(\tau-t))\rho_{22}(t) \end{pmatrix} \right\} \quad (3.96)$$

Afterwards, we get

$$\{\{\rho_{11}(s),\rho_{12}(s)\},\{\rho_{21}(s),\rho_{22}(s)\}\} =$$

$$\{\{\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{11}(0)+2\tau_0^3\theta_n^2\rho_{22}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}, \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{12}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}\}$$

$$\{\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{21}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}, \frac{2\tau_0^3\theta_n^2\rho_{11}(0)+\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{22}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}\}\}$$

$$+\{\{\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{11}(t)+2\tau_0^3\theta_n^2\rho_{22}(t)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}, \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{12}(t)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}\}$$

$$\{\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{21}(t)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}, \frac{2\tau_0^3\theta_n^2\rho_{11}(t)+\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{22}(t)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}\}\} \quad (3.97)$$

After we make addition we find matrix C. The elements of matrix C are as follows:

$$\begin{aligned}
C_{11} &= \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{11}(t)+2\tau_0^2\theta_n^2\rho_{22}(t)+\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{11}(0)+2\tau_0^2\theta_n^2)\rho_{22}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2} \\
C_{12} &= \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)(\rho_{12}(t)+\rho_{12}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2} \\
C_{21} &= \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)(\rho_{21}(t)+\rho_{21}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}
\end{aligned}$$

$$C_{22} = \frac{\tau_0(2\tau_0^2\theta_n^2\rho_{11}(t))((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{22}(t)+2\tau_0^2\theta_n^2)\rho_{11}(0)+((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{22}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2} \quad (3.98)$$

and

$$\{\{\rho_{11}(s),\rho_{12}(s)\},\{\rho_{21}(s),\rho_{22}(s)\}\} \quad =$$

$$\{\{\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{11}(t)+2\tau_0^2\theta_n^2\rho_{22}(t)+\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{11}(0)+2\tau_0^2\theta_n^2)\rho_{22}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}$$

$$,\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)(\rho_{12}(t)+\rho_{12}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2},\frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)(\rho_{21}(t)+\rho_{21}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2}$$

$$\frac{\tau_0(2\tau_0^2\theta_n^2\rho_{11}(t))((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{22}(t)+2\tau_0^2\theta_n^2)\rho_{11}(0)+((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{22}(0)}{(1+\tau_0 s)^3+4\tau_0^2(1+\tau_0 s)\theta_n^2} \tag{3.99}$$

If we solve the equation (3.99) we find;

$$\rho_{11}(s)= \frac{\tau_0(1+\tau_0(-1+(2+\tau_0(-1+s))s+2\tau_0\theta_n^2))\rho_{11}(0)+2\tau_0^3\theta_n^2 rho_{22}(0)}{(1+\tau_0(-1+s))((1+\tau_0(-1+s))(1+\tau_0 s)+4\tau_0^2\theta_n^2)} \tag{3.100}$$

$$\rho_{12}(s)= \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{12}(0)}{(1+\tau_0(-1+s))(1+\tau_0 s)^2+2\tau_0^2(2+\tau_0(-1+2s))\theta_n^2} \tag{3.101}$$

$$\rho_{21}(s)= \frac{\tau_0((1+\tau_0 s)^2+2\tau_0^2\theta_n^2)\rho_{21}(0)}{(1+\tau_0(-1+s))(1+\tau_0 s)^2+2\tau_0^2(2+\tau_0(-1+2s))\theta_n^2} \tag{3.102}$$

$$\rho_{22}(s)= \frac{2\tau_0^3\theta_n^2\rho_{11}(0)+\tau_0(1+\tau_0(-1+(2+\tau_0(-1+s))s+2\tau_0\theta_n^2))\rho_{22}(0)}{(1+\tau_0(-1+s))((1+\tau_0(-1+s))(1+\tau_0 s)+4\tau_0^2\theta_n^2)} \tag{3.103}$$

if the take inverselaplace transform of equations (3.100)and (3.103) we find

$$\rho_{11}(t) = \frac{1}{2}\Bigg(\rho_{11}(0) + e^{-\frac{t}{2\tau_0}}\Bigg(\cosh[2g_0\sqrt{n+1}Z_n t] + \frac{sinh[2g_0\sqrt{n+1}Z_n t]}{2g_0\tau_0\sqrt{n+1}Z_n}\Bigg) (\rho_{11}(0) - \rho_{22}(0)) + \rho_{22}(0)\Bigg) \quad (3.104)$$

$$\rho_{22}(t) = \frac{1}{2}\Bigg(\rho_{11}(0) - e^{-\frac{t}{2\tau_0}}\Bigg(\cosh[2g_0\sqrt{n+1}Z_n t] + \frac{sinh[2g_0\sqrt{n+1}Z_n t]}{2g_0\tau_0\sqrt{n+1}Z_n}\Bigg) (\rho_{11}(0) - \rho_{22}(0)) + \rho_{22}(0)\Bigg) \quad (3.105)$$

in which $Z_n = \left(\frac{1}{16g_0^2 t_0^2(n+1)} - 1\right)^{1/2}$

Now we find $W_n = \rho_{11} - \rho_{22}$ again.

$$W_n = (cos[2g_0\sqrt{n+1}Z_n t]\frac{1}{2g_0\tau_0\sqrt{n+1}Z_n} sin[2g_0\sqrt{n+1}Z_n t])e^{-\frac{t}{\tau_0}} W_n(0) \quad (3.106)$$

We find the same solution with equation (3.90).

CHAPTER 4
ENTANGLEMENT IN TWO ATOMS-ONE FIELD INTERACTION UNDER RANDOM PHASE TELEGRAPH NOISE

In this chapter, we review the entanglement properties of a two-atom system, which are inside an optical cavity in an stochastic interaction with field and studied by the Jaynes-Cummings Model. The phase telegraph noise is seen as a noise term and an exact solution about this model is obtained. This solution presents the resulting decoherence effects of the noise on the system's entanglement properties. It shows that the single atoms are not in an entanglement with cavity field under the noise. However, a strong atom-atom entanglement is observed in stationary state. It is observed that, in the steady state of atom-atom entanglement a relatively strong noise is cooperative.This chapter is based on the article of Özel, Yılmaz, Kayhan [74].

4.1 The Formulation Of The Problem

We have considered the interaction of two-level atoms. These are described by spin-1/2 operators $S_{\pm}^{(1,2)}, S_z^{(1,2)}$ and have a single-mode of quantized radiation field which is described by the annihilation and the creation operators $a, a^{\dagger}$, respectively. We suppose that the field shows resonance with the atomic transition frequency ω_0. As a result, the Hamiltonian of the system takes the form of ($\hbar = 1$).

$$H = \omega_0 a^{\dagger} a + \sum_{i=1,2} \omega_0 S_z^i + (g^*(t) S_+^i a + g(t) S_-^i a^{\dagger}) \tag{4.1}$$

where $g(t)$ is the time dependent coupling coefficient between the atom and the field. In order to apply the phase telegraph noise into the problem, $g(t)$ is regarded to be stochastic and described as

$$g(t) = g_0 e^{i\phi(t)} \tag{4.2}$$

where $g(0)$ is a positive real constant amplitude and $\phi(t)$ is a stochastic variable which fluctuates between various arbitrary phases in a manner of jumps. The jumps are separated by an average time interval called the mean dwell time. $\phi(t)$s in the neighboring intervals are not correlated. As a result, the probability of finding ϕ stays the same at any instant of time. Hence, $\phi(t)$ can undergo continuous random change of Markov process and this situation allows us to take the average over the stochastic fluctuations.

