DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

LATTICE, QUANTUM ANALYSIS AND APPLICATIONS TO DISCRETE D’ALEMBERT PROBLEMS

by
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İZMİR

LATTICE, QUANTUM ANALYSIS AND APPLICATIONS TO DISCRETE D'ALEMBERT PROBLEMS

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We have read the thesis entitled **“LATTICE, QUANTUM ANALYSIS AND APPLICATIONS TO DISCRETE D’ALEMBERT PROBLEMS”** completed by **ZEHRA TUNCER** under supervision of **ASSOC. PROF. DR. BURCU SİLİNDİR YANTIR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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LATTICE, QUANTUM ANALYSIS AND APPLICATIONS TO DISCRETE D'ALEMBERT PROBLEMS

ABSTRACT

In this thesis, we study on lattice and quantum numbers. First, similar to the ordinary calculus, we build h-nabla, h-delta derivatives, h-binomial, h-exponential, h-trigonometric, h-hyperbolic functions and we introduce h-nabla antiderivative. Since h-analysis recovers real numbers as h tends to zero, the study on h-analysis can be called as "*discretization*" (h-discretization). As an application, we construct h-analogue of problems of wave equations in other words, h-D'Alembert problems and we find h-D'Alembert solutions in terms of h-binomial functions. Also we define h-analyticity and using h-analogue of Cauchy-Kovalevskaya Theorem we present some examples which have unique and analytic solutions. Next, we construct quantum analysis with related q-derivative, $\frac{1}{q}$-derivative and q-integral, $\frac{1}{q}$-integral properties. Finally we construct q and $\frac{1}{q}$-D'Alembert problems and their solutions. By using q-analogue of Cauchy Kovalevskaya Theorem, we examine the uniqueness and q-analyticity of these solutions.

Keywords: Lattice numbers, quantum numbers, lattice derivatives, quantum derivatives, h-Cauchy problems, q-Cauchy problems

LATİS, KUANTUM ANALİZ VE FARK D'ALEMBERT PROBLEMLERİNE UYGULAMALARI

ÖZ

Bu tezde latis ve kuantum sayıları üzerine çalıştık. İlk olarak, normal kalkülüse benzer şekilde, h-nabla, h-delta türevleri, h-binom, h-üstel, h-trigonometrik, h-hyperbolik fonksiyonları inşa ettik ve h-nabla ters türevi tanıttık. h sıfıra giderken h-analiz real sayıları kapladığından, h-analiz üzerindeki bu çalışma "*ayrıklaştırma*" (h-ayrıklaştırması) olarak isimlendirilebilir. Bunun uygulaması olarak, dalga denklemi problemlerinin h-analogunu bir diğer deyişle, h-D'Alembert problemlerini oluşturduk ve h-binom fonksiyonları cinsinden h-D'Alembert çözümlerini bulduk. Ayrıca h-analitikliği tanımladık ve Cauchy-Kovalevskaya Teoreminin h-versiyonunu kullanarak tek ve analitik çözümlere sahip olan bazı örnekler sunduk. Daha sonra kuantum analizle ilgili olan q-türev, $\frac{1}{q}$-türev ve q-integral, $\frac{1}{q}$-integral özelliklerini inşa ettik. Son olarak da q ve $\frac{1}{q}$-D'Alembert problemlerini ve onların çözümlerini inşa ettik. Cauchy-Kovalevskaya Teoreminin q-analogunu kullanarak bu çözümlerin teklik ve q-analitikliklerini inceledik.

Anahtar kelimeler: Latis sayıları, kuantum sayıları, latis türevleri, kuantum türevleri, h-Cauchy problemleri, q-Cauchy problemleri

CONTENTS

CHAPTER ONE
INTRODUCTION

Discrete mathematics can be considered as the branch of math which deal with mathematical constructions that are basically discrete rather than continuous. The set of objects studied in discrete mathematics have the same cardinality with the natural numbers, including rational numbers but not real numbers.

The idea, that was originated in 1769 with Euler's polygonal method, of using difference equations to approach the solutions of differential equations was given by Cauchy about 1840. The subject seemed to be not interesting until around the end of the 19th century, when Lipschitz, Runge, and Kutta improved the procedures. The influential application of linear difference equations to the computation of special functions was given in 1952 with Miller's algorithm for Bessel's functions. J. Wimp (Wimp, 1984) discussed the improvement of this computation and related algorithms with Olver, Clenshaw, Gautschi, and others. Also they discussed some examples of calculation with nonlinear difference equations. Moreover, with this improvement linear difference equations were comparable to linear differential equations (for more details see Hartman, 1978). During the 1950's, simple nonlinear difference equations were used by ecologists for stability of iteration on populations.

Quantum analysis is an area related with abstract and applied mathematics. Works on q-difference equations were appeared at the beginning of 20th century by especially F. H. Jackson (Jackson, 1910), R.D. Carmichael (Carmichael, 1912), and other authors such as Picard, Ramanujan. From the 30s to the begining of the 80s, remarkable interest was not observed in this area. But after the 70s, significant and surprising studies such as principally new difference calculus; q-calculus (Kac, 2001), q-combinatorices (Charalambides, 2016), q-integrable systems (Silindir, 2012 and Silindir & Soyoğlu, 2015), and related subjects (Nalci & Pashaev, 2010 and Pashaev & Nalci 2014) were presented.

The wave equation was appeared in the 18th century, when calculus was developed and it was possible to connect the regularities observed in waves to a mathematical format. The wave equation was discovered by Brook (Brook, 1717) using laws of motion (formulated by Newton). The one-dimensional wave equation is defined by

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.1}$$

which is a second-order linear partial differential equation that is used for the explanation of waves. A general solution of the wave equation is given by

$$u(x,t) = f(x+ct) + g(x-ct), \tag{1.2}$$

where the functions f and g are arbitrary twice continuously differentiable (Jean le Rond, 1747). In other words, solutions (1.2) of the one dimensional wave equations describe sums of two distinct waves one of which travels left and one to the right both at the same speed c. Here "travel" means that these two waves do not interact with each other, so their shapes do not change.

For an initial value problem defined on (1.1) together with $u(x,0) = F(x)$ and $u_t(x,0) = G(x)$, the solution is

$$u(x,t) = \frac{1}{2}[F(x+ct) + G(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} G(x')dx' \tag{1.3}$$

and it is called D'Alembert formula.

In this thesis we will do *discretization* that will be done in two ways; h-discretization and q-discretization. When we use h-discretization terminology, we mean that as $h \to 0$, equations on lattice numbers give real (ordinary) equations. In the same way, as $q \to 1$, we recover continuous equations. Within this context, in Chapter 2 lattice analysis (i.e. h-analysis) and in Chapter 4 quantum analysis (i.e. q-analysis) will be studied by giving some related properties of them.

In Chapter 2, we present first *the time scale calculus* which is developed by Hilger

(Hilger, 1988). We give some time scale examples such as lattice numbers denoted by

$$h\mathbb{Z} := \{hn : h \in \mathbb{R}^+,\ \ n \in \mathbb{Z}\} \tag{1.4}$$

and quantum numbers denoted by

$$\mathbb{K}_q := \{q^k : q \in \mathbb{R},\ \ q \neq 1,\ \ k \in \mathbb{Z}\} \cup \{0\}. \tag{1.5}$$

We present lattice differentials and define corresponding lattice derivatives as Δ_h^x and ∇_h^x. Regarding to these lattice derivatives, we give product and quotient rules for two functions on $h\mathbb{Z}$. With these definitions and results, it is easy to say that there are some similarities and differences between lattice analysis and ordinary calculus. For example; h-derivative of product of two functions which is

$$\Delta_h^x(fg)(x) = f(x+h)d_hg(x) + g(x)d_hf(x)$$

can be given. Also we show that as $h \to 0$, lattice numbers recovers $\mathbb{R}$. We present h-binomial function and its some properties. By using these properties we calculate h-delta and h-nabla derivatives of h-binomial. We generalize Taylor's Formula for any function f then by utilizing this formula we present h-Taylor's Formula from which analyticity definition for $h\mathbb{Z}$ can be given. Then we define two h-exponential functions by using the h-Taylor's Formula. We introduce h-trigonometric, h-hyperbolic functions, and calculate their lattice derivatives. Next, we introduce h-nabla antiderivative, definite h-nabla integral, and construct Fundamental Theorem of Calculus on $h\mathbb{Z}$. Similarly, one can construct h-delta antiderivative and its properties (more details see Kac, 2001).

In Chapter 3, we aim to construct h-D'Alembert problem. For this purpose, we calculate h-derivatives of h-binomials $(x+t)_h^n$ and $(x-t)_h^n$ with respect to x and t. We define h-Laurient Series representation by

$$f(x \pm t) := \sum_{n=-\infty}^{\infty} a_n(x \pm t)_h^n. \tag{1.6}$$

We build h-version of wave equation which is defined as follows

$$(\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t)u(x,t) = 0 \tag{1.7}$$

whose solution is

$$u(x,t) = f(x+t) + g(x-t), \tag{1.8}$$

where the functions f and g which are in C^2 defined as Laurient Series representations. It is easy to say that as $h \to 0$, h-wave equation turns out to be wave equation (1.1) whose solution is (1.2) with $c = 1$. Note that the solution (1.8) is so similar to a solution of wave equation (1.2) but here the functions f and g which are defined in (1.6) are Laurient series. Then we introduce h-D'Alembert problem with the initial conditions as follows

$$\begin{aligned}
&\Box u = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t) = 0, \quad \forall x, t \in h\mathbb{Z}, \\
&u(x,0) = \phi(x), \\
&\nabla_h^t u(x,0) = \psi(x).
\end{aligned} \tag{1.9}$$

We show that h-D'Alembert solution is given by

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_a^b \psi(x')d_{\bar{h}}x', \tag{1.10}$$

where $a := x - t$ and $b := x + t$. Here $\forall x, t \in h\mathbb{Z} \quad \phi \in C^2$, $\psi \in C^1$ on the region $\Omega = h\mathbb{Z} \times h\mathbb{Z}$. Then we examine the uniqueness and analyticity of these h-D'Alembert solutions by using the Cauchy Kovelavskaya Theorem, (Gantumur, 2011).

Chapter 4 is concerned with quantum analysis. Despite the similarities with lattice analysis, quantum analysis is quite different. We first present q-differential and q-derivative (Kac, 2001). According to the q-derivative we derive product and quotient rules for q-derivative. We show that there are lots of similarities and differences between quantum analysis and ordinary calculus. For example; q-derivative of q-trigonometric functions defined as $\sin_{\mathrm{q}} x$ and $\cos_{\mathrm{q}} x$ are $\cos x$ and

$\sin x$, respectively. However

$$d_q(fg)(x) = f(qx)d_qg(x) + g(x)d_qf(x), \quad x \in \mathbb{K}_q. \tag{1.11}$$

Whereas $h \to 0$, real numbers are recovered by lattice analysis (h-discretization), here as $q \to 1$, we recover real numbers and this process is called q-discretization. We develop quantum calculus, many important notions and results. For example; when we take q-derivative of $f(x) = x^n$, we get

$$\Delta_q x^n = \frac{(qx)^n - x^n}{(q-1)x} = \frac{x^n(q^n-1)}{(q-1)x} = \frac{q^n-1}{q-1}x^{n-1}. \tag{1.12}$$

Then we define

$$[n]_q := \frac{q^n-1}{q-1} = 1 + q + ... + q^{n-1} \tag{1.13}$$

as the q-analogue of integer n. So we can say the function $f(x) = x^n$ as a good function in $\mathbb{K}_q$. Because we have

$$\Delta_q x^n = [n]_q x^{n-1} \tag{1.14}$$

that can be written in a nice manner. But a bad one in h-analysis since

$$\Delta_h^x x^n = \frac{(x+h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)}{2}hx^{n-2} + ... + hx^{n-1}. \tag{1.15}$$

Next, we define q-analogue of binomials $(x \pm a)_q^n$, $(a - x)_q^n$, and derive related properties. Then by using the generalized Taylor's Formula given in Chapter 2, we present q-Taylor's Formula. Then we find q-Taylor's Formula of $(x+a)_q^n$ at $c = 0$ as

$$(x+a)_q^n := \sum_{j=0}^{n} \binom{n}{j}_q q^{\frac{j(j-1)}{2}} a^j x^{n-j}. \tag{1.16}$$

The formula (1.16) is called as a Gauss' binomial formula, (Kac, 2001). We also find q-Taylor's Formula of $f(x) = \dfrac{1}{(1-x)_q^n}$ at $c = 0$ as

$$\frac{1}{(1-x)_q^n} = \sum_{j=0}^{N} (\Delta_q^j f)(0) \frac{(x-0)_q^j}{[j]_q!} = \sum_{j=0}^{n} \frac{[n]_q[n+1]_q...[n+j-1]_q}{[j]_q!} x^j \tag{1.17}$$

which is called Heine's binomial formula, (Kac, 2001). Next, we define q-exponential functions based on Gauss' binomial formula and Heine's binomial formula. Also we show q-exponential function defined as $y(x) = e_q^x$ is a solution for the problem given as follows

$$\begin{aligned} \Delta_q^x y(x) &= y(x), \\ y(0) &= 1, \end{aligned} \tag{1.18}$$

and other q-exponential function defined as $y(x) = E_q^x$, is a solution for the problem given as follows

$$\begin{aligned} &\Delta_q^x y(x) = y(qx), \\ &y(0) = 1. \end{aligned} \tag{1.19}$$

Clearly, we can see that q-exponential functions are related. Next, regarding to the q-exponential functions, we define q-trigonometric functions as $\sin_q x, \cos_q x, \mathrm{Sin}_q x, \mathrm{Cos}_q x$, and derive q-derivative of these functions using appropriate a exponential functions. Then, we present q-antiderivative, definite q-integral, and Fundamental Theorem of Calculus on $\mathbb{K}_q$ for q-integral. In Section 4.7, similar to the q-derivative, we define $\frac{1}{q}$-derivative, $\frac{1}{q}$-binomial, and we calculate $\frac{1}{q}$-derivative of this binomial. Next, we introduce $\frac{1}{q}$-antiderivative, definite $\frac{1}{q}$-integral, and construct Fundamental Theorem of Calculus on $\mathbb{K}_q$ for $\frac{1}{q}$-integral.

In Chapter 5, similar to the h-Cauchy problem we introduce $\frac{1}{q}$-Cauchy problem given by

$$\begin{aligned} &[(\Delta_q^t)^2 - c^2(\Delta_{\frac{1}{q}}^x)^2]u(x,t) = 0, \quad \forall x, t \in \mathbb{K}_q, \\ &u(x,0) = \Phi(x), \\ &\Delta_{\frac{1}{q}}^t u(x,0) = \Psi(x), \end{aligned} \tag{1.20}$$

whose solution is

$$u(x,t) = \frac{1}{2}[\Phi(x+ct) + \Phi(x-ct)] + \frac{1}{2c}\int_a^b \Psi(x')d_{\frac{1}{q}}x', \tag{1.21}$$

where $a := x - ct$, $b := x + ct$. Here $\forall x,\ t \in \mathbb{K}_q,\ \phi \in C^2,\ \psi \in C^1$ on $\Omega = \mathbb{K}_q \times \mathbb{K}_q$. Note that as $q \to 1$, $\frac{1}{q}$-D'Alembert problem turns out to be D'Alembert problem. Then we define $\frac{1}{q}$-analyticity and by $\frac{1}{q}$-analogue of Cauchy Kovalevskaya Theorem. We present some examples that have unique and analytic solutions. Similarly, we present q-D'Alembert problem given in Nalcı Tumer, 2017

$$\begin{aligned}
&[(\Delta^t_{\frac{1}{q}})^2 - c^2(\Delta^x_q)^2]u(x,t) = 0 \quad, \forall x, t \in \mathbb{K}_q,\\
&u(x,0) = \Phi(x),\\
&\Delta^t_q u(x,0) = \Psi(x),
\end{aligned} \tag{1.22}$$

whose solution is

$$u(x,t) = \frac{1}{2}[\Phi(x-ct) + \Phi(x+ct)] + \frac{1}{2c}\int_a^b \Psi(x')d_q x', \tag{1.23}$$

where $a = x - ct$, $b = x + ct$. Here $\forall x,\ t \in \mathbb{K}_q,\ \phi \in C^2,\ \psi \in C^1$ on $\Omega = \mathbb{K}_q \times \mathbb{K}_q$.

CHAPTER TWO
LATTICE ANALYSIS

We start this chapter by presenting the notion of time scales introduced by Hilger (Hilger, 1988). We collect below some definitions and results from the book Bohner, 2002.

Definition 2.1. *A time scale* $\mathbb{T}$ *is a nonempty closed subset of* $\mathbb{R}$.

Definition 2.2. *Let* $\mathbb{T}$ *be a time scale. For* $t \in \mathbb{T}$*, we define forward shift operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{s \in \mathbb{T}, \ \ s > t\}$$

and backward shift operator $\rho : \mathbb{T} \to \mathbb{T}$ *by*

$$\rho(t) := \sup\{s \in \mathbb{T}, \ \ s < t\}.$$

Definition 2.3. *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\mu(t) := \sigma(t) - t.$$

Example 2.1. *Quantum numbers, denoted by* $\mathbb{K}_q$*, is defined by*

$$\mathbb{K}_q := \{q^k : q \in \mathbb{R}, \ \ q \neq 1, \ \ k \in \mathbb{Z}\} \cup \{0\}$$

and it is a time scale. Further, consider the time scale $\mathbb{T} = \mathbb{K}_q$*. Then forward shift operator is*

$$\sigma(t) = \inf\{q^k : k \in [m+1, \infty)\} = qt$$

and backward shift operator is

$$\rho(t) = \sup\{q^k : k \in (-\infty, m-1]\} = \frac{q}{t}$$

for $t = q^m \in \mathbb{K}_q$. *Consequently*

$$\mu(t) = \sigma(t) - t = (q-1)t.$$

Example 2.2. *Lattice numbers, denoted by* $h\mathbb{Z}$*, is defined by*

$$h\mathbb{Z} := \{hn : h \in \mathbb{R}^+, \;\; n \in \mathbb{Z}\}$$

and it is a time scale. Moreover, consider the time scale $\mathbb{T} = h\mathbb{Z}$. *Then forward shift operator is*

$$\sigma(t) = \inf\{t + nh : n \in \mathbb{N}\} = t + h$$

and backward shift operator is

$$\rho(t) = \sup\{t - nh : n \in \mathbb{N}\} = t - h$$

for $t \in h\mathbb{Z}$. *Consequently*

$$\mu(t) = \sigma(t) - t = t + h - t = h.$$

Example 2.3. *Consider the time scale* $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$. *Then*

$$\sigma(t) = \inf\{2^n : n \in [m+1, \infty)\} = 2t$$

and similarly

$$\rho(t) = \frac{t}{2}$$

for $t = 2^m \in \mathbb{T}$. *Consequently*

$$\mu(t) = t.$$

Example 2.4. *Consider* $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$. *We have*

$$\sigma(n^2) = \inf\{n^2 : n \in \mathbb{N}_0\} = (n+1)^2.$$

Then,

$$\mu(n^2) = \sigma(n^2) - n^2 = 2n + 1.$$

Since $t = n^2 \in \mathbb{N}_0$, *we get*

$$\sigma(t) = (\sqrt{t} + 1)^2 \ \ \textit{and} \ \ \mu(t) = 1 + 2\sqrt{t}.$$

Example 2.5. *If* $\mathbb{T} = \mathbb{R}$ *is considered, forward shift operator is*

$$\sigma(t) = \text{inf}\{s \in \mathbb{R} : s > t\} = \text{inf}(t, \infty) = t$$

and similarly backward shift operator is

$$\rho(t) = t.$$

Then

$$\mu(t) = 0.$$

Remark 2.1. *Quantum numbers has non-uniform step-size*

$$\mu(x) = qx - x, \ \ \forall x \in \mathbb{K}_q.$$

However, $h\mathbb{Z}$ *has uniform step-size*

$$\mu(x) = (x + h) - x = x - (x - h) = h \in \mathbb{R}^+.$$

2.1 *h*-derivative

Till the end of the Section 2.5 , we use definitions and results from Kac, 2001. Also we introduce some new identities hold in $h\mathbb{Z}$ in Section 2.2 and other sections.

Definition 2.4. *Let* f *be an arbitrary function on* $h\mathbb{Z}$. *The* h*-delta differential of the* *function f on $h\mathbb{Z}$ is defined by*

$$d_h f(x) \coloneqq f(x+h) - f(x), \quad x \in h\mathbb{Z}. \tag{2.1}$$

Similarly,

$$\bar{d}_h f(x) \coloneqq f(x) - f(x-h), \quad x \in h\mathbb{Z} \tag{2.2}$$

denotes the h-nabla differential of the function f on $h\mathbb{Z}$.

Remark 2.2. *Lattice-differentials (or h-differentials) are different from the ordinary calculus because of the lack of the symmetry in the differential of the product of two functions. For example*

$$\begin{aligned} d_h(fg)(x) =& f(x+h)g(x+h) - f(x)g(x) \\ =& f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x) \\ =& f(x+h)d_h g(x) + g(x)d_h f(x). \end{aligned}$$

Now, with these h-differentials, we can define the corresponding h-derivatives.

Definition 2.5. *Let f be an arbitrary function on $h\mathbb{Z}$. The delta h-derivative of $f(x)$ denoted by Δ_h is*

$$\Delta_h^x f(x) \coloneqq \frac{f(x+h) - f(x)}{h}, \tag{2.3}$$

and the nabla h-derivative of the function $f(x)$ denoted by ∇_h is

$$\nabla_h^x f(x) \coloneqq \frac{f(x) - f(x-h)}{h}. \tag{2.4}$$

Note that continuous limits of discrete derivatives $\Delta_h^x x$ and $\nabla_h^x x$ recover classical derivative i.e.

$$\lim_{h\to 0} \Delta_h^x f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{df(x)}{dx}$$

and similarly

$$\lim_{h\to 0} \nabla_h^x f(x) = \frac{df(x)}{dx}.$$

Theorem 2.1. *The operators $\Delta_h : h\mathbb{Z} \to h\mathbb{Z}$ and $\nabla_h : h\mathbb{Z} \to h\mathbb{Z}$ are linear.*

Proof. Let f, g be arbitrary functions on $h\mathbb{Z}$. Then for all a, b scalars, we get

$$\begin{aligned}\Delta_h^x[af(x) + bg(x)] &= \frac{af(x+h) + bg(x+h) - (af(x) + bg(x))}{h} \\ &= a\Delta_h^x f(x) + b\Delta_h^x g(x).\end{aligned}$$

Similarly, the operator ∇_h^x is linear. □

Proposition 2.1. *Let f, g be arbitrary functions whose h-delta derivatives exist on $h\mathbb{Z}$. Then the product rule for h-delta derivative arises in two forms:*

$$\textbf{\textit{(i)}} \quad \Delta_h^x[f(x)g(x)] = f(x)\Delta_h^x g(x) + g(x+h)\Delta_h^x f(x), \tag{2.5}$$

$$\textbf{\textit{(ii)}} \quad \Delta_h^x[f(x)g(x)] = f(x+h)\Delta_h^x g(x) + g(x)\Delta_h^x f(x). \tag{2.6}$$

Proof. (i) By using the Definition 2.5, we get

$$\Delta_h^x[f(x)g(x)] = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

Without loss of generality, add and subtract $f(x)g(x+h)$ on the numerator, we obtain

$$\begin{aligned}\Delta_h^x[f(x)g(x)] &= f(x)\frac{g(x+h) - g(x)}{h} + g(x+h)\frac{f(x+h) - f(x)}{h} \\ &= f(x)\Delta_h^x g(x) + g(x+h)\Delta_h^x f(x).\end{aligned}$$

(ii) Similarly, add and subtract $f(x+h)g(x)$ on the numerator of $\Delta_h^x f(x)g(x)$, we get

$$\begin{aligned}\Delta_h^x[f(x)g(x)] &= f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \\ &= f(x+h)\Delta_h^x g(x) + g(x)\Delta_h^x f(x).\end{aligned}$$

□

Proposition 2.2. *Let f, g be arbitrary functions whose h-nabla derivatives exist on $h\mathbb{Z}$. Then the product rule for h-nabla derivative arises in two forms:*

$$\textbf{\textit{(i)}} \quad \nabla_h^x[f(x)g(x)] = f(x)\nabla_h^x g(x) + g(x-h)\nabla_h^x f(x), \tag{2.7}$$

$$\textbf{\textit{(ii)}} \quad \nabla_h^x[f(x)g(x)] = f(x-h)\nabla_h^x g(x) + g(x)\nabla_h^x f(x). \tag{2.8}$$

Proof. The proof is similar to the one given for Proposition 2.1. □

Proposition 2.3. *Let f, g be arbitrary functions whose h-delta derivatives exist on $h\mathbb{Z}$. Then the quotient rule for h-delta derivative arises in two forms:*

$$\textbf{\textit{(i)}} \quad \Delta_h^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{g(x)\Delta_h^x f(x) - f(x)\Delta_h^x g(x)}{g(x)g(x+h)}, \tag{2.9}$$

$$\textbf{\textit{(ii)}} \quad \Delta_h^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{g(x+h)\Delta_h^x f(x) - f(x+h)\Delta_h^x g(x)}{g(x)g(x+h)}. \tag{2.10}$$

Proof. (i) By using the Definition 2.5, we get

$$\Delta_h^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}.$$

Without loss of generality, add and subtract $f(x)g(x)$ to the numerator, we have

$$\begin{aligned}\Delta_h^x\Big[\frac{f(x)}{g(x)}\Big] &= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{hg(x)g(x+h)} \\ &= \frac{g(x)\Delta_h^x f(x) - f(x)\Delta_h^x g(x)}{g(x)g(x+h)}.\end{aligned}$$

(ii) Similarly, add and subtract $f(x+h)g(x+h)$ to the numerator of $\Delta_h^x \frac{f(x)}{g(x)}$, we get

$$\begin{aligned}\Delta_h^x\Big[\frac{f(x)}{g(x)}\Big] &= \frac{g(x+h)[f(x+h) - f(x)] - f(x+h)[g(x+h) - g(x)]}{hg(x)g(x+h)} \\ &= \frac{g(x+h)\Delta_h^x f(x) - f(x+h)\Delta_h^x g(x)}{g(x)g(x+h)}.\end{aligned}$$

□

Proposition 2.4. *Let f, g be arbitrary functions whose h-nabla derivatives exist on $h\mathbb{Z}$.*

Then the quotient rule for h-nabla derivative arises in two forms:

$$\textbf{(i)} \quad \nabla_h^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{g(x)\nabla_h^x f(x) - f(x)\nabla_h^x g(x)}{g(x)g(x-h)}, \tag{2.11}$$

$$\textbf{(ii)} \quad \nabla_h^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{g(x-h)\nabla_h^x f(x) - f(x-h)\nabla_h^x g(x)}{g(x)g(x-h)}. \tag{2.12}$$

Proof. We can prove this proposition similar to the proof of Proposition 2.3. □

Remark 2.3. *(i) When $h \to 0$, the product rules for h-derivatives of two functions turn to the ordinary product rule since*

$$\begin{aligned}\lim_{h\to 0} \Delta_h^x[(fg)(x)] &= f(x)\lim_{h\to 0}\Delta_h^x g(x) + \lim_{h\to 0} g(x+h)\Delta_h^x f(x)\\ &= f(x)g'(x) + g(x)f'(x).\end{aligned}$$

Similarly,

$$\lim_{h\to 0} \nabla_h^x[(fg)(x)] = f(x)g'(x) + g(x)f'(x).$$

(ii) When $h \to 0$, the quotient rules for h-derivatives of two functions turn to the ordinary quotient rule since

$$\begin{aligned}\lim_{h\to 0} \Delta_h^x[\frac{f(x)}{g(x)}] &= g(x)\lim_{h\to 0}\frac{\Delta_h^x f(x)}{g(x+h)g(x)} - f(x)\lim_{h\to 0}\frac{\Delta_h^x g(x)}{g(x+h)g(x)}\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.\end{aligned}$$

Similarly,

$$\lim_{h\to 0} \nabla_h^x[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Remark 2.4. *We see that $h\mathbb{Z} \to \mathbb{R}$ as $h \to 0$ in other words, whenever we use* ***h-discretization*** *terminology we mean the study on $h\mathbb{Z}$.*

2.2 *h*-binomial

Definition 2.6. *We define h-analogue of a binomial $(x-t)^n$ as a polynomial*

$$(x-t)_h^n := (x-t)(x-(t+h))(x-(t+2h))\dots(x-(t+(n-1)h)), \tag{2.13}$$

where $(x-t)_h^0 = 1$, t is any constant and $n \geq 1$.

