LANGLANDS FUNCTORIALITY PRINCIPLE

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Table of Contents

List of Symbols

K, L, F, E	:	Mostly local / global fields.
v	:	Valuation on a local field.
O_v	:	Valuation ring of the valued field (K, v).
$\mathfrak{p}_v$	:	Maximal ideal of the valuation ring.
κ_v	:	Residue field of the valued field (K, v).
π_K	:	Uniformizer element of the valued field (K, v).
U_v	:	Unit group of the valued field (K, v).
$\|\|\cdot\|\|_v$	:	Absolute value on thevalued field (K, v).
$\widehat{K}$	:	Completion of the field K.
$e_{L/K}$	:	Ramification index of the local field extension L/K.
$f_{L/K}$	:	Residual degree of the local field extension L/K.
K^{nr}	:	Maximal unramified extension of the local field K .
$\overline{K}$	:	Fixed algebraic closure of the field K.
K^{sep}	:	Fixed seperable closure of the field K.
$Gal(K^{sep}/K)$	:	Absolute Galois group of the field K.
I_K	:	Inertia group of the field K.
Art_K	:	Artin reciprocity law of the field K.
$N_{L/K}$	:	Norm group of the extension L/K.
$V_{G_K \to G_L}$	:	Transfer homomorphism from K to L.
$\mathfrak{h}_K$	:	Set of finite (henselian) places of the field K.
$\mathfrak{a}_K$	:	Set of infinite (archimedean) places of the field K.
$\mathbb{A}_K$	:	Adele ring of the global field K.
$\mathbb{J}_K$	:	Idele group of the global field K.
W_K	:	Weil group of the field K.

$Frob_K$	:	Geometric Frobenius substitution on the field K.
Fr_K	:	Geometric Frobenius element on the field K.
G	:	A locally profinite / algebraic group.
Z_G	:	Centre of the group G.
WD_K	:	Weil-Deligne group of a non-archimedean local field K.
K^{tame}	:	Maximal tamely ramified extension of the local field K.
$R(G)$	:	Radical of an algebraic group G.
$R_u(G)$	:	Unipotent radical of an algebraic group G.
T	:	An algebraic torus.
$\mathbb{G}_m$	:	Multiplicative group scheme.
B	:	Usually a Borel subgroup.
P	:	Parabolic subgroup.
$\mathfrak{g}$	:	Lie algebra.
$\Psi(G,T)$	:	Root data of a given tuple (G,T).
$U(\mathfrak{g}_{\mathbb{C}})$	:	Universal enveloping algebra of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$.
δ_P	:	Modular quasi-character function attached to parabolic subgroup P.
$\mathcal{A}_G$	:	Space of automorphic forms on G.
$C_c^\infty(G)$	:	Space of compactly supported smooth functions on G.
$H(G)$	:	Hecke algebra of G.
${}^LG$	:	Langlands L-group of G.
$L(s,\Pi,r)$	:	Automorphic L-function attached to the pair (Π,r).
$\mathbb{L}_F$	:	Automorphic Langlands group of a global field F.

Abstract

In this thesis we aimed to study functoriality principle of Langlands which is an important part of the Langlands programme, regarded as the grand unified theory connecting number theory and harmonic analysis.

With the introduction, this thesis consists of eight chapters. In the first chapter, we studied basics of local and global abelian class field theories in which one can see Langlands conjectures as non-abelian versions of these two theories.

In the second chapter we investigated Weil groups and Weil-Deligne representations which are generalizations of Galois groups and Galois representations in various important cases. In particular their relation with ℓ-adic Galois representations are investigated.

In the third and fourth chapter, we studied basic theory of algebraic groups and automorphic representations both archimedean and non-archimedean setting which constitutes primary objects to state Langlands conjectures. Their relation with Lie algebras and root systems are investigated. Also Hecke algebras associated to locally profinite groups studied and Flath's decomposition theorem is proved.

In the fifth and sixth chapter, L -groups of Langlands associated to connected reductive groups constructed and L-homomorphism between them are defined. L-functions and ε-factors associated to Weil-Deligne and automorphic representations are studied. We defined automorphic L-functions and studied the Langlands-Shahidi method with the basic theory of Eisenstein series.

In the seventh chapter, local and global functoriality principle of Langlands are stated. The automorphic Langlands group of a global field defined and its excepted properties are stated. Last but not least, some examples of functoriality principle provided.

Keywords: Functoriality, Reciprocity, Weil-Deligne Representation, Reductive group, Automorphic representation, L-group, Automorphic L-function.

Özet

Bu tezde, sayılar teorisi ile harmonik analizi birleştiren büyük birleşik teori olarak kabul edilen, Langlands programının önemli bir parçası olan Langlands fonktörsellik ilkesini çalışmayı amaçladık.

Giriş kısmı ile birlikte, bu tez sekiz bölümden oluşmaktadır. Birinci bölümde, Langlands varsayımlarını abelyen olmayan durumları olarak görebileceğimiz yerel ve global abelyen sınıf cisim kuramlarının temel kavramları çalışıldı.

İkinci bölümde Galois gruplarını ve Galois temsillerinin çeşitli önemli durumlarda genellemeleri olan Weil grupları ve Weil-Deligne temsillerini inceledik. Özellikle ℓ-adik Galois temsilleri ile ilişkileri araştırıldı.

Üçüncü ve dördüncü bölümde, Langlands sanılarında ifade etmede öncelikli objeler olan cebirsel grupların temel teorisi ve otomorfik temsillerin arşimetyen ve arşimetyen olmayan teorisi çalışıldı. Bu objelerin Lie cebirleri ve kök sistemleri ile ilişkisi incelendi. Ayrıca yerel olarak profinite olan grupların Hecke cebirleri çalışıldı ve Flath'ın ayrışma teoremi ispatlandı.

Beşinci ve altınca bölümlerde, bağlantılı indirgeyici gruplarla ilişkili Langlands'ın L-grubunu inşa edildi ve aralarındaki L-homomorfizmaları tanımlandı. Weil-Deligne ve otomorfik temsillere ilişkin L-fonksiyonları ve ε-faktörleri tanımlandı. Otomorfik L-fonksiyonları tanımlandı ve Eisenstein serilerinin temel teorisi ile birlikte, Langlands-Shahidi methodu çalıştıldı.

Yedinci bölümde, Langlands'ın lokal ve global fonktörsellik ilkeleri belirtildi. Bir global cismin otomorfik Langlands grubu tanımlandı ve sağlaması beklenen özellikleri belirtildi. En son olarak, fonktörsellik ilkesinin bazı örnekleri verildi.

Anahtar Sözcükler: Fonktörsellik, Karşılıklılık, Weil-Deligne Temsili, İndirgeyici grup, Otomorfik temsil, L-grubu, Otomorfik L-fonksiyonu.

Chapter 1

Introduction

Historically, the Langlands programme considered to started with a seventeen page latter by Robert Langlands to French mathematician Andre Weil in January 1967. Roughly speaking, in this letter, Robert Langlands initially presented his ideas on comparision of automorphic representations of different groups by means of non-abelian harmonic analysis with infinite dimensional representations of Lie groups. This was the first time the functoriality appears in the mathematical literature. But this was not the only purpose of Langlands. At that time algebraic number theorists was searching for non-abelian version of class field theory. Emil Artin proposed non-abelian reciprocity laws by means of Artin L-series and Artin representations, which generalizations of proposal on functoriality principle, contains this approach as a very special case. Moreover, he formed his ideas by the influence of Hecke's L-series as well as those of Artin's L-series. One of the important developments in this vein was the concept of the L-group, which provides a meaningful notion of automorphic L-functions, which has Euler product expression, and effectuates generalization of Hecke and Artin's work. In this thesis, we follow ideas of Langlands, as well as other mathematicians, in order to understand automorphic forms by the aid of number theoretic and representation theoretic tools. Our study possesses both algebraic and analytic structures, as well as detailed study of L-functions attached to various groups. We also show the compatibility of local / global reciprocity laws of Langlands with the general principle of Functoriality and how to obtain Artin's non-abelian reciprocity law from the machinery of the Lie theory of algebraic groups and the theory of Galois representations. Indeed, this particular phenomena shows us why the Langlands program offers a reasonable non-abelian generalization of class field theory.

1.1 Review of Local Class Field Theory

In this part, we introduce basic concepts of local class field theory. Our main goal is to express the Artin reciprocity law, constitutes fundamental theorem of this section.

Definition 1.1.1. *Let K be a field. The map $v : K \to \mathbb{R} \cup \{\infty\}$ is called a valuation if the following conditions are satisfied :*

1) For $x \in K$, $v(x) = \infty$ if and only if $x = 0_K$.

2) For $x, y \in K$,

$$v(xy) = v(x) + v(y).$$

3) (Ultrametric property) For $x, y \in K$,

$$v(x+y) \geq min\{v(x), v(y)\}.$$

The pair (K, v) is called a valued field. If $v(K^{\times}) \subseteq \mathbb{Z}$, then $v : K \to \mathbb{R} \cup \{\infty\}$ is called a discrete valuation . In particular, if $v(K^{\times}) = \mathbb{Z}$, then it is called normalized.

Remark 1.1.1. *1) By the second property in the definition, we have*

$$v(1_K) = v(1_K) + v(1_K)$$

and hence $v(1_K) = 0$. Therefore we have

$$v(1_K) = v((-1_K) \cdot (-1_K)) = v(-1_K) + v(-1_K) = 0.$$

so $v(-1_K) = -v(-1_K)$. Thus we have $v(-1_K) = v(1_K) = 0$.

2) Let $x, y \in K$, so that $v(y) > v(x)$. Then we have

$$v(x+y) \geq min\{v(x), v(y)\} = v(x) = v(x+y-y) \geq min\{v(x+y), v(-y)\}$$

Note that if $v(x+y) > v(-y)$, then

$$v(x) = v(x+y-y) \geq min\{v(x+y), v(-y)\} = v(-y) = v(-1_K) + v(y) = v(y) > v(x)$$

which is a contradiction. Therefore $min\{v(x+y), v(-y)\} = v(x+y)$ *and*

$$v(x+y) = v(x).$$

Definition 1.1.2. *(Neukirch, 1999) The subring of K given by*

$$O_v = \{x \in K : v(x) \geq 0\}$$

called valuation ring of K. It is a local ring with only maximal ideal

$$\mathfrak{p}_v = \{x \in K : v(x) > 0\}.$$

In particular, $O_v - \mathfrak{p}_v$ *is the group of invertible elements of* O_v*, denoted by* U_v*, and is called the unit group of K. The quotient*

$$\kappa_v = O_v/\mathfrak{p}_v$$

is called residue field of (K, v)*. One may likewise put symbols* $\mathfrak{p}_K$*,* O_K*,* κ_K *instead of* $\mathfrak{p}_v$*,* O_v*,* κ_v *respectively.*

Remark 1.1.2. *Indeed, to show the valuation ring* O_v *is a local ring, we need to show that the subring* $\mathfrak{p}_v$ *consists of all non-invertible elements of* O_v *. But any* $x \in O_v^\times$ *satisfies* $v(x) \geq 0$ *and*

$$v(x^{-1}) = v(1_K) - v(x) = -v(x) \geq 0.$$

so $v(x) = 0$*. Hence* O_v *is a local ring with unique maximal ideal* $\mathfrak{p}_v$*.*

Assumption Henceforth, we will assume the valuation on (K, v) under consideration is discrete and normalized. Hence there exists an element $\pi_K \in K$ so that

$$v(\pi_K) = 1.$$

We call such an element π_K, a uniformizing element (or a uniformizer) of K.

Remark 1.1.3. *Each* $x \in K^\times$ *has a unique decomposition as* $x = u \cdot \pi_K^n$ *for some* $u \in U_v$ *and* $n \in \mathbb{Z}$*. Indeed if* $v(x) = n$*, then* $v(x \cdot \pi_K^{-n}) = 0$ *implies* $x \cdot \pi_K^n \in U_v$ *and* $x = u \cdot \pi_K^n$ *for some* $u \in U_v$*. Moreover if,* $x = u_1 \cdot \pi_K^n = u_2 \cdot \pi_K^m$*, for* $u_1, u_2 \in U_v$ *and* $n, m \in \mathbb{Z}$*, then*

$u_2^{-1} \cdot u_1 \cdot \pi_K^{n-m} = 1_K$ and by remark 1.1.1, we have $v(u_2^{-1} \cdot u_1 \cdot \pi_K^{n-m}) = v(1_K) = 0$, therefore

$$v(u_2^{-1}) + v(u_1) + v(\pi_K^{n-m}) = -v(u_2) + v(u_1) + n - m = 0.$$

But since $u_1, u_2 \in U_v$, so $v(u_1) = v(u_2) = 0$, and hence $n = m$. This implies $u_1 = u_2$ and we see that any $x \in K^\times$ has one and only one factorization in the shape $x = u \cdot \pi_K^n$ for an appropriate $u \in U_v$ and $n \in \mathbb{Z}$.

Theorem 1.1.1. *Suppose (K, v) be a valued field. Then the valuation ring*

$$O_v = \{x \in K : v(x) \geq 0\}.$$

is a principal ideal domain. Moreover the nonzero ideals of O_v are given by

$$\mathfrak{p}_v^n = \pi_K^n \cdot O_v = \{x \in K : v(x) \geq n\}, \quad n \geq 1$$

where π_K is a uniformizer of K and an isomorphism

$$\mathfrak{p}_v^n / \mathfrak{p}_v^{n+1} \simeq O_v / \mathfrak{p}$$

of $O_v/\mathfrak{p}$-modules.

Proof. Let $0 \neq I$ be an ideal of O_v. By the well-ordering principle of positive integers, let $0 \neq x \in I$ element with smallest appropriate value $v(x) = n$ among the elements of I. Then $v(x \cdot \pi_K^{-n}) = 0$ implies $x \cdot \pi_K^{-n} = u$ for some $u \in U_v$ and hence $\pi_K^n \cdot O_v \subseteq I$. Conversely, if $y = u_1 \cdot \pi_K^m \in I$ be an arbitrary element, with $u_1 \in U_v$, then by assumption we have $m = v(y) \geq v(x) = n$. Therefore,

$$y = (u_1 \cdot \pi_K^{m-n}) \cdot \pi_K^n \in \pi_K^n \cdot O_v$$

so $\pi_K^n \cdot O_v = I$. Define a map of rings

$$\upsilon_n : \mathfrak{p}_v^n / \mathfrak{p}_v^{n+1} \to O_v / \mathfrak{p}_v$$

by the rule

$$a \cdot \pi_K^n \to a \; mod \; \mathfrak{p}_v = \bar{a}$$

Clearly it is a module homomorphism and is surjective. To see the injectivity, suppose $a \; mod \; \mathfrak{p}_v = \bar{a} = b \; mod \; \mathfrak{p}_v = \bar{b}$. We may assume $\bar{a}, \bar{b} \neq 0$. This equivalently to say

$a, b \in U_v$. Then since $O_v/\mathfrak{p}_v$ is a field, we have $\bar{a} \cdot \bar{b}^{-1} = 1 mod\ \mathfrak{p}_v$, in the level of coset representatives and hence $a \cdot b^{-1} = 1 + \pi_K \cdot u$ for some $u \in U_v$. Therefore

$$a \cdot \pi_K^n = b \cdot \pi_K^n + \pi_K^{n+1} \cdot u$$

so we have

$$a \cdot \pi_K^n \equiv b \cdot \pi_K^n \ \ mod \ \ \mathfrak{p}_v^{n+1}$$

This shows the injectivity and hence $\upsilon_n : \mathfrak{p}_v^n/\mathfrak{p}_v^{n+1} \to O_v/\mathfrak{p}_v$ is an isomorphism.

□

Definition 1.1.3. *Let (K, v) be a valued field. We define higher unit groups U_v^n of (K, v) as*

$$U_v^n = 1 + \mathfrak{p}_v^n = \{1 + \pi_K^n \cdot x \mid x \in O_v\}$$

for all $n \in \mathbb{N}$.

Proposition 1.1.1. *Let (K, v) be a valued field. Then one has following isomorphisms*

$$O_v^\times/U_v^n \xrightarrow{\sim} (O_v/\mathfrak{p}_v^n)^\times, \quad U_v^n/U_v^{n+1} \xrightarrow{\sim} O_v/\mathfrak{p}_v$$

Proof. For the first assertion, consider the following canonical surjective homomorphism

$$f_n : O_v^\times \to (O_v/\mathfrak{p}_v^n)^\times$$

$$x \to x\ mod\ \mathfrak{p}_v^n$$

Note that, the inclusion $\mathfrak{p}_v^n \subset \mathfrak{p}_v$ implies $U_v = O_v - \mathfrak{p}_v \subset O_v - \mathfrak{p}_v^n$, so f_n is not the zero homomorphism. In particular, for some $x \in O_v^\times$, one has $x^{-1} \in O_v^\times$, therefore $x^{-1}\ mod\ \mathfrak{p}_v^n \in (O_v/\mathfrak{p}_v^n)^\times$. If $x \in ker(f_n)$ then $x \equiv 1 mod\ \mathfrak{p}_v^n$ hence $x = 1 + \pi_K^n \cdot y$, for some $y \in O_v$. This shows that $ker(f_n) = U_v^n$. For the second, consider the canonical surjective homomorphism

$$f^n : U_v^n \to O_v/\mathfrak{p}_v$$

$$1 + \pi_K^n \cdot x \to x\ mod\ \mathfrak{p}_v$$

If $1 + \pi_K^n \cdot x \in ker(f^n)$ then $x \in \mathfrak{p}_v$ so $x = \pi_K \cdot y$ for some $y \in O_v$. This implies that $1 + \pi_K^n \cdot x = 1 + \pi_K^n \cdot \pi_K \cdot y = 1 + \pi_K^{n+1} \cdot y$. Thus $ker(f^n) = U_v^{n+1}$.

□

Let (K,v) be a valued field with $d \in (0,1)$ be some real number. We define the mapping

$$||\cdot||_v : K \to \mathbb{R}$$

$$||x||_v = d^{v(x)}$$

which, by definition, satisfies $\forall x, y \in K$,

$$||x+y||_v = d^{v(x+y)} \leq d^{min(v(x),v(y))} = max\{||x||_v, ||y||_v\}.$$

For $d \in (0,1)$ we have

$$||0_K||_v = d^{v(0_K)} = 0(= d^{\infty}).$$

and

$$||xy||_v = d^{v(xy)} = d^{v(x)+v(y)} = d^{v(x)} \cdot d^{v(y)} = ||x||_v \cdot ||y||_v$$

Therefore, the mapping $||\cdot||_v$, defines a non-archimedean absolute value on K. By using it, one considers the following metric on (K,v),

$$d_v : K \times K \to \mathbb{R}.$$

$$d_v(x,y) = ||x-y||_v, \quad \forall x, y \in K.$$

In particular, for any $x \in K$ and $0 < \varepsilon \in \mathbb{R}$, consider the open ball of radius ε, as usual by

$$B_\varepsilon(x) = \{y \in K \mid ||x-y||_v < \varepsilon\}$$

where the family $\{B_\varepsilon(x) \mid \varepsilon \in \mathbb{R}_{>0}, x \in K\}$ forms a basis for the topology. We call the topology, induced from metric d_v, the v-adic topology on (K,v). In particular for any $k \in \mathbb{Z}$, we can re-characterize O_v-submodules of K as

$$\pi_K^k \cdot O_v = \{x \in K \mid v(x) \geq k > k-1\} = \{x \in K \mid ||x||_v < d^{k-1}\} = B_{d^{k-1}}(0_K).$$

O_v-submodules $\pi_K^k \cdot O_v$ of K are both open and closed subsets with respect to v-adic topology and effectuates basis of open sets around 0_K. Therefore K equipped with v-adic topology is a non-discrete, totally disconnected topological field. In particular, higher unit groups U_v^n of (K,v) effectuates basis of open sets of $1_K \in K^\times$ and defines the following descending filtration on U_v,

$$U_v \supseteq U_v^1 \supseteq U_v^2 \supseteq U_v^3 \supseteq \cdots \supseteq \{1\}$$

Definition 1.1.4. *Let (K,v) be some valued field, endowed via v-adic topology. The sequence $(a_k)_{k\in\mathbb{N}}$ with entries in K called convergent if for some $n_0 \in \mathbb{N}$, $(a_k)_{k\in\mathbb{N}}$ bounded from below by some $\alpha \in \mathbb{Z}$ i.e $v(a_n) > \alpha$ for all $n \geq n_0$. In particular it is called a Cauchy sequence if for every $r \in \mathbb{Z}$, there exists $n_\varepsilon \in \mathbb{N}$ so that $v(a_n - a_m) \geq r$ for all $n,m \geq n_\varepsilon$. In particular, (K,v) is called complete, if all Cauchy sequences in K has a limit in K.*

Definition 1.1.5. *Let (K,v) be some valued field. A discretely valued field $(\hat{K},\hat{v})$ is called completion of K if it is complete, $\hat{v}|_K = v$ and K is a dense subset of $\hat{K}$ with respect to $\hat{v}$.*

Proposition 1.1.2. *(Fesenko & Vostokov, 1993) Every discrete valuation field K has a completion which is unique up to an isometric isomorphism over K.*

Lemma 1.1.1. *The set $A(K)$ of all Cauchy sequences in the valued field (K,v), forms a ring with respect to componentwise addition and multiplication. The sequences $(a_k)_{k\in\mathbb{N}} \in A(K)$ tending to 0 forms a maximal ideal $M(K)$ of $A(K)$. The quotient*

$$\widehat{K} = A(K)/N(K)$$

is a discretely valued field with its discrete valuation $\hat{v} : \widehat{K} \to \mathbb{Z}$ defined as

$$\hat{v}((a_n)_{\mathbb{N}} + N(K)) = \lim_{n\to\infty} v(a_n)$$

for $(a_k)_{k\in\mathbb{N}} \in A(K)$. Moreover, $\widehat{K}$ is complete with respect to $\hat{v} : \widehat{K} \to \mathbb{Z}$. The pair $(\widehat{K},\hat{v})$ is called the v-adic completion of K.

Proof. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in K$ be two Cauchy sequences. Hence for all $r \in \mathbb{R}$ there exists $n_a \in \mathbb{N}$ and $n_b \in \mathbb{N}$ so that for all $n,m \geq n_a$ and $j,k \geq n_b$, we have

$$v(a_n - a_m) \geq n_a, \quad v(b_j - b_k) \geq n_b$$

We set $n_0 = min\{n_a,n_b\}$. Then for the sequence $(a_n + b_n)_{n\in\mathbb{N}} \in K$, we have

$$v((a_n + b_n) - (a_m + b_m)) = v((a_n - a_m) + (b_n - b_m)) \geq min\{v(a_n - a_m), v(b_n - b_m)\}$$

Since both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences, hence for every $r\in\mathbb{R}$, and $k,l\geq n_0$ we have

$$v((a_k+b_k)-(a_l+b_l))\geq r$$

This shows that componentwise sum of two Cauchy sequence is a Cauchy sequence. Since $(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}\in K$ are Cauchy sequences, for all $\alpha,\beta\in\mathbb{Z}$, there exists $n_a,n_b\in\mathbb{N}$ so that we have $v(a_n-a_m)\geq\alpha$ and $v(b_n-b_m)\geq\beta$ for $n,m\geq max\{n_a,n_b\}$. Therefore we have

$$v(a_n\cdot b_n-a_m\cdot b_m)\geq min\{v(a_n-a_m)+v(b_n),v(b_n-b_m)+v(a_m)\}$$

$$\geq min\{\alpha+v(b_n),v(a_m)+\beta\}$$

Hence we see that the componentwise product of two Cauchy sequence is also a Cauchy sequence. Now let $(c_n)_{n\in\mathbb{N}}\in A(K)$ be a Cauchy sequence so that $(c_n)_{n\in\mathbb{N}}\notin N(K)$. This means there is some $n_0\geq 0$ so that $c_n\neq 0$ for all $n\geq n_0$. Construct a sequence $(d_n)_{n\in\mathbb{N}}\in A(K)$ with the following formula :

$$d_n=0,\ \ n<n_0,\quad d_n=c_n^{-1},\ \ n\geq n_0$$

Then we have

$$1-(c_n)\cdot(d_n)=(1,1,\cdots,1,0,0,0,\cdots)\in N(K)$$

which last non-zero entry is (n_0-1)-th entry. This shows that $(1-(c_n)\cdot(d_n))_{n\in\mathbb{N}}$ is a null-sequence and hence $N(K)$ forms a maximal ideal of $A(K)$.

□

Remark 1.1.4. *Let (K,v) be some valued field and let $(\widehat{K},\hat{v})$ be its v-adic completion. Then :*

- $\hat{v}(\widehat{K}^{\times})=\hat{v}(K^{\times})$.

- *For any $n>0$, one has*

$$O_{\hat{v}}/\mathfrak{p}_{\hat{v}}^n\simeq O_v/\mathfrak{p}_v^n$$

where $O_{\hat{v}}$ is the valuation ring of $(\widehat{K},\hat{v})$ with $\mathfrak{p}_{\hat{v}}$ be its maximal ideal.

- *Let $R \subset O_v$ be some coset representatives of κ_v. For $x \in \widehat{K}$ there exists one and only one expression*

$$x = \sum_{n \geq n_0} t_n \pi_K^n$$

for $n_0 \in \mathbb{Z}$ and $t_n \in R$.

Indeed, assume $x = \pi_K^{n_0} \cdot u$ with $u \in O_{\hat{v}}$. Since $O_{\hat{v}}/\mathfrak{p}_{\hat{v}}^n \simeq O_v/\mathfrak{p}_v^n$, the class $u \ mod \ \mathfrak{p}_{\hat{v}}$ has a unique representative $0 \neq t_0 \in R$. Therefore we have $u = t_0 + \pi_K \cdot s_1$ for some $s_1 \in O_{\hat{v}}$. Assume that there exists elements $t_0, \cdots, t_{n-1} \in R$ satisfying

$$u = t_0 + t_1 \cdot \pi_K + \cdots + t_{n-1} \cdot \pi_K^{n-1} + \pi_K^n \cdot s_n$$

for some $s_n \in O_{\hat{v}}$ and that t_i's uniquely induced from that phrase . Then any representative $t_n \in R$ of $s_n \ mod \ \pi_K O_{\hat{v}} \in O_{\hat{v}}/\mathfrak{p}_{\hat{v}} \simeq O_v/\mathfrak{p}$ is again determined uniquely by u and hence we have $s_n = t_n + \pi_K \cdot s_{n+1}$ for some $s_{n+1} \in O_{\hat{v}}$. Therefore we have

$$u = t_0 + t_1 \cdot \pi_K + \cdots + t_{n-1} \cdot \pi_K^{n-1} + t_n \cdot \pi_K^n + \pi_K^{n+1} \cdot s_{n+1}$$

By this process we obtain an infinite series $\sum_{k=0}^{\infty} t_k \cdot \pi_K^k$ which is determined uniquely by u. Since the reminder term tends to 0, it converges to u, hence

$$x = \sum_{n \geq n_0} t_n \pi_K^n$$

Definition 1.1.6. *A discrete valuation field K, which is complete and has finite residue field is called a nonarchimedean local field. $\mathbb{R}$ and $\mathbb{C}$ are called archimedean local fields.*

Remark 1.1.5. *Now suppose K be a local field and let L/K be some finite extension. We define normalized discrete valuation on L by*

$$v_L : L \to \mathbb{Z} \cup \{\infty\}$$

$$x \to \frac{1}{f_{L/K}}(N_{L/K}(x)), \quad x \in L$$

for an appropriate constant $f_{L/K}$. In particular L is a local field with respect to valuation v_L. We have the following properties :

1) $v_L(K^\times)$ *is an additive subgroup of* $v_L(L^\times)$*.*

2) The residue field κ_L *is a finite extension of* κ_K*. Moreover one has* $f_{L/K} = [\kappa_L : \kappa_K]$*.*

Definition 1.1.7. *Given any finite extension* L/K*, the index*

$$e_{L/K} = (v_L(L^\times) : v_L(K^\times))$$

is called the ramification index of L/K*. The number of the residue field extension*

$$f_{L/K} = [\kappa_L : \kappa_K]$$

is called the residual degree of L/K*.*

Remark 1.1.6. *Given any finite extension* L/K*, we have*

$$\pi_K = \pi_L^{e_{L/K}} \cdot t, \quad t \in U_L$$

and the fundamental equality

$$e_{L/K} \cdot f_{L/K} = [L : K]$$

In particular, for a chain $K \subset L \subset M$ *of finite extensions of local fields, following transitivity relations holds*

$$e_{M/K} = e_{M/L} \cdot e_{L/K}$$

$$f_{M/K} = f_{M/L} \cdot f_{L/K}$$

In addition if $e_{L/K} = 1$ *(resp.* $f_{L/K} = 1$*), the extension* L/K *is unramified (resp. totally ramified). In particular* L/K *is tamely ramified (resp. wildly ramified) if* κ_L/κ_K *is a seperable extension and* $char(\kappa_K) \nmid e_{L/K}$ *(resp.* $char(\kappa_K) | e_{L/K}$*).*

Remark 1.1.7. *Suppose* K *be a local field. For* $n \in \mathbb{Z}_+$*, there is a unique unramified extension* K_n^{nr} *of* K *with degree* n*. Since* $e_{K_n^{nr}/K} = 1$*, we have the equality* $[K_n^{nr} : K] = [\kappa_{K_n^{nr}} : \kappa_K]$*. In particular, the following mapping*

$$Gal(K_n^{nr}/K) \to Gal(\kappa_{K_n^{nr}}/\kappa_K)$$

$$\varphi \to \bar{\varphi}$$

$$\varphi(x)\ mod\ \mathfrak{p}_{K_n^{nr}} = \bar{\varphi}(x\ mod\ \mathfrak{p}_{K_n^{nr}})$$

is an isomorphism. Moreover if $q = p^f = |\kappa_K|$, *then* $Gal(\kappa_{K_n^{nr}}/\kappa_K)$ *generated by*

$$Frob_n(x) = x^q$$

for $x \in \kappa_{K_n^{nr}}$.

Definition 1.1.8. *For a local field* K *and* K_n^{nr}/K *unique unramified extension with degree* n*, the generator* $Frob_n^{-1}$ *of* $Gal(K_n^{nr}/K)$ *is called geometric Frobenius element of the unramified extension* K_n^{nr}/K*. In particular, the generator* $Frob_n$ *of the Galois group* $Gal(\kappa_{K_n^{nr}}/\kappa_K)$ *is called the arithmetic Frobenius element of the unramified extension* K_n^{nr}/K.