4.2 The Solution

We consider the system to which the atoms are trapped inside a single mode optical cavity. Firstly, the cavity field is set in the vacuum state $|0\rangle$, atom 1 in the excited state $|e\rangle$, and atom 2 in the ground state $|g\rangle$ [75, 76]. In this case, the initial density matrix of the system becomes

$$\rho(0) = |0\rangle\langle 0| \otimes |eg\rangle\langle eg| \tag{4.3}$$

The exact analytic solution of the system of the time-dependent relevant density matrix in the interaction picture can be get from the method

$$\rho(\tau)e^{\frac{\tau}{\tau_0}} = \int U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)dQ(\phi) + \frac{1}{\tau_0}\int U(\phi;\tau,t)\rho(t)U^{-1}(\phi;\tau,t)dQ(\phi)dt \tag{4.4}$$

where τ_0 is the mean dwell time and $dQ(\phi) = \frac{d\phi}{2\pi}$. In this integral equation, the statistical average is taken over the stochastic variable $\phi(t)$.

Our aim is obtain the general solution of the density matrix. In order to solve the problem we will use the Laplace transformation techniques.

First we find H_{int} matrix. The subspace basis are $|0ee\rangle, |0eg\rangle, |0ge\rangle, |0gg\rangle, |1ee\rangle,$ $|1eg\rangle, |1ge\rangle, |1gg\rangle$. Using these basis we can write elements of H_{int} matrix like this;

$$
\begin{aligned}
H_{11} &= \langle 0ee|H_{int}|0ee\rangle \qquad H_{12} = \langle 0ee|H_{int}|0eg\rangle .....H_{18} = \langle 0ee|H_{int}|1gg\rangle \\
H_{21} &= \langle 0eg|H_{int}|0ee\rangle \qquad H_{22} = \langle 0eg|H_{int}|0eg\rangle .....H_{28} = \langle 0eg|H_{int}|1gg\rangle \\
H_{31} &= \langle 0ge|H_{int}|0ee\rangle \qquad H_{32} = \langle 0ge|H_{int}|0eg\rangle .....H_{38} = \langle 0ge|H_{int}|1gg\rangle \\
H_{41} &= \langle 0gg|H_{int}|0ee\rangle \qquad H_{42} = \langle 0gg|H_{int}|0eg\rangle .....H_{48} = \langle 0gg|H_{int}|1gg\rangle \\
H_{51} &= \langle 1ee|H_{int}|0ee\rangle \qquad H_{52} = \langle 1ee|H_{int}|0eg\rangle .....H_{58} = \langle 1ee|H_{int}|1gg\rangle \\
H_{61} &= \langle 1eg|H_{int}|0ee\rangle \qquad H_{62} = \langle 1eg|H_{int}|0eg\rangle .....H_{68} = \langle 1eg|H_{int}|1gg\rangle \\
H_{71} &= \langle 1ge|H_{int}|0ee\rangle \qquad H_{72} = \langle 1ge|H_{int}|0eg\rangle .....H_{78} = \langle 1ge|H_{int}|1gg\rangle \\
H_{81} &= \langle 1gg|H_{int}|0ee\rangle \qquad H_{82} = \langle 1gg|H_{int}|0eg\rangle .....H_{88} = \langle 1gg|H_{int}|1gg\rangle \quad (4.5)
\end{aligned}
$$

We can write H_{int} matrix as follows by using the above equation (4.5)

$$
H_{int} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & e^{-i\phi(t)} & e^{-i\phi(t)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i\phi(t)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i\phi(t)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{i\phi(t)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{i\phi(t)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{i\phi(t)} & e^{i\phi(t)} & 0
\end{pmatrix} \tag{4.6}
$$

We find unitary matrix $U(\phi;\tau,0)$ below

$$
\begin{aligned}
U(\phi;\tau,0) \;=\; & \{\cos(\sqrt{2}\tau),0,0,0,0,-\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}},-\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}},0\} \\
& \{0,\cos(\frac{\tau}{\tau_0})^2,-\sin(\frac{\tau}{\tau_0})^2,0,0,0,0,-\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}}\} \\
& \{0-\sin(\frac{\tau}{\tau_0})^2,\cos(\frac{\tau}{\tau_0})^2,0,0,0,0,-\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}}\} \\
& \{0,0,0,1,0,0,0,0\},\{0,0,0,0,1,0,0,0\} \\
& \{\frac{\sin\sqrt{2}\tau(-i\cos(\phi)+\sin(\phi))}{\sqrt{2}},0,0,0,0,\cos(\frac{\tau}{\sqrt{2}})^2,-\sin(\frac{\tau}{\sqrt{2}})^2,0\} \\
& \{\frac{\sin\sqrt{2}\tau(-i\cos(\phi)+\sin(\phi))}{\sqrt{2}},0,0,0,0,-\sin(\frac{\tau}{\sqrt{2}})^2,\cos(\frac{\tau}{\sqrt{2}})^2,0\} \\
& \{0,\frac{\sin\sqrt{2}\tau(-i\cos(\phi)+\sin(\phi))}{\sqrt{2}},\frac{\sin\sqrt{2}\tau(-i\cos(\phi)+\sin(\phi))}{\sqrt{2}} \\
& ,0,0,0,0,\cos(\sqrt{2}\tau)\}
\end{aligned}
\tag{4.7}
$$

We take inverse of U and find $U^{-1}(\phi;\tau,0)$;

$$
\begin{aligned}
U^{-1}(\phi;\tau,0) \;=\; & \{\cos(\sqrt{2}\tau),0,0,0,0,\frac{\sin(\sqrt{2}\tau)(i\cos(\phi)+\sin(\phi))}{\sqrt{2}} \\
& ,\frac{\sin(\sqrt{2}\tau)(i\cos(\phi)+\sin(\phi))}{\sqrt{2}},0\} \\
& \{0,\cos(\frac{\tau}{\sqrt{2}})^2,-\sin(\frac{\tau}{\sqrt{2}})^2,0,0,0,0,\frac{\sin(\sqrt{2}\tau)(i\cos(\phi)+\sin(\phi))}{\sqrt{2}}\} \\
& \{0,-\sin(\frac{\tau}{\sqrt{2}})^2,\cos(\frac{\tau}{\sqrt{2}})^2,0,0,0,0,\frac{\sin(\sqrt{2}\tau)(i\cos(\phi)+\sin(\phi))}{\sqrt{2}}\} \\
& \{0,0,0,1,0,0,0,0\},\{0,0,0,0,1,0,0,0\} \\
& \{\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}},0,0,0,0,\cos(\frac{\tau}{\sqrt{2}})^2,-\sin(\frac{\tau}{\sqrt{2}})^2,0\} \\
& \{\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}},0,0,0,0,-\sin(\frac{\tau}{\sqrt{2}})^2,\cos(\frac{\tau}{\sqrt{2}})^2,0\} \\
& \{0,\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}},\frac{ie^{-i\phi}\sin(\sqrt{2}\tau)}{\sqrt{2}},0,0,0,0,\cos(\sqrt{2}\tau)\}
\end{aligned}
\tag{4.8}
$$