Remark 2.5. *When $t = 0$, the Definition 2.6 becomes*

$$(x-0)_h^n := (x)_h^n = \begin{cases} x(x-h)(x-2h)\dots(x-(n-1)h) & , \ n > 0 \\ x & , \ n = 1 \\ 1 & , \ n = 0. \end{cases}$$

Note that $(x)_h^n \neq x^n$.

Remark 2.6. *Clearly we have*

$$\lim_{h\to 0}(x \pm t)_h^n = (x \pm t)^n.$$

Proposition 2.5. *For any integer m, n, we have*

$$(x-t)_h^{m+n} = (x-t)_h^m(x-(t+mh))_h^n. \tag{2.14}$$

Proof. There are five cases.

(i) For $m = 0$ and $n = 0$, it is obvious.

(ii) For $m > 0$ and $n > 0$, by Definition 2.5, we have

$$\begin{aligned}(x-t)_h^{m+n} =& (x-t)(x-(t+h))\dots(x-(t+(m-1)h))(x-(t+mh)) \\ & (x-(t+(m+1)h))\dots(x-(t+(m+(n-1)h))) \\ =& (x-t)_h^m(x-(t+mh))_h^n.\end{aligned}$$

Note that

$$(x-t)_h^{-n} = \frac{1}{(x-(t-nh))_h^n,} \quad \text{for any positive integer } n. \tag{2.15}$$

It yields by substituting m by $-n$ in the Case (ii).

(iii) For $m<0, n>0$, consider $m=-m'<0$. Using the equation (2.15), we get

$$\begin{aligned}
&(x-t)_h^m(x-(t+mh))_h^n = (x-t)_h^{-m'}(x-(t-m'h))_h^n = \frac{(x-(t-m'h))_h^n}{(x-(t-m'h))_h^{m'}} \\
&= \begin{cases} (x-t)(x-(t+h))...(x-(t+(n-m'-1)h)) & , n \geq m' \\ \dfrac{1}{(x-(t-m'h+nh))_h^{m'-n}} & , m' > n \end{cases} \\
&= (x-t)_h^{n-m'} = (x-t)_h^{m+n}.
\end{aligned}$$

(iv) For $m>0, n<0$, consider $n=-n'<0$. Using the equation (2.15), we get

$$\begin{aligned}
&(x-t)_h^m(x-(t+mh))_h^n = (x-t)_h^m(x-(t+mh))_h^{-n'} \\
=&\frac{(x-t)_h^m}{(x-(t+(m-n')h))_h^{n'}} \\
=&\begin{cases} \dfrac{(x-t)_h^{m-n'}(x-(t+(m-n')h))_h^{-n'}}{(x-(t+(m-n')h))_h^{n'}} & , \quad m \geq n' \\ \dfrac{(x-t)_h^m}{(x-(t+(m-n')h))_h^{n'-m}(x-(t+(m-n'+n'-m)))_h^m} & , \quad n' > m \end{cases} \\
=&\begin{cases} (x-t)_h^{m-n'} & , \quad m \geq n' \\ \dfrac{1}{(x-(t+(m-n')h))_h^{n'-m}} & , \quad n' > m \end{cases} \\
=&(x-t)_h^{m-n'} = (x-t)_h^{m+n}.
\end{aligned}$$

(v) For $m < 0$ and $n < 0$, consider $m = -m' < 0$ and $n = -n' < 0$. We get

$$\begin{aligned}
&(x-t)_h^m(x-(t+mh))_h^n = (x-t)_h^{-m'}(x-(t-m'h))_h^{-n'} \\
=&\frac{1}{(x-(t-m'h))_h^{m'}(x-(t-(m'+n')h))_h^{n'}} \\
=&\frac{1}{(x-(t+n'h-(m'+n')h))_h^{m'}(x-(t-(m'+n')h))_h^{n'}} \\
=&\frac{1}{(x-(t-(m'+n')h))_h^{n'+m'}} \\
=&(x-t)_h^{-n'-m'} = (x-t)_h^{m+n},
\end{aligned}$$

where we used the fact that $n', m' > 0$.

□

Proposition 2.6. *For any integer* n*, the h-delta derivative of a binomial* $(x-t)_h^n$ *with respect to x is*

$$\Delta_h^x(x-t)_h^n = n(x-t)_h^{n-1}. \tag{2.16}$$

Proof. We have two cases.

(i) For $n \geq 0$, by the h-delta derivative Definition 2.6, we have

$$\begin{aligned}
\Delta_h^x(x-t)_h^n &= \Delta_h^x(x-t)(x-(t+h))(x-(t+2h))\dots(x-(t+(n-1)h)) \\
&= \frac{1}{h}[(x+h-t)(x+h-(t+h))(x+h-(t+2h)).. \\
&..(x+h-(t+(n-1)h)) - (x-t)(x-(t+h))(x-(t+2h)).. \\
&..(x-(t+(n-1)h))] \\
&= \frac{1}{h}(x-t)(x-(t+h))...(x-(t+(n-2)h))nh = n(x-t)_h^{n-1}.
\end{aligned}$$

(ii) For $n < 0$, set $n = -n' < 0$. By the equation (2.15) and h-delta quotient rule,

we have

$$\begin{aligned}
\Delta_h^x (x-t)_h^n = \Delta_h^x (x-t)_h^{-n'} &= \Delta_h^x \frac{1}{(x-(t-n'h))_h^{n'}} \\
&= \frac{-\Delta_h^x (x-(t-n'h))_h^{n'}}{(x-(t-n'h))_h^{n'} (x+h-(t-n'h))_h^{n'}} \\
&= \frac{-n'(x-(t-n'h))_h^{n'-1}}{(x-(t-n'h))_h^{n'} (x-(t-(n'+1)h))_h^{n'}} \\
&= \frac{-n'}{(x-(t-h))(x-(t-(n'+1)h))_h^{n'}} \\
&= \frac{-n'}{(x-(t-(n'+1)h))_h^{n'+1}} \\
&= -n'(x-t)_h^{-(n'+1)} = n(x-t)_h^{n-1},
\end{aligned}$$

where we used the facts $n' > 0$ and Proposition 2.5.

$\square$

2.3 Generalized Taylor's Formula

Let us recall the analyticity definition on ordinary calculus from Wade, 2000.

Definition 2.7. *A real valued function f is said to be analytic on (a, b) if and only if given $x_0 \in (a, b)$ there is a power series centered at x_0 that converges to f near x_0, namely there exist coefficients $\{a_k\}_{k=0}^{\infty}$ and points $c, d \in (a, b)$ such that $c < x_0 < d$ and*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for all $x \in (c, d)$.

Theorem 2.2. *Let c, d be extended real numbers with $c < d$, let $x_0 \in (c, d)$. Suppose that $f : (c, d) \to \mathbb{R}$. If $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ for each $x \in (c, d)$, then $f \in C^{\infty}(c, d)$ and*

$$a_k = \frac{f^{(k)}(x_0)}{k!}, \quad k = 0, 1, \ldots$$

Theorem 2.3. *Suppose that I is an open interval centered at c and*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k, \quad x \in I. \tag{2.17}$$

If $x_0 \in I$ and $r > 0$ satisfy $(x_0 - r, x_0 + r) \subseteq I$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

for all $x \in (x_0 - r, x_0 + r)$. In particular, if f is a C^∞ function whose Taylor series expansion converges to f on some open interval J, then f is analytic on J.

Counter Example: Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & , \quad x \neq 0 \\ 0 & , \quad x = 0. \end{cases}$$

Since $f \in C^\infty$, we can always compute its Taylor Series. It can be shown that $f^{(n)}(0)$ exists for all $n \geq 0$. Then Taylor series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + 0 + ... + 0 \neq f(x).$$

Hence $f(x) \in C^\infty$ but it is not analytic.

Such analyticity definition coincides with the analyticity in complex analysis.

In general, in ordinary calculus we deal with elementary functions; trigonometric, exponential, logarithmic, inverse trigonometric functions all of which are analytic functions. Similarly in lattice analysis, we will deal with elemantary functions; exponential, trigonometric, and hyperbolic functions.

Now, we present generalized Taylor's Formula, (Kac, 2001) from which we will derive h-Taylor's Formula and elementary functions on $h\mathbb{Z}$.

Theorem 2.4. *Let a be a number, D be a linear operator on the vector space of* *polynomials, and* $\{P_0(x), P_1(x), P_2(x), ...\}$ *be a sequence of polynomials s.t.*

(i) $P_0(a) = 1$ *and* $P_n(a) = 0$, $\forall n \geq 1$,

(ii) deg $P_n = n$,

(iii) $DP_n(x) = P_{n-1}(x)$ *for any* $n \geq 1$ *and* $D(1) = 0$.

Then, for any polynomial $f(x)$ *of degree* N*, we have the generalized Taylor's Formula;*

$$f(x) = \sum_{n=0}^{N} (D^n f)(a) P_n(x). \tag{2.18}$$

Proof. Let V be the space of polynomials of degree not larger than N with dimension $\dim V = N + 1$. Consider

$$B = \{P_0(x), P_1(x), ..., P_i(x), P_j(x), ..., P_N(x)\}$$

and note that $i < j$. By condition (ii), B is linearly independent. Moreover, $|B| = N+1$ so B is a basis for V, since $\dim V = N + 1$. Then any $f \in V$ is a linear combination of the elements of B as $\text{span} B = V$. We can write

$$f(x) = \sum_{i=0}^{N} c_i P_i(x), \quad \text{for some unique constants} \quad c_i. \tag{2.19}$$

Consider

$$f(a) = \sum_{i=0}^{N} c_i P_i(a) = c_0 P_0(a) + c_1 P_1(a) + ... + c_N P_N(a).$$

By the first condition,

$$f(a) = c_0.$$

Acting the linear operator D on (2.19);

$$\begin{aligned}Df(x) =& D(\sum_{i=0}^{N} c_i P_i(x)) = \sum_{i=0}^{N} c_i DP_i(x),\\ =& c_0 DP_0(x) + c_1 DP_1(x) + ... + c_N DP_N(x)\\ =& c_0 DP_0(x) + c_1 P_0(x) + c_2 P_1(x) + ... + c_N P_{N-1}(x).\end{aligned}$$

For $x = a$, we get

$$Df(a) = c_1 P_0(a) + c_2 P_1(a) + ... + c_N P_{N-1}(a) = c_1.$$

Similarly,

$$D^2 f(x) = \sum_{i=2}^{N} c_i P_{i-2} \quad \text{and} \quad D^2 f(a) = c_2 P_0(a) + ... + c_N P_{N-2}(a) = c_2.$$

Thus, one can inductively calculate uniquely $D^n f(a) = c_n$, $0 \leq n \leq N$. Thus

$$f(x) = \sum_{n=0}^{N} D^n f(a) P_n(x).$$

$\square$

Remark 2.7. *If* $D := \dfrac{d}{dx}$ *which is a linear operator and* $P_n(x) = \dfrac{(x-a)^n}{n!}$ *in the generalized Taylor Formula given in Theorem* 2.4*, we get*

$$\sum_{n=0}^{N} (D^n f)(a) P_n(x) = \sum_{n=0}^{N} \frac{f^n(a)}{n!} (x-a)^n. \tag{2.20}$$

2.4 *h*-Taylor's Formula

Theorem 2.5. *For any polynomial* $f(x)$ *with deg*$f(x) = N$*, h-Taylor series expansion is given as*

$$f(x) = \sum_{j=0}^{N} (\Delta_h^x f)^j (c) \frac{(x-c)_h^j}{j!} \tag{2.21}$$

where c is any number and $(\Delta_h^x f)^j$ is the jth order of $\Delta_h^x f$.

Proof. By Theorem 2.4, we know the generalized Taylor's Formula

$$f(x) = \sum_{n=0}^{N} (D^n f)(c) P_n(x). \tag{2.22}$$

We choose the linear operator $D = \Delta_h$ and $P_n(x) = \dfrac{(x-c)_h^n}{n!}$. Let us show that chosen $P_n(x)$ satisfy the three conditions of Theorem 2.4. Clearly,

(i) $P_0(c) = 1$ and $P_n(c) = 0, \ \forall n \geq 1$,

(ii) deg$P_n(x) = n$,

(iii) $\Delta_h^x P_n(x) = \Delta_h^x \Big[\dfrac{(x-c)_h^n}{n!}\Big] = \dfrac{n(x-c)_h^{n-1}}{n!} = \dfrac{(x-c)_h^{n-1}}{(n-1)!} = P_{n-1}(x).$

Then substituting the chosen operator and $P_n(x)$ in the equation (2.22) we get

$$f(x) = \sum_{j=0}^{N} (\Delta_h^x f)^j (c) \frac{(x-c)_h^j}{j!}.$$

□

Example 2.6. *Find the h-Taylor series expansion for $f(x) = (x-a)_h^n$ about $c = 0$ where $n \in \mathbb{Z}^+$.*

Solution. Let us find $(\Delta_h^x f)^j(x)$ for $j \leq n$. By Proposition 2.6, we have

$$(\Delta_h^x f)(x) = \Delta_h^x (x-a)_h^n = n(x-a)_h^{n-1},$$
$$(\Delta_h^x f)^2(x) = \Delta_h^x n(x-a)_h^{n-1} = n(n-1)(x-a)_h^{n-2}.$$

Then inductively we obtain

$$(\Delta_h^x f)^j(x) = n(n-1)...(n-(n-j))(x-a)_h^{n-j}.$$

Hence the h-Taylor series expansion of $f(x) = (x-a)_h^n$ at $c = 0$ is

$$(x-a)_h^n = \sum_{j=0}^{n} n(n-1)...(n-(j-1))(-a)_h^{n-j}\frac{(x)_h^j}{j!} = \sum_{j=0}^{n}\binom{n}{j}(-a)_h^{n-j}(x)_h^j,$$

where $(-a)_h^{n-j} = (-1)^{n-j+1}(a+h)(a+2h)...(a+(n-j+1)h)$ and
$(x)_h^j = x(x-h)(x-2h)...(x-(j-1)h)$. $\square$

Definition 2.8. *If there exists an h-power (h-Taylor) series expansion converging to $f(x)$ itself then f is said to be h-analytic function.*

2.5 *h*-exponential, *h*-trigonometric and *h*-hyperbolic functions

In this section, we aim to present h-analogue of an exponential function

$$f(x) = e_h^x.$$

Three conditions of Theorem 2.4 that $f(x)$ must satisfy are

(i) $f(0) = 1$

(ii) $\Delta_h^x f(x) = f(x)$

(iii) f must have h-Taylor series expansion (2.21) about $c = 0$, with $N = \infty$ for small h.

Since (i) and (ii) imply that

$$(\Delta_h^x f)^j(0) = 1, \quad j = 0, 1, 2 \ldots$$

then,

$$f(x) = \sum_{j=0}^{\infty} \frac{(x-0)_h^j}{j!} = \sum_{j=0}^{\infty} \frac{x(x-h)...(x-(j-1)h)}{j!}$$
$$= \sum_{j=0}^{\infty} \frac{\frac{x}{h}(\frac{x}{h}-1)...(\frac{x}{h}-j+1)h^j}{j!} = \sum_{j=0}^{\infty} \binom{\frac{x}{h}}{j} h^j.$$

By ordinary binomial expansion

$$f(x) = e_h^x = (1+h)^{\frac{x}{h}} = \sum_{j=0}^{\infty} \binom{\frac{x}{h}}{j} h^j.$$

Thus, let us introduce h-exponential as follows.

Definition 2.9. *We define h-exponential function by*

$$e_h^x := (1+h)^{\frac{x}{h}}. \tag{2.23}$$

Remark 2.8. *Note that*

$$\lim_{h\to 0} e_h^x = \lim_{h\to 0} (1+h)^{\frac{x}{h}} = e^x.$$

Remark 2.9. *Note that h-exponential function e_h^x is h-analytic since there exist an h-Taylor series expansion converging to e_h^x itself.*

Example 2.7. *Note that the h-delta derivative of e_h^x is again itself, i.e.*

$$\Delta_h^x e_h^x = e_h^x,$$

since

$$\Delta_h^x e_h^x = \frac{(1+h)^{\frac{x+h}{h}} - (1+h)^{\frac{x}{h}}}{h} = \frac{1}{h}(1+h)^{\frac{x}{h}}((1+h)-1) = (1+h)^{\frac{x}{h}} = e_h^x.$$

But the h-nabla derivative of e_h^x is quite different. Because

$$\nabla_h^x e_h^x = \frac{(1+h)^{\frac{x}{h}} - (1+h)^{\frac{x-h}{h}}}{h} = \frac{1}{h}(1+h)^{\frac{x}{h}}(1-(1+h)^{-1}) = (1+h)^{\frac{x-h}{h}} = e_h^{x-h}.$$

In other words

$$\nabla_h^x e_h^x = e_h^{x-h}.$$

Definition 2.10. *We introduce another h-exponential function as follows*

$$E_h^x := \Big(\frac{1}{1-h}\Big)^{\frac{x}{h}}. \tag{2.24}$$

Remark 2.10. *Since* $\lim_{h\to\infty}(1+\frac{x}{h})^h = e^x$*, with some necessary arrangement, we have*

$$\lim_{h\to 0} E_h^x = \lim_{h\to 0}\Big(\frac{1}{1-h}\Big)^{\frac{x}{h}} = e^x.$$

Proposition 2.7. *We have the following relation between h-exponential functions*

$$e_h^x E_{-h}^{-x} = 1. \tag{2.25}$$

Remark 2.11. *By Proposition 2.7, we can say that* E_h^x *is also h-analytic, since* e_h^x *is h-analytic.*

Example 2.8. *The h-nabla derivative of* E_h^x *is again itself, i.e.*

$$\nabla_h^x E_h^x = E_h^x,$$

since

$$\nabla_h^x E_h^x = \frac{\Big(\frac{1}{1-h}\Big)^{\frac{x}{h}} - \Big(\frac{1}{1-h}\Big)^{\frac{x-h}{h}}}{h} = \frac{1}{h}\Big(\frac{1}{1-h}\Big)^{\frac{x}{h}}\Big(1-\Big(\frac{1}{1-h}\Big)^{-1}\Big) = \Big(\frac{1}{1-h}\Big)^{\frac{x}{h}} = E_h^x.$$

But the h-delta derivative of E_h^x *is quite different. Because*

$$\Delta_h^x E_h^x = \frac{\Big(\frac{1}{1-h}\Big)^{\frac{x+h}{h}} - \Big(\frac{1}{1-h}\Big)^{\frac{x}{h}}}{h} = \frac{1}{h}\Big(\frac{1}{1-h}\Big)^{\frac{x}{h}}\Big(\frac{1}{1-h}-1\Big) = \Big(\frac{1}{1-h}\Big)^{\frac{x+h}{h}} = E_h^{x+h}.$$

In other words

$$\Delta_h^x E_h^x = E_h^{x+h}.$$

Definition 2.11. *The h-trigonometric functions are defined as follows;*

$$\mathrm{Sin_h}x := \frac{E_h^{ix} - E_h^{-ix}}{2i}, \tag{2.26}$$
$$\mathrm{Cos_h}x := \frac{E_h^{ix} + E_h^{-ix}}{2}, \tag{2.27}$$
$$\sin_h x := \frac{e_h^{ix} - e_h^{-ix}}{2i}, \tag{2.28}$$
$$\cos_h x := \frac{e_h^{ix} + e_h^{-ix}}{2}. \tag{2.29}$$

Proposition 2.8. *For the h-trigonometric functions, we have the following equalities*

$$\textit{(i)}\quad \nabla_h^x \mathrm{Sin_h}x = \mathrm{Cos_h}x, \tag{2.30}$$
$$\textit{(ii)}\quad \nabla_h^x \mathrm{Cos_h}x = -\mathrm{Sin_h}x, \tag{2.31}$$
$$\textit{(iii)}\quad \Delta_h^x \mathrm{Sin_h}x = \mathrm{Cos_h}(x+h), \tag{2.32}$$
$$\textit{(iv)}\quad \Delta_h^x \mathrm{Cos_h}x = -\mathrm{Sin_h}(x+h), \tag{2.33}$$
$$\textit{(v)}\quad \Delta_h^x \mathrm{sin_h}x = \mathrm{cos_h}x, \tag{2.34}$$
$$\textit{(vi)}\quad \Delta_h^x \mathrm{cos_h}x = -\mathrm{sin_h}x, \tag{2.35}$$
$$\textit{(vii)}\quad \nabla_h^x \mathrm{sin_h}x = \mathrm{cos_h}(x-h), \tag{2.36}$$
$$\textit{(viii)}\quad \nabla_h^x \mathrm{cos_h}x = -\mathrm{sin_h}(x-h). \tag{2.37}$$

Proof. **(i)** Using Definition 2.11, we have

$$\nabla_h^x \mathrm{Sin_h}x = \frac{\nabla_h^x E_h^{ix} - \nabla_h^x E_h^{-ix}}{2i} = \frac{iE_h^{ix} + iE_h^{-ix}}{2i} = \mathrm{Cos_h}x.$$

(ii) Similar to the proof of **(i)**.

(iii) Using Definition 2.11, we have

$$\Delta_h^x \mathrm{Sin_h}x = \Delta_h^x\left(\frac{E_h^{ix} - E_h^{-ix}}{2i}\right) = \frac{iE_h^{i(x+h)} + iE_h^{-i(x+h)}}{2i} = \mathrm{Cos_h}(x+h).$$

(iv) Similar to the proof of **(iii)**.

(v) Using Definition 2.11, we have

$$\Delta_h^x \mathrm{sin_h} x = \Delta_h^x \left(\frac{e_h^{ix} - e_h^{-ix}}{2i}\right) = \frac{\Delta_h^x e_h^{ix} - \Delta_h^x e_h^{-ix}}{2i} = \frac{ie_h^{ix} + ie_h^{-ix}}{2i} = \mathrm{cos_h} x.$$

(vi) Similar to the proof of **(v)**.

