L_ζ-MODULES AND A THEOREM OF JON CARLSON

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By

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ABSTRACT

L_ζ-MODULES AND A THEOREM OF JON CARLSON

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In this thesis, we study L_ζ-modules, and using some exact sequences involving L_ζ-modules, we give an alternative proof to a theorem by Jon Carlson which says that any $\mathbb{Z}G$-module is a direct summand of a module which has a filtration by modules induced from elementary abelian subgroups.

Keywords: L_ζ-modules, cohomology, projective module, injective module, projective resolutions, elementary abelian p-subgroups, exact sequences.

ÖZET

L_ζ-MODÜLLERİ VE JON CARLSON'IN BİR THEOREMİ

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Bu tezde L_ζ-modüllerini inceledik ve L_ζ-modüllerini içeren bazı tam dizileri kullanarak, herhangi bir $\mathbb{Z}G$-modülün, temel Abel altgruplardan genişletilmiş modüllerle filitre edilmiş bir modülün direk toplam terimi olduğunu söyleyen Jon Carlson'a ait bir teoremi değişik bir yoldan ispatladık.

Anahtar sözcükler: L_ζ-modülleri, kohomoloji, projektif modül, injektif modül, projektif çözücüler, temel Abel p-altgruplar, tam diziler.

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Chapter 1

Introduction

Let G be a finite group and R be a commutative ring with identity. The cohomology of a group G with coefficients in a RG-module N, where RG is the group algebra, is the cohomology of the cochain complex of the RG-modules:

$$0 \to \mathrm{Hom}_{RG}(P_0, N) \to \mathrm{Hom}_{RG}(P_1, N) \to \dots$$

obtained by applying $\mathrm{Hom}_{RG}(-, N)$ to a projective resolution of the trivial RG-module R. We will denote the cohomology of a group G with coefficients in N as $H^n(G, N)$. The most important cases for the ground ring R of the group ring RG is $R = \mathbb{Z}$ or a field, denoted by k, of characteristic p dividing the order of G. Note that, by Maschke's theorem, the group algebra kG is semisimple when the characteristic p of k does not divide the order of G. In this case, all kG-modules will be projective and hence the cohomology of G will be trivial. That is why, we assume that the characteristic of k divides the order of G.

In [19], Quillen proves a conjecture of Atiyah and Swan which says that the Krull dimension of the mod p cohomology ring of a compact Lie group G equals to the maximum rank of an elementary abelian p-subgroup. Another result in the same paper states that the minimal prime ideals of the mod p cohomology ring of a compact Lie group G are in one to one correspondence with the conjugacy classes of maximal elementary abelian p-subgroups. Using Quillen's work

Chouinard [13] proved that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup is projective. These are some results which emphasize the importance of the elementary abelian p-subgroups of a finite group G for its cohomology and its module category.

Another result in this direction is given by Jon Carlson in [8] which says that any $\mathbb{Z}G$-module M is a direct summand of a module which has a filtration by modules induced from elementary abelian subgroups. The main theorem of [8] is the following:

Theorem 1.0.1 *There exists an integer τ, depending only on G, and there exists a finitely generated $\mathbb{Z}G$-module V such that the direct sum $\mathbb{Z} \oplus V$ has a filtration*

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = \mathbb{Z} \oplus V$$

with the property that for each $i = 1, 2 \ldots, \tau$, there is an elementary abelian subgroup $E_i \subseteq G$ and a $\mathbb{Z}E_i$-module W_i such that

$$L_i/L_{i-1} \cong W_i^{\uparrow G}.$$

The modules $V, W_1, \ldots, W_\tau$ can be assumed to be free as $\mathbb{Z}$-modules.

Here $W_i^{\uparrow G}$ denotes the induced module $W_i^{\uparrow G} = kG \otimes_{kH} W_i$ for kH-module W_i where H is a subgroup of G.

When the coefficient ring $\mathbb{Z}$ is replaced by a field of characteristic p, there is a similar filtration coming from the modules induced from elementary abelian p-subgroups. This result is used to prove some other results, such as Chouinard's result [13] and a theorem of Alperin and Evens [3] on the complexity of modules. Also note that Theorem 1.0.1 is the main ingredient of Carlson and Thevenaz's [11]) work on endo-permutation modules.

Associated to a cohomology class $\zeta \in H^n(G, k)$, there is a module L_ζ defined as the kernel of representing homomorphism $\hat{\zeta} : \Omega^n(k) \to k$ (see Theorem 4.1.11). A module of this form is called L_ζ-module or Carlson's L_ζ-module, referring to Carlson's great contribution to the study of these modules. The main applications of L_ζ-modules can be found in the subject of varieties of modules (see for example [10]).

In this thesis, our main goal is to give an alternative proof of Theorem 1.0.1 using L_ζ-modules. The main idea of the proof is to use Serre's theorem (see page 43) together with the following propositions:

Proposition 1.0.2 ([4]) *If $\zeta_1 \in H^r(G,k)$ and $\zeta_2 \in H^s(G,k)$, then there is an exact sequence*

$$0 \to \Omega^r(L_{\zeta_2}) \to L_{\zeta_1\cdot\zeta_2} \oplus (proj) \to L_{\zeta_1} \to 0.$$

Here the notation $(proj)$ means that the statement is true after adding a suitable projective summand. We will be using this notation throughout the thesis.

Proposition 1.0.3 *Let G be a 2-group. If ζ is a cohomology class in $H^1(G,k)$, then $L_\zeta \cong \Omega(k_H)^{\uparrow G}$ where H is the kernel of ζ.*

Proposition 1.0.4 *Let G be a finite p-group where $p > 2$. If ζ is a cohomology class in $H^1(G,k)$, then $L_{\beta(\zeta)} \oplus (proj)$ has a filtration $\{0\} = M_0 \subseteq M_1 \subseteq M_2 = L_{\beta(\zeta)} \oplus (proj)$ with the property $M_2/M_1 \cong k_H^{\uparrow G}$ and $M_1/M_0 \cong \Omega(k_H)^{\uparrow G}$ here H is the kernel of ζ.*

Here β denotes the Bokstein operator in the group cohomology. We prove these propositions in Chapter 4, and give the alternative proof in Chapter 5. In Chapter 5, we also give two generalizations of Carlson's theorems:

Given kG-modules $M_0, \ldots, M_{n-1}$, we say that a module K has a filtration with sections isomorphic to the Heller shifts of M_i's , for $i = 0, \ldots, n-1$, if there is a filtration $\{0\} = K_0 \subseteq \cdots \subseteq K_n = M$ with the property $K_i/K_{i-1} \cong \Omega^{t_i}(M_{i-1})$ for $i = 1, \ldots, n$.

Theorem 1.0.5 *Let A and B be kG-modules, and E be an n-fold extension*

$$E : 0 \to B \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to A \to 0$$

with extension class $\alpha \in \mathrm{Ext}^n_{kG}(A, B)$. *Suppose that*

$$\tilde{E} : 0 \to \Omega^{-(n-1)}(B) \to M \to A \to 0$$

is the extension associated to α *under the isomorphism* $\mathrm{Ext}^n_{kG}(A, B) \cong \mathrm{Ext}(A, \Omega^{-(n-1)}(B))$. *Then,* $M \oplus (proj)$ *has a filtration with sections isomorphic to Heller shifts of* $M_i^{'}s$ *for* $i = 0, \ldots, n-1$.

Theorem 1.0.6 *Let* ζ *be the cohomology class in* $H^n(G, k) = \mathrm{Ext}^n_{kG}(k, k)$ *which is represented by the extension*

$$E : 0 \to k \to M_{n-1} \to \cdots \to M_0 \to k \to 0.$$

Then $L_\zeta \oplus (proj)$ *has a filtration with sections isomorphic to Heller shifts of* M_i*'s.*

Note that the modular version (over a field of characteristic p) of Carlson's theorem follows from these theorems.

The thesis is organized as follows:

In chapter 2, we give some background material from homological algebra which contains definitions of cohomology, projective resolutions and some basic theorems of cohomology theory for an arbitrary ring R with identity.

In chapter 3, we study the group algebra kG, projective and injective kG-modules and then we focus on group cohomology including the relation between cohomology and extensions, first cohomology $H^1(G, N)$, the existence of minimal projective resolutions.

Chapter 4 includes the syzygies and the proof of well-definedness of L_ζ-modules and an exact sequence of L_ζ which has an important role in chapter 5.

In chapter 5, we give a survey of the paper [8] and then write the alternative proof of Carlson's theorem. We also give generalizations of Carlson's theorem.

In chapter 6, we summarize the proof of the theorem in integral cohomology.

Chapter 2

Preliminaries on Homological Algebra

We compute the cohomology of a finite group G using projective resolutions of the trivial RG-module R, where RG is the group algebra and R is the ground ring which is commutative with identity. The most important cases for R is $R = \mathbb{Z}$ or R is a field, especially a field of characteristic p where p is a prime number. In this chapter, our main interest is the cohomology of a cochain complex of R-modules for any ring with identity. We give the general theory of the homology and the cohomology of a chain complex and a cochain complex of R-modules to obtain main applications to group algebra which are used in cohomology theory of groups. To get more details about the materials in this chapter, we refer the reader to [4], [7], [16].

2.1 Complexes and Homology

Definition 2.1.1 *A complex C of R-modules is a family*
$C = \{C_n, \partial_n\}$, $n \in \mathbb{Z}$, where each C_n is an R-module and $\partial_n : C_n \to C_{n-1}$ is R-module homomorphism, satisfying $\partial_n \circ \partial_{n+1}$=0. Here ∂_n is called the differential

of the complex. Thus a complex C has the form

$$\dots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow C_{-1} \longrightarrow \dots .$$

In this complex, instead of using lower indices, it is often convenient to write C^n for C_{-n} and $\delta^n : C^n \to C^{n+1}$ in place of $\partial_{-n} : C_{-n} \to C_{-n-1}$ for $n \geq 0$.

Definition 2.1.2 *A complex C of R-modules is positive if $C_n = 0$ for $n < 0$. The positive complex is called chain complex. It looks like*

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow 0.$$

A complex C of R-modules is negative if $C_n = 0$ for $n > 0$. The negative complex is called cochain complex and has the form:

$$0 \longrightarrow C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \dots \longrightarrow C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \dots$$

The condition $\partial_n \circ \partial_{n+1}=0$ for all integers n gives that Im $\partial_{n+1} \subseteq$ ker ∂_n. The homology and similarly the cohomology measures the differences between Im ∂_{n+1} and ker ∂_n as follows.

Definition 2.1.3 *The homology of a chain complex C is defined as*

$$H_n(C) = H_n(C, \partial_*) = \ker\ (\partial_n : C_n \to C_{n-1})/\mathrm{Im}\ (\partial_{n+1} : C_{n+1} \to C_n).$$

The cohomology of a cochain complex C is defined as

$$H^n(C) = H^n(C, \delta^*) = \ker\ (\delta^n : C^n \to C^{n+1})/\mathrm{Im}\ (\delta^{n-1} : C^{n-1} \to C^n).$$

An n-cycle of C is an element of $Z_n(C) := \ker\ (\partial_n : C_n \to C_{n-1})$ and an n-boundary is an element of $B_n(C) := \mathrm{Im}\ (\partial_{n+1} : C_{n+1} \to C_n)$. Similarly an n-cocycle is an element of $Z^n(C) := \ker\ (\delta_n : C^n \to C^{n+1})$ and an n-coboundary is an element of $B_n(C) := \mathrm{Im}\ (\delta^{n-1} : C^{n-1} \to C^n)$. If $x \in C_n$ is such that $\partial_n(x) = 0$ then $x \in Z_n(C)$ and $[x]$ is the image of x in $H_n(C)$ and $[x]$ is called homology class. Two n-cycles x_1, x_2 are in the same homology class, that is $[x_1] = [x_2]$, if and only if $x_1 - x_2 \in \mathrm{Im}\ \partial_{n+1}$. And also if $x \in C_n$, then we say that x has dimension n.

Definition 2.1.4 *If C and D are chain complexes (respectively cochain complexes), a chain map (respectively cochain map) $f : C \to D$ is a family of module homomorphisms $f_n : C_n \to D_n$ (respectively $f^n : C^n \to D^n$), $n \in \mathbb{Z}$, such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, "\partial'_{n-1}"] & D_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

That is $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n (Respectively

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n+1} \arrow[r, "\delta^{n+1}"] \arrow[d, "f^{n+1}"] & C^n \arrow[r, "\delta^n"] \arrow[d, "f^n"] & C^{n-1} \arrow[r, "\delta^{n-1}"] \arrow[d, "f^{n-1}"] & C^{n-2} \arrow[r] \arrow[d, "f^{n-2}"] & \cdots \\
\cdots \arrow[r] & D^{n+1} \arrow[r, "\delta^{n'}"] & D^n \arrow[r, "\delta^{n'}"] & D^{n-1} \arrow[r, "\delta^{n-1'}"] & D^{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

that is $\delta^{n'} \circ f^n = f^{n-1} \circ \delta^n$).

Lemma 2.1.5 *A chain map $f : C \to D$ induces a homomorphism $f_* : H_n(C) \to H_n(D)$ defined by $f_*([x]) = [f_n(x)]$ for $x \in Z_n(C)$ and similarly a cochain map $f : C \to D$ induces a homomorphism $f^* : H^n(C) \to H^n(D)$ defined by $f^*([x]) = [f^n(x)]$ for $x \in Z^n(C)$.*

Definition 2.1.6 *Let $f, f' : C \to D$ be chain maps. We say that f and f' are chain homotopic (written $f \simeq f'$), if there are module homomorphisms $h_n : C_n \to D_{n+1}$ such that $f_n - f'_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ holds for all $n \in \mathbb{Z}$ for the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "{f_{n+1}, f'_{n+1}}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "{f_n \mid f'_n}"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "{f_{n-1}, f'_{n-1}}"] \arrow[dl, "h_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "{f_{n-2}, f'_{n-2}}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, "\partial'_{n-1}"] & D_{n-2} \arrow[r] & \cdots
\end{tikzcd}.$$

Definition 2.1.7 *We say that C and D are chain homotopy equivalent (written $C \simeq D$), if there are chain maps $f : C \to D$ and $f^{'} : D \to C$ such that $f \circ f^{'} \simeq Id_D$ and $f^{'} \circ f \simeq Id_C$. The chain maps f and $f^{'}$ are called chain equivalences.*

We have similar definitions for cochain complexes.