We write $\rho(0)$, $\rho(t)$ and $\rho(\tau)$ below

$$\rho(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.9}$$

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t) & \rho_{15}(t) & \rho_{16}(t) & \rho_{17}(t) & \rho_{18}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t) & \rho_{25}(t) & \rho_{26}(t) & \rho_{27}(t) & \rho_{28}(t) \\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t) & \rho_{35}(t) & \rho_{36}(t) & \rho_{37}(t) & \rho_{38}(t) \\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(t) & \rho_{45}(t) & \rho_{46}(t) & \rho_{47}(t) & \rho_{48}(t) \\ \rho_{51}(t) & \rho_{52}(t) & \rho_{53}(t) & \rho_{54}(t) & \rho_{55}(t) & \rho_{56}(t) & \rho_{57}(t) & \rho_{58}(t) \\ \rho_{61}(t) & \rho_{62}(t) & \rho_{63}(t) & \rho_{64}(t) & \rho_{65}(t) & \rho_{66}(t) & \rho_{67}(t) & \rho_{68}(t) \\ \rho_{71}(t) & \rho_{72}(t) & \rho_{73}(t) & \rho_{74}(t) & \rho_{75}(t) & \rho_{76}(t) & \rho_{77}(t) & \rho_{78}(t) \\ \rho_{81}(t) & \rho_{82}(t) & \rho_{83}(t) & \rho_{84}(t) & \rho_{85}(t) & \rho_{86}(t) & \rho_{87}(t) & \rho_{88}(t) \end{pmatrix} \tag{4.10}$$

$$\rho(\tau) = \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) & \rho_{13}(\tau) & \rho_{14}(\tau) & \rho_{15}(\tau) & \rho_{16}(\tau) & \rho_{17}(\tau) & \rho_{18}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) & \rho_{23}(\tau) & \rho_{24}(\tau) & \rho_{25}(\tau) & \rho_{26}(\tau) & \rho_{27}(\tau) & \rho_{28}(\tau) \\ \rho_{31}(\tau) & \rho_{32}(\tau) & \rho_{33}(\tau) & \rho_{34}(\tau) & \rho_{35}(\tau) & \rho_{36}(\tau) & \rho_{37}(\tau) & \rho_{38}(\tau) \\ \rho_{41}(\tau) & \rho_{42}(\tau) & \rho_{43}(\tau) & \rho_{44}(\tau) & \rho_{45}(\tau) & \rho_{46}(\tau) & \rho_{47}(\tau) & \rho_{48}(\tau) \\ \rho_{51}(\tau) & \rho_{52}(\tau) & \rho_{53}(\tau) & \rho_{54}(\tau) & \rho_{55}(\tau) & \rho_{56}(\tau) & \rho_{57}(\tau) & \rho_{58}(\tau) \\ \rho_{61}(\tau) & \rho_{62}(\tau) & \rho_{63}(\tau) & \rho_{64}(\tau) & \rho_{65}(\tau) & \rho_{66}(\tau) & \rho_{67}(\tau) & \rho_{68}(\tau) \\ \rho_{71}(\tau) & \rho_{72}(\tau) & \rho_{73}(\tau) & \rho_{74}(\tau) & \rho_{75}(\tau) & \rho_{76}(\tau) & \rho_{77}(\tau) & \rho_{78}(\tau) \\ \rho_{81}(\tau) & \rho_{82}(\tau) & \rho_{83}(\tau) & \rho_{84}(\tau) & \rho_{85}(\tau) & \rho_{86}(\tau) & \rho_{87}(\tau) & \rho_{88}(\tau) \end{pmatrix} \tag{4.11}$$

We find $D = U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)$ and then find the integral of matrix D

$$D_{int} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{\tau}{\sqrt{2}})^4 & -\frac{1}{4}\sin(\sqrt{2}\tau)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4}\sin(\sqrt{2}\tau)^2 & \sin(\frac{\tau}{\sqrt{2}})^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}sin(\sqrt{2}\tau)^2 \end{pmatrix} \tag{4.12}$$

We multiply D_{int} with $e^{\frac{-\tau}{\tau_0}}$ and find a new matrix if the Laplace transform of new matrix is taken we find

$$\begin{aligned} \mathcal{L}(D_{int}) &= \{0,0,0,0,0,0,0,0\} \\ &\{0, \frac{-1}{8}\tau_0(-\frac{3}{1+\tau_0 s} - \frac{4(1+\tau_0 s)}{1+2\tau_0 s+\tau_0^2(2+s^2)} - \frac{1+\tau_0 s}{1+2\tau_0 s+\tau_0^2(8+s^2)}) \\ &- \frac{\tau_0^3}{(1+\tau_0 s)(8\tau_0^2+(1+\tau_0 s)^2)}, 0,0,0,0,0\} \\ &\{0, -\frac{\tau_0^3}{(1+\tau_0 s)(8\tau_0^2+(1+\tau_0 s)^2)} \\ &\frac{6\tau_0^5}{(1+\tau_0 s)(1+2\tau_0 s+\tau_0^2(2+s^2))(1+2\tau_0 s+\tau_0^2(8+s^2))}, 0,0,0,0,0\} \\ &\{0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0,0\} \\ &\{0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0, \frac{2\tau_0^3}{(1+\tau_0 s)(8\tau_0^2+(1+\tau_0 s)^2)}\} \end{aligned} \tag{4.13}$$

We find $E = U(\phi;\tau,t)\rho(t)U^{-1}(\phi;\tau,t)$ and take the integral of matrix E and afterwards we get E_{int}. We multiply E_{int} with $e^{\frac{-\tau}{\tau_0}}$ and find a new matrix. We take Laplace transform of the new matrix. If the Laplace of the new matrix is taken we find $\mathcal{L}(E_{int})$.

