EXTREME BEHAVIOR OF LEX IDEALS ON BETTI NUMBERS

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By

Hubeyb Üsame Gürdoğan

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Müfit Sezer(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Cansu Betin

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Özgün ünlü

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

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Hubeyb Üsame Gürdoğan
M.S. in Mathematics
Supervisor: Assoc. Prof. Müfit Sezer
July, 2013

This paper mainly deals with the finitely generated graded modules of the polynomial ring $\mathbf{k}[x_1, x_2, ..., x_n]$. Free resolutions is an important tool to understand the structure of these modules. Betti numbers are an useful invariant that encodes the free resolutions. Our concentration accumulates on proving that the lex ideals provides an upper bound for Betti numbers of the graded ideals with the same Hilbert function in the polynomial ring $\mathbf{k}[x_1, x_2, ..., x_n]$. The material of this thesis is contemporary classical and includes the detailed study of the material that is scattered throughout the sources cited in the bibliography list.

Keywords: Lex Ideals, Betti Numbers.

ÖZET

LEKS İDEALLERİNİN BETTİ SAYILARI ÜZERİNDEKİ UÇ DAVRANIŞI

Hubeyb Üsame Gürdoğan
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Tez Yöneticisi: Doç. Dr. Müfit Sezer
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Bu tezde ana konu $\mathbf{k}[x_1, x_2, ..., x_n]$ olarak verilen n boyutlu polinom halkasının sonlu üretilmiş serbest modülleridir. Serbest çözünürlükler bu modüllerin yapısını anlamak için önemli bir araçtır. Betti sayıları ise serbest çözünürlükleri kodlayan faydalı bir sabit olarak bilinmektedir. Biz bu çalışmada lex ideallerinin Betti sayılarına odaklaniyoruz. Bu sayıların aynı Hilbert fonksiyonuna sahip idealler içinde üst sınır oluşturduğunun ispatını tekrar üretiyoruz. Bu tezdeki bilgiler (modern)-klasik olup kaynakçada belirtilen kaynaklardaki materyalin ayrıntılarının çalışılmasıyla ortaya konmuştur.

Anahtar sözcükler: Leks İdealler, Betti Sayıları.

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Contents

Introduction

This paper mainly deals with the finitely generated graded modules of the polynomial ring $\mathbf{k}[x_1, x_2, ..., x_n]$. It was Hilbert's idea to associate a (free) resolution to a module and resolutions proved to be important tools to understand the structure of the module. Betti numbers are useful invariants that encode the free resolutions. Our main target is to prove that the lex ideals provides an upper bound for Betti numbers of the graded ideals in the polynomial ring $\mathbf{k}[x_1, x_2, ..., x_n]$ among all the ideal with the same Hilbert function. The material of this thesis is somewhat contemporary classical and contains material that can be found scattered throughout the sources cited in the bibliography list.

In Chapter 1 we make a brief introduction to basic concepts of bits of commutative algebra involved in this thesis such as graded modules, free resolutions and Betti numbers. And we recall some foundational theorems about them. The proofs of the theorems in this section are omitted since they are not directly related with the scope of this paper.

In second chapter we review Gr$\ddot{o}$bner basis and introduce the generic initial ideals. We demonstrate that the Hilbert function of a graded ideal is same with the Hilbert function of its initial ideal. Also at the end of the chapter we present a famous theorem stating that the generic initial ideal of all ideals in S are Borel. These two results play an important role in the the proof of the main theorem of this paper.

It is proven by Bigatti-Hulett that the Betti numbers of lex ideals are the upper bound for the Betti numbers of all ideals. But this result was powered by the Macaulay's theorem which was proven far before. In Chapter 3 we introduce the lex monomial spaces, lex ideals, Borel monomial spaces and other related objects. We quote the Macaulay's theorem and postpone the proof of it to preceding chapters. The remarkable theorem of M. Green stating that the Betti numbers of the initial ideal is greater than the Betti numbers of the original ideal is represented in this chapter.

Chapter 4 is devoted for introducing all the required tools and results for proving the Macaulay's Theorem. Those will also be useful for the main theorem of the paper. The technique of multicompression is introduced and used to characterize the lex ideals and Borel ideals. Some of the results presented in this chapter will be proved in the next chapter along with the proof of Macaulay's theorem since they are closely related.

Chapter 5 is devoted to the proof of Macaulay's theorem. It is worth noting that almost all results given in chapter 4 also take a key part in the inductive proof of the Macaulay's theorem. If the Macaulay's theorem was not true, whole chapter 4 would collapse. In the preceding chapters we will make use of some of these results to prove a continuation of Macaulay's theorem.

Chapter 6 provides the necessary motivation to come of with the idea of the main theorem of this paper. It is stated and its proof is postponed due to the essential preparations will be done in the next chapter.

Our main theorem is directly related with understanding the Betti numbers of a graded ideal. Betti numbers of an ideal encodes the structure of its minimal free resolution. There is no precise way to calculate Betti numbers of an arbitrary graded ideal but it is possible to calculate it for the Borel monomial ideals via the Eliahou-Kervaire resolution which will be the main endeavour of the Chapter 7.

