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THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

LACUNARY WEAK STATISTICAL CONVERGENCE

Shahram Ahmed MUSTAFA

Master's Thesis

Department of Mathematics
Program of Analysis and Functions Theory

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I hereby declare that I wrote this Master's Thesis titled “LACUNARY WEAK STATISTICAL CONVERGENCE” in consistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Firat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, express all institutions or organizations or persons who supported the thesis financially. I have never used the data and information I provide here in order to get a degree in any way.

9 January 2020

Shahram Ahmed MUSTAFA

PREFACE

In this thesis, we provide lacunary weak statistical convergence of order α and weak N_{θ}^{α}-Summability of order for $0 < \alpha \leq 1$. The aim of this idea introduces the Δ-lacunary weak statistical convergence of order α and weak $N_{\theta}^{\alpha}(\Delta)$-Summability of order α for $0 < \alpha \leq 1$. We also give some inclusion relations between these concepts.

First and foremost, praise and thanks to God who guides us through the way, special thanks to my supervisor Prof. Dr. Çiğdem A. BEKTAŞ for her great efforts of supervising and leading me to accomplish this fine project to my mother and my father for their warm kindness, encouragements and for their great love, my friends, they were great source of support and encouragement, I thank them all and wish them all the best in their lives. To people who gave me something to light our pathway, I thank them for believing me.

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In this thesis, we give the concepts of statistical convergence of order α for $0 < \alpha \leq 1$, strong p-Cesàro summability of order α for $0 < \alpha \leq 1$ and weak statistical convergence. Also, some relations between statistical convergence of order α and strong p-Cesàro summability of order α are given for $0 < \alpha \leq 1$. Furthermore, we provide lacunary weak statistical convergence of order α and weak N_θ^α-Summability of order for $0 < \alpha \leq 1$. In the last section, we introduce the Δ-lacunary weak statistical convergence of order α and weak $N_\theta^\alpha(\Delta)$-Summability of order α for $0 < \alpha \leq 1$. We also give some inclusion relations between these concepts.

Keywords: Statistical convergence, statistical convergence of order α, lacunary weak statistical convergence of order α, weak N_θ^α-Summability of order α, Δ-lacunary weak statistical convergence of order α.

ÖZET

Lacunary Zayıf İstatistiksel Yakınsaklık

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Bu tezde α. dereceden istatistiksel yakınsaklık ve α. dereceden kuvvetli p-Cesaro toplanabilirlik kavramları gösterildi. Aynı zamanda α. dereceden istatistiksel yakınsaklık ve α. dereceden kuvvetli p-Cesaro toplanabilirlik arasındaki bazı bağıntılar verildi. Dahası α. dereceden lacunary zayıf istatistiksel yakınsaklığı ve $0 < \alpha \leq 1$ için zayıf N_θ^α- yakınsaklığı göstereceğiz. En son kısımda α. dereceden Δ-lacunary zayıf istatistiksel yakınsaklığı ve $0 < \alpha \leq 1$ için zayıf $N_\theta^\alpha(\Delta)$-yakınsaklığı göstereceğiz. Bu yakınsaklık tiplerinin bazı özelliklerini ve içerme bağıntılarını ispatlayacağız.

Anahtar kelimeler: İstatistiksel yakınsaklık, α. dereceden istatistiksel yakınsaklık, α. dereceden lacunary zayıf istatistiksel yakınsaklık, α. dereceden zayıf $N_\theta^\alpha(\Delta)$-yakınsaklık, α. dereceden Δ-lacunary zayıf istatistiksel yakınsaklık.

LIST OF SYMOBLS

$\mathbb{N}$	: Set of natural numbers,
$\mathbb{R}$	: Set of real numbers,
$\mathbb{C}$	: Complex number,
w	: Space of all sequences,
X^*	: Continuous dual of X,
X^{**}	: Second continuous dual of X,
$a.a.k$	: Almost for all k,
c	: Set of convergent sequence,
c_0	: Set of null sequences,
L_p	: Function space,
l_∞	: Set of bounded sequences,
S	: Set of statistically convergent sequences,
S_0	: Set of statistically null sequences,
S_θ	: Set of lacunary statistically convergent,
WS	: Set of weak statistically convergent,
$W\text{-}St$	: Weak lacunary statistically convergent,
WN_θ^α	: Weak N_θ-summable of order α,
S_a	: Set of sequences which are statistically convergent of order α,
WS_θ^α	: Lacunary weakly statistically convergent to l,
$WS_\theta^\alpha(\Delta)$	: Δ-Lacunary weakly statistically convergent to l.

1. INTRODUCTION

In 1935, Zygmund [1] provided the statistical convergence's idea in the first edition published in Warsaw. After that Fast [2] and Steinhaus [3] reintroduced this idea formally and then Schoenberg [4] independently reintroduced this idea. Consistently and under various names, statistical convergence has explored in number theory, trigonometric series, ergodic theory, Fourier analysis, measure theory, Banach spaces, and turnpike theory. Later It was investigated further from the viewpoint of linked with summability theory and sequence space by Fridy [5], Connor [6], Rath and Tripathy [7], Şengül and Et [8], Fridy and Orhan [9], Móricz [10], Salat [11], Bhardwaj and Bala [12], Mursaleen [13] etc. In the study of Strong Integral Summability, some generalizations on statistical convergence have emerged in recent years and the construction of ideals of bounded continuous functions on locally compact space. In fact, statistical convergence is closely linked to the theory of convergence in probability.

The statistical convergence depends on the density of the subsets of $\mathbb{N}$.
Let $E \subset \mathbb{N}$. Then $\delta(E)$ denotes the natural density of E which defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in E\}|$$

whenever the limit exists. $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n.

A sequence (x_k) is named statistically convergence to a number l if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0.$$

We write $S - \lim_{k\to\infty} x_k = l$ in this case (see [5]). This principle is used to solve many of the problems in trigonometric series, Banach spaces, fuzzy set theory, ergodic theory. Several authors have also researched related topics with the concept of summability theory such as Rath and Tripathy [7], Şengül and Et [8].

Gadjiev and Orhan [14] have given the idea of statistical convergence of β degree with $\beta \in (0,1)$. Then Çolak [15] has studied the strong p-Cesàro summability and statistical convergence of order α.

1.1 Some Basic Definitions

Definition 1.1.1 A vector space is a set X in which two operations addition and scalar multiplication are defined such that for each $x, y \in X$ there correspond a vector $x + y \in X$ and

$$x + y = y + x, \qquad x + (y + z) = (x + y) + z \qquad \forall\, x, y, z \in X$$

where $(X, +)$ is a group with the natural element denoted by θ and inverse denoted by $-$,

$$x + (-x) = x - x = \theta.$$

Scalar multiplication of $x \in X$ by elements $k \in F$, denoted by $kx \in X$, and

$$k(ax) = (ka)x, \quad k(x + y) = kx + ky,$$

$$(k + a)x = kx + ax, \qquad \forall\, x, y \in X,\ k, a \in F.$$

Moreover $1x = x$ for all $x \in X$, 1 being the unit in F.
It follows from the definition that

$$0x = 0, \qquad (-1)x = -x.$$

Definition 1.1.2 A metric space (X, d) consists of a non-empty set X and a function $d: X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

(i) $d(x, y) = 0$ if and only if $x = y$. (Positivity)

(ii) $d(x, y) = d(y, x)$. (Symmetry)

(iii) $d(x, y) \leq d(x, z) + d(z, y)$. (Triangle Inequality)

A function d satisfying conditions (i)-(iii), is called a metric on X.

Definition 1.1.3 Let X be a linear space with a field F. A mapping $\|\,.\,\|: X \longrightarrow \mathbb{R}$ is claimed to be a norm on X if the following conditions are holds:

(i) $\|x\| \geq 0, \qquad \forall\, x \in X,$

(ii) $\|x\| = 0 \Leftrightarrow x = \theta,$

(iii) $\| \alpha x \| = | \alpha | \| x \|$, for all $\alpha \in F$ (Homogeneity of norm)

(iv) $\| x + y \| \leq \| x \| + \| y \|$, for all $x, y \in X$ (Triangle inequality).

Definition 1.1.4 [16] The sequence (a_n) is convergent to a number a if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon,$$

for $n \geq N$, and it is denoted by $\lim_{n\to\infty} a_n = a$.

Definition 1.1.5 [16] A sequence (a_n) is bounded if there is $K > 0$ such that

$$|a_n| < K \qquad \forall\, n \in \mathbb{N}.$$

Definition 1.1.6 [17] We say that a sequence (a_n) of $\mathbb{R}$ is Cauchy provided that for every $\varepsilon > 0$, there exists a natural number N such that $\forall\, n,\, m \geq N$,

$$|a_n - a_m| < \varepsilon.$$

Definition 1.1.7 [18] The normed space X is complete (or Banach space) if every Cauchy sequence in X is convergent.

Definition 1.1.8 If X and Y are normed linear spaces, then a linear operator $T: X \to Y$ is called a bounded linear operator if there is an $M \in \mathbb{R}$, $M \geq 0$ such that for every $x \in X$ we have that:

$$\|T(x)\| \leq M\|x\|.$$

We denoted the set of all bounded linear operators from X to Y by $B(X, Y)$.

Definition 1.1.9 [19] A linear functional f is a linear operator with domain $\mathcal{D}(f)$ in a vector space X and range in the scalar field

$$f: \mathcal{D}(f) \to K$$

where R is real line and C is a complex plane, $K = R$ if X is real and $K = C$ if X is a complex.