We sum $\mathcal{L}(D_{int})$ and $\mathcal{L}(E_{int})$ and find a very big matrix. We take inverse Laplace of the big matrix and find density matrix $\rho(t)$ The elements of density matrix are;

$$\rho_{22}(t) = \frac{1}{16}\left(6 + 2e^{\frac{-t}{2\tau_0}}\left(5\cosh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}] + 5\frac{\sinh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}]}{\sqrt{1-64\tau_0^2}}\right)\right) \tag{4.14}$$

$$\rho_{23}(t) = \frac{1}{16}\Bigg(-2 + 2e^{\frac{-t}{2\tau_0}}\Bigg(\cosh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}] + \frac{\sinh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}]}{\sqrt{1-64\tau_0^2}}\Bigg)\Bigg) \tag{4.15}$$

$$\rho_{32}(t) = \frac{1}{16}\Bigg(-2 + 2e^{\frac{-t}{2\tau_0}}\Bigg(\cosh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}] + \frac{\sinh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}]}{\sqrt{1-64\tau_0^2}}\Bigg)\Bigg) \tag{4.16}$$

$$\rho_{33}(t) = \frac{1}{16}\Bigg(6 + 2e^{\frac{-t}{2\tau_0}}\Bigg(-3\cosh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}] - 3\frac{\sinh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}]}{\sqrt{1-64\tau_0^2}}\Bigg)\Bigg) \tag{4.17}$$

$$\rho_{88}(t) = \frac{1}{4}\Bigg(1 - e^{\frac{-t}{2\tau_0}}\cosh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}] - \frac{e^{\frac{-t}{2\tau_0}}\sinh[\frac{\sqrt{1-64\tau_0^2}t}{2\tau_0}]}{\sqrt{1-64\tau_0^2}}\Bigg) \tag{4.18}$$

The general solution is

$$\begin{aligned}\rho(t) &= \rho_{22}(t)|0ge\rangle\langle 0ge| + (\rho_{23}(t)|0ge\rangle\langle 0ge| + h.c.) \\ &\quad + \rho_{33}(t)|0ge\rangle\langle 0ge| + \rho_{88}(t)|1gg\rangle\langle 1gg|\end{aligned} \tag{4.19}$$

4.3 Results And Discussion

We use the (4.19) in order to investigate the effects of the phase telegraph noise on the entanglement properties of the system. We obtain the reduced density matrix $\rho^{af,1}(t)$ of the subsystem for the atom 1 and the field as follows by tracing out degree of freedom of the second atom:

$$\rho^{af,1}(t) = \rho_{22}(t)|0g\rangle\langle 0g| + \rho_{33}(t)|0e\rangle\langle 0e| + \rho_{88}(t)|1g\rangle\langle 1g| \tag{4.20}$$

and by repeating this process for the first atom in order to obtain the density matrix for the atom 2 and the field subsystem

$$\rho^{af,2}(t) = \rho_{22}(t)|0e\rangle\langle 0e| + \rho_{33}(t)|0g\rangle\langle 0g| + \rho_{88}(t)|1g\rangle\langle 1g| \tag{4.21}$$

As $\rho_{23}(t)$ and $\rho_{32}(t)$ are equal, it can be suggested that $\rho^{af,1}(t)$ and $\rho^{af,2}(t)$ gives the same density matrix. So

$$\rho^{af,1}(t) = \rho^{af,2}(t) \tag{4.22}$$

By tracing out the degree of freedom of the cavity field, we get the reduced density matrix $\rho^{aa}(t)$, for the subsystem containing the atom as

$$\begin{aligned} \rho^{aa}(t) &= \rho_{22}(t)|ge\rangle\langle ge| + (\rho_{23}(t)|ge\rangle\langle eg| + h.c.) \\ &\quad + \rho_{33}(t)|eg\rangle\langle eg| + \rho_{88}(t)|gg\rangle\langle gg| \end{aligned} \tag{4.23}$$

The dimension of the atom-atom subsystems are $2 \otimes 2$. For these systems the degree of the entanglement can be quantified by the Wootters's concurrence *C(t)* [77] which is defined as

$$C(t) = max\{0, \lambda_1(t) - \lambda_2(t) - \lambda_3(t) - \lambda_4(t)\} \tag{4.24}$$

where $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t) \geq \lambda_4(t)$ are the square roots of the eigenvalues of the matrix

$R(t) = \rho(t)(\sigma_y \otimes \sigma_y)\rho(t)^*(\sigma_y \otimes \sigma_y)$. Here $\rho(t)$ is density matrix. Finding *R(t)* we first write Pauli spin matrix σ_y.

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.25}$$

Afterwards, we find tensor product of σ_y and σ_y

$$(\sigma_y \otimes \sigma_y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.26}$$

We will calculate $C(t)$ of these subsystems for checking entanglement properties.

First the atom 1 and the field subsystem will be calculated. We can write the equation (4.20) as a matrix, as in the following

$$\rho^{af,1}(t) = \begin{pmatrix} \rho_{22}(t) & 0 & 0 & 0 \\ 0 & \rho_{33}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{88}(t) \end{pmatrix} \tag{4.27}$$

We calculate conjugate of (4.27) If we multiply equation (4.26), (4.27) and conjugate of $\rho^{af,1}(t)$ we find *R(t)*. Later we find eigenvalues of *R(t)* and we calculate concurrence of *R(t)*. We find **C(t)** and we plot it.

$$\rho^{af,2}(t) = \begin{pmatrix} \rho_{33}(t) & 0 & 0 & 0 \\ 0 & \rho_{22}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{88}(t) \end{pmatrix} \tag{4.28}$$

We calculate conjugate of (4.28) If we multiply equation (4.26), (4.28) and conjugate of $\rho^{af,2}(t)$ we find *R(t)*. Later we find eigenvalues of *R(t)* and we calculate concurrence of *R(t)*. We find **C(t)**.

The degree of the entanglement is calculated and found to zero, $C(t) = 0$ from the equations (4.20) and (4.21). So there is no entanglement present between the field and the individual atoms during the time evolution of the system due to the noise. As a result, it can be stated that the atoms don't entangle with the cavity field. At the beginning of the interaction, there may be minor entanglement between the field and

the atoms but it disappears in a very short time.

Then we calculate the concurrence of atom-atom subsystem. From the equation (4.23) we can write the reduced density matrix $\rho^{aa}(t)$, for the subsystem which contains two atoms. We write $\rho^{aa}(t)$ below

$$\rho^{aa}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{32}(t) & \rho_{33}(t) & 0 \\ 0 & 0 & 0 & \rho_{88}(t) \end{pmatrix} \tag{4.29}$$

After that, we calculate conjugate of $\rho^{aa}(t)$. If we multiply equation (4.26), (4.29) and conjugate of $\rho^{aa}(t)$ we find *R(t)* for $\rho^{aa}(t)$. Later we find eigenvalues of *R(t)* and we calculate concurrence of *R(t)*. In the atom-atom subsystem entanglement the situation is very different. This can seen from (4.29) results. The effects of random phase telegraph noise on the entanglement properties of the atom-atom subsystem are shown in Fig.4.1.

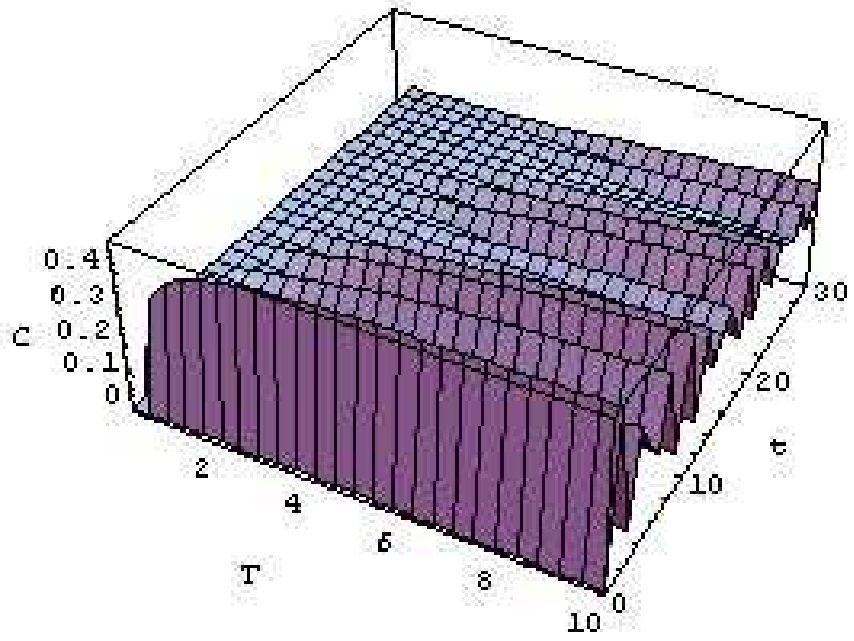

Figure 4.1: Concurrence C as the function of the time $g_0 t$ and the mean dwell time