In the tree of the results presented in this paper there is almost no isolated result. In other words there are almost no side work that is not related with the main theorem of this paper. In chapter 8 we prove that the lex ideals provide the upper bounds of the Betti numbers among the graded ideals.

Chapter 1

Preliminary Results

Let $S = k[x_1, x_2, ..., x_n]$ be a polynomial ring over a field k. The elements of the form $x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ are called monomials and $\alpha_1 + \alpha_2 + ... + \alpha_n$ is called the degree of that monomial. Also S_i is to denote the k-vector space spanned by all monomials of degree i. In particular $S_0 = k$.

Definition 1.0.1. An S-module M is called graded if it can be written as $\bigoplus_{i \in Z} M_i$ over k where the modules M_i satisfy $S_i M_j \subseteq M_{i+j}$ $\forall i,\ j \in Z$. M_i are called homogenous components and elements of them are called homogeneous. For any element $m \in M_i$ we define $deg(m) = i$. In this setup for all $m \in M = \bigoplus_{i \in Z} M_i$ there exists a unique set $\{m_i \in M_i \big| i \in Z\}$ s.t $m = \sum_{i \in Z} m_i$. In this equality m_i are called homogeneous component of m of degree i.

According to this definition the ring S is graded in itself since it is obviously seen that $S = \bigoplus S_i$ over k and $S_i S_j \subseteq S_{i+j}$. Also since ideals of S can also regarded as S-modules the above definition determines the requirements for ideals to be graded.

Definition 1.0.2. Let N and T be graded S-modules. And let $\varphi : N \to T$ be a module homomorphism. If exist a fixed i such that $deg(\varphi(m)) = deg(m) + i$ $\forall m \in N$, we call φ a graded module homomorphism of degree i.

Definition 1.0.3. Let M be a graded S-module. Then for $p \in Z$ denote by $M(-p)$ the graded S-module such that $M(-p)_i = M_{i-p}$. It is called the module M shifted by p degrees.

Remark 1.0.4. The module $S(-p)$ is the free S-module generated by one element of degree p.

Proposition 1.0.5. *Let M be a free graded S-module. Then $M \cong \bigoplus\limits_{i} S(-a_i)$ for some set of a_i's.*

Proof. Let $\{m_1, m_2, ..., m_j\}$ be homogeneous generators of M where $deg(m_i) = a_i$. Then since M is free we have $M = \bigoplus\limits_{i=1}^{j} S(m_i)$. Then by the previous remark we get $M = \bigoplus\limits_{i=1}^{j} S(-a_i)$. □

Definition 1.0.6. A complex F over S is a sequence of homomorphisms of S-modules;

$$F : ... \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow ... \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow ...$$

where $d_i d_{i-1} = 0$ is satisfied. The maps d_i are called the differential of the complex. Moreover if $F_i = 0$ for all $i < 0$, F is called, left complex.

Definition 1.0.7. Let F be a complex defined in the previous definition. Then we define and denote the homology of the complex F in degree i as $H_i(F) = \frac{ker(d_i)}{im(d_{i+1})}$.

Definition 1.0.8. Let F be a left complex defined before. And let U be a finitely generated S-module. Then F is called a free resolution of the module U if the following conditions hold;

(1) F_i are finitely generated free S-modules for all i.

(2) $H_i(F) = 0$ for all $i > 0$ i.e. the sequence is exact.

(3) $H_0(F) = \frac{F_0}{im(d_1)} \cong U$.

Also if the degree of the differential is zero the resolution is called graded.

construction 1.0.9. Let U be a finitely generated graded S-module. Then there exists a graded free resolution of U as we will construct here. Let $u_1, u_2, ..., u_r$ be a basis of U and let $a_1, a_2, ..., a_r$ be their degrees. Let $F_0 = S(-a_1) \oplus S(-a_2) \oplus ... \oplus S(-a_r)$ and let $e_1, e_2, ..., e_r$ be the generators of each summand. Set $d_1(e_j) = u_j$. This is a homomorphism of degree 0. Then by first isomorphism theorem we get $\frac{F_0}{ker(d_1)} = F_0 \cong U$.

Now we need to construct F_i for $i > 0$ properly. We do this by arguing an induction on i. Assume F_i and d_i are defined. Set $ker(d_i) = U_i$. It is not hard to see that $ker(d_i)$ is a graded S-module since F_i is a graded module and d_i is a graded module homomorphism. So let $l_1, l_2, ..., l_{r_i}$ be the homogeneous generators of it and $c_1, c_2, ..., c_{r_i}$ be the degrees of them. Let $F_{i+1} = S(-c_1) \oplus S(-c_2) \oplus ... \oplus S(-c_{r_i})$ and $f_1, f_2, ..., f_{r_i}$ be the generators of each summand respectively. Now set the map $d_{i+1}(f_j) = l_j$. This is a homomorphism of degree 0. And clearly $im(d_{i+1}) = ker(d_i)$ which implies $H_i(F) = 0$ ie the exactness. So we are done.

$$\begin{tikzcd} F_{i+1} \arrow[r, "d_{i+1}"] \arrow[dr] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & F_i \arrow[r, "d_i"] & F_{i-1} \\ & ker(d_i) \arrow[u] & \end{tikzcd}$$