(vii) Using Definition 2.11, we have

$$\nabla_h^x \mathrm{sin_h} x = \nabla_h^x \left(\frac{e_h^{ix} - e_h^{-ix}}{2i}\right) = \frac{ie_h^{i(x-h)} + ie_h^{-i(x-h)}}{2i} = \mathrm{cos_h}(x-h).$$

(viii) Similar to the proof of **(vii)**. □

Proposition 2.9. *We have the following identity*

$$\mathrm{sin_h} x \mathrm{Sin_{-h}} x + \mathrm{cos_h} x \mathrm{Cos_{-h}} x = 1 \tag{2.38}$$

which is the h-analogue of identity $\sin^2 x + \cos^2 x = 1$ *in ordinary calculus.*

Proof. Since $e_h^x E_{-h}^{-x} = 1$, we have

$$\begin{aligned}
\mathrm{sin_h} x \mathrm{Sin_{-h}} x + \mathrm{cos_h} x \mathrm{Cos_{-h}} x =& \frac{e_h^{ix} - e_h^{-ix}}{2i} \frac{E_{-h}^{ix} - E_{-h}^{-ix}}{2i} + \frac{e_h^{ix} + e_h^{-ix}}{2} \frac{E_{-h}^{ix} + E_{-h}^{-ix}}{2} \\
=& \frac{e_h^{ix} E_{-h}^{ix} - e_h^{ix} E_{-h}^{-ix} - e_h^{-ix} E_{-h}^{ix} + e_h^{-ix} E_{-h}^{-ix}}{4i^2} \\
+& \frac{e_h^{ix} E_{-h}^{ix} + e_h^{ix} E_{-h}^{-ix} + e_h^{-ix} E_{-h}^{ix} + e_h^{-ix} E_{-h}^{-ix}}{4} \\
=& \frac{2e_h^{ix} E_{-h}^{-ix} + 2e_h^{-ix} E_{-h}^{ix}}{4} = 1.
\end{aligned}$$

□

Definition 2.12. *The h-hyperbolic functions are defined as follows:*

$$\mathrm{Sinh_h}x := \frac{E_h^x - E_h^{-x}}{2}, \tag{2.39}$$
$$\mathrm{Cosh_h}x := \frac{E_h^x + E_h^{-x}}{2}, \tag{2.40}$$
$$\mathrm{sinh_h}x := \frac{e_h^x - e_h^{-x}}{2}, \tag{2.41}$$
$$\mathrm{cosh_h x} := \frac{e_h^x + e_h^{-x}}{2}. \tag{2.42}$$

Proposition 2.10. *For the h-hyperbolic functions, we have the following equalities*

(i) $$\nabla_h^x \mathrm{Sinh_h}x = \mathrm{Cosh_h}x, \tag{2.43}$$

(ii) $$\nabla_h^x \mathrm{Cosh_h}x = -\mathrm{Sinh_h}x, \tag{2.44}$$

(iii) $$\Delta_h^x \mathrm{Sinh_h}x = \mathrm{Cosh_h}(x+h), \tag{2.45}$$

(iv) $$\Delta_h^x \mathrm{Cosh_h}x = -\mathrm{Sinh_h}(x+h), \tag{2.46}$$

(v) $$\nabla_h^x \mathrm{sinh_h}x = \mathrm{cosh_h}(x-h), \tag{2.47}$$

(vi) $$\nabla_h^x \mathrm{cosh_h}x = \mathrm{sinh_h}(x-h), \tag{2.48}$$

(vii) $$\Delta_h^x \mathrm{sinh_h}x = \mathrm{cosh_h}x, \tag{2.49}$$

(viii) $$\Delta_h^x \mathrm{cosh_h}x = \mathrm{sinh_h}x. \tag{2.50}$$

Proof is similar to the proof given for h-trigonometric functions.

Remark 2.12. *Note that since h-exponential functions are h-analytic, h-trigonometric and h-hyperbolic functions are also h-analytic.*

2.6 *h*-nabla antiderivative

In this section, we introduce h-nabla antiderivative of $f(x)$ denoted by $F(x)$. Let $f(x)$ be an arbitrary function and $F(x)$ be h-nabla antiderivative of $f(x)$, i.e.

$$\nabla_h^x F(x) = f(x). \tag{2.51}$$

Define the operator

$$\hat{N}_h F(x) := F(x-h). \tag{2.52}$$

Using the definition of h-nabla derivative of $F(x)$, we get

$$\nabla_h^x F(x) = \frac{F(x) - F(x-h)}{h}. \tag{2.53}$$

Rewriting (2.53) using the operator $\hat{N}_h$, we obtain

$$\nabla_h^x F(x) = \frac{F(x) - \hat{N}_h F(x)}{h} = \frac{(1-\hat{N}_h)F(x)}{h}.$$

Since $F(x)$ is h-nabla antiderivative of $f(x)$, we can write

$$(I - \hat{N}_h)F(x) = hf(x).$$

Applying $(I - \hat{N}_h)^{-1}$ to find $F(x)$, we get

$$F(x) = h(I - \hat{N}_h)^{-1} f(x). \tag{2.54}$$

Then by using the geometric series, we can write

$$F(x) = h \sum_{n=0}^{\infty} \hat{N}_h^n f(x), \tag{2.55}$$

where $\hat{N}_h^n$ is nth order of operator s.t. $\hat{N}_h^n = \hat{N}_h(\hat{N}_h{}^{n-1})$. Note that we have

$$\hat{N}_h^n f(x) = f(x - nh), \quad \forall n \geq 0.$$

Then, $F(x)$ becomes

$$F(x) = h \sum_{n=0}^{\infty} f(x - nh). \tag{2.56}$$

Definition 2.13. *We define h-nabla antiderivative of $f(x)$ denoted by $F(x)$ as follows:*

$$F(x) = \int f(x) d_{\bar{h}} x := h \sum_{n=0}^{\infty} f(x - nh). \tag{2.57}$$

Definition 2.14. *The definite h-nabla integral is defined as*

$$\int_0^b f(x)d_{\bar{h}}x := h\sum_{n=0}^{\infty} f(b-nh). \tag{2.58}$$

We also have

$$\int_a^b f(x)d_{\bar{h}}x := \int_0^b f(x)d_{\bar{h}}x - \int_0^a f(x)d_{\bar{h}}x. \tag{2.59}$$

Proposition 2.11. *Let f be an arbitrary function and $b-a \in h\mathbb{Z}$, then*

$$\int_a^b f(x)d_{\bar{h}}x = \begin{cases} h\sum\limits_{n=0}^{\frac{b-a}{h}-1} f(b-nh) & , a<b \\ -h\sum\limits_{n=0}^{\frac{a-b}{h}-1} f(a-nh) & , a>b \\ 0 & , a=b. \end{cases}$$

Proof. (i) Let $a<b$. By using Definition 2.14, we have

$$\begin{aligned} \int_a^b f(x)d_{\bar{h}}x &= \int_0^b f(x)d_{\bar{h}}x - \int_0^a f(x)d_{\bar{h}}x \\ &= h\sum_{n=0}^{\infty} f(b-nh) - h\sum_{n=0}^{\infty} f(a-nh) \\ &= h\sum_{n=0}^{\frac{b-a}{h}-1} f(b-nh) = h[f(b)+f(b-h)+...+f(a+h)]. \end{aligned}$$

(ii) Let $a>b$. By using Definition 2.14, we get

$$\begin{aligned} \int_a^b f(x)d_{\bar{h}}x &= -\int_b^a f(x)d_{\bar{h}}x = -\int_0^b f(x)d_{\bar{h}}x + \int_0^a f(x)d_{\bar{h}}x \\ &= -h\sum_{n=0}^{\infty} f(b-nh) + h\sum_{n=0}^{\infty} f(a-nh) \\ &= -h\sum_{n=0}^{\frac{a-b}{h}-1} f(a-nh) = -h[f(a)+f(a-h)+...+f(b+h)]. \end{aligned}$$

(iii) For the case $a=b=0$ there is nothing to prove.

□

2.7 Fundamental Theorem of *h*-Calculus

In this section, we introduce Fundamental Theorem of h-Calculus for h-nabla antiderivative.

Theorem 2.6. *Let $b-a \in h\mathbb{Z}$. If $F(x)$ is h-nabla antiderivative of $f(x)$ then*

$$\int_a^b f(x)d_{\bar{h}}x = F(b) - F(a), \tag{2.60}$$

where $0 \leq a < b$.

Proof. By the Proposition 2.11, we have

$$\int_a^b f(x)d_{\bar{h}}x = h \sum_{n=0}^{\frac{b-a}{h}-1} f(b-nh). \tag{2.61}$$

Since $F(x)$ is h-nabla antiderivative of $f(x)$, we have

$$\nabla_h^x F(x) = \frac{F(x) - F(x-h)}{h} = f(x)$$

which implies

$$hf(x) = \nabla_h^x F(x) = F(x) - F(x-h).$$

Then (2.61) becomes

$$\int_a^b f(x)d_{\bar{h}}x = \sum_{n=0}^{\frac{b-a}{h}-1} \nabla_h^x F(x)\Big|_{x=b-nh} = \sum_{n=0}^{\frac{b-a}{h}-1} [F(b-nh) - F(b-(n+1)h)]$$

$$=[F(b) + F(b-h) + ... + F(a+h)] - [F(b-h) + F(b-2h) + ... + F(a)]$$

$$=F(b) - F(a).$$

$\square$

Corollary 2.1. *Let $b-a \in h\mathbb{Z}$ and $F(x)$ be h-nabla antiderivative of $f(x)$, then we have*

$$\int_a^b \nabla_h^x F(x)d_{\bar{h}}x = F(b) - F(a), \tag{2.62}$$

where $0 \leq a < b$.

Proof. We can write Fundamental theorem of h-Calculus for h-nabla antiderivative by setting $\nabla_h^x F(x)$ instead of $f(x)$. $\square$

Remark 2.13. *In Kac, 2001 h-delta antiderivative of $f(x)$ denoted by $F(x)$ is introduced as follows*

$$F(x) = \int_a^b f(x) d_h x = -h \sum_{n=0}^{\infty} f(x+nh) = \begin{cases} h \displaystyle\sum_{n=0}^{\frac{b-a}{h}-1} f(a+nh) & , a < b \\ -h \displaystyle\sum_{n=0}^{\frac{a-b}{h}-1} f(b+nh) & , a > b \\ 0 & , a=b. \end{cases}.$$

Remark 2.14. *h-delta antiderivative and its properties are discussed in Kac, 2001.*

CHAPTER THREE
h-CAUCHY PROBLEMS

In this Chapter, we present properties of h-binomial. Then we construct h-wave equation that tends to usual wave equation as $h \to 0$. Next, we introduce h-Cauchy (or h-D'Alembert) problems and their solutions in terms of h-binomials. Then we introduce analyticity of function on $h\mathbb{Z}$ from which will state h-analogue of Cauchy Kovalevskaya Theorem and we give some examples that have unique and analytic solutions.

Proposition 3.1. *The h-nabla derivative of a binomial $(x-t)_h^n$ with respect to t is*

$$\nabla_h^t (x-t)_h^n = -n(x-t)_h^{n-1}, \quad n \in \mathbb{Z}. \tag{3.1}$$

Proof. Case (i) For $n \geq 0$, by Definition 2.6, we have

$$\begin{aligned}
\nabla_h^t (x-t)_h^n &= \nabla_h^t ((x-t)(x-(t+h))(x-(t+2h))\dots(x-(t+(n-1)h))) \\
&= \frac{1}{h}[(x-t)(x-(t-h))(x-(t-2h))\dots(x-(t-(n-1)h)) \\
&- (x-(t-h))(x-t)(x-(t+h))\dots(x-(t-h+(n-1)h))] \\
&= \frac{1}{h}(x-t)(x-(t+h))\dots(x-(t+(n-2)h))[-nh] \\
&= -n(x-t)_h^{n-1}.
\end{aligned}$$

Case (ii) For $n < 0$, set $n = n' < 0$. By the equation (2.15) and h-nabla quotient rule,

we have

$$
\begin{aligned}
\nabla_h^t (x-t)_h^{-n'} &= \nabla_h^t \frac{1}{(x-(t-n'h))_h^{n'}} = \frac{-\nabla_h^t (x-(t-n'h))_h^{n'}}{(x-(t-n'h))_h^{n'}(x-(t-h-n'h))_h^{n'}} \\
&= \frac{n'(x-(t-n'h))_h^{n'-1}}{(x-(t-n'h))_h^{n'}(x-(t-(n'+1)h))_h^{n'}} \\
&= \frac{n'}{(x-(t+(n'-1)h-n'h))(x-(t-(n'+1)h))_h^{n'}} \\
&= \frac{n'}{(x-(t-h))_h^1(x-(t-(n'+1)h))_h^{n'}} = \frac{n'}{(x-(t-(n'+1)h))_h^{n'+1}} \\
&= n'(x-t)_h^{-(n'+1)} = -n(x-t)_h^{n-1}.
\end{aligned}
$$

□

Proposition 3.2. *The h-nabla derivative of a binomial $(x+t)_h^n$ with respect to x is*

$$
\nabla_h^x (x+t)_h^n = n(x+t)_h^{n-1}, \quad n \in \mathbb{Z}. \tag{3.2}
$$

Proof. Case (i) For $n \geq 0$, by Definition 2.6, we have

$$
\begin{aligned}
\nabla_h^x (x+t)_h^n &= \nabla_h^x (x+t)(x+(t+h))(x+(t+2h))\dots(x+(t+(n-1)h)) \\
&= \frac{1}{h}[(x+t)(x+t+h)(x+t+2h)\dots(x+t+(n-1)h) \\
&- (x-h+t)(x-h+t+h)(x-h+t+2h)\dots(x+t+(n-2)h)] \\
&= \frac{1}{h}(x+t)(x+(t+h))\dots(x+(t+(n-2)h))[(n-1)h+h] \\
&= n(x+t)_h^{n-1}.
\end{aligned}
$$

Case (ii) For $n < 0$, set $n = n' < 0$. By the equation (2.15) and h-nabla quotient rule,

we have

$$
\begin{aligned}
\nabla_h^x (x+t)_h^{-n'} &= \nabla_h^x \frac{1}{(x+(t-n'h))_h^{n'}} = \frac{-\nabla_h^x (x+(t-n'h))_h^{n'}}{(x+(t-n'h))_h^{n'}(x-h-(t-n'h))_h^{n'}} \\
&= \frac{-n'(x+(t-n'h))_h^{n'-1}}{(x+(t-n'h))_h^{n'}(x+(t-(n'+1)h))_h^{n'}} \\
&= \frac{-n'}{(x+t-h)_h^1 (x+t-(n'+1)h)_h^{n'}} \\
&= \frac{-n'}{(x+t-(n'+1)h)_h^{n'+1}} = -n'(x+t)_h^{-(n'+1)} = n(x+t)_h^{n-1}.
\end{aligned}
$$

□

Proposition 3.3. *The h-nabla derivative of the polynomial* $(x+t)_h^n$ *with respect to t is*

$$
\nabla_h^t (x+t)_h^n = n(x+t)_h^{n-1}, \quad n \in \mathbb{Z}. \tag{3.3}
$$

Proof. Case (i) For $n \geq 0$, by the h-nabla derivative Definition 2.5, we have

$$
\begin{aligned}
\nabla_h^t (x+t)_h^n &= \nabla_h^t (x+t)(x+t+h)(x+t+2h)\dots(x+t+(n-1)h) \\
&= \frac{1}{h}[(x+t)(x+t+h)(x+t+2h)\dots(x+t+(n-1)h) \\
&- (x+(t-h))(x+t)(x+t+h)\dots(x+(t-h)+(n-1)h)] \\
&= \frac{1}{h}(x+t)(x+t+h)\dots(x+(t+(n-2)h))[(n-1)h+h] \\
&= n(x+t)_h^{n-1}.
\end{aligned}
$$

Case (ii) For $n < 0$, set $n = n' < 0$. By the equation (2.15) and h-nabla quotient rule,

we have

$$
\begin{aligned}
\nabla_h^t (x+t)_h^{-n'} &= \nabla_h^t \frac{1}{(x+(t-n'h))_h^{n'}} = \frac{-\nabla_h^t (x+t-n'h)_h^{n'}}{(x+t-n'h)_h^{n'}(x+(t-h)-n'h)_h^{n'}} \\
&= \frac{-n'(x+t-n'h)_h^{n'-1}}{(x+t-n'h)_h^{n'}(x+t-(n'+1)h)_h^{n'}} \\
&= \frac{-n'}{(x+t+(n'-1)h-n'h)(x+t-(n'+1)h)_h^{n'}} \\
&= \frac{n'}{(x+t-h)(x+t-(n'+1)h)_h^{n'}} = \frac{-n'}{(x+t-(n'+1)h)_h^{n'+1}} \\
&= -n'(x+t)_h^{-(n'+1)} = n(x+t)_h^{n-1}.
\end{aligned}
$$

□

Remark 3.1. *It follows from the Propositions* 2.6*,* 3.1*,* 3.2*, and* 3.3 *that we have*

$$(\Delta_h^x + \nabla_h^t)(x-t)_h^n = 0, \tag{3.4}$$

$$(\nabla_h^x - \nabla_h^t)(x+t)_h^n = 0. \tag{3.5}$$

Definition 3.1. *We define*

$$f(x \pm ct) \coloneqq \sum_{n=-\infty}^{\infty} a_n (x \pm ct)_h^n \tag{3.6}$$

as an h-Laurient Series representation of function f.

Corollary 3.1. *The general solution of h-Cauchy problem*

$$\Box u(x,t) = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t) u(x,t) = 0 \tag{3.7}$$

is

$$u(x,t) = f(x+t) + g(x-t), \tag{3.8}$$

where the functions $f, g \in C^2(\Omega)$ on $\Omega = h\mathbb{Z} \times h\mathbb{Z}$ that are given by

$$f(x+t) \coloneqq \sum_{n=-\infty}^{\infty} a_n (x+t)_h^n, \quad g(x-t) \coloneqq \sum_{n=-\infty}^{\infty} b_n (x-t)_h^n.$$

Remark 3.2. *As we mention in the Introduction, as $h \to 0$, the h-wave equation* (3.7) *turns out to be usual wave equation*

$$u_{tt} - u_{xx} = 0.$$

Theorem 3.1. *The h-Cauchy problem with the initial conditions*

$$\begin{aligned} &\Box u = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t) = 0, \quad \forall x, t \in h\mathbb{Z}, \\ &u(x,0) = \phi(x), \\ &\nabla_h^t u(x,0) = \psi(x), \end{aligned} \tag{3.9}$$

has solution

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_a^b \psi(x') d_{\bar{h}} x', \tag{3.10}$$

where $a := x - t$ and $b := x + t$. Here $x, t \in h\mathbb{Z}$ $\phi \in C^2(\Omega)$, $\psi \in C^1(\Omega)$ on the region $\Omega = h\mathbb{Z} \times h\mathbb{Z}$.

Proof. The Corollary 3.1 implies the general solution as follows:

$$u(x,t) = f(x+t) + g(x-t).$$

Imposing the first initial condition on $u(x,t)$, we have

$$u(x,0) = f(x) + g(x) = \phi(x).$$

Taking h-nabla derivative of $u(x,0)$ with respect to x we get

$$\nabla_h^x f(x) + \nabla_h^x g(x) = \nabla_h^x \phi(x). \tag{3.11}$$

Now, imposing the other initial condition on $u(x,t)$ we get

$$\nabla_h^t u(x,t)|_{t=0} = \psi(x).$$

Let $\xi \coloneqq x+t$ and $\zeta \coloneqq x-t$ so that we have $u(\xi,\zeta)=f(\xi)+g(\zeta)$. Then

$$\nabla_h^t u(x,t)=\nabla_h^\xi f(\xi)\nabla_h^t\xi(t)+\nabla_h^\zeta g(\zeta)\nabla_h^t\zeta(t)=\nabla_h^\xi f(\xi)-\nabla_h^\zeta g(\zeta).$$

When $t=0,\quad \xi=\zeta=x$. So, we have

$$\nabla_h^t u(x,t)\Big|_{t=0}=\nabla_h^x f(x)-\nabla_h^x g(x)=\psi(x). \tag{3.12}$$

By the equalities (3.11) and (3.12), we have

$$\nabla_h^x f(x)=\frac{1}{2}\nabla_h^x\phi(x)+\frac{1}{2}\psi(x), \tag{3.13}$$
$$\nabla_h^x g(x)=\frac{1}{2}\nabla_h^x\phi(x)-\frac{1}{2}\psi(x). \tag{3.14}$$

Applying h-nabla integral to (3.13) and (3.14) with respect to x over $[0,x]$, respectively, we obtain

$$\int_0^x \nabla_h^{x'} f(x')d_{\bar{h}}x'=\frac{1}{2}\Big[\int_0^x \nabla_h^{x'}\phi(x')d_{\bar{h}}x'+\int_0^x \psi(x')d_{\bar{h}}x'\Big],$$

which implies

$$f(x)=\frac{1}{2}[\phi(x)-\phi(0)]+\int_0^x \psi(x')d_{\bar{h}}x'+f(0) \tag{3.15}$$

and

$$\int_0^x \nabla_h^{x'} g(x')d_{\bar{h}}x'=\frac{1}{2}\Big[\int_0^x \nabla_h^{x'}\phi(x')d_{\bar{h}}x'+\int_0^x \psi(x')d_{\bar{h}}x'\Big],$$

which implies

$$g(x)=\frac{1}{2}[\phi(x)-\phi(0)]+\int_0^x \psi(x')d_{\bar{h}}x'+g(0), \tag{3.16}$$

using Corollary 2.1. Hence,

$$u(x,t)=\frac{1}{2}[\phi(x+t)+\phi(x-t)]+\frac{1}{2}\int_a^b \psi(x')d_{\bar{h}}x' \tag{3.17}$$

is a solution to h-initial value problem (3.9). Here $a = x - t$, $b = x + t$. Note that we neglect the integral constant $f(0) + g(0) - \phi(0)$ and in order to use Corollary of Fundamental Theorem of h-Calculus and to satisfy h-initial value problem (3.9), we assume $\phi \in C^2(\Omega)$ and $\psi \in C^1(\Omega)$ on the region $\Omega = h\mathbb{Z} \times h\mathbb{Z}$. □

Remark 3.3. *Clearly, as $h \to 0$, a solution of the h-Cauchy problem*

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_a^b \psi(x')d_{\bar{h}}x' \tag{3.18}$$

tends to

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(x')dx'$$

which is a solution of D'Alembert problem given by

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \Phi(x), \quad -\infty < x < \infty, \\ &u_t(x,0) = \Psi(x), \quad -\infty < x < \infty. \end{aligned} \tag{3.19}$$

Now, let us recall the Cauchy-Kovalevskaya Theorem, (Gantumur, 2011).

Theorem 3.2. *Consider*

$$\begin{aligned} &u'(t) = f(u(t)), \\ &u(0) = 0. \end{aligned} \tag{3.20}$$

If f is analytic in a neighborhood of origin then there exists unique and analytic solution $u = \phi(t)$ of the Cauchy problem (3.20).