Proposition 2.1.8 *If $f, f^{'} : C \to D$ are chain homotopic, then*

$$f_* = f^{'}_* : H_n(C) \to H_n(D).$$

A homotopy equivalence $C \simeq D$ induces an isomorphism $H_n(C) \cong H_n(D)$ for all $n \in \mathbb{Z}$.

□

The cohomological version of the above proposition is the following.

Proposition 2.1.9 *If $f, f^{'} : C \to D$ are cochain homotopic, then*

$$f^* = (f^{'})^* : H^n(C) \to H^n(D).$$

A homotopy equivalence $C \simeq D$ induces an isomorphism $H^n(C) \cong H^n(D)$ for all $n \in \mathbb{Z}$.

□

Each R-module M may be thought as a trivial positive complex. That is $M_0 = M$ and $M_n = 0$ for $n \neq 0$ and $\partial = 0$.

Definition 2.1.10 *Let M be an R-module and C be a chain complex. A contracting homotopy for the chain map $\varepsilon : C \to M$ is a chain map $f : M \to C$ together with $\varepsilon \circ f = \mathrm{Id}_M$ and a homotopy $s : \mathrm{Id} \simeq f \circ \varepsilon$. That means a contracting homotopy consists of module homomorphisms $f : M \to C_0$ and $s_n : C_n \to C_{n+1}$, $n = 0, 1, 2 \ldots$ such that $\varepsilon \circ f = \mathrm{Id}$, $\partial_1 \circ s_0 + f \circ \varepsilon = \mathrm{Id}_{C_0}$ and $\partial_{n+1} \circ s_n + s_{n-1} \circ \partial_n = \mathrm{Id}$ for $n > 0$.*

Remark 2.1.11 *If $\varepsilon : C \to M$ has a contracting homotopy then we have $\varepsilon_* : H_0(C) \cong M$ for $n = 0$ and $H_n(C) = 0$ for $n > 0$. Contracting homotopy measures the exactness of the complex $\varepsilon : C \to M$.*

Similar things are valid for the cohomology of cochain complex.

Definition 2.1.12 *A short exact sequence*

$$0 \to C^{'} \to C \to C^{''} \to 0$$

of chain complexes consists of chain maps $C^{'} \to C$ and $C \to C^{''}$ such that for each n,

$$0 \longrightarrow C^{'}_n \xrightarrow{g_n} C_n \xrightarrow{f_n} C^{''}_n \longrightarrow 0$$

is a short exact sequence.

Proposition 2.1.13 *Let*

$$0 \longrightarrow C^{'} \xrightarrow{f} C \xrightarrow{g} C^{''} \longrightarrow 0$$

be a short exact sequence of chain complexes, then there is a long exact sequence

$$\ldots \longrightarrow H_{n+1}(C^{''}) \xrightarrow{\partial} H_n(C^{'}) \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C^{''}) \xrightarrow{\partial} \ldots$$

where ∂ is the connecting homomorphism.

□

The definition of the connecting homomorphism and the proof of this proposition can be found in [[4], Ch.2, pg. 27].

We have a similar exact sequence for cohomology:

Proposition 2.1.14 *Let*

$$0 \longrightarrow C^{'} \xrightarrow{f} C \xrightarrow{g} C^{''} \longrightarrow 0$$

a short exact sequence of cochain complexes, then there is a long sequence

$$\ldots \longrightarrow H^n(C') \xrightarrow{g^*} H^n(C) \xrightarrow{f^*} H^n(C'') \xrightarrow{\delta} H^{n+1}(C') \longrightarrow \ldots$$

where δ is connecting homomorphism.

□

2.2 Projective Resolutions and Cohomology

Definition 2.2.1 *An R-module P is called projective if for every homomorphism $f : P \to B$ and every epimorphism $g : A \to B$, there is a homomorphism $h : P \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

Definition 2.2.2 *An R-module I is called injective if for every homomorphism $\beta : A \to I$ and every momomorphism $\gamma : A \to B$, there is homomorphism $\alpha : B \to I$ such that the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\gamma"] \arrow[d, "\beta"'] & B \arrow[dl, "\alpha"] \\ & I & \end{tikzcd}$$

Definition 2.2.3 *A projective resolution of an R-module M is a long exact sequence*

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

where each P_i is a projective R-module.

Remark 2.2.4 *Since every module is a homomorphic image of a free module and every free module is projective, projective resolution always exists.*

Theorem 2.2.5 *(Comparison Theorem) Any homomorphism of modules*

$$M \xrightarrow{f} N$$

can be extended to a chain map of projective resolutions with the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] \arrow[dl, "h_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r, "\partial'_{n+1}"] & Q_n \arrow[r, "\partial'_n"] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Given any two such chain maps f_n and f'_n, there is a chain homotopy $h_n : P_n \to Q_{n+1}$ so that $f_n - f'_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ where $\partial_n : P_n \to P_{n-1}$ and $\partial'_n : Q_n \to Q_{n-1}$ are differentials of the resolutions.

□

Proof: We will prove the theorem using induction on n. Note that, f_0 exists since P_0 is a projective module. Assume that $f_0, f_1, ..., f_{n-1}$ are defined. For f_{n-1} we have $\partial'_{n-1} \circ f_{n-1} \circ \partial_n = f_{n-2} \circ \partial_{n-1} \circ \partial_n = 0$. Thus $f_{n-1} \circ \partial_n \in \ker \partial'_{n-1} = \mathrm{Im}\ \partial'_n$. Consider the diagram:

$$\begin{tikzcd}
 & P_n \arrow[d, "f_{n-1} \circ \partial_n"'] & \\
Q_n \arrow[r, "\partial'_n"] & \mathrm{Im}(\partial'_n) \arrow[r] & 0.
\end{tikzcd}$$

Since P_n is projective there exists a module homomorphism $f_n : P_n \to Q_n$ with the property $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$.

For the chain homotopy we get the proof again by induction on n. The map $h_0 : P_0 \to Q_1$ exists because P_0 is projective. Assume that $h_0, ..., h_{n-1}$ are defined. Consider h_{n-1}. We have $\partial'_n \circ (f_n - f'_n - h_{n-1} \circ \partial_n) = (f_{n-1} - f'_{n-1} - \partial'_n \circ h_{n-1}) \circ \partial_n =$

$h_{n-2} \circ \partial_{n-1} \circ \partial_n = 0$ which means $(f_n - f_n^{'} - h_{n-1} \circ \partial_n) \in \ker \partial_n^{'} = \operatorname{Im} \partial_{n+1}^{'}$. Again we have the diagram

$$\begin{tikzcd} & P_n \arrow[d, "f_n - f_n^{'} - h_{n-1}\circ\partial_n"'] & \\ Q_{n+1} \arrow[r, "\partial_{n+1}^{'}"] & \operatorname{Im}(\partial_{n+1}^{'}) \arrow[r] & 0. \end{tikzcd}$$

Thus there exists $h_n : P_n \to Q_{n+1}$ with $f_n - f_n^{'} - h_{n-1} \circ \partial_n = \partial_{n+1}^{'} \circ h_n$, because P_n is a projective module.

$\square$

Definition 2.2.6 *If N is right R-module and*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \longrightarrow M \to 0$$

is a projective resolution of a left R-module M, then we have a chain complex

$$\ldots \longrightarrow N \otimes_R P_{n+1} \xrightarrow{Id\otimes\partial_{n+1}} N \otimes_R P_n \xrightarrow{Id\otimes\partial_n} N \otimes_R P_{n-1} \longrightarrow \ldots$$

$\operatorname{Tor}_n^R(N, M)$ *is defined as the homology of this complex:*

$$\operatorname{Tor}_n^R(N, M) := H_n(N \otimes P, Id \otimes \partial_*)$$

Definition 2.2.7 *If N is a left R-module and*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \to M \to 0$$

is a projective resolution of a left R-module M, then we have a cochain complex

$$0 \longrightarrow \operatorname{Hom}_R(P_0, N) \xrightarrow{\delta^0} \operatorname{Hom}_R(P_1, N) \xrightarrow{\delta^1} \operatorname{Hom}_R(P_2, N) \to \cdots$$

Ext_R^n*(M,N) is defined as the cohomology of this complex:*

$$\operatorname{Ext}_R^n(M, N) := H^n(\operatorname{Hom}_R(P, N), \delta^*)$$

In this definition, for $n = 0$, we have $\operatorname{Tor}_0^R(N, M) = N \otimes_R M$ and $\operatorname{Ext}_R^0(M, N) = \operatorname{Hom}_R(M, N)$.

Proposition 2.2.8 *If M is projective R-module and N is any R-module, then* $\mathrm{Ext}_R^n(M,N) = 0 = \mathrm{Tor}_n^R(M,N)$ *for all n.*

$\mathrm{Tor}_n^R(-,-)$ and $\mathrm{Ext}_R^n(-,-)$ preserve direct sum.

Proposition 2.2.9 *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of left R-modules.*
i)If N is a right R-module, then there is a long exact sequence

$$\cdots \to \mathrm{Tor}_n^R(N,M_1) \to \mathrm{Tor}_n^R(N,M_2) \to \mathrm{Tor}_n^R(N,M_3) \to \dots$$
$$\to N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \to 0$$

ii)If N is a left R-module, there is a long exact sequence

$$0 \to \mathrm{Hom}_R(N,M_1) \to \mathrm{Hom}_R(N,M_2) \to \mathrm{Hom}_R(N,M_3) \to$$
$$\cdots \to \mathrm{Ext}_R^n(N,M_1) \to \mathrm{Ext}_R^n(N,M_2) \to \mathrm{Ext}_R^n(N,M_3) \to \dots.$$

□

$N \otimes_R -$ or $- \otimes_R N$ are covariant functors. $\mathrm{Hom}_R(N,-)$ is a covariant functor, but $\mathrm{Hom}_R(-,N)$ is a contravariant functor.

Proposition 2.2.10 *Let*

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

be a short exact sequence of right R-modules.
i) N is a left R-module. Then there is a long exact sequence

$$\cdots \to \mathrm{Tor}_n^R(M_0,N) \to \mathrm{Tor}_n^R(M_1,N) \to \mathrm{Tor}_n^R(M_2,N) \to \dots$$
$$\to M_0 \otimes_R N \to M_1 \otimes_R N \to M_2 \otimes_R N \to 0$$

ii)Let

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

be a short exact sequence of left R-modules and N is a left R-modules. Then there is a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N) \to \mathrm{Hom}_R(M_0, N) \to ..\\ \cdots \to \mathrm{Ext}^n_R(M_2, N) \to \mathrm{Ext}^n_R(M_1, N) \to \mathrm{Ext}^n_R(M_0, N) \to \dots \end{aligned}$$

□

2.3 The Künneth Theorem

Let C and D be chain complexes of right, respectively left, R-modules. We can construct a new complex in the following form

$$(C \otimes_R D)_n = \bigoplus_{i+j=n} (C_i \otimes_R C_j)$$

The differential $\partial_n : (C \otimes_R D)_n \to (C \otimes_R D)_{n-1}$ is given by

$$\partial_n(x \otimes y) = \partial_i(x) \otimes y + (-1)^i x \otimes \partial_j(y)$$

for $x \in C_i$ and $y \in D_j$ and we have $\partial_n \circ \partial_{n+1} = 0$. This formula shows that the tensor product $x_1 \otimes x_2$ of cycles is a cycle in $C \otimes D$ and the tensor product of a cycle and a boundary is a boundary. Thus if x_1 and x_2 are cycles in C and D respectively then we have a well defined group homomorphism
$\rho : H_i(C) \otimes_R H_j(D) \to H_{i+j}(C \otimes_R D)$ such that $\rho : [x_1] \otimes [x_2] \mapsto [x_1 \otimes x_2]$.

Definition 2.3.1 *A left R-module N is called flat if for any long exact sequence of right R-modules*

$$\cdots \to M_n \to M_{n-1} \to M_{n-2} \to \dots$$

the sequence

$$\cdots \to M_n \otimes_R N \to M_{n-1} \otimes_R N \to M_{n-2} \otimes_R N \to \dots$$

is also exact.