Definition 1.0.10. Let F be a graded free resolution of a graded S-module U. Then F is called minimal if the following holds for all $i \geq 0$;

$$d_i(F_i) \subseteq (x_1, x_2, ..., x_n)F_{i-1}.$$

Theorem 1.0.11. *Let F be a graded free resolution of an S-module U constructed in 1.0.9. This resolution is minimal if and only if the chosen system of homogeneous generators of $ker(d_i)$ in construction 1.0.9 is minimal.*

Theorem 1.0.12. *Up to isomorphism there exist a unique minimal graded free resolution of the graded finitely generated S-module U.*

With the help of the preceding theorem we can make a well defined invariant of graded finitely generated S-modules.

Definition 1.0.13. Let U be a graded finitely generated S-module. Let F be a minimal graded free resolution of U. Then since F_i are free it can be written as

$F_i = \bigoplus_{p \in Z} S(-p)^c_{i,p}$ in a unique way by 1.12. We define the graded Betti numbers of U as the number of summands in F_i of the form $S(-p)$. We denote it as $\beta^S_{i,p}(U) = c_{i,p}$.

Theorem 1.0.14 (5, Theorem 1.34)**.** *Let I be a finitely generated ideal in S. Then we have $\beta^S_{i,j}(I) = \beta_{i,j+1}(S/I)$ $\forall i, j$.*

Using this result we understand that betti number of I is the one degree shifted of the betti number of S/I. For notational simplicity we use $\beta^S_{i,j}(I)$ to mean both of them. Betti numbers somehow encodes the structure of finitely generated graded modules. The main struggle of this paper is mostly related with calculating Betti numbers of finitely generated graded ideals of S.

Chapter 2

Gr$\ddot{o}$bner Basis and the Generic Initial Ideals

Definition 2.0.15. Let M be a graded finitely generated S-module. It decomposes as direct sum of its components $M = \bigoplus\limits_{i \in N} M_i$. The Hilbert function of M is defined from N to N by $i \longrightarrow dim_k(M_i)$. Also we denote $|M_i| = dim_k(M_i)$.

Definition 2.0.16. Let m and m' be two monomials in S. We set the rules of degree lex order that determine which one of them is greater in the following steps;

1 $x_1 \underset{lex}{>} x_2 \underset{lex}{>} ... \underset{lex}{>} x_n$.

2 If $deg(m) > deg(m')$ then $m \underset{lex}{>} m'$ and visa versa.

3 If $deg(m) = deg(m')$ write $m = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ and $m' = x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n}$. Let i be the smallest number s.t $\alpha_i \neq \beta_i$ and $\alpha_j = \beta_j$ for all j less than i. If $\alpha_i > \beta_i$ then $m \underset{lex}{>} m'$ and visa versa.

For notational simplicity we stick to the notation " $>$ " instead of " $\underset{lex}{>}$ " for the rest of the paper.

Definition 2.0.17. Let J be an ideal in S and let $f \in J$. We call the greatest monomial in f respect to lex order as initial element of f and denote it as $in(f)$.

Initials of all elements in J generate a monomial ideal. It is called the initial ideal of J and denoted as $in(J) = \langle in(f) \big| f \in J \rangle$.

Definition 2.0.18. Let J be an ideal in S, then a set G in J is called a Grobner basis if the initial terms of the elements in G generates the initial ideal of J;

$$in(J) = \langle in(f) \big| f \in G \rangle.$$

Theorem 2.0.19. *Grobner basis G of the ideal J generates J.*

Proof. Lets assume that $f \in J/\langle G \rangle$ be the one with the smallest initial term. Since $in(G)$ generates $in(J)$ it also generates $in(f)$. So there is an element $g \in G$ and a monomial $m \in S$ s.t. $in(g)m = in(f)$ which implies $in(mg) = in(f)$. Then $f - mg$ has initial term less than $in(f)$ and clearly $f - mg \in J/\langle G \rangle$ since $mg \in \langle G \rangle$, $f \in J/\langle G \rangle$. These together leads to a contradiction with the smallest initial term assumption on f. Therefore we conclude that there is no element in J that lies out of $\langle G \rangle$, which ends the proof. □

Theorem 2.0.20. *There exist a finite Gröbner basis of all ideals J of S and same Grobner basis works with respect to every monomial order.*

Theorem 2.0.21. *Let J be an ideal in S. The monomials not in $in(J)$ form a basis of the* **k***-vector space S/J.*

Proof. Let $m_1, m_2, ..., m_p$ be monomials not in $in(I)$. We first prove that they are linearly independent. Assume not, i.e. $\exists \alpha_i \in \mathbf{k}/0$ s.t. $\alpha_1 m_1 + \alpha_2 m_2 + ... + \alpha_p m_p \in J$. Then we get $in(\alpha_1 m_1 + \alpha_2 m_2 + ... + \alpha_p m_p) \in in(J)$. But the initial term of it is a scalar multiple of the monomials which are not in $in(J)$. This is a contradiction. So we have $m_1, m_2, ..., m_p$ are linearly independent.