Remark 3.4. *Cauchy-Kovalevskaya Theorem states that Cauchy problem* (3.19) *has a unique and analytic solution provided that ϕ, ψ are analytic functions in the neighborhood of initial points. We can clearly discuss Cauchy-Kovalevskaya Theorem on $h\mathbb{Z}$ as $h\mathbb{Z} \subset \mathbb{R}$. Here if the initial conditions ϕ, ψ are h-analytic functions, then by Cauchy-Kovalevskaya Theorem, the problem* (3.9) *has a unique and analytic solution $u(x,t)$ in the neighborhood of initial point.*

Example 3.1. *Consider the following h-Cauchy problem with the initial conditions*

$$\begin{aligned}
&\Box u = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t)u(x,t) = 0, \quad \forall x, t \in h\mathbb{Z}, \\
&u(x,0) = E_h^x = \Big(\frac{1}{1-h}\Big)^{\frac{x}{h}} = \phi(x), \\
&\nabla_h^t u(x,0) = E_h^x = \psi(x).
\end{aligned} \tag{3.21}$$

Then solution is

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_a^b \psi(x')d_{\bar{h}}x',$$

where $a = x - t$ *and* $b = x + t$*. Here*

$$\int_a^b \psi(x')d_{\bar{h}}x' = \int_a^b E_h^{x'} d_{\bar{h}}x' = \int_a^b [\nabla_h^{x'}(E_h^{x'})]d_{\bar{h}}x' = E_h^{x'}\Big|_{x'=a}^{x'=b},$$

by using the Corollary 2.1 *and the fact* $\nabla_h^x E_h^x = E_h^x$*. Hence,*

$$\begin{aligned}
u(x,t) =& \frac{1}{2}\Big[\Big(\frac{1}{1-h}\Big)^{\frac{x+t}{h}} + \Big(\frac{1}{1-h}\Big)^{\frac{x-t}{h}}\Big] + \frac{1}{2}\Big[\Big(\frac{1}{1-h}\Big)^{\frac{x+t}{h}} - \Big(\frac{1}{1-h}\Big)^{\frac{x-t}{h}}\Big] \\
=& \Big(\frac{1}{1-h}\Big)^{\frac{x+t}{h}} = \Big(\frac{1}{1-h}\Big)^{\frac{x}{h}}\Big(\frac{1}{1-h}\Big)^{\frac{t}{h}} = E_h^x E_h^t.
\end{aligned}$$

Now, we will check whether solution $u(x,t) = E_h^x E_h^t$ *satisfies the problem* $\Box u = 0$ *and the initial conditions.*

$$\Box u(x,t) = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t)(E_h^x E_h^t) = 0,$$

since $\nabla_h^t(E_h^x E_h^t) = E_h^x E_h^t$ *and* $\nabla_h^x(E_h^x E_h^t) = E_h^x E_h^t$*. Also*

$$\begin{aligned}
&u(x,0) = E_h^x E_h^t|_{t=0} = (\frac{1}{1-h})^{\frac{x}{h}} = \phi(x), \\
&\nabla_h^t u(x,t)|_{t=0} = \nabla_h^t(E_h^x E_h^t)|_{t=0} = E_h^x = \psi(x).
\end{aligned}$$

Note that since the initial condition E_h^x *is h-analytic, by Cauchy-Kovalevskaya Theorem the problem* (3.21) *has a unique and analytic solution* $u(x,t) = E_h^x E_h^t$*.*

Example 3.2. *Consider the following h-Cauchy problem with the initial conditions*

$$
\begin{aligned}
&\Box u = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t)u(x,t) = 0, \quad \forall x, t \in h\mathbb{Z}, \\
&u(x,0) = \mathrm{Cosh_h} x = \phi(x), \\
&\nabla_h^t u(x,0) = \mathrm{Sinh_h} x = \psi(x).
\end{aligned} \tag{3.22}
$$

Then solution is

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_a^b \psi(x')d_{\bar{h}}x',$$

where $a = x - t$ *and* $b = x + t$. *Here*

$$\int_a^b \psi(x')d_{\bar{h}}x' = \int_a^b \mathrm{Sinh_h} x' d_{\bar{h}}x' = \int_a^b (\nabla_h^{x'}\mathrm{Cosh_h} x')d_{\bar{h}}x' = \mathrm{Cosh_h} x'\Big|_{x'=a}^{x'=b},$$

by using the Corollary 2.1 *and the fact* $\nabla_h^x \mathrm{Sinh_h} x = \mathrm{Cosh_h} x$. *Then,*

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}[\mathrm{Cosh_h}(x+t) + \mathrm{Cosh_h}(x-t)] + \frac{1}{2}[\mathrm{Cosh_h}(x+t) - \mathrm{Cosh_h}(x-t)] \\
&= \mathrm{Cosh_h}(x+t).
\end{aligned}
$$

Let us check whether there is such $u(x,t) = \mathrm{Cosh_h}(x+t)$ *that satisfies the problem* $\Box u(x,t) = 0$ *and the initial conditions or not;*

$$\Box u(x,t) = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t)\mathrm{Cosh_h}(x+t) = 0,$$

since $\nabla_h^x \mathrm{Cosh_h}(x+t) = \mathrm{Sinh_h} x$ *and* $\nabla_h^t \mathrm{Cosh_h}(x+t) = \mathrm{Sinh_h} x$. *Also we have*

$$
\begin{aligned}
&u(x,0) = \mathrm{Cosh_h}(x) = \phi(x) \\
&\nabla_h^t u(x,t)|_{t=0} = \nabla_h^t \mathrm{Cosh_h}(x+t)|_{t=0} = \mathrm{Sinh_h} x = \psi(x).
\end{aligned}
$$

Note that the initial conditions $\mathrm{Cosh_h} x$ *and* $\mathrm{Sinh_h} x$ *are h-analytic. Thus* (3.22) *has a unique and analytic solution by Cauchy-Kovalesvkaya Theorem.*

Example 3.3. *Consider the following h-initial value problem*

$$\begin{aligned}
&\Box u = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t) = 0, \quad \forall x, t \in h\mathbb{Z},\\
&u(x,0) = \phi(x) = (x)_h^3,\\
&\nabla_h^t u(x,0) = \psi(x) = 3(x)_h^2,
\end{aligned}$$

where $(x)_h^3 = (x+0)_h^3 = x(x+h)(x+2h)$. *Then solution is*

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[(x+t)_h^3 + (x-t)_h^3] + \frac{1}{2}\int_a^b 3(x')_h^2 d_{\bar{h}} x'\\
&= \frac{1}{2}[(x+t)_h^3 + (x-t)_h^3] + \frac{1}{2}[(x+t)_h^3 - (x-t)_h^3] = (x+t)_h^3.
\end{aligned}$$

Here we used

$$\int_a^b 3(x')_h^2 d_{\bar{h}} x' = \int_a^b [\nabla_h^{x'}(x')_h^3] d_{\bar{h}} x' = (x')_h^3\Big|_{x'=a}^{x'=b} = (b)_h^3 - (a)_h^3,$$

and $a = x - t$, $b = x + t$ *by the Corollary* 2.1. *Note that solution* $u(x,t) = (x+t)_h^3$ *satisfies the problem* $\Box u(x,t) = 0$ *and the initial conditions. Because*

$$\Box u = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t)(x+t)_h^3 = (\Delta_h^x + \nabla_h^t)(3(x+t)_h^2 - 3(x+t)_h^2) = 0$$

and

$$\begin{aligned}
&u(x,0) = (x)_h^3 = \phi(x),\\
&\nabla_h^t u(x,0) = \nabla_h^t (x+t)_h^3|_{t=0} = 3(x+t)_h^2|_{t=0} = 3(x)_h^2 = \psi(x).
\end{aligned}$$

Remark 3.5. *As a general case, the h-Cauchy problem with the initial conditions*

$$\begin{aligned}
&\Box u(x,t) = (\Delta_h^x + \nabla_h^t)(\nabla_h^x - \nabla_h^t) u(x,t) = 0,\\
&u(x,0) = \phi(x) = (x)_h^n,\\
&\nabla_h^t u(x,0) = \psi(x) = n(x)_h^{n-1},
\end{aligned} \tag{3.23}$$

has solution

$$u(x,t) = (x+t)_h^n.$$

CHAPTER FOUR
QUANTUM ANALYSIS

4.1 *q*-derivative

In this Chapter, we build quantum analysis by defining q-derivative, $\frac{1}{q}$-derivative, and constructing $\frac{1}{q}$-integral, Jackson integral, and related properties in similar a way to the lattice analysis. Till the Section 4.6, we collect the notions and results on q-Calculus from Kac, 2001.

Definition 4.1. *Let f be an arbitrary function. The q-differential of f on $\mathbb{K}_q$ is defined by*

$$d_q f(x) := f(qx) - f(x), \quad x \in \mathbb{K}_q. \tag{4.1}$$

Remark 4.1. *Quantum differential is different from the ordinary calculus, due to the lack of the symmetry in the differential of the product of two functions. Because*

$$\begin{aligned} d_q(fg)(x) =& f(qx)g(qx) - f(x)g(x) \\ &= f(qx)g(qx) - f(x)g(x) + f(qx)g(x) - f(qx)g(x) \\ &= f(qx)d_q g(x) + g(x)d_q f(x). \end{aligned}$$

With this quantum differential, we define the corresponding quantum derivative.

Definition 4.2. *The q-derivative of the function $f(x)$ is*

$$\Delta_q^x f(x) := \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x}, \quad x \in \mathbb{K}_q. \tag{4.2}$$

Note that continuous limit of q-derivative $\Delta_q^x x$ recover classical derivative i.e.

$$\lim_{q \to 1} \Delta_q^x f(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{x(q-1)} = \frac{qf'(x) - f'(x)}{q-1} = \frac{df(x)}{dx}.$$

Theorem 4.1. *The operator $\Delta_q^x : \mathbb{K}_q \longrightarrow \mathbb{K}_q$ is linear.*

Proof. Let f, g be arbitrary functions on $\mathbb{K}_q$. Then for all a, b scalars, we have

$$\begin{aligned}\Delta_q^x[af(x)+bg(x)] =& \frac{af(qx)+bg(qx)-(af(x)+bg(x))}{(q-1)x} \\ =& a\frac{f(qx)-f(x)}{(q-1)x} + b\frac{g(qx)-g(x)}{(q-1)x} = a\Delta_q^x f(x) + b\Delta_q^x g(x).\end{aligned}$$

□

Proposition 4.1. *For $n \geq 1$, the q-derivative of the function $f(x) = x^n$ is*

$$\Delta_q^x x^n = \frac{q^n-1}{q-1}x^{n-1}. \tag{4.3}$$

Proof. By Definition 4.2, we get

$$\Delta_q^x x^n = \frac{(qx)^n - x^n}{(q-1)x} = \frac{x^n(q^n-1)}{(q-1)x} = \frac{q^n-1}{q-1}x^{n-1}.$$

□

Remark 4.2. *As $q \to 1$, the limit of q-derivative of x^n gives ordinary derivative of x^n. Because*

$$\lim_{q\to 1}\Big(\frac{q^n-1}{q-1}\Big)x^{n-1} = x^{n-1}\lim_{q\to 1}\frac{(q-1)(q^{n-1}+q^{n-2}+...+1)}{(q-1)} = nx^{n-1}.$$

Therefore, we may define q-analoque of an integer n as follows.

Definition 4.3. *The q-analoque of an integer n is*

$$[n]_q := \frac{q^n-1}{q-1} = \frac{(q-1)(q^{n-1}+q^{n-2}+...+1)}{(q-1)} = 1+q+...+q^{n-1}, \tag{4.4}$$

where $q \neq 1$.

Remark 4.3. *By definition of q-analogue of integer n,* (4.3) *becomes*

$$\Delta_q^x x^n = [n]_q x^{n-1}. \tag{4.5}$$

Proposition 4.2. *Let $f, g : \mathbb{K}_q \to \mathbb{R}$ be arbitrary functions whose q-derivatives exist on $\mathbb{K}_q$. Then the q-product rule arises in two forms:*

$$\text{(i)}\ \Delta_q^x[(fg)(x)] = f(qx)\Delta_q^x g(x) + g(x)\Delta_q^x f(x) \tag{4.6}$$

$$\text{(ii)}\ \Delta_q^x[(fg)(x)] = f(x)\Delta_q^x g(x) + g(qx)\Delta_q^x f(x). \tag{4.7}$$

Proof. (i) Using Definition 4.2, we get

$$\Delta_q^x[(fg)(x)] = \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x}.$$

Without loss of generality, add and subtract $f(qx)g(x)$ to the numerator of $\Delta_q^x(fg)(x)$, then we get

$$\begin{aligned}
\Delta_q^x[(fg)(x)] =& \frac{f(qx)g(qx) + f(qx)g(x) - f(qx)g(x) - f(x)g(x)}{(q-1)x} \\
=& f(qx)\frac{g(qx) - g(x)}{(q-1)x} + g(x)\frac{f(qx) - f(x)}{(q-1)x} \\
=& f(qx)\Delta_q^x g(x) + g(x)\Delta_q^x f(x).
\end{aligned}$$

(ii) Similarly, add and subtract $f(x)g(qx)$ to the numerator, then we get

$$\begin{aligned}
\Delta_q^x[(fg)(x)] &= \frac{f(qx)g(qx) - f(x)g(x) + f(x)g(qx) - f(x)g(qx)}{(q-1)x} \\
&= f(x)\Delta_q^x g(x) + g(qx)\Delta_q^x f(x).
\end{aligned}$$

□

Proposition 4.3. *Let f, g be arbitrary functions on $\mathbb{K}_q$ whose q-derivatives exist. Then the q-quotient rule arises in two forms:*

$$\text{(i)}\ \Delta_q^x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta_q^x f(x) - f(x)\Delta_q^x g(x)}{g(qx)g(x)}, \tag{4.8}$$

$$\text{(i)}\ \Delta_q^x\left[\frac{f(x)}{g(x)}\right] = \frac{g(qx)\Delta_q^x f(x) - f(qx)\Delta_q^x g(x)}{g(qx)g(x)}. \tag{4.9}$$

Proof. (i) Using Definition 4.2, we have

$$\Delta_q^x\Big[\frac{f(x)}{g(x)}\Big] = \frac{\frac{f(qx)}{g(qx)} - \frac{f(x)}{g(x)}}{(q-1)x}.$$

Without loss of generality, add and subtract $f(x)g(x)$ to the numerator of $\Delta_q^x\Big[\frac{f(x)}{g(x)}\Big]$, we get

$$\begin{aligned}\Delta_q^x\Big[\frac{f(x)}{g(x)}\Big] =& \frac{f(qx)g(x) - f(x)g(qx) + f(x)g(x) - f(x)g(x)}{g(qx)g(x)(q-1)x} \\ =& \frac{g(x)[f(qx) - f(x)] - f(x)[g(qx) - g(x)]}{g(qx)g(x)(q-1)x} = \frac{g(x)\Delta_q^x f(x) - f(x)\Delta_q^x g(x)}{g(qx)g(x)}.\end{aligned}$$

(ii) Similarly, add and subtract $f(qx)g(qx)$ to the numerator, then we obtain

$$\begin{aligned}\Delta_q^x\Big[\frac{f(x)}{g(x)}\Big] &= \frac{f(qx)g(x) - f(x)g(qx) + f(qx)g(qx) - f(qx)g(qx)}{g(qx)g(x)(q-1)x} \\ &= \frac{g(qx)\Delta_q^x f(x) - f(qx)\Delta_q^x g(x)}{g(qx)g(x)}.\end{aligned}$$

□

Remark 4.4. *(i) When $q \to 1$, the product rule of two functions in quantum calculus turns to the ordinary product rule. Because*

$$\lim_{q\to 1}\Delta_q^x[(fg)(x)] = f(x)\lim_{q\to 1}\Delta_q^x g(x) + \lim_{q\to 1} g(qx)\Delta_q^x f(x) = f(x)g'(x) + g(x)f'(x).$$

(ii) When $q \to 1$, the quotient rule of two functions in quantum calculus turns to the ordinary quotient rule. Because

$$\begin{aligned}\lim_{q\to 1}\Delta_q^x\Big[\frac{f(x)}{g(x)}\Big] &= g(x)\lim_{q\to 1}\frac{\Delta_q^x f(x)}{g(qx)g(x)} - f(x)\lim_{q\to 1}\frac{\Delta_q^x g(x)}{g(qx)g(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.\end{aligned}$$

Remark 4.5. *We do not have a general chain rule for q-derivatives. For instance, $u(x) = x + x^2$ and $u(x) = \sin(x)$, $u(qx)$ cannot be expressed in terms of u in a simple manner.*

Remark 4.6. *We see that as $q \to 1$ we recover $\mathbb{R}$. Whenever we use* ***q-discretization*** *terminology, we work on $\mathbb{K}_q$.*

4.2 *q*-binomials and *q*-Taylor's Formula

Definition 4.4. *The q-analoque of a binomial $(x \pm a)^n$ is the polynomial*

$$(x \pm a)_q^n := \begin{cases} 1 & \text{if} \quad n = 0, \\ (x \pm a)(x \pm qa)(x \pm q^2 a) \dots (x \pm q^{n-1} a) & \text{if} \quad n \geq 1. \end{cases} \tag{4.10}$$

where a is any constant.

Proposition 4.4. *For any two integers m, n, we have*

$$(x - a)_q^{m+n} = (x - a)_q^m (x - q^m a)_q^n. \tag{4.11}$$

Proof. There are five cases.

(i) For $m = 0$ and $n = 0$, it is obvious.

(ii) For $m > 0$ and $n > 0$, by Definition 4.4, we have

$$\begin{aligned}(x - a)_q^{m+n} =& (x - a)(x - qa) \dots (x - q^{m-1} a)(x - q^m a) \\ & (x - q^{m+1} a) \dots (x - q^{m+n-1} a) \\ =& (x - a)_q^m (x - q^m a)(x - q(q^m a)) \dots (x - q^{n-1}(q^m) a) \\ =& (x - a)_q^m (x - q^m a)_q^n.\end{aligned}$$

Note that

$$(x - a)_q^{-n} = \frac{1}{(x - q^{-n} a)_q^n}, \text{ for any positive integer } n. \tag{4.12}$$

It result by substituting m by $-n$ in the Case (ii).

(iii) For $m < 0$ and $n > 0$, consider $m = -m' < 0$, $n > 0$. Using the equation (4.12), we have

$$
\begin{aligned}
(x-a)_q^m (x-q^m a)_q^n =& (x-a)_q^{-m'} (x-q^{-m'}a)_q^n = \frac{(x-q^{-m'}a)_q^n}{(x-q^{-m'}a)_q^{m'}} \\
=& \begin{cases} (x-q^{m'}q^{-m'}a)...(x-q^{n-1}q^{-m'}a) & , \quad n \geq m' \\ \dfrac{1}{(x-q^n q^{-m'}a)(x-q^{m'-1}q^{-m'}a)} & , \quad m' > n \end{cases} \\
=& \begin{cases} (x-q^{m'}(q^{-m'}a))_q{}^{n-m'} & , \quad n \geq m' \\ \dfrac{1}{(x-q^n(q^{-m'}a))_q{}^{m'-n}} & , \quad m' > n \end{cases} \\
=& (x-a)_q^{n-m'} = (x-a)_q^{m+n}.
\end{aligned}
$$

(iv) For $m > 0$ and $n < 0$, consider $m > 0$ and $n = -n' < 0$. We get

$$
\begin{aligned}
&(x-a)_q^m (x-q^m a)_q^n = (x-a)_q^m (x-q^m a)^{-n'} = \frac{(x-a)_q^m}{(x-q^{m-n'}a)_q^{n'}} \\
=& \begin{cases} \dfrac{(x-a)_q^{m-n'}(x-q^{m-n'a})}{(x-q^{m-n'}a)_q^{n'}} & , \quad m \geq n' \\ \dfrac{(x-a)_q^m}{(x-q^{m-n'}a)_q^{n'-m}(x-q^{n'-m}(q^{m-n'}a))_q^m} & , \quad m < n' \end{cases} \\
=& \begin{cases} (x-a)_q^{m-n'} & , \quad m \geq n' \\ \dfrac{1}{(x-q^{m-n'}a)_q^{n'-m}} & , \quad m < n' \end{cases} \\
=& (x-a)_q^{m-n'} = (x-a)_q^{m+n}.
\end{aligned}
$$

(v) For $m < 0$ and $n < 0$, consider $m = -m' < 0$ and $n = -n' < 0$. We get

$$
\begin{aligned}
(x-a)_q^m (x-q^m a)_q^n =& (x-a)_q^{-m'} (x-q^{-m'}a)_q^{-n'} \\
=& \frac{1}{(x-q^{-m'}a)_q^{m'}(x-q^{-n'-m'}a)_q^{n'}} \\
=& \frac{1}{(x-q^{-n'-m'}a)_q^{n'}(x-q^{n'}(q^{-m'-n'}a))_q^{m'}} \\
=& \frac{1}{(x-q^{-n'-m'}a)_q^{n'+m'}} \\
=& (x-a)_q^{-n'-m'} = (x-a)_q^{m+n},
\end{aligned}
$$

where we used the fact that $n', m' > 0$.

□

Theorem 4.2. *For any integer n, we have*

$$\Delta_q^x (x-a)_q^n = [n]_q (x-a)_q^{n-1}. \tag{4.13}$$

Proof. There are three cases.

(i) For $n = 0$, since $[0]_q = 0$ by Definition 4.4, (4.13) is true.

(ii) For $n > 0$, let us prove by induction on k. Clearly, (4.13) is true for $n = 1$. Assume that (4.13) holds for k. We will prove that it also holds for $k+1$. By Proposition 4.4, we get

$$(x-a)_q^{k+1} = (x-a)_q^k (x - q^k a)_q^1.$$

Then consider

$$\begin{aligned}
\Delta_q^x (x-a)_q^{k+1} =& \Delta_q^x [(x-a)_q^k (x-q^k a)] \\
=& (qx - q^k a) \Delta_q^x (x-a)_q^k + (x-a)_q^k \Delta_q^x (x - q^k a) \\
=& q[k]_q (x-a)_q^{k-1} (x - q^{k-1} a) + (x-a)_q^k \\
=& q[k]_q (x-a)_q^k + (x-a)_q^k \\
=& (q[k]_q + 1)(x-a)_q^k = [k+1]_q (x-a)_q^k,
\end{aligned}$$

where we used q-product rule (4.6), inductive hypothesis, and the fact that:

$$q[k]_q + 1 = q\frac{q^k - 1}{q-1} + 1 = \frac{q^{k+1}-1}{q-1} = [k+1]_q.$$

(iii) For $n < 0$, set $n = -n' < 0$. Then

$$\begin{aligned}
\Delta_q^x (x-a)_q^n =& \Delta_q^x (x-a)_q^{-n'} = \Delta_q^x [\frac{1}{(x - q^{-n'} a)_q^{n'}}] \\
=& \frac{-\Delta_q^x (x - q^{-n'} a)_q^{n'}}{(qx - q^{-n'} a)_q^{n'} (x - q^{-n'} a)_q^{n'}}.
\end{aligned}$$

Here since

$$
\begin{aligned}
(qx - q^{-n'}a)_q^{n'} =& (qx - q^{-n'}a)(qx - q^{-n'+1}a)...(qx - q^{-n'+(n'-1)}a) \\
=& q(x - q^{-n'-1}a)q(x - q^{-n'}a)...q(x - q^{(-n'-1)+(n'-1)}a) \\
=& q^{n'}(x - q^{-n'-1}a)_q^{n'}
\end{aligned}
$$

and $n' > 0$, we obtain

$$
\begin{aligned}
\Delta_q^x (x-a)_q^n =& \frac{-[n']_q (x - q^{-n'}a)_q^{n'-1}}{q^{n'}(x - q^{-n'-1}a)_q^{n'}(x - q^{-n'}a)_q^{n'}} = \frac{-[n']_q}{q^{n'}(x - q^{-1}a)} \\
=& \frac{1 - q^{n'}}{q-1} \frac{q^{-n'}}{(x - q^{-1}a)(x - q^{-n'-1}a)_q^{n'}} \\
=& \frac{q^{-n'} - 1}{q-1} \frac{1}{(x - q^{-n'-1}a)_q^{n'+1}} = \frac{q^{n-1}}{q-1}(x-a)_q^{n-1} = [n]_q (x-a)_q^{n-1},
\end{aligned}
$$

where we used Proposition 4.4 and the equation (4.12).

□

Definition 4.5. *The q-analogue of a binomial* $(a \pm x)^n$ *is the polynomial*

$$
(a \pm x)_q^n := \begin{cases} (a \pm x)(a \pm qx)(a \pm q^2 x)...(a \pm q^{n-1}x) & \textit{if} \quad n \geq 1, \\ 1 & \textit{if} \quad n = 0. \end{cases} \tag{4.14}
$$

Corollary 4.1. *For any integer* m *and* n*, we have*

$$
(a - x)_q^{m+n} = (a - x)_q^m (a - q^m x)_q^n. \tag{4.15}
$$

Proof. There are five cases.

(i) For $m = 0$ and $n = 0$, it is obvious.

(ii) For $m, n > 0$, by Definition 4.5

$$\begin{aligned}(a-x)_q^{m+n} &= (a-x)_q^m (a-q^m x)(a-q^{m+1}x)...(a-q^{m+n-1}x)\\ &= (a-x)_q^m (a-q^m x)_q^n.\end{aligned}$$

Note that

$$(a-x)_q^{-n} = \frac{1}{(a-q^{-n}x)_q^n}, \text{ for any positive integer } n. \tag{4.16}$$

It result by replacing m by $-n$ in the Case (ii).

(iii) For $m < 0$ and $n > 0$, say $m = -m' < 0$ and $n \geq 0$. Then we have,

$$\begin{aligned}(a-x)_q^m (a-q^m x)_q^n =&(a-x)_q^{-m'} (a-q^{-m'}x)_q^n = \frac{(a-q^{-m'}x)_q^n}{(a-q^{-m'}x)_q^{m'}}\\ =&\begin{cases} (a-q^{m'}(q^{-m'}x))_q^{n-m'} & , \quad n \geq m' \\ \dfrac{1}{(a-q^n(q^{-m'}x))_q^{m'-n}} & , \quad m' > n \end{cases}\\ =&(a-x)_q^{n-m'} = (a-x)_q^{n+m}.\end{aligned}$$

(iv) For $n < 0$ and $m > 0$ say $n = -n' < 0$. By using the equation (4.16),

$$(a-x)_q^m (a-q^m x)_q^n = (a-x)_q^m (a-q^m x)_q^{-n'} = \frac{(a-x)_q^m}{(a-q^{m-n'}x)_q^{n'}}$$

$$=\begin{cases} \dfrac{(a-x)_q^{m-n'}(a-q^{m-n'}x)_q^{n'}}{(a-q^{m-n'}x)_q^{n'}} & , \quad m > n' \\ \dfrac{(a-x)_q^m}{(a-q^{m-n'}x)_q^{n'-m}(a-q^{n'-m}(q^{m-n'}x))_q^m} & , \quad n' \geq m. \end{cases}$$

$$=(a-x)_q^{m-n'} = (a-x)_q^{m+n}.$$

(v) For $m < 0$ and $n < 0$, say $m = -m' < 0$ and $n = -n' < 0$. Then we get

$$\begin{aligned}(a-x)_q^m(a-q^mx)_q^n =& (a-x)_q^{-m'}(a-q^{-m'}x)_q^{-n'} \\ =& \frac{1}{(a-q^{-m'}x)_q^{m'}(a-q^{-n'-m'}x)_q^{n'}} \\ =& \frac{1}{(a-q^{-n'-m'}x)_q^{n'}(a-q^{n'}(q^{-m'-n'}x)_q^{m'}} \\ =& \frac{1}{(a-q^{-n'-m'}x)_q^{n'+m'}} = (a-x)_q^{-n'-m'} = (a-x)_q^{n+m}.\end{aligned}$$

□

Corollary 4.2. *For any integer n, we have*

$$(a-x)_q^n = (-1)^n q^{\frac{n(n-1)}{2}}(x-q^{-n+1}a)_q^n. \tag{4.17}$$

Proof. We have two cases.