Definition 1.1.10 [19] The space l_p of sequence in real number for any $1 \leq p < \infty$ is a sequence $x = (x_i) = (x_1, x_2, \dots)$ of numbers such that $|x_1|^p + |x_2|^p + \cdots$ convergence thus,

$$\sum_{i=2}^{\infty} |x_i|^p < \infty$$

and the metric is described by

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}.$$

Also where l_p $-$norm is described by

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}.$$

Definition 1.1.11 (Hölder's inequality). Let $1 < p, 1/p + 1/q = 1$, $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n \geq 0$. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}.$$

Also,

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k\right) \max b_k .$$

Definition 1.1.12 Assuming X is a vector space and $X^* = \{f | f: X \to F\}$ where f is a bounded linear functional and $F = \mathbb{R}$ or $\mathbb{C}$. Then X^* is referred to as the continuous dual of X. We can define it is continuous dual, $(X^*)^* = X^{**}$, which is called the second continuous dual of X. An important property of X^{**} is that there exists an injective mapping $C: X \to X^{**}$ called the canonical mapping of X into X^{**}. This embedding is always a linear isometry. If it is also surjective i.e., if it maps X onto X^*, then X is said to be reflexive and we write $X = X^*$

Definition 1.1.13 [19] For any fixed real number $p \geq 1$, the space L_p is the completion of the normed space which consists of all continuous real-valued functions on $[a, b]$ with the norm defined by

$$\|x\|_p = \left(\int_a^b x(t)^p dt\right)^{1/p}.$$

The subscript p is supposed to remind us that this norm depends on the choice of p.

2. STATISTICAL CONVERGENCE AND STRONG p-CESÀRO SUMMABILITY OF ORDER α.

Definition 2.1 [20] Assuming $E \subset \mathbb{N}$, then the natural density of E is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in E\}|$$

in the case this limit exists, and $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n.

Clearly $\delta(\mathbb{N}) = 1$ and $\delta(E) = \delta(\mathbb{N}) - \delta(E^c) = 1 - \delta(E^c)$ for every $E \subset \mathbb{N}$ and also $\delta(E) = 0$ if E is a finite subset of $\mathbb{N}$.

Definition 2.2 Let $x = (x_k) \in w$. The sequence (x_k) is called statistically convergent to the number l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0$$

i.e. for $a.a.k.$ $|x_k - l| < \varepsilon$. We write $S - \lim x_k = l$, in this situation. And we denote the set of all statistical convergence sequences by S.

Lemma 2.3 [11] If $S - \lim x_k = l$ and $S - \lim y_k = m$ and $c \in \mathbb{R}$, then

(i) $S - \lim(x_k + y_k) = l + m$,

(ii) $S - \lim(cx_k) = cl$.

Lemma 2.4 [21] A sequence (x_k) of numbers is statistically bounded if and only if there exists $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and (x_{k_n}) is bounded.

Lemma 2.5 [22] Assume that $x_k \leq y_k$, for $a.a.k.$ If $S - \lim x_k$ and $S - \lim y_k$ exist, then $S - \lim x_k \leq S - \lim y_k$.

Definition 2.6 Let $\alpha \in (0,1]$. We define the α-density of the subset E of $\mathbb{N}$ by

$$\delta_\alpha(E) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : k \in E\}|.$$

In the case this limit exists, where $|\{k \leq n : k \in E\}|$ is denoted the number of elements of E which are less than or equal to n.

Now, we give the following notation. Let $x = (x_k)$ be any sequence which satisfies the property $p(k)$ for all of its members (x_k) for all k apart from a set of α-density zero satisfies the property $p(k)$ for all k apart from a set of α-density zero, then we say that (x_k)satisfies $p(k)$ for "almost all k according to α" and we abbreviate this by "$a.a.k(\alpha)$".

Obviously $\delta_\alpha(E) = 0$ for any finite subset $E \subset \mathbb{N}$ and $\delta_\alpha(E) = 1 - \delta_\alpha(E^c)$ does not hold for $\alpha \in (0,1]$, in general (for more details see [15]), in the case $\alpha = 1$ the equality is always true. Note that the α-density is similar to the natural density if $\alpha = 1$.

Lemma 2.7 Assuming $E \subseteq \mathbb{N}$. Then, $\delta_\beta(E) \leq \delta_\alpha(E)$ if $0 < \alpha \leq \beta \leq 1$.

Proof: Suppose $0 < \alpha \leq \beta \leq 1$. Since $n^\alpha \leq n^\beta$ so that $\frac{1}{n^\beta} \leq \frac{1}{n^\alpha}$ for each $n \in \mathbb{N}$, then

$$\frac{1}{n^\beta}|\{k \leq n: k \in E\}| \leq \frac{1}{n^\alpha}\ |\{k \leq n: k \in E\}|.$$

From this inequality, we have $\delta_\beta(E) \leq \delta_\alpha(E)$.

Now, let $0 < \alpha \leq \beta \leq 1$. Then, from Lemma 2.7 we have that if E has zero α-density then E has zero β-density, and if E has zero α-density for some $0 < \alpha \leq 1$, then it has zero natural density [15].

Definition 2.8 Suppose $x = (x_k) \in w$ and $\alpha \in (0,1]$, the sequence (x_k) is called statistically convergent of order α to the number l if

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n: |x_k - l| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. That is for $a.a.k\ (\alpha)\ |x_k - l| < \varepsilon$. In which case we write $S^\alpha - \lim x_k = l$. We denote the set of all statistically convergent sequence of order α by S^α.

Theorem 2.9 [15] Let $0 < \alpha \leq 1$ and $x = (x_k),\ y = (y_k)$ be sequences of $\mathbb{C}$.

(i) If $S^\alpha - \lim x_k = x_0$ and $c \in \mathbb{C}$, then $S^\alpha - \lim cx_k = cx_0$.

(ii) If $S^\alpha - \lim x_k = x_0$ and $S^\alpha - \lim y_k = y_0$, then $S^\alpha - \lim(x_k + y_k) = x_0 + y_0$.

Proof:

(i) For the case $c = 0$ the proof is clear**.** Assuming $c \neq 0$ then the proof of (i) is derived from

$$\frac{1}{n^\alpha}|\{k \leq n: |cx_k - cx_0| \geq \varepsilon\}| \leq \frac{1}{n^\alpha}\left|\left\{k \leq n: |x_k - x_0| \geq \frac{\varepsilon}{|c|}\right\}\right|$$

and that of (ii) follows from

$$\frac{1}{n^\alpha}|\{k \leq n: |x_k + y_k - (x_0 + y_0)| \geq \varepsilon\}|$$
$$\leq \frac{1}{n^\alpha}\left|\left\{k \leq n: |x_k - x_0| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{n^\alpha}\left|\left\{k \leq n: |y_k - y_0| \geq \frac{\varepsilon}{2}\right\}\right|.$$

It is clear to see that $c \subset S^{\alpha}$ for every $\alpha \in (0,1]$. But the converse is not true in general. For example, define the sequence $x = (x_k)$ as

$$x_k = \begin{cases} 1, & k = n^3, \\ 0, & k \neq n^3. \end{cases} \tag{2.1}$$

Then it is clear that $x \notin c$ but $x \in S^{\alpha}$ for $\alpha > {}^{1}/_{3}$ with S^{α}-$\lim x_k = 0$.

Definition 2.10 Assuming $\alpha \in (0,1]$ and $p > 0$. A sequence $x = (x_k)$ is named strongly p-Cesàro summable of order α to l, if

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |x_k - l|^p = 0.$$

The set of all strongly p-Cesàro summable sequence of order α denoted by w_p^{α} and in the case $l = 0$ we write w_{op}^{α}. That is

$$w_p^{\alpha} = \left\{ x = (x_k) \colon \lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |x_k - l|^p = 0 \text{ for some } l \right\},$$

$$w_{op}^{\alpha} = \left\{ x = (x_k) \colon \lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |x_k - 0|^p = 0 \text{ for some } l \right\}.$$

Theorem 2.11 [15] If $0 < \alpha \leq \beta \leq 1$, then $S^{\alpha} \subseteq S^{\beta}$ and the inclusion is strict for some α and β such that $\alpha < \beta$.

Proof: Let $0 < \alpha \leq \beta \leq 1$, then

$$\frac{1}{n^{\beta}} \left|\{k \leq n : |x_k - l| \geq \varepsilon\}\right| \leq \frac{1}{n^{\alpha}} \left|\{k \leq n : |x_k - l| \geq \varepsilon\}\right|$$

for every $\varepsilon > 0$ and this gives that $S^{\alpha} \subseteq S^{\beta}$. For the strict inclusion we the sequence $x = (x_k)$ as

$$x_k = \begin{cases} k, & k = n^2, \\ 0, & k \neq n^2. \end{cases} \tag{2.2}$$

Then, $S^{\beta} - \lim x_k = 0$, i.e. $x \in S^{\beta}$ for $1/2 < \beta \leq 1$, but $x \notin S^{\alpha}$ for $0 < \alpha \leq 1/2$.

We obtain the following result by taking $\beta = 1$ in Theorem 2.8.

Corollary 2.12 [15] Suppose $\alpha \in (0,1]$. Then $S^{\alpha} \subseteq S$ and the inclusion is strict.

The following result is obtained by Theorem 2.8.

Corollary 2.13 [15]

(i) $S^{\alpha} = S^{\beta}$ if and only if $\alpha = \beta$.

(ii) $S^{\alpha} = S$ if and only if $\alpha = 1$.

The proof is omitted because it is clear from the definition.