Let $T = \{m_1, m_2, ...m_p\}$. We need to prove that $span(T) = S/J$. It is equivalent to prove that $\{T, J\}$ span S. Assume not. Let $f \notin span(T, J)$ and has the minimal initial term among the polynomials not in the span of $\{T, J\}$. If $in(f) \notin in(J)$ then $in(J) \in span(T)$. Then we have $f - in(f) \notin span(T, J)$. But $f - in(f)$ has initial term less than the initial term of f which contradicts with the assumption on f. So $in(f) \in in(J)$. Then $\exists g \in J$ such that $in(g) = in(f)$.

Therefore $f - g \notin span(T, J)$ and has the initial term less than the initial term of f. Again it contradicts with the assumption on f. Hence $span(T, J) = S$. $\square$

Corollary 2.0.22. *Let I be a graded ideal in S. Then S/I and $S/in(I)$ have the same Hilbert function.*

Proof. By the previous theorem the monomials not in $in(I)$ spans S/I. Also by the same theorem we have that the monomials not in $in(I) = in(in(I))$ spans $S/in(I)$. Hence the Hilbert function of S/I and $S/in(I)$ are same. $\square$

Lemma 2.0.23. *Let $0 \longrightarrow K \xrightarrow{\gamma} M \xrightarrow{\phi} N \longrightarrow 0$ be a short exact sequence of graded finitely generated S-modules and homomorphisms of degree 0. Then*

$$|M_q| = |K_q| + |N_q| \; for \; all \; q \geq 0.$$

Proof. Since the degree of the homomorphisms are zero, for each $q \geq 0$ we have the short exact sequence of $\mathbf{k}$-vector spaces

$$0 \longrightarrow K_q \xrightarrow{\gamma} M_q \xrightarrow{\phi} N_q \longrightarrow 0.$$

Let the base set of K_q, M_q, N_q be β_K, β_M, β_N respectively. Since γ is a one to one homomorphism we have $\beta_M \cong \beta_K \sqcup \beta$. Also since $im(\gamma) = ker(\phi)$ and ϕ is surjective we have $\phi\big|_\beta$ is a one to one and onto homomorphism. So we get $\beta \cong \beta_N$. Hence $\beta_M \cong \beta_K \sqcup \beta_N$ which implies $|M_q| = |K_q| + |N_q|$ for all $q \geq 0$. We are done. $\square$

Corollary 2.0.24. *Let I be a graded ideal in S. Then I and $in(I)$ have the same Hilbert function.*

Proof. We have $S/I = \bigoplus\limits_i S_i/I_i$ a graded ring with the grades shown in equation. Then we have the following exact sequences of graded ideals and homomorphisms of degree zero

$$0 \longrightarrow I \underset{incl.}{\xrightarrow{\gamma}} S \underset{proj.}{\xrightarrow{\phi}} S/I \longrightarrow 0.$$

$$0 \longrightarrow in(I) \underset{incl.}{\xrightarrow{\gamma}} S \underset{proj.}{\xrightarrow{\phi}} S/in(I) \longrightarrow 0.$$

Now by using the previous lemma and the corollary we have the following implications

$$\begin{aligned} |I_q| + |(S/I)_q| &= |S_q| = |(in(I))_q| + |(S/in(I))_q| \\ |I_q| &= |(in(I))_q| \ \ \forall q \geq 0. \end{aligned}$$

Hence we proved that I and $in(I)$ have the same Hilbert function. □

Theorem 2.0.25 (Weispfenning Theorem)**.** *Let y be a set of variables different from $x_1, ..., x_n$. Consider the extension ring $\mathbf{k}[x, y] = S[y]$. For every ideal J in $S[y]$ and for every homomorphism $\phi : \mathbf{k}[y] \to \mathbf{k}$ there is a finite set C of polynomials $p(x, y) \in J$ such that $\phi(C)$ is a Gröbner basis for the ideal $\phi(J)$ in S. C is called comprehensive Gröbner basis.*

Definition 2.0.26. For a $g \in Gl_n(\mathbf{k})$ and a polynomial $p(x_1, x_2, ..., x_n) \in S$ we define the act of g on p by

$$g \cdot p = p(gx_1, gx_2, ..., gx_n), \ \ where \ \ gx_i = \sum_{j=1}^{n} g_{ij} x_j.$$

Also we define $g \cdot I = \{g \cdot p \mid p \in I\}$.

Definition 2.0.27. The general linear group is defined and denoted as;

$$GL_n(\mathbf{k}) = \{ \text{ invertible } n \times n \text{ matrices } \}.$$

Lets define a equivalence relation on $GL_n(\mathbf{k})$ by imposing the condition $in(g \cdot I) = in(g' \cdot I)$ for g and g' to be equivalent. It is easy to see that this is indeed an equivalence relation. This gives us a partition of the group $GL_n(\mathbf{k})$ into equivalence classes.

Definition 2.0.28. For $g \in GL_n(\mathbf{k})$ let $\mathbf{k}[g] = \mathbf{k}[g_{11}, g_{12}, ..., g_{nn}]$ where g is considered to be variable. Then a subset G of $GL_n(\mathbf{k})$ is called Zariski closed if it is a zero set of an ideal in $\mathbf{k}[g]$. Also G is called Zariski open if complement of G is Zariski closed.