Suppose K be a local field and L/K be unramified extension of degree n. Furthermore, suppose M/L be an unramified extension of degree m. Then by Remark 1.1.5, we have a transitivity on ramification degrees

$$e_{M/K} = e_{M/L} \cdot e_{L/K} = 1 \cdot 1 = 1$$

Therefore M/K is an unramified extension of degree $n \cdot m$. In particular,

$$K^{nr} = \bigcup_{n \geq 1} K_n^{nr}$$

be an unramified extension of K. Note that, we have $K_n^{nr} \subset K_{n'}^{nr}$ if and only if $n \mid n'$. In particular we have a projective system of Galois groups

$$(Gal(K_n^{nr}/K), p_n^m : Gal(K_m^{nr}/K) \to Gal(K_n^{nr}/K))_{n,m \in \mathbb{N}}$$

where $p_n^m : Gal(K_m^{nr}/K) \to Gal(K_n^{nr}/K)$ is a connecting morphism defined as the restriction homomorphism, when $n|m$, for all $n, m \in \mathbb{N}$. By taking projective limit of this system, we have

$$Gal(K^{nr}/K) = \varprojlim Gal(K_n^{nr}/K) \xrightarrow{\sim} \varprojlim Gal(\kappa_{K_n^{nr}}/\kappa_K) \xrightarrow{\sim} \varprojlim \mathbb{Z}/n\mathbb{Z} \simeq \hat{\mathbb{Z}}$$

Definition 1.1.9. *The field*

$$K^{nr} = \bigcup_{n \geq 1} K_n^{nr}$$

is called the maximal unramified extension of K. In particular topological generator $Frob_K = (Frob_n)_{n\geq 1}$ of $Gal(K^{nr}/K)$ is called the Frobenius element of K^{nr}/K.

Remark 1.1.8. *By infinite Galois theory, the subgroup of $Gal(K^{sep}/K)$, which corresponds to maximal unramified extension K^{nr}/K is*

$$I_K = Gal(K^{sep}/K^{nr})$$

which is called the inertia subgroup of the absolute Galois group $Gal(K^{sep}/K)$.

1.1.1 Local Artin Reciprocity

In this subsection we will introduce the local Artin reciprocity principle. In other words, we will study the fundamental theorem of local class field theory.

Let K be a non-archimedean local field. The local Artin reciprocity law is a topological group homomorphism

$$Art_K : K^\times \to Gal(K^{ab}/K)$$

satisfying the following properties :

(1) For each finite extension L/K with $\alpha \in L^\times$, one has

$$Art_L(\alpha)|_{K^{ab}} = Art_K(N_{L/K}(\alpha)).$$

Here $N_{L/K} : L^\times \to K^\times$ is the norm homomorphism from L to K.

(2) For each finite extension L/K with $\alpha \in K^\times$, one has the equality

$$V_{G_K\to G_L}(Art_K(\alpha)) = Art_L(\alpha).$$

Here the map $V_{G_K\to G_L} : Gal(K^{ab}/K) \to Gal(L^{ab}/L)$ is the transfer "Verlagerung" homomorphism from K to L.

(3) For any $\sigma \in Aut(K^{sep})$ and $\alpha \in K^\times$, one has the following :

$$Art_{K^\sigma}(\alpha^\sigma) = \sigma \cdot Art_K(\alpha) \cdot \sigma^{-1}.$$

(4) For each finite abelian extension L/K, composition homomorphism

$$\theta_{L/K} : K^{\times} \xrightarrow{Art_K} Gal(K^{ab}/K) \xrightarrow{res_L} Gal(L/K)$$

is surjective and $ker(\theta_{L/K}) = N_{L/K}\,(L^{\times})$. Therefore

$$\theta_{L/K} : K^{\times}/N_{L/K}\,(L^{\times}) \xrightarrow{\sim} Gal(L/K).$$

(5) By taking projective limits in (4) on both sides

$$\varprojlim_{L/K} \theta_{L/K} : \varprojlim_{L/K} K^{\times}/N_{L/K}\,(L^{\times}) \xrightarrow{\sim} \varprojlim_{L/K} Gal(L/K)$$

one has topological group isomorphism

$$\widehat{Art_K} : \widehat{K^{\times}} \xrightarrow{\sim} \varprojlim_{L/K} K^{\times}/N_{L/K}(L^{\times}) \xrightarrow{\sim} Gal(K^{ab}/K)$$

(6) There exists bijections between finite index subgroups of $K^{\times}$, finite index open subgroups of $Gal(K^{ab}/K)$ and finite extensions of K in K^{sep} given by

$$Ker(\theta_{L/K}) \leftrightarrows Gal(K^{ab}/L) \leftrightarrows L/K$$

(7) (Ramification Theorem) There exists a bijection between higher unit groups of K and higher ramification subgroups in $Gal(K^{ab}/K)$ as

$$\{U_K^i \leq U_K \leq K^{\times} \mid 0 \leq i \in \mathbb{Z}\} \leftrightarrows \{Gal(K^{ab}/K)^v \leq Gal(K^{ab}/K) \mid -1 \leq v \in \mathbb{R}\}$$

1.2 Review of Global Class Field Theory

In this part, we shall introduce basic concepts of global class field theory. By using local theory developed in previous section, we shall express global Artin reciprocity principle which stands as a fundamental theorem of this part.

Definition 1.2.1. *A global field is either a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_q(t)$, with $q = p^r$ for some prime $p \in \mathbb{Z}$, $r \in \mathbb{N}$. In this instance, we say it is a number field or a global function field respectively.*

Suppose K be a global field. Let $\mathfrak{h}_K$, $\mathfrak{a}_K$ denote sets of finite (henselian) and infinite (archimedean) places of K respectively. If K is a global function field, then $\mathfrak{a}_K = \emptyset$. In particular, for any $v \in \mathfrak{h}_K \cup \mathfrak{a}_K$, we denote by K_v as the associated local field. If v is non-archimedean, one shall denote valuation ring of K_v by O_v and for an archimedean v, we simply define $K_v = O_v$.

Definition 1.2.2. *Let K be a global field. The adele ring $\mathbb{A}_K$ of K is the topological ring defined as the restricted product*

$$\mathbb{A}_K := \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K}' (K_v, O_v)$$

$$= \{(x_v)_v \in \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K} K_v \mid x_v \in O_v \; for\ all\ but\ finitely\ many\ v \in \mathfrak{h}_K \cup \mathfrak{a}_K\}$$

where addition and multiplication are defined as the component-wise addition and multiplication. In particular the idele group of K is a topological group defined as the restricted product

$$\mathbb{J}_K = \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K}' (K_v^\times, O_v^\times)$$

$$= \{(x_v)_v \in \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K} K_v^\times \mid x_v \in O_v^\times \; for\ all\ but\ finitely\ many\ v \in \mathfrak{h}_K \cup \mathfrak{a}_K\}$$

which can be seen as the subgroup $\mathbb{A}_K^\times$ of units of $\mathbb{A}_K$ endowed with the restricted product topology (which is not the topology induced from the restricted product topology on $\mathbb{A}_K$).

For each place $v \in \mathfrak{h}_K \cup \mathfrak{a}_K$ of K, natural embeddings $K \hookrightarrow K_v$ determines an embedding

$$K \hookrightarrow \mathbb{A}_K$$

$$x \to (x, x, \cdots)$$

The image of K in $\mathbb{A}_K$ determines the subring of principal adeles of K. In particular for each $v \in \mathfrak{h}_K \cup \mathfrak{a}_K$, we can extend the absolute value $||\cdot||_v$ on K_v to an absolute value $||\cdot||$ on $\mathbb{A}_K$ by defining :

$$||x||_v = ||x_v||_v, \quad x = (x_v)_v \in \mathbb{A}_K$$

Definition 1.2.3. *The adelic absolute value $||\cdot||$ on $\mathbb{A}_K$ is defined by*

$$||x|| = \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K} ||x||_v \in \mathbb{R}_{\geq 0}$$

for every $x = (x_v)_v \in \mathbb{A}_K$. Note that this is a well defined definition, as the product is convergent, because $||x||_v \leq 1$ for almost all v.

Let L/K be a finite seperable extensions of the global field K. The natural embedding $K \hookrightarrow \mathbb{A}_K$ makes $\mathbb{A}_K$ a topological K-vector space. Thus the base change

$$\mathbb{A}_K \otimes_K L$$

is also a topological L-vector space. Since it has finite dimension as a K-vector space, the topology on $\mathbb{A}_K \otimes_K L$ is the product topology on $\mathbb{A}_K^{[L:K]}$.

Theorem 1.2.1. *Suppose L/K be a finite seperable extension of the global field K. Then there is an isomorphism of topological rings*

$$\mathbb{A}_L \simeq \mathbb{A}_K \otimes_K L$$

and the diagram below commutes :

$$\begin{tikzcd} L \arrow[r] \arrow[d] & K \otimes_K L \arrow[d] \\ \mathbb{A}_L \arrow[r] & \mathbb{A}_K \otimes_K L \end{tikzcd}$$

Proof. (Sketch) By the definition of adele ring $\mathbb{A}_K$, we have the following isomorphism

$$\mathbb{A}_K \otimes_K L = \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K}^{\prime} (K_v, O_v) \otimes_K L \xrightarrow{\sim} \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K}^{\prime} (K_v \otimes_K L, O_v \otimes_{O_K} O_L)$$

$$(x_v)_v \otimes y \to (x_v \otimes y)_v$$

of topological rings which is also a homeomorphism of topological rings. On the other hand, for any place $w \in \mathfrak{h}_L \cup \mathfrak{a}_L$ of L, lying over a place v of K, $w|v$, one has

$$K_v \otimes_K L \simeq \prod_{w|v} L_w, \quad O_v \otimes_{O_K} O_L \simeq \prod_{w|v} O_w$$

Therefore we have the following equivalent expressions

$$\mathbb{A}_K \otimes_K L \simeq \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K}^{\prime} (K_v \otimes_K L, O_v \otimes_{O_K} O_L) \simeq \prod_{w \in \mathfrak{h}_L \cup \mathfrak{a}_L}^{\prime} (L_w, O_w) = \mathbb{A}_L$$

which are topological ring isomorphisms. For the second claim, we simply chase the given diagram as if $y \in L$, then its image in $\mathbb{A}_K \otimes_K L$ is $1 \otimes y = (1, 1, \cdots) \otimes y$. In particular, by the isomorphism $\mathbb{A}_L \simeq \mathbb{A}_K \otimes_K L$ of topological rings, the image of $1 \otimes y = (1, 1, \cdots) \otimes y$ in $\mathbb{A}_L$ is $(y, y, \cdots)$. This shows the commutativity of the diagram.

$\square$

Suppose K be a global field. Then we have surjective homomorphism

$$\varphi : \mathbb{J}_K \to I_K$$

$$x = (x_v)_v \to \prod_v \mathfrak{p}_v^{v(x_v)}$$

where I_K is the group of invertible fractionals of O_K which is by definition the group of non-zero O_K-submodules of K and $\mathfrak{p}_v$ is the prime ideal in O_K corresponding the valuation v. Note that, since for almost all v, $x_v \in O_v^\times$ and hence $v(x_v) = 0$ for almost all v. This observation yields that the product is finite over non-zero prime ideals of O_K.

Definition 1.2.4. *Let K be a global field. The quotient group $\mathbb{I}_K/K^\times$ is called idele class group of K.*

In particular following diagram commutes :

$$\begin{tikzcd}
1 \arrow[r] & K^\times \arrow[r] \arrow[d, "x \to (x)"] & \mathbb{I}_K \arrow[r] \arrow[d, "\varphi"] & \mathbb{I}_K/K^\times \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{P}_K \arrow[r] & I_K \arrow[r] & Cl_K \arrow[r] & 1
\end{tikzcd}$$

where $\mathcal{P}_K$ is the group of principal fractional ideals in O_K which is by definition the group of non-zero O_K-modules generated by a single element and $Cl_K = I_K/\mathcal{P}_K$ is the ideal class group of K.

Definition 1.2.5. *Let L/K be a finite seperable extension of the global field K. The idele norm is a group homomorphism $N_{L/K} : \mathbb{I}_L \to \mathbb{I}_K$ defined for each $(y_w)_w \in \mathbb{I}_L$, by the formula*

$$N_{L/K}((y_w)_w) = ((x_v)_v)$$

$$x_v = \prod_{w|v} N_{L_w/K_v}(y_w)$$

where $N_{L_w/K_v} : L_w \to K_v$ is the norm of finite seperable extensions L_w/K_v for each place w and v of L and K with $w|v$ respectively.

In particular by restricting the idele norm $N_{L/K} : \mathbb{J}_L \to \mathbb{J}_K$ onto the subgroup of principle ideles $L^\times \subset \mathbb{J}_L$, we obtain nothing but the usual norm $N_{L/K} : L^\times \to K^\times$ and the following diagram commutes :

$$\begin{tikzcd} L^\times \arrow[r] \arrow[d, "N_{L/K}"'] & \mathbb{J}_L \arrow[r] \arrow[d, "N_{L/K}"] & I_L \arrow[d, "N^\star_{L/K}"] \\ K^\times \arrow[r] & \mathbb{J}_L \arrow[r] & I_K \end{tikzcd}$$

where $N^\star_{L/K} : I_L \to I_K$ is the usual ideal norm defined by $N^\star_{L/K}(\mathfrak{q}) = \mathfrak{p}^{[O_L/\mathfrak{q}:O_K/\mathfrak{p}]}$, for non-zero prime ideals $\mathfrak{q}$ in O_L with $\mathfrak{p} = \mathfrak{q} \cap O_K$.

1.2.1 Global Artin Reciprocity

In this subsection we will introduce the global Artin reciprocity principle. In other words, we will study the fundamental theorem of global class field theory.

Let K be a global field. The global Artin reciprocity law is the unique topological group homomorphism

$$Art_K : \mathbb{J}_K/K^\times \to Gal(K^{ab}/K)$$

with properties :

(1) For each finite extension L/K with $\alpha \in \mathbb{J}_L/L^\times$, one has

$$Art_L(\alpha)|_{K^{ab}} = Art_K(N_{L/K}(\alpha)).$$

(2) For each finite extension L/K with $\alpha \in \mathbb{J}_K/K^\times$, the following equality

$$V_{G_K \to G_L}(Art_K(\alpha)) = Art_L(\alpha)$$

holds. Here the map $V_{G_K \to G_L} : Gal(K^{ab}/K) \to Gal(L^{ab}/L)$ is the transfer ”Verlagerung” homomorphism from K to L.

(3) For any $\sigma \in Aut(K^{sep})$ and $\alpha \in \mathbb{J}_K/K^\times$, the following relation holds :

$$Art_{K^\sigma}(\alpha^\sigma) = \sigma \cdot Art_K(\alpha) \cdot \sigma^{-1}.$$

(4) For each finite abelian L/K, the composition

$$\theta_{L/K} : \mathbb{J}_K/K^\times \xrightarrow{Art_K} Gal(K^{ab}/K) \xrightarrow{res_L} Gal(L/K)$$

is surjective with $ker(\theta_{L/K}) = N_{L/K}\,(\mathbb{J}_L/L^\times)$. There exists an isomorphism

$$\theta_{L/K} : \mathbb{J}_K/K^\times/N_{L/K}\,(\mathbb{J}_L/L^\times) \xrightarrow{\sim} Gal(L/K).$$

(5) By taking projective limits

$$\varprojlim_{L/K} \theta_{L/K} : \varprojlim_{L/K} \mathbb{J}_K/K^\times/N_{L/K}\,(\mathbb{J}_L/L^\times) \xrightarrow{\sim} \varprojlim_{L/K} Gal(L/K)$$

we have a topological group isomorphism

$$\widehat{Art_K} : \widehat{\mathbb{J}_K/K^\times} \xrightarrow{\sim} \varprojlim_{L/K} \mathbb{J}_K/K^\times/N_{L/K}\,(\mathbb{J}_L/L^\times) \xrightarrow{\sim} Gal(K^{ab}/K).$$

6) For any finite index open subgroup $H \leq \mathbb{J}_K/K^\times$, there is a unique finite abelian extension L/K so that $N_{L/K}(\mathbb{J}_L/L^\times) = H$. In particular one has an inclusion reversing bijection

$$H = N_{L/K}(\mathbb{J}_L/L^\times) = ker(\theta_{L/K}) \Leftrightarrow Gal(K^{ab}/L) \leftrightarrows L/K$$

between finite index open subgroups H of $\mathbb{J}_K/K^\times$ and finite abelian extensions L/K of K in K^{ab}.

(7) (Ramification Theorem) Let K be a number field. Let $\mathcal{M}_K = \mathfrak{h}_K \cup \mathfrak{a}_K$ be the set of all places of K and $\mathfrak{m} : \mathcal{M}_K \to \mathbb{Z}_{\geq 0}$ be a modulus for K, for which $\mathfrak{m} = \prod_{v \in \mathcal{M}_K} v^{e_v}$ with $e_v \leq 1$ when v is archimedean and $e_v = 0$ for v, a infinite complex prime. We define the following subgroups; $U_{\mathfrak{m},v} := U_v$ if $e_v = 0$, $U_{\mathfrak{m},v} := U_v^{e_v}$ if $n_\mathfrak{p} > 0$ and $v \in \mathfrak{h}_K$ and $U_{\mathfrak{m},v} = \mathbb{R}_{>0}$ if $e_v > 0$ and v is a real place and put $U_\mathfrak{m} = \prod_{v \in \mathfrak{h}_K \cup \mathfrak{a}_K} U_{\mathfrak{m},\mathfrak{p}} \leq \mathbb{J}_K$. Under the canonical surjection $\pi_K : \mathbb{J}_K \to \mathbb{J}_K/K^\times$, the image $\pi_K(U_\mathfrak{m}) = \overline{U_\mathfrak{m}}$ is a finite index open subgroup of $\mathbb{J}_K/K^\times$ and there exist an abelian $K_\mathfrak{m}/K$ in K^{ab} so that one has the following isomorphisms

$$C_K/\overline{U_\mathfrak{m}} \xrightarrow{\sim} \mathbb{J}_K/(U_\mathfrak{m} \cdot K^\times) \xrightarrow{\sim} Gal(K_\mathfrak{m}/K)$$

Chapter 2

Weil Groups and Weil-Deligne Representations

Here, we shall axiomatically define Weil groups of both local / global fields which occupy an important place in Langlands philosophy. We will study some primary principles of representation theory of these groups in order to derive machinery for representation theoretic features on Galois side. Finally, we shall introduce Weil-Deligne groups for non-archimedean local fields and study the representations of these groups.

2.1 Axiomatic Construction of Weil Groups

Let G be some topological group. Let G^c denotes the closure of the first derived subgroup of G with $G^{ab} = G/G^c$ be the maximal abelian Hausdorff quotient of G.

Definition 2.1.1. *Let H be some closed subgroup in G of finite index. The map*

$$t_H^G : G^{ab} \to H^{ab}.$$

is said transfer homomorphism if for a given section $s : H\backslash G \to G$, one has

$$t_H^G(gG^c) = \prod_{x \in H\backslash G} h_{g,x} \ \ (mod\ H^c)$$

with $g \in G$ and $h_{g,x} \in H$ so that $s(x) \cdot g = h_{g,x} \cdot s(x \cdot g)$.

Remark 2.1.1. *Indeed transfer homomorphism $t_H^G : G^{ab} \to H^{ab}$ does not depend on any choice of section $s : H\backslash G \to G$. In fact, let $s' : H\backslash G \to G$ be another section of the map* *of sets*

$$prj_H^G : G \to H\backslash G, \quad g \to Hg.$$

Fact is, G acts on set of right cosets $H\backslash G$ by $g_1 \cdot (Hg_2) = H(g_1g_2)$, for $g_1, g_2 \in G$. Let $C = \{g_1, g_2, \cdots, g_n\}$ be some set of representatives for cosets in $H\backslash G$ i.e $G = \bigcup_{i=1}^{n} Hg_i$ where $(G : H) = n$. For simplicity, let us interpret the set C with the set $\{1, 2, \cdots n\}$. Then by the definition of the transfer homomorphism, we have $h_{g,i} = \frac{s(Hg_i)g}{s(Hg_ig)}$ and $h'_{g,i} = \frac{s'(Hg_i)g}{s'(Hg_ig)}$. where $h_{g,i} = h_{g,Hg_i}$ and $h'_{g,i} = h'_{g,Hg_i}$ are elements of H. Since s and s' are sections of the mapping $prj_H^G : G \to H\backslash G$, we have $s \circ prj_H^G = s' \circ prj_H^G = id_G$. Hence for any $g_i \in C$, we calculate

$$s \circ prj_H^G(g_i) = s(prj_H^G(g_i)) = s(Hg_i) = g_i$$

and

$$s' \circ prj_H^G(g_i) = s'(prj_H^G(g_i)) = s'(Hg_i) = g_i$$

Therefore we have

$$h_{g,i} = \frac{s(Hg_i)g}{s(Hg_ig)} = \frac{s'(Hg_i)g}{s'(Hg_ig)} = h'_{g,i}$$

for every $i \in \{1, 2, \cdots n\}$. Thus we see that transfer homomorphism $t_H^G : G^{ab} \to H^{ab}$ is independent of the choice of the section $s : H\backslash G \to G$.

Let K be a local or a global field and $\overline{K}$ be a fixed a seperable algebraic closure of K. Let $R_K = \{K \subset E \subset \overline{K} \mid [E : K] < \infty\}$ be the set of all finite and seperable extensions of K in $\overline{K}$.

Definition 2.1.2. *A Weil group for $\overline{K}/K$ is a tripartite $(W_K, \phi^K, (Art_E)_{E \in R_K})$, where*

(1) W_K is a certain topological group.

(2) $\phi^K : W_K \to Gal(\overline{K}/K)$ is a continuous homomorphism with a dense image.

(3) For each $E \in R_K$, Art_E is an isomorphism induced from local/global Artin reciprocity law, as for local field extensions E over K,

$$Art_E : E^\times \xrightarrow{\sim} W_E^{ab}$$

and for global field extensions E over K

$$Art_E : \mathbb{J}_E/E^\times \xrightarrow{\sim} W_E^{ab}.$$

Let $W_E = (\phi^K)^{-1}(Gal(\overline{K}/E))$. *As* $\phi^K : W_K \to Gal(\overline{K}/K)$ *is continuous,* W_E *is an open subgroup of* W_K. *For simplicity, we will use the following notation, by the aid of Artin reciprocity law, for our fields* $E \in R_K$,

$$S_E = \begin{cases} E^\times & \textit{if } E \textit{ is local} \\ \mathbb{I}_E/E^\times & \textit{if } E \textit{ is global} \end{cases}$$

In order W_K *to form a Weil group, these components must satisfy the following conditions :*

(W1) Suppose E/K *be some finite extension of local fields. The composition*

$$Art_E : E^\times \xrightarrow{\sim} W_E^{ab} \xrightarrow{(\phi^K)|_{W_E}^{ab}} G_E^{ab}$$

is the local Artin reciprocity law. In particular if E/K *is some finite extension of global fields then composition*

$$Art_E : \mathbb{I}_E/E^\times \xrightarrow{\sim} W_E^{ab} \xrightarrow{(\phi^K)|_{W_E}^{ab}} G_E^{ab}$$

is the global Artin reciprocity law.

(W2) Suppose $w \in W_K$ *and* $\sigma = \phi^K(w) \in Gal(\overline{K}/K)$. *Then for every* $E \in R_K$, *below diagram commutes :*

$$\begin{tikzcd} S_E \arrow[r] \arrow[d, "ind.by\ \sigma"'] & W_E^{ab} \arrow[d, "conj.by\ w"] \\ S_{E^\sigma} \arrow[r] & W_{E^\sigma}^{ab} \end{tikzcd}$$

(W3) For any $E, L \in R_K$ *with* $L \subset E$, *the following diagram*

$$\begin{tikzcd} S_L \arrow[r] \arrow[d, "ind.by\ L \subset E"'] & W_L^{ab} \arrow[d, "t_L^E"] \\ S_E \arrow[r] & W_E^{ab} \end{tikzcd}$$

is commutative where

$$t_L^E : W_L^{ab} \to W_E^{ab}$$

is the transfer homomorphism in the sense of Definition 2.1.1.

(W4) The map

$$W_K \to \lim_{\leftarrow} W_K/W_E^c$$

is an isomorphism of topological groups where projective limit taken over elements $E \in R_K$, ordered by inclusion. More precisely, if $E, L \in R_K$ with $L \subset E$, then connecting homomorphisms are given by usual restriction maps :

$$pr_L^E : W_K/W_E^c \to W_K/W_L^c$$

Therefore W_K is the projective limit of the inverse system

$$((W_K/W_E^c)_{E\in R_K}, (pr_L^E)_{L\subset E}))$$

Remark 2.1.2. *By having a dense image of the homomorphism $\phi^K(W_K) \subset Gal(\overline{K}/K)$, we shall mean that ϕ^K induces a bijection between homogeneous spaces*

$$W_K/W_E \xrightarrow{\sim} Gal(\overline{K}/K)/Gal(\overline{K}/E) \simeq Hom_K(E, \overline{K}).$$

If E/K is a Galois extension, then the continuous homomorphism $\phi^K : W_K \to Gal(\overline{K}/K)$ induces an isomorphism

$$W_K/W_E \xrightarrow{\sim} Gal(E/K)$$

Theorem 2.1.1. *Let W_K be a Weil group for $\overline{K}/K$. Then for each $E, L \in R_K$, with $L \subset E$, the following diagram*

$$\begin{tikzcd} S_E \arrow[r, "Art_E"] \arrow[d, "Norm\ N_{E/L}"'] & W_E^{ab} \arrow[d, "ind.by\ W_E \subset W_L"] \\ S_L \arrow[r, "Art_L"] & W_L^{ab} \end{tikzcd}$$

commutes.

Proof. We have

$$W_L/W_E \xrightarrow{\sim} Gal(\overline{K}/L)/Gal(\overline{K}/E) \simeq Hom_L(E, \overline{K})$$

Thus W_E is of finite index in W_L hence W_E^{ab} is a finite index normal subgroup of W_L^{ab}.

The composition map

$$g_E : W_E^{ab} \xrightarrow{ind.by\ inclusion} W_L^{ab} \xrightarrow{transfer} W_E^{ab}$$

takes any element $w \in W_E$ into the product of its conjugates by the representatives of the quotient W_L/W_E, i.e

$$g_E(w \cdot W_E^c) = \prod_{\sigma \in Hom_L(E,\overline{K})} w^\sigma$$

Therefore the left vertical arrow is compatible with the norm map $N_{L/K} : S_E \to S_L$. Since both horizontal arrows are isomorphisms by Artin reciprocity, this shows that aforementioned diagram is commutative.

□

2.2 Weil Groups of Local Fields

In this part, we shall investigate algebraic and topological properties of Weil groups of local fields. We will treat non-archimedean and archimedean cases seperately.

Suppose K be some non-archimedean local field with $G_K = Gal(\overline{K}/K)$ be the absolute Galois group of K for a fixed seperable algebraic closure $\overline{K}$. Let K^{nr} be the maximal unramified extension of K, and $I_K = Gal(\overline{K}/K^{nr})$ be the inertia subgroup of G_K. As in Chapter 1, κ_K be the residue field of K of order $q_K = p^r$ for some prime $p \in \mathbb{Z}$ and $r \in \mathbb{N}$.

In the first chapter, we observe that, for every $n \in \mathbb{Z}_+$, there exists a unique cyclic unramified extension K_n^{nr} of K of degree n and isomorphisms

$$Gal(K_n^{nr}/K) \xrightarrow{\sim} Gal(\kappa_{K_n^{nr}}/\kappa_K) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}.$$

Therefore by taking projective limits on both sides, under the divisibility of the extension degrees, we have

$$Gal(K^{nr}/K) = \varprojlim Gal(K_n^{nr}/K) \xrightarrow{\sim} \varprojlim Gal(\kappa_{K_n^{nr}}/\kappa_K) \xrightarrow{\sim} Gal(\overline{\kappa_K}/\kappa_K) \xrightarrow{\sim} \varprojlim \mathbb{Z}/n\mathbb{Z} \simeq \widehat{\mathbb{Z}}$$

so

$$\eta : Gal(K^{nr}/K) \xrightarrow{\sim} Gal(\overline{\kappa_K}/\kappa_K)$$

and one has below exact sequence

$$1 \longrightarrow I_K \longrightarrow Gal(\overline{K}/K) \xrightarrow{\pi} \widehat{\mathbb{Z}} \longrightarrow 1.$$

Definition 2.2.1. *Let K be a non-archimedean local field. The Weil group W_K of K as an abstract group is defined by*

$$\pi^{-1}(\mathbb{Z}) \subset Gal(\overline{K}/K).$$

Remark 2.2.1. *If we consider $Frob_{\kappa_K} \in Gal(\overline{\kappa_K}/\kappa_K)$, which defined as $Frob_{\kappa_K}(x) = x^{q_K}$ for $x \in \overline{\kappa_K}$, as the element $+1$ in $\widehat{\mathbb{Z}}$, then the subgroup generated by $+1$ is $\mathbb{Z}$ and it is a dense subgroup of $\widehat{\mathbb{Z}}$. Moreover let $Frob_K \in Gal(K^{nr}/K)$ be the Frobenius element that corresponds to $+1$. By the aid of the presentation $Gal(K^{nr}/K) \xrightarrow{\sim} \widehat{\mathbb{Z}}$, we can conclude that W_K is the subgroup of $Gal(\overline{K}/K)$, which image of its elements acts on $\overline{\kappa_K}$ by integer powers of $\eta(Frob_K) = Frob_{\kappa_K} \in Gal(\overline{\kappa_K}/\kappa_K)$ i.e*

$$W_K = \{\pi^{-1}(Frob_K^n) \mid n \in \mathbb{Z}\}.$$

In what follows we have the following short exact sequence

$$1 \to I_K \to W_K \xrightarrow{v} \mathbb{Z} \to 1$$

We call the element $Frob_K \in Gal(K^{nr}/K)$ a geometric Frobenius substitution on K^{nr}. In particular the preimage $Fr_K = \pi^{-1}(Frob_K)$ of $Frob_K \in Gal(K^{nr}/K)$ under the homomorphism $\pi : Gal(\overline{K}/K) \to Gal(K^{nr}/K)$ is called a geometric Frobenius element of W_K. Note that a geometric Frobenius substitution on K^{nr} is unique, but by the short exact sequence above, a geometric Frobenius element Fr_K of W_K is unique up to multiplication by element of I_K. Here $v : W_K \to \mathbb{Z}$ is valuation map on W_K defined as $v(Fr_K^n x) = n$, for $x \in I_K$ and $n \in \mathbb{Z}$. In what follows, the mapping v induces a non-archimedean multiplicative valuation on W_K by

$$||.||_v : W_K \to \mathbb{R}_{>0}, \quad ||w||_v = q_K^{-v(w)}$$

for $w \in W_K$. If E/K be some finite extension with $E \subset \overline{K}$, one has following commutative diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_E & \longrightarrow & W_E & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

Definition 2.2.2. *The topology on the Weil group W_K of K is given by product topology of $I_K \times \mathbb{Z}$, where I_K is declared to be an open subgroup of W_K which inherits the profinite topology induced from $Gal(\overline{K}/K)$ and $\mathbb{Z}$ has discrete topology.*

Remark 2.2.2. *Note that the topology introduced in Definition 2.2.2 on W_K is not the same as the induced topology on W_K from G_K.*

Remark 2.2.3. *In particular the group homomorphism $\delta :< Frob_K > \rightarrow W_K$ defined by the rule $\delta(Frob_K) = Fr_K \cdot I_K$, induces a splitting of the short exact sequence*

$$1 \to I_K \to W_K \xrightarrow{v} \mathbb{Z} \to 1$$

Hence this splitting induces a semi-direct product $W_K = I_K \rtimes \mathbb{Z}$ with the action

$$\tau :< Frob_K > \rightarrow Aut(I_K), \quad \tau(Frob_K)(x) = Fr_K \cdot x \cdot Fr_K^{-1}$$

for $x \in I_K$. Not that, by Remark 2.2.1, the choice of the geometric Frobenius element $Fr_K = \pi^{-1}(Frob_K)$ is not unique hence this splitting of the Weil group W_K is not unique. Hence by Remark 2.2.1, every $w \in W_K$ has unique expression $w = Fr_K^n x$ for some $x \in I_K$ and $n \in \mathbb{Z}$. Therefore we have

$$W_K = \bigcup_{n \in \mathbb{Z}} Fr_K^n I_K$$

Theorem 2.2.1. *Let L/K be some finite extension with $L \subset \overline{K}$. Suppose $\phi^K : W_K \to Gal(\overline{K}/K)$ be a continuous dense embedding defined by the rule $(\phi^K)^{-1}(Gal(\overline{K}/L)) = W_L$. Then we have*

$$W_L = Gal(\overline{K}/L) \cap W_K$$

In particular W_L is an open subgroup of W_K and inherits a subspace topology from W_K.