Theorem 2.14 [15**]** Assuming $0 < \alpha \leq 1$ and $x = (x_k) \in S^{\alpha}$ such that $S^{\alpha} - \lim x_k = l$. Then, there is a subsequence $y = (y_k)$ of $x = (x_k)$ such that $\lim y_k = l$.

Theorem 2.15 [15] Assuming $0 < \alpha \leq \beta \leq 1$ and $p > 0$. Then, $w_p^{\alpha} \subseteq w_p^{\beta}$ and the inclusion remains strict for some $\alpha < \beta$.

Proof: Suppose that $x = (x_k) \in w_p^{\alpha}$, $0 < \alpha \leq \beta \leq 1$, and $p > 0$, then we have

$$\frac{1}{n^{\beta}} \sum_{k=1}^{n} |x_k - l|^p \leq \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |x_k - l|^p$$

and this gives that $w_p^{\alpha} \subseteq w_p^{\beta}$.

For the strict inclusion, if we define the sequence $x = (x_k)$ as in (2.2), then we have

$$\frac{1}{n^{\beta}} \sum_{k=1}^{n} |x_k - 0|^p \leq \frac{\sqrt{n}}{n^{\beta}} = \frac{1}{n^{\beta - \frac{1}{2}}}$$

Since $\frac{1}{n^{\beta-\frac{1}{2}}} \to 0$ as $n \to \infty$, then $w_p^{\beta} - \lim x_k = 0$, i.e. $x \in w_p^{\beta}$ for $1/2 < \beta \leq 1$, but since

$$\frac{\sqrt{n} - 1}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |x_k - 0|^p$$

and $\frac{\sqrt{n}-1}{n^{\alpha}} \to \infty$ as $n \to \infty$ for $0 < \alpha < 1/2$, so that $x \notin w_p^{\alpha}$. This completes the proof.

The consequence below is a result of Theorem 2.15.

Corollary 2.16 Let $0 < \alpha \leq \beta \leq 1$ and $p > 0$. Then

(i) $w_p^{\alpha} = w_p^{\beta}$ if and only if $\alpha = \beta$.

(ii) $w_p^{\alpha} \subseteq w_p$ for each $\alpha \in (0, 1]$ and $p > 0$.

The outcome below is a simple result of Hölder's inequality, an extension of a Maddox result in [23].

Theorem 2.17 Suppose $\alpha \in (0,1]$ and $0 < p < q < \infty$. Then, $w_q^{\alpha} \subset w_p^{\alpha}$.

If we take $\alpha = 1$ in Theorem 2.14, then we get a result of Maddox [23]: $w_q \subset w_p$ if $0 < p < q < \infty$.

Theorem 2.18 [15] Suppose that $0 < \alpha \leq \beta \leq 1$, $p > 0$, and $x = (x_k) \in w$. If $x \in w_p^\alpha$ then $x \in S^\beta$.

Proof: For any sequence $x = (x_k)$ and $\varepsilon > 0$, we can write

$$\sum_{k=1}^{n} |x_k - l|^p \geq |\{k \leq n : |x_k - l|^P \geq \varepsilon\}| \varepsilon^p$$

and so that

$$\frac{1}{n^\alpha} \sum_{k=1}^{n} |x_k - l|^p \geq \frac{1}{n^\alpha} |\{k \leq n : |x_k - l|^p \geq \varepsilon\}| \varepsilon^p$$

$$\geq \frac{1}{n^\beta} |\{k \leq n : |x_k - l|^p \geq \varepsilon\}| \varepsilon^p.$$

So that $x \in w_p^\alpha$ implies $x \in S^\beta$.

The following result is obtained if we take $\beta = \alpha$ in Theorem 2.18.

Corollary 2.19 Assume that $\alpha \in (0,1]$, $0 < p \leq \infty$ and $x = (x_k) \in w$. If $x \in w_p^\alpha$, then $x \in S^\alpha$.

Remark [15] In general, the reverse of Theorem 2.18 does not hold. We see that a bounded sequence and statistically convergent sequence of order α does not have to be strongly p-Cesàro summable of order α, in general for $0 < \alpha < 1$. For this, define $x = (x_k)$ as

$$x_k = \begin{cases} \frac{1}{\sqrt{k}}, & k \neq m^3 \\ 1, & k = m^3. \end{cases}$$

It is clearly that $x \in l_\infty$ and $x \in S^\alpha$ for each $1/3 < \alpha \leq 1$. First recall that for every positive integer $n \geq 2$, we have

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}.$$

Define $H_n = \{k \leq n : k \neq m^3, m = 1, 2, 3, \cdots\}$ and take $p = 1$. Since

$$\sum_{k=1}^{n} |x_k|^P = \sum_{k=1}^{n} |x_k| = \sum_{k \in H_n, 1 \leq k \leq n}^{n} |x_k| + \sum_{k \notin H_n, 1 \leq k \leq n}^{n} |x_k|$$

$$= \sum_{k \in H_n, 1 \leq k \leq n} \frac{1}{\sqrt{k}} + \sum_{k \notin H_n, 1 \leq k \leq n} 1 > \sum_{k=1}^{1} \frac{1}{\sqrt{k}} > \sqrt{n}$$

we have

$$\frac{1}{n^{\alpha}}\sum_{k=1}^{n}|x_k|^P=\frac{1}{n^{\alpha}}\sum_{k=1}^{n}|x_k|>\frac{1}{n^{\alpha}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}>\frac{1}{n^{\alpha}}\sqrt{n}=\frac{1}{n^{\alpha-\frac{1}{2}}}\to\infty \text{ as } n\to\infty$$

and so that $x\notin w_p^{\alpha}$ for $0<\alpha<1/2$ if $p=1$. Therefore $x\in S^{\alpha}-w_p^{\alpha}$ for $1/3<\alpha<1/2$ if $p=1$ [15].

Corollary 2.20 Let $\alpha\in(0,1]$ and $p>0$. Then, $w_p^{\alpha}\subset S$ and the inclusion is strict for $\alpha\in(0,1]$.

Proof: We have $w_P^{\alpha}\subset S$ from Corollary 2.19 and Corollary 2.12. For the strict inclusion if we take the sequence $x=(x_k)$ defined in (2.1), then clearly $S-\lim x_k=0$, i.e. $x\in S$ but $x\notin w_p^{\alpha}$ for $0<\alpha\leq 1/3$ and $p=1$. Indeed it is easy to see that

$$\frac{1}{n^{\alpha}}\sum_{k=1}^{n}|x_k-0|^p=\frac{1}{n^{\alpha}}\sum_{k=1}^{n}|x_k|\geq\frac{\sqrt[3]{n}-1}{n^{\alpha}}$$

Since $\frac{\sqrt[3]{n}-1}{n^{\alpha}}\to\infty$ if $0<\alpha<1/3$ and $\frac{\sqrt[3]{n}-1}{n^{\alpha}}\to 1$ if $\alpha=1/3$, then $x\notin w_p^{\alpha}$ for $0<\alpha\leq 1/3$ and $p=1$. Consequently, $x\in S-w_p^{\alpha}$ for $0<\alpha\leq 1/3$ and $p=1$. This completes the proof.

From Corollary 2.12, Corollary 2.20 and Theorem 2.1 in [6] we obtain that $S^{\alpha}\cap l_{\infty}\subset w_p\subset S$ for each α, where $0<\alpha\leq 1$ and $0<p<\infty$.

Definition 2.21 Let X be a normed linear space and (x_k) be any sequence in X. If $(f(x_k))$ is convergent to $f(x)$ for all $f\in X^*$, then (x_k) is weakly convergent to $x\in X$ (or $W-\lim x_k=x$).

Lemma 2.22 [11] A sequence (x_k) of numbers is statistically convergent to l if and only if there exists $K=\{k_1<k_2<\cdots\}\subset\mathbb{N}$ such that $\delta(K)=1$ and $\lim_{n\to\infty}x_{k_n}=l$.

Definition 2.23 [24] Suppose X is a normed space and (x_k) is a sequence in X. Then (x_k) is named weak statistically convergent to $x\in X$ (or $WS-$convergent) if for each $\varphi\in X^*$ and every $\varepsilon>0$

$$\delta(\{k\leq n:|\varphi(x_k-x)|\geq\varepsilon\})=0.$$

We write WS–$\lim x_k=x$ or $x_k\xrightarrow{WS}x$ in this case. And we denote the set of all weak statistically convergent sequence by WS.

Theorem 2.24 [12] Suppose X is a normed space and suppose (x_k) is a sequence in X. If $W-\lim x_k=x$, then $WS-\lim x_k=x$. The converse does not have to be true in general.

Proof: If $W-\lim x_k=x$, then $(f\ (x_k))$ is convergent to $f\ (x)$, for all $f\in X^*$ which implies that $WS-\lim x_k=x$.

The example below shows that the converse is not correct.

Example 2.25 Let $(x_k) \in\ l_p\ (1 < p < \infty)$ be defined by

$$x_j^{(k)} = \begin{cases} m, & \text{if } j \le k, k = m^2 \\ \frac{1}{k}, & \text{if } j \le k, k \ne m^2 \\ 0, & \text{otherwise.} \end{cases}$$

For $k \ne m^2$ and arbitrary $f \in l_p^*$, there is unique $y \in l_q$ such that by Hölder's inequality

$$|f(x_k)| = \left|\sum_{j=1}^{\infty} x_j^{(k)} y_j\right| \le \left(\sum_{j=1}^{\infty} |x_j^k|^p\right)^{\frac{1}{p}} . \left(\sum_{j=1}^{\infty} |y_j|^q\right)^{\frac{1}{q}}$$

$$\le \left(\sum_{j=1}^{k} \frac{1}{k^p}\right)^{\frac{1}{p}} . M^{\frac{1}{q}} . \left(\frac{M}{k}\right)^{\frac{1}{q}} \to 0$$

as $k \to \infty$, for some positive constant M. Hence, $S - \lim f(x_k) = 0$, by Lemma 2.22, which in turn implies that $WS - \lim x_k = 0$. For $k = m^2$ and hence $m = \sqrt{k}$ consider the functional f_1 defined on l_p by $f_1(x) = x_1$, where $x = (x_k) \in l_p$. Clearly, $f_1(x_k) = x_1^k = \sqrt{k}$ for $k \to \infty$, $f_1(x_k) \to \infty$. Hence, (x_k) is not weakly convergent.