Lemma 2.0.29. *For any ideal I there are finitely many equivalence classes in $GL_n(\mathbf{k})$. One of these classes is a nonempty Zariski open subset inside of $GL_n(\mathbf{k})$.*

Proof. For notational simplicity let $x = (x_1, x_2, ..., x_n)$. And let $I = \langle p_1(x), p_2(x), ..., p_r(x) \rangle$. As in the statement of Weispfenning Theorem introduce a new set of variables $\{g_{11}, g_{12}, ..., g_{nn}\}$ call it g and consider the extension $S[g] = \mathbf{k}[g, x]$. Let J be an ideal of $\mathbf{k}[g, x]$ which is generated by elements $g.p_1(x), ..., g.p_r(x)$. By Weispfenning Theorem we have a comprehensive gr$\ddot{o}$bner basis C of J. Consider $\phi_h(J) = \langle h.p_1(x), ..., h.p_r(x) \rangle = h \cdot I$ where the value of $g = h \in GL_n(\mathbf{k})$ is inserted in J. By the definition of C we have $\phi_h(C)$ as a gr$\ddot{o}$bner basis of $h \cdot I$. Then $in(h \cdot I) = in(\phi_h(C))$. So we can read off all equivalence classes of $GL_n(\mathbf{k})$ by observing coefficients of polynomials in C. These coefficients are polynomials in $\mathbf{k}[g]$. By requiring that $\det g \neq 0$ (since g is invertible in $GL_n(\mathbf{k})$) and imposing the conditions " $= 0$" and " $\neq 0$" on these coefficient polynomials in all possible ways, we can read off all possible initial ideals $in(g.I)$. Since C is finite by definition, there are finite number of coefficients, so finite number of possibilities. Hence the number of distinct initial ideals $in(g.I)$ is finite which means there are finitely many equivalence classes in $GL_n(\mathbf{k})$.

Now let us prove that there is a unique Zariski open equivalence classes U specified by imposing the condition " $\neq 0$" on all leading coefficients of the polynomials in C. Firstly, U^c is actually the zero set of the principal ideal T in $\mathbf{k}[g]$ generated by the polynomial obtained by multiplying all leading coefficients of the polynomials in C. So U^c is Zariski closed, then U is Zariski open. Furthermore, U is clearly an equivalence class. For uniqueness; let V be a Zariski open equivalence class other than U, then there is an ideal T' in $\mathbf{k}[g]$ such that $var(T') = V^c$. Also we know that $var(T) = U^c$. Since U and V are clearly disjoint we get

$$\begin{aligned} GL_n(\mathbf{k}) &= U^c \cup V^c \\ &= var(T) \cup var(T') \\ &= var(TT'). \end{aligned}$$

Let $P_1(g) \in TT'$. Let $P_2(g)$ be an arbitrary one of the coefficient polynomials of g_{11} in $P_1(g)$. Similarly choose $P_3(g)$ as an arbitrary one of the coefficient polynomials of g_{12} in $P_2(g)$. Continuing this way we generate the arbitrarily chosen polynomials $P_1, P_2, ..., P_{n^2}$. If P_t is proven to be identically zero, then since the choice of P_t was arbitrary we get P_{t-1} is also zero identically.(*)

We have $P_1(g) = 0 \ \forall g \in Gl_n(\mathbf{k})$. We are going to prove that $P_2(g) = 0$ $\forall g \in Gl_n(\mathbf{k})$. $D(g)$ denotes the determinant polynomial of g. Now choose an arbitrary $g \in Gl_n(\mathbf{k})$ and change it by setting g_{11} to a variable z and leaving other entries as they are. Then $P_1(g)$ and $D(g)$ become polynomials of z. Set them to $p_1(z)$ and $d(z)$ respectively. We have $p_1(z) = 0 \ \forall z \in k$ s.t. $d(z) \neq 0$. Since $d(z)$ have only finitely many roots, there are infinitely many $z \in k$ s.t. $p_1(z) = 0$. Then since p_1 has only finitely many roots it becomes a zero polynomial. Since the choice of g was arbitrary we get all coefficient of g_{11} in $P_1(g)$ vanishes for all $g \in Gl_n(\mathbf{k})$. Then we get $P_2(g) = 0 \ \forall g \in Gl_n(\mathbf{k})$.

Applying same procedure successively it is easy to get $P_{n^2} = 0 \ \forall g \in Gl_n(\mathbf{k})$. But P_{n^2} is a polynomial in only variable g_{nn}. And for any value of g_{nn} in k there is an invertible matrix g that accepts g_{nn} as its entry. So P_{n^2} is zero for all $g_{nn} \in k$ which implies that $P_{n^2} = 0$ is identically zero. Then using (*) successively we get P_1 is identically zero. Since the choice of P_1 was arbitrary we get $TT' = 0$ which will result in one of T or T' to be zero. It is clear from the definition of T that $T \neq 0$. So we get $T' = 0$ which yields $V^c = GL_n(\mathbf{k})$. Then we get $V = 0$ which is a clear contradiction. We are done. □

Definition 2.0.30. The initial ideal $in(g \cdot I)$ is called the generic initial ideal if it is constant when g takes its values on a Zariski open subset V of $GL_n(\mathbf{k})$. It is denoted by $gin(I) = in(g \cdot I)$.