Proof. By the definition of the embedding ϕ^K, we have $W_L = (\phi^K)^{-1}(Gal(\overline{K}/L))$ hence

$$W_L \subset Gal(\overline{K}/L) \cap W_K$$

We will examine converse direction in two separate cases :
Case 1 : L/K be unramified. We have $L \subset K^{nr}$ and

$$I_K = Gal(\overline{K}/K^{nr}) \subset Gal(\overline{K}/L)$$

In particular we have the following isomorphisms

$$Gal(\overline{K}/K)/Gal(\overline{K}/K^{nr}) \xrightarrow{\sim} \widehat{\mathbb{Z}}, \quad Gal(\overline{K}/L)/Gal(\overline{K}/L^{nr}) \xrightarrow{\sim} \widehat{\mathbb{Z}}.$$

Herewith inclusion $Gal(\overline{K}/L) \subset Gal(\overline{K}/K)$ induces a homomorphism $f_{L/K} : \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$ which is multiplication by $m = [L : K]$. Therefore $(f_{L/K})^{-1}(\mathbb{Z}) = \mathbb{Z}$ and hence

$$Gal(\overline{K}/L) \cap W_K \subset W_L$$

For the second assertion, let $Fr_K \in W_K$ be a choice of a geometric Frobenius element. Hence Fr_K^m be a geometric Frobenius element of W_L. By Remark 2.2.3, we have $W_K = I_K \rtimes \mathbb{Z}$ and hence $W_L = I_L \rtimes m\mathbb{Z}$ is a splitting of W_L with respect to $Fr_K^m \in W_L$. Thus W_L is an open subgroup of W_K and inherits a subspace topology from W_K.

Case 2 : L/K be totally ramified. We have

$$I_L = Gal(\overline{K}/L) \cap Gal(\overline{K}/K^{nr})$$

In particular $Gal(\overline{K}/L) \cdot Gal(\overline{K}/K^{nr}) = Gal(\overline{K}/K)$. Therefore the induced map

$$f_{L/K} : \widehat{\mathbb{Z}} \simeq Gal(\overline{K}/L)/Gal(\overline{K}/L) \cap Gal(\overline{K}/K^{nr}) \to Gal(\overline{K}/K)/Gal(\overline{K}/K^{nr}) \simeq \widehat{\mathbb{Z}}$$

just the identity homomorphism. Hence $f_{L/K}^{-1}(\mathbb{Z}) = \mathbb{Z}$ and we have

$$Gal(\overline{K}/L) \cap W_K \subset W_L$$

Note that since L/K is totally ramified, for any geometric Frobenius element $Fr_K \in W_K$ we have $Fr_K \in W_L$. Hence we have $W_L = I_L \rtimes \mathbb{Z}$. This shows that W_L is an open subgroup of W_K and inherits the subspace topology from W_K. □

Remark 2.2.4. *Now we shall concretely describe Weil groups of local archimedean fields. To begin with, let K be an archimedean local field. We have two possibilities, namely $K = \mathbb{C}$ or $K = \mathbb{R}$.*

Case $K = \mathbb{C}$: We define Weil group of $\mathbb{C}$ as an abstract group $W_{\mathbb{C}} = \mathbb{C}^{\times}$. In this case $\phi^{\mathbb{C}}$ be the trivial map and the reciprocity homomorphism $Art_{\mathbb{C}} : \mathbb{C}^{\times} \to W_{\mathbb{C}}^{ab} = \mathbb{C}^{\times}$ is the identity homomorphism.

Case $K = \mathbb{R}$: We define Weil group of $\mathbb{R}$ as an abstract group $W_{\mathbb{R}} = \mathbb{C}^{\times} \cup j \cdot \mathbb{C}^{\times}$ where $j \in End(\mathbb{C}^{\times})$ with $j^2 = -1$ and $j \cdot p \cdot j^{-1} = \overline{p}$, for $id \neq p \in Gal(\mathbb{C}/\mathbb{R})$. In this case the map $\phi^{\mathbb{R}} : W_{\mathbb{R}} \to Gal(\mathbb{C}/\mathbb{R})$ carries $\mathbb{C}^{\times}$ to 1 and $j \cdot \mathbb{C}^{\times}$ to c. The reciprocity homomorphism $Art_{\mathbb{R}} : \mathbb{R}^{\times} \to W_{\mathbb{R}}^{ab}$ can be described as :

$$Art_{\mathbb{R}}(-1) = j \cdot W_{\mathbb{R}}^{c}$$

$$Art_{\mathbb{R}}(x) = \sqrt{x} \cdot W_{\mathbb{R}}^{c}, \quad x \in \mathbb{R}, \; x > 0$$

where $W_{\mathbb{R}}^{c}$ consists elements $x \in \mathbb{C}$ with $||x||_{\mathbb{C}} = N_{\mathbb{C}/\mathbb{R}}(x) = 1$.

2.3 Basic Theory of Locally Profinite Groups

In this part we shall study representation theoretic properties of Weil groups. To accomplish this we will first recall basic notions of locally profinite groups.

Definition 2.3.1. *A topological group G is called locally profinite if every open subset containing the identity element includes an open compact subgroup.*

Remark 2.3.1. *Any locally profinite group is locally compact. In particular, if $U \subset G$ be an open subset containing 1_G, then it contains a compact open normal subgroup in G. To see this, it is sufficient to consider U be a compact open subgroup. The subgroup*

$$V = \cap_{x \in G} xUx^{-1}$$

is compact, open and normal in G contained in U. In particular any closed subgroup of a locally profinite group is also locally profinite.

Example 2.3.1. *Suppose K be a non-archimedian local field. The linear group $GL_n(F)$ is locally profinite. The subgroups*

$$K = GL_n(O_K), \quad K_m = 1 + p_K^m M_n(O_F), \quad m \geq 1$$

are compact open . Furthermore these forms basis of open neighbourhoods of the identity matrix I_n in $GL_n(F)$.

Definition 2.3.2. *A Haussdorf, compact and totally disconnected topological group is called profinite. Equivalently, it can be writeable as projective limit of finite discrete groups.*

Lemma 2.3.1. *Every profinite group is locally profinite.*

Proof. For some given profinite group G let $G = \varprojlim_{i \in I} G_i$ over some directed index set I. Let $1_G \in U \subset G$ be an open neighbourhood. We set $V \subset \prod_{i \in I} G_i$ be an open subset with $V \cap G = U$. Then for a finite index set $I_1 \subset I$, the open set V contains an open subset of the form

$$\prod_{i \in I_1} U_i \times \prod_{i \in I \setminus I_1} G_i$$

We put

$$V_1 = \prod_{i \in I_1} \{1\} \times \prod_{i \in I \setminus I_1} G_i$$

We have $V_1 \subset V$ and by construction V_1 is an open subgroup of $\prod_{i\in I} G_i$. Therefore $U_1 = V_1 \cap G$ is an open subgroup with $U_1 \subset U$. Since an arbitrary open subgroup is also closed, we see that U_1 is closed and by the compactness of G, U_1 is a compact open set containing the identity element.

□

Example 2.3.2. *Let K be a nonarchimedean local field. Then by Lemma 2.3.1, the absolute Galois group*

$$Gal(K^{sep}/K) = \lim_{\leftarrow} Gal(E/K)$$

is a locally profinite group. In particular the Weil group W_K of K is a locally profinite group. To see this, since W_K has a topology given by the product topology on $I_K \times \mathbb{Z}$ where I_K is open in W_K equipped with profinite topology and $\mathbb{Z}$ equipped with discrete topology, therefore any open neighbourhood U of the identity in I_K or $\mathbb{Z}$ is a compact subgroup in each case. Hence W_K is a locally profinite group.

Definition 2.3.3. *Suppose G be a locally profinite group and V be some topological vector space over $\mathbb{C}$. A complex representation (π, V) of G is a continuous group morphism*

$$\pi : G \to GL(V)$$

If it inherits an inner product which is Hermitian in which for each $g \in G$, $\pi(g)$ is a unitary operator, we say it is unitary.

Definition 2.3.4. *Let G be a profinite group and (π, V) be representation of G. If for any $v \in V$, there exists an open compact subgroup K of G with subject to the condition $\pi(k).v = v$ for all $k \in K$, then (π, V) is called smooth.*

Remark 2.3.2. *Let V^K be the K-fixed elements in V. Then to give a smooth representation (π, V) of G is the same as put*

$$V^{sm} = \bigcup_K V^K$$

where the union is over all open compact subgroups K.

Definition 2.3.5. *Let G be a profinite group and (π, V) be some smooth representation of G.Then it is called admissible if the subspace V^K is finite dimensional for all open compact subgroups K of G.*

Proposition 2.3.1. *(Bushnell & Henniart, 2006) Suppose G be a locally profinite group. and (π, V) be smooth representation of G. Then the assertions are equivalent :*

(1) V as a G-space, is a sum of irreducible subspaces.

(2) There exists a collection of irreducible spaces so that V as a G-space, is a direct sum of this collection.

(3) An arbitrary subspace of V inherits a complementary subspace in V.

If any of the equivalent assertions holds, then (π, V) is called semisimple.

Theorem 2.3.1. *Suppose G be a profinite group and (π, V) be an irreducible smooth representation. Then V is finite dimensional.*

Proof. By assumption for $0 \neq v \in V$, there is an open compact subgroup K with $v \in V^K$. Since G is profinite and hence is compact, G/K is a finite, so $\{\pi(g) \cdot v \mid g \in G/K\}$ is a nonempty finite G-subspace. But V is irreducible, so we have $V = \{\pi(g) \cdot v \mid g \in G/K\}$. Let $K_1 = \bigcap_{g \in G/K} gKg^{-1}$ be a open subgroup of which is normal in K. Since G/K is finite, K_1 has a finite index in K. Thus for an arbitrary $\pi(g) \cdot v \in V$ and $gKg^{-1} \in K_1$, one has $\pi(gKg^{-1})\pi(g)v = \pi(gK)v = \pi(g)\pi(K)v = \pi(g)v$. This shows that $K_1 \subset ker\, \pi$. Therefore (π, V) is an irreducible non-trivial representation of G/K and hence V is finite dimensional.

□

Proposition 2.3.2. *Suppose G is a profinite group and (π, V) a smooth finite dimensional representation. Then (π, V) is semisimple.*

Proof. By Lemma 2.3.1, G is a locally profinite. Then by Remark 2.3.1, for $v \in V$, there is a normal open compact subgroup K, so that $v \in V^K$. Therefore the subspace $\langle \mathbb{C}v \rangle \subset V$ is irreducible as a representation of G/K and hence is finite dimensional and semisimple. Since $V = \sum_{i \in I} \mathbb{C}v$, by Proposition 2.3.1 we obtained that (π, V) is a semisimple.

□

Lemma 2.3.2. *(Bushnell & Henniart, 2006) Suppose G be locally profinite and (π, V) a smooth irreducible representation. Then $End_G(V) = \mathbb{C}$.*

Lemma 2.3.3. *Suppose G be a locally profinite group and (π, V) a smooth irreducible representation. Then, action of the centre Z_G is given by the character*

$$\omega_\pi : Z_G \to \mathbb{C}^\times$$

$$\pi(z) \cdot v = \omega_\pi(z) \cdot v.$$

$\forall v \in V$ *and* $\forall z \in Z_G$.

Proof. By Lemma 2.3.2, it exist and given by central homomorphism $\omega_\pi : Z_G \to \mathbb{C}^\times$ in which acts like a character $\pi(z) \cdot v = \omega_\pi(z) \cdot v$. If K be arbitrary compact open subgroup of G, with $V^K \neq 0$, $Z_G \cap K$ is also open compact subgroup of Z_G which ω_π is trivial. This implies that $\omega_\pi : Z_G \to \mathbb{C}^\times$ is a smooth representation of Z_G, hence a character of Z_G.

□

Remark 2.3.3. *Suppose G be locally profinite. For an irreducible smooth representation (π, V) of G, mapping $\omega_\pi : Z_G \to \mathbb{C}^\times$ is regarded as central character.*

Corollary 1. *Let G be a commutative locally profinite group. Then smooth irreducible representations of G are one dimensional.*

Lemma 2.3.4. *(Bushnell & Henniart, 2006) Suppose K be a non-archimedean field and (ρ, V) is a smooth irreducible representation of the Weil group W_K. Then V is finite dimensional.*

Definition 2.3.6. *Suppose K be a non-archimedean field. Then*

$$\chi : W_K \to \mathbb{C}.$$

is said unramified if its restriction to inertia subgroup I_K is trivial.

Remark 2.3.4. *Suppose K be a non-archimedean local field. If χ is a character of $K^\times$ then by composing it with Art_K, we have the induced character $\chi \circ Art_K$ of W_K. In the same vein, χ is unramified if and only if $\chi \circ Art_K$ is unramified i.e $\chi \circ Art_K|_{I_K} = id$.*

2.3.1 Additive and Inductive Maps

In this subsection we shall investigate the category of semisimple representations of Weil groups in order to get a more concrete description of these representations. To accomplish this, we shall use an idea from algebraic K-theory and define a particular group associated to Weil groups.

Remark 2.3.5. *Let K be local / global field. By $Rep_n(W_K)$ (resp. $Rep_n^{ss}(W_K)$), we shall indicate set of isomorphism classes of n-dimensional smooth (resp. semisimple) representations of W_K. By inclusion $Rep_n^{ss}(W_K) \subset Rep_n(W_K)$ and we set*

$$Rep^{ss}(W_K) = \bigcup_{n \geq 1} Rep_n^{ss}(W_K)$$

Definition 2.3.7. *Let $\mathbb{Z}^{\oplus Rep_n^{ss}(W_K)}$ be the free abelian group on $Rep_n^{ss}(W_K)$. Let H be the subgroup of $\mathbb{Z}^{\oplus Rep_n^{ss}(W_K)}$ spanned by isomorphism classes $[\rho]-[\rho_1]-[\rho_2]$ of short exact sequence of representations*

$$0 \to \rho_1 \to \rho \to \rho_2 \to 0$$

of the Weil group W_K. The group $\mathbb{Z}^{\oplus Rep_n^{ss}(W_K)}/H$ is called Grothendieck group of W_K, indicated by $K_0(W_K)$.

Remark 2.3.6. *By Proposition 2.3.1, any semisimple smooth representation of W_K is a direct sum of irreducible subrepresentations. Hence we deduce that the Grothendieck group $K_0(W_K)$ of W_K is isomorphic to the commutative group freely spanned by equivalence classes of irreducible smooth representations of W_K.*

Remark 2.3.7. *Suppose X be some commutative group and $\lambda : Rep_n^{ss}(W_K) \to X$ be a mapping. Then λ extends to a group homomorphism $\tilde{\lambda} : K_0(W_K) \to X$ if and only if for all triples (ρ, V),(ρ_1, V_1), (ρ_2, V_2) of representations of the Weil group W_K with*

$$0 \to \rho_1 \to \rho \to \rho_2 \to 0$$

the equality $\lambda(\rho) = \lambda(\rho_1) + \lambda(\rho_2)$ holds. In this case the mapping $\lambda : Rep_n^{ss}(W_K) \to X$ is called additive.

Lemma 2.3.5. *(Bushnell & Henniart, 2006) Let G be a locally profinite group and H be a finite index open subgroup of G. Then,*

(1) If (π, V) is a smooth representation of G, then V is a G-semisimple representation if and only if it is a H-semisimple representation.

(2) Let (χ, W) be a semisimple representation of H. Then the induced representation $Ind_H^G\, \sigma$ is G-semisimple.

Example 2.3.3. *Let K be non-archimedean local field, $\overline{K}$ be fixed seperable algebraic closure and L/K be a finite extension with $L \subset \overline{K}$. Then by Lemma 2.3.5, induction of a semisimple representation of W_L is a semisimple representation of W_K and the following map*

$$Ind_{L/K} : Rep_n^{ss}(W_L) \to K_0(W_K)$$

$$(\rho, V) \to [Ind_{W_L}^{W_K} \rho]$$

is additive.

Example 2.3.4. *The function $dim : Rep_n^{ss}(W_K) \to \mathbb{Z}$ is additive. Hence by Remark 2.3.7 it defines a homomorphism of abelian groups*

$$dim : K_0(W_K) \to \mathbb{Z}$$

The kernel of this homomorphism, is equivalence classes of representations of the form $[V_1] - [V_2]$, with $dim\ V_1 = dim\ V_2$. We denote it by $\widehat{K_0(W_K)}$.

Remark 2.3.8. *Let K be local / global field. By Artin reciprocity $Art_K : S_K \to W_K^{ab}$, we can identify the set of quasi-characters of S_K and of W_K^{ab}. Then for any $s \in \mathbb{C}$,*

$$\omega_s : W_K \to \mathbb{C}$$

$$\omega_s(w) = ||w||_K^s$$

is a quasi-character of W_K induced from absolute value on

$$S_K = \begin{cases} K^\times & \text{if } K \text{ is local} \\ \mathbb{I}_K/K^\times & \text{if } K \text{ is global} \end{cases}$$

Remark 2.3.9. *Let K be local / global field and $\overline{K}$ be a seperable algebraic closure. By construction, we can embed isomorphism classes of representations of $Gal(\overline{K}/K)$ into the isomorphism classes of representations of W_K*

$$Rep_n(G_K) \to Rep_n(W_K)$$

for all $n \geq 1$. We will call representations lying in the target of this map is of Galois type.

Lemma 2.3.6. *Let K be local / global field and (ρ, V) be continuous complex representation of W_K. Then ρ is identity on some open subgroup of I_K.*

Proof. In our proof we will basically use the fact that there are no small subgroups of general linear groups over complex numbers. Whereby ρ is a continuous representation, $\rho^{-1}(U)$ is an open set comprising identity of W_K. But the identity element in W_K has neighbourhood basis made up of open subgroups of I_K, thus $J_K \subset \rho^{-1}(U)$ for some open subgroup J_K of I_K. Therefore $\rho(J_K)$ is a subgroup of $GL(V)$ comprised in U, hereby it is trivial since $GL(V)$ has no small subgroups.

□

Corollary 2. *A representation (ρ, V) of W_K is of Galois type if and only if $\rho(W_K)$ is finite.*

Proposition 2.3.3. *The Grothendieck group $K_0(W_K)$ of W_K is spanned by elements*

$$Ind_{L/K}([\chi])$$

where L/K be some finite extension with χ be quasi-character of W_L. In particular, the subgroup $\widehat{K_0(W_K)}$ spanned by the elements

$$Ind_{L/K}([\chi_1] - [\chi_2]).$$

with χ_1, χ_2 are characters of the Weil group W_L.

Proof. By the Example 2.3.4, it is sufficient to prove allegation for $\widehat{K_0(W_K)}$ because we already have the identity

$$K_0(W_K) = \widehat{K_0(W_K)} + \mathbb{Z} \cdot [1]$$

Let $\widehat{K_0(W_K)}_*$ be the subgroup of $\widehat{K_0(W_K)}$ spanned by the elements of the form

$$Ind_{L/K}([\chi_1] - [\chi_2])$$

Then by Proposition 1.5 in (Deligne, 1973a), we have $Rep_0(G_K) \subset \widehat{K_0(W_K)}_*$. The identity

$$Ind_{L/K}(\rho \otimes Res_{L/K}\, \chi) = (Ind_{L/K}\, \rho) \otimes \chi$$

for each character χ of W_K and representation ρ of W_L implies that

$$\widehat{K_0(W_K)}_* \cdot \chi \subset \widehat{K_0(W_K)}_*.$$

Hence to prove the proposition we have to show that for each irreducible representation ρ of W_K, $[\rho] - (dim\, \rho)[1] \in \widehat{K_0(W_K)}_*$. By 2.2.3 in (Tate, 1979), for each representation ρ

of W_K there exists a finite extension L/K, and primitive irreducible representation ρ_L of W_L i.e of the form $\rho_L = \sigma \otimes \chi$ where σ is of Galois type and χ is a quasi-character so that $\rho = Ind_{L/K}\, \rho_L$. Then we have equality,

$$[\rho] - (dim\, \rho) \cdot [1] = Ind_{L/K}([\rho_L] - (dim\, \rho_L) \cdot [1] + (dim\, \rho_L) \cdot (Ind_{L/K}[1_L] - [L:K] \cdot [1_K]$$

The latter obviously of Galois type, so by the transitivity of the induction, it suffices to prove case ρ irreducible and primitive. By 2.2.4 in (Tate, 1979), any primitive irreducible representation of W_K are twists of Galois type representation by quasi-characters i.e $\rho = \sigma \otimes \chi$, where σ is of Galois type and χ is a quasi-character. In particular, if $dim\, \rho = dim\, \sigma = n$ then by a quick algebraic manipulation we have

$$[\rho] - n \cdot [1] = ([\sigma] - n \cdot [1]) \cdot [\chi] + n([\chi] - [1])$$

By definition we have $[\sigma] - n \cdot [1] \in \widehat{K_0(W_K)}$ and hence $[\rho] - n \cdot [1] \in \widehat{K_0(W_K)}_*$, which concludes the proof.

$\square$

2.4 Weil-Deligne Group and its Representations

In this section we will try to expand the category of Weil group representations in line with our purposes. The main idea to doing this is to obtain the correct formulation of the Langlands reciprocity principle. The inadequacy of the category of representations of Weil groups in this formulation is due to fact that it does not contain a class of representations which corresponds to representations of the p-adic linear groups. For this reason, P. Deligne was able to give the correct formulation of the reciprocity principle by inventing so called Weil-Deligne group associated to non-archimedean local fields, using A. Grothendieck' works on ℓ-adic Galois representations. Throughout this section, K will be a non-archimedean local field.

Definition 2.4.1. *Suppose E be some field of characteristic 0. A Weil-Deligne representation over E is a tripartite (ρ, V, N) which $\rho : W_K \to GL(V)$ a continuous representation of W_K over E and N be E-linear endomorphism of V so that*

$$\rho(w) \cdot N \cdot \rho(w)^{-1} = ||w||_K \cdot N.$$

Here

$$||w||_K = ||Art_K(w)||_K = q_K^{-v(w)}.$$

for the valuation mapping $v : W_K \to \mathbb{Z}$, $v(w) = v(Fr_K^n \cdot x) = n$, $\forall w = Fr_K^n \cdot x \in W_K$ and $x \in I_K$. In particular it is called Frobenius semisimple if ρ is a semisimple representation.

Remark 2.4.1. *Let (ρ_1, V_1, N_1) and (ρ_2, V_2, N_2) be Weil-Deligne representations of K. Then their tensor product*

$$(\rho, V, N) = (\rho_1, V_1, N_1) \otimes (\rho_2, V_2, N_2)$$

is again a Weil-Deligne representation of W_K, given by the following formulas :

$$\rho(w)(v_1 \otimes v_2) = \rho_1(w) \cdot v_1 \otimes \rho_2(w) \cdot v_2, \quad N(v_1 \otimes v_2) = N_1 \cdot v_1 \otimes v_2 + v_1 \otimes N_2 \cdot v_2$$

for $w \in W_K$ and $v_i \in V_i$, for $i = 1, 2$. In particular, we can form a Weil-Deligne representation of W_K by setting $(\rho, V, N) = Hom((\rho_1, V_1, N_1), (\rho_2, V_2, N_2))$ where $V = Hom_E(V_1, V_2)$ and

$$(\rho(w) \cdot f)(v_1) = \rho_2(w)(f(\rho_1(w)^{-1} \cdot v_1)), \quad (N \cdot f)(v_1) = N_2(f(v_1)) - f(N_1(v_1))$$

for $w \in W_K$, $v_i \in V_i$, for $i = 1, 2$.

Remark 2.4.2. *Given Weil-Deligne representation (ρ, V, N) of W_K, the kernel ker N of N is a W_K-stable subspace of V. Hence if (ρ, V, N) is irreducible, one has $N = 0$. Hence we see that an irreducible Weil-Deligne representation (ρ, V, N) is nothing but an irreducible representation of W_K.*

Example 2.4.1. *Suppose (ρ, V, N) be Weil-Deligne representation of W_K over $\mathbb{C}$ with $dim_{\mathbb{C}}\, V = 1$. Since N is nilpotent and 1-to 1, thus $N = 0$. This shows that any 1-dimensional Weil-Deligne representation is nothing but a continuous homomorphism*

$$\rho : W_K \to \mathbb{C}^{\times}$$

Definition 2.4.2. *The Weil-Deligne group WD_K of K is a group scheme over $\mathbb{Q}$,*

$$WD_K = W_K \ltimes \mathbb{G}_a$$

where W_K acts on $\mathbb{G}_a$ by the rule

$$w \cdot z \cdot w^{-1} = ||w||_K \cdot z, \quad w \in W_K,\ z \in \mathbb{G}_a$$

For any field E of characteristic 0, the E-valued points of WD_K are given by $W_K \ltimes E$ with composition

$$(w_1, z_1) \cdot (w_2, z_2) = (w_1 \cdot w_2, ||w_2||_K^{-1} \cdot z_1 + z_2)$$

for $w_1, w_2 \in W_K$ and $z_1, z_2 \in E$.

Remark 2.4.3. *Let E be a field with $\mathbb{Q} \subset E$ and V be a vector space over E. A homomorphism of group schemes*

$$\tilde{\rho} : W_K \ltimes_{\mathbb{Q}} E \to GL(V)$$

determines and is determined by a Weil-Deligne representation (ρ, V, N) via the formula

$$\tilde{\rho}((w, z)) = exp(z \cdot N) \cdot \rho(w)$$

for $w \in W_K$ and $z \in E$. Therefore to give a representation of WD_K as a group scheme, is same to give a Weil-Deligne representation (ρ, V, N) of W_K.

Remark 2.4.4. *In particular if $E = \mathbb{C}$, then we can re-define a complex representation as a continuous morphism*

$$\tilde{\sigma} : WD_K \to GL(V)$$

so that constriction $\tilde{\sigma}|_{\mathbb{C}}$ be complex analytic. (Rohrlich, 1994).

Proposition 2.4.1. *To give a complex representation $(\tilde{\sigma}, V)$ of WD_K is equivalent to give a complex Weil-Deligne representation (ρ, V, N) of W_K.*

Proof. Suppose (ρ, V, N) be complex Weil-Deligne representation of W_K. Assume $dim\, V = n$ for some $n \geq 1$. By fixing a basis of V, we may also assume $End(V) = GL_n(\mathbb{C})$. Let $exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ be the exponential map. Then $N \in M_n(\mathbb{C})$ and the following formula defines a representation $\tilde{\sigma}$ of WD_K on V as

$$\tilde{\sigma}((w,z)) = \sigma(w) \cdot exp(z \cdot N), \quad w \in W_K,\ z \in \mathbb{C}$$

Conversely let $(\tilde{\sigma}, V)$ be a complex representation of WD_K. Put

$$\sigma = \tilde{\sigma}|_{W_K}$$

and

$$N = \frac{log(\tilde{\sigma}(z))}{z}$$

for $z \in \mathbb{C}^{\times}$. We will verify that the triple (σ, V, N) is indeed defines a Weil-Deligne representation. If U is a unipotent element of $Aut(V)$, then

$$log\, U = (U-1) - \frac{(U-1)^2}{2} + \frac{(U-1)^3}{3} - \frac{(U-1)^4}{4} + \frac{(U-1)^5}{5} + \cdots$$

is the ordinary power series in $U - 1$ such that $log\, U$ is nilpotent. We have to check that $\tilde{\sigma}(z)$ is unipotent for all $z \in \mathbb{C}$. Any unipotent automorphism has one and only unipotent n-th root for every $n \geq 1$, so has a one and only one unipotent r-th power for every $r \in \mathbb{Q}$. By assumption $\tilde{\sigma}$ is a homomorphism hence

$$log\, \tilde{\sigma}(r \cdot z_0) = log(\tilde{\sigma}(z_0)^r) = r \cdot log\, \tilde{\sigma}(z_0)$$

for $z_0 \in \mathbb{C}$ and $r \in \mathbb{Q}$. In particular, By continuity implies $log\, \tilde{\sigma}(r \cdot z_0) = r \cdot log\, \tilde{\sigma}(z_0)$ for every $r \in \mathbb{R}$ and since $\tilde{\sigma}|_{\mathbb{C}}$ is analytic the relation satisfied for every $r \in \mathbb{C}$. Putting $r = z$ and $z_0 = 1$, we deduce that the fraction $\frac{log(\tilde{\sigma}(z))}{z}$ is independent of the choice $z \in \mathbb{C}^{\times}$. By definition, we have the following relation $w \cdot z \cdot w^{-1} = ||w||_K \cdot z$ for all $w \in W_K$ and $z \in \mathbb{C}$. In particular, for $w = Frob_K^{-1}$, this relation implies $\tilde{\sigma}(z)$ and $\tilde{\sigma}(z)^{q_K}$ are equivalent transformations. By recursive repetitions, we see that if λ is an eigenvalue of $\tilde{\sigma}(z)$, then it is also an eigenvalue $\lambda^{q_K^n}$ for every $n \geq 1$. In particular if $dim\ \tilde{\sigma} = d$, then $\tilde{\sigma}(z)$ possess at most d distinct eigenvalues in which for $m_0, n_0 \in \mathbb{N}$ with $0 \leq m_0 \leq n_0 \leq d$, we have $\lambda^{q_K^{m_0}} = \lambda^{q_K^{n_0}}$. Hence

$$k = \prod_{0 \leq m_0 \leq n_0 \leq d} (q^{n_0} - q^{m_0})$$

is a positive integer, and every eigenvalue of $\tilde{\sigma}(z)$ is an k-th root of unity. We repeat this process for the element $\tilde{\sigma}(\frac{z}{k})$ instead of $\tilde{\sigma}(z)$, and by setting $\tilde{\sigma}(z) = \tilde{\sigma}(\frac{z}{r})^r$, we see that any eigenvalue of $\tilde{\sigma}(z)$ is equal to 1 and hence $\tilde{\sigma}(z)$ is a unipotent transformation. This completes proof of the proposition.