3. LACUNARY WEAK STATISTICAL CONVERGENCE AND WEAK N_θ-SUMMABILITY OF ORDER α

We are going to provide the principles of WS_θ^α-statistical convergence and weak N_θ-summability of order α for $\alpha \in (0,1]$ in this section. We are also going to give some characteristics of these principles with some relations between them. In this study, we write X to denote a normed linear space and we write X^* to denote continuous dual and also $\theta = (k_r)$ denotes a lacunary sequence such that $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$. Also, the ratio k_r/k_{r-1} abbreviated by q_r.

Definition 3.1 [25] Let $K \subset \mathbb{N}$. The lacunary density (or θ-density) of K is defined by

$$S_\theta(K) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : k \in K\}|.$$

Definition 3.2 [26] Suppose θ is a lacunary sequence. A sequence (x_k) is named lacunary statistically convergent to l, or S_θ −convergent to l if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. In this case, we write $S_\theta - \lim_{k\to\infty} x_k = l$.

Definition 3.3 [27] Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named lacunary weakly statistically convergent sequence of order α (or WS_θ^α-statistically convergent) to l if, for any $\varphi \in X^*$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}| = 0 \qquad (3.1)$$

for every $\varepsilon > 0$. Let $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, that is, $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \dots, h_r^\alpha, \dots)$. In this case, we write $WS_\theta^\alpha - \lim_{k\to\infty} x_k = l$. The space of these sequences is denoted by WS_θ^α. Instead of WS_θ^α we write WS_θ if $\alpha = 1$. And instead of WS_θ^α we write WS if $\alpha = 1$ and $\theta = (2^r)$.

Definition 3.4 [27] Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named weakly N_θ-summable of order α to l if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} |\varphi(x_k - l)| = 0$$

and it is denoted by $WN_\theta^\alpha - \lim_{k\to\infty} x_k = l$. The class of WN_θ^α-summable sequences of order α is symbolized by WN_θ^α. Instead of WN_θ^α we write WN_θ if $\alpha = 1$. We write W^α when $\theta = (k_r)$. Instead of W^α we write W_0^α if $l = 0$. The class of weakly N_θ-summable sequences of order α to 0 is symbolized by $WN_{\theta,0}^\alpha$.

Definition 3.5 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named strongly lacunary statistically convergent of order α to l, if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r: \|x_k - l\| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. We write $S_\theta^\alpha - \lim_{k\to\infty} x_k = l$ in this situation.

Definition 3.6 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named an WS_θ^α-Cauchy sequence if there is a subsequence $(x_{k'(r)})$ of (x_k) such that $k'(r) \in I_r$ for every r, $W - \lim_{r\to\infty} x_{k'(r)} = l$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r: \left|\varphi(x_k - x_{k'(r)})\right| \geq \varepsilon\}\right| = 0$$

for each $\varphi \in X^*$ and every $\varepsilon > 0$. This definition is similar to the definition WS_θ-Cauchy for the case $\alpha = 1$.

It is clear that the WS_θ^α-statistically convergent is defined for $\alpha \in (0,1]$. But for $\alpha > 1$ it is not defined. For this, if we take the functional $\varphi(x) > 0$ on X, then for any number l we have

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r: |\varphi(x_k - l)| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. This means that the $WS_\theta^\alpha - \lim_{k\to\infty} x_k$ is not unique for $\alpha > 1$.

Theorem 3.7 [28] A sequence (x_k) is a weakly lacunary statistically convergent sequence if and only if (x_k) is a weakly lacunary statistically Cauchy.

Proof: Assuming the sequence (x_k) is weakly lacunary statistically Cauchy. Then for every $\varepsilon > 0$, we have

$$|\{k \in I_r: \ |f(x_k - x)| \geq \varepsilon\}|$$

$$\leq \left|\left\{k \in I_r : \left|f\left(x_k - x_{k'(r)}\right)\right| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{k \in I_r : \left|f\left(x_{k'(r)} - l\right)\right| \geq \frac{\varepsilon}{2}\right\}\right|,$$

hence we get that the sequence (x_k) is weakly lacunary statistically convergent. Let (x_k) be weakly lacunary statistically convergent to x and write $M_j = \{k \in \mathbb{N}: |\varphi(x_k - x)| < 1/j\}$ for each $j \in \mathbb{N}, M_j \supseteq M_{j+1}$ and $|M_j \cap I_r|/h_r \to 1$ as $r \to \infty$. Choose m_1 such that $r \geq m_1$ implies

$|M_1 \cap I_r|/h_r > 0$, i.e., $M_1 \cap I_r \neq \emptyset$. Next, choose $m_1 < m_2$ such that $r \geq m_2$ implies $M_2 \cap I_r \neq \emptyset$. Then for every r satisfying $m_1 \leq r \leq m_2$, choose $k'(r) \in I_r$ such that $k'(r) \in I_r \cap M_1$. In this way, choose $m_{l+1} > m_1$ such that $r > m_{l+1}$ implies $M_{l+1} \cap I_r \neq \emptyset$. Then for all r satisfying $m_l \leq r < m_{l+1}$, choose $k'(r) \in I_r \cap M_l$, i.e., $\left|f(x_{k'(r)} - x)\right| < \frac{1}{l}$. Hence we get $k'(r) \in I_r$ for every r, and $w - \lim x_{k'(r)} = x$. Also, we have for every $\varepsilon > 0$,

$$\frac{1}{h_r}\left|\left\{k \in I_r : \left|f(x_k - x_{k'(r)})\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{h_r}\left|\left\{k \in I_r : \left|f\left(x_{k(r)} - x\right)\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{h_r}\left|\left\{k \in I_r : \left|f\left(x_{k'(r)} - x\right)\right| \geq \frac{\varepsilon}{2}\right\}\right|,$$

whence (x_k) is a weakly lacunary statistically Cauchy sequence.

Theorem 3.8 [27] Let $\alpha \in (0,1]$ and $(x_k),(y_k) \in X$.

(i) If $WS_\theta^\alpha - \lim_{k\to\infty} x_k = l$, then l must be unique.

(ii) If $WS_\theta^\alpha - \lim_{k\to\infty} x_k = l_0$ and c being a scalar, then $WS_\theta^\alpha - \lim_{k\to\infty} cx_k = cl_0$.

(iii) If $WS_\theta^\alpha - \lim_{k\to\infty} x_k = l_0$ and $WS_\theta^\alpha - \lim_{k\to\infty} y_k = l_1$, then $WS_\theta^\alpha - \lim_{k\to\infty}(x_k + y_k) = l_0 + l_1$.

Theorem 3.9 [27] Let $\theta = (k_r)$ and $\theta' = (s_r)$ be lacunary sequences such that for each $r \in \mathbb{N}$, $I_r \subset J_r$ and $0 < \alpha \leq \beta \leq 1$.

(i) If

$$\lim_{r\to\infty} \inf \frac{h_r^\alpha}{\ell_r^\beta} > 0, \tag{3.2}$$

then $WS_{\theta'}^\beta \subseteq WS_\theta^\alpha$.

(ii) If

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^\beta} = 1, \tag{3.3}$$

then $WS_\theta^\alpha \subseteq WS_{\theta'}^\beta$.

Proof:

(i) Assume $I_r \subset J_r$ for each $r \in \mathbb{N}$ and let (3.2) is satisfied. For any $\varepsilon > 0$,

$$\{k \in J_r : |\varphi(x_k - l)| \geq \varepsilon\} \supseteq \{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}$$

and so

$$\frac{1}{\ell_r^{\beta}}|\{k \in J_r : |\varphi(x_k - l)| \geq \varepsilon\}| \geq \frac{h_r^{\alpha}}{\ell_r^{\beta}} \frac{1}{h_r^{\alpha}}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|$$

for each $r \in \mathbb{N}$, where $h_r = k_r - k_{r-1}$, $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$ and $\ell_r = s_r - s_{r-1}$. By using (3.2) and taking the limit as $r \to \infty$ on both sides in the last inequality we obtain that $WS_{\theta}^{\beta} \subseteq WS_{\theta}^{\alpha}$.

(ii) Assume that $(x_k) \in WS_{\theta}^{\alpha}$ and (3.3) is satisfied. Since $I_r \subset J_r$, then for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\ell_r^{\beta}}|\{k \in J_r : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&= \frac{1}{\ell_r^{\beta}}|\{s_{r-1} < k \leq k_{r-1} : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&+ \frac{1}{\ell_r^{\beta}}|\{k_r < k \leq s_r : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&+ \frac{1}{\ell_r^{\beta}}|\{k_{r-1} < k \leq k_r : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^{\beta}} + \frac{s_r - k_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&= \frac{\ell_r - h_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&\leq \frac{\ell_r - h_r^{\beta}}{h_r^{\beta}} + \frac{1}{h_r^{\beta}}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}| \\
&\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) + \frac{1}{h_r^{\alpha}}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|
\end{aligned}$$

for each $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{\ell_r}{h_r^{\beta}} = 1$ by (3.3) the first term and since $(x_k) \in WS_{\theta}^{\alpha}$ the second term tends to 0 as $r \to \infty$ $\left(\left(\frac{\ell_r}{h_r^{\beta}} - 1\right) \geq 0\right)$. This implies that $WS_{\theta}^{\alpha} \subseteq WS_{\theta'}^{\beta}$.