Theorem 2.3.2 *(The Künneth Theorem) Let C be a chain complex of right R-modules and D be a chain complex of left R-modules. If the cycles $Z_n(C)$ and the boundaries $B_n(C)$ are flat modules for all n, then there is a short exact sequence of R-modules*

$$0 \to \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D) \to H_n(C \otimes_R D) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0.$$

Proof: [[4], Ch.2, pg. 39] Consider $Z(C)$ and $B(C)$ as the complexes of flat modules with zero boundaries. Since $Z(C)$ is flat, we have

$$(Z(C) \otimes_R Z(D))_n = \ker(1 \otimes \partial : (Z(C) \otimes_R D)_n \to (Z(C) \otimes_R D)_{n-1})$$

and

$$(Z(C) \otimes_R B(D))_n = \mathrm{Im}(1 \otimes \partial : (Z(C) \otimes_R D)_{n+1} \to (Z(C) \otimes_R D)_n).$$

Thus

$$H_*(Z(C) \otimes_R D) = Z(C) \otimes_R H_*(D).$$

Since $B(C)$ is flat, similarly we have

$$H_*(B(C) \otimes_R D) = B(C) \otimes_R H_*(D).$$

Consider the short exact sequence of complexes:

$$0 \longrightarrow B(C) \xrightarrow{i} Z(C) \longrightarrow H(C) \longrightarrow 0$$

We tensor this exact sequence with $H(D)$. By definition of flat module, we have $\mathrm{Tor}_1^R(Z(C), H_*(D)) = 0$. So the long exact sequence in Proposition 2.2.10 becomes

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}_1^R(H_*(C), H_*(D)) &\longrightarrow H_*(B(C) \otimes_R D) \\ \xrightarrow{i_*} H_*(Z(C) \otimes_R D) &\longrightarrow H_*(C) \otimes_R H_*(D) \longrightarrow 0. \end{aligned} \tag{2.1}$$

Consider

$$0 \to Z(C) \to C \to B(C)[-1] \to 0$$

where [-1] means a shift of degree -1, so that $(B(C)[-1])_n = B_{n-1}(C)$. We tensor this short exact sequence with D. Since $\mathrm{Tor}_1^R(B(C), D) = 0$, we get a short exact sequence

$$0 \to Z(C) \otimes_R D \to C \otimes_R D \to (B(C) \otimes_R D)[-1] \to 0.$$

By Proposition 2.1.13, we have

$$\begin{array}{ccccccc} \ldots & \longrightarrow & H_*(B(C)\otimes_R D) & \xrightarrow{i_*} & H_*(Z(C)\otimes_R D) & \longrightarrow & H_*(C\otimes_R D) \\ & \longrightarrow & H_*(B(C)\otimes_R D)[-1] & \xrightarrow{i_*} & H_*(Z(C)\otimes_R D)[-1] & \longrightarrow & \ldots. \end{array}$$

This long exact sequence gives

$$0 \to \mathrm{Coker(i_*)} \to H_*(C\otimes_R D) \to \ker(i_*)[-1] \to 0.$$

Using the exact sequence (2.1), we get

$$0 \to H_*(C)\otimes_R H_*(D) \to H_*(C\otimes_R D) \to \mathrm{Tor}_1^R(H_*(C), H_*(D))[-1] \to 0.$$

$\square$

Let C be a chain complex such that $Z_n(C)$ and $H_n(C)$ are projective. Then the exact sequence

$$0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$$

splits, and hence $B_n(C)$ is projective. Since projective modules are also flat, $Z_n(C)$, $H_n(C)$ and $B_n(C)$ are flat and by the definition of a flat module $\mathrm{Tor}_1^R(H_i(C), H_j(D)) = 0$. Using the Künneth Theorem, we obtain the following corollaries.

Corollary 2.3.3 *If $Z_n(C)$ and $H_n(C)$ are projective R-modules for all n, then*

$$H_n(C\otimes_R D) \cong \bigoplus_{i+j=n} H_i(C)\otimes_R H_j(D).$$

Corollary 2.3.4 *If $Z_n(C)$ and $H_n(C)$ are projective R-modules and either C or D exact, then so is $C\otimes_R D$.*

Chapter 3

Group Cohomology

Let G be a finite group and k be a field of characteristic p. In this chapter we give some properties of the projective and the injective kG-modules. We give the definition of the group cohomology and study the relation between the cohomology and extensions, in particular, we study the first cohomology $H^1(G, -)$. Using the existence of the projective cover of a kG-module M, we give the existence of the minimal projective resolution of M which we will use to define the syzygies and L_ζ-modules later in chapter 4.

3.1 The Group Algebra kG

Definition 3.1.1 *Let G be a finite group with elements $\{g_1, \dots, g_n\}$ and k be a field of characteristic p. The group ring kG is the set of all formal finite sums*

$$\{\sum_{i=1}^{n} a_i g_i, a_i \in k\}$$

with addition and multiplication defined by

$$\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i$$

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_h h) = \sum_{g,h\in G} a_g b_h (gh).$$

Since k is a field, kG is a vector space with basis $g_1, \ldots, g_n$. The scalar multiplication is defined $\lambda u = \sum_{i=1}^{n}(\lambda a_i)g_i$ for $\lambda \in k$. So kG is an algebra which we call the group algebra kG. The group algebra kG has a multiplicative identity $1 = 1_k 1_G$. For any kG-module M, we define the k-dual $M^* = \mathrm{Hom}(M, k)$ as the kG-module of the k linear homomorphisms from M to the trivial module k. M^* is a kG-module with G-action $(gf)(m) = f(g^{-1}m)$ for $g \in G, f \in M^*, m \in M$.

We know list some of the basic properties of kG.

Proposition 3.1.2 *$kG \cong kG^*$ as kG-modules, that is, kG is a Frobenius algebra.*

Proof: For proof see [[9], pg. 8].

Proposition 3.1.3 *kG is an injective kG-module, that is, kG is self-injective.*

Corollary 3.1.4 *Every finitely generated injective kG-module is projective, and every finitely generated projective kG-module is injective.*

Proposition 3.1.5 *A kG-module M is projective if and only if M is a direct summand of a free module.*

Proposition 3.1.6 *If P is a projective kG-module and M is any kG-module, then $P \otimes M$ is a projective kG-module.*

Proof: See [[9], pg. 11].

The following propositions show one of the useful properties of the projective and the injective modules.

Proposition 3.1.7 *Given an exact sequence of the form*

$$0 \to A \to B \to C \oplus P \to 0$$

where P is a projective kG-module, there is a kG-module B' such that $B \cong B' \oplus P$ and

$$0 \to A \to B' \to C \to 0$$

is exact.

Proof: Using the given exact sequence

$$0 \to A \to B \to C \oplus P \to 0,$$

one gets the commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \ker(\pi_2 \circ \mathrm{g}) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\iota_1"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "g"] \arrow[d] & C \oplus P \arrow[r] \arrow[d, "\pi_2"'] & 0 \\
 & & P \arrow[r, equal] & P &
\end{tikzcd}$$

Consider the exact sequence

$$0 \longrightarrow \ker(\pi_2 \circ \mathrm{g}) \longrightarrow B \xrightarrow{\pi_2 \circ g} P \longrightarrow 0.$$

Since P is projective the exact sequence splits and we have

$$B \cong \ker(\pi_2 \circ \mathrm{g}) \oplus P.$$

The proposition follows by taking $B' \cong \ker(\pi_2 \circ \mathrm{g})$.

$\square$

Corollary 3.1.8 *If*

$$0 \to A \to B \oplus P_1 \to C \oplus P_2 \to 0$$

is an exact sequence of kG-modules where B is projective free and P_1, P_2 are projective kG-modules, then the sequence

$$0 \to A \to B \oplus P \to C \to 0$$

is exact for some projective kG-module P.

Proposition 3.1.9 *Given an exact sequence*

$$0 \to I \oplus A \to B \to C \to 0$$

where I is an injective kG-module , there exists a kG-module B' such that $B \cong B' \oplus I$ and

$$0 \to A \to B' \to C \to 0$$

is exact.

Proof: This is the dual argument of Proposition 3.1.7.

$\square$

Remark 3.1.10 *Since any projective kG-module is injective, we cancel the projective modules from the left and the right side of an exact sequence.*

Definition 3.1.11 *Let M be a kG-module, H a subgroup of G, and L be a kH-module. We denote the restriction of M to H as $M \downarrow_H$. The induced module $L \uparrow^G$ as a kG-module is defined as $L \uparrow^G := kG \otimes_{kH} L$ and here kG acts by left multiplication.*

Proposition 3.1.12 *If P is a projective kG-module and H is a subgroup of G, then $P \downarrow_H$ is a projective kH-module.*

Proof: See [[2], Ch.2, pg. 33].

Proposition 3.1.13 *If H is a subgroup of G and L is a projective kH-module, then $L \uparrow^G$ is a projective kG-module.*

Proof: See [[2], Ch.3, pg. 57].

3.2 Cohomology of Groups and Extensions

Definition 3.2.1 *Let M and N be finitely generated kG-modules. Let*

$$P_* \xrightarrow{\varepsilon} M$$

be any projective resolution of M. Applying $\mathrm{Hom}_{kG}(-, N)$ we get the complex

$$0 \to \mathrm{Hom}_{kG}(P_0, N) \to \mathrm{Hom}_{kG}(P_1, N) \to \dots$$

Then $\mathrm{Ext}^n_{kG}(M, N)$ is defined as the cohomology of the complex in the following way.

$$\mathrm{Ext}^n_{kG}(M, N) := H^n(\mathrm{Hom}_{kG}(P_*, N)).$$

If $M = k$ is the trivial kG-module then we have a special notation $H^n(G, N) := \mathrm{Ext}^n_{kG}(k, N)$ and it is called " The Cohomology of G with coefficients in N".

If we have $N = k$, then $H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$.

Note that $\mathrm{Ext}^n_{kG}(-, -)$ does not depend on the choice of the projective resolution. (See [[9], Ch.2, pg. 29])

Let $U^n(M, N)$ be the set of all exact sequences of finitely generated kG-modules of the form

$$E : 0 \to N \to B_{n-1} \to \cdots \to B_0 \to M \to 0.$$

We call the exact sequence

$$E : 0 \to N \to B_{n-1} \to \cdots \to B_0 \to M \to 0$$

an n-fold extension of M by N.

Define a relation $\equiv$ on $U^n(M, N)$ by $E_1 \equiv E_2$ if there is a chain map Θ_*

$$\begin{array}{ccccccccccccc}
E_1 : 0 & \longrightarrow & N & \longrightarrow & B_{n-1} & \longrightarrow & \dots & \longrightarrow & B_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \theta_{n-1}} & & & & \downarrow{\scriptstyle \theta_0} & & \| & & \\
E_2 : 0 & \longrightarrow & N & \longrightarrow & C_{n-1} & \longrightarrow & \dots & \longrightarrow & C_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

The relation $\equiv$ is not an equivalence relation, because it is not symmetric. To have an equivalence relation define $\sim$ as follows. $E_1 \sim E_2$ provided there exists a chain $F_0, ...F_m \in U^n(M,N)$ with $E_1 = F_0$, $E_2 = F_m$ and for each $i = 1, ..., m$ either $F_{i-1} \equiv F_i$ or $F_i \equiv F_{i-1}$. We can denote the equivalence classes of an exact sequence E by $class(E)$. There is an addition which makes $U^n(M,N)/\sim$ an abelian group. We have the following:

Theorem 3.2.2 *Let M and N be kG-modules. Then there is an isomorphism*

$$\mathrm{Ext}^n_{kG}(M,N) \cong U^n(M,N)/\sim .$$

Proof: Let

$$P_* \xrightarrow{\epsilon} M$$

be a projective resolution. For a given $E \in U^n(M,N)$, we get a chain map μ_*.

$$\begin{tikzcd}
{} \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "0"'] & P_n \arrow[r] \arrow[d, "\mu_n"'] & P_{n-1} \arrow[r] \arrow[d, "\mu_{n-1}"'] & \dots \arrow[r] & P_0 \arrow[r] \arrow[d, "\mu_0"'] & M \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & N \arrow[r] & B_{n-1} \arrow[r] & \dots \arrow[r] & B_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

From the diagram one gets $\mu_n \circ \partial_{n+1} = 0$ which means $\mu_n : P_n \to N$ is a cocycle. The assignment $class(E) \mapsto [\mu_n]$ gives a well defined homomorphism θ from $U^n(M,N)/\sim$ to $\mathrm{Ext}^n_{kG}(M,N)$. Conversely given $\zeta \in \mathrm{Ext}^n_{kG}(M,N)$, choose a cocycle $\hat{\zeta} : P_n \to N$ representing ζ. We have a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & P_{n+1} \arrow[r] \arrow[d, "0"'] & P_n \arrow[r, "\partial_n"] \arrow[d, "\hat{\zeta}"'] & P_{n-1} \arrow[r] \arrow[d, "g"'] & P_{n-2} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & M \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & N \arrow[r, "f"] & B \arrow[r, "h"] & P_{n-2} \arrow[r] & \dots \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where B is the pushout of the first square. This gives a well defined map ϕ on the opposite direction. It is easy to see that θ and ϕ are inverses to each other.

$\square$

3.3 Low-Dimensional Cohomology and Group Extensions

Definition 3.3.1 *An extension of a group G by a group N is a short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1. \tag{3.1}$$

Another extension

$$1 \longrightarrow N \longrightarrow E' \longrightarrow G \longrightarrow 1 \tag{3.2}$$

of G by N is said to be equivalent to (3.1) if there is a map $E \to E'$ making the diagram

$$\begin{tikzcd} & & E \arrow[dd] \arrow[dr] & & & \\ 1 \arrow[r] & N \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

commute. Such a map is necessarily an isomorphism. The main problem in the theory of group extensions is to classify the extensions of G by N up to equivalence. In fact, we are looking for all possible ways of building a group E with N as a normal subgroup and G as the quotient. This problem is closely related to the cohomology $H^i(G, -)$ for $i = 1, 2, 3$. For this section, we consider only the case where N is an abelian group written additively. In this case, G has an action on N, that is N is a G-module.

Definition 3.3.2 *A function $d : G \to N$ is called derivation if it satisfies $d(gh) = d(g) + g \cdot d(h)$ for all $g, h \in G$.*

A function $p : G \to N$ of the form $p : g \mapsto g \cdot a - a$ is called principal derivation for $g \in G$ and for some fixed $a \in N$.

There is an isomorphism between the first cohomology and the quotient group

$$H^1(G, N) \cong \mathrm{Der}(G, N)/P(G, N)$$

where $\mathrm{Der}(G, N)$ is the abelian group of derivations and $P(G, N)$ is the group of principal derivations.

In chapter 5, we will use Serre's theorem so that we will need the first cohomology of a p-group. For this reason we need the following:

Definition 3.3.3 *If G is a group, Frattini subgroup $\Phi(G)$ is defined as the intersection of all the maximal subgroups of G.*

Lemma 3.3.4 ([18]) *If G is a finite p-group, then $G/\Phi(G)$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$.*

Proposition 3.3.5 ([4], Ch.3, pg. 86) *Let G be a p-group. There is a natural isomorphism*

$$H^1(G, k) = \mathrm{Ext}^1_{kG}(k, k) \cong \mathrm{Hom}(G/\Phi(G), k^+)$$

where k^+ denotes the additive group of k. Thus if $G/\Phi(G)$ is elementary abelian of rank n, then $\mathrm{Ext}^1_{kG}(k, k)$ is an n-dimensional vector space over k.

Proof: A representation of G over k is a group homomorphism $\phi : G \to GL_n(k)$ where $GL_n(k)$ is the group of non-singular $n \times n$ matrices over k, for some n. The vector space k^n is a kG-module with G-action $(\sum_i r_i g_i)x = \sum_i r_i \phi(g_i)(x)$ where $x \in k^n$. This gives a one to one correspondence between the representations and finitely generated kG-modules.