By the last part of the proof for the previous lemma we see that there is a unique Zariski open equivalence class U. Taking $V = U$ we assure $in(g \cdot I)$ is constant on V. As a result we proved the existence of $gin(I)$. Uniqueness of $gin(I)$ follows from the uniqueness of U in the previous lemma.

Theorem 2.0.31 (2, Theorem 15.20)**.** *The generic initial ideal $gin(I)$ is Borel for all ideals $I \subset S$.*

Chapter 3

Lex Ideals

Definition 3.0.32. A k-vector subspace A_q of S_q generated by monomials is called an S_q-monomial space. $\{A_q\}$ is denoted as the set of monomials in A_q and (A_q) is the ideal generated by the elements in A_q. Clearly $dim_k A_q = |\{A_q\}| := |A_q|$.

Definition 3.0.33. The lex segment $M_{q,p}$ is the S_q-monomial space spanned by the greatest p monomials in S_q respect to lex order.

Definition 3.0.34. An S_q-monomial space A_q deserves the name lex if $A_q = M_{q,|A_q|}$. Also $M_{q,|A_q|}$ is called the lexification of A_q in the case it is not lex.

Definition 3.0.35. Let A_q be a S_q-monomial space. By S_1A_q we mean a k vector space generated by $\{S_1A_q\}$. With this interpretation S_1A_q equals to $(A_q)_{q+1}$.

Definition 3.0.36. Let A_q and T_q be S_q monomial spaces. We say T_q is lex greater than A_q if when we order the monomials $\{T_q\}$ and $\{A_q\}$ lexicographically, and then compare the two ordered sets lexicographically, and get $\{T_q\}$ is greater.

Theorem 3.0.37. *If a monomial space M_q is lex in S_q then S_1M_q is also lex in S_{q+1}.*

Proof. Let $m \in S_{q+1}$ such that $m \geq m'x_i$ for a monomial $m' \in M_q$. Let x_j be the lex last variable dividing $m'x_i$ and u be the lex last variable dividing m. In that case we have $\frac{m}{u} \geq \frac{m'x_i}{x_j} \geq m'$. Then since M_q is lex we must have $\frac{m}{u} \in M_q$ which forces m to be an element in S_1M_q. This proves that S_1M_q is lex in S_{q+1}. $\square$

Definition 3.0.38. A monomial ideal L in S is called lex ideal if $\forall q \in Z^+$ L_q is a lex monomial space.

Proposition 3.0.39. *Let M be a graded ideal. Then there exist a minimal system of homogeneous generators U of M where any proper subset of it does not generate U. Moreover for all q we have $dim_k U_q$ is fixed among the any other minimal system of homogeneous generators.*

Proposition 3.0.40. *Let M be a graded ideal and let U be a minimal system of homogenous generators. Then we have $U_q \subseteq M_q/S_1M_{q-1}$ and $\beta^S_{0,q}(M) = |U_q| = |M_q| - |S_1M_{q-1}|$. Also for the case M is a monomial ideal we have $U_q = \{M_q\}/\{S_1M_{q-1}\}$.*

Proof. Let m be an arbitrary element in U_q. Assume $m \notin M_q/S_1M_{q-1}$ ie, $m \in S_1M_{q-1}$. Then since M_{q-1} is generated by $U_1, U_2, ..., U_{q-1}$ we get m is generated by $U_1, U_2, ..., U_{q-1}$. Therefore $U/\{m\}$ is still a minimal system of homogeneous generators. Which contradicts the minimality of U. So we get $U_q \subseteq M_q/S_1M_{q-1}$. Since elements in M_q/S_1M_{q-1} are not generated by lower degrees, all must be generated by U_q. Therefore M_q/S_1M_{q-1} is a k-vector space with the basis set U_q. Then we get;

$$\beta^S_{0,q}(M) = |U_q| = dim_k(U_q) = dim_k(M_q/S_1M_{q-1}) = dim_k(M_q) - dim_k(S_1M_{q-1})$$
$$= |M_q| - |S_1M_{q-1}|$$

For the case M is a monomial ideal the basis set of M_q/S_1M_{q-1} is clearly the monomials in it. Which equals to the set $\{M_q\}/\{S_1M_{q-1}\}$.(Here $\{M_q\}$ and $\{S_1M_{q-1}\}$ are set of monomials generating M_q and S_1M_{q-1} respectively) $\square$

Proposition 3.0.41. *Let L be a monomial ideal in S. The following assertions are equivalent.*

(a) Let p be a number such that L has no minimal monomial generators of degree greater than p. For each $q \leq p$ we have L_q is lex.

(b) Let L be minimally generated by monomials $l_1, l_2, ..., l_r$. If m is a monomial satisfying $m > l_i$ with $deg(m) = deg(l_i)$ for some $1 \leq i \leq r$ then $m \in L$.