The following result is obtained from Theorem 3.9.

Corollary 3.10 [27] Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for each $r \in \mathbb{N}$. If (3.2) is hold, then

(i) $WS_{\theta'}^{\alpha} \subseteq WS_{\theta}^{\alpha}$ for $0 < \alpha \leq 1$,

(ii) $WS_{\theta'} \subseteq WS_{\theta}^{\alpha}$ for $0 < \alpha \leq 1$,

(iii) $WS_{\theta'} \subseteq WS_{\theta}$.

If (3.3) is hold, then

(i) $WS_{\theta}^{\alpha} \subseteq WS_{\theta'}^{\alpha}$ for $0 < \alpha \leq 1$,

(ii) $WS_{\theta}^{\alpha} \subseteq WS_{\theta'}$ for $0 < \alpha \leq 1$,

(iii) $WS_{\theta} \subseteq WS_{\theta'}$.

Corollary 3.11 [27] Suppose $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. Then $WS_{\theta}^{\alpha} \subseteq WS_{\theta}^{\beta}$.

Corollary 3.12 [27] Suppose $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. Then $WS_{\theta}^{\alpha} \subseteq WS_{\theta}$.

Corollary 3.13 [27]

(i) $WS_{\theta}^{\alpha} = WS_{\theta}^{\beta}$ if $\alpha = \beta$,

(ii) $WS_{\theta}^{\alpha} = WS_{\theta}$ if $\alpha = 1$.

Theorem 3.14 [27] Suppose $\theta = (k_r)$ is a lacunary sequence, $0 < \alpha \leq 1$ and $\lim_{r\to\infty} \inf \frac{h_r^{\alpha}}{h_r} > 0$. If there exists $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta_{\theta}(K) = 1$ and $\lim_{k\in K} \varphi(x_k - l) = 0$ for any $\varphi \in X^*$, then (x_k) is $WS_{\theta}^{\alpha} -$ convergent to l.

Proof: Assuming $\lim_{r\to\infty} \inf \frac{h_r^{\alpha}}{h_r} > 0$ holds and there exists a set $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ with $\delta_{\theta}(K) = 1$ and $\lim_{k\in K} \varphi(x_k - l) = 0$ for $\varphi \in X^*$. Then there is a number $N \in \mathbb{N}$ such that $|\varphi(x_k - l)| < \varepsilon$ for all $k \geq N$ and for any $\varphi \in X^*$.

If we take $M_{\varepsilon} = \{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\} \subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, k_{N+3}, \dots\}$, then $\delta_{\theta}(M_{\varepsilon}) = 0$. Using Corollary 3.10 (ii) we get that (x_k) is WS_{θ}^{α}-convergent to l.

Theorem 3.15 [27] Let $x = (x_k) \in w$. If $W - \lim_{k\to\infty} x_k = l$, then $WS_{\theta} - \lim_{k\to\infty} x_k = l$.

Proof: Let $W - \lim_{k\to\infty} x_k = l$, then for each $\varepsilon > 0$ and any $\varphi \in X^*$ if there is a number $N \in \mathbb{N}$ such that

$$|\varphi(x_k - l)| < \varepsilon$$

for each $k \geq N$. So, the set $M(\varepsilon) = \{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}$ is finite; for which $\delta_{\theta}\big(M(\varepsilon)\big) = 0$. This shows that $WS_{\theta} - \lim_{k\to\infty} x_k = l$.

In general, the opposite of the above result is not correct. This fact can be shown in the example below.

Example 3.16 [27] Consider the sequence which belongs to l_p where $1 < p < \infty$ and it is defined as follows:

$$x_r(k) = \begin{cases} \sqrt{r}, & k \in I_r \text{ and } r = m^2; \\ 0 \quad , & k \notin I_r \text{ and } r = m^2; \\ \dfrac{1}{r} \quad , & k \in I_r \text{ and } r \neq m^2; \\ 0 \quad , & k \notin I_r \text{ and } r \neq m^2 \end{cases}$$

where $I_r = (k_{r-1}, k_r]$. Let $k = \{r \in \mathbb{N}: r \neq m^2, m = 1,2, \dots\}$, then using the inequality of Hölder's from the structure of the sequence, for $n \in K$ and any $\varphi \in l_p^*$ there is a unique $y \in l_q$ such that

$$|\varphi(x_r)| = \left|\sum_{k=1}^{\infty} x_r(k) y(k)\right| \leq \left(\sum_{k=1}^{\infty} |x_r(k)|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y(k)|^p\right)^{q}$$

$$= \left(\sum\nolimits_{k \in I_r} \frac{1}{r^p}\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y(k)|^q\right)^{\frac{1}{q}} \leq \left(\frac{k_r - k_{r-1}}{r^p}\right)^{\frac{1}{p}} K^{\frac{1}{q}} \to 0$$

as $r \to \infty$ for a constant $K > 0$. By Theorem 3.15 we get $WS_\theta - \lim_{r\to\infty} x_r = 0$.

For $r \in K^C$, take the functional defined by $\varphi_k(x) = x(k)$ where $(x_r) \in l_p$. Clearly,

$$\varphi_k(x_r) = x_r(k) = \sqrt[3]{r} \to \infty$$

as $r \to \infty$. $W - \lim_{r\to\infty} x_r \neq 0$.

Theorem 3.17 [27] Assume that $\theta = (k_r)$ is a lacunary sequence, $0 < \alpha \leq 1$, $x = (x_k) \in w$ and $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{h_r} > 0$. If $W - \lim_{r\to\infty} x_k = l$, then $WS_\theta^\alpha - \lim_{r\to\infty} x_k = l$.

It can be easily proved from Corollary 3.10 (ii) and Theorem 3.15, so it is omitted.

Theorem 3.18 [27] Let $0 < \alpha \leq 1$. Then $S_\theta^\alpha \subset WS_\theta^\alpha$ for any lacunary sequence $\theta = (k_r)$ with the same limit in X.

Proof: Let $(x_k) \in X$ such that $x_k \xrightarrow{S_\theta^\alpha} l$. Then for each $\varepsilon > 0$,

$$\frac{1}{h_r^\alpha} |\{k \in I_r : \|x_k - l\| \geq \varepsilon\}| = 0.$$

Now, for each $\varepsilon > 0$ and any $\varphi \in X^*$, the expression

$$\frac{1}{h_r^\alpha} |\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}| \leq \frac{1}{h_r^\alpha} |\{k \in I_r : \|\varphi\| \|x_k - l\| \geq \varepsilon\}|$$

$$= \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : \|x_k - l\| \geq \frac{\varepsilon}{\|\varphi\|}\right\}\right|$$

gives immediately $\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}| = 0$.

The reverse of Theorem 3.18 need not be correct, in general. This fact can be shown in the example below.

Example 3.19 [29] Consider the space $L_p(-1,1)$ for $p > 1$ and $\liminf \frac{h_r^\alpha}{h_r} > 0$. Then define $x_n\colon (-1,1) \to \mathbb{R}$ by

$$x_n(t) = \begin{cases} n^{1/p}, & t \in \left[0, \frac{1}{n}\right] \\ 0, & \text{otherwise} \end{cases}$$

then we have $W - \lim_{k\to\infty} x_k = 0$ in $L_p(-1,1)$. We have $WS_\theta^\alpha - \lim_{k\to\infty} x_k = 0$ from Theorem 3.17. Next we show that $S_\theta^\alpha - \lim_{k\to\infty} x_k$ does not hold. For $0 < \varepsilon < 1$ since $\|x_k\|_{L_p(-1,1)} = 1$ we have

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha}|\{k \in I_r : \ \|x_k - 0\| \geq \varepsilon\}| \neq 0.$$

Hence $S_\theta^\alpha - \lim_{k\to\infty} x_k = 0$ does not happen.

Theorem 3.20 [27] Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$. If $\liminf q_r > 1$, then $WS^\alpha \subset WS_\theta^\alpha$.

Proof: Assuming $\liminf q_r > 1$, then we can choose $\delta \in (0,\infty)$ such that $1 + \delta \leq q_r$ for sufficiently large r, so that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \text{ and } \frac{1}{k_r^\alpha} \geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $(x_k) \in WS^\alpha$, then for sufficiently large r and for any $\varepsilon > 0$, we may write

$$\frac{1}{k_r^\alpha}|\{k \leq k_r : |\varphi(x_k - l)| \geq \varepsilon\}| \geq \frac{1}{k_r^\alpha}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|$$

$$\geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|.$$

So, the sufficiency is proved.

Theorem 3.21 [27] Assume that $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$. If $\limsup q_r < \infty$, then $WS_\theta^\alpha \subset WS$.

Proof: If $\limsup q_r < \infty$, then we can choose a number $H > 0$ such that $H > q_r$ for all r. Let $(x_k) \in WS_\theta^\alpha$, and let $N_r = |\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|$. Then by using (3.1), given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{N_r}{h_r^\alpha} < \varepsilon \text{ and } \frac{N_r}{h_r} < \varepsilon$$

where $0 < \alpha \leq 1$. The rest of the proof is shown in Lemma 3 of [9].