Consider the representation $\phi : G \to GL_2(k)$. An extension $0 \to k \to M \to k \to 0$ of kG-modules has a matrix representation of the form

$$\begin{pmatrix} 1 & \alpha(g) \\ 0 & 1 \end{pmatrix}$$

where $\alpha : G \to k^+$ is a homomorphism of groups from G to the additive group of k. By the help of this matrix representations, we have a one to one correspondence between $\mathrm{Ext}^1_{kG}(k, k)$ and $\mathrm{Hom}(G, k^+)$. Desired result follows from the fact that the kernel of α must contain $\Phi(G)$, since k^+ is abelian of exponent p and ker α is a maximal subgroup.

$\square$

3.4 Minimal Projective and Injective Resolutions

Definition 3.4.1 *A projective cover of a kG-module M is a projective module P_M together with a surjective homomorphism $\varepsilon : P_M \to M$ satisfying the following property:*

If $\theta : Q \to M$ is a surjective homomorphism from a projective kG-module Q onto M, then there is an injective homomorphism $\sigma : P_M \to Q$ such that the diagram commutes:

$$\begin{tikzcd} & P_M \arrow[dl, "\sigma"'] \arrow[d, "\varepsilon"] \\ Q \arrow[r, "\theta"] & M \end{tikzcd}$$

By definition, if

$$P_M \xrightarrow{\varepsilon} M$$

is a projective cover of M, then no proper projective submodule of P_M is mapped onto M. And projective cover, if they exist, are unique up to isomorphism.

Theorem 3.4.2 *Let M be a finitely generated kG-module. Then M has projective cover.*

Proof: Choose P_M to be a projective kG-module of smallest k-vector space such that there exist $P_M \twoheadrightarrow M$. Suppose we are given Q and θ as in the definition above. P_M and Q are projective there is a commutative diagram

$$\begin{tikzcd} P_M \arrow[d, shift right, "\sigma"'] \arrow[dr, "\varepsilon"] & \\ Q \arrow[u, shift right, "\tau"'] \arrow[r, "\theta"] & M \end{tikzcd}.$$

Let φ:=$\tau \circ \sigma$:$P_M \to P_M$.

To complete the proof it is enough to prove that φ is an automorphism. Since P_M is finite dimensional by Fitting's Lemma(see [[4], Ch.1, pg. 7]) P_M= ker $\varphi^n \oplus$ Im φ^n for sufficiently large n. Since P_M is projective ker φ^n and Im φ^n are projective. By the commutativity of the diagram, we have $\varepsilon \circ \varphi^n$=$\varepsilon$. By minimality, we have ker φ^n=0. That is φ is an automorphism. So, σ is injective as desired, and P_M is a projective cover by the definition.

$\square$

Definition 3.4.3 *Let M be a kG-module. A kG-module I containing M is called an injective hull of M if the following two conditions hold.*
i) I is injective with injective homomorphism $M \hookrightarrow I$
ii)There is no injective kG-module $\tilde{I}$ with $M \subset \tilde{I} \subset I$.

We denote the injective hull as $I := I(M)$.

Theorem 3.4.4 *For any kG-module M, injective hull always exists.*

(See [[4], Ch.1, pg. 9])

Definition 3.4.5 *A projective resolution*

$$... \to P_n \to P_{n-1} \to ... \to P_0 \to M \to 0$$

or in short writing

$$P_* \xrightarrow{\varepsilon} M$$

is called minimal projective resolution if there is another projective resolution

$$Q_* \xrightarrow{\theta} M$$

of M*, then there is an injective chain map* $\mu_* : (P_* \twoheadrightarrow M) \hookrightarrow (Q_* \twoheadrightarrow M)$ *and a surjective chain map* $\mu_*^{'} : (Q_* \twoheadrightarrow M) \twoheadrightarrow (P_* \twoheadrightarrow M)$ *such that both* μ_* *and* $\mu_*^{'}$ *lift the identity on* M*.*

Minimal projective resolutions always exists. Let

$$P_0 \xrightarrow{\varepsilon} M$$

be a projective cover of M, $P_1 \twoheadrightarrow \ker \varepsilon$ a projective cover of $\ker \varepsilon$ and repeating the same procedure, we get the minimal resolution. In the similar way, let

$$M \xrightarrow{\theta} I_0$$

is an injective hull of M, $\operatorname{coker} \theta \hookrightarrow I_1$ an injective hull of $\operatorname{coker} \theta$ and repeating the same procedure, we get an injective resolution and such a resolution is called minimal injective resolution. The advantage of using a minimal resolution is that if W is any simple module, then the differentials in the complexes $\mathrm{Hom}_{kG}(P_*, W)$ and $P_* \otimes_{kG} W$ are trivial. For this reason

$$\mathrm{Tor}_n^{kG}(M, W) = P_n \otimes_{kG} W$$

$$\mathrm{Ext}_{kG}^n(M, W) = \mathrm{Hom}_{kG}(P_n, W)$$

for any kG-module M. In particular, $\dim_k H^n(G, k) = \dim_k \mathrm{Hom_{kG}}(P_n, k)$.

Chapter 4

Carlson's L_ζ-Modules

Throughout the following G is a finite group, k is a field of characteristic p, and p is a prime number. In this chapter, we give the definition of the syzygies $\Omega^n(M)$ and some properties of $\Omega^n(M)$ and then we introduce Carlson's L_ζ-modules. We give some exact sequences involving L_ζ-modules. These will be used in the alternative proof of Carlson's theorem in Chapter 5.

4.1 The Syzygies $\Omega^n(M)$

Definition 4.1.1 *Let $\varepsilon : P_M \twoheadrightarrow M$ be the projective cover of M. We define $\Omega(M)$ as the kernel of ε and inductively $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for $n > 0$. Similarly let $\theta : M \hookrightarrow I(M)$ be the injective hull of M.We define $\Omega^{-1}(M)$ as cokernel of θ and inductively $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-(n-1)}(M))$ for $n > 0$. For $n = 0$ we let $\Omega^0(M) := \Omega^{-1}(\Omega(M))$, so that $M \cong \Omega^0(M) \oplus (proj)$. This module $\Omega^n(M)$ is called n-th syzygy of M, or n-th Heller shift of M.*

Lemma 4.1.2 *Let $P_* \twoheadrightarrow M$ be a minimal projective resolution*

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \to 0.$$

Let $M \hookrightarrow I_*$ be a minimal injective resolution

$$0 \to M \xrightarrow{\theta} I_0 \xrightarrow{\delta^0} I_{-1} \longrightarrow \cdots \longrightarrow I_{-(n-1)} \xrightarrow{\delta^n} I_{-n} \to \cdots .$$

Then $\Omega^n(M) = \ker\, \partial_{n-1} = \mathrm{Im}\ \partial_n$ for $n > 0$ where $\partial_n : P_n \to P_{n-1}$ is the differential and $\Omega^{-n}(M) = \ker\ \delta^{n+1}$.

The followings follow easily from the definition.

Proposition 4.1.3 *The modules $\Omega^n(M)$ are well defined up to isomorphism.*

Proof: See [[9], pg. 14].

Proposition 4.1.4 *M is projective kG-module if and only if $\Omega(M) = 0$.*

Proof: See [[9], pg. 15].

Proposition 4.1.5 *For any kG-module M, $\Omega(M)$ has no nonzero injective submodule.*

Proof: Assume $X \subset \Omega(M)$ is nonzero injective submodule. Then $P_M = X \oplus Y$ where Y is another injective submodule. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] & & \\
0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d] & P_M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] & Y \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

It shows that M has projective cover which is contained in P_M, which is a contradiction.

$\square$

Remark 4.1.6 *Since a projective kG-module is injective, $\Omega(M)$ has no projective submodule.*

Proposition 4.1.7 *Let M and N be kG-modules and m,n be integers. Then*

$$\Omega^n(M) \otimes \Omega^m(N) \cong \Omega^{n+m}(M \otimes_k N)) \oplus (proj).$$

Proof: Let

$$\ldots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a minimal resolution of M. When we tensor this resolution with N we get the exact sequence

$$\ldots \longrightarrow P_n \otimes N \xrightarrow{\partial_n \otimes \mathrm{Id}} P_{n-1} \otimes N \longrightarrow \ldots \longrightarrow M \otimes N \longrightarrow 0$$

This doesn't have to be a minimal projective resolution of $M \otimes N$. So

$$\Omega^m(M) \otimes N \cong \Omega^m(M \otimes N) \oplus (proj).$$

And similarly we have the isomorphism

$$\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^n(\Omega^m(M) \otimes N) \oplus (proj).$$

Thus we get

$$\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^{n+m}(M \otimes N) \oplus (proj).$$

$\square$

Proposition 4.1.8 *If H is a subgroup of G and L is a kH-module, then*

$$(\Omega^n(L))^{\uparrow G} \cong \Omega^n(L^{\uparrow G}) \oplus (proj)$$

for all $n \in \mathbb{Z}$. If G is a p-group, then the isomorphism is true without a projective summand.

Proof: Let

$$P_* \xrightarrow{\varepsilon} L$$

be a minimal resolution for kH-module L. By tensoring with $kG \otimes_{kH} -$, we get the projective resolution

$$kG \otimes_{kH} P_* \xrightarrow{\mathrm{Id} \otimes \varepsilon} kG \otimes_{kH} L \tag{4.1}$$

of the induced module $kG \otimes_{kH} L = L^{\uparrow G}$. This resolution doesn't have to be minimal, thus we have

$$(\Omega^n(L))^{\uparrow G} \cong \Omega(L^{\uparrow G}) \oplus (proj).$$

For p-groups, result follows from the fact that the projective resolution in (4.1) is minimal.

□

Recall that $I := I(M)$ is the injective hull of a kG-module M and we have the following exact sequence

$$0 \to M \to I(M) \to I(M)/M \to 0$$

and the cokernel of the injective hull $M \hookrightarrow I(M)$ is denoted by $\Omega^{-1}(M) = I(M)/M$.

Proposition 4.1.9 *Let N be an injective kG-module and let $f : M \to N$ be an injective homomorphism. Then, there exists an injective kG-module W such that $N \cong I(M) \oplus W$ and* $\mathrm{coker} f \cong \Omega^{-1}(M) \oplus W$.

Proof: By definition of the injective module and the injective hull, f can be extended to an injective homomorphism $\tilde{f} : I(M) \hookrightarrow N$ and we get an exact sequence

$$0 \to I(M) \to N \to N/I(M) \to 0.$$

Since $I(M)$ is injective the exact sequence splits. Since N and $I(M)$ are injective $N/I(M) = W$ is injective. We get the required module W. Let $U = \mathrm{coker} f$. We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & I(M) & \longrightarrow & \Omega^{-1}(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & U & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W. & &
\end{array}$$

From the diagram one gets $U = \Omega^{-1}(M) \oplus W$, since W is a projective module and the exact sequence $0 \to W \to U \to \Omega^{-1}(M) \to 0$ splits.

□

Proposition 4.1.10 *For each n , there is an isomorphism*

$$\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}^1_{kG}(\Omega^{n-1}(k),k).$$

Proof: Consider the following short exact sequence

$$0 \to \Omega(k) \to P_0 \to k \to 0.$$

This gives the long exact sequence

$$\dots \to \mathrm{Ext}^{n-1}_{kG}(k,k) \to \mathrm{Ext}^n_{kG}(P_0,k) \to \mathrm{Ext}^{n-1}_{kG}(\Omega(k),k) \to \mathrm{Ext}^n_{kG}(k,k) \to \dots$$

Since $\mathrm{Ext}^n_{kG}(P_0,k) = 0$ for all $n \geq 0$ we have the isomorphism

$$\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}^{n-1}_{kG}(\Omega(k),k). \tag{4.2}$$

Similarly the short exact sequence for $i = 1 \dots n-2$

$$0 \to \Omega^{i+1}(k) \to P_i \to \Omega^i(k) \to 0$$

gives the long exact sequence

$$\begin{aligned} \dots \to \mathrm{Ext}^{n-i-1}_{kG}(\Omega^i(k),k) \to \mathrm{Ext}^{n-i-1}_{kG}(P_i,k) \to \mathrm{Ext}^{n-i-1}_{kG}(\Omega^{i+1}(k),k) \to \dots \\ \to \mathrm{Ext}^{n-i}_{kG}(\Omega^i(k),k) \to \mathrm{Ext}^{n-i}_{kG}(\Omega^{i+1}(k),k) \to \mathrm{Ext}^{n-i}_{kG}(P_i,k) \to \dots \end{aligned}$$

Since $\mathrm{Ext}^{n-i}_{kG}(P_i,k) = 0$ for all $i = 1 \dots n-2$, we get the isomorphism

$$\mathrm{Ext}^{n-i}_{kG}(\Omega^i(k),k) \cong \mathrm{Ext}^{n-i-1}_{kG}(\Omega^{i+1}(k),k). \tag{4.3}$$

(4.1) and (4.2) give the isomorphism

$$\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}_{kG}(\Omega^{n-1}(k),k).$$

$\square$

Theorem 4.1.11 ([9], pg. 16) *Let M, N be kG-modules, $n \in \mathbb{Z}^+$*
i) Every cohomology element $\zeta \in \mathrm{Ext}^n_{kG}(M,N)$ is represented by a homomorphism $\hat{\zeta} : \Omega^n(M) \to N$.
ii)Every homomorphism $\hat{\zeta} : \Omega^n(M) \to N$ represents a cohomology class $(\hat{\zeta}) \in \mathrm{Ext}^n_{kG}(M,N)$.
iii)Two such homomorphisms $\hat{\zeta}, \tilde{\zeta}$ represent the same class if and only if $\hat{\zeta} - \tilde{\zeta}$ factors through a projective kG-module.