□

2.4.1 Connection with ℓ-adic Galois Representations

In this part we shall investigate the relationship between complex representations of the Weil-Deligne groups and particular Galois representations. We would like to draw attention to the fact that this phenomena would give Galois action on étale cohomology of smooth projective varieties in which compatibility of representation spaces, beside good reduction case, yet mysterious. (Tate, 1979).

Definition 2.4.3. *Let ℓ be some prime withal $\ell \neq char(\kappa_K)$. An ℓ-adic Galois representation is a continuous morphism*

$$\sigma_\ell : Gal(\overline{K}/K) \to GL(V_\ell).$$

with $dim_{\mathbb{Q}_\ell} V_\ell < \infty$.

Remark 2.4.5. *We first fix an embedding $j : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Let $\sigma_\ell : Gal(\overline{K}/K) \to GL(V_\ell)$ be some ℓ-adic Galois representation. To pair (σ_ℓ, V_ℓ), we attach the tuple (σ^ℓ, N_ℓ) with $\sigma^\ell : W_K \to GL(V_\ell)$ be a representation of W_K which is trivial on some open subgroup J_K of I_K and N_ℓ be nilpotent endomorphism on V_ℓ which satifsfies*

$$\sigma^\ell(w) \cdot N_\ell \cdot \sigma^\ell(w)^{-1} = ||w||_K \cdot N_\ell$$

for all $w \in W_K$. By combining σ^ℓ with the mapping $GL(V_\ell) \hookrightarrow GL(\mathbb{C} \otimes_j V_\ell)$, we form a *complex representation*

$$\sigma_{\ell,j} : W_K \xrightarrow{\sigma^\ell} GL(V_\ell) \hookrightarrow GL(\mathbb{C} \otimes_j V_\ell)$$

In particular, nilpotent endomorphism N_ℓ extends, by using the mapping $End(V_\ell) \hookrightarrow End(\mathbb{C} \otimes_j V_\ell)$ to a nilpotent endomorphism

$$N_{\ell,j} \in End(\mathbb{C} \otimes_j V_\ell)$$

By construction the triple

$$(\sigma_{\ell,j}, \mathbb{C} \otimes_j V_\ell, N_{\ell,j})$$

is a complex Weil-Deligne representation of W_K.

Remark 2.4.6. *Let ℓ be a prime with $\ell \neq p = char(\kappa_K)$. Let $t_\ell : I_K \to \mathbb{Q}_\ell$ be a non-zero homomorphism. Note that such a homomorphism t_ℓ exists, since we have the following isomorphism*

$$Gal(\overline{K}/K^{nr})/Gal(\overline{K}/K^{tame}) \xrightarrow{\sim} \prod_{\ell \neq p} \mathbb{Z}_\ell$$

where K^{tame} be the maximal tamely ramified extension of K. Indeed for any $n \geq 1$, with $p \nmid n$, there exists finite Galois extension E_n/K^{nr} withal cyclic Galois group, which for uniformizer $\pi_K \in O_K$, one has $E_n = K^{nr}(\pi_K^{\frac{1}{n}})$. Thereby degree of any finite Galois extension of K^{nr} is coprime to p. Hence we deduce that the group $P_K = Gal(\overline{K}/K^{tame})$ is a pro p-group, which is called the wild inertia group of K. Therefore up to $\mathbb{Q}_\ell^\times$-multiple, $t_\ell : I_K \to \mathbb{Q}_\ell$ is a projection onto $\mathbb{Z}_\ell$.

Proposition 2.4.2. *Suppose $\sigma_\ell : Gal(\bar{K}/K) \to GL(V_\ell)$ be an ℓ-adic representation and let $Frob_K \in Gal(\overline{K}/K)$ be a Frobenius element. Then;*

(1) There exists only one nilpotent endomorphism N_ℓ on V_ℓ with

$$\sigma_\ell(k) = exp(t_\ell(k) \cdot N_\ell)$$

$\forall k \in J_K$, where $J \subset I_K$ is an open subgroup and

$$\sigma_\ell(w) \cdot N_\ell \cdot \sigma_\ell(w)^{-1} = ||w||_K \cdot N_\ell$$

$\forall w \in W_K$. Particularly $N_\ell = 0$ if and only if there exists an open subgroup of J_K of I_K so

that $\sigma_\ell|_{J_K} = id$.

(2) The mapping $\sigma^\ell : W_K \to GL(V_\ell)$ *defined by the following formula*

$$\sigma^\ell(Frob_K^n \cdot \varphi) = \sigma_\ell(Frob_K^n \cdot \varphi) \cdot exp(-t_\ell(\varphi)N_\ell), \quad \varphi \in I_K, n \in \mathbb{Z}$$

is a continuous representation of W_K.

(3) One has the following relation

$$\sigma^\ell(w) \cdot N_\ell \cdot \sigma^\ell(w)^{-1} = ||w||_K \cdot N_\ell$$

for all $w \in W_K$.

Proof. (1) Since I_K is a compact group, the image $\sigma_\ell(I_K)$ stabilizes a lattice L in V_ℓ. Chancing K by a finite extension we may assume that $\sigma_\ell(I_K)$ stabilizes $L\ mod\ \ell^2$. Then $\sigma_\ell(I_K)$ is a pro-ℓ-group, and since $ker\ t_\ell$ is a prime to ℓ therefore the image $t_\ell(I_K)$ is also a pro-ℓ-group. We choose $c \in \mathbb{Q}_\ell$ so that $c \cdot t_\ell(I_K) = \mathbb{Z}_\ell$. Then there exists some $\alpha \in GL(V_\ell)$ stabilizing $L\ mod\ \ell^2$ so that

$$\sigma_\ell(\varphi) = \alpha^{c \cdot t_\ell(\varphi)} = exp(t_\ell(\varphi) \cdot N_\ell)$$

for all $\varphi \in I_K$ where $N_\ell = c \cdot log\ \alpha$. In particular we have

$$N_\ell^{\sigma_\ell(Frob_K)} = \sigma_\ell(Frob_K) \cdot N_\ell \cdot \sigma_\ell(Frob_K)^{-1} = q_K^{-1} \cdot N_\ell$$

Therefore we see that the set of eigenvalues of N_ℓ does not change under multiplication by q_K^{-1}. But since q_K is not a root of unity, so the only eigenvalue of N_ℓ is 0 and hence N_ℓ is a nilpotent endomorphism of V_ℓ with

$$\sigma_\ell(w) \cdot N_\ell \cdot \sigma_\ell(w)^{-1} = ||w||_K \cdot N_\ell$$

for all $w \in W_K$. Second assertion is clear and follows by the properties of the *exp* function.

(2) Let $w_n, w_m \in W_K$ so that $w_n = Frob_K^n \cdot \varphi_n, w_m = Frob_K^m \cdot \varphi_m$, for $\varphi_n, \varphi_m \in I_K$ and

$n, m \in \mathbb{Z}$. Then we have

$$\sigma^{\ell}(w_n \cdot w_m) = \sigma_{\ell}(Frob_K^n \cdot \varphi_n \cdot Frob_K^n \cdot \varphi_m) \cdot exp(-t_{\ell}(\varphi_n \cdot \varphi_m) \cdot N_{\ell})$$

$$= \sigma_{\ell}(Frob_K^n) \cdot \sigma_{\ell}(\varphi_n) \cdot \sigma_{\ell}(Frob_K^m) \cdot \sigma_{\ell}(\varphi_m) \cdot exp(-t_{\ell}(\varphi_n \cdot \varphi_m) \cdot N_{\ell})$$

$$= \sigma_{\ell}(Frob_K^n) \cdot \sigma_{\ell}(Frob_K^m) \cdot \sigma_{\ell}(\varphi_n) \cdot \sigma_{\ell}(\varphi_m) \cdot exp((-t_{\ell}(\varphi_n) - t_{\ell}(\varphi_m)) \cdot N_{\ell})$$

$$= \sigma_{\ell}(Frob_K^n) \cdot \sigma_{\ell}(\varphi_n) \cdot \sigma_{\ell}(Frob_K^m) \cdot \sigma_{\ell}(\varphi_m) \cdot exp(-t_{\ell}(\varphi_n) \cdot N_{\ell}) \cdot exp(-t_{\ell}(\varphi_m) \cdot N_{\ell})$$

$$= \sigma_{\ell}(Frob_K^n) \cdot \sigma_{\ell}(\varphi_n) \cdot exp(-t_{\ell}(\varphi_n) \cdot N_{\ell}) \cdot \sigma_{\ell}(Frob_K^m) \cdot \sigma_{\ell}(\varphi_m) \cdot \cdot exp(-t_{\ell}(\varphi_m) \cdot N_{\ell})$$

$$= \sigma_{\ell}(Frob_K^n \cdot \varphi_n) \cdot exp(-t_{\ell}(\varphi_n) \cdot N_{\ell}) \cdot \sigma_{\ell}(Frob_K^m \cdot \varphi_m) \cdot exp(-t_{\ell}(\varphi_m) \cdot N_{\ell})$$

$$= \sigma^{\ell}(w_n) \cdot \sigma^{\ell}(w_m)$$

thus σ^{ℓ} is a homomorphism and by part (1), is trivial on open subgroup of I_K therefore defines an ℓ-adic representation of W_K.

(3) Let $w \in W_K$ so that $w = Frob_K^n \cdot \varphi_n$, for $\varphi_n \in I_K$ and $n \in \mathbb{Z}$. Substituting formula for σ^{ℓ} given in (2), and by the second statement of (1) we have

$$\sigma^{\ell}(Frob_K^n \cdot \varphi_n) \cdot N_{\ell} \cdot \sigma^{\ell}(Frob_K^n \cdot \varphi_n)^{-1}$$

$$= \sigma_{\ell}(Frob_K^n \cdot \varphi_n) \cdot exp(-t_{\ell}(\varphi_n) N_{\ell}) \cdot N_{\ell} \cdot \sigma_{\ell}(Frob_K^n \cdot \varphi_n)^{-1} \cdot exp(t_{\ell}(\varphi_n) N_{\ell})$$

$$= \sigma_{\ell}(Frob_K^n \cdot \varphi_n)) \cdot N_{\ell} \cdot \sigma_{\ell}(Frob_K^n \cdot \varphi_n)^{-1}$$

$$= ||w||_K \cdot N_{\ell}$$

□

Chapter 3

Reductive Groups and Root Systems

Here, we shall study a particular class of algebraic groups, which can be determined by their root datum. We will investigate structure theory of these groups by studying associated root datum and root systems. Finally, we will introduce the class of parabolic subgroups which has a particular importance in terms of representation theory. Throughout the chapter, unless otherwise stated we assume any coefficient field K has characteristic 0.

3.1 Basic Notions of Algebraic Groups

Definition 3.1.1. *An algebraic variety G is called an algebraic group if it is equipped with,*

1) (id) : an element $e \in G$,
2) (mult) : a morphism $\mu : G \times G \to G$, with rule $(t,s) \to t \cdot s$
3) (inv) : a morphism $i : G \to G$, denoted $t \to t^{-1}$

which effectuates a group structure. We say, G a K-group if as an algebraic variety, it is defined over K with μ and i are defined over K. In particular, morphism among algebraic groups is a regular mapping therewithal a group morphism. In the case the underlying variety of G is affine, it is called an affine algebraic group.

Remark 3.1.1. *Given an algebraic group G/K, one has $e \in G(K)$. More precisely, the* *singleton $\{e\}$ constitute the image of*

$$\alpha \circ \delta : G \to G$$

where $\delta(t) = (t,t)$ and $\alpha(c,d) = c \cdot d^{-1}$, $t,c,d \in G(K)$. By G°, we express connected component of G containing the identity element e.

Lemma 3.1.1. *Suppose G/K be an algebraic group and H be a subgroup of G which may not be a closed subgroup. Suppose U and V be dense open subsets of G. Then one has,*
1) $U \cdot V = G$
2) The closure $\overline{H}$ of H is a subgroup of G. Besides, whether $H \subset G(K^{sep})$ and be invariant under $Gal(K^{sep}/K)$-action, then $\overline{H}$ is defined over K.

Proof. 1) Let $g \in G$. Then $\exists t \in U \cap g \cdot V^{-1}$, which is $t = g \cdot v^{-1}$, hence $g = u \cdot v \in U \cdot V$.
2) Since the mapping $g \to g^{-1}$ is a homeomorphism, one has

$$\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$$

If $s \in H$, then $s \cdot \overline{H} = \overline{s \cdot H} = \overline{H}$ hence $H \cdot \overline{H} = \overline{H}$. If $\overline{s} \in \overline{H}$, then $H \cdot \overline{s} \subset \overline{H}$ and hence $\overline{H} \cdot \overline{s} = \overline{H \cdot \overline{s}} \subset \overline{H}$. Therefore we see $\overline{H} \cdot \overline{H} = \overline{H}$, thus $\overline{H}$ is a group. By assumption, $H \subset \overline{H} \cap G(K^{sep})$ so that H is $Gal(K^{sep}/K)$-stable and dense in $\overline{H}$. Let $J = \bigcap_{h \in H} \mathfrak{m}_h = I(\overline{H}) = I(H)$ be the ideal of functions vanishing on $\overline{H}$ and $\mathfrak{m}_h$ be the maximal ideal of the germ at $h \in H$. Since $H \subset \overline{H} \cap G(K^{sep})$, J is defined over K^{sep} as a subspace of $K[G]$. If $\sigma \in Gal(K^{sep}/K)$, then we have

$$J^{\sigma}_{K^{sep}} = \bigcap \mathfrak{m}^{\sigma}_h = \bigcap \mathfrak{m}_{\sigma(h)} = \bigcap \mathfrak{m}_h = J_{K^{sep}}$$

Hence J is defined over K then so $\overline{H}$ is. □

Corollary 3. *Let G_1/K, G_2/K be algebraic groups with $m : G_1 \to G_2$ be a morphism. Then,*
1) $m(G_1)$ is closed in G_2 and is defined over K if m is defined over K.
2) One has $m(G_1^{\circ}) = m(G_1)^{\circ}$.
3) $dim\ G_1 = dim\ ker(m) + dim\ f(G_1)$.

Suppose G/K be affine algebraic group with coordinate ring $K[G]$. We will transport ingredients of structure in terms of $K[G]$. To accomplish put :

$$e : K[G] \to K,\ e(f) = f(e)$$

$$\mu_0 : K[G] \to K[G] \otimes_K K[G]$$

with if $\mu_0(f) = \sum_i g_i \otimes h_i$ then $f(x \cdot y) = \sum_i g_i(x) \cdot h_i(y)$. Also we let

$$i : G \to G : \ \ i^0 : K[G] \to K[G],\ (i^0 f)(x) = f(e)$$

In order to formulate group axioms let us introduce mappings

$$p : G \to G : \ \ p^0 : K[G] \to K[G]$$

$$p : t \to g : \ \ p^0(f)(g) = f(e)$$

Remark 3.1.2. *In fact K-algebra morphism p^0 is the composition of $e : K[G] \to K$ with inclusion $K \hookrightarrow K[G]$. Therefore, in terms of $K[G]$, the algebraic group G is determined by $(K[G], e, \mu_0, i^0)$ with obeying to axioms above.*

Example 3.1.1. *The additive group $\mathbb{G}_a$ over a field K. As a coordinate ring we have $K[\mathbb{G}_a] = K[T]$ with*

$$\mu^0(T) = (t \otimes 1) + (1 + \otimes T),\ \ i^0(T) = -T,\ \ e(T) = 0$$

In particular we have the multiplicative group $\mathbb{G}_m$ over K and its coordinate ring $K[\mathbb{G}_m] = K[T,Z]/ < TZ - 1 >= K[T,T^{-1}]$ with

$$e(T) = 1,\ \ \mu^0(T) = T \otimes T,\ \ i^0(T) = T^{-1}$$

Example 3.1.2. *The general linear group GL_n over a field K for some $n \geq 1$. As a coordinate ring we have $K[GL_n] = K[X_{11}, X_{12}, \cdots, X_{nn}, D^{-1}]$ withal $D = det\ (X_{ij})_{1 \leq i,j \leq n}$. Therefore GL_n is an affine open subset of K^{n^2}. We have*

$$e(X_{ij}) = \delta_{ij},\ \ \mu^0(X_{ij}) = \sum_k X_{ik} \otimes X_{kj},\ \ i^0(X_{ij}) = (-1)^{i+j} \cdot D^{-1} \cdot det\ (X_{ks})_{k \neq j, s \neq i}$$

Definition 3.1.2. *An affine algebraic group which isomorphic to, closed subgroup of GL_n ($\exists n \geq 1$) is called linear algebraic group. A morphism $G \to GL_n$ is called a rational (linear) representation of G.*

Definition 3.1.3. *Let G/K be linear algebraic group. Given $g \in G(K)$ called semisimple (resp. unipotent) whether $j(g)$ is , for any (hence all) rational representation $j : G \hookrightarrow GL_n$.*

Theorem 3.1.1. *(Additive Jordan Decomposition) Suppose $G/\overline{K}$ be linear algebraic group. For every $g \in G(\overline{K})$ there exists unique $g_s, g_n \in G(\overline{K})$ which g_s be semisimple and g_n be nilpotent, with $g_s \cdot g_n = g_n \cdot g_s$ and so that $g = g_s + g_n$.*

Proof. Let $g \in G(\overline{K})$ be given. Let us consider $det(X - g) = \prod_{i=0}^{n} (X - \alpha_i)^{m_i}$ with for $i \neq j$, $m_i \neq m_j$ and set $V = ker\ (g - \alpha_i \cdot I_n)^{m_i}$. Then we have $V = \bigcup_{i=0}^{n} V_i$. Let $g = h + k$ with h semisimple and k nilpotent so that $h \cdot k = k \cdot h$. Then h commutes with g, and so it commutes with $(g - \alpha_i \cdot I_n)^{m_i}$. Therefore the element h leaves every V_i invariant. We observe that $g - h = k$ is nilpotent so g and h have the same eigenvalues on V_i. But since g has only one on V_i, namely α_i, and h is semisimple, we have $b|_{V_i} = \alpha_i \cdot I_n|_{V_i}$. Hence we see that h is precisely determined and hence $k = g - h$ is too. In particular if we put g_s by $g_s|_{V_i} = \alpha_i \cdot I_n|_{V_i}$ and $g_n = g - g_s$, then these elements satisfies required properties of the theorem.

□

Corollary 4. *(Multiplicative Jordan Decomposition) Suppose $G/\overline{K}$ be linear algebraic group. Then for every $g \in G(\overline{K})$, there exists unique $g_s, g_u \in G(\overline{K})$ which g_s be semisimple and g_u be unipotent with*

$$g = g_s \cdot g_n = g_n \cdot g_s$$

Proof. For any $g \in G(\overline{K})$, by additive Jordan decomposition, we have a unique semisimple $g_s \in G(\overline{K})$ and a unique nilpotent $g_n \in G(\overline{K})$ elements with $g_s \cdot g_n = g_n \cdot g_s$ and $g = g_s + g_n$. Let us introduce the element $g_u = I_n + g_s^{-1} \cdot g_n$. Then since g_s and g_n commutes we see that $g_u \in G(\overline{K})$ is a unipotent element and $g = g_s \cdot g_n = g_n \cdot g_s$. For the uniqueness, suppose $g = h \cdot k = k \cdot h$ with h is semisimple and $m = k - I_n$ is nilpotent. Then $h \cdot m$ is nilpotent and commutes with h, therefore by the uniqueness of the additive Jordan decomposition, we have $g = h + h \cdot m$ as an additive Jordan decomposition. Therefore we have $h = g_s$ and $h \cdot m = g_n$. □

Definition 3.1.4. *Let G be a connected linear algebraic group. The radical of G, with notation $R(G)$, is the maximal normal connected solvable subgroup in G. In particular, set of unipotent elements in $R(G)$ is called unipotent radical of G, with notation $R_u(G)$.*

Definition 3.1.5. *A connected linear algebraic group G called semisimple whether $R(G) = 1$ and called reductive if $R_u(G) = 1$.*

Remark 3.1.3. *For any linear algebraic group G, by definition one has $R_u(G) \subset R(G)$. Therefore any semisimple linear algebraic group is reductive.*

Remark 3.1.4. *Suppose G be a linear algebraic group. The quotients $G/R(G)$, $G/R_u(G)$ are semisimple and reductive groups respectively. In particular, by looking at derived series in $R(G)$ and descending central series in $R_u(G)$, we deduce that G is semisimple (resp. reductive) whether G has nary connected abelian (resp. unipotent abelian) normal subgroup other than trivial subgroup.*

Example 3.1.3. *The group GL_n is reductive. It is not a semisimple algebraic group because the centre $Z(GL_n)$ is a connected abelian normal subgroup of GL_n. In particular radical of GL_n is $Z(GL_n)$. Hence by Remark 3.1.4, subgroup $GL_n/Z(GL_n) = SL_n$, is semisimple and hence is reductive by Remark 3.1.3.*

Theorem 3.1.2. *(Borel, 1969). 1) (Levi Decomposition) Suppose G/K be connected algebraic group. There exists some reductive subgroup L of G with*

$$G = L \cdot R_u(G).$$

as a semi-direct product. L is called the Levi subgroup of G.

2) If G is reductive, then :

$$G = G' \cdot R(G).$$

as an almost direct product i.e the intersection $G' \cap R(G)$ is finite, where $G' = [G,G]$.

3.1.1 Algebraic Tori and Borel Subgroups

In this subsection we shall investigate a particular class of algebraic groups which acting as building block in the theory of algebraic groups. We also expose class of Borel subgroups which we shall investigate their algebro-geometric and representation theoretic properties. Unless otherwise stated, throughout this subsection K will be field with characteristic 0 with $\overline{K}$ some fixed seperable algebraic closure.

Definition 3.1.6. *An algebraic torus T/K of rank $n \in \mathbb{Z}_{>0}$, is an algebraic group with*

$$T_{\overline{K}} \xrightarrow{\sim} \mathbb{G}_m^n.$$

Example 3.1.4. *Let $T = Res_{\mathbb{Q}[i]/Q}\,\mathbb{G}_m$ be algebraic group over $\mathbb{Q}$, defined as for $\mathbb{Q}$-algebra B, by the formula :*

$$T(B) = Res_{\mathbb{Q}[i]/\mathbb{Q}}\,\mathbb{G}_m(B) = (B \otimes_{\mathbb{Q}} \mathbb{Q}[i])^{\times} = (B[T]/(T^2+1))^{\times}$$

Let $Nm_{\mathbb{Q}[i]/Q} : T \to \mathbb{G}_{m,\mathbb{Q}}$ be the norm map, defined as for any $\mathbb{Q}$-algebra B,

$$Nm_{\mathbb{Q}[i]/Q}(B) : T(B) \to B^{\times}, \quad e + fT \to e^2 + f^2$$

In particular we have

$$ker(Nm_{\mathbb{Q}[i]/Q}) = \{e + fT \mid T^2 = -1, e, f \in B, e^2 + f^2 = 1\}$$

$$\simeq \{\begin{pmatrix} e & f \\ -f & e \end{pmatrix} \mid e, f \in B, e^2 + f^2 = 1\} \simeq SO_2(B)$$

which algebraic group isomorphism is morphism $e + fT \to \begin{pmatrix} e & f \\ -f & e \end{pmatrix}$. In particular we have

$$SO_{2,\mathbb{Q}[i]} \simeq Spec(\mathbb{Q}[i](e,f)/(e^2+f^2=1)), \quad \mathbb{G}_{m,(\mathbb{Q}[i]} = Spec((\mathbb{Q}[i](T,Z)/(TZ=1))$$

and isomorphism $\mathbb{Q}[i](e,f)/(e^2+f^2=1) \to (\mathbb{Q}[i](T,Z)/(TZ=1)$ is given as putting $T = e + fi$, $Z = e - fi$. Hence we have $SO_{2,\mathbb{Q}[i]} \simeq \mathbb{G}_{m,(\mathbb{Q}[i]}$. On the other hand, by definition we have an isomorphism $T = Res_{\mathbb{Q}[i]/\mathbb{Q}}\,\mathbb{G}_m \simeq GL_{1,\mathbb{Q}} \times GL_{1,\mathbb{Q}}$ as algebraic groups over $\mathbb{Q}$. Hence we see that kernel $ker(Nm_{\mathbb{Q}[i]/\mathbb{Q}})$ is an algebraic torus over $\mathbb{Q}$.

Definition 3.1.7. *Suppose G/K be some algebraic group. A character is a morphism $G \to \mathbb{G}_m$ of algebraic groups i.e an member of the set*

$$X^*(G) = Hom(G, \mathbb{G}_m)$$

In particular, a co-character of G is a morphism $\mathbb{G}_m \to G$ i.e member of the set

$$X_*(G) = Hom(\mathbb{G}_m, G)$$

Remark 3.1.5. *From the structure of character and co-character groups, for an* *algebraic torus T, we have the following dual pairing*

$$\langle , \rangle : X^*(T) \times X_*(T) \to Hom(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$$

given by

$$\langle \alpha, \beta \rangle = m \quad if \quad (\alpha \circ \beta)(x) = x^m.$$

Definition 3.1.8. *An algebraic torus T/K is called split if there is a K-isomorphism*

$$T \xrightarrow{\sim} \mathbb{G}_m^n, \quad n \in \mathbb{Z}_{>0}$$

Equivalently, T is split over K if $X^*(T)_K \xrightarrow{\sim} \mathbb{Z}^{rank_K(T)}$ for some $n \in \mathbb{Z}_{>0}$. It is called anisotropic if $X^*(T)_K = \{id\}$.

Remark 3.1.6. *Any algebraic torus T inherits a decomposition*

$$T = T^{ani} \cdot T^{spl}$$

where T^{spl}, T^{ani} be maximal split and anisotropic parts in T with intersection $T^{ani} \cap T^{spl}$ is finite.

Theorem 3.1.3. *Suppose G/K be connected reductive group with $G = R(G) \cdot G'$ decomposition of G as in Theorem 3.1.2. Then $R(G)$ is an algebraic torus with $X^*(G)$ be of finite index in $X^*(R(G))$.*

Proof. We will first show whether G be semisimple require $X^*(G) = 1$. If G be simple then for character $\chi : G \to GL(1)$, $ker\ (\chi)$ is a closed normal subgroup. Using dimension formula $dim\ G = dim\ Im(\chi) + dim\ ker(\chi)$ in Corollary 3, we see that $dim\ ker(\chi) \geq 1$. Hence by assumption we have $G = ker\ (\chi)$. Since G is semisimple, then by definition there exist an isogeny

$$\prod_{i=0}^{k} G_i \to G$$

where each factor G_i is a simple algebraic group . Therefore we have an injection

$$X^*(G) \hookrightarrow \prod_{i=0}^{k} X^*(G_i)$$

which implies $X^*(G) = 1$. Now assume G is reductive. Then by Theorem 3.1.2, there exists an isogeny $R(G) \times G^{'} \to G$ which induces a homomorphism

$$X^*(G) \hookrightarrow X^*(R(G)) \times X^*(G^{'})$$

Hence $X^*(G)$ is a finite index subgroup of $X^*(R(G))$. $\square$

Definition 3.1.9. *Suppose G/K be some connected linear algebraic group. A closed subgroup $B \subset G$ is called Borel subgroup whether $B/\overline{K}$ be connected, maximal, solvable in $G/\overline{K}$. In particular closed subgroup $P \subset G$ called parabolic whether the homogeneous space G/P is a complete algebraic variety.*

Theorem 3.1.4. *(Borel, 1969) Suppose G be connected linear algebraic group with $B \subset G$ Borel subgroup. Then whole Borel subgroups be conjugate with so G/B is a projective variety.*

Remark 3.1.7. *By Theorem 6.8 in (Borel, 1969), for any linear algebraic group G and a closed subgroup H, G/H is a smooth quasi-projective variety. Therefore we deduce that G/H is complete if and only if it is a projective variety.*

Lemma 3.1.2. *Suppose G be some connected linear algebraic group with P some closed subgroup. Then P is a parabolic subgroup if and only if it inherits some Borel subgroup.*

Proof. Assume $B \subset P$ for some Borel B. Then by Theorem 3.1.4 and Remark 3.1.7, G/B be complete and $f : G/B \to G/P$ is a surjective morphism. Thereof G/P is complete hence P be parabolic subgroup of G. Conversely by Theorem 10.4 in (Borel, 1969), the action of B on G/P has a fixed point and hence a conjugate of B is in P. By Theorem 3.1.4, we have $B \subset P$.

$\square$

Proposition 3.1.1. *Suppose G/K be connected linear algebraic group. Then the set of maximal tori in G equal to set of maximal tori in the various Borel subgroups in G, and they are conjugate.*

Proof. Any maximal algebraic torus T of G connected, solvable, hence encompassed in some Borel B. Obviously T also maximal algebraic torus in B. By Theorem 10.6 in (Borel, 1969), one has $B = T \cdot B_u$ where B_u is the set of unipotent elements in B. By Theorem 3.1.4, all Borel subgroups of G are conjugate, therefore any maximal algebraic torus in any Borel subgroup is an element in set of maximal tori in G.

$\square$

Remark 3.1.8. *By Proposition 3.1.3, we see that the rank of a maximal algebraic torus is an invariant of G. We say G be of rank n if the rank of some (hence each) maximal algebraic tori is n.*

Definition 3.1.10. *A reductive linear algebraic group G/K is called split over K whether it inherits some split maximal torus $T \subset G$ over K.*

Remark 3.1.9. *More generally, a linear algebraic group G/K is said split, if it contains Borel subgroup B that is split over K. Here, solvable linear algebraic group B split over K, whether it is defined over K with has a composition series*

$$\{e_G\} = B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0 = B$$

with B_i's are connected algebraic subgroups of B defined over K and each quotient group B_i/B_{i+1} is isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_m$ over K. Since any algebraic torus is a connected solvable linear algebraic group, this observation explains the main idea in Definition 3.1.8.

3.2 Root Data and Root Systems

Here we shall take a look on basic weight theory of Lie algebras and reveal the connection between connected reductive algebraic groups and root datum. Throughout the section, K will be field with characteristic 0 with $\overline{K}$ will be seperable closure.

Definition 3.2.1. *Suppose G/K be an algebraic group. The space of left invariant derivations of G, is defined by*

$$L_G = \{D \in Der_K(K[G]) \mid D \circ \lambda_g = \lambda_g \circ D, \ \forall g \in G\}.$$

is the Lie algebra of G, with notation $\mathfrak{g} := Lie(G)$. Here $Der_K(K[G])$ is space of K-derivations of $K[G]$ endowed with action by left translations as

$$(t,p) \to \lambda_t(p)(: \lambda_t(p)(s) = p(t^{-1} \cdot s)), \ t \in G.$$

From now on, we will assume G/K be a connected split reductive group.