As seen in the graphic above, the entanglement between the atoms reaches maximum level and it disappears with oscillations after a certain time. Another result is that, the average dwell time T increases and in line with this, the strength of the entanglement and the appearance time increases. The effects of the noise on the interaction decrease at the same time. There is no energy dissipation in the system. As a result, this dissipation mechanism is different from the general dissipation mechanism, which affects the system energy. The system monitors all phase changes and may react to them as a decrease factor. There is a relation between the entanglement time of the atoms and average dwell time. As a result, the sufficiently strong noise helps to create fixed state atom-atom entanglement. If average mean time goes to infinity, the effect of the noise over the atoms decreases and the entanglement dynamics become periodic. There is a monotonous relation between the entanglement of the atoms and the mean dwell time of phase telegraph noise. In the current study there is no energy dissipation from the system and the noise cannot be controlled. The noise depends completely on the stochastic behavior of the system, not the environmental effect. In quantum information processing, noise plays a constructive role, but entanglement arises from a controllable situation. We can say that there is no entanglement between the individual atoms and the field in time due to the noise

CHAPTER 5

ENTANGLEMENT OF THREE ATOM- ONE FIELD INTERACTION UNDER RANDOM PHASE TELEGRAPH NOISE

In this chapter we increases the number of atom interacting with the field and study one-field three-atom interaction by Jaynes-Cummings model under random phase telegraph noise. We show that under the noise the atoms and the field do not entangled with each other during the time evolution of the system. First we calculate fidelity of the subsystem to find if the system is entangled or not. We use fidelity because the system is three-partite. We know by the definition of the fidelity that, F is non-negative and it can not be larger than 1.

In section 5.1, the formulation of the problem is presented briefly. The analytical solution of the systems of three atoms in an optical cavity under random phase telegraph noise is obtained. In section 5.2 solution of the problem presented. In section 5.3 the result are discussed.

5.1 The Formulation of the Problem

The Hamiltonian of the system with rotating wave approximation under the assumed conditions is ($\hbar = 1$).

$$\begin{aligned} H &= \omega_0 S_z^1 + \omega_0 S_z^2 + \omega_0 S_z^3 + \omega_0 a^\dagger a + g^*(t) S_+^1 a + g(t) S_-^1 a^\dagger \\ &\quad + g^*(t) S_+^2 a + g(t) S_-^2 a^\dagger + g^*(t) S_+^3 a + g(t) S_-^3 a^\dagger \end{aligned} \tag{5.1}$$

We use the interaction part of the Hamiltonian. The interaction part of the Hamiltonian is;

$$H_{int}(t) = g^*(t) S_+ a + g(t) S_- a^\dagger + g^*(t) S_+ b + g(t) S_- b^\dagger \tag{5.2}$$

where $g(t)$ is the time-dependent coupling coefficient between the atom and the fields, and it is considered to be completely stochastic:

$$g(t) = g_0 e^{i\phi(t)} \tag{5.3}$$

g_0 is a real number. In our calculation we have $g_0 = 1$.

5.2 Solution

First we obtain H_{int} matrix with basis $\{|0egg\rangle, |0geg\rangle, |0gge\rangle, |1ggg\rangle\}$. We find the elements of H_{int} matrix below

$$\begin{aligned}
H_{11} &= \langle 0egg|H_{int}|0egg\rangle = 0 \qquad H_{12} = \langle 0egg|H_{int}|0geg\rangle = 0 \\
H_{13} &= \langle 0egg|H_{int}|0gge\rangle = 0 \qquad H_{14} = \langle 0egg|H_{int}|1ggg\rangle = g^*(t) \\
H_{21} &= \langle 0geg|H_{int}|0egg\rangle = 0 \qquad H_{22} = \langle 0geg|H_{int}|0geg\rangle = 0 \\
H_{23} &= \langle 0geg|H_{int}|0gge\rangle = 0 \qquad H_{24} = \langle 0geg|H_{int}|1ggg\rangle = g^*(t) \\
H_{31} &= \langle 0gge|H_{int}|0egg\rangle = 0 \qquad H_{32} = \langle 0gge|H_{int}|0geg\rangle = 0 \\
H_{33} &= \langle 0gge|H_{int}|0gge\rangle = 0 \qquad H_{34} = \langle 0gge|H_{int}|1ggg\rangle = g^*(t) \\
H_{41} &= \langle 1ggg|H_{int}|0egg\rangle = g(t) \quad H_{42} = \langle 1ggg|H_{int}|0geg\rangle = g(t) \\
H_{43} &= \langle 1ggg|H_{int}|0gge\rangle = g(t) \quad H_{44} = \langle 1ggg|H_{int}|1ggg\rangle = 0
\end{aligned} \tag{5.4}$$

From the equations above we obtain H_{int} matrix:

$$H_{int} = \begin{pmatrix} 0 & 0 & 0 & g^*(t) \\ 0 & 0 & 0 & g^*(t) \\ 0 & 0 & 0 & g^*(t) \\ g(t) & g(t) & g(t) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi(t)} \\ 0 & 0 & 0 & e^{-i\phi(t)} \\ 0 & 0 & 0 & e^{-i\phi(t)} \\ e^{i\phi(t)} & e^{i\phi(t)} & e^{i\phi(t)} & 0 \end{pmatrix} \tag{5.5}$$

From the other chapters, we now assume that the unitary matrix equal to

$$U = e^{(-iH_{int}\tau)} \tag{5.6}$$

We find unitary matrix $U(\phi;\tau,0)$ below

$$\begin{aligned}
U(\phi;\tau,0) &= \{\{\frac{1}{3}(2+\cos(\sqrt{3}\tau)), \frac{1}{3}(-1+\cos(\sqrt{3}\tau)) \\
&, \frac{1}{3}(-1+\cos(\sqrt{3}\tau)), -\frac{ie^{-i\phi}\sin(\sqrt{3}\tau)}{\sqrt{3}}\} \\
&\{\frac{1}{3}(-1+\cos(\sqrt{3}\tau)), \frac{1}{3}(2+\cos(\sqrt{3}\tau)) \\
&, \frac{1}{3}(-1+\cos(\sqrt{3}\tau)), -\frac{ie^{-i\phi}\sin(\sqrt{3}\tau)}{\sqrt{3}}\} \\
&\{\frac{1}{3}(-1+\cos(\sqrt{3}\tau)), \frac{1}{3}(-1+\cos(\sqrt{3}\tau)) \\
&, \frac{1}{3}(2+\cos(\sqrt{3}\tau)), -\frac{ie^{-i\phi}\sin(\sqrt{3}\tau)}{\sqrt{3}}\} \\
&\{\frac{\sin(\sqrt{3}\tau)(-i\cos(\phi)+\sin(\phi))}{\sqrt{3}}, \frac{\sin(\sqrt{3}\tau)(-i\cos(\phi)+\sin(\phi))}{\sqrt{3}}, \\
&\frac{\sin(\sqrt{3}\tau)(-i\cos(\phi)+\sin(\phi))}{\sqrt{3}}, cos(\sqrt{3}\tau)\}
\end{aligned} \tag{5.7}$$