Proof: $\zeta \in \mathrm{Ext}^n_{kG}(M,N) = H^n(\mathrm{Hom}_{kG}(P_*,N)) = \ker\,\delta^n/\mathrm{Im}\,\delta^{n-1}$. Let

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] & P_n \arrow[rr, "\partial_n"] \arrow[dr] & & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & & \Omega^n(M) \arrow[ur] & & & & &
\end{tikzcd}$$

be the minimal projective resolution. Let $f \in \mathrm{Hom}_{kG}(P_n,N)$ be an n-cocycle representing ζ. Then $(\delta^n f(x)) = f(\partial_{n+1}(x)) = 0$. Thus f induces a map $\hat{f}$: $P_n/\mathrm{Im}(\partial_{n+1}) \cong \Omega^n(M) \to N$. Denote $\hat{f}$ by $\hat{\zeta}$.

The remaining part of the proof is a detailed version of the proof in [[9], pg. 16].

ii) Consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] & P_n \arrow[r, "\partial_n"] \arrow[dr, "g"'] \arrow[dd, "\zeta'"] & P_{n-1} \arrow[r] & \cdots \\
& & & \Omega^n(M) \arrow[u, "\iota"] \arrow[dl, "\hat{\zeta}"] & \\
& & N & &
\end{tikzcd}$$

For the given $\hat{\zeta} : \Omega^n(M) \to N$ we have $\hat{\zeta} \circ g = \hat{\zeta} \circ \partial_n : P_n \to N$ represents a cohomology element denoted by $class(\hat{\zeta}) := class(\hat{\zeta} \circ g) \in \mathrm{Ext}^n_{kG}(M,N)$ because $\hat{\zeta} \circ \partial_n \circ \partial_{n+1} = 0$ which says that $\hat{\zeta} \circ \partial_n$ is a cocycle.

iii) If $class(\hat{\zeta}) = class(\tilde{\zeta})$, then $(\hat{\zeta} - \tilde{\zeta}) \circ g = f \circ \partial_n$ for some $f : P_{n-1} \to N$. So $\hat{\zeta} - \tilde{\zeta} = f \circ \iota$ factors through P_{n-1}, where $\iota : \Omega^n(M) \hookrightarrow P_{n-1}$ is the inclusion. Conversely suppose $\varphi := \hat{\zeta} - \tilde{\zeta} : \Omega^n(M) \to N$ factors through a projective module P. It is enough to show that φ is a coboundary that is it factors through P_{n-1}, because x is a coboundary if $x = \delta^{n-1}(y) = y \circ \partial_n$ for some $y \in \mathrm{Hom}_{kG}(P_{n-1},N)$. Say that $\varphi = \beta \circ \alpha$. For this consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[rr] & & P_{n-1} \arrow[r] \arrow[dd, "\psi"] & \cdots \\
& \Omega^n(M) \arrow[ur, "\iota"] \arrow[dl, "\varphi"] \arrow[dr, "\alpha"] & & \\
N & & P \arrow[ll, "\beta"] &
\end{tikzcd}$$

Since P is projective, it is injective thus the homomorphism ψ exists with $\psi \circ \iota = \alpha$ which means $\varphi = \beta \circ \psi \circ \iota$ factors through P_{n-1}.

$\square$

4.2 Definition of the L_ζ-Modules

Definition 4.2.1 *Let ζ be a cohomology class in $H^n(G,k) - \{0\}$ for $n \geq 1$ and $\hat{\zeta} : \Omega^n(k) \to k$ be the homomorphism representing ζ. We define L_ζ as the kernel of the homomorphism $\hat{\zeta}$. When $\zeta = 0$, we set $L_\zeta = \Omega(k) \oplus \Omega^n(k)$.*

Consider the minimal projective resolution of k and the representing homomorphism $\hat{\zeta} : \Omega^n(k) \to k$. We have the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] & P_n \arrow[r, "\partial_{n-1}"] \arrow[dr] \arrow[dd] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r, "\varepsilon"] & k \arrow[r] & 0 \\
& & & \Omega^n(k) \arrow[u] \arrow[dl, "\hat{\zeta}"] & & & & \\
& & k & & & & &
\end{tikzcd}$$

Using the above diagram one gets the following diagram:

$$\begin{tikzcd}
& L_\zeta \arrow[r, "="] \arrow[d] & L_\zeta \arrow[d] & & & & & \\
0 \arrow[r] & \Omega^n(k) \arrow[r] \arrow[d, "\hat{\zeta}"] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d, "="] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "="] & k \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & k \arrow[r] & P_{n-1}/L_\zeta \arrow[r] \arrow[d] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & k \arrow[r] & 0 \\
& & \Omega^{n-1}(k) \arrow[ur] & & & & &
\end{tikzcd} \tag{4.4}$$

In the diagram, P_{n-1}/L_ζ is the pushout of the diagram. For the cohomology class ζ we find a short exact sequence $0 \to k \to P_{n-1}/L_\zeta \to \Omega^{n-1}(k) \to 0$ in $\mathrm{Ext}_{kG}(\Omega^{n-1}(k), k)$ using the representing homomorphism $\hat{\zeta}$. Thus we conclude the following lemma.

Lemma 4.2.2 *If ζ is in $\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}_{kG}(\Omega^{n-1}(k), k)$, then ζ, as an element of $\mathrm{Ext}_{kG}(\Omega^{n-1}(k), k)$, is represented by the following extension*

$$0 \to k \to P_{n-1}/L_\zeta \to \Omega^{n-1}(k) \to 0.$$

Corollary 4.2.3 *If ζ is in $\mathrm{Ext}^n_{kG}(k,k)$, then we have an exact sequence*

$$0 \to \Omega(k) \to L_\zeta \oplus (proj) \to \Omega^n(k) \to 0.$$

with extension class corresponding to ζ under the isomorphism $\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}^n_{kG}(\Omega^n(k), \Omega(k))$.

Proof: Since P_{n-1} is projective kG-module, it is injective and P_{n-1} is not necessarily injective hull of the module L_ζ. Using Proposition 4.1.9, we deduce that $P_{n-1}/L_\zeta \cong \Omega^{-1}(L_\zeta) \oplus (proj)$. Consider the exact sequence

$$0 \to k \to \Omega^{-1}(L_\zeta) \oplus (proj) \to \Omega^{n-1}(k) \to 0.$$

Tensoring this exact sequence with $\Omega(k)$, we get

$$0 \to \Omega(k) \oplus (proj) \to L_\zeta \oplus (proj) \to \Omega^n(k) \oplus (proj) \to 0.$$

Using Proposition 3.1.7, one gets the desired exact sequence.

□

Remark 4.2.4 *This corollary explains why we defined L_ζ as $\Omega(k) \oplus \Omega^n(k)$ when $\zeta = 0$.*

Lemma 4.2.5 *L_ζ is well-defined up to isomorphism.*

Proof: Let $\zeta \in \mathrm{Ext}^n_{kG}(k,k)$ and let $\hat{\zeta}$ be the homomorphism representing ζ. From Lemma 4.2.2 we have the exact sequence

$$0 \to k \to P_{n-1}/L_\zeta \to \Omega^{n-1}(k) \to 0.$$

Let $\tilde{\zeta}$ be the another representative homomorphism for ζ, that is we have

$$\Omega^n(k) \xrightarrow{\tilde{\zeta}} k.$$

Let $\ker(\tilde{\zeta}) := L'_\zeta$, in the similar way we have the exact sequence $0 \to k \to P_{n-1}/L'_\zeta \to \Omega^{n-1}(k) \to 0$. Since $\tilde{\zeta}$ and $\hat{\zeta}$ represents the same cohomology class we have the equivalence of the exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k & \longrightarrow & P_{n-1}/L_\zeta & \longrightarrow & \Omega^{n-1}(k) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & P_{n-1}/L'_\zeta & \longrightarrow & \Omega^{n-1}(k) & \longrightarrow & 0
\end{array}$$

This gives that $P_{n-1}/L'_\zeta \cong P_{n-1}/L_\zeta$. Since $P_{n-1}/L_\zeta \cong \Omega^{-1}(L_\zeta) \oplus (proj)$ and $P_{n-1}/L'_\zeta \cong \Omega^{-1}(L'_\zeta) \oplus (proj)$, we have $\Omega^{-1}(L_\zeta) \oplus (proj) \cong \Omega^{-1}(L'_\zeta) \oplus (proj)$. Tensoring these with $\Omega(k)$, we get

$$\Omega(\Omega^{-1}(L_\zeta)) \cong \Omega(\Omega^{-1}(L'_\zeta))$$

and this gives $\Omega^0(L_\zeta) \cong \Omega^0(L'_\zeta)$. Since both L_ζ and L'_ζ are projective free we obtain

$$L_\zeta \cong L'_\zeta$$

as desired.

□

□

Let G be a finite p-group. By Proposition 3.3.5, we have

$$H^1(G, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Der}(G, \mathbb{Z}/p\mathbb{Z})/P(G, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}(G/\Phi(G), \mathbb{Z}/p\mathbb{Z}).$$

In other words for any nonzero $\zeta \in H^1(G, \mathbb{Z}/p\mathbb{Z})$, we have a corresponding homomorphism whose kernel is a maximal subgroup of G. We call this maximal subgroup the kernel of ζ.

Proposition 4.2.6 (Carlson [8]) *Let $p = 2$, $\zeta \in H^1(G, \mathbb{Z}/p\mathbb{Z})$ and H be the kernel of ζ. Then the exact sequence*

$$0 \longrightarrow k \longrightarrow k_H^{\uparrow G} \xrightarrow{\varepsilon} k \longrightarrow 0$$

has the extension class $\zeta \in \mathrm{Ext}^1_{kG}(k, k) = H^1(G, k)$.

Proof: This essentially follows from Proposition 3.3.5. Let f_ζ be the corresponding homomorphism of ζ. Since $\ker f_\zeta = H$, then ζ corresponds to an extension whose restriction to H splits. Since

$$E : 0 \longrightarrow k \longrightarrow k_H^{\uparrow G} \xrightarrow{\varepsilon} k \longrightarrow 0$$

splits as an exact sequence of kH-modules. Its extension class must be ζ.

$\square$

Using this result we can compute L_ζ for $\zeta \in H^1(G,k)$ where G is a 2-group.

Proposition 4.2.7 *Let G be a 2-group. If ζ is a cohomology class in $H^1(G,k)$, then $L_\zeta \cong \Omega(k_H)^{\uparrow G}$.*

Proof: Consider the commutative diagram

$$\begin{tikzcd}
 & L_\zeta \arrow[r, equal] \arrow[d] & L_\zeta \arrow[d] & & \\
0 \arrow[r] & \Omega(k) \arrow[r] \arrow[d, "\hat{\zeta}"'] & P_0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & P_0/L_\zeta \arrow[r] & k \arrow[r] & 0.
\end{tikzcd}$$

We get that $0 \to k \to P_0/L_\zeta \to k \to 0$ represents the class ζ and by Proposition 4.2.6, we have the equivalence of the exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] \arrow[d, equal] & P_0/L_\zeta \arrow[r] \arrow[d, "\cong"'] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & k_H^{\uparrow G} \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

It follows from the commutative diagram that we have an exact sequence of the form $0 \to L_\zeta \to P_0 \to k_H^{\uparrow G} \to 0$ and $\Omega^{-1}(L_\zeta) \oplus (proj) \cong k_H^{\uparrow G}$ since P_0 is not necessarily the injective hull of L_ζ. Taking first Heller shift of $\Omega^{-1}(L_\zeta) \oplus (proj) \cong k_H^{\uparrow G}$, we get

$$L_\zeta \cong \Omega(k_H^{\uparrow G}).$$

Note that for p-groups $\Omega(k_H^{\uparrow G}) \cong \Omega(k_H)^{\uparrow G}$ by Proposition 4.1.8 . Thus we have

$$L_\zeta \cong \Omega(k_H)^{\uparrow G}.$$

$\square$

Proposition 4.2.8 (Carlson [8]) *Let G be a p-group where $p > 2$, then the cohomology class $\beta(\zeta) \in H^2(G,k)$ is represented by an exact sequence of the form*

$$0 \longrightarrow k \longrightarrow k_H^{\uparrow G} \xrightarrow{\phi} k_H^{\uparrow G} \longrightarrow k \longrightarrow 0$$

where ϕ is multiplication by $x-1$ for x is not in H and where H is the kernel of the class ζ.

Proof: Similar to 4.2.6.

□

Here β denotes the Bockstein operator in the group cohomology. Recall that β is the connecting homomorphism of the sequence.

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Using this result we get that $L_{\beta(\zeta)} \oplus (proj)$ has a filtration.

Proposition 4.2.9 *Let G be a finite p-group where $p > 2$. If $\zeta \in H^1(G,k)$, then $L_{\beta(\zeta)} \oplus (proj)$ has a filtration $\{0\} = M_0 \subseteq \cdots \subseteq M_2 = L_{\beta(\zeta)} \oplus (proj)$ with the property $M_2/M_1 \cong k_H^{\uparrow G}$ and $M_1/M_0 \cong \Omega(k_H)^{\uparrow G}$.*

Proof: For $\beta(\zeta) \in H^2(G,k)$ we have the following commutative diagram:

$$\begin{tikzcd}
 & L_{\beta(\zeta)} \arrow[r, equal] \arrow[d] & L_{\beta(\zeta)} \arrow[d] & & & \\
0 \arrow[r] & \Omega^2(k) \arrow[r] \arrow[d, "\zeta"'] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & P_1/L_{\beta(\zeta)} \arrow[r] & P_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Since ζ is represented by $0 \to k \to k_H^{\uparrow G} \to k_H^{\uparrow G} \to k \to 0$ we have

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] \arrow[d, equal] & k_H^{\uparrow G} \arrow[r] \arrow[d] & k_H^{\uparrow G} \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & P_1/L_{\beta(\zeta)} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & k \arrow[r] & 0 \\
 & & (proj) \oplus \Omega^{-1}(k_H^{\uparrow G}) \arrow[r, equal] & \Omega^{-1}(k_H^{\uparrow G}) \oplus (proj). & &
\end{tikzcd}$$

This diagram gives us the exact sequence

$$0 \to k_H^{\uparrow G} \to \Omega^{-1}(L_{\beta(\zeta)}) \oplus (proj) \to \Omega^{-1}(k_H^{\uparrow G}) \oplus (proj) \to 0.$$

If we tensor this exact sequence with $\Omega(k)$ and using Proposition 3.1.7, we get

$$0 \to \Omega(k_H^{\uparrow G}) \to L_{\beta(\zeta)} \oplus (proj) \to k_H^{\uparrow G} \to 0.$$

$L_{\beta(\zeta)} \oplus (proj)$ has a filtration in the following way: Set $M_2 = L_{\beta(\zeta)} \oplus (proj)$ and $M_1 = \Omega(k_H)^{\uparrow G}$, then $M_2/M_1 \cong k_H^{\uparrow G}$.