Remark 3.2.1. *We have an Ad-action of G on $\mathfrak{g}$ by*

$$Ad(g)(X) = g \cdot X \cdot g^{-1}, \ \forall g \in G, \forall X \in \mathfrak{g}.$$

Suppose T be some maximal algebraic torus in G. Image $Ad(T)$ consist diagonalizable (semisimple) commutative endomorphisms of $\mathfrak{g} \to \mathfrak{g}$. In particular they are diagonalizable simultaneously. The characters of T are the associated eigenvalues. Therefore in terms of eigenspaces,

$$\mathfrak{g} = \mathfrak{g}_0^T \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha^T.$$

In particular for any $0 \neq \alpha \in X^*(T)$, with $\mathfrak{g}_\alpha \neq 0$ called root of G, and α- eigenspace $\mathfrak{g}_\alpha^T$ is called the root space associated to $\alpha \in X^*(T)$. In particular, only finitely many α's occur in the direct sum. We shall denote all roots of G by $\Phi(G,T)$.

Definition 3.2.2. *Suppose $V/\mathbb{R}$ be some vector space, $dim_{\mathbb{R}}\, V < \infty$ with $\Phi \subset V$. The pair (Φ, V) is called root system whether :*

(1) $|\Phi| < \infty$, spans V and $0 \notin \Phi$.

(2) For any $\alpha \in \Phi$, there exists an involution s_α of V with $s_\alpha(\alpha) = -\alpha$, so that whose restriction on a codimension 1 subspace of V is identity and $s_\alpha(\Phi) = \Phi$. Such an element s_α is called a reflection of V along α.

(3) For any $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta = r \cdot \alpha$, $\exists r \in \mathbb{Z}$.

Remark 3.2.2. *In particular, we can give a concrete formula for the reflection s_α along $\alpha \in \Phi$ as*

$$s_\alpha(\gamma) = \gamma - 2 \cdot \frac{(\gamma, \alpha)}{(\alpha, \alpha)} \cdot \alpha.$$

where $(,)$ is the standart Euclidian inner product on V.

Definition 3.2.3. *Let (Φ, V) be some root system. A subset $\triangle \subset \Phi$ called whether $\triangle$ constitutes a base for V withal for all $\alpha \in \Phi$ can be uniquely written as*

$$\sum_{k=1}^{n} b_k \cdot \alpha_k$$

where $b_k \in \mathbb{Z}$ and have the same sign for all $k = 1, 2, .., n$. We denote the set of positive (res. negative) roots as Φ^+(res. Φ^-). Obviously $\Phi = \Phi^+ \cup \Phi^-$.

Given (Φ, V), define :

$$f : \Phi \to V^\vee = Hom(V, \mathbb{R}), \;\; \alpha \to \alpha^\vee$$

with subject to the following conditions : $\alpha^\vee(\alpha) = 2$, $\alpha^\vee(\Phi) \in \mathbb{Z} \cap \mathbb{R}$, for each $\alpha \in \Phi$, $s'_\alpha(\Phi) = \Phi$. The image is referred coroots and is denoted by $\Phi^\vee$.

Theorem 3.2.1. *(Springer, 1979) The tuple $(\Phi^\vee, V^\vee)$ defines a root system.*

Remark 3.2.3. *Suppose G/K be an split reductive group with T be maximal split algebraic torus in G defined over K. Then then for*

$$V =< \Phi(G,T) > \otimes_{\mathbb{Z}} \mathbb{R}$$

the tuple $(V, \Phi(G,T))$ defines root system.

Let $\mathfrak{X} = X^*(T)$ and set $\Phi = \Phi(G,T)$. We consider the quadruple,

$$\Psi(G,T) = (\mathfrak{X}, \Phi, \mathfrak{X}^\vee, \Phi^\vee).$$

For an arbitrary $\alpha \in \Phi$, let

$$s_\alpha : \mathfrak{X} \to \mathfrak{X} \qquad s_{\alpha^\vee} : \mathfrak{X}^\vee \to \mathfrak{X}^\vee$$

are linear maps defined as

$$s_\alpha(\beta) = \beta - < \beta, \alpha^\vee > \cdot \alpha$$

$$s_{\alpha^\vee}(\gamma) = \gamma - < \gamma, \alpha > \cdot \alpha^\vee.$$

Definition 3.2.4. *The quadruple $\Psi(G,T) = (\mathfrak{X}, \Phi, \mathfrak{X}^\vee, \Phi^\vee)$ is called root datum whether:*

(1) For any $\alpha \in \Phi$, one has $< \alpha, \alpha^\vee >= 2$;

(2) For any $\alpha \in \Phi$, one has $s_\alpha(\Phi) \subset \Phi$ with $\langle s_\alpha \mid \alpha \in \Phi \rangle$ is finite, generated by the family $\{s_\alpha\}_{\alpha \in \Phi}$.

We say it is reduced if $\alpha \in \Phi$ implies $2\alpha \notin \Phi$.

Remark 3.2.4. *Suppose G/K be some connected reductive group with (B,T) be Borel pair. By Theorem 3.1.4 and Proposition 3.1.1, all Borel pairs are conjugate under the action of G^{ad}. If $ad(g)\cdot(B,T) = (B',T')$ for some $g \in G^{ad}$, then, it induces an isomorphism between associated root data, which is independet of the choice of $g \in G^{ad}$. Therefore, we could identify all root data. We shall denote the root datum obtained in this way by $\Psi(G)$. In particular, the $Aut(G)$ acts on $\Psi(G)$, with G^{ad} acts trivially.*

Theorem 3.2.2. *(Springer, 1979) The map*

$$G \to \Psi(G,T) = (X^*(T), \Phi, X_*(T), \Phi^\vee)$$

induces bijection among $\overline{K}$-isomorphism clases of connected reductive groups withal the set of isomorphism classes of reduced based root data. In particular, an isomorphism between connected reductive groups determined uniquely up to inner automorphism induced by the map on root data.

Definition 3.2.5. *A dual group for G is the complex reductive group $\widehat{G}$ together withal an isomorphism :*

$$i : \Psi(\widehat{G}) \xrightarrow{\sim} \Psi(G)^\vee.$$

Example 3.2.1. *Consider the general linear algebraic group $G = GL_n$ over $\mathbb{C}$. The subgroup of diagonal matrices*

$$T = \{A_{z_1\cdots z_n} = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \mid z_i \in \mathbb{C}^\times\}$$

is a maximal split torus in GL_n. The maps

$$\mathbb{Z}^n \to Hom(T, \mathbb{G}_m) \qquad \mathbb{Z}^n \to Hom(\mathbb{G}_m, T)$$

$$(a_1,...,a_n) \to (A_{z_1\cdots z_n} \to z_1^{a_1}\cdots z_n^{a_n}) \qquad (a_1,...,a_n) \to (z \to A_{z_1^{a_1}\cdots z_n^{a_n}})$$

induces isomorphisms between $\mathbb{Z}^n$ and both character and cocharacter group of T. Note that the characters

$$\chi_{i,j} : A_{z_1\cdots z_n} \to z_i \cdot z_j^{-1}$$

for every tuple with $1 \le i \ne j \le n$, are roots of GL_n respect to T. Hence the roots spaces $gl_{n,\chi_{i,j}}$ are spanned by the matrices whatever entries are 0 except for (i,j)-th component.

In particular the coroot $\chi_{i,j}^{\vee}$ is the map which assigns any $z \in \mathbb{C}^{\times}$ to diagonal matrix, i-th entry is z, j- th entry is z^{-1} and 1 in all other entries.

3.3 Structure of Parabolic Subgroups

In this subsection we shall explore the relationship between parabolic subgroups of connected reductive groups and the associated root systems. This structural correspondence will help us to track down various representation theoretic and analytic concepts in detail.

Definition 3.3.1. *A connected reductive group is called quasi-split, whether it contains Borel subgroup. In this case, Borel subgroups are called minimal parabolic subgroups. A parabolic subgroup which contains (fixed) minimal parabolic subgroup is called standard parabolic subgroup.*

Theorem 3.3.1. *(Borel, 1969) Suppose G be some connected reductive group. Any minimal parabolic of G inherits centralizer of some split algebraic tori.*

By the previous theorem, in order to characterize all minimal parabolics of G, is sufficient to arrange some maximal split tori T to characterize minimal ones comprising $C_G(T)$. By eigenspace decomposition, for $\Phi^+ \subset \triangle$, there exists unique minimal parabolic which unipotent radical U compensates,

$$Lie\ U = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha}.$$

In particular, let $S \subset \triangle$ be a subset of the base and let

$$\Phi(S) = < S >_{\mathbb{Z}} \cap \Phi.$$

Let B be a fixed minimal parabolics in G. Then there is a unique parabolic P_S, so that $B \subset P_S$ and its unipotent radical N_S satisfies

$$Lie\ N_S = \bigoplus_{\alpha \in \Phi^+ - (\Phi(S) \cap \Phi^+)} g_{\alpha}$$

Theorem 3.3.2. *(Borel, 1969) There exists a bijection among ensemble of bases $\triangle \subset \Phi(G,T)$ with minimal parabolis subgroups G withal $Z(T) \subset P$. Likewise, there exists a bijection among subsets of the base $\triangle$ with standart parabolics in G.*

Remark 3.3.1. *By Theorem 3.3.2, we can introduce a subgroup of G which is some sort of dual of a given parabolic subgroup P, so called opposite parabolic subgroup. Suppose P be some standard parabolic in G inheriting the centralizer $C_G(T)$. If P is minimal, we define P^-, minimal parabolic associated to $-\triangle$ by Theorem 3.3.2. Suppose $S \subset \triangle$. If P is not minimal, again by Theorem 3.3.2, we may assume $P \subset P_S$. Then put P_S^- to be parabolic of G, inheriting P^-, associated to the subset*

$$-S = \{\alpha \in \Phi \mid -\alpha \in S\} \subset -\triangle\,.$$

Example 3.3.1. *Let $G = GL_n$ be the general linear group. Then,*

$$B = \{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & \ddots * \end{pmatrix} \} \subset GL(n)$$

is a Borel subgroup in GL_n. By Theorem 3.1.2, one has $B = T \cdot U$, where T be diagonal matrices, withal U be upper triangular ones, withal cross entries are 1. By Example 3.2.1, roots of GL_n are given by characters of T of the following form,

$$\chi_{i,j} : A_{z_1 \cdots z_n} \to z_i \cdot z_j^{-1}$$

for every tuple $1 \leq i \neq j \leq n$. In particular,

$$\triangle = \{\chi_{i,i+1} \mid i \in [1; n-1]\}.$$

and $\Phi^+ = \{\chi_{i,j} \mid i < j\}$ the positive roots in $\triangle$. Moreover, we have bijective correspondence between

$$\{S \subset \triangle\} \xrightarrow{\sim} \{n_1, n_2, \cdots, n_k \in \mathbb{Z}^+ \mid \sum_{i=0}^{k} n_i = n\}$$

between subsets S of the base and partitions of n, where n_1 is the first integer, which χ_{n_1,n_1+1} is not the element of the subset S, in the same way n_k is the k-th index which χ_{n_k,n_k+1} is not the element of S. Therefore by Theorem 3.3.2, we see that the set of standart parabolic subgroups of GL_n are given by partitions of n. Therefore standard parabolic subgroup P_k in GL_n inheriting B is the following type :

$$P_k = \{ \begin{pmatrix} GL_{n_1} & * & \cdots & & * \\ & GL_{n_2} & * & \cdots & \vdots \\ & & & \ddots & * \\ & & & & GL_{n_k} \end{pmatrix} \in GL_n \mid n_1 + n_2 + \cdots n_k = n \} \subset GL_n$$

Chapter 4

Automorphic Representations

In this chapter, we shall introduce theory of representations on adelic algebraic groups, which constitutes main substances of global Langlands philosophy. Our presentation will be divided into two parts, namely archimedean and non-archimedean theory. Finally, we will prove Flath's theorem, which provides a simple description of some particular class of representations in terms of restricted tensor products.

4.1 $(\mathfrak{g}, K)$-modules

This part, we introduce a particular invariant spaces and the notion of admissiblity as in Chapter 2 in the full generality. Throughout the section, G will be a connected linear algebraic group over an archimedean field E with K will be some maximal compact subgroup of $G(E)$.

Definition 4.1.1. *Suppose (π, V) be a representation of K. If $[\beta]$ denotes an equivalence class of some irreducible representation of K, then consider the subspace*

$$V([\beta]) = \{v \in V \mid \langle \pi(t) \cdot v \ : t \in K \rangle_{\mathbb{C}} \simeq \beta\}$$

We call these subspaces β-isotypic. In particular an element $v \in V$ is called K-finite if $v \in V([\beta])$ for an equivalence class $[\beta]$ of irreducible representation of K. Decomposition

$$V_{fin} := \bigoplus_{[\beta]} V([\beta])$$

is usually called space of K-finite vectors of V, where summation ranges over all $[\sigma]$.

Definition 4.1.2. *A given representation of* $G(E)$ *is called admissible if for each irreducible representation* β*, associated* β*-isotypic space* $V([\beta])$ *is of finite dimensional.*

Definition 4.1.3. *Let V be a complex vector space. Then it is called a* $(\mathfrak{g}, K)$*-module if it is equipped with an action of the Lie algebra* $\mathfrak{g} = Lie\ G$ *and of the K, both denoted by* π*, so that :*

(1) V is a direct sum of finite dimensional representations of K.

(2) For any $X \in Lie\ K$ *and* $v \in V$ *the limit*

$$\frac{d}{dt}(\pi(exp(t \cdot X)) \cdot v)|_{t=0} = \lim_{h \to 0} \frac{(\pi(exp(h \cdot X)) \cdot v - v)}{h}$$

on the right exists and is equal to $\pi(X) \cdot v$.

(3) For any $t \in K$, $X \in \mathfrak{g}$*, one has*

$$\pi(t) \cdot \pi(X) \cdot \pi(t^{-1}) \cdot v = \pi(Ad(t) \cdot X) \cdot v$$

Moreover V is called admissible if the isotropic subspaces $V([\beta])$ *are finite dimensional, for each equivalence class* $[\beta]$ *of irreducible representation of K.*

Remark 4.1.1. *Suppose* $\mathfrak{g}$ *denotes a Lie algebra over E. One has the complexified Lie algebra as*

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

Note that, any associative algebra becomes a complex Lie algebra by the bracket $[X,Y] = X \cdot Y - Y \cdot X$*. In particular, the universal enveloping algebra* $U(\mathfrak{g}_{\mathbb{C}})$ *of* $\mathfrak{g}_{\mathbb{C}}$ *is an associated unital* $\mathbb{C}$*-algebra together with a homomorphism* $\imath : \mathfrak{g}_{\mathbb{C}} \to U(\mathfrak{g}_{\mathbb{C}})$ *of complex Lie algebras in the sense that for any Lie algebra homomorphism* $f : \mathfrak{g}_{\mathbb{C}} \to A$ *there exists a unique extension of f to a homomorphism of complex Lie algebras* $\hat{f} : U(\mathfrak{g}_{\mathbb{C}}) \to A$ *with* $f = \hat{f} \circ i$*. In particular, any complex representation of* $\mathfrak{g}$*, uniquely considered as a* $U(\mathfrak{g}_{\mathbb{C}})$*-module. We shall denote the center of* $U(\mathfrak{g}_{\mathbb{C}})$ *by* $Z(\mathfrak{g}_{\mathbb{C}})$.

Example 4.1.1. *Let* $G = GL_2$ *be the general linear group defined over* $\mathbb{R}$*. We have*

$\mathfrak{g} = Lie\ G = Mat_2(\mathbb{R})$. *As a decomposition we have*

$$Mat_2(\mathbb{R}) = \mathbb{R}z \oplus \mathbb{R}h \oplus \mathbb{R}a^+ \oplus \mathbb{R}a^-$$

where

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad a^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In particular, the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ *of* $Mat_2(\mathbb{C})$ *is generated by these element as a* $\mathbb{C}$*-algebra. By direct calculation, we have*

$$[h, a^+] = 2 \cdot a^+, [h, a^-] = -2 \cdot a^-, [a^+, a^-] = h$$

We put

$$\Delta = -\frac{h^2 + 2 \cdot a^+ \cdot a^- + 2 \cdot a^- \cdot a^+}{4} \in U(\mathfrak{g}_{\mathbb{C}})$$

Moreover, $\Delta \in U(\mathfrak{g}_{\mathbb{C}})$ *commutes with basis elements of* $U(\mathfrak{g}_{\mathbb{C}})$ *and hence* $\Delta \in Z(\mathfrak{g}_{\mathbb{C}})$*. In particular, we have* $Z(\mathfrak{g}_{\mathbb{C}}) = \langle \Delta, z \rangle_{\mathbb{C}}$*. We set*

$$k_\theta = \begin{pmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{pmatrix}$$

For ever pair $(s, \mu) \in \mathbb{C}^2$*, let* $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ *be a central character defined by*

$$z \to \mu, \quad \Delta \to s(1-s)$$

For any $\varepsilon \in \{0, 1\}$*, we have a* $(\mathfrak{g}, K)$*-module* $(\pi_{s,\mu,\varepsilon}, V_{s,\mu,\varepsilon})$*, with central character* χ*, which is given by as follows : As a representation space, we have*

$$V = \bigoplus_{t \equiv \varepsilon\ mod\ 2} \mathbb{C} \cdot v_t$$

with actions of the basis elements as

$$\pi(k_\theta) \cdot v_t = e^{it\theta} \cdot v_t,\ z \cdot v_t = \mu \cdot v_t,\ ,a^+ \cdot v_t = (s + \frac{t}{2}) \cdot v_{t+2},\ a^- \cdot v_t = (s - \frac{t}{2}) \cdot v_{t-2},\ \Delta \cdot v_t = s(1-s) \cdot v_t$$

and $\pi \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot v_t = v_{-t}$*. For* $k \in \mathbb{Z}$ *with* $k \equiv \varepsilon$ *mod 2,* $s \neq \frac{k}{2}$ *,* $(\pi_{s,\mu,\varepsilon}, V_{s,\mu,\varepsilon})$ *is an irreducible* $(\mathfrak{g}, K)$*-module.*

Let G be a connected reductive group over E, $\mathfrak{g} = Lie\ G$ be the Lie algebra of G and T be maximal algebraic torus of G, with Lie algebra $\mathfrak{t}$. In particular $\mathfrak{t}$ a Cartan subalgebra in $\mathfrak{g}$, that is a self normalizing nilpotent subalgebra. From this data, we obtain the following root system :

$$\Phi := \Phi(G \times_{Spec(\mathbb{R})} Spec(\mathbb{C}), T \times_{Spec(\mathbb{R})} Spec(\mathbb{C}))$$

Let $\triangle \subset \Phi$ be the base of this root system and $\Phi^+ \subset \Phi$ be the positive roots. Set for Harish-Chandra homomorphism as :

$$\chi : Z(\mathfrak{g}) \xrightarrow{proj} U(\mathfrak{t}) \longrightarrow Z(\mathfrak{t}).$$

$$X \xrightarrow{proj} X \longrightarrow X - \rho(X).$$

with

$$\rho := \frac{1}{2} \cdot \sum_{\alpha \in \Phi^+} \alpha$$

Theorem 4.1.1. *(Wallach, 1988) The map χ is a homomorphism, with $Im(\chi) \subset U(\mathfrak{t})^W$, where $W = W(G,T)(\mathbb{C}) = (N_G(T)/Z_G(T))(\mathbb{C})$ is the Weyl group of T in G acting on $\mathfrak{t}_{\mathbb{C}}$, and induces the isomorphism*

$$\chi : Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{t})^W$$

Remark 4.1.2. *By the Theorem 4.1.1, we can explicitly describe characters of the $\mathbb{C}$-algebra $Z(\mathfrak{g})$. Let $\phi \in Hom_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C})$. Since $\mathfrak{t}_{\mathbb{C}}$ is a commutative Lie algebra, the map $\phi : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ is a Lie algebra morphism. By Remark 4.1.1, one may amplify ϕ uniquely to some Lie algebra mapping $\hat{\phi} : U(\mathfrak{t}_{\mathbb{C}}) \to \mathbb{C}$. Composing $\hat{\phi}$ with χ in Theorem 4.1.2, we obtain a character of $Z(\mathfrak{g})$,*

$$\phi_\chi := \hat{\phi}|_{U(\mathfrak{t})^W} \circ \chi : Z(\mathfrak{g}) \to \mathbb{C}$$

Theorem 4.1.2. *Any Lie algebra homomorphism morphism $Z(\mathfrak{g}) \to \mathbb{C}$ is in shape ϕ_χ for appropriate $\phi \in Hom_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C})$. In particular, for any pair of elements $\phi_1, \phi_2 \in Hom_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C})$, the associated maps $\phi_{1,\chi}, \phi_{2,\chi}$ are equal if and only if they occur common W-orbit.*

Proof. The Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ associated to a maximal algebraic torus T is a commutative Lie algebra and hence its universal enveloping algebra $U(\mathfrak{t}_{\mathbb{C}})$ is nothing but the complex symmetric algebra $Sym(\mathfrak{t}_{\mathbb{C}})$. In particular $Sym(\mathfrak{t}_{\mathbb{C}})$ is a polynomial algebra on the set $Hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}},\mathbb{C})$ i.e $Sym(\mathfrak{t}_{\mathbb{C}}) \xrightarrow{\sim} \mathbb{C}[Hom_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}},\mathbb{C})]$ Therefore we have

$$Spec(\mathbb{C}[Hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}},\mathbb{C})],\mathbb{C}) \xrightarrow{\sim} Hom(U(\mathfrak{t}_{\mathbb{C}}),\mathbb{C})$$

The set $Spec(\mathbb{C}[Hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}},\mathbb{C})],\mathbb{C})$ is nothing but the maximal ideals of $Hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}},\mathbb{C})$ and by Hilbert's Nullstellensatz, it is identified with $Hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}},\mathbb{C})$. By looking at W-invariant elements, theorem follows.

□

Remark 4.1.3. *In particular, we say a $\mathfrak{g}$-module V adopt some infinitesimal character if the action of $Z(\mathfrak{g}_{\mathbb{C}})$ on V is given by a character. By Theorem 4.1.2 and Theorem 4.1.3, any such character can be seen as an element of $Hom_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}},\mathbb{C})$.*

Lemma 4.1.1. *(Harish - Chandra, 1968). Suppose V be some complex vector space with countable dimension. For any $T \in End_{\mathbb{C}}(V)$, there exists some $c \in \mathbb{C}$, so that $T - c \cdot I_V \in End_{\mathbb{C}}(V)$ is not invertible.*

Proof. Assume that, $T - c \cdot I_V \in End_{\mathbb{C}}(V)$ is invertible for all $c \in \mathbb{C}$. Then for any polynomial $P(x) \in \mathbb{C}[x]$, $P(T)$, as an element of $End_{\mathbb{C}}(V)$, is invertible on V. Hence for any rational function $R = \frac{P}{Q} \in \mathbb{C}(x)$, we may able to define $R(T)$ by setting $P(T) \cdot Q(T)^{-1}$. Therefore we have the induced linear mapping

$$\mathbb{C}(x) \to End_{\mathbb{C}}(V)$$

For $0 \neq v \in V$ and $0 \neq R = \frac{P}{Q} \in \mathbb{C}(x)$, $P(T) \cdot v = 0$ implies $R(T) \cdot v = 0$. Hence the mapping $\mathbb{C}(x) \to End_{\mathbb{C}}(V)$ is injective. This contradicts with our assumption, since $\mathbb{C}(x)$ has an uncountable dimension over $\mathbb{C}$.

□

Lemma 4.1.2. *(Dixmier's Lemma) Suppose V be some complex vector space with $S \subset End_{\mathbb{C}}(V)$ acting irreducibly on V. Any $T \in End_{\mathbb{C}}(V)$ commuting each element in S, one has $T = c \cdot I_V$, for some $c \in \mathbb{C}$.*

Proof. By Lemma 4.1.1, there is $c \in \mathbb{C}$, with $T - c \cdot I_V$ is not invertible. By assumption T commutes with every element of S, therefore S preserves $ker(T - c \cdot I_V)$ and $Im(T - c \cdot I_V)$. Since S acts irreducibly, either of the subspaces must be 0. Hence $T = c \cdot I_V$.

□

4.2 Non-Archimedean Representation Theory

Suppose G be some connected reductive group upon some non-archimedean F with parabolic P. Suppose $P = M \cdot N$ be the Levi partition of P, where M be the Levi factor of P. Let δ_P be the modular quasi-character function attached to P defined by the rule :

$$\delta_P := \delta_{P(F)} : P(F) \to \mathbb{R}^{\times}$$

$$\delta_P(x) := |det(Ad_{Lie\ P}(x) : Lie\ P(F) \to Lie\ P(F))|$$

Definition 4.2.1. *Suppose (σ, V) be irreducible smooth representation of $M(F)$. Parabolically induced representation $I(\sigma) := Ind_P^G(\sigma)$ of $G(F)$ is the smooth representation on locally constant mappings,*

$$Ind_P^G(V) := \{\psi : G(F) \to V \mid \psi(m \cdot n \cdot g) = \delta_P(m)^{\frac{1}{2}} \cdot \sigma(m) \cdot \psi(g)\}.$$

$\forall (m,n,g) \in M(F) \times N(F) \times G(F)$, where :

$$I(\sigma)(g) \cdot \psi(x) = \psi(x \cdot g)$$

for $g, x \in G(F)$ and $\psi \in Ind_P^G(V)$.

Definition 4.2.2. *The functor*

$$Ind_P^G : Rep_{sm}\, M(F) \to Rep_{sm}\, G(F)$$

is called parabolic induction.

Definition 4.2.3. *An admissible (resp. smooth) representation (π, V) of $G(F)$ called supercuspidal (resp. quasi-cuspidal) whether all matrix coefficients of π have compactly support respect to center $Z_G(F)$.*

Proposition 4.2.1. *Let (σ, V) be admissible representation on $M(F)$. Then $I(\sigma)$ is also admissible.*

Proof. Suppose $K \subset G(F)$ be some open subgroup. To deduce proposition, need to show that space $Ind_P^G(V)^K$ is finite dimensional. By the definition of the parabolic induction any function $\varphi \in Ind_P^G(V)^K$ is determined by the set of representatives R for coset space

$$M(F)N(F) \backslash G(F)/K$$

By Iwasawa decomposition, coset space is finite. In particular, for any $r \in R$, one has $\varphi(r) \in V^{M(F)\cap rKr^{-1}}$. By assumption (σ, V) is admissible thus the space $V^{M(F)\cap xKx^{-1}}$ is finite dimensional. Thereby $Ind_P^G(V)^K$ is too hence $Ind_P^G(V)$ be an admissible representation.

$\square$

Now we shall introduce left adjoint functors to parabolic induction Ind_P^G. Our main objective is to show that, one can construct admissible representations from supercuspidal representations via parabolic induction. To accomplish this, we introduce Jacquet modules as follows :

Suppose (σ, V) be smooth representation of G. Consider below subspace

$$V(N) := \langle \pi(t)v - v \mid v \in V, t \in N \rangle.$$

with the quotient $V_N = V/V(N)$. By Theorem 3.1.2, the Levi subgroup $M(F)$ acts on unipotent radical $N(F)$ by conjugation and hence $M(F)$ acts on the quotient V_N. Thus we can the Jacquet functor on smooth representations as :

$$J_N : Rep_{sm}(G(F)) \to Rep_{sm}(M(F))$$

$$(\pi, V) \to (\pi_N := \pi|_{M(F)} \otimes \delta_P^{\frac{1}{2}}, V_N)$$

Definition 4.2.4. *The smooth representation (π_N, V_N) of $M(F)$ is called the Jacquet module of the smooth representation (σ, V) of $G(F)$.*

Proposition 4.2.2. *Suppose (π_1, V), (π_2, W) be smooth representation of G and $M(F)$ respectively. Then :*

$$Hom_G(V, Ind_P^G(W)) \xrightarrow{\sim} Hom_{M(F)}(V_N, W).$$

Proof. The mapping $I(1) : Ind_P^G(W) \to W$, $\varphi \to \varphi(1)$ for $\in G$ is surjective. Since N acts trivially on $Ind_P^G(W)$, any $G(F)$-stable mapping $V \to Ind_P^G(W)$ factors through V_N. Hence we have $M(F)$-stable mapping

$$\widehat{I(1)} : Hom_{G(F)}(V, Ind_P^G(W)) \xrightarrow{\sim} Hom_{M(F)}(V_N, W)$$

To prove the proposition, we have to show that $\widehat{I(1)}$ admits an inverse. Let $\phi : V_N \to W$

be a $M(F)$- intertwining mapping for $\widehat{I(1)}$. Then we set

$$\widehat{\phi} : V \to Ind_P^G(W),\ \phi \to (g \to \phi(g \cdot \varphi))$$

and hence the mapping $\phi \to \widehat{\phi}$ is the inverse of $\widehat{I(1)}$. □

Theorem 4.2.1. *(Getz & Hahn, 2019) Suppose (R,U) be admissible representation of G. Then under Jacquet functor $J_N : Rep_{sm}(G(F)) \to Rep_{sm}(M(F))$, $J_N((r,U))$ is also admissible. In particular, an irreducible smooth representation (R,U) of G is quasi-cuspidal whether for any parabolic $P \leq G$, with Levi decomposition $P = MN$, the Jacquet module vanishes i.e $V_N = 0$. In particular, any irreducible quasi-cuspidal representation is supercusipdal.*

Theorem 4.2.2. *Suppose (R,U) be irreducible smooth representation on $G(F)$. Then for some parabolic $P \subset G$, there is some supercuspidal representation (S,U_1) of $M(F)$ with non-zero mapping $U \to Ind_P^G(U_1)$.*

Proof. We will proceed by induction. Theorize $dim\, G = 1$, so G be an algebraic torus, hence theorem holds. Now assume that $dim\, G > 1$ and for every $P \subset G$, mapping $U \to Ind_P^G(U_1)$ be absent. By Proposition 4.2.2, we have $Hom_{M(F)}(U_N, U_1) = 0$. Hence by Theorem 4.2.1, (R,U) is quasi-cuspidal. If (S,U_1) be smooth representation of the Levi subgroup with a non-zero intertwining map $U \to Ind_P^G(U_1)$, then by Proposition 4.2.2, we have a non-zero mapping $U_N \to U_1$, as $M(F)$-representations, hence by induction hypothesis, there is some parabolic subgroup $H \subset M$ and supercuspidal representation (P,T) of the Levi subgroup M_S, with a non-zero intertwining mapping

$$U_N \to U_1 \to Ind_H^M(T)$$

By Proposition 4.2.2, we have a non-zero mapping

$$V \to Ind_P^G \circ Ind_H^M(T) \xrightarrow{\sim} Ind_{HN}^G(T)$$

Hence we have the assertion.

□

4.3 Classical Automorphic Forms and Representations

Here we exhibit automorphic forms and automorphic representations associated to adelic points of arbitrary reductive groups. We shall give two description of automorphic forms which usage of either them depends on the considered context. Throughout the section, G be reductive group over global field E. Our main weld in this part is (Getz & Hahn, 2019).

We display with $\mathfrak{a}_E$, all archimedean places of E and let $E_\infty = \prod_{v\in\mathfrak{a}_E} E_v$. Note that $E_\infty = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Adjust some maximal compact $K_\infty \leq (Res_{E/\mathbb{Q}}\, G)(\mathbb{R}) = G(E_\infty)$ and open subgroup $K^\infty \leq G(A_E^\infty)$. Suppose $j : G \hookrightarrow GL_n$ be closed embedding over E. This induces the following embedding :

$$\overline{j} : G \hookrightarrow GL_{2n}, \quad \overline{j}(g) = \begin{pmatrix} j(g) & \\ & j^{-t}(g) \end{pmatrix}$$

By using these embedding, we define the following norm on $G(E_\infty)$:

$$||g|| := ||g||_j = \prod_{v\in\mathfrak{a}_E} (\max_{1\leq i,j\leq 2n} |g_{ij}|_v)$$

We shall first study at infinity in order to pass adelic setting.