Theorem 3.22 Assume that $\theta = (k_r)$ is a lacunary and $0 < \alpha \leq 1$. A sequence (x_k) is WS_θ^α-Cauchy sequence if and only if (x_k) is an WS_θ^α-convergent.

Proof: It is seen in a similar way, given by Theorem 2 of [30].

Theorem 3.23 [27] Assume that $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. If $\liminf_r q_r > 1$, then $W^\alpha \subset WN_\theta^\alpha$.

Proof: Let $\liminf_r q_r > 1$, then we can choose $\delta \in (0, \infty]$ such that $q_r \geq 1 + \delta$ for all $r \geq 1$. Then for $(x_n) \in W_0^\alpha$, we have

$$\tau_r^\alpha = \frac{1}{h_r^\alpha} \sum_{i=1}^{k_r} |\varphi(x_i)| - \frac{1}{h_r^\alpha} \sum_{i=1}^{k_{r-1}} |\varphi(x_i)|$$

$$= \frac{k_r^\alpha}{h_r^\alpha} \left(\frac{1}{k_r^\alpha} \sum_{i=1}^{k_r} |\varphi(x_i)| \right) - \frac{k_{r-1}^\alpha}{h_r^\alpha} \left(\frac{1}{k_{r-1}^\alpha} \sum_{i=1}^{k_{r-1}} |\varphi(x_i)| \right).$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r^\alpha}{h_r^\alpha} \leq \frac{(1+\delta)^\alpha}{\delta^\alpha} \text{ and } \frac{k_{r-1}^\alpha}{h_r^\alpha} \leq \frac{1}{\delta^\alpha}.$$

The terms $\frac{1}{k_r^\alpha} \sum_{i=1}^{k_r} |\varphi(x_i)|$ and $\frac{1}{k_{r-1}^\alpha} \sum_{i=1}^{k_{r-1}} |\varphi(x_i)|$ both converge to 0. This mean τ_r^α converges to 0, that is $(x_n) \in WN_{\theta,0}^\alpha$.

Theorem 3.24 [27] Assume that $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. If $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then $WN_\theta \subset W^\alpha$.

Proof: Suppose that $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then there is $M > 0$ and $\frac{k_r}{k_{r-1}^\alpha} < M$ for $r \geq 1$. Let $(x_n) \in WN_{\theta,0}$ and $\varepsilon > 0$, then we can take $R > 0$ and $K > 0$ such that $sup_{i \geq R} \tau_i < \varepsilon$ and $\tau_i < K$ for all $i \in \mathbb{N}$. Then if t is an integer with the condition $k_{r-1} < t < k_r$, where $r > R$, then we have

$$\frac{1}{t^\alpha} \sum_{i=1}^{t} |\varphi(x_i)| \leq \frac{1}{k_{r-1}^\alpha} \sum_{i=1}^{k_r} |\varphi(x_i)| = \frac{1}{k_{r-1}^\alpha} \left(\sum_{I_1} |\varphi(x_i)| + \sum_{I_2} |\varphi(x_i)| + \cdots + \sum_{I_r} |\varphi(x_i)| \right)$$

$$= \frac{k_1}{k_{r-1}^\alpha} \tau_1 + \frac{k_2 - k_1}{k_{r-1}^\alpha} \tau_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}^\alpha} \tau_R + \frac{k_{R+1} - k_R}{k_{r-1}^\alpha} \tau_{R+1} + \cdots$$

$$+\frac{k_r-k_{r-1}}{k_{r-1}^{\alpha}}\tau_r \le \left(\sup_{i\ge 1}\tau_i\right)\frac{k_R}{k_{r-1}^{\alpha}}+\left(\sup_{i\ge R}\tau_i\right)\frac{k_r-k_R}{k_{r-1}^{\alpha}} < K\frac{k_R}{k_{r-1}^{\alpha}}+\varepsilon M.$$

Since $k_{r-1}^{\alpha}\to\infty$ as $t\to\infty$, we have $\frac{1}{t^{\alpha}}\sum_{i=1}^{t}|\varphi(x_i)|\to 0$. Hence $(x_n)\in W_0^{\alpha}$.

Theorem 3.25 [27] Suppose $\theta=(k_r)$ and $\theta'=(s_r)$ are lacunary sequences such that $I_r\subseteq J_r$ for $r\in\mathbb{N}$ and $0<\alpha\le\beta\le 1$.

(i) $WN_{\theta'}^{\beta}\subset WN_{\theta}^{\beta}$ if (3.2) holds.

(ii) $WN_{\theta}^{\alpha}\subset WN_{\theta'}^{\beta}$ if (3.3) holds.

Proof:

(i) Omitted.

(ii) Assuming that $(x_k)\in WN_{\theta}^{\alpha}$ and that (3.3) holds. Since φ is bounded there exists $M>0$ such that $|\varphi(x_k-l)|\le M$ for each k. Since $I_r\subseteq J_r$ and $h_r\le \ell_r$ for every $r\in\mathbb{N}$, then we have

$$\frac{1}{\ell_r^{\beta}}\sum_{k\in J_r}|\varphi(x_k-l)|=\frac{1}{\ell_r^{\beta}}\sum_{k\in J_r-I_r}|\varphi(x_k-l)|+\frac{1}{\ell_r^{\beta}}\sum_{k\in I_r}|\varphi(x_k-l)|$$

$$\le\left(\frac{\ell_r-h_r}{\ell_r^{\beta}}\right)M+\frac{1}{\ell_r^{\beta}}\sum_{k\in I_r}|\varphi(x_k-l)|$$

$$\le\left(\frac{\ell_r-h_r^{\beta}}{h_r^{\beta}}\right)M+\frac{1}{h_r^{\beta}}\sum_{k\epsilon I_r}|\varphi(x_k-l)|$$

$$\le\left(\frac{\ell_r}{h_r^{\beta}}-1\right)M+\frac{1}{h_r^{\alpha}}\sum_{k\in I_r}|\varphi(x_k-l)|$$

for every $r\in\mathbb{N}$. Therefore $(x_k)\in WN_{\theta'}^{\beta}$.

The result below is obtained from Theorem 3.25.

Corollary 3.26 [27] Suppose $\theta=(k_r)$ and $\theta'=(s_r)$ are two lacunary sequences with the condition $I_r\subseteq J_r$ for every $r\in\mathbb{N}$. If (3.2) holds, then we have the following inclusions

(i) $WN_{\theta'}^{\alpha}\subseteq WN_{\theta}^{\alpha}$, for each $0<\alpha\le 1$,

(ii) $WN_{\theta'}\subseteq WN_{\theta}^{\alpha}$, for each $0<\alpha\le 1$,

(iii) $WN_{\theta'}\subseteq WN_{\theta}$.

If (3.3) holds, then

(i) $WN_{\theta}^{\alpha}\subset WN_{\theta'}^{\alpha}$, for each $0<\alpha\le 1$,

(ii) $WN_{\theta}^{\alpha}\subset WN_{\theta'}$, for each $0<\alpha\le 1$,

(iii) $WN_{\theta}\subset WN_{\theta'}$.

Theorem 3.27 [27] Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$.

(i) $WN_{\theta'}^{\beta} \subseteq WS_{\theta}^{\alpha}$ if (3.2) holds,

(ii) $WS_{\theta}^{\alpha} \subseteq WN_{\theta'}^{\beta}$ if (3.3) holds.

Proof:

(i) For every $\varepsilon > 0$ and sequence (x_k), we have

$$\sum_{k\in J_r} |\varphi(x_k - l)| = \sum_{\substack{k\in I_r \\ |\varphi(x_k-l)|\geq\varepsilon}} |\varphi(x_k - l)| + \sum_{\substack{k\in I_r \\ |\varphi(x_k-l)|<\varepsilon}} |\varphi(x_k - l)|$$

$$\geq \sum_{\substack{k\in I_r \\ |\varphi(x_k-l)|\geq\varepsilon}} |\varphi(x_k - l)| \geq |\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|\varepsilon$$

and so that

$$\frac{1}{\ell_r^{\beta}} \sum_{k\in J_r} |\varphi(x_k - l)| \geq \frac{1}{\ell_r^{\beta}} |\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|\varepsilon$$

$$\geq \frac{h_r^{\alpha} 1}{\ell_r^{\beta}\, h_r^{\alpha}} |\{k \in I_r : |\varphi(x_k - l)| \geq \varepsilon\}|\varepsilon.$$

So, if $(x_k) \in WN_{\theta'}^{\beta}$, then $(x_k) \in WS_{\theta}^{\alpha}$ since (3.2) holds.