(i) For $n \geq 1$, by Definition 4.5,

$$\begin{aligned}(a-x)_q^n =& (a-x)(a-qx)(a-q^2x)...(a-q^{n-1}x) \\ =& (a-x)q(q^{-1}a-x)q^2(q^{-2}a-x)...q^{n-1}(q^{-n+1}a-x) \\ =& (-1)^n q^{\frac{n(n-1)}{2}}(x-a)(x-q^{-1}a)...(x-q^{-n+1}a) \\ =& (-1)^n q^{\frac{n(n-1)}{2}}(x-q^{-n+1}a)_q^n.\end{aligned}$$

(ii) For $n < 0$, say $n = -n' < 0$. By the equation (4.16),

$$\begin{aligned}
(a-x)_q^{-n} = (a-x)_q^{-n'} =& \frac{1}{(a-q^{-n'}x)_q^{n'}} \\
=& \frac{1}{(a-q^{-n'}x)(a-q^{-n'+1}x)...(a-q^{-n'+n'-1}x)} \\
=& \frac{1}{q^{-n'}(q^{n'}a-x)...q^{-n'+n'-1}(q^{n'-n'+1}a-x)} \\
=& \frac{q^{n'}q^{n'-1}q^{n'-2}...q^{n'-n'+1}}{(-1)^{n'}(x-q^{n'}a)(x-q^{n'-1}a)...(x-q^{n'-(n'-1)}a)} \\
=& \frac{(-1)^{-n'}q^{n'^2-(1+2+...(n'-1))}}{(x-q^{n'-(n'-1)}a)_q^{n'}} \\
=& (-1)^{-n'}(x-q^{n'+1}a)_q^{-n'}q^{\frac{n'(n'+1)}{2}} \\
=& (-1)^n q^{\frac{n(n-1)}{2}}(x-q^{-n+1}a)_q^n.
\end{aligned}$$

□

Corollary 4.3. *For any integer n, we have the followings:*

$$\textit{(i)} \quad \Delta_q^x \frac{1}{(x-a)_q^n} = [-n]_q(x-q^na)_q^{-n-1}, \tag{4.18}$$

$$\textit{(ii)} \quad \Delta_q^x (a-x)_q^n = -[n]_q(a-qx)_q^{n-1}, \tag{4.19}$$

$$\textit{(iii)} \quad \Delta_q^x \frac{1}{(a-x)_q^n} = \frac{[n]_q}{(a-x)_q^{n+1}}. \tag{4.20}$$

Proof. (i) By using the equation (4.12) and Theorem 4.2, we get

$$\Delta_q^x \frac{1}{(x-a)_q^n} = \Delta_q^x \frac{1}{(x-q^{-n}(q^na))_q^n} = \Delta_q^x (x-q^na)_q^{-n} = [-n]_q(x-q^na)_q^{-n-1}.$$

(ii) By using Corollary 4.2 and Theorem 4.2, we get

$$\begin{aligned}
\Delta_q^x (a-x)_q^n =& (-1)^n q^{\frac{n(n-1)}{2}} \Delta_q^x (x-q^{-n+1}a)_q^n \\
=& (-1)^n q^{\frac{n(n-1)}{2}} [n]_q (x-q^{-n+1}a)_q^{n-1} \\
=& -[n]_q(-1)^{n-1}q^{n-1}q^{\frac{(n-1)(n-2)}{2}}(x-q^{-n+2}(q^{-1}a))_q^{n-1} \\
=& -[n]_q q^{n-1}(-1)^{n-1}q^{\frac{(n-1)(n-2)}{2}}(x-q^{-(n-1)+1}(q^{-1}a))_q^{n-1} \\
=& -[n]_q q^{n-1}(q^{-1}a-x)_q^{n-1} = -[n]_q(a-qx)_q^{n-1}.
\end{aligned}$$

(iii) By using q-quotient rule (4.8) and Theorem 4.2, we get

$$\Delta_q^x \frac{1}{(a-x)_q^n} = \frac{-\Delta_q^x (a-x)_q^n}{(a-qx)_q^n (a-x)_q^n} = \frac{[n]_q (a-qx)_q^{n-1}}{(a-x)_q^n (a-qx)_q^n}$$
$$= \frac{[n]_q}{(a-x)_q^n (a-q^n x)} = \frac{[n]_q}{(a-x)_q^{n+1}}.$$

□

Now we are ready to apply Theorem 2.3 to $D \equiv \Delta_q^x$. But first we will give the following definition.

Definition 4.6. *The q-analogue of $n!$ is*

$$[n]!_q := \begin{cases} 1 & \textit{if} \quad n = 0, \\ [n]_q [n-1]_q \dots [1]_q & \textit{if} \quad n = 1, 2, 3, \dots \end{cases} \tag{4.21}$$

Remark 4.7. *Let us try to construct the sequence of polynomials $\{P_0(x), P_1(x), P_2(x), \dots\}$ satisfying the conditions of Theorem 2.3 with respect to $D \equiv \Delta_q^x$. If $a = 0$ in the Theorem 2.3, we can choose*

$$P_n(x) = \frac{x^n}{[n]!_q}. \tag{4.22}$$

Let us check whether chosen $P_n(x)$ satisfy the three conditions of Theorem 2.3. Clearly

(a) $P_0(0) = 1$ *and* $P_n(0) = 0 \ \forall n \geq 1$,

(b) $degP_n(x) = n$,

(c) $\Delta_q^x P_n(x) = \Delta_q^x [\frac{x^n}{[n]!_q}] = \frac{[n]_q x^{n-1}}{[n]_q [n-1]_q \dots [1]_q} = \frac{x^{n-1}}{[n-1]!_q} = P_{n-1}(x).$

But, if $a \neq 0$, the choice of $P_n(x)$ satisfying the three conditions of Theorem 2.3 is not simple, like $P_n(x) = \frac{(x-a)^n}{[n]!_q}$. For example; $\Delta_q^x \frac{(x-a)^2}{[2]!_q} \neq P_1(x) = (x-a)$. Now, let us try to find the first few terms of the sequence of polynomials $P_n(x)$ to obtain the *general formula i.e. the q-version of Taylor's formula. We have*

$$P_0(x) = 1.$$

To obtain $\Delta_q^x P_1(x) = P_0(x) = 1$ *and* $P_1(a) = 0$, *we must have*

$$P_1(x) = (x - a).$$

To obtain $\Delta_q^x P_2(x) = P_1(x) = (x - a)$ *and* $P_2(a) = 0$, *we must have*

$$\begin{aligned} P_2(x) =& \frac{x^2}{[2]_q} - ax - \frac{a^2}{[2]_q} + a^2 = \frac{x^2 - (q+1)ax - a^2 + a^2(q+1)}{[2]_q} \\ =& \frac{x^2 - a(q+1)x + a^2 q}{[2]_q} = \frac{(x-a)(x-qa)}{[2]_q}. \end{aligned}$$

Similarly,

$$P_3(x) = \frac{(x-a)(x-qa)(x-q^2a)}{[2]_q[3]_q}.$$

and so on. In this way, we obtain

$$P_n(x) = \frac{(x-a)(x-qa)(x-q^2a)...(x-q^{n-1}a)}{[n]!_q} = \frac{(x-a)_q^n}{[n]!_q}. \tag{4.23}$$

(4.23) *agrees with* (4.22) *when* $a = 0$.
Now, check the validity of condition (c), i.e. $\Delta_q^x P_n(x) = P_{n-1}(x)$. *We have*

$$\Delta_q^x P_n(x) = \Delta_q^x \frac{(x-a)_q^n}{[n]!_q} = \frac{1}{[n]!_q}[n]_q (x-a)_q^{n-1} = \frac{(x-a)_q^{n-1}}{[n-1]!_q} = P_{n-1}(x).$$

Since the equation (4.23) satisfies the three conditions of the Theorem 2.4 with respect to Δ_q^x, we can write the following theorem.

Theorem 4.3. *(q-Taylor's formula) For any polynomial* $f(x)$ *with deg*$f(x) = N$, *we have*

$$f(x) = \sum_{j=0}^{N} (\Delta_q^x f)^j(c) \frac{(x-c)_q^j}{[j]!_q}, \tag{4.24}$$

where c *is any number, and* $(\Delta_q^x f)^j$ *is the* j*th order of* $\Delta_q^x f$.

Example 4.1. *Find the q-Taylor series expansion of* $f(x) = x^n$ *at* $c = 1$ *where* $n \in \mathbb{Z}^+$.

Solution. Let us find all $(\Delta_q^x f)^j(x)$'s for $j \leq n$.

$$\begin{aligned}
\Delta_q^x f(x) =& \Delta_q^x x^n = [n]_q x^{n-1}, \\
(\Delta_q^x)^2 x^n =& \Delta_q^x(\Delta_q^x x^n) = [n]_q[n-1]_q x^{n-2}, \\
(\Delta_q^x)^3 x^n =& [n]_q[n-1]_q[n-2]_q x^{n-3}.
\end{aligned}$$

In this way, we obtain

$$(\Delta_q^x x^n)^j = [n]_q[n-1]_q \ldots [n-(j-1)]_q x^{n-j} \ \text{ for } \ j \leq n.$$

Then, at $c = 1$ we have

$$(\Delta_q^x f)^j(1) = [n]_q[n-1]_q \ldots [n-(j-1)]_q.$$

As a result, the q-Taylor's Formula for x^n about $c = 1$ is

$$x^n = \sum_{j=0}^{n} \frac{[n]_q[n-1]_q \ldots [n-(j-1)]_q}{[j]!_q}(x-1)_q^j = \sum_{j=0}^{n} \binom{n}{j}_q (x-1)_q^j, \tag{4.25}$$

where

$$\binom{n}{j}_q := \frac{[n]_q[n-1]_q \ldots [n-(j-1)]_q}{[j]!_q} = \frac{[n]!_q}{[j]!_q[n-j]!_q}, \tag{4.26}$$

are defined as q-binomial coefficients. □

Note that as $q \to 1$, the q-binomial coefficients (4.26) gives the ordinary binomial coefficients, so (4.25) becomes

$$x^n = \sum_{j=0}^{n} \binom{n}{j} (x-1)^j,$$

classical ordinary binomial formula of x^n.

4.3 Gauss' binomial formula and Heine's binomial formula

In this section, we present Gauss' binomial formula and Heine's binomial formula given in Kac, 2001 by using q-Taylor's Formula.

Example 4.2. *Find a q-Taylor series expansion of* $f(x) = (x+a)_q^n$ *about* $c = 0$, *where* $n \in \mathbb{Z}^+ \cup \{0\}$.

Solution. Let us find all $(\Delta_q^x)^j$'s, for $j = 1, 2, \dots$

$$\begin{aligned}(\Delta_q^x f)(x) =& \Delta_q^x (x+a)_q^n = [n]_q (x+a)_q^{n-1},\\ (\Delta_q^x f)^2(x) =& [n]_q [n-1]_q (x+a)_q^{n-2}.\end{aligned}$$

In this way, we obtain

$$(\Delta_q^x f)^j(x) = [n]_q [n-1]_q ... [n-(j-1)]_q (x+a)_q^{n-j}.$$

For $c = 0$, we get

$$\begin{aligned}(\Delta_q^x f)^j(c)\Big|_{c=0} &= (\Delta_q^x f)^j(0) = [n]_q [n-1]_q ... [n-j+1]_q a(qa)(q^2 a)...(q^{n-j-1}a)\\ &= [n]_q [n-1]_q ... [n-j+1]_q a^{n-j} q^{\frac{(n-j+1)(n-j)}{2}}.\end{aligned}$$

Hence, by Theorem 4.3, the q-Taylor Formula of $f(x)$ about $c = 0$ is

$$f(x) = \sum_{j=0}^{n} \frac{(\Delta_q^x f)^j(0)}{[j]_q!} (x-0)_q^j = \sum_{j=0}^{n} \binom{n}{j}_q q^{\frac{(n-j)(n-j+1)}{2}} a^{n-j} x^j. \tag{4.27}$$

By replacing j by $n-j$, we get

$$\binom{n}{n-j}_q = \frac{[n]_q!}{[j]_q! [n-j]_q!} = \binom{n}{j}_q. \tag{4.28}$$

So, (4.27) is equivalent to

$$f(x) = (x+a)_q^n = \sum_{j=0}^{n} \binom{n}{j}_q q^{\frac{j(j-1)}{2}} a^j x^{n-j}.$$

Definition 4.7. *We define Gauss' binomial formula as follows*

$$(x+a)_q^n := \sum_{j=0}^{n} \binom{n}{j}_q q^{\frac{j(j-1)}{2}} a^j x^{n-j}, \;\; n \geq 1. \tag{4.29}$$

Note that in quantum calculus, multiplication is not commutative for instance composition of operators. We can explain this non-commutativity by considering two linear operators $\hat{x}$ and $\hat{M}_q$ on the vector space of polynomials such as

$$\hat{x}[f(x)] = xf(x) \quad \text{and} \quad \hat{M}_q[f(x)] = f(qx).$$

For any function $f(x)$,

$$\hat{M}_q\hat{x}[f(x)] = \hat{M}_q(xf(x)) = qxf(qx) = qx\hat{M}_q(f(x)) = q\hat{x}\hat{M}_q f(x).$$

As a result,

$$\hat{M}_q\hat{x} = q\hat{x}\hat{M}_q.$$

Definition 4.8. *Let $x, y \in \mathbb{K}_q$. If $xy = qyx$ where $q \neq 1$ is a number, then x, y are said to be "q-commuting variables".*

Remark 4.8. *Let x and y be q-commuting variables. Then,*

$$y^k x = q^k x y^k, \tag{4.30}$$

since $y^k x = qy^{k-1}xy = q^2y^{k-2}xy^2 = ... = q^k x y^k$.

Standard binomial expansion does not hold for quantum numbers. However it holds by q-binomial coefficient provided that x, y are q-commuting variables. This result is presented in Kac, 2001.

Theorem 4.4. *If x and y are q-commuting variables, i.e. $xy = qyx$, then we have*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}. \tag{4.31}$$

Proof. We will prove by induction on n. For $n = 0$, the equation holds obviously. Assume that the equation holds for $n \geq 1$. Then writing $(x+y)^{n+1} = (x+y)(x+y)^n$, we have

$$\begin{aligned}
(x+y)^{n+1} =& \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j} x + \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j+1} \\
=& \sum_{j=0}^{n} \binom{n}{j}_q x^j (q^{n-j} x y^{n-j}) + \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j+1} \\
=& \sum_{j=1}^{n+1} \binom{n}{j-1}_q x^{j-1} q^{n-j+1} x y^{n-j+1} + \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j+1} \\
=& y^{n+1} + \sum_{j=1}^{n} [q^{n-j+1} \binom{n}{j-1}_q + \binom{n}{j}_q] x^j y^{n-j+1} + x^{n+1} \\
=& \sum_{j=0}^{n+1} \binom{n+1}{j}_q x^j y^{n-j+1},
\end{aligned}$$

where we used the inductive hypothesis, the equation (4.30) in step 4 and in the last step, ”q- Pascal identity” which will be given in Proposition 4.5. □

Note that binomial formula (4.31) is valid only if x and y are q-commuting variables.

In order to obtain Heine’s binomial formula, we need to derive q-Pascal identity for similar to Pascal Rule in combinatorics

$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}, \quad 1 \leq j \leq n-1.$$

Proposition 4.5. *In quantum Calculus, q-Pascal Rule arises in two forms*

$$\binom{n}{j}_q := \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q \tag{4.32}$$

or

$$\binom{n}{j}_q := q^{n-j} \binom{n-1}{j-1}_q + \binom{n-1}{j}_q, \tag{4.33}$$

where $1 \leq j \leq n-1$.

Proof. Consider

$$\binom{n}{j}_q = \frac{[n]_q!}{[j]_q![n-j]_q!} = \frac{[n]_q[n-1]_q!}{[j]_q![n-j]_q!}. \tag{4.34}$$

Note that for any $1 \leq j \leq n-1$,

$$[n]_q = 1+q+...+q^{n-1} = (1+q+...q^{j-1})+q^j(1+q+...+q^{n-j-1}) = [j]_q+q^j[n-j]_q.$$

So, the equation (4.34) becomes

$$\begin{aligned}\binom{n}{j}_q =& \frac{[n-1]_q!([j]_q+q^j[n-j]_q)}{[j]_q![n-j]_q!}\\ =& \frac{[n-1]_q!}{[j-1]_q![n-j]_q!} + q^j \frac{[n-1]_q!}{[j]_q![n-j-1]_q!} = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q\end{aligned}$$

which gives (4.32). Now, replace j by $n-j$ on the right of the last equation, we get

$$\binom{n}{j}_q = \binom{n-1}{n-j-1}_q + q^{n-j}\binom{n-1}{n-j}_q = \binom{n-1}{j}_q + q^{n-j}\binom{n-1}{j-1}_q,$$

which gives (4.33). □

Example 4.3. *Let* $f(x) = \dfrac{1}{(1-x)_q^n}$. *Find the q-Taylor expansion of* $f(x)$ *about* $c = 0$.

Solution. Firstly, let us find q-derivatives of $f(x)$. By Corollary 4.3-(iii), we have

$$\Delta_q^x f(x) = \frac{-[n]_q}{(1-x)_q^{n+1}},$$
$$(\Delta_q^x f)^2(x) = \Delta_q^x \frac{-[n]_q}{(1-x)_q^{n+1}} = -[n]_q \Delta_q^x \frac{1}{(1-x)_q^{n+1}} = \frac{[n]_q[n+1]_q}{(1-x)_q^{n+2}}.$$

In this manner, we obtain

$$(\Delta_q^x f)^j(x) = \frac{[n]_q[n+1]_q...[n+(j-1)]_q}{(1-x)_q^{n+j}}, \quad \forall j \geq 0.$$

Then at $c = 0$, we have

$$(\Delta_q^x f)^j(c)|_{c=0} = [n]_q[n+1]_q...[n+j-1]_q \quad \text{for any} \quad j \geq 1.$$

Hence the q-Taylor's Formula of $f(x) = \dfrac{1}{(1-x)_q}$ at $x = 0$ is

$$\frac{1}{(1-x)_q^n} = \sum_{j=0}^{n} (\Delta_q^x f)^j(0) \frac{(x-0)_q^j}{[j]_q!} = \sum_{j=0}^{n} \binom{n}{j}_q x^j.$$

$\square$

Definition 4.9. *We define Heine's binomial formula as follows*

$$\frac{1}{(1-x)_q^n} := \sum_{j=0}^{n} \binom{n}{j}_q x^j, \quad n \geq 1. \tag{4.35}$$

4.4 *q*-exponential functions and *q*-trigonometric functions

In this section, we will define two q-exponential functions based on Gauss binomial formula (4.29) and Heine's binomial formula (4.35).

Remark 4.9. *When $n \to \infty$ and $|q| < 1$, we have*

$$\lim_{n\to\infty} [n]_q = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q} \tag{4.36}$$

and

$$\begin{aligned}\lim_{n\to\infty} \binom{n}{j}_q &= \lim_{n\to\infty} \frac{[n]_q!}{[j]_q![n-j]_q!} = \lim_{n\to\infty} \frac{[n]_q[n-1]_q...[n-j+1]_q}{[j]_q!} \\ &= \lim_{n\to\infty} \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-j+1})}{(1-q^j)(1-q^{j-1})...(1-q)} \\ &= \frac{1}{(1-q^j)(1-q^{j-1})...(1-q)}.\end{aligned} \tag{4.37}$$

Proposition 4.6. *The following identities hold*

$$\textbf{\textit{(i)}}\ (1+x)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{(1-q)(1-q^2)...(1-q^j)} \tag{4.38}$$

$$\textbf{\textit{(ii)}}\ \frac{1}{(1-x)_q^\infty} = \sum_{j=0}^{\infty} \frac{x^j}{(1-q)(1-q^2)...(1-q^j)}. \tag{4.39}$$

Remark 4.10. *Identities* (4.38) *and* (4.39) *are called as Euler's first (E_1) and second identities (E_2), since they are discovered by Euler long before Gauss and Heine.*

Proof. (i) Consider the Gauss' binomial formula (4.29) as $n \to \infty$.

$$\lim_{n\to\infty}(1+x)_q^n = \lim_{n\to\infty}\sum_{j=0}^{n} q^{\frac{j(j-1)}{2}}\binom{n}{j}_q x^j. \tag{4.40}$$

By Remark 4.9, we have

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{(1-q)(1-q^2)...(1-q^j)}.$$

(ii) Similarly, consider the Heine's binomial formula (4.35) as $n \to \infty$. That is

$$\lim_{n\to\infty}\frac{1}{(1-x)_q^n} = \lim_{n\to\infty}\sum_{j=0}^{n}\binom{n}{j}_q x^j.$$

By Remark 4.9, we have

$$\frac{1}{(1-x)_q^\infty} = \sum_{j=0}^{\infty}\frac{x^j}{(1-q)(1-q^2)...(1-q^j)}.$$

□

Now, reconsider E_2. Without loss of generality, dividing numerator and denumerator of right hand side by $(1-q)^j$, we get

$$\frac{1}{(1-x)_q^\infty} = \sum_{j=0}^{\infty}\frac{(\frac{x}{1-q})^j}{(\frac{1-q}{1-q})(\frac{1-q^2}{1-q})...(\frac{1-q^j}{1-q})} = \sum_{j=0}^{\infty}\frac{(\frac{x}{1-q})^j}{[1]_q[2]_q...[j]_q} = \sum_{j=0}^{\infty}\frac{(\frac{x}{1-q})^j}{[j]_q!}, \tag{4.41}$$

where we used the definition of q-analogue of n given by (4.4) and q-analoque of $n!$ given by (4.21).

Note that the formula (4.41) resembles the Taylor's expansion of the classical exponential function

$$e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}.$$

If we set $y = \dfrac{x}{1-q}$, then (4.41) becomes

$$\frac{1}{(1-y(1-q))_q^\infty} = \sum_{j=0}^{\infty} \frac{y^j}{[j]_q!}.$$

Now, we are ready to define q-analogue of exponential function.

Definition 4.10. *A q-analogue of the exponential function e^x is*

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} = \frac{1}{(1-(1-q)x)_q^\infty}. \tag{4.42}$$

Remark 4.11. *When $q \to 1$, e_q^x turns to the ordinary exponential function.*

Now, let us reconsider E_1,

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{(1-q)(1-q^2)...(1-q^j)}.$$

Similarly, dividing numerator and denumerator of right hand side by $(1-q)^j$, we get

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{(\frac{x}{1-q})^j}{[j]_q!}.$$

Setting $y = \dfrac{x}{1-q}$, we obtain

$$(1+(1-q)y)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{y^j}{[j]_q!}.$$

Due to the similarity of right hand side to Taylor's expansion of exponential function, we can give the following definition.

Definition 4.11. *Another q-analogue of exponential function is defined as*

$$E_q^x := \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]_q!} = (1+(1-q)x)_q^\infty. \tag{4.43}$$

Also one can write

$$E_q^{\frac{x}{1-q}} = (1+x)_q^\infty. \tag{4.44}$$

The classical exponential function is unchanged under differentiation. In quantum calculus q-exponential function e_q^x is also unchanged under the operator Δ_q^x for q-differentiation. But E_q^x is a slightly different.