Definition 4.3.1. *Let $\Gamma := j^{-1}(GL(O_E))$ subgroup of $G(E)$. A smooth function $f : G(E_\infty) \to \mathbb{C}$ called automorphic form whether :*

1) f is (left) Γ-invariant : $f(ts) = f(s), \quad t \in \Gamma$.

2) (K_{max}-finite) The subspace of K_∞-translates $\{g \to \varphi(g\cdot k) \mid k \in K_\infty\}$ is finite dimensional.

3) ($Z(\mathfrak{g}_\mathbb{C})$-finite) The subspace $\langle Z(\mathfrak{g}_\mathbb{C})\cdot f\rangle$ is finite dimensional.

4) f is of moderate growth i.e there exists $k, m \in \mathbb{R}_{>0}$ with

$$|f(s)| \leq m\cdot ||s||^r.$$

Remark 4.3.1. *In particular, it may appropriate per distinguish space, which* *annihilated on an arbitrary ideal $J \triangleleft Z(\mathfrak{g})$. We denote this subspace by $A(\Gamma, J)$. Furthermore, we have the following decomposition*

$$A(\Gamma, J) = \bigoplus_{\sigma} A(\Gamma, J, \sigma)$$

where the summation ranges over all irreducible representations of K_∞.

Now we shall consider adelic automorphic forms which in some way similar with archimedean case. We consider the closed embedding $\bar{\imath} : G \hookrightarrow SL_{2n}$, defined by the same formula for $\bar{j} : G \hookrightarrow GL_{2n}$ in archimedean case. Moreover, for $S \subset \mathfrak{a}_E \cup \mathfrak{h}_E$, we define following norm on $G(\mathbb{A}_E)$:

$$||g||_S = \prod_{v \in S} ||g||_v.$$

We will omit the notation S in the norm above if the above product is over all places of E.

Definition 4.3.2. *A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_E^\infty)$-module (π, V) is both $(\mathfrak{g}, K_\infty)$ and $G(\mathbb{A}_E^\infty)$-module in which both actions commutes. In particular, it is admissible whether any base subgroup K^∞ of $G(\mathbb{A}_E^\infty)$, subspace of K^∞-fixed vectors V^{K^∞} be an admissible $(\mathfrak{g}, K_\infty)$ -module.*

Remark 4.3.2. *Let K_{max} be maximal compact subgroup in $G(\mathbb{A}_E)$. Then $K_{max} = K_\infty \times K_{max}^\infty$ where K_{max}^∞ be maximal compact in $G(\mathbb{A}_E^\infty)$. A mapping $p : G(A_E) \to \mathbb{C}$ said K-finite whether K_{max}-subspace*

$$\{g \to p(g \cdot t) \mid t \in K_{max}\}.$$

is finite dimensional.

Definition 4.3.3. *A smooth function $f : G(\mathbb{A}_E) \to \mathbb{C}$ is called automorphic form on G if :*

1) $f(x \cdot y) = f(x), \quad \forall x \in G(\mathbb{A}_E)$ and $\forall y \in G(E)$.

2) (K_{max}-finite) The subspace of K_{max}-translates $\{g \to \varphi(g \cdot k) \mid k \in K_{max}\}$ is finite dimensional.

3) ($Z(\mathfrak{g}_\mathbb{C})$-finite) The subspace $\langle Z(\mathfrak{g}_\mathbb{C}) \cdot f \rangle$ is finite dimensional.

4) f is of moderate growth i.e there exists $k, m \in \mathbb{R}_{>0}$, so that

$$|f(g)| \leq m \cdot ||g||^r.$$

We shall display automorphic forms on G by $\mathcal{A}(G)$.

Remark 4.3.3. *Besides $\mathcal{A}(G)$ endowed as a $(\mathfrak{g}, K_\infty)$-module and action of $G(\mathbb{A}_E^\infty)$ by right translation of elements as :*

$$(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_E^\infty) \times \mathcal{A}(G) \to \mathcal{A}(G)$$

$$(X, t, g, \varphi) \to (s \to X \cdot \varphi(s \cdot t \cdot g))$$

By construction, both actions of $(\mathfrak{g}, K_\infty)$ and of $G(\mathbb{A}_E^\infty)$ commutes, therefore $\mathcal{A}(G)$ effectuate $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_E^\infty)$-module. However, the $G(\mathbb{A}_E^\infty)$-module structure on $\mathcal{A}(G)$ does not extend to a $G(\mathbb{A}_E)$-module structure. Indeed for any $f \in \mathcal{A}(G)$ and $g \in G(E_\infty)$, the mapping $x \to f(xg)$ is only $gK_\infty g^{-1}$-finite.

Definition 4.3.4. *A representation of $G(\mathbb{A}_E)$ be automorphic if it is irreducible, admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_E^\infty)$-module that is equivalent to some subquotient of $\mathcal{A}(G)$.*

Definition 4.3.5. *An automorphic form $f \in \mathcal{A}(G)$ is called cuspidal whether any parabolic subgroup $P \subset G$ along unipotent radical N_P, the following vanishing condition*

$$\int_{N_P(E)\backslash N_P(A_E)} f(n \cdot g) dn = 0$$

holds for all $g \in G(A_E)$. We shall denote the subspace of cuspidal forms as $\mathcal{A}_{cusp} := \mathcal{A}_{cusp}(G)$. In particular, a cuspidal representation of $G(\mathbb{A}_E)$ is an automorphic representation of $G(\mathbb{A}_E)$ that equivalent to some subquotient of $\mathcal{A}_{cusp}$.

4.4 Flath's Theorem

Here we shall prove Flath's Theorem which provides a factorization of a given automorphic representation in terms of local representations. In order to do this, we shall introduce the concept of Hecke algebra and the associated Hecke modules. Unless otherwise stated, G will be a unimodular, locally profinite topological group.

Definition 4.4.1. *A mapping $m : G \to \mathbb{C}$ is called smooth whether it is (right) invariant* *via some base subgroup. We denote space of all smooth functions on G by $C^\infty(G)$. In particular we display as $C_c^\infty(G)$, subspace of compact supported smooth mappings on G.*

Remark 4.4.1. *Suppose μ be some fixed Haar measure on G. We define binary operation by following formula :*

$$m_1 \star m_2(t) := \int_G m_1(s)\cdot m_2(s^{-1}\cdot t)d\mu(s), \;\; m_1, m_2 \in C_c^\infty(G).$$

It is straightforward to grasp, operation $\star$ on $C_c^\infty(G)$ be associative. It is called a convolution on G.

Definition 4.4.2. *The pair $H(G) := (C_c^\infty(G), \star)$ is said to Hecke algebra of G.*

Remark 4.4.2. *In general $H(G)$ does not have a unit element but it does have many idempotent elements. To see this, suppose K_1 be an open compact in G. Then $e_K \in H(G)$ defined as :*

$$e_K(g) = \begin{cases} \mu(K_1)^{-1} & g \in K_1 \\ 0 & g \notin K_1 \end{cases}$$

an idempotent element of $H(G)$.

Proposition 4.4.1. *For $m \in H(G)$, $e_{K_1} \star m = m$ whether $m(k\cdot h) = m(h)$, $\forall h \in G$ and $\forall k \in K_1$.*

Proof. By calculating,

$$e_{K_1} \star m(k\cdot h) = \int_G e_{K_1}(s)\cdot m(s^{-1}\cdot k\cdot h)\cdot d\mu(s).$$

$$= \int_G e_{K_1}(k\cdot s)\cdot m(s^{-1}\cdot h)\cdot d\mu(s).$$

$$= \int_G e_{K_1}(s)\cdot m(s^{-1}\cdot h)\cdot d\mu(s) = e_{K_1} \star m(h).$$

Hence whether $m(k\cdot h) = m(h)$ then integral is equal to $m(h)$ thus we deduce the assertion.

□

Remark 4.4.3. *Suppose (R, U) be smooth representation of G and let $m \in H(G)$, $u \in U$. We define action of $H(G)$ by putting*

$$\pi(m)\cdot u = \int_G f(h)\cdot \pi(h)\cdot u\cdot d\mu(h).$$

The mapping

$$C_c^\infty(G)\otimes U \to C_c^\infty(G,U)$$

$$\sum_i m_i \otimes u_i \to (h \to \sum_i m_i(h)\cdot u_i)$$

is an isomorphism of $\mathbb{C}$*-vector spaces. Therefore* $m(h)\cdot R(h)\cdot u \in C_c^\infty(G,U)$ *and hence* $\pi(m)\cdot u \in U$*. Besides this action is smooth i.e* $H(G)\star U = U$*.*

Proposition 4.4.2. *Suppose* U *be smooth* G*-module. Then* U *is irreducible* $\Longleftrightarrow U^K$ *is irreducible as* $H(G,K_1)$*-module in which* $H(G,K_1) = e_{K_1}\star H(G)\star e_{K_1}$*, the* $\mathbb{C}$*-subalgebra of* $H(G)$*, for any open compact* $K_1 \subset G$*.*

Proof. Assume contrary, U be reducible. Then by Remark 4.4.3 we have $U = W_1 \oplus W_2$ as $H(G)$-module. Then we have $U^K = W_1^K \oplus W_2^K$ as $H(G,K_1))$-modules for any open compact subgroup K_1 of G. By smoothness the spaces W_1^K and W_2^K are nonzero then so U^K. This contradicts reducibility of U. Conversely, let us assume U is irreducible, and $U^K = W_1 \oplus W_2$ as a $H(G,K_1))$-module for some open compact subgroup K_1. Since V is irreducible, $V_1 \neq 0$ implies

$$W_1 = H(G,K_1))\cdot W_1 = e_{K_1}\star H(G)\star e_{K_1}\cdot W_1 = e_{K_1}\star H(G)\cdot W_1 = e_{K_1}\cdot U = U^K$$

Hence we deduce that $W_2 = 0$.

□

Remark 4.4.4. *Let* I *be a countable set and* $I_1 \subset I$ *be some finite subset. Suppose* $\{V_v \mid v \in I\}$ *be ensemble of complex spaces indexed by I. Every* $v \in I - I_1$*, let* $w_{0,v} \in W_v$ *be a non-zero vector. For every subset* S *of* I *with* $I_1 \subset S \subset I$*, we set* $W_S = \prod_{v\in S} W_v$*. Then for every* $S \subset S_1 \subset I$*, we define the mapping* $f_S^{S_1} : W_S \to W_{S_1}$ *by the rule*

$$\bigotimes_{v\in S}\phi_v \to \bigotimes_{v\in S}\phi_v \bigotimes (\bigotimes_{v\in S_1 - S} w_{0,v})$$

The restricted tensor product of the family $(W_v)_{v\in I}$ *is defined by*

$$W = {\bigotimes_{v\in I}}' W_v = \lim_{\to,S} W_v$$

whereby transition mappings given by the mappings $f_S^{S_1} : W_S \to W_{S_1}$ *for every subset* $S, S_1 \subset I$ *with* $S \subset S_1$*.*

Definition 4.4.3. *A $C_c^\infty(G(\mathbb{A}_E^\infty))$-module V is called factorizable if it has a restricted tensor product decomposition*

$$V \simeq {\bigotimes_{v \in M_E}}' V_v$$

with respect to family of subspaces $\{W_v \subset V_v \mid v \in M_E\}$ so that $dim(W_v^{K_v}) = 1$, for a choice of a maximal compact K_v in $G(E_v)$, $\forall v \in M_E$.

Theorem 4.4.1. *(Satake Isomorphism) Suppose G be split reductive group over non-archimedean local field F_1 with K_1 be maximal compact open subgroup of $G(F_1)$. Then,*

$$C_c^\infty(G(F_1), K_1) \xrightarrow{\sim} \mathbb{C}[\widehat{T}]^{W(\widehat{G},\widehat{T})(\mathbb{C})}$$

for some maximal dual algebraic torus $\widehat{T} \leq \widehat{G}$.

Theorem 4.4.2. *Under same suppositions of previous theorem, $C_c^\infty(G(F_1), K_1)$ is a commutative ring.*

Proof. Since T is a commutative algebraic group, then its complex dual $\hat{T}$ is also a commutative complex algebraic group and hence the group algebra $\mathbb{C}[\hat{T}]$ is a commutative ring. Hence by Satake isomorphism, $C_c^\infty(G(F_1), K_1)$ is commutative. $\square$

Theorem 4.4.3. *Suppose $G = G_1 \times G_2$ with each component be locally profinite. Then,*

(1) Given U_i be irreducible admissible representation on G_i, $i = 1, 2$, whence so $U_1 \otimes U_2$ is.

(2) Given U be irreducible admissible representation on G, there is an irreducible admissible representation V_i on G_i, which $V \simeq V_1 \otimes V_2$.

Proof. (1) By assumption both U_1 and U_2 are admissible G_1 and G_2 modules respectively, then for compact open subgroups K_i of G_i, for $i = 1, 2$, we have $dim(V_1^{K_1} \times V_2^{K_2}) = dim(V_1^{K_1}) \times dim(V_2^{K_2}) < \infty$ and hence $V_1 \times V_2$ is an admissible G-module. By Proposition 4.4.2 for any open compact $K_1 \times K_2$ of $G_1 \times G_2$, $H(G, K_i)$-module $V_i^{K_i}$, irreducible and finite dimensional. Hence $V_1 \otimes V_2$ is an irreducible $H(G, K_1) \otimes H(G, K_2)$. But since

$$H(G_1 \times G_2, K_1 \times K_2) = H(G_1, K_1) \otimes H(G_2, K_2)$$

$$(V_1 \otimes V_2)^{K_1 \times K_2} = V_1^{K_1} \otimes V_2^{K_2}$$

thus by Proposition 4.4.2, $V_1 \otimes V_2$ be irreducible.
(2) Suppose V be an admissible G-module. By smoothness we can pick a compact open subgroup of the form $K = K_1 \times K_2$ with $V^K \neq 0$. By assumption V^K is finite, whence for finite dimensional $H(G,K)$-modules V_{i,K_i} one has $V^K \xrightarrow{\sim} V_{1,K_1} \otimes V_{2,K_2}$. Hence we put

$$V_1 = \lim_{\rightarrow, K_1} V_{1,K_1} \quad V_2 = \lim_{\rightarrow, K_2} V_{2,K_2}$$

we obtain the decomposition $V \simeq V_1 \otimes V_2$ as $H(G)$-modules. V_1 and V_2 are obviously admissible and by Proposition 4.4.2 them irreducible.

□

Theorem 4.4.4. *(Flath) Irreducible admissible representations of $H(G(\mathbb{A}_E^\infty))$ are factorizable.*

Proof. Let V be some admissible irreducible representation of $H(G(\mathbb{A}_E^\infty))$. For every $S \subset M_E$, comprising infinite ones and of which the local group G_{E_v} is ramified, we consider $K^S := \prod_{v \notin S} K_v \subset G(\mathbb{A}_E^S)$ where K_v be hyperspecial subgroup of G_{E_v} for $v \notin S$. Let

$$H(G(\mathbb{A}_E^\infty)) \simeq \bigotimes_{v \nmid \infty}{}' H(G(E_v)).$$

be presentation of $H(G(\mathbb{A}_E^\infty))$. Suppose $A_S := H(G(E_S^\infty)) \otimes e_{K^S}$ be a $\mathbb{C}$-subalgebra of $H(G(\mathbb{A}_E^\infty))$ where $e_{K^S} = \otimes_{v \notin S} e_{K_v} = \frac{1}{meas(K^S)} \cdot id_{K^S}$ is the idempotent element associated to open compact subgroup K^S as in Remark 4.4.2 and id_{K^S} be characteristic function on K^S. By Theorem 4.4.3 we obtain the isomorphism $V^{K^S} \simeq \otimes_{v \in S - \{\infty\}} (V_v \otimes V^S)$, which V^S be complex space equipped with a trivial action of the idempotent e_{K^S} with $dim_{\mathbb{C}} V^S = 1$. Since V irreducible and admissible via $H(G(\mathbb{A}_E^\infty))$-action, we have

$$V = \bigcup_S V^{K_S} \simeq \lim_{\rightarrow, S} \bigotimes_{v \in S - \{\infty\}} (V_v \otimes V^S).$$

with transition mappings as in Remark 4.4.4. In particular the presentation of the Hecke algebra $H(G(\mathbb{A}_E^\infty))$ implies

$$H(G(\mathbb{A}_E^\infty)) = \bigcup_S A_S = \lim_{\rightarrow, S} A_S$$

By construction, V is stable under the action of $\lim_{\rightarrow, S} A_S$. This shows that V is factorizable as a $H(G(\mathbb{A}_E^\infty))$-module.

$\square$

Corollary 5. *Suppose* (π, V) *be automorphic representation of* $G(\mathbb{A}_E)$. *Then* π *decomposes into restricted product of local irreducible admissbile representations*

$$\pi = \otimes_v^{'} \pi_v$$

of $G(E_v)$ *for each place* v *of* E.

Chapter 5

L-Groups and L-Homomorphisms

Here we shall introduce a particular class of groups and homomorphisms which are central objects in Langlands philosophy. In order to define L-groups, we shall primarily investigate Galois actions on root datums associated to connected reductive groups and their maximal subtori. We also give a general form of the local Langlands correspondence by the aid of parametrizing morphisms.

5.1 Construction of the L-groups

Let G be a connected reductive group over a local or global field k. We choose $\bar{k}$, some seperable algebraic closure and set $G_k = Gal(\bar{k}/k)$. Over $\bar{k}$, G splits and by Theorem 3.2.2 it is determined the by root data. We pick a Borel subgroup $B/\bar{k}$ and some maximal algebraic tori $T/\bar{k}$ with $T \subset B$, which both splits over $\bar{k}$. The pair (B,T) is called a Borel pair. By Theorem 3.3.2, a selection of a subset roots $\triangle \subset \Phi(G,T)$ that are simple in the system of roots associated to G and T is the same as equivalent to selection of some B with $T \subset B$ over $\bar{k}$. Moreover one has the following bijection

$$(G,B,T) \xrightarrow{\sim} \Psi_0(G,T) = (X^*(T), \triangle, X_*(T), \triangle^\vee)$$

In particular, again by Theorem 3.2.2, the based dual root data $\Psi_0(G,T)^\vee = (X_*(T), \triangle^\vee, X^*(T), \triangle)$ determines unique complex reductive group $\widehat{G}$, up to inner inner automorphism, together with the isomorphism of root datum

$$\Psi_0(\widehat{G},\widehat{T}) \xrightarrow{\sim} \Psi_0(G,T)^\vee$$

Remark 5.1.1. *All the Borel pairs (B,T) become conjugate with the action of adjoint group G^{ad}. In particular if $ad(g)$ takes (B_1,T_1) to (B_2,T_2), then it induces an isomorphism on the associated root datum which does not depends on the choice of $g \in G^{ad}$. Therefore we can match all the root datums induced by Borel pairs (B,T). We denote the root datum obtained in this way by $\Psi(G)$. Therefore the automorphism group $Aut(G)$ acts on $\Psi(G)$ with the adjoint group G^{ad} acts trivially.*

Remark 5.1.2. *For $\alpha \in \Phi(G,T)$, there exists one and only one subgroup N_α of G with $N_G(N_\alpha) = T$ and Lie $N_\alpha = \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the α-eigenspace of $Ad : G \to GL(\mathfrak{g})$. The group N_α named as the root group associated to $\alpha \in \Phi(G,T)$.*

Definition 5.1.1. *A pinning of G is given triple*

$$(B,T,\{X_\alpha\}_{\alpha\in\triangle})$$

in which $X_\alpha \in \mathfrak{g}_\alpha - \{0\}$ for all $\alpha \in \Phi(G,T)$. The automorphism group $Aut(B,T,\{X_\alpha\}_{\alpha\in\triangle})$ of the pinning $(B,T,\{X_\alpha\}_{\alpha\in\triangle})$, is the family of automorphisms, which conserve the Borel pair (B,T) and $\{X_\alpha\}_{\alpha\in\triangle}$.

Remark 5.1.3. *By Theorem 3.2.2, we can lift automorphisms of*

$$\Psi(G,T) = (X^*(T),\Phi(G,T),X_*(T),\Phi(\widehat{G},\widehat{T}))$$

to automorphisms of the pair (G,T). Therefore for a pinning $(B,T,\{X_\alpha\}_{\alpha\in\triangle})$ of G, this process induces

$$Aut(B,T,\{X_\alpha\}_{\alpha\in\triangle}) \xrightarrow{\sim} Aut((X^*(T),\triangle,X_*(T),\triangle^\vee))$$

Lemma 5.1.1. *The short exact sequence*

$$1 \to Inn(G(\bar{k})) \xrightarrow{i} Aut(G(\bar{k})) \xrightarrow{r} Aut(\Psi_0(G,T)) \to 1$$

splits.

Proof. The mapping $i : Inn(G(\bar{k})) \to Aut(G(\bar{k})$ is clearly an injective homomorphism. By Theorem 3.1.4 and Proposition 3.1.2, each Borel pairs pair (B,T) conjugate by some element $G(\bar{k})$. Therefore for any $\varphi \in Aut(G(\bar{k}))$ and for $g \in G(\bar{k})$ so that $\varphi \circ ad(g)$ preserves this Borel pair, induced from the base $\triangle \subset \Phi(G,T)$. Hence $\varphi \circ ad(g)$ defines an automorphism of the based root datum $\Psi_0(G,T)$ defined by $\triangle \subset \Phi(G,T)$. Since it stabilizes B, by taking $\chi \cdot g$ instead of g, $\chi \in T(\overline{k})$, one may assume $\varphi \circ ad(g)$ maintains simple ones. Hence one has the mapping

$$r : Aut(G(\overline{k})/Inn(G(\overline{k})) \to Aut(\Psi_0(G,T))$$

By Theorem 3.2.2, r is surjective. In particular for given $\varphi' \in Aut(\Psi(G,T))$, there exists $\varphi \in Aut(G)$, so that $\varphi(T) = T$ and $r(\varphi) = \varphi'$. Moreover if $\varphi \in Aut(G)$ preserves the simple roots then its lift φ' is uniquely determined. Therefore we have $Im(i) = ker(r)$.

$\square$

Now we shall introduce the notion of L-group. By Lemma 5.1.1, homomorphism $Gal(\overline{k}/k) \to Aut(G(\overline{k}))$ induces a Galois action on the based data $\Psi_0(G,T) = (X^*(T), \triangle, X_*(T), \triangle^\vee)$. Likewise, one has dual based data $\Psi_0(\widehat{G},\widehat{T}) = (X_*(T), \triangle^\vee, X^*(T), \triangle)$. Hence we can find a Borel subgroup $\widehat{B}$ of $\widehat{G}$ with

$$(\widehat{G},\widehat{B},\widehat{T}) \xrightarrow{\sim} \Psi_0(\widehat{G},\widehat{T}) = (X_*(T), \triangle^\vee, X^*(T), \triangle)$$

Therefore we have a homomorphism on dual side

$$Gal(\overline{k}/k) \to Aut(\Psi_0(\widehat{G},\widehat{T}))$$

Moreover a choice of a pinning $(\widehat{B},\widehat{T},\{X_{\alpha^\vee}\}_{\alpha^\vee \in \triangle^\vee})$ of $\widehat{G}$, induces a mapping

$$Aut(\Psi_0(\widehat{G},\widehat{T})) \to Aut(\widehat{G})$$

Therefore we obtain a Galois action $Gal(\overline{k}/k) \to Aut(\widehat{G})$ on $\widehat{G}$.

Definition 5.1.2. *The L-group of G is*

$${}^L G := \widehat{G}(\mathbb{C}) \rtimes Gal(\bar{k}/k)$$

Remark 5.1.4. *In particular, the complex dual group $\widehat{G}$ is the identity component of L-group ${}^L G$, henceforth -it is demonstrated as ${}^L G^0$. Moreover, since we have a canonical homomorphism $W_k \to Gal(\overline{k}/k)$, hence we can construct the Weil form of ${}^L G$ as*

$${}^L G = {}^L G^0 \rtimes W_k.$$

Example 5.1.1. *Let $G = GL_n$ over a local or global field k. Note that a standard Borel subgroup of G is given by upper triangular matrices in G and by Theorem 3.1.4, any*

Borel subgroup of G is conjugate to it. Hence $G = GL_n$ is a split group over k. As a maximal algebraic torus in G, we may take diagonal matrices D_n, so by Proposition 3.1.2, any maximal algebraic tori is conjugate to D_n. We have the identifications $X^*(D_n) = X_*(D_n) = \mathbb{Z}^n$ with standart symbol$(e_i, e_j) = \delta_{ij}$. Let

$$\triangle = \triangle^\vee = \{e_i - e_{i+1} \mid 1 \le i \le n-1\}$$

then $i : \Psi(G) \to \Psi(G)^\vee \simeq \Psi(\widehat{G})$ is nothing but identity mapping. Hence ${}^LG^0 = GL_n(\mathbb{C})$. Since G is split, the Galois action on $\widehat{G}$ is trivial so we have

$${}^LG = GL_n(\mathbb{C}) \times Gal(\bar{k}/k)$$

5.2 Admissible and *L*-Homomorphisms

In this subsection we shall define a special class of homomorphisms which provides us a key tool for the arithmetic parametrization of irreducible admissible representations. Set k for a local field and $\bar{k}$ be a seperable algebraic closure of k. We will denote the Weil-Deligne group of k by WD_k.

Definition 5.2.1. *A group morphism $\phi : WD_k \to {}^LG$, is called admissible if the following conditions are satisfied :*

1) The diagram in below is commutative :

$$\begin{tikzcd} {}^LH \arrow[r, "u"] \arrow[d, "prj"'] & {}^LG \arrow[d, "prj"] \\ Gal(\bar{k}/k) \arrow[r, "id"'] & Gal(\bar{k}/k) \end{tikzcd}$$

2) ϕ is a continuous mapping which preserves semisimple element, and elements of $\phi(\mathbb{G}_a) \subset {}^LG^0$ are unipotent.

3) If $\phi(WD_k)$ is contained in a Levi subgroup of a proper parabolic subgroup ${}^LP$ of ${}^LG$ then ${}^LP$ is relevant.

We shall denote by $\Phi(G)$, all admissible homomorphisms $\phi : WD_k \to {}^LG$ of G, up to inner action by elements of ${}^LG^\circ$

Remark 5.2.1. *Suppose G is a reductive group k. By (Borel, 1979a), for any $\phi \in \Phi(G)$ there exists an character ω_ϕ of the centre $Z(G)$ associated to ϕ. In particular as in (Borel, 1979a), for any cohomology class $\varphi \in H^1(WD_k; Z({}^LG^\circ))$, there exists character η_φ of $G(k)$ associated to φ. For $\phi \in \Phi(G)$, we write $\phi = (\phi_1, \phi_2)$ with $\phi_1(w) \in {}^LG^\circ$, $\phi_2(w) \in Gal(\bar{k}/k)$ for any $w \in WD_k$. In particular, ϕ_1 is a cocycle on WD_k with values in ${}^LG^\circ$ and the mapping $\phi = (\phi_1, \phi_2) \to \phi_1$ defines an embedding $\Phi(G) \hookrightarrow H^1(WD_k; {}^LG^\circ)$. Moreover the cohomology group $H^1(WD_k, Z({}^LG^\circ))$ acts on $H^1(WD_k, {}^LG^\circ)$ by preserving $\Phi(G)$. Also, set $A(G) = A(G(k))$ be set of equivalence classes of irreducible, admissible complex representations.*

Local Langlands Correspondence. Let G be a connected reductive group over a local field k. Then there exist a surjective map $LLC_k : A(G) \to \Phi(G)$ with for any $\phi \in \Phi(G)$, the fibers $A_\phi := LLC_k^{-1}(\phi)$ are finite and provides a partition of $A(G)$ into finite disjoint sets

$$A(G) := \bigcup_{\phi \in \Phi(G)} A_\phi$$

satisfying the following conditions :

1) For $\pi \in A_\phi$, central character ω_π is equal to ω_φ.

2) (Compatibility with twisting) For any cohomology class $\alpha \in H^1(WD_k, Z({}^LG^\circ))$ and the associated character η_α of $G(k)$ one has $A_{\alpha\cdot\varphi} = \{\pi \cdot \eta_\alpha \mid \pi \in A_\varphi\}$.

3) An element $\pi \in A_\phi$ is L^2 sense integrable mod $Z(G)$ if and only if all element in A_ϕ are L^2 sense integrable mod $Z(G)$. This amount to say, $\phi(WD_k)$ does not occur in some proper Levi subgroup, of ${}^LG$.

4) An element $\pi \in A_\phi$ is to be tempered is the same as all elements in A_ϕ are tempered. This is equivalent to $\phi(W_k)$ to be bounded.

5) Let H be connected reductive group over k. $\vartheta : H(k) \to G(k)$ is a morphism of algebraic groups over k, with abelian kernel and co-kernel, then ϑ induces a canonical map ${}^L\vartheta : {}^LG \to {}^LH$ and if we set $\tilde{\phi} := {}^L\vartheta \circ \phi$ for $\phi \in \Phi(G)$, then for any $H(k)$-module

$\pi \in A_\phi(G)$, one has a direct sum

$$\pi = \bigoplus_{\varsigma \in A_{\tilde{\phi}}(H)} \varsigma$$

Remark 5.2.2. *A fiber A_ϕ, in the local Langlands conjecture, for $\phi \in \Phi(G)$, is called an L-packet. We say elements occurs in the same L-packet are L-indistinguisable.*

Remark 5.2.3. *For any $\phi \in \Phi(G)$, elements in A_ϕ are regulated by*

$$C_\phi = S_\phi / Z({}^L G) \cdot S_\phi^0.$$

Here $S_\phi = C_{{}^L G}(Im(\phi))$ and S_ϕ^0 be connected component of S_ϕ. (Arthur, 2006).

Example 5.2.1. *Let $G = GL_n$ over a local field k. In particular for every finite set $S \subset {}^L G$, the centralizer $Z_{{}^L G}(S)$ is connected. Thus $C_\phi = 1$ for all $\phi \in \Phi(G)$ and we deduce that all L-packets A_ϕ's are singletons.*

Definition 5.2.2. *Let H and G be two reductive algebraic groups over k. A homomorphism between L-groups $u : {}^L H \to {}^L G$, is called an L-homomorphism if*

1) The diagram in below is commutative :

$$\begin{tikzcd} {}^L H \arrow[r, "u"] \arrow[d, "prj"'] & {}^L G \arrow[d, "prj"] \\ Gal(\bar{k}/k) \arrow[r, "id"'] & Gal(\bar{k}/k) \end{tikzcd}$$

2) u is continuous and restriction of u to the connected component ${}^L H^0$ of ${}^L H$ defines an analytic mapping

$$u|_{res} : {}^L H^0 \to {}^L G^0$$

Remark 5.2.4. *If G is quasisplit over a local field k, then for any admissible homomorphism φ_H of H, the composite*

$$\varphi_G := u \circ \varphi_H : WD_k \to {}^L H \xrightarrow{u} {}^L G$$

is an admissible homomorphism of G. Therefore $u : {}^L H \to {}^L G$ induces mapping

$$\Phi(u) : \Phi(H) \to \Phi(G), \;\; \varphi_H \to \varphi_G := u \circ \varphi_H$$

between equivalence classes of admissible homomorphisms of H and G.