(ii) Let $WS_{\theta}^{\alpha} - \lim_{k\to\infty} x_k = l$. Since φ is bounded then it is clear that there is $M > 0$ such that $|\varphi(x_k - l)| \leq M$ for all k, so that for every $\varepsilon > 0$, we have

$$\frac{1}{\ell_r^{\beta}} \sum_{k\in J_r} |\varphi(x_k - l)| = \frac{1}{\ell_r^{\beta}} \sum_{k\epsilon J_r - I_r} |\varphi(x_k - l)| + \frac{1}{\ell_r^{\beta}} \sum_{k\in I_r} |\varphi(x_k - l)|$$

$$\leq \left(\frac{\ell_r - h_r}{\ell_r^{\beta}}\right) M + \frac{1}{\ell_r^{\beta}} \sum_{k\epsilon I_r} |\varphi(x_k - l)| \leq \left(\frac{\ell_r - h_r^{\beta}}{\ell_r^{\beta}}\right) M + \frac{1}{\ell_r^{\beta}} \sum_{k\epsilon I_r} |\varphi(x_k - l)|$$

$$\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) M + \frac{1}{h_r^{\beta}} \sum_{\substack{k\in I_r \\ |\varphi(x_k-l)|\geq\varepsilon}} |\varphi(x_k - l)| + \frac{1}{h_r^{\beta}} \sum_{\substack{k\in I_r \\ |\varphi(x_k-l)|\geq\varepsilon}} |\varphi(x_k - l)|$$

$$\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) M + \frac{M}{h_r^{\beta}} |\{k\epsilon I_r : |\varphi(x_k - l)| \geq \varepsilon\}| + \frac{h_r}{h_r^{\beta}} \varepsilon$$

$$\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) M + \frac{M}{h_r^{\alpha}} |\{k \epsilon I_r : |\varphi(x_k - l)| \geq \varepsilon\}| + \frac{h_r}{h_r^{\beta}} \varepsilon$$

for all $r \in \mathbb{N}$. Using (3.3) we obtain that $WN_{\theta'}^{\beta} - \lim_{k\to\infty} x_k = l$, whenever $WS_{\theta}^{\alpha} - \lim_{k\to\infty} x_k = l$.

The result below is obtained from Theorem 3.27.

Corollary 3.28 Assume that $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If (3.2) holds, then

(i) $WN_{\theta'}^{\alpha} \subseteq WS_{\theta}^{\alpha}$,

(ii) $WN_{\theta'} \subseteq WS_{\theta}^{\alpha}$,

(iii) $WN_{\theta'} \subseteq WS_{\theta}$.

If (3.3) holds, then

(i) $WS_{\theta}^{\alpha} \subseteq WN_{\theta'}^{\alpha}$,

(ii) $WS_{\theta}^{\alpha} \subseteq WN_{\theta'}$,

(iii) $WS_{\theta} \subseteq WN_{\theta'}$.

4. Δ-LACUNARY WEAK STATISTICAL CONVERGENCE AND WEAK $N_\theta(\Delta)$-SUMMABILITY OF ORDER α

In this section, we are going to introduce the principles of Δ-lacunary weakly statistically convergent sequence of order α (or $WS_\theta^\alpha(\Delta)$-statistically convergent) and Δ-lacunary weak N-summability of order α (or $WN_\theta^\alpha(\Delta)$-summability) for $\alpha \in (0, 1]$. We are also going to give some characteristics of these two principles with some relations between them.

Definition 4.1 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named Δ-lacunary weakly statistically convergent sequence of order α (or $WS_\theta^\alpha(\Delta)$-statistically convergent) to l if, for any $\varphi \in X^*$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r: |\varphi(\Delta x_k - l)| \geq \varepsilon\}| = 0 \tag{4.1}$$

for every $\varepsilon > 0$. Let $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, that is, $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \dots, h_r^\alpha, \dots)$. In this case, we write $WS_\theta^\alpha(\Delta) - \lim_{k\to\infty} x_k = l$. The space of these sequences is denoted by $WS_\theta^\alpha(\Delta)$. Instead of $WS_\theta^\alpha(\Delta)$ we write $WS_\theta(\Delta)$ if $\alpha = 1$. And instead of $WS_\theta^\alpha(\Delta)$ we write $WS(\Delta)$ if $\alpha = 1$ and $\theta = (2^r)$.

Definition 4.2 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named weakly $N_\theta(\Delta)$-summable of order α to l if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} |\varphi(\Delta x_k - l)| = 0$$

and it is denoted by $WN_\theta^\alpha(\Delta) - \lim_{k\to\infty} x_k = l$. The class of $WN_\theta^\alpha(\Delta)$-summable sequences of order α is symbolized by $WN_\theta^\alpha(\Delta)$. Instead of $WN_\theta^\alpha(\Delta)$ we write $WN_\theta(\Delta)$ if $\alpha = 1$. We write $W^\alpha(\Delta)$ when $\theta = (k_r)$. Instead of $W^\alpha(\Delta)$ we write $W_0^\alpha(\Delta)$ if $l = 0$. The class of weakly $N_\theta(\Delta)$-summable sequence of order α to 0 is symbolized by $WN_{\theta,0}^\alpha(\Delta)$.

Definition 4.3 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named Δ-strongly lacunary statistically convergent of order α to l, if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r: \|\Delta x_k - l\| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. We write $S_\theta^\alpha(\Delta) - \lim_{k\to\infty} x_k = l$ in this situation.

Definition 4.4 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named an $WS_\theta^\alpha(\Delta)$-Cauchy sequence if there is a subsequence $(x_{k'(r)})$ of (x_k) such that $k'(r) \in I_r$ for every r, $W(\Delta) - \lim_{r\to\infty} x_{k'(r)} = l$

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r:\left|\varphi(\Delta x_k-\Delta x_{k'(r)})\right|\geq\varepsilon\}\right|=0$$

for each $\varphi\in X^*$ and every $\varepsilon>0$. This definition is similar to the definition WS_θ-Cauchy for the case $\alpha=1$.

It is clear that the $WS_\theta^\alpha(\Delta)$-statistically convergent is defined for $\alpha\in(0,1]$. But for $\alpha>1$ it is not defined. For this, if we take the functional $\varphi(x)>0$ on X, then for any number l we have

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}|\{k\in I_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}|=0$$

for every $\varepsilon>0$. This means that the $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k$ is not unique for $\alpha>1$.

Theorem 4.5 Let $\alpha\in(0,1]$ and $(x_k),(y_k)\in X$.

(i) If $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k=l$, then l must be unique.

(ii) If $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k=l_0$ and c being a scalar, then $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}cx_k=cl_0$.

(iii) If $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k=l_0$ and $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}y_k=l_1$, $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}(x_k+y_k)=l_0+l_1$.

Theorem 4.6 [27] Let $\theta=(k_r)$ and $\theta'=(s_r)$ be lacunary sequences such that for each $r\in\mathbb{N}$, $I_r\subset J_r$ and $0<\alpha\leq\beta\leq1$.

(i) If

$$\lim_{r\to\infty}\inf\frac{h_r^\alpha}{\ell_r^\beta}>0, \tag{4.2}$$

then $WS_{\theta'}^\beta(\Delta)\subseteq WS_\theta^\alpha(\Delta)$.

(ii) If

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^\beta}=1, \tag{4.3}$$

then $WS_\theta^\alpha(\Delta)\subseteq WS_{\theta'}^\beta(\Delta)$.

Proof:

(i) Assuming $I_r\subset J_r$ for each $r\in\mathbb{N}$ and let (4.2) be satisfied. For any $\varepsilon>0$,

$$\{k\in J_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}\supseteq\{k\in I_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}$$

and so

$$\frac{1}{\ell_r^{\beta}}|\{k \in J_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}| \geq \frac{h_r^{\alpha}}{\ell_r^{\beta}}\frac{1}{h_r^{\alpha}}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

for each $r \in \mathbb{N}$, where $h_r = k_r - k_{r-1}$, $I_r = (k_{r-1}, k_r]$, $j_r = (s_{r-1}, s_r]$ and $\ell_r = s_r - s_{r-1}$. By using (4.2) and taking the limit as $r \to \infty$ on both sides in the last inequality we obtain that $WS_{\theta}^{\beta}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$.

(ii) Assume that $(x_k) \in WS_{\theta}^{\alpha}(\Delta)$ and (4.3) is satisfied. Since $I_r \subset J_r$, then for every $\varepsilon > 0$, we have

$$\frac{1}{\ell_r^{\beta}}|\{k \in J_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

$$= \frac{1}{\ell_r^{\beta}}|\{s_{r-1} < k \leq k_{r-1} : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

$$+ \frac{1}{\ell_r^{\beta}}|\{k_r < k \leq s_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

$$+ \frac{1}{\ell_r^{\beta}}|\{k_{r-1} < k \leq k_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

$$\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) + \frac{1}{h_r^{\alpha}}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{\ell_r}{h_r^{\beta}} = 1$ by (4.3) the first term and since $(x_k) \in WS_{\theta}^{\alpha}(\Delta)$ the second term tends to 0 as $r \to \infty$ $\left(\left(\frac{\ell_r}{h_r^{\beta}} - 1\right) \geq 0\right)$. This implies that $WS_{\theta}^{\alpha}(\Delta) \subseteq WS_{\theta'}^{\beta}(\Delta)$. We have the following results from Theorem 4.6.

Corollary 4.7 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for each $r \in \mathbb{N}$. If (4.2) is hold, then,

(i) $WS_{\theta'}^{\alpha}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$ for $0 < \alpha \leq 1$,

(ii) $WS_{\theta'}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$ for $0 < \alpha \leq 1$,

(iii) $WS_{\theta'}(\Delta) \subseteq WS_{\theta}(\Delta)$.

If (4.3) is hold, then

(i) $WS_{\theta}^{\alpha}(\Delta) \subseteq WS_{\theta'}^{\alpha}(\Delta)$ for $0 < \alpha \leq 1$,

(ii) $WS_{\theta}^{\alpha}(\Delta) \subseteq WS_{\theta'}(\Delta)$ for $0 < \alpha \leq 1$,

(iii) $WS_\theta(\Delta) \subseteq WS_{\theta'}(\Delta)$.

Corollary 4.8 Suppose $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. Then $WS_\theta^\alpha(\Delta) \subseteq WS_\theta^\beta(\Delta)$.

Corollary 4.9 Suppose $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. Then $WS_\theta^\alpha(\Delta) \subseteq WS_\theta(\Delta)$.