Proposition 4.7. *For q- exponential functions e_q^x and E_q^x, we have*

$$\textit{(i)} \quad \Delta_q^x e_q^x = e_q^x, \tag{4.45}$$

$$\textit{(ii)} \quad \Delta_q^x E_q^x = E_q^{qx}. \tag{4.46}$$

Proof. (i) By Definition 4.10, we have

$$\Delta_q^x e_q^x = \sum_{j=0}^{\infty} \frac{1}{[j]_q!} \Delta_q^x (x^j) = \sum_{j=1}^{\infty} \frac{[j]}{[j]_q!} x^{j-1} = \sum_{j=1}^{\infty} \frac{x^{j-1}}{[j-1]_q!} = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} = e_q^x.$$

(ii) Similarly by Definition 4.11, we have

$$\begin{aligned} \Delta_q^x E_q^x &= \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{1}{[j]_q!} \Delta_q^x (x^j) = \sum_{j=1}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{[j]_q!} [j]_q x^{j-1} \\ &= \sum_{j=1}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^{j-1}}{[j-1]_q!} = \sum_{j=0}^{\infty} q^{\frac{j(j+1)}{2}} \frac{x^j}{[j]_q!} = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{q^j x^j}{[j]_q!} = E_q^{qx}. \end{aligned}$$

□

Proposition 4.8. *If x and y satisfy the commutativity relation, $xy = qyx$, then the additive property is valid for q-analogue exponential function e_q^x, i.e. we have*

$$e_q^{x+y} = e_q^x e_q^y. \tag{4.47}$$

Proof. Let us consider right hand side.

$$e_q^x e_q^y = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} \sum_{k=0}^{\infty} \frac{y^k}{[k]_q!} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k}{[j]_q![k]_q!} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{[j+k]_q!}{[j]_q![k]_q!} \frac{x^j y^k}{[j+k]_q!}.$$

Say $n = j + k$, so $k = n - j$, then by using Theorem 4.4 we get

$$e_q^x e_q^y = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{[n]_q!}{[j]_q![n-j]_q!} \frac{x^j y^{n-j}}{[n]_q!} = \sum_{n=0}^{\infty}(\sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}) \frac{1}{[n]_q!} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{[n]_q!}$$
$$= e_q^{x+y}.$$

□

Remark 4.12. *Due to the fact that x, y are not q-commutativity, we generally do not have*

$$e_q^x e_q^y \neq e_q^y e_q^x.$$

Definition 4.12. *For any positive integer n, the $\frac{1}{q}$-analogue of n is*

$$[n]_{\frac{1}{q}} := \frac{1-(\frac{1}{q})^n}{1-\frac{1}{q}} = \frac{(q^n-1)q}{(q-1)q^n} = \frac{1}{q^{n-1}}[n]_q. \tag{4.48}$$

Corollary 4.4. *The q-exponential functions e_q^x and E_q^x have the following relations*

$$\textbf{\textit{(i)}}\quad e_q^x E_q^{-x} = 1, \tag{4.49}$$

$$\textbf{\textit{(ii)}}\quad e_{\frac{1}{q}}^x = E_q^x. \tag{4.50}$$

Proof. (i) By Definition 4.10 and 4.11, we can write

$$\frac{1}{E_q^{-x}} = \frac{1}{(1-(1-q)x)_q^{\infty}} = e_q^x. \tag{4.51}$$

(ii) Consider $e^x_{1/q}$.

$$
\begin{aligned}
e^x_{\frac{1}{q}} &= \sum_{j=0}^{\infty} \frac{x^j}{[j]!_{\frac{1}{q}}} = \sum_{j=0}^{\infty} \frac{x^j}{[j]!_{\frac{1}{q}}[j-1]!_{\frac{1}{q}}...[1]!_{\frac{1}{q}}} = \sum_{j=0}^{\infty} \frac{x^j}{\frac{1-\frac{1}{q^j}}{1-\frac{1}{q}}\frac{1-\frac{1}{q^{j-1}}}{1-\frac{1}{q}}...\frac{1-\frac{1}{q}}{1-\frac{1}{q}}} \\
&= \sum_{j=0}^{\infty} \frac{(1-\frac{1}{q})^j x^j}{(1-\frac{1}{q^j})(1-\frac{1}{q^{j-1}})...(1-\frac{1}{q})} \\
&= \sum_{j=0}^{\infty} \frac{(q-1)^j x^j q^j q^{j-1}...q}{(q^j-1)(q^{j-1}-1)...(q-1)q^j} = \sum_{j=0}^{\infty} \frac{1}{[j]_q!} x^j q^{\frac{j(j-1)}{2}} = E^x_q.
\end{aligned}
$$

□

Example 4.4. *Let us find the solution for the following initial value problem*

$$
\begin{aligned}
\Delta^x_q y(x) &= ay(x) \qquad (4.52)\\
y(0) &= 1.
\end{aligned}
$$

Solution. By the Definition of q-derivative (4.2), we have

$$\Delta^x_q y(x) = \frac{y(qx)-y(x)}{(q-1)x} = ay(x)$$

which implies

$$y(qx) = y(x) + a(q-1)xy(x). \qquad (4.53)$$

Let us assume that $y(x) = \sum_{n=0}^{\infty} c_n x^n$ be the solution. Substituting this in (4.53) we get

$$\sum_{n=0}^{\infty} c_n (qx)^n = \sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} (q-1)ac_n x^{n+1}. \qquad (4.54)$$

Expanding both sides we get

$$(c_0 + c_1 qx + c_2 q^2 x^2 + ...) = (c_0 + c_1 x + c_2 x^2 + ...) + (q-1)ac_0 x + (q-1)ac_1 x^2 + ...$$

Let us investigate each coefficient of x^i, $i \geq 1$. We obtain

$$c_1 q = c_1 + (q-1)ac_0$$

for $i = 1$ which implies that

$$c_1 = ac_0.$$

For $i = 2$, we have

$$c_2 q^2 + c_2 = (q-1)ac_1,$$

which implies

$$c_2 = \frac{(q-1)a^2 c_0}{(q^2-1)}.$$

For $i = 3$, we get

$$c_3 = \frac{(q-1)^2 a^3}{(q^2-1)(q^3-1)} c_0 = \frac{q-1}{q^3-1} \frac{q-1}{q^2-1} a^3 c_0.$$

So we can guess the form of c_n as

$$c_n = c_0 \Big(\prod_{k=1}^{n} \frac{1-q}{1-q^k} \Big) a^n. \tag{4.55}$$

Let us check whether (4.55) is true or not by using the mathematical induction on n. The basis step is for $n = 1$,

$$c_1 = c_0 \frac{1-q}{1-q} a.$$

Then, $c_1 = c_0 a$ is true. Now, assume (4.55) holds for $n = k$. We will show that it also holds for $n = k+1$. We can write

$$c_{k+1} q^{k+1} = c_{k+1} + (q-1)ac_k$$

which implies

$$c_{k+1} = \frac{1-q}{1-q^{k+1}} ac_k.$$

By inductive hypothesis,

$$c_{k+1} = \frac{1-q}{1-q^{k+1}} a c_0 \prod_{j=1}^{k} \frac{1-q}{1-q^j} a^k = \prod_{j=1}^{k+1} \frac{1-q}{1-q^j} a^{k+1} c_0.$$

Hence (4.55) is true for all $n \geq 0$. By setting $n = k+1$, we obtain

$$c_n = c_0 \prod_{j=1}^{n} \frac{1-q}{1-q^j} a^n = c_0 \frac{1-q}{1-q} \frac{1-q}{1-q^2} \cdots \frac{1-q}{1-q^n} a^n = c_0 \frac{1}{[1]_q} \frac{1}{[2]_q} \cdots \frac{1}{[n]_q} a^n = c_0 \frac{a^n}{[n]_q!}.$$

Thus, we have the solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_0 \frac{a^n}{[n]_q!} x^n = c_0 \sum_{n=0}^{\infty} \frac{(ax)^n}{[n]_q!}.$$

Imposing the initial condition $y(0) = 1$ we get

$$y(x) = \sum_{n=0}^{\infty} \frac{(ax)^n}{[n]_q!}. \tag{4.56}$$

Note that one can write (4.56) in the form of q-exponential

$$y(x) = \sum_{n=0}^{\infty} \frac{(ax)^n}{[n]_q!} = e_q^{ax}.$$

Hence, the initial value problem (4.52) has a unique solution $y(x) = e_q^{qx}$.

Remark 4.13. *Example 4.4 is q-analogue of the problem*

$$y'(x) = ax,$$
$$y(0) = 1$$

whose unique solution is $y(x) = e^{ax}$.

By Example 4.4, we can give the following proposition.

Proposition 4.9. *We have*

$$\Delta_q^x e_q^{ax} = a e_q^{ax}, \tag{4.57}$$

where a is any constant.

Example 4.5. *Let us consider the initial value problem*

$$\begin{aligned}\Delta_q^x y(x) &= ay(qx), \qquad (4.58)\\ y(0) &= 1.\end{aligned}$$

Solution. Using q-derivative Definition 4.2, we have

$$\Delta_q^x y(x) = \frac{y(qx) - y(x)}{(q-1)x} = ay(qx),$$

and this implies

$$y(x) = y(qx) - a(q-1)xy(qx). \qquad (4.59)$$

Setting $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we get

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n (qx)^n - \sum_{n=0}^{\infty} (q-1)aq^n x^{n+1} c_n. \qquad (4.60)$$

Investigating each coefficient of x^i, $i \geq 1$, for $i = 1$, we have

$$c_1 = c_1 q + (1-q)ac_0$$

equivalently $c_1 = ac_0$.

For $i = 2$, we have

$$c_2 = c_2 q^2 + (1-q)qac_1$$

which implies

$$c_2 = \frac{(1-q)}{(1-q^2)} qa^2 c_0.$$

Intuitively we have the form of coefficients c_n as follows

$$c_n = c_0 \left(\prod_{k=1}^{n} \frac{1-q}{1-q^k}\right) q^{\frac{n^2-n}{2}} a^n. \qquad (4.61)$$

Similarly by induction on n, we can prove (4.61). The basis step is obvious. Assume (4.61) is valid for $n = k$. Now let us consider for $n = k + 1$. Then we have

$$c_{k+1} = c_{k+1}q^{k+1} + (1-q)q^k a c_k$$

whic implies

$$\begin{aligned} c_{k+1} =& \frac{1-q}{1-q^{k+1}} q a c_k = \frac{1-q}{1-q^{k+1}} q^k a c_0 \prod_{j=1}^{k} \frac{1-q}{1-q^j} q^{\frac{k^2-k}{2}} a^k \\ =& c_0 \prod_{j=1}^{k+1} \frac{1-q}{1-q^j} a^{k+1} q^{\frac{k^2+k}{2}} = c_0 \prod_{j=1}^{k+1} \frac{1-q}{1-q^j} a^{k+1} q^{\frac{(k+1)^2-(k+1)}{2}}. \end{aligned}$$

Therefore (4.61) is true for all $n \geq 0$. Another way to write (4.61) is that

$$\begin{aligned} c_n =& c_0 \prod_{j=1}^{n} \frac{1-q}{1-q^j} q^{\frac{n^2-n}{2}} a^n = c_0 \frac{q-1}{q-1} . \frac{q-1}{q^2-1} ... \frac{q-1}{q^n-1} q^{\frac{n(n-1)}{2}} a^n \\ =& c_0 \frac{q-1}{q-1} . \frac{q-1}{q^2-1} ... \frac{q-1}{q^n-1} q^{1+2+3+...+(n-1)} a^n \\ =& c_0 \frac{(q-1)q}{(q-1)q} . \frac{(q-1)q^2}{(q^2-1)q} ... \frac{(q-1)q^n}{(q^n-1)q} a^n = c_0 \frac{a^n}{[n]!_{\frac{1}{q}}}. \end{aligned}$$

Thus, we have the solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_0 a^n \frac{x^n}{[n]!_{\frac{1}{q}}}.$$

Employing the initial condition $y(0) = 1$, we obtain the unique solution for inital value problem (4.58) as

$$y(x) = E_q^{ax}. \tag{4.62}$$

Here note that we used the below equality

$$\sum_{n=0}^{\infty} \frac{x^n}{[n]!_{\frac{1}{q}}} = e_{\frac{1}{q}}^{ax} = E_q^{ax}. \tag{4.63}$$

By Example 4.5, we can give the following proposition.

Proposition 4.10. *We have*

$$\Delta_q^x E_q^{ax} = aE_q^{ax}, \tag{4.64}$$

where a is any constant.

Definition 4.13. *The q-trigonometric functions are defined as follows*

$$\sin_q x := \frac{e_q^{ix} - e_q^{-ix}}{2i}, \tag{4.65}$$

$$\cos_q x := \frac{e_q^{ix} + e_q^{-ix}}{2}, \tag{4.66}$$

$$\mathrm{Sin}_q x := \frac{E_q^{ix} - E_q^{-ix}}{2i}, \tag{4.67}$$

$$\mathrm{Cos}_q x := \frac{E_q^{ix} + E_q^{-ix}}{2}. \tag{4.68}$$

Corollary 4.5. *We have the following identities:*

$$\mathrm{Sin}_q x = \sin_{\frac{1}{q}} x \quad \textit{and} \quad \mathrm{Cos}_q x = \cos_{\frac{1}{q}} x.$$

Proof. Since $E_q^x = e_{\frac{1}{q}} x$, we get

$$\mathrm{Sin}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2i} = \frac{e_{\frac{1}{q}}^{ix} - e_{\frac{1}{q}}^{-ix}}{2i} = \sin_{\frac{1}{q}} x.$$

Similarly we get

$$\mathrm{Cos}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2} = \frac{e_{\frac{1}{q}}^{ix} - e_{\frac{1}{q}}^{-ix}}{2} = \cos_{\frac{1}{q}} x.$$

□

Definition 4.14. *We have also*

$$\mathrm{Sin}_{\frac{1}{q}} x := \frac{E_{\frac{1}{q}}^{ix} - E_{\frac{1}{q}}^{-ix}}{2i}, \tag{4.69}$$

$$\mathrm{Cos}_{\frac{1}{q}} x := \frac{E_{\frac{1}{q}}^{ix} + E_{\frac{1}{q}}^{-ix}}{2}. \tag{4.70}$$

Remark 4.14. *Since $e_q^x e_q^y = e_q^{x+y}$ only if $yx = qxy$, the additive property does not* *hold for q-trigonometric functions in general but we can develop identities for other combinations.*

Proposition 4.11. *We have the following identity*

$$\cos_q x \operatorname{Cos}_q x + \sin_q x \operatorname{Sin}_q x = 1. \tag{4.71}$$

Proof. By Definition 4.13, consider $\cos_q x \operatorname{Cos}_q x + \sin_q x \operatorname{Sin}_q x$,

$$\begin{aligned}\cos_q x \operatorname{Cos}_q x + \sin_q x \operatorname{Sin}_q x =& \frac{e_q^{ix} + e_q^{-ix}}{2}\frac{E_q^{ix} + E_q^{-ix}}{2} + \frac{e_q^{ix} - e_q^{-ix}}{2i}\frac{E_q^{ix} - E_q^{-ix}}{2i} \\ =& \frac{2e_q^{ix}E_q^{-ix} + 2e_q^{-ix}E_q^{ix}}{4}.\end{aligned}$$

Since $e_q^x E_q^{-x} = 1$, we obtain

$$\cos_q x \operatorname{Cos}_q x + \sin_q x \operatorname{Sin}_q x = 1.$$

□

Note that the q-analogue of the identity $\sin^2 x + \cos^2 x = 1$ is

$$\cos_q x \operatorname{Cos}_q x + \sin_q x \operatorname{Sin}_q x = 1.$$

Proposition 4.12. *The q-derivatives of the q-trigonometric functions are as follows*

$$\textbf{\textit{(i)}} \quad \Delta_q^x \sin_q x = \cos_q x, \tag{4.72}$$

$$\textbf{\textit{(ii)}} \quad \Delta_q^x \cos_q x = -\sin_q x, \tag{4.73}$$

$$\textbf{\textit{(iii)}} \quad \Delta_q^x \operatorname{Sin}_q x = \operatorname{Cos}_q x, \tag{4.74}$$

$$\textbf{\textit{(iv)}} \quad \Delta_q^x \operatorname{Cos}_q x = -\operatorname{Sin}_q x. \tag{4.75}$$

Proof. (i) Consider the q-derivative definition of $\sin_q x$. By Proposition 4.9, we get

$$\Delta_q^x \sin_q x = \Delta_q^x\Big(\frac{e_q^{ix} - e_q^{-ix}}{2i}\Big) = \frac{ie_q^{ix} + ie_q^{-ix}}{2i} = \frac{e_q^{ix} + e_q^{-ix}}{2} = \cos_q x.$$

(ii) Similarly, consider

$$\Delta_q^x \mathrm{cos_q} x = \Delta_q^x \Big(\frac{e_q^{ix} + e_q^{-ix}}{2} \Big) = \frac{ie_q^{ix} - ie_q^{-ix}}{2} = \frac{-e_q^{ix} + e_q^{-ix}}{2i} = -\mathrm{sin_q} x.$$

(iii) By Corollary 4.5, we get

$$\Delta_q^x \mathrm{Sin_q} x = \Delta_q^x \mathrm{sin}_{\frac{1}{q}} x = \mathrm{cos}_{\frac{1}{q}} x = \mathrm{Cos_q} x.$$

(iv) Similarly, by Corollary 4.5, we get

$$\Delta_q^x \mathrm{Cos_q} x = \Delta_q^x \mathrm{cos}_{\frac{1}{q}} x = -\mathrm{sin}_{\frac{1}{q}} x = -\mathrm{Sin_q} x.$$

□

Proposition 4.13. *We have the following relation between q-trigonometric functions:*

$$\mathrm{sin_q} x \mathrm{Cos_q} x = \mathrm{cos_q} x \mathrm{Sin_q} x.$$

Proof. Consider

$$\begin{aligned}
\mathrm{sin_q} x \mathrm{Cos_q} x - \mathrm{cos_q} x \mathrm{Sin_q} x =& \frac{e_q^{ix} - e_q^{-ix}}{2i} \frac{E_q^{ix} + E_q^{-ix}}{2} - \frac{e_q^{ix} + e_q^{-ix}}{2} \frac{E_q^{ix} - E_q^{-ix}}{2i} \\
=& \frac{e_q^{ix} E_q^{ix} + e_q^{ix} E_q^{-ix} - e_q^{-ix} E_q^{ix} - e_q^{-ix} E_q^{-ix}}{4i} \\
-& \frac{e_q^{ix} E_q^{ix} - e_q^{ix} E_q^{-ix} - e_q^{-ix} E_q^{ix} + e_q^{-ix} E_q^{-ix}}{4i} = 0.
\end{aligned}$$

Namely, we have $\mathrm{sin_q} x \mathrm{Cos_q} x - \mathrm{cos_q} x \mathrm{Sin_q} x = 0$. Thus

$$\mathrm{sin_q} x \mathrm{Cos_q} x = \mathrm{cos_q} x \mathrm{Sin_q} x.$$

□

Remark 4.15. *Note that by Proposition 4.13, the identity*

$$\sin x \cos x + \cos x \sin x = \sin 2x$$

does not hold in quantum case.

4.5 Jackson's integral

In this section, we will construct $F(x)$ as q-antiderivative of $f(x)$. Let $f(x)$ be an arbitrary function and $F(x)$ be q-antiderivative of $f(x)$, i.e.

$$\Delta_q^x F(x) = f(x).$$

To construct q-antiderivative $F(x)$, recall the operator

$$\hat{M}_q F(x) = F(qx).$$

Using the q-derivative definition of $F(x)$, we obtain

$$\Delta_q^x F(x) = \frac{F(qx) - F(x)}{(q-1)x}. \tag{4.76}$$

Rewriting (4.76) using the operator $\hat{M}_q$, we get

$$\Delta_q^x F(x) = \frac{\hat{M}_q F(x) - F(x)}{(q-1)x} = \frac{(\hat{M}_q - I)F(x)}{(q-1)x}.$$

Since $F(x)$ is q-antiderivative of $f(x)$, we can write

$$(\hat{M}_q - I)F(x) = f(x)(q-1)x. \tag{4.77}$$

Then apply $(I - \hat{M}_q)^{-1}$ to both sides of (4.77), we get

$$(I - \hat{M}_q)^{-1}(\hat{M}_q - I)F(x) = (q-1)(I - \hat{M}_q)^{-1} x f(x)$$

which implies

$$F(x) = (1-q)(I - \hat{M}_q)^{-1} x f(x). \tag{4.78}$$

Utilizing geometric series, we can rewrite (4.78) as

$$F(x) = (1-q)\frac{1}{1-\hat{M}_q}xf(x) = (1-q)\sum_{n=0}^{\infty}(\hat{M}_q)^n(xf(x)),$$

where $\hat{M}_q^n$ is nth order of operator s.t. $\hat{M}_q^n = \hat{M}_q(\hat{M}_q^{n-1})$. Note that we have

$$(\hat{M}_q)^n(xf(x)) = q^n xf(q^n x), \ \ \forall n \geq 0.$$

Then $F(x)$ becomes

$$F(x) = (1-q)\sum_{n=0}^{\infty} q^n xf(q^n x). \tag{4.79}$$

Hence, we can give the following definition.

Definition 4.15. *Let f be an arbitrary function. The q-antiderivative $F(x)$ of $f(x)$ is*

$$F(x) := \int f(x)d_q x = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x). \tag{4.80}$$

This series is also called Jackson's integral of $f(x)$, (Kac, 2001).

We will give the following theorem to understand under what conditions Jackson integral converges to q-antiderivative of $f(x)$.

Theorem 4.5. *Suppose $0 < q < 1$. If $|f(x)x^{\alpha}|$ is bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then Jackson integral defined by* (4.80) *converges to q-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.*

Proof. Aim 1: Jackson integral converges to $F(x)$.
Let $|f(x)x^{\alpha}|$ be bounded on $(0, A]$. So, there exists $M \in (0, A]$ such that

$$|f(x)x^{\alpha}| < M.$$

For any $0 < x \leq A$, $j \geq 0$, we can write

$$|q^j f(q^j x)| < q^j M(q^j x)^{-\alpha} = x^{-\alpha} M q^{j(1-\alpha)}$$

since $|f(x)| < Mx^{-\alpha}$ and $0 < q < 1$. Hence, we get

$$|q^j f(q^j x)| < Mx^{-\alpha}(q^{1-\alpha})^j.$$

Consider the series

$$\sum_{j=0}^{\infty} Mx^{-\alpha}(q^{1-\alpha})^j. \tag{4.81}$$

Since $1 - \alpha > 0$ and $0 < q < 1$, we have $|q^{1-\alpha}| < 1$. So (4.81) is a geometric series and so it is convergent. By comparison test,

$$\sum_{j=0}^{\infty} |q^j f(q^j x)| \quad \text{is convergent.}$$

So, the right hand side of (4.80) converges pointwise to a function $F(x)$ that is,

$$|F(x)| = |(1-q)x \sum_{j=0}^{\infty} q^j f(q^j x)|.$$

<u>Aim 2:</u> $F(x)$ is continuous at $x = 0$, i.e. $\lim_{x\to 0} F(x) = F(0)$.
Obviously $F(0) = 0$. For all $\epsilon > 0$, suppose $|x - 0| < \delta$, such that $|x| < \delta$. We obtain

$$\begin{aligned} |F(x)| =& |(1-q)x \sum_{j=0}^{\infty} q^j f(q^j x)| < (1-q)|x| \sum_{j=0}^{\infty} |q^j f(q^j x)| \\ <& (1-q)|x| \sum_{j=0}^{\infty} Mx^{-\alpha}(q^{1-\alpha})^j = (1-q)|x| \frac{Mx^{-\alpha}}{1-q^{1-\alpha}} \\ =& (1-q)M \frac{x^{1-\alpha}}{1-q^{1-\alpha}} < (1-q)M \frac{\delta^{1-\alpha}}{1-q^{1-\alpha}} < \epsilon. \end{aligned}$$

Since $|q^{1-\alpha}| < 1$. Thus for every $\epsilon > 0$, there exist a $\delta > 0$ such that $|x| = x < \delta$, we have $|F(x) - F(0)| < \epsilon$.

Aim 3: $F(x)$ is a q-antiderivative of $f(x)$, i.e. $\Delta_q^x F(x) = f(x)$.

$$\begin{aligned}\Delta_q^x F(x) =&\Delta_q^x[(1-q)x\sum_{j=0}^{\infty} q^j f(q^j x)] = (1-q)\sum_{j=0}^{\infty} q^j \Delta_q^x[xf(q^j x)]\\ =&(1-q)[\sum_{j=0}^{\infty} q^j x\Delta_q^x f(q^j x) + \sum_{j=0}^{\infty} q^j f(q^{j+1}x)\Delta_q^x x]\\ =&\sum_{j=0}^{\infty} q^j[f(q^{j+1}x) - f(q^j x)] - (1-q)\sum_{j=0}^{\infty} q^j f(q^{j+1}x)\\ =&\sum_{j=0}^{\infty} q^{j+1} f(q^{j+1}x) - \sum_{j=0}^{\infty} q^j f(q^j x) = \sum_{j=1}^{\infty} q^j f(q^j x) - \sum_{j=0}^{\infty} q^j f(q^j x)\\ =&q^0 f(q^0 x) = f(x)\end{aligned}$$

□

Definition 4.16. *Suppose* $0 < a < b$*. The definite* q*-integral is defined as*

$$\int_0^b f(x)d_qx := (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b) \tag{4.82}$$

and we have also

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{4.83}$$

4.6 Fundamental Theorem of *q*-Calculus

Theorem 4.6. *Let* $F(x)$ *be* q*-antiderivative of* $f(x)$*, i.e.* $F(x) = \int_a^x f(t)d_qt$ *and* F *is continuous at* $x = 0$*. Then we have*

$$\int_a^b f(x)d_qx = F(b) - F(a), \tag{4.84}$$

where $0 \leq a < b$ *.*

Proof. Since $F(x)$ is continuous at $x = 0$, by Definition 4.5, we have

$$F(x) = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x) + F(0).. \tag{4.85}$$

By the equation (4.83), we have

$$\begin{aligned}\int_a^b f(x)d_qx = &\int_0^b f(x)d_qx - \int_0^a f(x)d_qx = (1-q)[b\sum_{n=0}^{\infty} q^n f(q^n x) - a\sum_{n=0}^{\infty} q^n f(q^n a)]\\ =&[F(b)-F(0)] - [(F(a)-F(0))] = F(b)-F(a).\end{aligned}$$

□

Corollary 4.6. *If $f'(x)$ exists in a neighborhood of $x = 0$ and is continuous at $x = 0$, then we have*

$$\int_a^b \Delta_q^x f(x)d_qx = f(b) - f(a). \tag{4.86}$$

Proof. The aim is to show that $\Delta_q^x f(x)$ is continuous. Consider

$$\lim_{x\to 0}\Delta_q^x f(x) = \lim_{x\to 0}\frac{f(qx)-f(x)}{(q-1)x} = \lim_{x\to 0}\frac{qf'(x)-f'(x)}{(q-1)} = f'(0).$$

Since $\Delta_q^x f(x)$ is continuous at $x = 0$, we can use the Fundamental Theorem of q-Calculus by setting $\Delta_q^x f(x)$ instead of $f(x)$. □

4.7 $\frac{1}{q}$-derivative and its properties

For the $\frac{1}{q}$-derivative and its related properties, q is replaced by $\frac{1}{q}$ and we get the following definitions and results.