$\square$

Propositions 4.2.7 and 4.2.9 will be used in the section 5.2 for the alternative proof of Carlson's theorem.

4.3 Some Exact Sequences

Let $\zeta \in \mathrm{Ext}^n_{kG}(k,k)$ and $\eta \in \mathrm{Ext}^m_{kG}(k,k)$. Let ζ and η be represented by

$$0 \to k \to B_{n-1} \to \cdots \to B_0 \to k \to 0$$

and

$$0 \to k \to C_{m-1} \to \cdots \to C_0 \to k \to 0$$

respectively. Then we can form their Yoneda splice as follows

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & B_{n-1} \arrow[r] & \cdots \arrow[r] & B_0 \arrow[rr] \arrow[dr] & & C_{m-1} \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & k \arrow[r] & 0 \\
& & & & & k \arrow[ur] & & & & &
\end{tikzcd}$$

to obtain an element $\zeta \cdot \eta \in \mathrm{Ext}^{n+m}_{kG}(k,k)$. Note that Yoneda splice is the same as the cup product. One can see the details about the cup product in [[4],Ch.3, pg. 51].

Proposition 4.3.1 *If $\zeta_1 \in H^r(G,k)$ and $\zeta_2 \in H^s(G,k)$, then there is an exact sequence*

$$0 \to \Omega^r(L_{\zeta_2}) \to L_{\zeta_1 \cdot \zeta_2} \oplus (proj) \to L_{\zeta_1} \to 0.$$

Proof: Using Corollary 4.2.3 for $\zeta_1 \in H^r(G,k) = \mathrm{Ext}_{kG}(\Omega^{r-1}(k),k)$, we have the following exact sequence

$$E_{\zeta_1} : 0 \to \Omega(k) \to L_{\zeta_1} \oplus (proj) \to \Omega^r(k) \to 0.$$

Similarly for ζ_2, we have

$$E_{\zeta_2} : 0 \to \Omega(k) \to L_{\zeta_2} \oplus (proj) \to \Omega^s(k) \to 0.$$

We tensor E_{ζ_2} with $\Omega^{r-1}(k)$, we get

$$E_{\zeta_2} \otimes \mathrm{id}_{\Omega^{r-1}(k)} : 0 \to \Omega^r(k) \oplus (proj) \to \Omega^{r-1}(L_{\zeta_2}) \oplus (proj) \to \Omega^{r+s-1}(k) \oplus (proj) \to 0$$

We delete $(proj)$s from the exact sequence and we get

$$E_{\zeta_2} : 0 \to \Omega^r(k) \to \Omega^{r-1}(L_{\zeta_2}) \oplus (proj) \to \Omega^{r+s-1}(k) \to 0$$

Using Yoneda splice of E_{ζ_1} and E_{ζ_2}, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega(k) \arrow[r] \arrow[dd, equal, "\|"] & L_{\zeta_1} \oplus proj \arrow[rr] \arrow[dd] \arrow[dr] & & \Omega^{r-1}(L_{\zeta_2}) \arrow[r, tail] \arrow[dd] & \Omega^{r+s-1}(k) \arrow[r] \arrow[dd, equal, "\|"] & 0 \\
 & & & \Omega^r(k) \arrow[ur] & & & \\
0 \arrow[r] & \Omega(k) \arrow[r] & I \arrow[rr] \arrow[dr] \arrow[dd] & & T \arrow[r] \arrow[dd] & \Omega^{r+s-1}(k) \arrow[r] & 0 \\
 & & & k \oplus proj \arrow[ur] & & & \\
 & & \Omega^{-1}(L_{\zeta_1}) \arrow[rr, dash, "="] & & \Omega^{-1}(L_{\zeta_1}) & &
\end{tikzcd} \tag{4.5}$$

where I is the injective hull of $L_{\zeta_1} \oplus (proj)$ and T is the pushout of the middle square. Consider the exact sequence

$$0 \to k \oplus (proj) \to T \to \Omega^{r+s-1}(k) \to 0.$$

We tensor it with $\Omega(k)$ and cancel the projective modules and we get

$$E : 0 \to \Omega(k) \to \Omega(T) \oplus (proj) \to \Omega^{r+s}(k) \to 0.$$

By Corollary 4.2.3, we have

$$L_{\zeta_1\zeta_2} \cong \Omega(T).$$

From the commutative diagram 4.2, we have

$$0 \to \Omega^{r-1}(L_{\zeta_2}) \oplus (proj) \to T \to \Omega^{-1}(L_{\zeta_1}) \to 0.$$

Tensoring with $\Omega(k)$ and using Proposition 3.1.7, we get

$$0 \to \Omega^{r}(L_{\zeta_2}) \to L_{\zeta_1\zeta_2} \oplus proj \to L_{\zeta_1} \to 0.$$

□

Proposition 4.3.2 *If H is a subgroup of G, then there is an isomorphism $L_\zeta \downarrow_H \cong L_{res^G_H(\zeta)} \oplus (proj)$.*

Proof: Consider the exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n(k) \xrightarrow{\hat{\zeta}} k \longrightarrow 0.$$

Applying res^G_H to this exact sequence, one gets

$$0 \longrightarrow L_\zeta \downarrow_H \longrightarrow \Omega^n(k) \oplus (proj) \xrightarrow{\hat{\zeta}\downarrow_H} k \longrightarrow 0.$$

On the other hand we have

$$0 \longrightarrow L_{res^G_H(\zeta)} \longrightarrow \Omega^n(k) \xrightarrow{\widehat{res^G_H(\zeta)}} k \longrightarrow 0.$$

Using these exact sequences we have the following commutative diagram

$$\begin{tikzcd}
 & proj \arrow[r, equal] \arrow[d] & proj \arrow[d] & & \\
0 \arrow[r] & L_\zeta \downarrow_H \arrow[r] \arrow[d] & \Omega^n(k) \oplus (proj) \arrow[r, "\hat{\zeta}\downarrow_H"] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_{res^G_H(\zeta)} \arrow[r] & \Omega^n(k) \arrow[r, "\widehat{res^G_H(\zeta)}"] & k \arrow[r] & 0.
\end{tikzcd}$$

Since projective kG-module is also injective, the exact sequence

$$0 \to proj \to L_\zeta \downarrow_H \to L_{res^G_H(\zeta)} \to 0$$

splits and we get $L_\zeta \downarrow_H \cong L_{res^G_H(\zeta)} \oplus (proj)$.

□

Chapter 5

Carlson's Theorem

Throughout the following G is a finite group, k is a field of characteristic p, and p is a prime number. In this chapter, we consider a theorem by Carlson which says that any kG-module M is a direct summand of a kG-module which has a filtration by modules induced from elementary abelian p-subgroups. The main result of this chapter is an alternative proof of Carlson's theorem using L_ζ-modules. This is also the main result of this thesis.

5.1 Main Points of Carlson's Proof

Theorem 5.1.1 (Carlson [8]) *There exists an integer $\tau = \tau(G,p)$, depending only on G and p and there exists a finitely generated kG-module V such that the direct sum $k \oplus V$ has a filtration*

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\tau = k \oplus V$$

with the property that for each $i = 1, 2, \ldots, \tau$, there is an elementary abelian p-subgroup $E_i \subseteq G$ and a kE_i-module W_i such that

$$M_i/M_{i-1} \cong W_i^{\uparrow G}.$$

Let $W_i^{\uparrow G}$ denotes the induced module $W_i^{\uparrow G} = kG \otimes_{kH} W_i$ for kH-module W_i where H is a subgroup of G. For V as in the theorem we say that $k \oplus V$ is filtered by modules induced from elementary abelian p-subgroups.
It is sufficient to prove the theorem in the case that the group G, is a p-group. So we have the following lemma.

Lemma 5.1.2 (Carlson [8]) *If the theorem 5.1.1 is true for all finite p-groups, then it is true for all finite groups.*

Proof: Let G be any finite group and let S be a Sylow p-subgroup of G. By hypothesis, there is an integer τ and a kS-module V such that $k_S \oplus V$ has a filtration

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = k_S \oplus V$$

where for each i, $L_i/L_{i-1} \cong W_i^{\uparrow S}$ for some kE_i-module W_i and some elementary abelian p-subgroup E_i of S. Now we can induce the entire system to G that means, $k_S^{\uparrow G} \oplus V^{\uparrow G}$ has a filtration

$$\{0\} = L_0^{\uparrow G} \subseteq L_1^{\uparrow G} \subseteq \cdots \subseteq L_\tau^{\uparrow G} = k_S^{\uparrow G} \oplus V^{\uparrow G}$$

such that $L_i^{\uparrow G}/L_{i-1}^{\uparrow G} = (W_i^{\uparrow S})^{\uparrow G} = W_i^{\uparrow G}$. Then we have the required result because of the fact that k is a direct summand of $k_S^{\uparrow G}$.

□

By the help of this lemma, for the remainder of section 1 and section 2 of this chapter, we assume that G is a finite p-group. The proof of Theorem 5.1.1 for p-groups is based on Serre's Theorem [20] which is used to distinguish the module theory for the group algebras of elementary abelian p-groups from that of other p-groups.

Theorem 5.1.3 (Serre [20]) *Let $k = \mathbb{Z}/p\mathbb{Z}$ be the field of p-elements. Suppose that G is a p-group which is not elementary abelian. Then there is a sequence $\zeta_1, \zeta_2, \ldots, \zeta_n \in H^1(G, k)$ of nonzero elements such that*

$$\zeta_1 \cdot \zeta_2 \cdots \zeta_n = 0$$

if $p = 2$ *and*

$$\beta(\zeta_1) \cdot \beta(\zeta_2) \cdots \beta(\zeta_n) = 0$$

if $p > 2$.

For the proof of the main theorem, Carlson prove the following lemma using Proposition 4.2.8 and the classes in the following corollary of Serre's theorem.

Corollary 5.1.4 *Suppose that G is a p-group which is not elementary abelian. Then there is a sequence of maximal subgroups $H_1, \ldots, H_n$ and an exact sequence*

$$E : 0 \to k \to C_{2n-1} \to \cdots \to C_1 \to C_0 \to k \to 0$$

such that
i) $C_{2i-2} \cong C_{2i-1} \cong k_{H_i}^{\uparrow G}$ for $i = 1, \ldots, n$ and
ii) the class of E in $\mathrm{Ext}^{2n}_{kG}(k,k)$ is zero.

In the paper the exact sequence E is used to obtain a complex C. The complex C is in the following form: $C_i = C_i$ for $i = 0, \ldots, 2n-1$ and $C_i = 0$ for $i < 0$ or $i > 2n-1$. The differentials $C_i \to C_{i-1}$ are the same as the ones in E for $1 \leq i \leq 2n-1$. Then the homology of this complex is $H_0(C) \cong H_{2n-1}(C) \cong k$ and $H_i(C) = 0$ for $i \neq 0$ or $i \neq 2n-1$. Consider the complex $C \otimes P$ where C is the complex defined above and P is a minimal projective resolution such that $(C \otimes P)_i = \bigoplus_{j=0}^{i} C_j \otimes P_{i-j}$ is projective for all i. By the help of Künneth theorem, one has $H_i(C \otimes P) \cong H_i(C)$ for all i. Let t be any natural number such that $t \geq 2n-1$. Consider the complex which is obtained from the complex $C \otimes P$

$$\Gamma(C \otimes P) : \cdots \to (C \otimes P)_{t+1} \to (C \otimes P)_t \to 0.$$

It is deduced that $H_i(\Gamma(C \otimes P)) = 0$, if $i \neq t$. And it is shown that $H_t(\Gamma(C \otimes P))$ is filtered by modules induced from the subgroups $H_1, \ldots H_n$. $H_t(\Gamma(C \otimes P))$ has a filtration in the following way:

Let $C_*^{(0)}$ be the complex with only one nonzero term $C_*^{(0)} = C_0$. There is an exact sequence of complexes

$$0 \to C_*^{(0)} \to C_* \to D_*^{(0)} \to 0$$

where $D_*^{(0)} = C_*/C_*^{(0)}$ coincides with C_* except in degree zero that means $D_0^{(0)} = 0$ and $D_n^{(0)} = C_n$ for $n \neq 0$. Using this sequence we get

$$0 \to \Gamma(C_*^{(0)} \otimes P_*) \to \Gamma(C_* \otimes P_*) \to \Gamma(D_*^{(0)} \otimes P_*) \to 0.$$

This exact sequence induces an exact sequence

$$0 \to H_t(\Gamma(C_*^{(0)} \otimes P_*)) \to H_t(\Gamma(C_* \otimes P_*)) \to H_t(\Gamma(D_*^{(0)} \otimes P_*)) \to 0.$$

Since $C_* \otimes P_*$ is a projective resolution for C_0 we get that $H_t(\Gamma(C_*^{(0)} \otimes P_*)) = \ker(\partial_{t-1}) \cong \Omega^t(k_{H_1}^{\uparrow G}) \oplus P$, where P is some projective kG-module. The above process is repeated several times. That means, let $C_*^{(1)}$ be the complex with only one nonzero term $C_1^{(1)}$ in degree 1. In the same way, we have

$$0 \to H_t(\Gamma(C_*^{(1)} \otimes P_*)) \to H_t(\Gamma(D_*^{(0)} \otimes P_*)) \to H_t(\Gamma(D_*^{(1)} \otimes P_*)) \to 0.$$

By the similar reason $H_t(\Gamma(C_*^{(1)} \otimes P_*)) \cong \Omega^{t-1}(k_{H_1}^{\uparrow G}) \oplus (proj)$. This process gives that $H_t(\Gamma(C_* \otimes P_*))$ has a filtration

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{2n} \cong H_t(\Gamma(C_* \otimes P_*))$$

with $L_i/L_{i-1} \cong \Omega^{t-i+1}(k_{H_m}^{\uparrow G})$ where $i = 2m-1-j$ for $j = 0, 1$. Desired condition follows from the fact that $H_t(\Gamma(C_* \otimes P_*)) = \Omega^t(k) \oplus \Omega^{t-2n+1}(k) \oplus (proj)$. We tensor the entire system with $\Omega^{-t}(k)$ and obtain that $k \oplus \Omega^{-2n+1}(k) \oplus (proj)$ has a filtration

$$\{0\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{2n} \cong H_t(\Gamma(C_* \otimes P_*))$$

with $N_i/N_{i-1} \cong \Omega^{-i+1}(k_{H_m}^{\uparrow G}) \cong (\Omega^{-i+1}(k_{H_m})^{\uparrow G}$ where $i = 2m-1-j$ for $j = 0, 1$.