Corollary 4.10

(i) $WS_\theta^\alpha(\Delta) = WS_\theta^\beta(\Delta)$ if $\alpha = \beta$,

(ii) $WS_\theta^\alpha(\Delta) = WS_\theta(\Delta)$ if $\alpha = 1$.

Theorem 4.11 Suppose $\theta = (k_r)$ is a lacunary sequence, $0 < \alpha \leq 1$ and $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{h_r} > 0$. If there exists $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta_\theta(\Delta)(K) = 1$ and $\lim_{k\in K} \varphi(\Delta x_k - l) = 0$ for any $\varphi \in X^*$, then (x_k) is $WS_\theta^\alpha(\Delta) -$ convergent to l.

Proof: Assume that $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{h_r} > 0$ holds and there exists a set $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ with $\delta_\theta(\Delta)(K) = 1$ and $\lim_{k\in K} \varphi(\Delta x_k - l) = 0$ for $\varphi \in X^*$. Then there is a number $N \in \mathbb{N}$ such that $|\varphi(\Delta x_k - l)| < \varepsilon$ for all $k \geq N$ and for any $\varphi \in X^*$.

If we take $M_\varepsilon = \{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\} \subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, k_{N+3}, \ldots\}$, then $\delta_\theta(\Delta)(M_\varepsilon) = 0$. Using Corollary 4.7 (ii) we get that (x_k) is $WS_\theta^\alpha(\Delta)$-convergent to l.

Theorem 4.12 Let $x = (x_k) \in w$. If $W(\Delta) - \lim_{k\to\infty} x_k = l$, then $WS_\theta(\Delta) - \lim_{k\to\infty} x_k = l$.

Proof: Let $W(\Delta) - \lim_{k\to\infty} x_k = l$, then for each $\varepsilon > 0$ and any $\varphi \in X^*$ if there is a number $N \in \mathbb{N}$ such that

$$|\varphi(\Delta x_k - l)| < \varepsilon$$

for each $k \geq N$. So the set $M(\varepsilon) = \{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}$ is finite; for which $\delta_\theta(\Delta)\big(M(\varepsilon)\big) = 0$. This shows that $WS_\theta(\Delta) - \lim_{k\to\infty} x_k = l$.

In general, the opposite of the above result is not correct. To show that we can use Example 3.16.

Theorem 4.13 Assume that $\theta = (k_r)$ is a lacunary sequence, $0 < \alpha \leq 1$, $x = (x_k) \in w$ and $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{h_r} > 0$. If $W(\Delta) - \lim_{r\to\infty} x_k = l$, then $WS_\theta^\alpha(\Delta) - \lim_{r\to\infty} x_k = l$.

It can be easily proved from corollary 4.7 (ii) and theorem 4.12

Theorem 4.14 Let $0 < \alpha \leq 1$. Then $S_\theta^\alpha(\Delta) \subset WS_\theta^\alpha(\Delta)$ for any lacunary sequence $\theta = (k_r)$ with the same limit in X.

Proof: Let $(x_k) \in X$ such that $x_k \xrightarrow{S_\theta^\alpha(\Delta)} l$. Then for each $\varepsilon > 0$,

$$\frac{1}{h_r^\alpha}|\{k \in I_r : \|\Delta x_k - l\| \geq \varepsilon\}| = 0.$$

Now, for each $\varepsilon > 0$ and any $\varphi \in X^*$, the expression

$$\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}| \leq \frac{1}{h_r^\alpha}|\{k \in I_r : \|\varphi\|\|\Delta x_k - l\| \geq \varepsilon\}|$$

$$= \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : \|\Delta x_k - l\| \geq \frac{\varepsilon}{\|\varphi\|}\right\}\right|,$$

gives immediately $\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(\Delta\, x_k - l)| \geq \varepsilon\}| = 0.$

The converse is not true, in general. To show that we can use Example 3.19.

Theorem 4.15 Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$. If $\liminf q_r > 1$, then, $WS^\alpha(\Delta) \subset WS_\theta^\alpha(\Delta)$.

Proof: Assuming $\liminf q_r > 1$, then we can choose $\delta(0, \infty)$ such that $1 + \delta \leq q_r$ for sufficiently large r, so that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \text{ and } \frac{1}{k_r^\alpha} \geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $(x_k) \in WS^\alpha(\Delta)$, then for sufficiently large r and for any $\varepsilon > 0$, and we have

$$\frac{1}{k_r^\alpha}|\{k \leq k_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}| \geq \frac{1}{k_r^\alpha}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

$$\geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|.$$

So, the sufficiency is proved.

Theorem 4.16 Assume that $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$. If $\limsup q_r < \infty$, then $WS_\theta^\alpha(\Delta) \subset WS(\Delta)$.

Proof: If $\limsup q_r < \infty$, then we can choose a number $H > 0$ such that $H > q_r$ for all r. Let $(x_k) \in WS_\theta^\alpha(\Delta)$, and let $N_r = |\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$. Then by using (4.1), given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{N_r}{h_r^\alpha} < \varepsilon \text{ and } \frac{N_r}{h_r} < \varepsilon$$

where $0 < \alpha \leq 1$. The rest of proof is seen in(Lemma 3) from [30].

Theorem 4.17 Assume that $\theta = (k_r)$ is a lacunary and $0 < \alpha \leq 1$. A sequence (x_k) is $WS_\theta^\alpha(\Delta)$-Cauchy sequence if and only if (x_k) is an $WS_\theta^\alpha(\Delta)$-convergent.

Theorem 4.18 Assume that $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. If $\liminf_r q_r > 1$, then $W^\alpha(\Delta) \subset WN_\theta^\alpha(\Delta)$.

The proof is similar to Theorem 3.23.

Theorem 4.19 Assume that $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. If $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then $WN_\theta(\Delta) \subset W^\alpha(\Delta)$.

Theorem 4.20 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$.

(i) $WN_{\theta'}^\beta(\Delta) \subset WN_\theta^\beta(\Delta)$ if (4.2) holds.

(ii) $WN_\theta^\alpha(\Delta) \subset WN_{\theta'}^\beta(\Delta)$ if (4.3) holds.

Proof:

(i) Omitted.

(ii) Assuming that $(x_k) \in WN_\theta^\alpha(\Delta)$ and that (4.3) holds. Since φ is bounded there exists $M > 0$ such that $|\varphi(\Delta x_k - l)| \leq M$ for each k. We can find inclusion from.

$$\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|\varphi(\Delta x_k - l)| = \frac{1}{\ell_r^\beta}\sum_{k\in J_r - I_r}|\varphi(\Delta x_k - l)| + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|\varphi(\Delta x_k - l)|$$

$$\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right)M + \frac{1}{h_r^\alpha}\sum_{k\in I_r}|\varphi(\Delta x_k - l)|$$

for every $r \in \mathbb{N}$. Therefore $(x_k) \in WN_{\theta'}^\beta(\Delta)$.

The result below is obtained from Theorem 4.20.

Corollary 4.21 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequences with the condition $I_r \subseteq J_r$ for every $r \in \mathbb{N}$. If (4.2) holds, then we have the following inclusions

(i) $WN_{\theta'}^\alpha(\Delta) \subseteq WN_\theta^\alpha(\Delta)$, for each $0 < \alpha \leq 1$,

(ii) $WN_{\theta'}(\Delta) \subseteq WN_\theta^\alpha(\Delta)$, for each $0 < \alpha \leq 1$,

(iii) $WN_{\theta'}(\Delta) \subseteq WN_\theta(\Delta)$.

If (4.3) holds, then

(i) $WN_\theta^\alpha(\Delta) \subset WN_{\theta'}^\alpha(\Delta)$, for each $\alpha \in (0, 1]$,

(ii) $WN_\theta^\alpha(\Delta) \subset WN_{\theta'}(\Delta)$, for each $\alpha \in (0, 1]$,

(iii) $WN_\theta(\Delta) \subset WN_{\theta'}(\Delta)$.

Theorem 4.22 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β are fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.

(i) $WN_{\theta'}^{\beta}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$ if (4.2) holds,

(ii) $WS_{\theta}^{\alpha}(\Delta) \subseteq WN_{\theta'}^{\beta}(\Delta)$ if (4.3) holds.

The proof is similar to Theorem 3.27.

Corollary 4.23 Assume that $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If (4.2) holds, then

(i) $WN_{\theta'}^{\alpha}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$,

(ii) $WN_{\theta'}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$,

(iii) $WN_{\theta'}(\Delta) \subseteq WS_{\theta}(\Delta)$.

If (4.3) holds, then

(i) $WS_{\theta}^{\alpha}(\Delta) \subseteq WN_{\theta'}^{\alpha}(\Delta)$,

(ii) $WS_{\theta}^{\alpha}(\Delta) \subseteq WN_{\theta'}(\Delta)$,

(iii) $WS_{\theta}(\Delta) \subseteq WN_{\theta'}(\Delta)$.

5. CONCLUSION

In this thesis, we have given the natural density, statistical convergence, weak convergence, weak statistical convergence, some concepts of statistical convergence of order α for $0 < \alpha \leq 1$, and strong p-Cesàro summability of order α for $0 < \alpha \leq 1$. We have also given, some relations between statistical convergence of order α and strong p-Cesàro summability of order α for $0 < \alpha \leq 1$.

Furthermore, we have given lacunary weak statistical convergence of order α and weak N_θ^α-summability for $0 < \alpha \leq 1$.

Finally, we established the Δ-lacunary weak statistical convergence of order α and weak $N_\theta^\alpha(\Delta)$-summability for $0 < \alpha \leq 1$. We have also given some inclusion relations between these concepts.

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