Definition 4.17. *The $\frac{1}{q}$-derivative of the function $f(x)$ is defined as*

$$\Delta_{\frac{1}{q}}^x f(x) := \frac{f(\frac{1}{q}x) - f(x)}{(\frac{1}{q}-1)x}, \quad x \in \mathbb{K}_q. \tag{4.87}$$

Example 4.6. *For $n \geq 1$, the $\frac{1}{q}$-derivative of $f(x) = x^n$ is*

$$\Delta^x_{\frac{1}{q}} x^n = \frac{(\frac{x}{q})^n - x^n}{(\frac{1}{q} - 1)x} = \frac{(x^n - q^n x^n)q}{(1-q)q^n x} = \frac{x^n(1-q^n)}{q^{n-1}(1-q)x} = \frac{1}{q^{n-1}}[n]_q x^{n-1}. \tag{4.88}$$

Remark 4.16. *By Definition* 4.12*, the equation* (4.88) *becomes*

$$\Delta^x_{\frac{1}{q}} x^n = [n]_{\frac{1}{q}} x^{n-1}.$$

Proposition 4.14. *The operator $\Delta_{\frac{1}{q}} : \mathbb{K}_{\frac{1}{q}} \to \mathbb{K}_{\frac{1}{q}}$ is linear.*

Proposition 4.15. *Let f, g be arbitrary functions whose $\frac{1}{q}$- derivatives exist. Then $\frac{1}{q}$-product rule arises in two forms:*

$$\textbf{\textit{(i)}} \quad \Delta^x_{\frac{1}{q}}[f(x)g(x)] = f(\frac{1}{q}x)\Delta^x_{\frac{1}{q}} g(x) + g(x)\Delta^x_{\frac{1}{q}} f(x) \tag{4.89}$$

$$\textbf{\textit{(ii)}} \quad \Delta^x_{\frac{1}{q}}[f(x)g(x)] = g(\frac{1}{q}x)\Delta^x_{\frac{1}{q}} f(x) + f(x)\Delta^x_{\frac{1}{q}} g(x). \tag{4.90}$$

Proposition 4.16. *Let f, g be arbitrary functions whose $\frac{1}{q}$-derivatives exist. Then $\frac{1}{q}$-quoetient rule arises in two forms:*

$$\textbf{\textit{(i)}} \quad \Delta^x_{\frac{1}{q}}[\frac{f(x)}{g(x)}] = \frac{g(x)\Delta^x_{\frac{1}{q}} f(x) - f(x)\Delta^x_{\frac{1}{q}} g(x)}{g(x)g(\frac{1}{q}x)} \tag{4.91}$$

$$\textbf{\textit{(ii)}} \quad \Delta^x_{\frac{1}{q}}[\frac{f(x)}{g(x)}] = \frac{g(\frac{1}{q}x)\Delta^x_{\frac{1}{q}} f(x) - f(\frac{1}{q}x)\Delta^x_{\frac{1}{q}} g(x)}{g(x)g(\frac{1}{q}x)}. \tag{4.92}$$

Example 4.7. *Let $f(x,t) = (x + q^n ct)$, where c is any arbitrary constant. Let us compute $\Delta^t_{\frac{1}{q}} f(x,t)$. By Definition* 4.17*, we have*

$$\Delta^t_{\frac{1}{q}}(x + q^n ct) = \frac{[x + q^{n-1}ct)] - [x + q^n ct]}{(\frac{1}{q} - 1)t} = \frac{q^n ct(q^{-1} - 1)}{(\frac{1}{q} - 1)t} = q^n c.$$

4.8 $\frac{1}{q}$-binomial and $\frac{1}{q}$-Taylor's Formula

We introduce another q-binomial as follows

Definition 4.18. *The $\frac{1}{q}$-analogue of a binomial $(x \pm ct)^n$ is a polynomial as*

$$(x \pm ct)^n_{\frac{1}{q}} := (x \pm ct)(x \pm q^{-1}ct)(x \pm q^{-2}ct)...(x \pm q^{-(n-1)}ct), \tag{4.93}$$

where $(x \pm ct)^0_{\frac{1}{q}} = 1$ and c is any constant. Also $(x+0)^n_{\frac{1}{q}} = x^n$.

Remark 4.17. *Note that*

$$\lim_{q \to 1}(x \pm ct)^n_{\frac{1}{q}} = (x \pm ct)^n.$$

Proposition 4.17. *For any integer m, n, we have*

$$(x - ct)^{m+n}_{\frac{1}{q}} = (x - ct)^m_{\frac{1}{q}}(x - (q^{-1})^m ct)^n_{\frac{1}{q}}. \tag{4.94}$$

Proof. The proof is similar to the one given in q-version. □

Remark 4.18. *One can write $\frac{1}{q}$-Taylor formula similar to the Theorem 4.3 that is*

$$f(x) = \sum_{j=0}^{N} (\Delta^x_{\frac{1}{q}})^j(c)\frac{(x-c)^j_q}{[j]_{\frac{1}{q}}!}, \tag{4.95}$$

where c is any constant and f is polynomial of degree N.

Remark 4.19. *We have already given $e^x_{\frac{1}{q}} = E^x_q$ in Corollary 4.4 and $\sin_{\frac{1}{q}} x$, $\cos_{\frac{1}{q}} x$ in Corollary 4.5. Similar to properties of q-analysis, one can discuss*

$$E^x_{\frac{1}{q}} = e^x_q, \tag{4.96}$$

$$\Delta^x_{\frac{1}{q}} e^x_{\frac{1}{q}} = e^x_{\frac{1}{q}}, \tag{4.97}$$

$$\Delta^x_{\frac{1}{q}} \sin_{\frac{1}{q}} x = \cos_{\frac{1}{q}} x, \tag{4.98}$$

$$\Delta^x_{\frac{1}{q}} \cos_{\frac{1}{q}} x = -\sin_{\frac{1}{q}} x, \tag{4.99}$$

$$\Delta^x_{\frac{1}{q}} \mathrm{Sin}_{\frac{1}{q}} x = \mathrm{Cos}_{\frac{1}{q}} x, \tag{4.100}$$

$$\Delta^x_{\frac{1}{q}} \mathrm{Cos}_{\frac{1}{q}} x = -\mathrm{Sin}_{\frac{1}{q}} x. \tag{4.101}$$

4.9 $\frac{1}{q}$-antiderivative and its properties

Let us construct $F(x)$ as $\frac{1}{q}$-antiderivative of $f(x)$. Recall the operator

$$\hat{M}_{\frac{1}{q}}F(x) = F(q^{-1}x).$$

Consider

$$\Delta^x_{\frac{1}{q}}F(x) = \frac{F(q^{-1}x) - F(x)}{(q^{-1}-1)x} = \frac{\hat{M}_{\frac{1}{q}}F(x) - F(x)}{(q^{-1}-1)x}. \tag{4.102}$$

Since $F(x)$ is $\frac{1}{q}$-antiderivative of $f(x)$, rewrite (4.102) and we get

$$(\hat{M}_{\frac{1}{q}} - I)F(x) = (q^{-1}-1)xf(x).$$

which implies

$$F(x) = (q^{-1}-1)(\hat{M}_{\frac{1}{q}} - I)^{-1}(xf(x)).$$

Similar to arguments in q-antiderivative, using geometric series we obtain

$$F(x) = (1-q^{-1})\sum_{n=0}^{\infty}(\hat{M}_{\frac{1}{q}})^n(xf(x)).$$

Note that $(\hat{M}_{\frac{1}{q}})^n(xf(x)) = q^{-n}xf(q^{-n}x), \quad \forall n \geq 0$. Thus

$$F(x) = (1-q^{-1})\sum_{n=0}^{\infty}(\hat{M}_{\frac{1}{q}})^n(xf(x)) = (1-q^{-1})x\sum_{n=0}^{\infty}q^{-n}f(q^{-n}x).$$

So, we can give the following definition;

Definition 4.19. *The $\frac{1}{q}$-antiderivative of $f(x)$ denoted by $F(x)$ is defined by*

$$F(x) := \int f(x)d_{\frac{1}{q}}x = (1-q^{-1})x\sum_{n=0}^{\infty}q^{-n}f(q^{-n}x). \tag{4.103}$$

Theorem 4.7. *Suppose $0 < \frac{1}{q} < 1$. If $|f(x)x^{\alpha}|$ is bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then $\frac{1}{q}$-integral defined by* (4.103) *converges to $\frac{1}{q}$-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.*

Proof. The proof is so similar to the proof of given in q-version. □

Definition 4.20. *Suppose $0 < a < b$. The definite $\frac{1}{q}$-integral is defined as*

$$\int_0^b f(x)d_{\frac{1}{q}}x := (1-q^{-1})b\sum_{j=0}^{\infty} q^{-j}f(q^{-j}b) \tag{4.104}$$

and we have also

$$\int_a^b f(x)d_{\frac{1}{q}}x := \int_0^b f(x)d_{\frac{1}{q}}x - \int_0^a f(x)d_{\frac{1}{q}}x. \tag{4.105}$$

4.10 Fundamental Theorem of $\frac{1}{q}$-Calculus

Theorem 4.8. *Let $F(x)$ be $\frac{1}{q}$-antiderivative of $f(x)$, i.e. $F(x) = \int_a^x f(t)d_{\frac{1}{q}}t$, and F is continuous at $x = 0$, then we have*

$$\int_a^b f(x)d_{\frac{1}{q}}x = F(b) - F(a) \tag{4.106}$$

where $0 \leq a < b$.

Proof is similar to q-version of Fundamental Theorem of Calculus.

Corollary 4.7. *Let $f'(x)$ exist in a neighborhood of $x = 0$ and is continuous at $x = 0$, then we have*

$$\int_a^b \Delta^x_{\frac{1}{q}} f(x)d_{\frac{1}{q}}x = f(b) - f(a). \tag{4.107}$$

CHAPTER FIVE
q -CAUCHY PROBLEMS

In this Chapter, we show properties of q-binomial and $\frac{1}{q}$-binomial. Then we construct $\frac{1}{q}$-wave equation which tends to usual wave equation as $q \to 1$. Next, we introduce $\frac{1}{q}$-initial value (or $\frac{1}{q}$-D'Alembert) problems and we find their solutions in terms of $\frac{1}{q}$-binomials. Then by defining $\frac{1}{q}$-analyticity and stating $\frac{1}{q}$-analogue of Cauchy Kovalevskaya Theorem, we examine the uniqueness and analyticity of solutions. Next, we give another q-discretized initial value problem, that is q-D'Alembert problems and we reconstruct their solution from Nalcı Tumer, 2017. Finally, we introduce a general result that combines these two Cauchy problems in quantum analysis.

Theorem 5.1. *The q-derivative of* $(x+ct)^n_{\frac{1}{q}}$ *with respect to* t *is*

$$\Delta_q^t (x+ct)^n_{\frac{1}{q}} = c[n]_{\frac{1}{q}} (x+ct)^{n-1}_{\frac{1}{q}}, \quad n \in \mathbb{Z}. \tag{5.1}$$

Proof. Case (i) For $n \geq 0$. We will prove by induction on n. The basis step is obvious. Assume (5.1) is true for $n = k$, we will show that (5.1) is also true for $n = k+1$. Consider

$$\begin{aligned}
\Delta_q^t (x+ct)^{k+1}_{\frac{1}{q}} =& \Delta_q^t [(x+ct)^k_{\frac{1}{q}} (x + q^{-k} ct)] \\
=& [\Delta_q^t (x+ct)^k_{\frac{1}{q}}](x + q^{-k} cqt) + (x+ct)^k_{\frac{1}{q}} \Delta_q^t (x + q^{-k} ct) \\
=& c[k]_{\frac{1}{q}} (x+ct)^{k-1}_{\frac{1}{q}} (x + q^{-(k-1)} ct) + (x+ct)^k_{\frac{1}{q}} q^{-k} c \\
=& c(1 + q^{-1} + ... + (q^{-1})^{k-1})(x+ct)^k_{\frac{1}{q}} + (x+ct)^k_{\frac{1}{q}} q^{-k} c \\
=& c(x+ct)^k_{\frac{1}{q}} (1 + q^{-1} + ... + (q^{-1})^{k-1} + (q^{-1})^k) = c[k+1]_{\frac{1}{q}} (x+ct)^k_{\frac{1}{q}}.
\end{aligned}$$

Here we used the q-product rule (4.6) and the statement of the assumption. Thus, (5.1) is true for $n = k+1$.

Case (ii) For $n < 0$. Say $n = -n' < 0$, then

$$\begin{aligned}
\Delta_q^t (x+ct)_{\frac{1}{q}}^{n} =& \Delta_q^t (x+ct)_{\frac{1}{q}}^{-n'} = \Delta_q^t \frac{1}{(x+(q^{-1})^{-n'})ct)_{\frac{1}{q}}^{n'}} = \frac{-\Delta_q^t (x+q^{n'}ct)_{\frac{1}{q}}^{n'}}{(x+q^{n'}ct)_{\frac{1}{q}}^{n'}(x+q^{n'}cqt)_{\frac{1}{q}}^{n'}} \\
=& \frac{-q^{n'}c[n']_{\frac{1}{q}}(x+q^{n'}ct)_{\frac{1}{q}}^{n'-1}}{(x+q^{n'}ct)_{\frac{1}{q}}^{n'}(x+q^{n'+1}ct)_{\frac{1}{q}}^{n'}} = \frac{-q^{n'}c\frac{(q^{-1})^{n'}-1}{q^{-1}-1}}{(x+q^{n'}(q^{-1})^{n'-1}ct)(x+q^{n'+1}ct)_{\frac{1}{q}}^{n'}} \\
=& c\frac{q^{n'}-1}{q^{-1}-1}\frac{1}{(x+(q^{-1})^{-(n'+1)}ct)_{\frac{1}{q}}^{n'+1}} = c\frac{q^{n'}-1}{q^{-1}-1}(x+ct)_{\frac{1}{q}}^{-n'-1} \\
=& c\frac{q^{-n}-1}{q^{-1}-1}(x+ct)_{\frac{1}{q}}^{n-1} = c[n]_{\frac{1}{q}}(x+ct)_{\frac{1}{q}}^{n-1}.
\end{aligned}$$

Here we used the q-quotient rule (4.8) and the fact $n' > 0$. □

Theorem 5.2. *The q-derivative of $(x+ct)_{\frac{1}{q}}^{n}$ with respect to x is*

$$\Delta_{\frac{1}{q}}^x (x+ct)_{\frac{1}{q}}^{n} = [n]_{\frac{1}{q}}(x+ct)_{\frac{1}{q}}^{n-1}, \quad n \in \mathbb{Z}. \tag{5.2}$$

Proof. Case (i) For $n \geq 0$. We will prove the proposition by induction on n. For $n = 0$, (5.2) trivially holds. We suppose that (5.2) holds for $n = k$. We will show that (5.2) also holds for $n = k+1$.

$$\begin{aligned}
\Delta_{\frac{1}{q}}^x (x+ct)_{\frac{1}{q}}^{k+1} =& \Delta_{\frac{1}{q}}^x [(x+ct)_{\frac{1}{q}}^{k}(x+(q^{-1})^k ct)] \\
=& \Delta_{\frac{1}{q}}^x (x+ct)_{\frac{1}{q}}^{k}(q^{-1}x+(q^{-1})^k ct) + (x+ct)_{\frac{1}{q}}^{k}\Delta_{\frac{1}{q}}^x (x+(q^{-1})^k ct) \\
=& [k]_{\frac{1}{q}}(x+ct)_{\frac{1}{q}}^{k-1}(q^{-1}x+(q^{-1})^k ct) + (x+ct)_{\frac{1}{q}}^{k} \\
=& q^{-1}[k]_{\frac{1}{q}}(x+ct)_{\frac{1}{q}}^{k-1}(x+(q^{-1})^{k-1}ct) + (x+ct)_{\frac{1}{q}}^{k} \\
=& q^{-1}[k]_{\frac{1}{q}}(x+ct)_{\frac{1}{q}}^{k} + (x+ct)_{\frac{1}{q}}^{k} = (x+ct)_{\frac{1}{q}}^{k}(1+q^{-1}[k]_{\frac{1}{q}}) \\
=& (x+ct)_{\frac{1}{q}}^{k}(1+q^{-1}...+(q^{-1})^k) = [k+1]_{\frac{1}{q}}(x+ct)_{\frac{1}{q}}^{k}.
\end{aligned}$$

Here we used the $\frac{1}{q}$-product rule (4.89) and the statement of the assumption. Hence, (5.2) holds for $n = k+1$.

Case (ii) For $n < 0$. Consider $n = -n' < 0$.

$$\begin{aligned}
\Delta^x_{\frac{1}{q}}(x+ct)^n_{\frac{1}{q}} =& \Delta^x_{\frac{1}{q}}(x+ct)^{-n'}_{\frac{1}{q}} = \Delta^x_{\frac{1}{q}} \frac{1}{(x+(q^{-1})^{-n'}ct)^{n'}_{\frac{1}{q}}} \\
=& \frac{-\Delta^x_{\frac{1}{q}}(x+(q^{-1})^{-n'}ct)^{n'}_{\frac{1}{q}}}{(x+(q^{-1})^{-n'}ct)^{n'}_{\frac{1}{q}}(q^{-1}x+(q^{-1})^{-n'}ct)^{n'}_{\frac{1}{q}}} \\
=& \frac{-[n']_{\frac{1}{q}}(q^{-1})^{-n'}(x+(q^{-1})^{-n'}ct)^{n'-1}_{\frac{1}{q}}}{(x+(q^{-1})^{-n'}ct)^{n'}_{\frac{1}{q}}(q^{-1}x+(q^{-1})^{-n'}ct)^{n'}_{\frac{1}{q}}} \\
=& \frac{-[n']_{\frac{1}{q}}(q^{-1})^{-n'}}{(x+(q^{-1})^{-n'+n'-1}ct)(q^{-1})^{n'}(x+(q^{-1})^{-n'-1}ct)^{n'}_{\frac{1}{q}}} \\
=& \frac{-[n']_{\frac{1}{q}}}{(x+(q^{-1})^{n'-1}ct)^{n'+1}_{\frac{1}{q}}} = -[n']_{\frac{1}{q}}(x+ct)^{-n'-1}_{\frac{1}{q}} \\
=& [n]_{\frac{1}{q}}(x+ct)^{n-1}_{\frac{1}{q}}.
\end{aligned}$$

Here we used $\frac{1}{q}$-quotient rule (4.91) and inductive hypothesis. □

Corollary 5.1. *We have the following equations*

$$(\Delta^t_q \pm c\Delta^x_{\frac{1}{q}})(x \mp ct)^n_{\frac{1}{q}} = 0, \quad n \in \mathbb{Z}.$$

Proof. By Theorem 5.1 and Theorem 5.2, clearly we have

$$(\Delta^t_q - c\Delta^x_{\frac{1}{q}})(x+ct)^n_{\frac{1}{q}} = (c[n]_{\frac{1}{q}} - c[n]_{\frac{1}{q}})(x+ct)^{n-1}_{\frac{1}{q}} = 0.$$

Similarly, we have

$$(\Delta^t_q + c\Delta^x_{\frac{1}{q}})(x-ct)^n_{\frac{1}{q}} = -c[n]_{\frac{1}{q}}(x-ct)^{n-1}_{\frac{1}{q}} + c[n]_{\frac{1}{q}}(x-ct)^{n-1}_{\frac{1}{q}} = 0.$$

□

Definition 5.1. *We define*

$$F(x \pm ct) := \sum_{n=-\infty}^{\infty} a_n (x \pm ct)^n_{\frac{1}{q}} \tag{5.3}$$

as a $\frac{1}{q}$-Laurient Series representation, where $F \in C^2(I)$, $I = \mathbb{K}_q \times \mathbb{K}_q$.

Corollary 5.2. *We can clearly state that*

$$[(\Delta_q^t)^2 - c^2(\Delta_{\frac{1}{q}}^x)^2]u(x,t) = 0 \tag{5.4}$$

has a general solution

$$u(x,t) = F(x - ct) + G(x + ct), \tag{5.5}$$

Here the functions F, G are $\frac{1}{q}$-Laurient Series representation which are defined in Definition 5.1.

Remark 5.1. *As we mention in the Introduction that, as $q \to 1$ the $\frac{1}{q}$-wave equation* (5.4) *turns out to be wave equation which defined as*

$$u_{tt} - c^2 u_{xx} = 0.$$

Theorem 5.3. *The homogenous $\frac{1}{q}$-Cauchy problem*

$$\begin{aligned}
&[(\Delta_q^t)^2 - c^2(\Delta_{\frac{1}{q}}^x)^2]u(x,t) = 0, \quad \forall x, t \in \mathbb{K}_q,\\
&u(x,0) = \Phi(x),\\
&\Delta_{\frac{1}{q}}^t u(x,0) = \Psi(x),
\end{aligned} \tag{5.6}$$

has solution

$$u(x,t) = \frac{1}{2}[\Phi(x + ct) + \Phi(x - ct)] + \frac{1}{2c}\int_a^b \Psi(s) d_{\frac{1}{q}} s, \tag{5.7}$$

where

$$\begin{aligned}
\int_a^b \Psi(s) d_{\frac{1}{q}} s &= \int_0^b \Psi(s) d_{\frac{1}{q}} s - \int_0^a \Psi(s) d_{\frac{1}{q}} s\\
&= (1 - q^{-1}) b \sum_{j=0}^{\infty} q^{-j} \Psi(q^{-j} b) - (1 - q^{-1}) a \sum_{n=0}^{\infty} q^{-j} \Psi(q^{-j} a)
\end{aligned}$$

and $a := x - ct$, $b := x + ct$. Here $x,\ t \in \mathbb{K}_q,\ \phi \in C^2(I),\ \psi \in C^1(I)$ on $I = \mathbb{K}_q \times \mathbb{K}_q$.

Proof. We know general solution for $\frac{1}{q}$-wave equation is

$$u(x,t) = F(x-ct) + G(x+ct).$$

We impose initial conditions on $u(x,t)$. Then, we get

$$\begin{aligned}
&u(x,0) = F(x) + G(x) = \phi(x),\\
&\Delta^t_{\frac{1}{q}} u(x,0) = \Delta^t_{\frac{1}{q}}[F(x-ct) + G(x+ct)]\Big|_{t=0} = -c\Delta^t_{\frac{1}{q}} F(x) + c\Delta^t_{\frac{1}{q}} G(x) = \psi(x).
\end{aligned}$$

Applying $\frac{1}{q}$-antiderivative to both sides of $\Delta^t_{\frac{1}{q}} u(x,0)$ over $[0,x]$. We obtain

$$-c\int_0^x \Delta^t_{\frac{1}{q}} F(s) d_{\frac{1}{q}} s + c\int_0^x \Delta^t_{\frac{1}{q}} G(s) d_{\frac{1}{q}} s = \int_0^x \psi(s) d_{\frac{1}{q}} s.$$

Using Fundamental Theorem of $\frac{1}{q}$-Calculus, we get

$$c[G(x) - G(0)] - c[F(x) - F(0)] = \int_0^x \Psi(s) d_{\frac{1}{q}} s$$

which implies

$$cG(x) - cF(x) = \int_0^x \psi(s) d_{\frac{1}{q}} s.$$

Note that we neglect the integration constant. We have also

$$F(x) + G(x) = \Phi(x).$$

Solving $F(x)$ and $G(x)$ in terms of $\phi(x)$ and $\psi(x)$ we get

$$\begin{aligned}
F(x) =& \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s) d_{\frac{1}{q}} s,\\
G(x) =& \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s) d_{\frac{1}{q}} s.
\end{aligned}$$

Thus, solution is

$$u(x,t) = \frac{1}{2}[\Phi(x-ct) + \Phi(x+ct)] + \frac{1}{2c}\int_a^b \Psi(s) d_{\frac{1}{q}} s$$

or equivalently

$$u(x,t) = \frac{1}{2}[\Phi(x-ct)+\Phi(x+ct)] \\ + \frac{1}{2c}[(1-q^{-1})b\sum_{j=0}^{\infty} q^{-j}\Psi(q^{-j}b) - (1-q^{-1})a\sum_{j=0}^{\infty} q^{-j}\Psi(q^{-j}a)],$$

where $a := x-ct$ and $b := x+ct$. In order to use Fundamental Theorem of $\frac{1}{q}$- Calculus and to satisfy initial value problem (5.6) , we assume $\phi \in C^2(I)$ and $\psi \in C^1(I)$ in the region $I = \mathbb{K}_q \times \mathbb{K}_q$. $\square$

Remark 5.2. *Clearly, as* $q \to 1$*, the solution of* $\frac{1}{q}$*-Cauchy problem:*

$$u(x,t) = \frac{1}{2}[\Phi(x-ct)+\Phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)d_{\frac{1}{q}}s$$

tends to the D'Alembert solution

$$u(x,t) = \frac{1}{2}[\Phi(x-ct)+\Phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)ds$$

of the D'Alembert problem given by

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \Phi(x), \quad -\infty < x < \infty, \\ &u_t(x,0) = \Psi(x), \quad -\infty < x < \infty. \end{aligned}$$

Definition 5.2. *The function* $f(x)$ *is said to be* $\frac{1}{q}$*-analytic function if there exists a q-power series (*$\frac{1}{q}$*-Taylor) expansion converging to* $f(x)$ *itself.*

Remark 5.3. *We can discuss Cauchy-Kovalevskaya Theorem, (Gantumur, 2011) on* $\mathbb{K}_q$ *as* $\mathbb{K}_q \subset \mathbb{R}$*, that is, if the initial conditions* ϕ *and* ψ *are* $\frac{1}{q}$*-analytic functions, then the problem* (5.6) *has unique, analytic solution in the neighborhood of initial points.*

Example 5.1. *Consider the* $\frac{1}{q}$*-initial value problem with the initial conditions*

$$\begin{aligned}&[(\Delta_q^t)^2 - c^2(\Delta_{\frac{1}{q}}^x)^2]u(x,t) = 0, \quad \forall x,t \in \mathbb{K}_q,\\&u(x,0) = x^2,\\&\Delta_{\frac{1}{q}}^t u(x,0) = 0.\end{aligned} \tag{5.8}$$

Then by Theorem 5.3, solution is

$$u(x,t) = \frac{1}{2}[(x-ct)_{\frac{1}{q}}^2 + (x+ct)_{\frac{1}{q}}^2] = x^2 + q^{-1}c^2t^2. \tag{5.9}$$

Clearly, (5.9) *satisfies the initial conditions of the problem* (5.8).