We take inverse of U and find $U^{-1}(\phi;\tau,0)$;

$$\begin{aligned}
U^{-1}(\phi;\tau,0) &= \{\frac{1}{3}(2+\cos(\sqrt{3}\tau)), \frac{1}{3}(-1+\cos(\sqrt{3}\tau)), \frac{1}{3}(-1+\cos(\sqrt{3}\tau)) \\
&\frac{\sin(\sqrt{3}\tau)(-i\cos(\phi)+\sin(\phi))}{\sqrt{3}}\} \\
&\{\frac{1}{3}(-1+\cos(\sqrt{3}\tau)), \frac{1}{3}(2+\cos(\sqrt{3}\tau)), \frac{1}{3}(-1+\cos(\sqrt{3}\tau)) \\
&\frac{\sin(\sqrt{3}\tau)(-i\cos(\phi)+\sin(\phi))}{\sqrt{3}}\} \\
&\{\frac{1}{3}(-1+\cos(\sqrt{3}\tau)), \frac{1}{3}(-1+\cos(\sqrt{3}\tau)), \frac{1}{3}(2+\cos(\sqrt{3}\tau)) \\
&\frac{\sin(\sqrt{3}\tau)(-i\cos(\phi)+\sin(\phi))}{\sqrt{3}}\} \\
&\{\frac{ie^{i\phi}\sin(\sqrt{3}\tau)}{\sqrt{3}}, \frac{ie^{i\phi}\sin(\sqrt{3}\tau)}{\sqrt{3}}, \frac{ie^{i\phi}\sin(\sqrt{3}\tau)}{\sqrt{3}}, \cos(\sqrt{3}\tau)\}
\end{aligned} \tag{5.8}$$

We write $\rho(0)$, $\rho(t)$ and $\rho(\tau)$ below

$$\rho(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.9}$$

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t) \\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t) \\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(t)) \end{pmatrix} \tag{5.10}$$

and

$$\rho(\tau) = \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) & \rho_{13}(\tau) & \rho_{14}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) & \rho_{23}(\tau) & \rho_{24}(\tau) \\ \rho_{31}(\tau) & \rho_{32}(\tau) & \rho_{33}(\tau) & \rho_{34}(\tau) \\ \rho_{41}(\tau) & \rho_{42}(\tau) & \rho_{43}(\tau) & \rho_{44}(\tau) \end{pmatrix} \tag{5.11}$$

We apply the appropriate form of equations (5.7) and (5.8) with the conditions in equations (5.9), (5.10) and (5.11) to our master equation. First of all, we make the integrating parts respect to the ϕ in the interval $[0, 2\pi]$. Afterwards, we divide the both sides of the equation $e^{+\frac{\tau}{\tau_0}}$. After division we use of the Laplace transformation respect to the $[\tau, s]$. Then we solve the equation. After then, in order to obtain the density matrix we use the inverselaplace transform. After simplifying our solution we have

$$\begin{aligned} \rho_{11}(t) &= \frac{1}{18}\Bigg(9 + e^{-\frac{t}{2\tau_0}}\Bigg(8cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) + 8cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) \\ &\quad + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} + \frac{sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \end{aligned} \tag{5.12}$$

$$\rho_{12}(t) = \frac{1}{36}\Bigg(-6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) + 2cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} + \frac{2sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \tag{5.13}$$

$$\rho_{13}(t) = \frac{1}{36}\Bigg(-6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) + 2cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} + \frac{2sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \tag{5.14}$$

$$\rho_{21}(t) = \frac{1}{36}\Bigg(-6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) + 2cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} + \frac{2sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \tag{5.15}$$

$$\rho_{22}(t) = \frac{1}{36}\Bigg(6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) - 4cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} - \frac{4sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \tag{5.16}$$

$$\rho_{23}(t) = \frac{1}{36}\Bigg(6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) - 4cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} - \frac{4sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \tag{5.17}$$

$$\rho_{31}(t) = \frac{1}{36}\Bigg(-6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) + 2cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} + \frac{2sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \quad (5.18)$$

$$\rho_{32}(t) = \frac{1}{36}\Bigg(6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) - 4cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} - \frac{4sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \quad (5.19)$$

$$\rho_{33}(t) = \frac{1}{36}\Bigg(6 + 2e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) - 4cosh(\frac{\sqrt{1-12\tau_0^2}t}{2\tau_0}) + \frac{sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}} - \frac{4sinh(\frac{\sqrt{1-12\tau_0^2}t}{\tau_0^2})}{\sqrt{1-12\tau_0^2}}\Bigg)\Bigg) \quad (5.20)$$

$$\rho_{44}(t) = \frac{1}{6}\Bigg(1 - e^{-\frac{t}{2\tau_0}}\Bigg(cosh(\frac{\sqrt{1-48\tau_0^2}t}{2\tau_0}) - \frac{e^{-\frac{t}{2\tau_0}}sinh(\frac{\sqrt{1-48\tau_0^2}t}{\tau_0^2})}{\sqrt{1-48\tau_0^2}}\Bigg)\Bigg) \quad (5.21)$$

The other elements are

$\rho_{14}(t) = 0, \rho_{24}(t) = 0, \rho_{34}(t) = 0, \rho_{41}(t) = 0, \rho_{42}(t) = 0, \rho_{43}(t) = 0$

Our general solution is

$$\begin{aligned}\rho(t) = {} & \rho_{11}(t)|0egg\rangle\langle 0egg| + \rho_{12}(t)|0egg\rangle\langle 0geg| + \rho_{13}(t)|0egg\rangle\langle 0gge| \\ & + \rho_{21}(t)|0geg\rangle\langle 0egg| + \rho_{22}(t)|0geg\rangle\langle 0geg| + \rho_{23}(t)|0geg\rangle\langle 0gge| \\ & + \rho_{31}(t)|0gge\rangle\langle 0egg| + \rho_{32}(t)|0gge\rangle\langle 0geg| + \rho_{33}(t)|0gge\rangle\langle 0gge| \\ & + \rho_{44}(t)|1ggg\rangle\langle 1ggg| \end{aligned} \quad (5.22)$$

5.3 Result and Discussion

Now we can obtain a subsystem which is containing field 1 and the atom by tracing out of the degree of freedom of the second field, by using the equation (5.22).