Proof. Here we divide our proof into two parts. In the first part we additionally prove that (a) and (b) are also equivalent to the assertion that L is a lex ideal.

(a $\Longrightarrow$ b) If L has no minimal generator of degree $q+1$. Then by previous proposition we get;

$$\{L_{q+1}\}/\{S_1 L_q\} = \varnothing \;\Rightarrow\; \{L_{q+1}\} = \{S_1 L_q\} \Rightarrow L_{q+1} = S_1 L_q.$$

Using this with 3.0.37 it is clear that L_{q+1} is lex if L_q is lex. Since it is given that L_q is lex for all $q < p$ we get L_q is lex for all q which implies that L is a lex ideal. (Here it is worthy to note that L is lex also clearly implies (a))Now let m be a monomial with $m > l_i$ and $deg(m) = deg(l_i)$. Then since L is a lex ideal $L_{deg(m)}$ is a lex monomial space which implies clearly that $m \in L_{deg(m)} \subset L$.

(b $\Longrightarrow$ a) Let $l_1, l_2, ..., l_r$ be written in an increasing order respect to lex. And let $deg(l_r) = p$. We need to prove that L_q is lex $\forall q \leq p$. Fix q less than p. Any monomial in L_q must be in the form $l_j s$ where $s \in S_{q-deg(l_j)}$. Assume that $\exists m \in S_q$ such that $l_j s < m$. Choose lex first $deg(l_j)$ divisors of m, let their multiplication be l. Then clearly $l > l_j$ which together with assertion b implies that $l \in L_{deg(l_j)}$. It results in $m = l\frac{m}{l} \in L_q$ since $\frac{m}{l} \in S_{q-degl_j}$. These all implies that L_q is a lex monomial space. We are done.

□

Theorem 3.0.42. *The following statements are equivalent.*

(1) For any q Let A_q be an S_q-monomial space and L_q be its lexification in S_q, then we have $|S_1 L_q| \leq |S_1 A_q|$.

(2) *For every graded ideal J there exists a lex ideal L having the same Hilbert function with J.*

Proof. **(1 $\implies$ 2)** For notational simplicity denote $K = in(J)$. By 2.0.24 we know that K is a monomial ideal having the same Hilbert function with J. Now let L_q be the lexification of K_q. Now let $L = \bigoplus L_q$. L has the same Hilbert function with K then so with J. It remains to prove that L is an ideal indeed. For that it clearly suffices to prove that $S_1 L_q \subseteq L_{q+1}$. Since L_q is lex, by 3.0.37 it follows that $S_1 L_q$ is lex also. Therefore if we prove that $|S_1 L_q| \leq |L_{q+1}|$ we are done. By applying the assertion **(1)** on the S_q-monomial space K_q we get $S_1 L_q \leq S_1 K_q$. Also since K is an ideal we have $S_1 K_q \subseteq K_{q+1}$ which implies $|S_1 K_q| \leq |K_{q+1}| = |L_{q+1}|$ clearly. Assembling the last two inequalities together $|S_1 L_q| \leq |S_1 K_q| \leq |K_{q+1}| = |L_{q+1}|$ we get what we want. As a result **(1)** implies that for every graded ideal J there exist a lex ideal L which has the same Hilbert function with J .

(2 $\implies$ 1) Consider the ideal (A_q). By assertion **(2)** there exists a lex ideal L with the same Hilbert function with (A_q). Now clearly L_q is the lexification of $A_q = (A_q)_q$. And since L and (A_q) have the same Hilbert function $|L_{q+1}| = |(A_q)_{q+1})| = |S_1 A_q|$ is obvious. Also since L is an ideal $|S_1 L_q| \leq |L_{q+1}|$ is obtained. Combining all together we get the desired inequality $|S_1 L_q| \leq |S_1 A_q|$. □

In 2.0.24 it is proven that $|I_q| = |(in(I))_q|$ for any $q \geq 0$ which can be rephrased as $\beta_{0,q}(J) \leq \beta_{0,q}(in(J))$ for all graded ideals J and all q. We represent the following generalization of it.

Theorem 3.0.43 (1, Corollary 1.21)**.** *For every graded ideal J we have $\beta_{i,i+j}(J) \leq \beta_{i,i+j}(in(J))$ for all i, j.*

Now we proceed to the theorem proven by Macaulay which boost power to almost all results that will be shown in this paper and probably to many ones not in this paper.

Theorem 3.0.44 (Macaulay)**.** *The equivalent statements in the Proposition 3.0.42 hold.*

The proof of the Maculay's theorem will be done in the chapter 5.

Definition 3.0.45. We say that an S_q-monomial space A_q is Borel if for an arbitrary monomial $m \in S_{q-1}$ $x_j m \in A_q$ implies that $x_i m \in A_q$ for all $1 \leq i \leq j$.

Proposition 3.0.46. *All lex S_q-monomial spaces are also Borel but the converse is not true.*

Proof. Let A_q be lex and $x_j m \in A_q$, then every monomials greater than $x_j m$ must be in A_q. Then $x_j m < x_i m$ $\forall i$ s.t. $1 \leq i \leq j$ we get that $x_i m \in A_q$ $\forall i$ s.t. $1 \leq i \leq j$ which makes A_q Borel. Then it is understood that every lex monomial spaces are also Borel.