Chapter 6

Objects attached to L-Homomorphisms

In this chapter we shall define analytic factors associated to L-groups and L-homomorphisms. Initially we will first define elementary factors associated to Weil group representations and Weil-Deligne representations which includes also non-abelian L functions of Artin as a special case. Finally we shall cover hypothetical automorphic L-functions and ε-factors associated to automorphic representations .

6.1 L-Functions and ε-Factors for representations associated to quasi-characters of local fields

In this part, we fix a local field K, and denote group of characters of K, under multiplication, by $K^\vee$. Recall that any representation of $K^\times$ given by character $\psi : K^\times \to \mathbb{C}^\times$, therefore the theory of 1-dimensional smooth representations (i.e continuous homomorphisms $W_K \to \mathbb{C}^\times$) of the Weil group W_K is nothing but the theory of quasi-characters of $K^\times$.

Definition 6.1.1. *Let $1 \neq \psi \in K^\vee$. The level of ψ is the smallest $m > 0$, in which $\mathfrak{p}_K \subset ker\ \psi$, i.e $\psi(\mathfrak{p}_K^m) = 1$.*

Proposition 6.1.1. *(Bushnell & Henniart, 2006) Let $1 \neq \psi \in K^\vee$ be a character of K with level m. Then,*

1) For any $a \in K$, the mapping $a\psi : x \to \psi(ax)$ is in $K^\vee$. In particular for $a \neq 0$, $a\psi$ possesses $m - v_K(a)$.

2) Mapping $a \to a\psi$ induces isomorphism $K \simeq K^{\vee}$.

We fix $1 \neq \psi \in K^{\vee}$ and an additive Haar measure μ on K.

Definition 6.1.2. *For any $\Phi \in C_c^{\infty}(K)$, the Fourier transform $\widehat{\Phi}$ given as*

$$\widehat{\Phi}(y) = \int_K \Phi(x) \cdot \psi(xy) \cdot d\mu(y), \;\; x \in K.$$

Proposition 6.1.2. *(Bushnell & Henniart, 2006). Let $\Phi \in C_c^{\infty}(K)$ with Fourier transform $\widehat{\Phi}$. Then one has the following results,*

1) The function $\widehat{\Phi}$ lies in $C_c^{\infty}(K)$.

2) One has,

$$\widehat{\widehat{\Phi}}(k) = r \cdot \Phi(-k), \;\; \Phi \in C_c^{\infty}(K), \; k \in K.$$

for some $r \in \mathbb{R}$.

3) Any $\psi \in K^{\vee}$, equipped with only one additive Haar measure μ_{ψ}, so that $r = 1$. Moreover $\mu_{\psi}(O_K) = q_K^{\frac{m}{2}}$, where m be level of ψ. In particular for $x \in K^{\times}$, withal $\mu_{x\psi} = |x|_K \cdot \mu_{\psi}$

Remark 6.1.1. *By Proposition 6.1.2 and by the aid of μ_{ψ}, one has,*

$$\widehat{\Phi}(l) = \int_K \Phi(k) \cdot \psi(kl) \cdot d\mu_{\psi}(l)$$

which implies

$$\widehat{\widehat{\Phi}}(k) = \Phi(-k), \;\; \Phi \in C_c^{\infty}(K), \; k \in K.$$

For an integer m, let 1_m denotes characteristic function on $\pi_k \cdot U_K = \mathfrak{p}_K^n \backslash \mathfrak{p}_K^{n+1}$. Then the function $\Phi_m = 1_m \cdot \Phi$ is an element of $\in C_c^{\infty}(K^{\times})$. In particular, for a sufficiently small m, the function Φ_m is identically zero. Hence we set

$$z_m = z_m(\Phi, \varphi) = \int_{\pi_k^m \cdot U_K} \Phi(x) \cdot \varphi(x) \cdot d\mu^*(x), \; m \in \mathbb{Z}$$

and we define the following :

$$Z(\Phi, \varphi, X) = \sum_{m \in \mathbb{Z}} z_m \cdot X^m \in \mathbb{C}((X)).$$

Moreover the induced map $C_c^\infty(K) \to \mathbb{C}((X))$, $\Phi \to Z(\Phi,\varphi,X)$ is a linear mapping since the coefficients z_m are defined in terms of integrals which linearity follows. We denote the image of $C_c^\infty(K) \to \mathbb{C}((X))$ by

$$\mathcal{Z}(\varphi) = \mathcal{Z}(\varphi,X) = \{Z(\Phi,\varphi,X) \mid \Phi \in C_c^\infty(K)\}$$

In particular for $t \in K^\times$ and $\Phi \in C_c^\infty(K)$, one has scaled mapping $a\Phi : k \to \Phi(t^{-1}k)$.

Proposition 6.1.3. *$\mathcal{Z}(\varphi)$ is a $\mathbb{C}[X,X^{-1}]$-module.*

Proof. By Proposition 6.1.1, if $\varphi \in K^\vee$ has level m, then the character $x \to \psi(tx)$ of K has level $m - v_K(t)$. Hence for $Z(t\Phi,\varphi,X) = \sum_{m\in\mathbb{Z}} c_m \cdot X^m \in \mathcal{Z}(\varphi)$, we calculate the coefficient c_m, $\forall m \in \mathbb{Z}$, by the change of variable $a^{-1}k \to l$ as

$$c_m = \int_{\pi_k^m \cdot U_K} (t\Phi)(k)\cdot\varphi(k)\cdot d\mu^*(k) = \int_{\pi_k^m \cdot U_K} \Phi(t^{-1}k)\cdot\varphi(k)\cdot d\mu^*(k)$$

$$\int_{\pi_k^{m-v_K(a)}\cdot U_K} \Phi(l)\cdot\varphi(tl)\cdot d\mu^*(tl) = \int_{\pi_k^{m-v_K(t)}\cdot U_K} \Phi(l)\cdot\varphi(t)\varphi(l)\cdot d\mu^*(l)$$

$$= \varphi(t)\cdot\int_{\pi_k^{m-v_K(t)}\cdot U_K} \Phi(l)\cdot\varphi(l)\cdot d\mu^*(l) = \varphi(a)\cdot z_{m-v_K(t)}$$

Therefore by a quick algebraic manipulation, we have the following

$$Z(t\Phi,\varphi,X) = \sum_{m\in\mathbb{Z}} \varphi(t)\cdot z_{m-v_K(t)}\cdot X^m = \varphi(t)\cdot X^{v_K(t)}\cdot\sum_{m\in\mathbb{Z}} z_{m-v_K(t)}\cdot X^{m-v_K(t)}$$

$$= \varphi(t)\cdot X^{v_K(t)}\cdot Z(\Phi,\varphi,X)$$

which shows the asseveration.

$\square$

Proposition 6.1.4. *(Bushnell & Henniart, 2006). Let φ be a quasi-character of $K^\times$. Then for the function*

$$P_\varphi(X) = \begin{cases} \frac{1}{1-\varphi(\pi_K)\cdot X} & \text{if } \varphi \text{ is unramified} \\ 1 & \text{otherwise} \end{cases}$$

one has

$$\mathcal{Z}(\varphi,X) = P_\varphi(X)\cdot\mathbb{C}[X,X^{-1}].$$

Lemma 6.1.1. *(Bushnell & Henniart, 2006). The space*

$$\Lambda = \{f : C_c^\infty(K) \to \mathbb{C}(X) \mid f(t\Phi) = \varphi(t) \cdot X^{\nu_K(t)} \cdot f(\Phi),\ \Phi \in C_c^\infty(k),\ t \in K^\times\}$$

is one-dimensional over $\mathbb{C}(X)$.

Theorem 6.1.1. *Suppose ψ is a character of K^+ with $\varphi \in K^\vee$. Then there exists only complex function $c(\varphi, \psi, X) \in \mathbb{C}(X)$, with*

$$Z(\widehat{\Phi}, \widehat{\varphi}, \frac{1}{q_K \cdot X}) = c(\varphi, \psi, X) \cdot Z(\Phi, \varphi, X)$$

Proof. $\mathbb{C}(X)$-vector space Λ in Lemma 6.1.1, covers mapping $\lambda_0 : \Phi \to Z(\Phi, \varphi, X)$, and by Proposition 6.1.4, λ_0 is non-zero. Therefore, by Proposition 6.1.2, for $\Phi \in C_c^\infty(K)$ and $a \in K^\times$, the change of variable $a^{-1} \cdot y \to c$, we have

$$\widehat{a\Phi}(x) = \int_F (a\Phi)(y) \cdot \psi(xy) \cdot d\mu_\psi(y) = \int_F \Phi(a^{-1}y) \cdot \psi(xy) \cdot d\mu_\psi(y) = \int_F \Phi(c) \cdot \psi(axc) \cdot d\mu_\psi(ca)$$

$$= |a|_K \cdot \int_F \Phi(c) \cdot \psi(axc) \cdot d\mu_\psi(c) = |a|_K \cdot \widehat{\Phi}(ax)$$

Hence we have $\widehat{a\Phi}(x) = |a|_K \cdot \widehat{\Phi}(ax) = |a|_K \cdot a^{-1} \cdot \widehat{\Phi}(x)$. In particular, the map

$$\lambda_1 : \Phi \to Z(\widehat{\Phi}, \widehat{\varphi}, \frac{1}{q_K \cdot X})$$

is an element of Λ. Hence by Lemma 6.1.1, Λ has dimension 1 over $\mathbb{C}(X)$, so $\exists!\ c(\varphi, \psi, X) \in \mathbb{C}(X)$ so that $Z(\widehat{\Phi}, \widehat{\varphi}, \frac{1}{q_K \cdot X}) = c(\varphi, \psi, X) \cdot Z(\Phi, \varphi, X)$. $\square$

Remark 6.1.2. *We introduce the following conventional notations as for a complex variable s,*

$$\zeta(\Phi, \varphi, s) = Z(\Phi, \varphi, q^{-s}),\ \ L(\varphi, s) = P_\varphi(q^{-s})^{-1},\ \ \gamma(\varphi, s, \psi) = c(\varphi, \psi, q^{-s})$$

where $q = q_K$. In particular,

$$\zeta(\Phi, \varphi, s) = \int_{K^\times} \Phi(x) \cdot \varphi(x) \cdot \omega_s(x) \cdot d\mu^*(x)$$

where by Proposition 6.1.4, it is convergent, both uniformly and absolutely for some $Re(s) > s_0$.

Definition 6.1.3. *Suppose $\varphi \in K^\vee$. Then the L-function associated to φ is defined by*

$$L(\varphi,s)=\begin{cases}\frac{1}{1-\varphi(\pi_K)\cdot q_K^{-s}} & \textit{if } \varphi \textit{ is unramified}\\ 1 & \textit{otherwise}\end{cases}$$

Definition 6.1.4. *Let φ be a quasi-character of $K^{\times}$, ψ be an additive character of K and let $d\mu$ be a self dual additive Haar measure on K. The function $\varepsilon(\varphi,s,\psi)\in\mathbb{C}(q^{-s})$ defined by the equation*

$$\varepsilon(\varphi,s,\psi)=\gamma(\varphi,s,\psi)\cdot\frac{L(\varphi,s)}{L(\widehat{\varphi},1-s)}$$

is called the ε-factor associated to the quasi-character φ.

Remark 6.1.3. *By using conventional notations in Remark 6.1.2, and Theorem 6.1.1, we can express the ε-factor associated to a quasi-character φ of $K^{\times}$, as,*

$$\varepsilon(\varphi,s,\psi)=\gamma(\varphi,s,\psi)\cdot\frac{L(\varphi,s)}{L(\widehat{\varphi},1-s)}=\frac{Z(\widehat{\Phi},\widehat{\varphi},q_K^{s-1})}{Z(\Phi,\varphi,q_K^{-s})}\cdot\frac{L(\varphi,s)}{L(\widehat{\varphi},1-s)}$$

hence we have

$$\frac{Z(\widehat{\Phi},\widehat{\varphi},q_K^{s-1})}{L(\widehat{\varphi},1-s)}=\varepsilon(\varphi,s,\psi)\cdot\frac{Z(\Phi,\varphi,q_K^{-s})}{L(\varphi,s)}$$

$$\frac{\zeta(\widehat{\Phi},\widehat{\varphi},1-s)}{L(\widehat{\varphi},1-s)}=\varepsilon(\varphi,s,\psi)\cdot\frac{\zeta(\Phi,\varphi,s)}{L(\varphi,s)}$$

This expression is called Tate's local functional equation associated to quasi-character φ of $K^{\times}$.

6.2 Local Non-Abelian Invariants Associated to Representations of W_K

In the previous section, we defined L functions and ε factors of quasi-characters of local field K. In particular by Artin reciprocity law, $Art_K^{-1}:K^{\times}\xrightarrow{\sim}W_K^{ab}$, and for a quasi-character φ of K, we have

$$L(\varphi\circ Art_K^{-1},s)=L(\varphi,s),\ \ \varepsilon(\varphi\circ Art_K^{-1},s,\psi)=\varepsilon(\varphi,s,\psi)$$

for some additive character ψ of K. Therefore we already defined L-functions and ε-factors for 1-dimensional smooth representations of W_K. Here we present L functions and ε-factors for arbitrary dimensional representations of W_K. Initially, we will at first treat the case K be non-archimedean.

Lemma 6.2.1. *Suppose (ρ, V) be smooth representation of W_K and $Frob_K \in W_K$ some Frobenius element. Then the automorphism*

$$\rho(Frob_K)|_{V^{I_K}}$$

does not depend on $Frob_K \in W_K$.

Proof. Inertia subgroup I_K is normal in W_K, and the subspace V^{I_K} is stable under the action W_K and hence a subrepresentation $(\rho|_{V^{I_K}}, V^{I_K})$ of W_K. In particular, if $Frob_K, Frob_K^1 \in W_K$ are two Frobenius elements, then there exists some $\sigma \in I_K$ so that $Frob_K^1 = \sigma \cdot Frob_K \cdot \sigma^{-1}$. Hence for any $v \in V^{I_K}$, we have

$$\rho|_{V^{I_K}}(Frob_K^1)(v) = \rho|_{V^{I_K}}(\sigma \cdot Frob_K \cdot \sigma^{-1})(v) = \rho|_{V^{I_K}}(Frob_K)(v)$$

□

Definition 6.2.1. *Suppose (ρ, V) be a finite dimensional smooth representation of W_K with $Frob_K \in W_K$ some Frobenius element in W_K. Then L-function associated (ρ, V) is given as*

$$L(\rho, s) = det(1 - \rho(Frob_K)|_{V^{I_K}} \cdot q^{-s})^{-1}.$$

Example 6.2.1. *Suppose (ρ, V) be a representation of W_K with $dim\, V = 1$. Then $\rho : W_K \to \mathbb{C}^\times$ is simply a quasi-character, and is smooth. if ρ is unramified, then restriction to I_K is identity so we have $\mathbb{C}^{I_K} = \mathbb{C}$. Therefore the L-function is given by*

$$L(\rho, s) = (1 - \rho(Frob_K) \cdot q^{-s})^{-1}$$

for some Frobenius element $Frob_K \in W_K$. If ρ is ramified then $\mathbb{C}^{I_K} = 0$, since there exists an element $\sigma \in I_K$ so that $\rho(\sigma)v \neq v$ for some $0 \neq v \in \mathbb{C}^{I_K}$. But $\rho(\sigma) = \rho(\sigma \cdot I_K) \cdot v = \rho(I_K) \cdot v = v$ which is a contradiction. Thus $L(\rho, s) = 1$. Hence we have nothing but the same L-function as in Definition 6.1.3.

Example 6.2.2. *Let (ρ, V) be an irreducible representation of W_K, with $dim\, V \geq 2$. The invariant subspace V^{I_K} is a subrepresentation of W_K. By irreducibility, we have either $V^{I_K} = V$ or $V^{I_K} = 0$. Assume $V^{I_K} = V$. Then we have $I_K \subseteq ker\, \rho$ and hence (ρ, V) is an irreducible representation of $W_K/I_K \simeq \mathbb{Z}$. But any eigenvector of $\sigma(1)$ generates a* *subrepresentation, so this contradicts with our assumption. Hence* $V^{I_K} = 0$ *and the L-function is* $L(\rho, s) = 1$.

Proposition 6.2.1. *Let* (ρ_1, V_1) *and* (ρ_2, V_2) *be finite dimensional smooth representations of* W_K*. Then we have the equality*

$$L(\rho_1 \oplus \rho_2, s) = L(\rho_1, s) \cdot L(\rho_2, s)$$

Proof. Let $Frob_K \in W_K$ be a Frobenius element. We have the following equality

$$(V_1 \oplus V_2)^{I_K} = V_1^{I_K} \oplus V_2^{I_K}$$

Therefore in terms of characteristic polynomials we have

$$det(1-(\rho_1 \oplus \rho_2)(Frob_K)|_{(V_1 \oplus V_2)^{I_K}} \cdot X) = det(1-\rho_1(Frob_K)|_{V_1^{I_K}} \cdot X) \cdot det(1-\rho_2(Frob_K)|_{V_2^{I_K}} \cdot X)$$

hence by evaluating both sides at q^{-s} and taking inverses we have the asseveration.

□

Let L/K be a finite seperable extension. For any $id \neq \psi : K \to \mathbb{C}$, set for $\psi_L := \psi \circ Tr_{L/K}$ as an induced additive character of L. Now we shall give a description of ε-factors associated to representations of W_K.

Theorem 6.2.1. *(Bushnell & Henniart, 2006) There exists a unique collection of mappings*

$$\gamma_L : Rep^{ss}(L) \to \mathbb{C}[q^{-s}, q^s]^{\times}$$

$$\rho \to \varepsilon(\rho, s, \psi_E)$$

possessing below features :

- *Given* $\varphi_L : L^{\times} \to \mathbb{C}$ *a quasi-character* L *, then one has*

$$\varepsilon(\varphi_L \circ Art_L, s, \psi_L) = \varepsilon(\varphi_L, s, \psi_L).$$

Note that this expression is nothing but the ε*-factor associated to a quasi-character of* W_L.

- *For $\rho_1, \rho_2 \in Rep^{ss}(L)$, then we have*

$$\varepsilon(\rho_1 \oplus \rho_2, s, \psi_L) = \varepsilon(\rho_1, s, \psi_L) \cdot \varepsilon(\rho_2, s, \psi_L)$$

- *For $\rho \in Rep^{ss}(L)$ with $K \subset M \subset L$ one has*

$$\frac{\varepsilon(Ind_{L/M}\, \rho, s, \psi_M)}{\varepsilon(\rho, s, \psi_L)} = \frac{\varepsilon(Ind_{L/M}\, 1_M, s, \psi_M)^n}{\varepsilon(1_L, s, \psi_L)^n}$$

where 1_M is the trivial representation of W_M.

Definition 6.2.2. *The element $\varepsilon(\rho, s, \psi) \in \mathbb{C}[q^{-s}, q^s]^\times$ for $\rho \in Rep^{ss}(K)$ is regarded by "Langlands-Deligne constant" of ρ, attached to $\psi : K^* \to \mathbb{C}^*$.*

6.2.1 L-Functions and ε-factors for Weil-Deligne Representations

In this part, we present some invariants attached to Weil-Deligne representations. Throughout this subsection K will be a non-archimedean local field.

Example 6.2.3. *Suppose $V = \mathbb{C}^n$ and $N \in Mat_n(\mathbb{C})$ standart Jordan block of rank $n-1$. This means if $\{e_1, e_2, \cdots, e_n\}$ be usual base of $\mathbb{C}^n$, we have $N \cdot e_n = 0$ and $N \cdot e_i = e_{i+1}$ for $1 \leq i \leq n$. Let $(\rho, \mathbb{C}^n)$ be a smooth representation of W_K with following relation :*

$$\rho(w) \cdot e_i = ||w||_K^i \cdot e_i$$

for $w \in W_K$ and $1 \leq i \leq n$. The representation $(\rho, \mathbb{C}^n)$ of W_K is said special representation of W_K and indicated by $sp(n)$.

Remark 6.2.1. *Let $(sp(n), \mathbb{C}^n)$ be the special representation of W_K. Since $ker\, N = \mathbb{C} \cdot e_n$ is a W_K-invariant subspace, we see that $sp(n)$ is reducible for $n > 2$. In particular, if $\mathbb{C}^n = U \oplus W$ be a non-trivial decomposition, then since N is nilpotent, both subspaces $U \cap ker\, N$ and $W \cap ker\, N$ are non-trivial. Thus $(sp(n), \mathbb{C}^n)$ is an indecomposable representation of W_K.*

Proposition 6.2.2. *Let (ρ, V, N) be Weil-Deligne representation over some characteristic 0 field. Then (ρ, V, N) is irreducible if and only if (ρ, V) is irreducible with $N = 0$.*

Lemma 6.2.2. *Let (ρ, V) be a irreducible finite dimensional irreducible representation of W_K. The representation $\rho \otimes sp(n)$ of W_K is indecomposable.*

Proof. Let $(\widehat{\rho}, \widehat{V}, N) = \rho \otimes sp(n)$. Then for any non-trivial decomposition $V = U \oplus W$, one has

$$ker\, N = (U \cap ker\, N) \oplus (W \cap ker\, N)$$

Since N nilpotent, the intersection must be non-trivial. In particular $ker\, N$ is a W_K-invariant subspace of $\widehat{V}$, and is equivalent to $\rho \otimes ||,||^{n-1}$ which is irreducible. This contradicts with the fact that decomposition $ker\, N = (U \cap ker\, N) \oplus (W \cap ker\, N)$ be non-trivial. Therefore $\rho \otimes sp(n)$ is an indecomposable Weil-Deligne representation of W_K. □

Proposition 6.2.3. *(Deligne, 1973b). Every semisimple indecomposable Weil-Deligne representation is equivalent to some representation $V \otimes sp(n)$, $n \in \mathbb{Z}_+$, where (ρ, V) be a smooth irreducible representation of W_K.*

Remark 6.2.2. *By Lemma 6.2.2 and Proposition 6.2.4, any such representation of W_K of the following form :*

$$(\rho, V, N) = \bigoplus_{i}^{n} \rho_i \otimes sp(n_i)$$

where (ρ_i, V_i) is an irreducible smooth representations of W_K and $sp(n_i)$ is the n_i-dimensional special representation of W_K, for $i = 1, 2, \cdot\cdot, n$.

Definition 6.2.3. *Let (ρ, V, N) be some WD-representation of W_K on a field E of characteristic 0 and $id \neq \psi : K \to \mathbb{C}^\times$ a character of K^+. Then L-function attached to (ρ, V, N) given by*

$$L((\rho, V, N), s) = det(1 - \rho(Frob_K)|_{V_N^{I_K}} \cdot q^{-s})^{-1}$$

where $V_N^{I_K} = (ker\, N)^{I_K} = \{v \in ker\, N \mid \rho(\sigma) \cdot v = v,\ \forall \sigma \in I_K\}$. Moreover the ε-factor $\varepsilon((\rho, V, N), \psi)$ associated to (ρ, V, N) is defined as

$$\varepsilon((\rho, V, N), \psi) = \varepsilon(\rho, s, \psi) \cdot det(-\rho(Frob_K)|_{V^{I_K}/V_N^{I_K}})$$

6.3 Automorphic *L*-Functions

In this part G will denote connected, quasi-split reductive group on some number field $F$and S specifies finite set of places containing archimedians and ramified ones for local representations. We adjust Borel subgroup B of G. By Theorem 3.1.2 and Proposition

3.1.2, we have a Levi decomposition $B = TU$, where T is a maximal algebraic tori in B with $U \subset B$ unipotent radical, both are defined over F. Set for P a parabolic subgroup in G with $B \subset P$. Suppose $P = MN$ be Levi decomposition of P with Levi factor M. One has $N \subset U$. For all $v \in S$, we put $G_v = G(F_v)$. By the same way, we shall denote by B_v, T_v, U_v, P_v, M_v and N_v the associated groups with F_v-points. If G is unramified over the completion F_v, of F at some place v, we put $K_v = G(O_{F_v})$ a standart maximal compact subgroup in G_v. We set $K_{\mathbb{A}_F} = \otimes_v K_v$. Then,

$$G(\mathbb{A}_F) = P(\mathbb{A}_F)K_{\mathbb{A}_F} = M(\mathbb{A}_F)\cdot N(\mathbb{A}_F)\cdot K_{\mathbb{A}_F}.$$

Suppose $\Pi = \otimes_v \Pi_v$ be some cuspidal representation of $G(\mathbb{A}_K)$. For $v \notin S$, the local representation Π_v determines a unique semisimple conjugacy class $\{c(\Pi_v)\} \subset {}^L G_v$. The conjugacy class $\{c(\Pi_v)\}$ associated to representation Π_v is called the Satake parameter for Π_v. In particular, for every v, there exists a natural group homomorphism $\xi_v : {}^L G_v \longrightarrow {}^L G$. Moreover, for a finite dimensional complex representation

$$r : {}^L G \longrightarrow GL_N(\mathbb{C})$$

of ${}^L G$, we may consider the composition $r_v := r \circ \xi_v : {}^L G_v \longrightarrow GL_N(\mathbb{C})$ which is a complex representation of the local L-group ${}^L G_v$.

Definition 6.3.1. *The local Langlands L-function $L(s, \Pi_v, r_v)$ associated to Π_v and r_v is defined by*

$$L(s, \Pi_v, r_v) = det(I_n - r_v(c(\Pi_v)) \cdot q_v^{-s})^{-1}$$

In particular the partial L-function is defined by

$$L_S(s, \Pi, r) = \prod_{v \notin S} L(s, \Pi_v, r_v).$$

Example 6.3.1. *Suppose $\Pi = \otimes_v \Pi_v$ be some cuspidal representation of $GL_2(A_{\mathbb{Q}})$. Let $c(\Pi_v) = diag(\alpha_v, \beta_v)$ be the unique semi-simple conjugacy class in $GL_2(\mathbb{C})$ associated to a local representation Π_v of $GL_2(\mathbb{Q}_v)$ for nearly whole places. Set*

$$Sym^m : GL_2(\mathbb{C}) \to GL_{m+1}(\mathbb{C})$$

denotes m-th symmetric power representation. For this Langlands L-function is

$$L(s,\Pi_v,Sym^m) = \prod_{i=0}^{m}(1-\alpha_v^{m-i}\cdot\beta_v^i\cdot q_v^{-s})^{-1}.$$

Theorem 6.3.1. *(Langlands, 1970). Suppose $\Pi = \otimes_v \Pi_v$ be some cuspidal representation of $G(\mathbb{A}_K)$. Then for $Re(s) \gg 0$, $L_S(s,\Pi,r) = \prod_{v\notin S} L(s,\Pi_v,r_v)$ is absolutely convergent and defines a holomorphic function there.*

Conjecture (Langlands, 1970). $L_S(s,\Pi,r)$ admits meromorphic continuation to $\mathbb{C}$ with providing standart functional equation.

Suppose $\psi = \otimes_v \psi_v$ be some character of $\mathbb{A}_F/F$. Indeed Langlands formulated more refined conjecture :

Conjecture (Langlands, 1970). For every $v \in S$, one may set for local L-function $L(s,\Pi_v,r)$ (of shape $P(q_v^{-s})^{-1}$, $P(X) \in \mathbb{C}[x]$ with $P(0) = 1$) and a local root number $\varepsilon(s,\Pi_v,r,\psi_v)$ (of the form $A\cdot q_v^{-B\cdot s}$) which,

$$L(s,\Pi,r) = \prod_{all\ v} L(s,\Pi_v,r_v).$$

admits meromorphic continuation to all $\mathbb{C}$, withal fitting into some functional equation of the form

$$L(s,\Pi,r) = \varepsilon(s,\Pi,r)\cdot L(1-s,\Pi,\widehat{r})$$

with

$$\varepsilon(s,\Pi,r) = \prod_{v} \varepsilon(s,\Pi,r_v,\psi_v)$$

and $\widehat{r}$ be contragredient of r given as $\widehat{r}(g) = {}^t r(g)^{-1}$.

Definition 6.3.2. *A complex function $L(s,\Pi,r)$ is called an automorphic L-function.*

In (Shahidi, 1988a) and (Shahidi, 1988a), F.Shahidi established a partial result in this direction using the method so-called Langlands-Shadidi method. What exactly he did was the calculation of the non-constant terms of Eisenstein series, associated cuspidal representations of the Levi subgroups in maximal parabolic subgroups of connected split semisimple algebraic groups. The observation that many automorphic L-functions come in sight in constant terms of Eisenstein series above was due to R.Langlands (Langlands, 1971), which he proved meromorphic contuniation of this particular Eisenstein series but not the associated functional equation. By using Langlands-Shadidi method, F.Shadidi was able to obtained the functional equations. Here and the following subsection we shall express Shahidi's result briefly.

Suppose A be some torus in Z_M, which splits. We put $\mathfrak{a} = Hom(X(M)_F, \mathbb{R})$. By Theorem 3.1.3, $X(M)_F$ has finite index in $X(A)_F$, and hence we have

$$\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} \simeq X(A)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\mathfrak{a}^*_{\mathbb{C}} := \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. Set γ for real Lie algebra of the split torus in $Z(G)$. The embedding of free abelian groups

$$X(M)_F \hookrightarrow X(M)_{F_v}$$

delivers embeddings $\mathfrak{a}_v \hookrightarrow \mathfrak{a}$, where $\mathfrak{a}_v = Hom(X(M)_{F_v}, \mathbb{R})$.

Definition 6.3.3. *The Harish-Chandra homomorphism $H_P : M(\mathbb{A}_F) \to \mathfrak{a}$ is defined as*

$$exp\langle \chi, H_P(m)\rangle = \prod_v |\chi(m_v)|_v.$$

for $\chi \in X(M)_F$, $m = \otimes_v m_v \in M(\mathbb{A}_F)$.

We could extend the Harish-Chandra homomorphism H_P to $G(\mathbb{A}_F)$, by using Iwasawa decomposition $G(\mathbb{A}_F) = M(\mathbb{A}_F) \cdot N(\mathbb{A}_F) \cdot K_{\mathbb{A}_F}$, which we let H_P be trivial on $N(\mathbb{A}_F)$ and $K_{\mathbb{A}_F}$. Define $H_{P_v} : M_v \to \mathfrak{a}_v$ by

$$q_v^{exp\langle \chi_v, H_{P_v}(m)\rangle} = |\chi_v(m_v)|_v$$

for $\chi_v \in X(M)_{F_v}$ and $m_v \in M_v$, where v is a finite place and define

$$exp\langle \chi_v, H_{P_v}(m)\rangle = |\chi_v(m_v)|_v$$

for $\chi_v \in X(M)_{F_v}$ and $m_v \in M_v$, where v is an infinite place. Hence we have

$$exp\langle \chi, H_P(m)\rangle = \prod_{v|\infty} exp\langle \chi_v, H_{P_v}(m)\rangle \cdot \prod_{v<\infty} q_v^{exp\langle \chi_v, H_{P_v}(m)\rangle}$$

for $\chi \in X(M)_F$, $m = \otimes_v m_v \in M(\mathbb{A}_F)$.