Example 5.2. *Consider the* $\frac{1}{q}$*-initial value problem with the initial conditions*

$$\begin{aligned}&[(\Delta_q^t)^2 - c^2(\Delta_{\frac{1}{q}}^x)^2]u(x,t) = 0, \quad \forall x,t \in \mathbb{K}_q,\\&u(x,0) = \cos_{\frac{1}{q}} x,\\&\Delta_{\frac{1}{q}}^t u(x,0) = \sin_{\frac{1}{q}} x.\end{aligned} \tag{5.10}$$

Then solution of the given $\frac{1}{q}$*-initial value problem is*

$$\begin{aligned}u(x,t) =& \frac{1}{2}[\cos_{\frac{1}{q}}(x+ct) + \cos_{\frac{1}{q}}(x-ct)] + \frac{1}{2c}\int_a^b \sin_{\frac{1}{q}} x' d_q x'\\=& \frac{1}{2}[\cos_{\frac{1}{q}}(x+ct) + \cos_{\frac{1}{q}}(x-ct)] + \frac{1}{2c}[-\cos_{\frac{1}{q}}(x+ct) + \cos_{\frac{1}{q}}(x-ct)]\\=& \frac{1}{2}\Big[(1-\frac{1}{c}\Big)\cos_{\frac{1}{q}}(x+ct) + \Big(1+\frac{1}{c}\Big)\cos_{\frac{1}{q}}(x-ct)\Big].\end{aligned} \tag{5.11}$$

Here we used

$$\int_a^b \sin_{\frac{1}{q}} x' d_q x' = -\int_a^b \Delta_{\frac{1}{q}}^{x'}\Big(\cos_{\frac{1}{q}}\Big) d_{\frac{1}{q}} x' = -\cos_{\frac{1}{q}}(b) + \cos_{\frac{1}{q}}(a)$$

by Corollary 4.6 *and* $a := x - ct$, $b := x + ct$. (5.11) *satisfies the initial conditions of* *the problem* (5.10)*. Because*

$$u(x,0)=\frac{1}{2}[\cos_{\frac{1}{q}}x-\frac{1}{c}\cos_{\frac{1}{q}}x+\cos_{\frac{1}{q}}x+\frac{1}{c}\cos_{\frac{1}{q}}x]=\cos_{\frac{1}{q}}x,$$
$$\Delta^t_{\frac{1}{q}}u(x,0)=\frac{1}{2}[-c\Big(1-\frac{1}{c}\Big)\sin_{\frac{1}{q}}x+c\Big(1+\frac{1}{c}\Big)\sin_{\frac{1}{q}}x]=\sin_{\frac{1}{q}}x.$$

Note that since $e^x_{\frac{1}{q}}$ is $\frac{1}{q}$-analytic, determined by its Taylor expansion, clearly so are $\cos_{\frac{1}{q}}x$ *and* $\sin_{\frac{1}{q}}x$*. Then the initial functions ϕ, ψ are $\frac{1}{q}$-analytic so, the problem* (5.10) *has a unique and analytic solution by Cauchy-Kovalevskaya Theorem.*

Example 5.3. *Consider the $\frac{1}{q}$-initial value problem with the initial functions*

$$\begin{aligned}&[(\Delta^t_q)^2-c^2(\Delta^x_{\frac{1}{q}})^2]u(x,t)=0,\quad \forall x,t\in\mathbb{K}_q,\\ &u(x,0)=\phi(x)=x^3,\\ &\Delta^t_{\frac{1}{q}}u(x,0)=\psi(x)=-[3]_{\frac{1}{q}}cx^2.\end{aligned}\tag{5.12}$$

Then solution is

$$u(x,t)=\frac{1}{2}[\phi(x-ct)+\phi(x+ct)]+\frac{1}{2c}\int_a^b\Psi(s)d_{\frac{1}{q}}s,$$

where $a=x-ct$ and $b=x+ct$. First way to find $\int_a^b\Psi(s)d_{\frac{1}{q}}s$ is the following:

$$\begin{aligned}\int_a^b\Psi(s)d_{\frac{1}{q}}s&=-c\int_a^b[3]_{\frac{1}{q}}s^2d_{\frac{1}{q}}s=-\int_a^b[\Delta^s_{\frac{1}{q}}(s)^3_{\frac{1}{q}}]d_{\frac{1}{q}}s=[(s)^3_{\frac{1}{q}}]\Big|^{s=b:=(x+ct)}_{s=a:=(x-ct)}\\ &=-c[(x+ct)^3_{\frac{1}{q}}-(x-ct)^3_{\frac{1}{q}}].\end{aligned}$$

Here we used the fact $\Delta^x_{\frac{1}{q}}[(x)^3_{\frac{1}{q}}]=[3]_{\frac{1}{q}}x^2$ and Corollary 4.6*. We can also find*

$\displaystyle\int_a^b \Psi(x) d_{\frac{1}{q}} x$ *by using* $\frac{1}{q}$*-antiderivative Definition* 4.19*, that is*

$$\begin{aligned}
\int_a^b \Psi(s) d_{\frac{1}{q}} s =& \int_0^b \Psi(s) d_{\frac{1}{q}} s - \int_0^a \Psi(s) d_{\frac{1}{q}} s \\
=&(1-q^{-1})b \sum_{j=0}^{\infty} q^{-j} \psi(q^{-j} b) - (1-q^{-1})a \sum_{j=0}^{\infty} q^{-j} \psi(q^{-j} a) \\
=& -(1-q^{-1})b \sum_{j=0}^{\infty} q^{-j} [3]_{\frac{1}{q}} c (q^{-j} b)^2 + (1-q^{-1})a \sum_{j=0}^{\infty} q^{-j} [3]_{\frac{1}{q}} c (q^{-j} a)^2 \\
=& -c \Big[(1-q^{-1}) b^3 [3]_{\frac{1}{q}} \sum_{j=0}^{\infty} (q^{-2})^j - (1-q^{-1}) a^3 [3]_{\frac{1}{q}} \sum_{j=0}^{\infty} (q^{-2})^j \Big] \\
=& -c \Big[(1-q^{-1}) b^3 (1 + q^{-1} + (q^{-1})^2) \frac{1}{1-q^{-2}} - (1-q^{-1}) a^3 [3]_{\frac{1}{q}} \frac{1}{1-q^{-2}} \Big] \\
=& -c(b^3 - a^3) = -c[(x+ct)^3_{\frac{1}{q}} - (x-ct)^3_{\frac{1}{q}}].
\end{aligned}$$

Here we used the definition of $[3]_q$ *and geometric series expansion. Thus, we have solution*

$$u(x,t) = \frac{1}{2}[(x-ct)^3_{\frac{1}{q}} + (x+ct)^3_{\frac{1}{q}}] - \frac{1}{2}[(x+ct)^3_{\frac{1}{q}} - (x-ct)^3_{\frac{1}{q}}] = (x-ct)^3_{\frac{1}{q}}.$$

Clearly, $u(x,t) = (x-ct)^3_{\frac{1}{q}}$ *satisfies the initial conditions of* (5.12) *since*

$$\begin{aligned}
&u(x,0) = x^3, \\
&\Delta^t_{\frac{1}{q}} u(x,0) = \Delta^t_{\frac{1}{q}} (x-ct)^3_{\frac{1}{q}} |_{t=0} = -[3]_{\frac{1}{q}} c (x-ct)^2_{\frac{1}{q}} |_{t=0} = \psi(x).
\end{aligned}$$

We can generalize this example to the case where $\phi(x) = x^n$.

Example 5.4. *Consider the below problem*

$$\begin{aligned}
&[(\Delta^t_q)^2 - c^2 (\Delta^x_{\frac{1}{q}})^2] u(x,t) = 0, \quad \forall x, t \in \mathbb{K}_q, \\
&\phi(x) = x^n, \\
&\psi(x) = -c[n]_{\frac{1}{q}} x^{n-1},
\end{aligned} \tag{5.13}$$

for a more general initial functions. Then $\frac{1}{q}$-Cauchy solution is

$$u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_a^b \Psi(x')d_{\frac{1}{q}}x',$$

where $a = x - ct$, $b = x + ct$ and

$$\begin{aligned}\int_a^b \Psi(x')d_{\frac{1}{q}}x' &= -c\int_a^b [n]_{\frac{1}{q}}(x')^{n-1}d_{\frac{1}{q}}x' = -c\int_a^b \left[\Delta^{x'}_{\frac{1}{q}}(x')^n_{\frac{1}{q}}\right]d_{\frac{1}{q}}x' \\ &= -c\left[(x+ct)^n_{\frac{1}{q}} - (x-ct)^n_{\frac{1}{q}}\right]\end{aligned}$$

since $\Delta^{x'}_{\frac{1}{q}}[(x')^n_{\frac{1}{q}}] = [n]_{\frac{1}{q}}(x')^{n-1}$ or equivalently we have

$$\begin{aligned}\int_a^b \Psi(x')d_{\frac{1}{q}}x' =& \int_0^b \Psi(x')d_{\frac{1}{q}}x' - \int_0^a \Psi(x')d_{\frac{1}{q}}x' \\ =&(1-q^{-1})b\sum_{j=0}^{\infty} q^{-j}\psi(q^{-j}b) - (1-q^{-1})a\sum_{j=0}^{\infty} q^{-j}\psi(q^{-j}a) \\ =&-(1-q^{-1})\left[b\sum_{j=0}^{\infty} q^{-j}c[n]_{\frac{1}{q}}(q^{-j}b)^{n-1} - a\sum_{j=0}^{\infty} q^{-j}c[n]_{\frac{1}{q}}(q^{-j}a)^{n-1}\right] \\ =&-c(1-q^{-1})[n]_{\frac{1}{q}}\left[\sum_{j=0}^{\infty}(q^{-n})^j\right](b^n - a^n) \\ =&-c(1-q^{-1})(1+q^{-1}+(q^{-1})^2 ... + (q^{-1})^{n-1})\left[\frac{1}{1-(q^{-1})^n}\right](b^n - a^n) \\ =&-c(b^n - a^n) = -c[(x+ct)^n_{\frac{1}{q}} - (x-ct)^n_{\frac{1}{q}}].\end{aligned}$$

Thus, solution is

$$u(x,t) = \frac{1}{2}\left[(x-ct)^n_{\frac{1}{q}} + (x+ct)^n_{\frac{1}{q}}\right] - \frac{c}{2c}\left[(x+ct)^n_{\frac{1}{q}} - (x-ct)^n_{\frac{1}{q}}\right] = (x-ct)^n_{\frac{1}{q}}.$$

Note that $u(x,t) = (x-ct)^n_{\frac{1}{q}}$ satisfies the inital conditions of the problem (5.13) *since*

$$\begin{aligned}&u(x,0) = (x-ct)^n_{\frac{1}{q}}\Big|_{t=0} = \phi(x), \\ &\Delta^t_{\frac{1}{q}}u(x,0) = \Delta^t_{\frac{1}{q}}(x-ct)^n_{\frac{1}{q}}\Big|_{t=0} = -[n]_{\frac{1}{q}}c(x-ct)^{n-1}_{\frac{1}{q}}\Big|_{t=0} = \psi(x).\end{aligned}$$

Theorem 5.4. *The $\frac{1}{q}$-derivative of the function $(x+ct)_q^n$ with respect to t is*

$$\Delta^t_{\frac{1}{q}}(x+ct)_q^n = c[n]_q(x+ct)_q^{n-1}, \quad n \in \mathbb{Z}. \tag{5.14}$$

Proof. Case (i) For $n \geq 0$. We will prove the proposition by induction on n. For $n = 1$, we get

$$\Delta^t_{\frac{1}{q}}(x+ct) = \frac{(x+\frac{1}{q}ct)-(x+ct)}{(\frac{1}{q}-1)t} = c.$$

Now, assume (5.14) holds for $n = k$. Let us try to show that it also holds for $n = k+1$. Consider

$$\begin{aligned}
\Delta^t_{\frac{1}{q}}(x+ct)_q^{k+1} =& \Delta^t_{\frac{1}{q}}[(x+ct)_q^k(x+q^kct)] \\
=& (x+ct)_q^k\Delta^t_{\frac{1}{q}}(x+q^kct) + (x+q^{k-1}ct)\Delta_{\frac{1}{q}}(x+ct)_q^k \\
=& (x+ct)_q^k\frac{x+q^{k-1}ct-x-q^kct}{(\frac{1}{q}-1)t} + (x+q^{k-1}ct)c[k]_q(x+ct)_q^{k-1} \\
=& (x+ct)_q^k\frac{ctq^k(q^{-1}-1)}{(\frac{1}{q}-1)t} + c[k]_qx+q^{k-1}ct)(x+ct)_q^{k-1} \\
=& cq^k(x+ct)_q^k + c[k]_q(x+ct)_q^k \\
=& c(x+ct)_q^k(q^k+[k]_q) = c(x+ct)_q^k(1+q+...+q^{k-1}+q^k) \\
=& c[k+1]_q(x+ct)_q^k.
\end{aligned}$$

Then (5.14) holds for $n = k+1$. Here we used product rule (4.89) and the statement of the assumption.

Case (ii) We need to prove for $n < 0$. Say $n = -n' < 0$. Consider

$$\begin{aligned}
\Delta^t_{\frac{1}{q}}(x+ct)_q^n =& \Delta^t_{\frac{1}{q}}(x+ct)_q^{-n'} = \Delta^t_{\frac{1}{q}}[\frac{1}{(x+q^{-n'}ct)_q^{n'}}] \\
=& \frac{(x+q^{-n'}ct)_q^{n'}\Delta^t_{\frac{1}{q}}(1) - \Delta^t_{\frac{1}{q}}(x+q^{-n'}ct)_q^{n'}}{(x+q^{-n'}ct)_q^{n'}(x+q^{-n'-1}ct)_q^{n'}} \\
=& \frac{-q^{n'}c[n']_q(x+q^{-n'}ct)_q^{n'-1}}{(x+q^{-n'ct})_q^{n'}(x+q^{-n'-1}ct)_q^{n'}} = \frac{-q^{-n'}c\frac{q^{n'}-1}{q-1}}{(x+q^{-1}ct)(x+q^{-n'-1}ct)_q^{n'}} \\
=& \frac{\frac{q^{-n'}-1}{q-1}c}{(x+q^{-n'-1}ct)_q^{n'+1}} = c[-n']_q\frac{1}{(x+q^{-(n'+1)}ct)_q^{n'+1}} \\
=& c[-n']_q(x+ct)_q^{-n'-1} = c[n]_q(x+ct)_q^{n-1}.
\end{aligned}$$

Here we used $\frac{1}{q}$-quotient rule (4.91), the fact $n' > 0$ and the Proposition 4.4. Thus, the equation (5.14) holds for any integer n. $\square$

Corollary 5.3. *We have the following equations:*

$$(\Delta^t_{\frac{1}{q}} \pm c^2\Delta^x_q)(x \mp ct)^n_q = 0, \quad n \in \mathbb{Z}. \tag{5.15}$$

Remark 5.4. *We can also give another q-initial value problem which is given by Nalcı Tumer & Oktay Pashaev (Nalcı Tumer, 2017) defined as*

$$\begin{aligned}
&[(\Delta^t_{\frac{1}{q}})^2 - c^2(\Delta^x_q)^2]u(x,t) = 0, \quad \forall x, t \in \mathbb{K}_q, \\
&u(x,0) = \Phi(x), \\
&\Delta^t_q u(x,0) = \Psi(x),
\end{aligned} \tag{5.16}$$

whose solution is

$$u(x,t) = \frac{1}{2}[\Phi(x-ct) + \Phi(x+ct)] + \frac{1}{2c}\int_a^b \Psi(s)d_qs, \tag{5.17}$$

where

$$\int_a^b \Psi(s)d_q = \int_0^b \Psi(s)d_q - \int_0^a \Psi(s)d_q = (1-q)b\sum_{j=0}^{\infty} q^j\Psi(q^jb) - (1-q)a\sum_{n=0}^{\infty} q^j\Psi(q^ja)$$

and $a := x - ct$, $b := x + ct$. Here $x,\ t \in \mathbb{K}_q,\ \phi \in C^2(I),\ \psi \in C^1(I)$ on $I = \mathbb{K}_q \times \mathbb{K}_q$.

Note that q-Cauchy problem can be solved in a similar way to $\frac{1}{q}$-Cauchy problem.

Remark 5.5. *Clearly, as $q \to 1$, the solution of q-Cauchy problem;*

$$u(x,t) = \frac{1}{2}[\Phi(x-ct) + \Phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)d_qs$$

tends to D'Alembert problem solution.

Definition 5.3. *The function $f(x)$ is said to be q-analytic function if there exists a q-power series (q-Taylor) expansion converging to $f(x)$ itself.*

Example 5.5. *Consider the q-initial value problem with the general initial conditions*

$$\begin{aligned}
&[(\Delta^t_{\frac{1}{q}})^2 - c^2(\Delta^x_q)^2]u(x,t) = 0, \quad \forall x, t \in \mathbb{K}_q,\\
&u(x,0) = \phi(x) = x^n, \qquad\qquad (5.18)\\
&\Delta^t_q u(x,0) = \psi(x) = -c[n]_q x^{n-1}.
\end{aligned}$$

Then q-D'Alembert solution is

$$u(x,t) = \frac{1}{2}[\phi(x-ct)^2 + \phi(x+ct)^2] + \frac{1}{2c}\int_a^b \Psi(x')d_qx',$$

where $a = x - ct$ *and* $b = x + ct$ *and*

$$\begin{aligned}
\int_a^b \Psi(x')d_qx' = -c\int_a^b [n]_q(x')^{n-1}d_qx = -c\int_a^b \Delta^{x'}_q[(x')^n_q]d_qx'\\
= -c[(x+ct)^n_q + (x-ct)^n_q]
\end{aligned}$$

since $\Delta^x_q[(x)^n_q] = [n]_q x^{n-1}$ *or equivalently we have*

$$\begin{aligned}
\int_a^b \Psi(x')d_qx' = &\int_0^b \Psi(x')d_qx' - \int_0^a \Psi(x')d_qx'\\
=&(1-q)b\sum_{j=0}^{\infty} q^j\psi(q^jb) - (1-q)a\sum_{j=0}^{\infty} q^j\psi(q^ja)\\
=&-(1-q)b\sum_{j=0}^{\infty} q^jc[n]_q(q^jb)^{n-1} + (1-q)a\sum_{j=0}^{\infty} q^jc[n]_q(q^ja)^{n-1}\\
=&-c(1-q)[n]_q\Big[\sum_{j=0}^{\infty}(q^n)^j\Big](b^n - a^n)\\
=&-c(1-q)(1+q+...+q^{n-1})[\frac{1}{1-q^n}](b^n - a^n) = -c(b^n - a^n)\\
=&-c[(x+ct)^n_q - (x-ct)^n_q].
\end{aligned}$$

Thus, solution is

$$u(x,t) = \frac{1}{2}[(x-ct)^n_q + (x+ct)^n_q] - \frac{c}{2c}[(x+ct)^n_q - (x-ct)^n_q] = (x-ct)^n_q.$$

Clearly, $u(x,t) = (x-ct)^n_q$ *satisfies the initial conditions of the problem* (5.18).

Because

$$u(x,0) = (x-0)_q^n = x^n = \phi(x),$$
$$\Delta_q^t u(x,0) = [n]_q c(x-ct)_q^{n-1}\Big|_{t=0} = [n]_q c x^{n-1} = \psi(x).$$

Taking these two q-Cauchy problems into account we state the following theorem.

Theorem 5.5. *The following problem*

$$[(\Delta_q^t)^2 - c^2(\Delta_{\bar{q}}^x)^2]u(x,t) = 0 \tag{5.19}$$

has the following solution

$$u(x,t) = f(x+ct) + g(x-ct)$$

if and only if $q\bar{q} = 1$ where

$$f(x+ct) \coloneqq a(x+ct)_{\bar{q}}^2 \quad \textit{and} \quad g(x+ct) \coloneqq b(x+ct)_{\bar{q}}^2.$$

Proof. ($\Leftarrow$) Assume $q\bar{q} = 1$, i.e. $\bar{q} = \frac{1}{q}$. By Corollary 5.2,

$$(x+ct)_{\frac{1}{q}}^2 + (x-ct)_{\frac{1}{q}}^2$$

satisfies the problem

$$[(\Delta_q^t)^2 - c^2(\Delta_{\bar{q}}^x)^2]u(x,t) = 0.$$

So we can say that

$$u(x,t) = f(x+ct) + g(x-ct)$$

is a solution for the problem (5.19) where

$$f(x+ct) \coloneqq a(x+ct)_{\bar{q}}^2 \quad \text{and} \quad g(x+ct) \coloneqq b(x+ct)_{\bar{q}}^2.$$

($\Rightarrow$) Assume the problem (5.19) has a solution

$$u(x,t) = f(x+ct) + g(x-ct).$$

Then we have

$$[(\Delta_q^t)^2 - c^2(\Delta_{\bar{q}}^x)^2][(x+ct)_{\bar{q}}^2 + (x-ct)_{\bar{q}}^2] = 0. \tag{5.20}$$

Using the Definition 4.2 and Definition 4.17 we get

$$\begin{aligned}
\Delta_q^t(x+ct)_{\bar{q}}^2 &= \Delta_q^t[(x+ct)(x+c\bar{q}t)] = c\bar{q}(x+ct) + c(x+q\bar{q}ct),\\
\Delta_{\bar{q}}^x(x+ct)_{\bar{q}}^2 &= \Delta_q^x[(x+ct)(x+c\bar{q}t)] = (1+\bar{q})(x+ct),
\end{aligned}$$

which implies that

$$\begin{aligned}
(\Delta_q^t - c\Delta_{\bar{q}}^x)(x+ct)_{\bar{q}}^2 &= (x+ct)[c\bar{q} + (1+\bar{q})] + c(x+q\bar{q}ct) &\quad (5.21)\\
&= -c(x+ct) + c(x+q\bar{q}ct) = -c^2t + c^2q\bar{q}t\\
&= -c^2t(1-q\bar{q}). &\quad (5.22)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\Delta_q^t(x-ct)_{\bar{q}}^2 &= \Delta_q^t[(x-ct)(x-c\bar{q}t)] = -c\bar{q}(x-ct) - c(x-q\bar{q}ct),\\
\Delta_{\bar{q}}^x(x-ct)_{\bar{q}}^2 &= \Delta_q^x[(x-ct)(x-c\bar{q}t)] = (1+\bar{q})(x-ct),
\end{aligned}$$

which implies that

$$\begin{aligned}
(\Delta_q^t + c\Delta_{\bar{q}}^x)(x-ct)_{\bar{q}}^2 &= (x-ct)[-c\bar{q} + c(1+\bar{q})] - c(x-q\bar{q}ct)\\
&= c(x-ct) - c(x-q\bar{q}ct) = -c^2t + c^2q\bar{q}t\\
&= -c^2t(1-q\bar{q}). &\quad (5.23)
\end{aligned}$$

By (5.22) and (5.23), the problem (5.20) turns to

$$(\Delta_q^t - c\Delta_{\bar{q}}^x)(x+ct)_{\bar{q}}^2 + (\Delta_q^t + c\Delta_{\bar{q}}^x)(x-ct)_{\bar{q}}^2 = -2c^2t(1-q\bar{q}) = 0.$$

Since $c \neq 0$ and $t \neq 0$, the term $(1 - q\bar{q})$ must be equal to 0. As a result $q\bar{q} = 1$. $\square$

We introduce a general one sided result which combines two cauchy problems in quantum analysis as follows

Proposition 5.1. *If $q\bar{q} = 1$ then the problem* (5.19) *has general solution*

$$u(x,t) = f(x+ct) + g(x-ct)$$

where

$$f(x+ct) \coloneqq \sum_{n=-\infty}^{\infty} a_n (x+ct)_{\bar{q}}^{n} \quad \textit{and} \quad g(x+ct) \coloneqq \sum_{n=-\infty}^{\infty} b_n (x+ct)_{\bar{q}}^{n}.$$

Proof. Assume $q\bar{q} = 1$, i.e., $\bar{q} = \frac{1}{q}$. By Corollary 5.2, we know the solution of the problem

$$[(\Delta_q^t)^2 - c^2(\Delta_{\bar{q}}^x)^2]u(x,t) = 0$$

which is

$$u(x,t) = f(x+ct) + g(x-ct)$$

where the functions $f(x+ct) \coloneqq a(x+ct)_{\bar{q}}^{n}$ and $g(x+ct) \coloneqq b(x+ct)_{\bar{q}}^{n}$. $\square$

CHAPTER SIX
CONCLUSION

In this thesis, we studied on lattice and quantum numbers. First similar to the ordinary calculus, we built lattice differentials and obtained corresponding lattice derivatives which were called h-delta and h-nabla derivatives. Then we gave h-binomial function and related properties. By utilizing those properties, we calculated h-delta and h-nabla derivatives of h-binomial function. There were differences and similarities between lattice analysis (h-analysis) and ordinary calculus. As h tends to zero, we showed that lattice analysis gave real numbers then we called the study on h-analysis as h-discretization. We generalized Taylor's Formula and from which we presented h-Taylor's Formula. We defined two h-exponential functions by using h-Taylor's formula. Then by using h-exponential functions we introduced h-trigonometric, h-hyperbolic functions and calculated their lattice derivatives. Next, we introduced h-nabla antiderivative, definite h-nabla integral and constructed Fundamental Theorem of h-Calculus. As an application, we introduced h-D'Alembert problems and found their solutions in terms of h-binomial functions. Also the uniqueness and analyticity of solutions were examined by stating h-analogue of Cauchy-Kovalevskaya Theorem.

We applied the same works done in lattice analysis for quantum numbers and we constructed quantum analysis (q-analysis) with related quantum derivatives, q-integral, $\frac{1}{q}$- integral with properties. Also we presented q-binomial, $\frac{1}{q}$-binomial functions, and related properties. We gave Taylor's Formula of the function f defined on quantum numbers from which we defined q-exponential functions. As an application of q-analysis, we introduced $\frac{1}{q}$-Cauchy problems and their solutions. We gave related examples and also similar to h-Cauchy problems, uniqueness and analyticity of solutions were examined. As a remark, we gave another q-initial value problem that was discussed before. Then we end up with by related examples.

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