By tracing out the degree of third atom we obtain the reduce density matrix $\rho^{faa,12}(t)$ of the subsystem for the atoms 1, 2 and the field as

$$\begin{aligned}\rho^{faa,12}(t) &= \langle e|\rho(t)|e\rangle + \langle g|\rho(t)|g\rangle \\ &= \rho_{11}(t)|0eg\rangle\langle 0eg| + \rho_{12}(t)|0eg\rangle\langle 0ge| + \rho_{21}(t)|0ge\rangle\langle 0eg| \\ &\quad + \rho_{22}(t)|0ge\rangle\langle 0ge| + \rho_{33}(t)|0gg\rangle\langle 0gg| + \rho_{44}(t)|1gg\rangle\langle 1gg| \quad (5.23)\end{aligned}$$

then we find it for the second atom to obtain the density matrix for the atoms 1, 3 and the field subsystem

$$\begin{aligned}\rho^{faa,13}(t) &= \langle e|\rho(t)|e\rangle + \langle g|\rho(t)|g\rangle \\ &= \rho_{11}(t)|0eg\rangle\langle 0eg| + \rho_{13}(t)|0eg\rangle\langle 0ge| + \rho_{22}(t)|0gg\rangle\langle 0gg| \\ &\quad + \rho_{31}(t)|0ge\rangle\langle 0eg| + \rho_{33}(t)|0ge\rangle\langle 0ge| + \rho_{44}(t)|1gg\rangle\langle 1gg| \quad (5.24)\end{aligned}$$

and repeating it for the first atom to obtain the density matrix for the atoms 2, 3 and the field subsystem

$$\begin{aligned}\rho^{faa,23}(t) &= \langle e|\rho(t)|e\rangle + \langle g|\rho(t)|g\rangle \\ &= \rho_{11}(t)|0gg\rangle\langle 0gg| + \rho_{22}(t)|0eg\rangle\langle 0eg| + \rho_{23}(t)|0eg\rangle\langle 0ge| \\ &\quad + \rho_{32}(t)|0ge\rangle\langle 0eg| + \rho_{33}(t)|0ge\rangle\langle 0ge| + \rho_{44}(t)|1gg\rangle\langle 1gg| \quad (5.25)\end{aligned}$$

By tracing out the degree of freedom of the cavity field, we obtain the reduced density matrix $\rho^{aaa}(t)$, for the subsystem containing three atoms as

$$
\begin{aligned}
\rho^{aaa}(t) &= \langle 0|\rho(t)|0\rangle + \langle 1|\rho(t)|1\rangle \\
&= \rho_{11}(t)|egg\rangle\langle egg| + \rho_{12}(t)|egg\rangle\langle geg| + \rho_{13}(t)|egg\rangle\langle gge| \\
&\quad + \rho_{21}(t)|geg\rangle\langle egg| + \rho_{22}(t)|geg\rangle\langle geg\| + \rho_{23}(t)|geg\rangle\langle gge| \\
&\quad + \rho_{31}(t)|gge\rangle\langle egg| + \rho_{32}(t)|gge\rangle\langle geg| + \rho_{33}(t)|gge\rangle\langle gge| \\
&\quad + \rho_{44}(t)|ggg\rangle\langle ggg| \qquad (5.26)
\end{aligned}
$$

We first checked if the genuine three-partite entanglement can appear in evolution of this system or not. For discriminating the genuine three-partite entangled states from other entangled states or separable states, we searched the state preparation fidelity to adopt and to study if the three-partite entangled states can be generated in the system. For a three-qubit state, the state preparation fidelity, F, is defined as

$$F(\rho) = \langle GHZ|\rho|GHZ\rangle \qquad (5.27)$$

,

where $|GHZ\rangle$ is the three-partite GHZ state. A sufficient condition to determine the genuine three-partite entanglement is whether $F(\rho) > \frac{1}{2}$ or not. We considered the state described by the density matrix $\rho(t)$ in equation (5.22) and chose the GHZ state to be $|GHZ\rangle$

$$|GHZ\rangle = \frac{1}{4}[(|g\rangle - i|e\rangle) \otimes (|g\rangle + |e\rangle) \otimes (|g\rangle - |e\rangle) - (|g\rangle + i|e\rangle) \otimes (|g\rangle - |e\rangle) \otimes (|g\rangle + |e\rangle) \qquad (5.28)$$

$$|GHZ\rangle = \frac{1}{2}(-|gge\rangle + |geg\rangle - i|egg\rangle + i|eee\rangle) \qquad (5.29)$$

If $|GHZ\rangle$ is equal to above equation $\langle GHZ|$ is equal to;

$$\langle GHZ| = \frac{1}{2}(-\langle gge| + \langle geg| + i\langle egg| - i\langle eee|) \tag{5.30}$$

For finding fidelity we have to find $\rho^{aaa}(t)|GHZ\rangle$

$$\begin{aligned}
\rho^{aaa}(t)|GHZ\rangle &= \Big(\rho_{11}(t)|egg\rangle\langle egg| + \rho_{12}(t)|egg\rangle\langle geg| + \rho_{13}(t)|egg\rangle\langle gge| \\
&\quad + \rho_{21}(t)|geg\rangle\langle egg| + \rho_{22}(t)|geg\rangle\langle geg| + \rho_{23}(t)|geg\rangle\langle gge| \\
&\quad + \rho_{31}(t)|gge\rangle\langle egg| + \rho_{32}(t)|gge\rangle\langle geg| + \rho_{33}(t)|gge\rangle\langle gge| \\
&\quad + \rho_{44}(t)|ggg\rangle\langle ggg|\Big)\Big(\frac{1}{2}(-|gge\rangle + |geg\rangle - i|egg\rangle + i|eee\rangle)\Big) \\
&= \frac{1}{2}\Big(- \rho_{13}(t)|egg\rangle - \rho_{23}(t)|geg\rangle - \rho_{33}(t)|gge\rangle \\
&\quad + \rho_{12}(t)|egg\rangle + \rho_{22}(t)|geg\rangle + \rho_{32}(t)|gge\rangle - i\rho_{11}(t) \\
&\quad |egg\rangle - i\rho_{21}(t)|geg\rangle - i\rho_{31}(t)|gge\rangle\Big)
\end{aligned} \tag{5.31}$$

$$\begin{aligned}
F(\rho^{aaa}(t)) = \langle GHZ|\rho^{aaa}(t)|GHZ\rangle &= \Big[\frac{1}{2}\Big(- \langle gge| + \langle geg| + i\langle egg| - i\langle eee|\Big)\Big] \\
&\quad \Big[\frac{1}{2}\Big(- \rho_{13}(t)|egg\rangle - \rho_{23}(t)|geg\rangle - \rho_{33}(t)|gge\rangle \\
&\quad + \rho_{12}(t)|egg\rangle + \rho_{22}(t)|geg\rangle + \rho_{32}(t)|gge\rangle \\
&\quad - i\rho_{11}(t)|egg\rangle - i\rho_{21}(t)|geg\rangle - i\rho_{31}(t)|gge\rangle\Big)\Big] \\
&= \frac{1}{4}\Big(\rho_{33}(t) - \rho_{32}(t) + i\rho_{31}(t) - \rho_{23}(t) + \rho_{22}(t) \\
&\quad - i\rho_{21}(t) - i\rho_{13}(t) + i\rho_{12}(t) + \rho_{11}(t)\Big)
\end{aligned} \tag{5.32}$$

The graphics of fidelity are below.