Proposition 6.3.1. *The product*

$$exp\langle\chi, H_P(m)\rangle = \prod_{v|\infty} exp\langle\chi_v, H_{P_v}(m)\rangle \cdot \prod_{v<\infty} q_v^{exp\langle\chi_v, H_{P_v}(m)\rangle}$$

is finite for$\chi \in X(M)_F$*,* $m = \otimes_v m_v \in M(\mathbb{A}_F)$*.*

Proof. For almost all v, $\chi \in X(M)_F$ is defined by a polynomial in O_v. Therefore for almost all v, $m_v \in M(O_v)$ and $\chi(m_v) \in O_v^*$. Hence $|\chi(m_v)|_v = 1$, except for some finite set. Hence above product is a finite.

□

Suppose $A_0 \subset T$ be some maximal torus , which splits. Put Φ be some set of roots of A_0. Whereas $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ be positive roots generating unipotent radical U with $\Phi^- = -\Phi^+$. We identify the unique reduced root of A in N by an element $\alpha \in \triangle$, where $\triangle \subset \Phi^+$ is the simple roots. Put ρ_P the half of the summation of roots generating N. Put

$$\tilde{\alpha} = \langle\rho_P, \alpha\rangle^{-1} \cdot \rho_P$$

For any pair of roots $\alpha, \beta \in \Phi^+$, we define a pairing $\langle\alpha, \beta\rangle$ by the following way. Suppose $\tilde{\Phi}^+$ be the non-restricted roots of T on U. Thus simple roots $\tilde{\triangle} \subset \tilde{\Phi}^+$ restricts to $\triangle$. Hence by identifying α and β to roots in $\tilde{\Phi}^+$, we put

$$\langle\alpha, \beta\rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

where (,) is the standard inner product on $\mathbb{R}^l$ and $l = |\tilde{\triangle}|$.

Now suppose $\Pi = \otimes_v \Pi_v$ given some cuspidal representation of $M(\mathbb{A}_F)$. Given $M(\mathbb{A}_F) \cap K_{\mathbb{A}_F}$-finite mapping, ϕ ion space V_Π of Π, one can elevate ϕ to mapping $\tilde{\phi}$ on $G(\mathbb{A}_F)$. (Shahidi, 1978). In particular we define the Eisenstein series $E(s, \tilde{\phi}, g, P)$ as

$$E(s, \tilde{\phi}, g, P) = \sum_{\gamma \in P(F)\backslash G(F)} \tilde{\phi}(\gamma \cdot g) \cdot exp\langle s \cdot \tilde{\alpha} + \rho_P, H_P(\gamma \cdot g)\rangle.$$

for $g \in G(\mathbb{A}_F)$.

Theorem 6.3.2. *(Langlands, 1976). The Eisenstein series* $E(s, \tilde{\phi}, g, P)$ *is absolutely convergent in sufficiently large* $Re(s) \gg 0$ *which expands to some meromorphic mapping on* $\mathbb{C}$*, admitting finitely poles on* $Re(s) > 0$*, which are simple poles on* $\mathbb{R}$*.*

Let W_{A_0} be Weyl group of A_0 in G. Suppose $\theta \subset \triangle$ be subset of the base generating M. We have $\triangle = \theta \cup \{\alpha\}$ where α is the element of the base $\triangle$, identified with the unique reduced root of A in N. Hence there exists only one $\overline{w} \in W_{A_0}$ with $\overline{w}(\theta) \subset \triangle$ and $\overline{w}(\alpha) \in \Phi^-$. For our purposes we fix a representative $w \in K_{\mathbb{A}_F} \cap G(F)$ for the element $\overline{w}$. For simplicity, let us denote every component of w again by w. Let

$$I(s,\Pi) = Ind_{M(\mathbb{A}_F)N(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \Pi \otimes exp\langle s \cdot \tilde{\alpha}, H_P(\ ,\ )\rangle \otimes 1$$

be the induced representation of $G(\mathbb{A}_F)$ from the parabolic subgroup $P(\mathbb{A}_F)$. We have the decomposition $I(s,\Pi) = \otimes_v I(s,\Pi_v)$ with each local representation is given by

$$I(s,\Pi_v) = Ind_{M_vN_v}^{G_v} \Pi_v \otimes q_v^{\langle s\cdot\tilde{\alpha}, H_P(\ ,\ )\rangle} \otimes 1$$

for each place v (Yang, 2009). Note that the quantity q_v should be replaced by exp if v is an archimedean place. We define $M' \subset G$ be the subgroup spanned by the element $\overline{w}(\theta)$. In particular there is some parabolic $B \subset P'$ so that $P' = M' \cdot N'$ with M' be the corresponding Levi subgroup. Let $f \in I(s,\Pi)$ and for $Re(s) \gg 0$, define

$$M(s,\Pi)f(g) = \int_{N'(\mathbb{A}_F)} f(w^{-1} \cdot n \cdot g) \cdot dn$$

for $g \in G(\mathbb{A}_F)$. For each v and for $Re(s) \gg 0$ we may define

$$A(s,\Pi_v,w)f_v(g) = \int_{N'_v} f_v(w^{-1} \cdot n \cdot g) \cdot dn$$

for $g \in G_v$ where $f_v \in I(s,\Pi_v)$. We call such an operator a local interwining operator. In particular we have the following decomposition

$$M(s,\Pi) = \otimes_v A(s,\Pi_v,w)$$

Theorem 6.3.3. *(Langlands, 1976). For $Re(s) \gg 0$, $M(s,\Pi)$ extended to meromorphic mapping on $\mathbb{C}$ admitting finitely many simple poles.*

6.3.1 Main Result of Shahidi

Suppose ${}^LP$ be a parabolic subgroup in the L-group ${}^LG$ with Levi decomposition ${}^LG = {}^LM\,{}^LN$. This give rise to an adjoint representation

$$r : {}^LM \to End({}^L\mathfrak{n})$$

given by adjoint action of ${}^LM$, to Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$. Let

$$r = r_1 \oplus r_2 \oplus \cdots \oplus r_n$$

be the decomposition of r. As in (Langlands, 1971; Shahidi,1988b), each of the irreducible constituent (r_i, V_i) given by

$$V_i = \{X_{\beta^\vee} \in {}^L\mathfrak{n} \mid \langle \tilde{\alpha}, \beta \rangle = i\}$$

Theorem 6.3.4. *(Lai, 1974). One has the following expression*

$$M(s,\Pi)f := (\otimes_{v\in S} A(s,\Pi_v,w)f_v) \bigotimes (\otimes_{v\notin S} \tilde{f}_v) \times \prod_{i=0}^{n} \frac{L_S(is,\Pi,r_i)}{L_S(1+is,\Pi,r_i)}$$

where $f = \otimes_v f_v \in I(s,\Pi)$ which for $v \notin S$, f_v is the only K_v-stable vector with $f_v(e_v) = 1$, $\tilde{f}_v$ is K_v-stable vector on $I(-s,\bar{w}(\Pi_v))$ with $\tilde{f}_v(e_v) = 1$.

For every archimedean v, let $\phi_v : W_{F_v} \to {}^LM_v$ be the admissible homomorphism associated to Π_v by local Langlands correspondence (Langlands, 1989). We have the inclusion homomorphisms $\mu_v : {}^LM_v \hookrightarrow {}^LM$ and we set

$$r_{i,v} := r_i \circ \mu_v : {}^LM_v \to End(V_i)$$

for $i = 1,2,\cdots,n$. Hence the composition $r_{i,v} \circ \phi_v$ a finite dimensional representation of W_{F_v} on V_i, for $i = 1,2,\cdots,n$. If $L(s, r_{i,v} \circ \phi_v)$ is the associated L-function, we put

$$L^S(s,\Pi,r_i) = \prod_{v|\infty} L(s, r_{i,v} \circ \phi_v) \times \prod_{v\notin S} L(s,\Pi_v, r_i \circ \mu_v).$$

Let $\rho : M(\mathbb{A}_F) \to M(\mathbb{A}_F)^{ad}$ be the projection onto the adjoint group. Main result of Shahidi is :

Theorem 6.3.5. *(Shahidi, 1988b). Let $\Pi = \otimes_v \Pi_v$ be some cuspidal representation of*

$M(A_F)^{ad}$. Then $L^S(s,\Pi,r_i\circ {}^L\rho)$ admits meromorphic continuation to s-plane. In particular, if Π is generic, then $L^S(s,\Pi,r_i\circ {}^L\rho)$ satisfies the following functional equation

$$L^S(s,\Pi,r_i\circ {}^L\rho)=\varepsilon_S(s,\Pi,r_i\circ {}^L\rho)\cdot L^S(1-s\Pi,\widehat{r_i\circ {}^L\rho})$$

where $\varepsilon_S(s,\Pi,r_i\circ {}^L\rho)$ is the root number associated to Π and $r_i\circ {}^L\rho$, and $\widehat{r_i\circ {}^L\rho}$ is the contragredient representation of $r_i\circ {}^L\rho$.

As an application of the theorem, we now study a particular example :

Example 6.3.2. *Suppose G is simply connected semisimple split group of type G_2 over F (Kim, 2004). Let $\mathbb{A}_\infty=\prod_{v|\infty}F_v$ and K_∞ be maximal compact in $G(\mathbb{A}_\infty)$. Set for a finite place v of F we let $K_v=G(O_v)$. Suppose T a split maximal torus in G with $B=TU$ a Borel subgroup. Since G is of type G_2, let $\triangle=\{\beta_1,\beta_6\}$ be the basis of system of roots Φ interrelated to (B,T) with long simple root β_1 and the short simple root β_6. The other roots are given by*

$$\beta_2=\beta_1+\beta_6,\quad \beta_3=2\beta_1+3\beta_6\quad \beta_4=\beta_1+2\beta_6\quad \beta_5=\beta_1+3\beta_6.$$

Let P be maximal parabolic subgroup in G generated by long β_1 with decomposition $P=MN$ where $M\simeq GL(2)$ as in Lemma 2.1 in (Shahidi, 1989). Hence we have

$$\mathfrak{a}^*=\mathbb{R}\beta_4,\quad \mathfrak{a}=\mathbb{R}\beta_4^\vee\quad \rho_P=\frac{5}{2}\beta_4$$

Let $\tilde\alpha=\beta_4$. Then the element $s\cdot\tilde\alpha$ for $s\in\mathbb{C}$ comes with $|det(m)|^s$. Suppose $\Pi=\otimes_v\Pi_v$ be cuspidal representation of $GL(2,\mathbb{A}_F)$. Assume the central character w_Π of Π trivial on the group $\mathbb{R}_+^*$. If $K_{\mathbb{A}_F}=K_\infty\times\prod_{v<\infty}K_v$, then for a K_{A_F}-finite function ϕ in the space V_Π of Π, by Theorem 6.3.2 the Eisenstein series $E(s,\tilde\phi,g,P)$ is absolutely convergent for $Re(s)\gg 0$ with lifts to some meromorphic function on $\mathbb{C}$. We know from (Langlands, 1976), that the discrete spectrum $L^2_{disc}(G(F)\backslash G(\mathbb{A}_F))_{(M,\Pi)}$ generated by the residues of Eisenstein series for $Re(s)>0$. Moreover poles overlap with constant coefficients. Hence it is sufficient to look constant coefficient throughout the parabolic subgroup P. Any $f\in I(s,\Pi)$, the constant coefficient in $E(s,f,g)$ throughout the group P is

$$E_0(s,f,g)=\sum_{\lambda\in A}M(s,\Pi,\lambda)\cdot f(g),\qquad A=\{1,s_6s_1s_6s_1s_6\}$$

where s_i is the reflection through β_i defined as

$$s_i(\beta) = \beta - \frac{2(\beta_i, \beta)}{(\beta_i, \beta_i)} \cdot \beta_i, \quad 1 \geq i \geq 6, \ \ \beta \in \Phi$$

We choose the Weyl group representatives to stand in $K_{\mathbb{A}_F} \cap G(F)$. We have

$$M(s, \Pi, w) \cdot f(g) = \int_{N_w^-(\mathbb{A}_F)} f(w^{-1}ng)\, dn = \prod_v \int_{N_w^-(F_v)} f_v(w_v^{-1} n_v g_v) dn_v$$

with $g = \otimes_v g_v \in G(A_F)$ and $f = \otimes_v f_v \in I(s, \Pi)$ so that f_v is the only K_v-stable element with $f_v(e_v) = 1$ for nearly all v and

$$N_w^- = \prod_{\alpha>0,\ w^{-1}\alpha<0} U_\alpha$$

with U_α is a one parameter unipotent subgroup. Consider the standart representation

$$S : GL_2(\mathbb{C}) \to GL_2(\mathbb{C})$$

and the third symmetric power representation of $GL_2(\mathbb{C})$,

$$Sym^3 :\ {}^L M = GL_2(\mathbb{C}) \to GL_4(\mathbb{C})$$

Furthermore let

$$(Sym^3)^0 = Sym^3 \bigotimes (\wedge^2 S)^{-1}$$

be the adjoint cube representation of $GL_2(\mathbb{C})$. By Theorem 6.3.4, for the automorphism $w = s_6 \cdot s_1 \cdot s_6 \cdot s_1 \cdot s_6$, we have

$$M(s, \Pi) \cdot f = (\otimes_{v \in S} M(s, \Pi_v, w) \cdot f_v) \otimes (\otimes_{v \notin S} \tilde{f}_v \times L_S(s, \tilde{\Pi}, (Sym^3)^0)$$

$$\times L_S(2s, \tilde{\Pi}, \wedge^2 S) \times L_S(1+s, \tilde{\Pi}, (Sym^3)^0)^{-1} \times L_S(1+2s, \tilde{\Pi}, \wedge^2 S)^{-1}.$$

In particular, if Π is non-monomial cuspidal representation of $M(\mathbb{A}_F) = GL_2(\mathbb{A}_F)$, then $L(s, \Pi, Sym^3)$ and $L(s, \Pi, (Sym^3)^0)$ are analytic on $\mathbb{C}$ (Kim & Shahidi, 1999), and hence satisfy :

$$L(s, \Pi, Sym^3) = \varepsilon(s, \Pi, Sym^3) \cdot L(1-s, \tilde{\Pi}, Sym^3).$$

$$L(s, \Pi, (Sym^3)^0) = \varepsilon(s, \Pi, (Sym^3)^0) \cdot L(1-s, \tilde{\Pi}, (Sym^3)^0).$$

Chapter 7

Statement of the Functoriality Principle

The principle of Functoriality is a profound proposal put forward by Robert Langlands in the late $60's$. Roughly speaking, the Functorialty principle aims to understand irreducible admissible or automorphic representations of a given group G, by transfering representations to other groups via local/global lifting maps. Such a transfer could have local/global compatibility in the sense of transfer parameters.

7.1 Local and Global Functoriality

Conjecture (Local Functoriality) Let k be a local field and H, G are connected reductive groups over k with G quasi-split. Then for any L-homomorphism

$$\rho : {}^L H \to {}^L G$$

a lifting of admissible representations of H to that of G exists. More precisely, for some given irreducible admissible representation Π of $H(k)$, one may acquire an irreducible admissible representation $\widehat{\Pi}$ of $G(k)$ via the following diagram :

$$\begin{tikzcd} & & WD_k \arrow[dl, "\phi_\Pi"'] \arrow[dr, "\phi_{\widehat{\Pi}}"] & & \\ \Pi \xrightarrow{LLC} & {}^L H \arrow[rr, "\rho"] & & {}^L G & \xrightarrow{LLC} \widehat{\Pi} \end{tikzcd}$$

Remark 7.1.1. *Suppose k be a non-archimedean with both H, G are quasi-split. Suppose for some finite extension K/k, both $H(O_K)$ and $G(O_K)$ are maximal compact subgroups. Let Π be an unramified representation of $H(k)$ with ϕ_Π is associated unramified admissible homomorphism. Then for each $\rho : {}^L H \to {}^L G$, composition*

$\widehat{\phi} = u \circ \phi$ *is an admissible homomorphism by Remark 5.1.8 and is unramified. It defines an L-packet* $A_{\widehat{\phi}}(G)$ *which contains a unique unramified representation* $\widehat{\Pi}$. *Representation* $\widehat{\Pi}$ *is called the natural lift of* Π.

Conjecture (Global Functoriality) Suppose F be some global field and let H, G are connected reductive groups over F with G quasi-split. Then for any automorphic representation $\Pi = \otimes_v \Pi_v$ of $H(\mathbb{A}_F)$, and for an L-homomorphism $\rho : {}^L H \to {}^L G$, there should some automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}_F)$with for semi-simple Hecke conjugacy classes one has

$$c_v(\Pi) = \rho(c_v(\pi)), \qquad v \notin S(\Pi) \cup S(\pi).$$

where $S(\Pi)$, $S(\pi)$ are finite set of ramified places in F for Π and π respectively.

Example 7.1.1. *Let* $H = \{1\}$ *and* $G = GL_n$ *be the general linear group both are defined over some number field F. Since G be split, Galois action on L-group of G is trivial. Also connected component of the L-group of H is trivial and the Galois action factors through a finite Galois extension* E/F. *Hence we have the following admissible homomorphism*

$$\rho : Gal(E/F) \to {}^L G = GL_n(\mathbb{C})$$

which is a simply a continuous Galois representation.

Conjecture (Strong Artin Conjecture) Let $G = GL_n$ over a number field F, for some $n \in \mathbb{N}$, and suppose E/F be some finite normal extension. Then for n-dimensional Galois representation

$$\rho : Gal(E/F) \to {}^L G = GL_n(\mathbb{C})$$

there should be an automorphic representation $\Pi = \otimes_v \Pi_v$ of $GL_n(\mathbb{A}_F)$ with

$$c_v(\Pi) = \rho(Frob_v), \quad v \notin S(E).$$

Remark 7.1.2. *For* $F = \mathbb{Q}$ *and* $n = 1$, *Strong Artin Conjecture is nothing but the Kronecker-Weber theorem. In particular, for an arbitrary number field F and* $n = 1$, *this is exactly the Artin reciprocity law and hence the fundamental theorem of abelian global class field theory.*

7.2 Automorphic Langlands Group

In this section we shall investigate arithmetic parametrization problem of automorphic representations. Throughout this section F will be a global field and G will be a connected reductive group over F. Our main aim is to understand the hypothetical structure of the automorphic Langlands group. It has arisen from the fact that $Gal(F^{sep}/F)$ and W_F are too small to parametrize automorphic representations. Our main reference is (İkeda, 2020).

Definition 7.2.1. *Let $\mathfrak{h}_F$ and $\mathfrak{a}_F$ denotes henselian and archimedean places of F respectively. For every $v \in \mathfrak{h}_F \cup \mathfrak{a}_F$, the automorphic Langlands group $\mathbb{L}_{F_v}$ of the local field F_v is defined as*

$$\mathbb{L}_{F_v} = W_{F_v} \times SU_2(\mathbb{R}), \text{ if } v \in \mathfrak{h}_F$$

$$\mathbb{L}_{F_v} = W_{F_v}, \text{ if } v \in \mathfrak{a}_F$$

Hence the local automorphic Langlands group is split extension of W_{F_v} by some compact, simply connected Lie group (either $SU_2(\mathbb{R})$ or $\{1\}$).

Remark 7.2.1. *Recall that W_F is a locally compact group endowed with a continuous dense mapping $\delta_F : W_F \to Gal(F^{sep}/F)$. In particular if $char(F) > 0$, then the automorphic Langlands group associated to F is simply the Weil group W_F of F.*

Definition 7.2.2. *Suppose F be a number field. Automorphic Langlands group $\mathbb{L}_F$ is a hypothetical locally compact topological group which is an infinite fibre product of the non-split extension*

$$1 \to K_c \to \mathbb{L}_F \to W_F \to 1$$

of W_F by a compact, semisimple, simply connected Lie group K_c, with following expected formal properties :

1) There exists a surjective topological group homomorphism

$$\mathbb{L}_F \xrightarrow{\chi_F} W_F$$

2) The kernel of the composed topological group homomorphism

$$\mathbb{L}_F \xrightarrow{\chi_F} W_F \xrightarrow{\delta_F} Gal(F^{sep}/F)$$

is connected. In particular, one has an isomorphism of abelianizations via χ_F *:*

$$\mathbb{L}_F^{ab} \xrightarrow{\sim} W_F^{ab} \simeq \mathbb{A}_F^\times / F^\times$$

3) The kernel of the composed topological group homomorphism

$$\mathbb{L}_F \xrightarrow{ab} \mathbb{L}_F^{ab} \xrightarrow{\sim} W_F^{ab} \xrightarrow{Art_F} \mathbb{A}_F^\times / F^\times \xrightarrow{log|.|} \mathbb{R}$$

is compact.

4) For each place $v \in \mathfrak{h}_F \cup \mathfrak{a}_F$, *the automorphic Langlands group* $\mathbb{L}_{F_v}$ *of the local field* F_v *determines a surjective continuous homomorphism :*

$$\mathbb{L}_{F_v} \xrightarrow{\chi_{F_v}} W_{F_v}$$

5) Every embedding $i_v : F^{sep} \hookrightarrow F_v^{sep}$ *determines an* $\mathbb{L}_F$*-conjugacy class of a continuous embedding* $i_v^L : \mathbb{L}_{F_v} \hookrightarrow \mathbb{L}_F$, *and extends the conjugacy classes of embeddings* $i_v^W : W_{F_v} \hookrightarrow W_F$ *and* $\tilde{i}_v : Gal(F_v^{sep}/F_v) \hookrightarrow Gal(F^{sep}/F)$ *via the following commutative diagram of continuous homomorphisms :*

$$\begin{array}{ccccc} \mathbb{L}_{F_v} & \xrightarrow{\chi_{F_v}} & W_{F_v} & \xrightarrow{\delta_{F_v}} & Gal(F_v^{sep}/F_v) \\ {\scriptstyle i_v^L}\downarrow & & {\scriptstyle i_v^W}\downarrow & & \downarrow{\scriptstyle \tilde{i}_v} \\ \mathbb{L}_F & \xrightarrow[\chi_F]{} & W_F & \xrightarrow[\delta_F]{} & Gal(F^{sep}/F) \end{array}$$

Conjecture (Global Langlands Reciprocity) Automorphic representations $\Pi = \otimes_v \Pi_v$ of $G(\mathbb{A}_F)$ can be parametrized by admissible homomorphisms $\phi : \mathbb{L}_F \to {}^L G$, with following properties:

1) There exists an L-packet A_ϕ which consist of automorphic representations of $G(\mathbb{A}_F)$ associated to ϕ.

2) L-packets uniquely determines a given automorphic representation $\Pi = \otimes_v \Pi_v$ of $G(\mathbb{A}_F)$ i.e every automorphic representation belongs to some A_ϕ for a unique ϕ. In particular all L-packets A_ϕ are disjoint.

Remark 7.2.2. *By using the local Langlands correspondence, we can give a reformulation of the global Langlands Functoriality as follows : Let* $\pi = \otimes_v \pi_v$ *be an automorphic representation of* $H(\mathbb{A}_F)$. *By the local Langlands correspondence, for each* π_v, *there is an admissible homomorphism* $\phi_v : WD_{F_v} \to {}^L H$ *.Then for a given L-homomorphism*

$$\rho : {}^L H \to {}^L G$$

by Remark 5.1.8, we have an admissible homomorphism of G *as* $\rho \circ \phi_v : WD_{k_v} \to {}^L G$. *Therefore* $\rho \circ \phi_v$ *induces an irreducible admissible representation* Π_v *of* $G(F_v)$. *By Flath's theorem* $\Pi = \otimes_v \Pi_v$ *is an irreducible admissible representation of* $G(\mathbb{A}_F)$. *Therefore the global Langlands functoriality conjecture asserts* Π *should be automorphic representation of* $G(\mathbb{A}_F)$.

Remark 7.2.3. *Using global Langlands reciprocity principle, we can formulate the global Langlands functoriality as follows : Given automorphic representation* $\pi = \otimes_v \pi_v$ *of* $H(\mathbb{A}_F)$ *we attach an admissible homomorphism* $\phi_\pi : \mathbb{L}_F \to {}^L H$ *by global Langlands reciprocity principle. Then for a given L-homomorphism*

$$\rho : {}^L H \to {}^L G$$

we have an admissible homomorphism $\rho \circ \phi_\pi : \mathbb{L}_F \to {}^L G$ *of* G. *Then the global Langlands functoriality claims that there exists an L-packet associated to* $\rho \circ \phi_\pi$.

7.3 Some Examples of Functoriality

Example 7.3.1. *Let* $H = GL_n \times GL_m$ *and* $G = GL_{mn}$ *be linear groups, both are defined over some number field* F. *Hence as L-groups we have*

$${}^L H = GL_n(\mathbb{C}) \times GL_m(\mathbb{C}), \quad {}^L G = GL_{nm}(\mathbb{C})$$

As an L-homomorphism we take the usual tensor product homomorphism

$$\otimes = \rho : GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \to GL_{nm}(\mathbb{C})$$

Suppose $\pi = \otimes_v \pi_v$, $\tau = \otimes_v \tau_v$ *are cuspidal representations of* $GL_n(\mathbb{A}_\mathbb{F})$ *and* $GL_m(\mathbb{A}_\mathbb{F})$ *respectively. By local Langlands correspondence for* GL_n *(Harris & Taylor, 2002;*

Langlands, 1989), we have an admissible homomorphisms

$$\phi_v : WD_{F_v} \to GL_n(\mathbb{C}), \;\; \psi_v : WD_{F_v} \to GL_m(\mathbb{C})$$

for every $v \in \mathfrak{h}_F \cup \mathfrak{a}_F$. *Let*

$$\rho \circ [\phi_v, \psi_v] : WD_{F_v} \to GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) = {}^L H \xrightarrow{\otimes} GL_{nm}(\mathbb{C}) = {}^L G$$

be admissible homomorphism of G defined by homomorphism

$$[\phi_v, \psi_v](w) = (\phi_v(w), \psi_v(w)) \quad w \in WD_{F_v}$$

of H. *Again, by local Langlands correspondence for* GL_n, $\theta_v = \rho \circ [\phi_v, \psi_v]$ *corresponds to some irreducible admissible representation* $\pi_v \boxtimes \tau_v$ *of* $GL_{mn}(F_v)$. *Put*

$$\pi \boxtimes \tau = \bigotimes_v \pi_v \boxtimes \tau_v$$

Hence the global Langlands functoriality assets that $\pi \boxtimes \tau$ *be an automorphic representation of* $GL_{nm}(\mathbb{A}_{\mathbb{F}})$.

Example 7.3.2. *Let* $H = GL_2$ *and* $G = GL_{n+1}$ *be linear groups, both are defined over some number field* F, *for* $n \geq 2$. *Suppose* $\Pi = \otimes_v \Pi_v$ *be an automorphic representation of* $H(\mathbb{A}_{\mathbb{F}})$. *By local Langlands correspondence for* GL_n *(Harris & Taylor, 2002; Langlands, 1989), for every* $v \in \mathfrak{h}_F \cup \mathfrak{a}_F$, *we have a Hecke parameter*

$$c(\Pi_v) = \{ \begin{pmatrix} \alpha_v & 0 \\ & \\ 0 & \beta_v \end{pmatrix} \in GL_2(\mathbb{C}) \}$$

as a semisimple conjugacy class in $GL_2(\mathbb{C})$. *As in Example 6.3.1, we take an L-homomorphism as the symmetric n-th power representation*

$$\rho = Sym^n : {}^L H = GL_2(\mathbb{C}) \to {}^L G = GL_{n+1}(\mathbb{C})$$

For each $v \in \mathfrak{h}_F \cup \mathfrak{a}_F$, *there is an irreducible admissible symmetric power representation* *of $GL_{n+1}(\mathbb{C})$, denoted by $Sym^n(\Pi_v)$, attached to the Hecke parameter*

$$\{\begin{pmatrix} \alpha_v^n & & \\ & \alpha_v^{n-1}\cdot\beta_v & \\ & & \ddots\ \beta_v^n \end{pmatrix} \in GL_{n+1}(\mathbb{C})\}$$

as a semisimple conjugacy class in $GL_{n+1}(\mathbb{C})$. Let

$$Sym^n(\Pi) = \otimes_v\, Sym^n(\Pi_v)$$

In particular $Sym^n(\Pi_v)$ is an irreducible admissible representation of $GL_{n+1}(\mathbb{A}_F)$. Langlands functoriality principle implies that $Sym^n(\Pi)$ be an automorphic representation of $GL_{n+1}(\mathbb{A}_F)$.

Example 7.3.3. *Let $H = GL_n$ and $G = GL_m$ be linear groups, both are defined over a number field F where $n \geq 2$ and $m = \frac{n(n-1)}{2}$. As an L-homomorphism we take the exterior square*

$$\wedge^2 : {}^L H = GL_n(\mathbb{C}) \to {}^L G = GL_m(\mathbb{C})$$

Let $\Pi = \otimes_v \Pi_v$ be a cuspidal representation of $GL_n(\mathbb{A}_F)$. Then for any $v \in \mathfrak{h}_F \cup \mathfrak{a}_F$, we can attach an admissible homomorphism $\phi_v : WD_{F_v} \to {}^L H = GL_n(\mathbb{C})$ to local representation Π_v. Since the local parametrization of unramified admissible representations is valid, and hence the admissible homomorphism

$$\psi_v = \wedge^2 \circ \phi_v : WD_{F_v} \to {}^L G = GL_m(\mathbb{C})$$

of G induces an irreducible admissible representation of $GL_m(F_v)$, say $\wedge^2\,\Pi_v$, for each unramified local representation Π_v. We set

$$\wedge^2\,\Pi = \bigotimes_v \wedge^2\Pi_v$$

In particular $\wedge^2\,\Pi$ is an irreducible admissible representation of $GL_m(\mathbb{A}_F)$. Langlands funtoriality principle implies $\wedge^2\,\Pi$ be an automorphic representation of $GL_m(\mathbb{A}_F)$.

Chapter 8

Conclusion

In this thesis, our main goal was understand and exhibit the functoriality principle of Langlands in detail. For this purpose, we developed different machineries both from algebraic number theory and harmonic analysis. We proved theorems about compatibility of Weil groups and ℓ-adic representations whose proofs are not available in the current literature. Furhermore, we provided detailed study of L-functions and ε-factors attached to various representations which we support our arguments with various different examples. Also we explained in detail, so called Langlands-Shahidi method, withal Eisenstein series, which we ensure an example related to some standard third symmetric power representation. Also after stating local and global functoriality principles, we defined hypotethical automorphic Langlands groups of number fields, which in global function field case is simply the Weil group, and using this group along with local and global Langlands reciprocity principles, we showed how one can obtain global Langlands functoriality principle by means of reciprocity principles. Finally, we provide some examples that are accommodate tensor structures, symmetric powers and exterior powers of certain groups.

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