5.2 An Alternative Proof of Carlson's Theorem Using L_ζ-Modules

The curicial part of Carlson's proof is the following lemma. In the lemma we have a filtration by modules induced from maximal subgroups of G. After proving the lemma, we will give the proof for the filtration by modules induced from elementary abelian p-subgroups.

Lemma 5.2.1 (Carlson,[8]) *Suppose that G is a p-group which is not elementary abelian. Then there are integers m and $n > 0$ and a sequence $H_1, \dots H_n$ of maximal subgroups of G and $t_1, \dots t_n$ of integers such that $k \oplus \Omega^m(k) \oplus (proj)$ has a filtration*

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \cong k \oplus \Omega^m(k) \oplus (proj)$$

where $L_i/L_{i-1} \cong (\Omega^{t_i}(k_{H_i}))^{\uparrow G}$.

We now give an alternative proof to this lemma using L_ζ-modules.

Proof: We have two cases:

Case 1: $p = 2$.

Let $\zeta_1, \dots, \zeta_n$ be the classes satisfying Serre's condition $\zeta_1 \cdots \zeta_n = 0$. Consider the following exact sequences obtained by applying Proposition 4.3.1,

$$0 \to \Omega^{n-1}(L_{\zeta_1}) \to L_{\zeta_1 \cdots \zeta_n} \oplus P_1 \to L_{\zeta_2 \cdots \zeta_n} \to 0$$

$$0 \to \Omega^{n-2}(L_{\zeta_2}) \to L_{\zeta_2 \cdots \zeta_n} \oplus P_2 \to L_{\zeta_3 \cdots \zeta_n} \to 0$$

$$0 \to \Omega^{n-3}(L_{\zeta_3}) \to L_{\zeta_3 \cdots \zeta_n} \oplus P_3 \to L_{\zeta_4 \cdots \zeta_n} \to 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \to \Omega(L_{\zeta_{n-1}}) \to L_{\zeta_{n-1} \cdot \zeta_n} \oplus P_{n-1} \to L_{\zeta_n} \to 0.$$

We add projective modules $\bigoplus_{k=i+1}^{n-1} P_i$ to the last two terms of each exact sequence for $i = 1, \dots, n-2$, then we obtain

$$0 \to \Omega^{n-i}(L_{\zeta_i}) \to L_{\zeta_i \cdots \zeta_n} \oplus \bigoplus_{k=i}^{n-1} P_k \to L_{\zeta_{i+1} \cdots \zeta_n} \oplus \bigoplus_{k=i+1}^{n-1} P_k \to 0$$

for $i = 1, \dots, n-1$. Using these exact sequences, we see that

$$L_{\zeta_1 \cdots \zeta_n} \oplus \bigoplus_{i=1}^{n-1} P_i$$

has a filtration in the following way:

Let $M_n = L_{\zeta_1\cdots\zeta_n} \oplus \bigoplus_{k=1}^{n-1} P_k$ and $M_1 = \Omega^{n-1}(L_{\zeta_1})$. Then we have $M_n/M_1 \cong L_{\zeta_2\cdots\zeta_n} \oplus \bigoplus_{k=2}^{n-1} P_k$. Choose M_2 such that $M_2/M_1 \cong \Omega^{n-2}(L_{\zeta_2})$. Then the exact sequence

$$0 \to \Omega^{n-2}(L_{\zeta_2}) \to L_{\zeta_2\cdots\zeta_n} \oplus \bigoplus_{k=2}^{n-1} P_k \to L_{\zeta_3\cdots\zeta_n} \oplus \bigoplus_{k=3}^{n-1} P_k \to 0$$

gives that $M_n/M_2 \cong L_{\zeta_3\cdots\zeta_n} \oplus \bigoplus_{k=3}^{n-1} P_k$ which will be the middle term of the next exact sequence. Inductively, we can define the modules

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n$$

such that $M_n/M_i \cong L_{\zeta_{i+1}\cdots\zeta_n} \oplus \bigoplus_{k=i+1}^{n-1} P_k$ and $M_i/M_{i-1} \cong \Omega^{n-i}(L_{\zeta_i})$ for $i = 1, \ldots, n$. To get the desired filtration we use Proposition 4.2.7 which says $\Omega^{n-i+1}(k_{H_i}^{\uparrow G}) \cong \Omega^{n-i}(L_{\zeta_i})$. Note that for p-groups we have $\Omega^n(k_{H_i}^{\uparrow G}) \cong (\Omega^n(k_{H_i}))^{\uparrow G}$.

So, we can conclude that

$$L_{\zeta_1} \cdots \zeta_n \oplus (proj) \cong \Omega(k) \oplus \Omega^n(k) \oplus (proj)$$

has a filtration with

$$M_i/M_{i-1} \cong (\Omega^{n-i+1}(k_{H_i}))^{\uparrow G}.$$

To get a filtration for $k \oplus V$, we tensor the entire system with $\Omega^{-1}(k)$ and obtain a filtration

$$\{0\} = L_0 \subseteq \cdots \subseteq L_n = k \oplus \Omega^{n-1}(k) \oplus (proj)$$

with

$$L_i/L_{i-1} \cong (M_i \otimes \Omega^{-1}(k))/(M_{i-1} \otimes \Omega^{-1}(k)) \cong M_i/M_{i-1} \otimes \Omega^{-1}(k) \cong (\Omega^{n-i}(k_{H_i}))^{\uparrow G}.$$

This completes the proof for $p = 2$

Case 2: $p > 2$. The proof is similar to the first case.

We consider again the classes $\zeta_1, \ldots, \zeta_n \in H^1(G, k)$ with $\beta(\zeta_1) \cdots \beta(\zeta_n) = 0$. As in the first case, we have the exact sequence

$$0 \to \Omega^{2n-2i}(L_{\beta(\zeta_i)}) \to L_{\beta(\zeta_i)\cdots\beta(\zeta_n)} \oplus P_i \to L_{\beta(\zeta_{i+1})\cdots\beta(\zeta_n)} \to 0 \qquad (5.1)$$

for $i = 1, \ldots, n-1$ By Proposition 4.2.9, we know that $\Omega^{2n-2i}(L_{\beta(\zeta_i)}) \oplus Q_i$ has a filtration for some projective module Q_i for each $i = 1, \ldots, n-1$. We use this proposition to complete the proof. That is why, we have to add Q_i to the first two terms of the exact sequence (5.1). We get

$$0 \to \Omega^{2n-2i}(L_{\beta(\zeta_i)}) \oplus Q_i \to L_{\beta(\zeta_i)\cdots\beta(\zeta_n)} \oplus P_i \oplus Q_i \to L_{\beta(\zeta_{i+1})\cdots\beta(\zeta_n)} \to 0. \quad (5.2)$$

Using the same process in the case $p = 2$, we add $\bigoplus_{k=i+1}^{n-1}(P_i \oplus Q_i)$ to the last two terms of the exact sequence (5.2) and get

$$0 \to \Omega^{2n-2i}(L_{\beta(\zeta_i)}) \oplus Q_i \to L_{\beta(\zeta_i)\cdots\beta(\zeta_n)} \oplus \bigoplus_{k=i}^{n-1}(P_i \oplus Q_i)$$
$$\to L_{\beta(\zeta_{i+1})\cdots\beta(\zeta_n)} \oplus \bigoplus_{k=i+1}^{n-1}(P_i \oplus Q_i) \to 0. \quad (5.3)$$

for $i = 1, \ldots, n-1$.

Using these exact sequences as in the first case, we get a filtration for the module as

$$M_n = L_{\beta(\zeta_1)\cdots\beta(\zeta_n)} \oplus \bigoplus_{i=1}^{n-1}(P_i \oplus Q_i)$$

with the property $M_i/M_{i-1} \cong \Omega^{2n-2i}(L_{\beta(\zeta_i)}) \oplus Q_i$.

Consider the exact sequence

$$0 \to \Omega^{2n-2i+1}(k_{H_i}^{\uparrow G}) \to \Omega^{2n-2i}(L_{\beta(\zeta_i)}) \oplus Q_i \to \Omega^{2n-2i}(k_{H_i}^{\uparrow G}) \to 0.$$

Let for each $i = 1, \ldots, n$, N_i be a kG-module satisfying $M_{i-1} \subseteq N_i \subseteq M_i$. Then $\Omega^{2n-2i}(L_{\beta(\zeta_i)}) \oplus Q_i$ has a filtration with $N_i/M_{i-1} \cong \Omega^{2n-2i+1}(k_{H_i}^{\uparrow G})$ and $M_i/N_i \cong \Omega^{2n-2i}(k_{H_i}^{\uparrow G})$.

We obtain a refined filtration

$$\{0\} = M_0 \subseteq N_1 \subseteq M_1 \subseteq N_2 \subseteq M_2 \subseteq \cdots \subseteq M_{i-1} \subseteq N_i \subseteq M_i \subseteq \cdots \subseteq M_n$$

such that $M_n = \Omega(k) \oplus \Omega^{2n}(k) \oplus (proj)$ where (proj) is sufficiently large and

$$M_i/N_i \cong (\Omega^{2n-2i}(k_{H_i})^{\uparrow G}$$

and

$$N_i/M_{i-1} \cong (\Omega^{2n-2i+1}(k_{H_i})^{\uparrow G}$$

for $i = 1, \ldots, n$. We tensor the entire system with $\Omega^{-1}(k)$ and get the desired result.

$\square$

If all the maximal subgroups $H_1, \ldots, H_n$ are elementary abelian, then the proof of Theorem 5.1.1 is complete. If H_i is not elementary abelian, then by induction, there exists a kH_i-module V such that $k_{H_i} \oplus V$ has a filtration $\{0\} = M_0 \subseteq M_1 \cdots \subseteq M_k = k_{H_i} \oplus V$ with the property that for each $j = 0, \ldots, k$, there is an elementary abelian subgroup $E_j \subseteq H_i$ and a kE_j-module V_j such that $M_j/M_{j-1} \cong V_j^{\uparrow H_i}$. If we induce the entire system to G, then $k_{H_i}^{\uparrow G} \oplus V^{\uparrow G}$ is filtered by the submodules $M_j^{\uparrow G}$ with $M_j^{\uparrow G}/M_{j-1}^{\uparrow G} \cong V_j^{\uparrow G}$. In the same way $\Omega^{t_i}(k_{H_i})^{\uparrow G} \oplus \Omega^{t_i}(V)^{\uparrow G}$ has a filtration by the modules $\Omega^{t_i}(V_j)^{\uparrow G}$. As a result $W = k \oplus \Omega^m(k) \oplus \Omega^{t_i}(V)^{\uparrow G} \oplus (proj)$ has a filtration $\{0\} = L_0 \subseteq \cdots \subseteq L_{i-1} \subseteq \hat{M}_1 \subseteq \cdots \subseteq \hat{M}_k = L_i \oplus \Omega^{k_i}(V)^{\uparrow G} \subseteq \cdots \subseteq L_n \oplus \Omega^{k_i}(V)^{\uparrow G} = W$ where for each j, $\hat{M}_j$ is an extension

$$0 \to L_{i-1} \to \hat{M}_i \to \Omega^{k_i}(M_j) \to 0$$

and $\hat{M}_j/\hat{M}_{j-1} \cong \Omega^{k_i}(M_j)/\Omega^{k_i}(M_{j-1}) \cong \Omega^{k_i}(V_j)^{\uparrow G}$. Similar process for each $i = 1, \ldots, n$ gives the desired conclusion.

5.3 Generalizations of Carlson's Theorem

Given kG-modules $M_0, \ldots, M_{n-1}$, we say that a module K has a filtration with sections isomorphic to the Heller shifts of M_i's , for $i = 0, \ldots, n-1$, if there is a filtration $\{0\} = K_0 \subseteq \cdots \subseteq K_n = M$ with the property $K_i/K_{i-1} \cong \Omega^{t_i}(M_{i-1})$ for $i = 1, \ldots, n$.