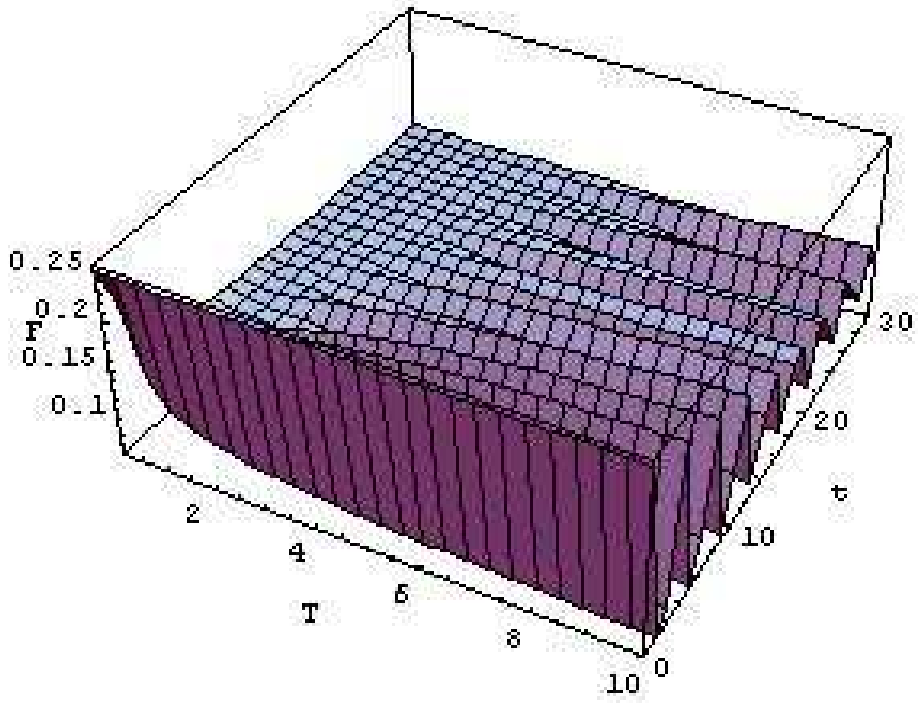

Figure 5.1: The state preparation fidelity F as a function of the time $g_0 t$ and mean dwell time T of the one-field three-atom system under RPTN

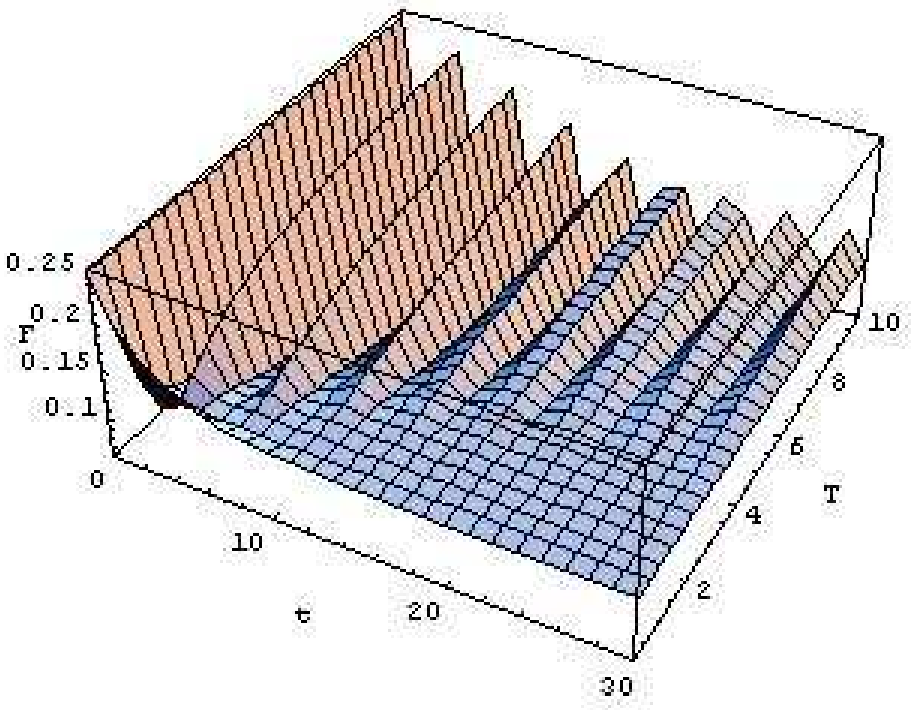

Figure 5.2: The state preparation fidelity F as a function of the time $g_0 t$ and mean dwell time T of the one-field three-atom system under RPTN

In figures 5.1 and 5.2 the state preparation fidelity F is plotted as a function of the time ($g_0 t$) and the mean dwell time τ_0. In the figure while the time increasing the entanglement of the system is decreases. So we can say that the effect of random phase telegraph noise is decreasing here.It can be seen that F cannot exceed $\frac{1}{2}$ during the system whatever the value τ takes.This factor shows that we cannot achieve the genuine three-partite entangled pure state. As a consequence, the one field and three atoms that are in a three-partite mixed state cannot become genuinely three-partite entangled when random phase telegraph noise is present. Thus, one state in this system remains separable during the interaction. In order to investigate the situation more explicitly, we investigate the entanglement dynamics in the sub-systems of this system.

CHAPTER 6
CONCLUSION

In this thesis, we analyzed analytically the interaction between the two-level atom and the single-mode cavity field in case there is a noise resulting from the stochastic phase fluctuations in the atom model proposed by the Jaynes-Cummings model. While examining this situation, we considered the random phase telegraph noise caused totally by the system's stochastic behavior. We also studied the visible effects of the interaction of this noise on the entanglement dynamics in the case of random phase telegraph noise. The entanglement dynamics of these systems in atom-field interaction have also been studied under this noise.

First of all, we observed that noise density of the entanglement is a non-monotonic function in a single atom-field interaction system. For optimal noise density the entanglement the entanglement degree decreases to minimum value and then reaches to a sufficiently high density. Secondly, in the systems which have two atoms and single field interaction, the effects of this noise on the systems' entanglement properties showed that the atoms and the fields are not in entanglement with each other. The noise does not allow the atoms to interact with fields or vice versa. However, the non-vanishing atom-atom entanglement can be reached and this entanglement has a higher scale than the atom-field entanglements. It has been showed that the atom-atom entanglement is rather sensitive to the noise. This system can track almost all of the phase changes of the noise and reacts to the entanglement level which is independent of the mean dwell time. There is correlation between the atom-atom entanglement and mean dwell time. The entanglement is the monotonic function of noise density. However, in the mean dwell time values which are sufficiently small, the decay in the entanglement stabilizes at a constant value and continues to exist in stationary state.

Apart from these, we also made a research on the effects of random phase telegraph noise on this interaction. In order to understand the effects of atom-field interaction on entanglement, we started to study on random phase telegraph noise. Finally, by taking the non-linearity effects into account, we reviewed the entanglement properties

of the Jaynes-Cummings model and proved that the entanglement does not disappear in both cases.

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