Theorem 5.3.1 *Let A and B be kG-modules, and E be an n-fold extension*

$$E : 0 \to B \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to A \to 0$$

with extension class $\alpha \in \mathrm{Ext}^n_{kG}(A, B)$. Suppose that

$$\tilde{E} : 0 \to \Omega^{-(n-1)}(B) \to M \to A \to 0$$

is the extension associated to α *under the isomorphism* $\mathrm{Ext}^n_{kG}(A,B) \cong \mathrm{Ext}(A,\Omega^{-(n-1)}(B))$*. Then,* $M \oplus (proj)$ *has a filtration with sections isomorphic to Heller shifts of* $M_i^{'}s$ *for* $i = 0, \ldots, n-1$*.*

Proof: Proof by induction on n. Let

$$E : 0 \to B \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to A \to 0$$

be an n-fold extension and M be the module as in the theorem. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "="] & M_{n-1} \arrow[rr] \arrow[d] & & M_{n-2} \arrow[r] \arrow[d] & M_{n-3} \arrow[r] \arrow[d, "="] & \cdots \arrow[r] & A \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & B \arrow[r] & I(M_{n-1}) \arrow[rr] \arrow[dr] \arrow[dd] & & K_{n-2} \arrow[r] \arrow[dd] & M_{n-3} \arrow[r] & \cdots \arrow[r] & A \arrow[r] & 0 \\
& & & \Omega^{-1}(B) \oplus (proj) \arrow[ur] & & & & & \\
& & \Omega^{-1}(M_{n-1}) \arrow[rr, "="] & & \Omega^{-1}(M_{n-1}) & & & &
\end{tikzcd}$$

where K_{n-2} is the push out and $I(M_{n-1})$ is the injective hull of M_{n-1}. This diagram gives the extension

$$\tilde{E} : 0 \to \Omega^{-1}(B) \oplus (proj) \to K_{n-2} \to M_{n-3} \to \cdots \to M_0 \to A \to 0$$

with the extension class $\tilde{\alpha}$ associate to α under the isomorphism $\mathrm{Ext}^{n-1}_{kG}(A, \Omega^{-1}(B)) \cong \mathrm{Ext}^n_{kG}(A,B)$. Let $0 \to \Omega^{-(n-1)}(B) \to K \to A \to 0$ be the extension corresponding to $\tilde{\alpha}$. By induction, $K \oplus (proj)$ has a filtration with sections isomorphic to the Heller shifts of $M_0, M_1, \ldots, M_{n-3}$ and K_{n-2}. We need to show that M has a filtration with sections isomorphic to the Heller shifts of $M_0, \ldots M_{n-2}, M_{n-1}$. Note that $K \cong M$, since α and $\tilde{\alpha}$ corresponds to equivalent extensions in $\mathrm{Ext}(A, \Omega^{-(n-1)}(B))$. So, M has a filtration with sections isomorphic to the Heller shifts of $M_0, M_1, \ldots, M_{n-3}$ and K_{n-2} On the other hand, we have the exact sequence $0 \to M_{n-2} \to K_{n-2} \to \Omega^{-1}(M_{n-1}) \to 0$. This gives that K_{n-2} has a filtration with sections isomorphic to the Heller shifts of M_{n-2} and M_{n-1}. Thus M has a filtration with sections isomorphic to the Heller shifts of $M_0, \ldots, M_{n-1}$.

□

Note that Carlson's theorem follows as a corollary. To see this, apply Theorem 5.3.1 to the extension

$$E : 0 \to k \to k_{H_n}^{\uparrow G} \to k_{H_n}^{\uparrow G} \to \cdots \to k_{H_1}^{\uparrow G} \to k_{H_1}^{\uparrow G} \to k \to 0$$

in Corollary 5.4.1 .

Using Theorem 5.3.1, we obtain another generalization of Carlson's Theorem:

Theorem 5.3.2 *Let ζ be the cohomology class in $H^n(G,k) = \mathrm{Ext}^n_{kG}(k,k)$ which is represented by the extension*

$$E : 0 \to k \to M_{n-1} \to \cdots \to M_0 \to k \to 0.$$

Then $L_\zeta \oplus (proj)$ has a filtration with sections isomorphic to Heller shifts of M_i's.

Proof: By Lemma 4.2.2, ζ is represented by

$$0 \to k \to P_{n-1}/L_\zeta \to \Omega^{n-1}(k) \to 0.$$

Tensoring this with $\Omega^{-(n-1)}(k)$ and cancelling projective modules, we get

$$0 \to \Omega^{-(n-1)}(k) \to \Omega^{-n}(L_\zeta) \oplus (proj) \to k \to 0 \tag{5.4}$$

Thus we can apply Theorem 5.3.1 with $M \cong \Omega^{-n}(L_\zeta) \oplus (proj)$, so $\Omega^{-n}(L_\zeta) \oplus (proj)$ has a filtration

$$\{0\} = T_0 \subseteq T_1 \subseteq \ldots T_n = \Omega^{-n}(L_\zeta) \oplus (proj)$$

with the property $T_i/T_{i-1} = \Omega^{-(i-1)}(M_{i-1})$. Tensoring the entire system with $\Omega^n(k)$ gives a filtration

$$\{0\} = \hat{T}_0 \subseteq \hat{T}_1 \subseteq \ldots \hat{T}_n = L_\zeta \oplus (proj)$$

for $L_\zeta \oplus (proj)$ with property $\hat{T}_i/\hat{T}_{i-1} \cong \Omega^{n-(i-1)}(M_{i-1})$.

□

Note that Carlson's theorem follows from Theorem 5.3.2 as a corollary.

Chapter 6

Carlson's Theorem in Integral Cohomology

In [8], it is shown that any $\mathbb{Z}G$-module M is a direct summand of a module that has a filtration by modules induced from elementary abelian subgroups. If the coefficient ring is a field of characteristic p, then only the elementary abelian p-subgroups are used. In this chapter, we see that most of the results in chapter 5 is true in integral cohomology. We give a summary of the proof of Carlson's theorem.

6.1 Carlson's Argument in Integral Cohomology

Throughout the following G is still a finite group. R denotes a general commutative coefficient ring with unit and we let again k be a field of characteristic $p > 0$. R (or k) denote the trivial RG-module (or kG-module). Carlson states the main theorem of the paper [8] for integer coefficients. In this chapter all these $\mathbb{Z}G$-modules that we are interested in will be $\mathbb{Z}G$-lattice. A $\mathbb{Z}G$-module is called $\mathbb{Z}G$-lattice if it is free as $\mathbb{Z}$-modules.

Theorem 6.1.1 *There exists an integer τ, depending only on G, and there exists a finitely generated $\mathbb{Z}G$-module V such that the direct sum $\mathbb{Z} \oplus V$ has a filtration*

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = \mathbb{Z} \oplus V$$

with the property that for each $i = 1, 2 \ldots, \tau$, there is an elementary abelian subgroup $E_i \subseteq G$ and a $\mathbb{Z}E_i$-module W_i such that

$$L_i/L_{i-1} \cong W_i^{\uparrow G}.$$

The modules $V, W_1, \ldots, W_\tau$ can be assumed to be free as $\mathbb{Z}$-modules.

As before $W^{\uparrow G}$ denote the induced module $W^{\uparrow G} \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} W$ for any $\mathbb{Z}H$-module W where H is a subgroup of V. In the theorem Carlson says that $\mathbb{Z} \oplus V$ is filtered by modules induced from elementary abelian subgroups. One of the implications of the theorem is that any RG-module M is direct summand of a module which is filtered by modules induced from elementary abelian subgroups.

Corollary 6.1.2 *For any RG-module M there is an RG-module M' such that $M \oplus M'$ has a filtration*

$$\{0\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\tau = M \oplus M'$$

where $N_i/N_{i-1} \cong U_i^{\uparrow G}$ for some RE_i-module U_i. Moreover, if M is finitely generated, then M' may also be taken to be finitely generated.

Proof: For the proof see [8]

$\square$

Lemma 6.1.3 ([8]) *If theorem 6.1.1 is true for all finite p-groups, then it is true for all finite groups.*

Proof: Let G be any finite group, p a prime dividing order of G, and let S be a sylow p-subgroup of G. By assumption, we have an integer τ and $\mathbb{Z}S$-module V such that $\mathbb{Z}_\mathbb{S} \oplus V$ has a filtration

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = \mathbb{Z}_\mathbb{S} \oplus V$$

where for each i, $L_i/L_{i-1} \cong W_i^{\uparrow G}$ for some $\mathbb{Z}E_i$-module W_i and some elementary abelian p-subgroup E_i of S. We can induce the entire system to G. Then we have that $\mathbb{Z_S}^{\uparrow G} \oplus V^{\uparrow G}$ has a filtration

$$\{0\} = L_0^{\uparrow G} \subseteq L_1^{\uparrow G} \subseteq \cdots \subseteq L_\tau^{\uparrow G} \cong \mathbb{Z_S}^{\uparrow G} \oplus V^{\uparrow G}$$

such that $L_i^{\uparrow G}/L_{i-1}^{\uparrow G} \cong (W_i^{\uparrow S})^{\uparrow G} = W_i^{\uparrow G}$. Since we consider $\mathbb{Z}G$-modules which are free as $\mathbb{Z}$-modules, all L_i's are free as $\mathbb{Z}$-modules and so are the modules $W_i^{\uparrow G}$ and $V^{\uparrow G}$.

Now suppose that $p_1, \ldots, p_r$ are the primes dividing the order of G and for each i, S_i is a sylow p_i-subgroup and V_i is a module such that $\mathbb{Z}_{S_i}^G \oplus V_i$ is filtered by modules induced from elementary abelian p_i-subgroups. Then the direct sum $\bigoplus_i (\mathbb{Z}_{S_i}^G \oplus V_i)$ can be filtered by modules induced from elementary abelian subgroups. Since the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is a direct summand of $\bigoplus_i \mathbb{Z}_{S_i}^G$ we get the required result.

□

For the remainder of the section we assume that G is a finite p-group. As in the chapter 5 proof based on Serre's theorem with some changes. We know that each cohomology class satisfying Serre's condition is represented by an exact sequence in the form

$$0 \to k \to k_H^{\uparrow G} \to k_H^{\uparrow G} \to k \to 0$$

where H is a maximal subgroup of G. This sequence can be lifted to the integral coefficients. For a maximal subgroup H of G there is an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_H^{\uparrow G} \to \mathbb{Z}_H^{\uparrow G} \to \mathbb{Z} \to 0$$

and reduction modulo p of this exact sequence represents the Bockstein element associated to H. Assume that

$$\beta_H : 0 \to \mathbb{Z} \to \mathbb{Z}_H^{\uparrow G} \to \mathbb{Z}_H^{\uparrow G} \to \mathbb{Z} \to 0$$

be the representative element in $H^2(G, \mathbb{Z})$. Thus there exists maximal subgroups $H_1, \ldots, H_n$ such that the product $\beta = \beta_1 \ldots \beta_n$ is contained in $pH^{2n}(G, \mathbb{Z})$ since its reduction modulo p is zero. The groups $H^i(G, \mathbb{Z})$ are $|G|$-torsion, for sufficiently large m we have $\beta^m = 0$. Thus Serre's theorem takes the following form in integral cohomology.

Corollary 6.1.4 *Suppose that G is a p-group which is not elementary abelian. Then there is a sequence of maximal subgroups $H_1, \dots, H_n$ of G and an exact sequence*

$$E : 0 \to \mathbb{Z} \to C_{2n-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

such that
i) $C_{2i-2} \cong C_{2i-1} \cong \mathbb{Z}_{H_i}^{\uparrow G}$ for $i = 1, \dots n$ and
ii) the class of E in $\operatorname{Ext}^{2n}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is zero.

Proof of the main theorem is nearly the same as the proof in chapter 5. There are some difficulties in integral cohomology. Here $\Omega(M)$ denotes the kernel of a surjective homomorphism from $P \twoheadrightarrow M$ a projective module to M where it is a $\mathbb{Z}G$-module free as $\mathbb{Z}$-module. Inductively $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$. In integral case, a projective cover need not might be unique up to isomorphism. So, $\Omega^n(M)$ may not be well defined up to isomorphism. It is well defined up to a projective module.

In integral case, to write the proof we need Propositions 3.1.7 and 3.1.9, so we need the following definition.

Definition 6.1.5 *A $\mathbb{Z}G$-module P is called weakly injective if every exact sequence $0 \to P \to M \to N \to 0$ of $\mathbb{Z}G$-lattices which splits as a sequence of $\mathbb{Z}$-modules splits over $\mathbb{Z}G$.*

$\mathbb{Z}G$ is a weakly injective module. Projective $\mathbb{Z}G$-modules are not injective, but they are weakly injective. That is why Propositions 3.1.7 and 3.1.9 are true in integral case.

For a given $\mathbb{Z}G$-lattice M, there is projective hence weakly injective, $\mathbb{Z}G$-lattice Q and an injection $Q \rightarrowtail M$. This is similar to injective hull for $\mathbb{Z}G$-lattices. We define $\Omega^{-1}(M)$ as the cokernel of the the injection. Inductively, we define $\Omega^{-n}(M) \cong \Omega^{-1}(\Omega^{-n+1}(M))$. And these are also well defined up to a projective module.

If $\zeta \in H^n(G, \mathbb{Z})$, we can represent ζ by a cocycle $\hat{\zeta} : \Omega^n(\mathbb{Z}) \to \mathbb{Z}$. By adding a free summand to $\Omega^n(\mathbb{Z})$, if necessary, we may assume that this map is surjective (even if $\zeta = 0$). We write $L_\zeta = \ker(\hat{\zeta})$ and get the exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n(\mathbb{Z}) \xrightarrow{\hat{\zeta}} \mathbb{Z} \longrightarrow 0$$

In integral cohomology L_ζ is well defined up to a projective module because $\Omega^n(\mathbb{Z})$ is well defined up to a projective module. We have the exact sequence

$$0 \to \Omega^s(L_{\zeta_2}) \to L_{\zeta_1\zeta_2} \oplus (proj) \to L_{\zeta_1} \to 0$$

for $\zeta_1 \in H^s(G, \mathbb{Z})$ and $\zeta_2 \in H^r(G, \mathbb{Z})$ from the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega^r(L_\eta) \arrow[r, equal] \arrow[d] & \Omega^r(L_\eta) \arrow[d] & & \\
0 \arrow[r] & L_{\zeta\cdot\eta} \oplus (proj) \arrow[r] \arrow[d] & \Omega^{r+s}(\mathbb{Z}) \oplus (proj) \arrow[r, "\hat{\zeta}\circ\hat{\eta}"] \arrow[d, "id\otimes\hat{\eta}"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_\zeta \arrow[r] \arrow[d] & \Omega^r(\mathbb{Z}) \arrow[r, "\hat{\zeta}"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that the propositions 3.1.6, 3.1.7 and 3.1.9 remain vaild for integer coefficients. And the propositions 4.1.7, 4.1.10, the lemma 4.2.2, the corollary 4.2.3, the remark 4.2.4 and the proposition 4.2.6 are also true in integral coefficient. In chapter 5, the proposition 5.1.6 is true in integer coefficients. And also for p-groups, if M is $\mathbb{Z}$-free $\mathbb{Z}H$-module where H is a subgroup of G, then $\Omega^n(M^{\uparrow G}) \cong (\Omega^n(M))^{\uparrow G}$. Using all these materials, as it is in chapter 5 we can write the proof for the main theorem in integral cohomology using L_ζ-modules.

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