For the converse we have a counter example. Let A_2 be the S_2-monomial space generated by the set of monomials $\{x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^2\}$ where $S = k[x_1, x_2, x_3, x_4]$. This is clearly a Borel S_2-monomial space but not lex because it does not include the element x_1x_4 which is greater than $x_2x_3 \in A_2$. $\square$

Chapter 4

Multicompression Technique

Let $A \subset \{x_1, x_2, ..., x_n\}$ and let C_q be a monomial space. We can write it as $C_q = \bigoplus_m m V_m$ where V_m is a monomial space in the ring $S/A^c = k[A]$, and m is a monomial in variables A^c. We will make our definition upon this construction

Definition 4.0.47. C_q is called A-multicompressed if all V_m are lex in $k[A]$. It is called multicompressed if C_q is A-multicompressed for all possible $A \subset \{x_1, x_2, ..., x_n\}$. It is called (j)-multicompressed if it is A-multicompressed for all sets $A \subset \{x_1, x_2, ..., x_n\}$ of size j.

Definition 4.0.48. If C_q is A-multicompressed when $A = \{x_1, x_2, ..., x_n\}/x_j$ then we call it j-compressed rather than calling it $\{x_1, x_2, ..., x_n\}/x_j$-multicompressed. In this case C_q looks more simple, $C_q = \bigoplus_{1 \leq i \leq q} x_j^{q-i} V_i$ where V_i are monomial spaces in S/x_j. And we call C_q a compressed monomial space if it is j-compressed for all j s.t $1 \leq j \leq n$.

Definition 4.0.49. Continuing on the construction done in the beginning of this section, we define the A-multicompression of the monomial space C_q as the monomial space $\bigoplus_m m L_m$ where L_m's are lexifications of V_m's in the ring $k[A]$. Also we define the j-compression of C_q with the same way.

Proposition 4.0.50. *If L_q is a lex monomial space then it is a A-multicompressed monomial space $\forall A \subset \{1, 2, ..., n\}$.*

Proof. Let $L_q = \bigoplus_{m \in k[A^c]} mV_m$ where V_m are monomial space in $(k[A])_{q-deg(m)}$. Assume that L_q is not A-multicompressed. This means V_m is not lex for some m. Then $\exists w \in V_m$ s.t. $\exists u \in (k[A])_{q-deg(m)}$ where $u > w$ but $u \notin (k[A])_{q-deg(m)}$. Then we get $mu > mw$ and $mu \notin L_q$ clearly. This contradicts with the lex property of L_q. So we proved that L_q is A-multicompressed for all $A \subset \{1, 2, ..., n\}$. □

Proposition 4.0.51. *If C_q is a (i)-multicompressed monomial space then it is (j)-multicompressed for all j s.t. $1 \leq j < i$.*

Proof. Fix a j s.t $1 \leq j < i$. Also let A be an arbitrary subset of $\{1, 2, ..., n\}$ having j elements. Complete A to another subset B of $\{1, 2, ..., n\}$ with i elements by adding required number of elements arbitrarily.
Since C_q is (i)-multicompressed it must be B-multicompressed also. Therefore we can write $C_q = \bigoplus_{m \in k[B^c]} mV_m$ where V_m are lex monomial spaces in $(k[B])_{q-deg(m)}$. Since $A \subset B$ and V_m are lex in $k[B]$ we get by 4.0.50 that V_m are A-multicompressed monomial space in $K[B]$. Then one can write $V_m = \bigoplus_{w \in k[B/A]} wK_w$ where K_w are lex in $(k[A])_{q-deg(m)-deg(w)}$. Therefore we get

$$C_q = \bigoplus_{\substack{m \in k[B^c] \\ w \in k[B/A]}} mwK_w = \bigoplus_{mw \in k[A^c]} (mw)K_w \text{ where } K_w \text{ are lex in } (k[A])_{q-deg(mw)}$$

So we proved that C_q is A-multicompressed. Since the choice of the set A and j are arbitrary we get C_q is (j)-multicompressed for all j satisfying $1 \leq j < i$. □

Theorem 4.0.52 (Mermin)**.** *Let C_q be a monomial space. We have the following statements that characterize the Borel monomial spaces and lex monomial spaces.*

(a) C_q is Borel $\Leftrightarrow$ C_q is a (2)-multicompressed monomial space.

(b) C_q is lex $\Leftrightarrow$ C_q is a (3)-multicompressed monomail space.

Proof. **(a)($\Leftarrow$)** Let C_q be (2)-multicompressed and $mx_i \in C_q$. To prove C_q is Borel, we need to show that $mx_j \in C_q$ for all $1 \leq j < i$. Let $A = \{x_i, x_j\}$. Since C_q is (2)-multicompressed it is A-multicompressed also. Therefore $C_q =$

$\bigoplus_{m'\in k[A^c]} m'V_{m'}$ where $V_{m'}$ are lex in $k[A]$. Now let $m = m'x_i^\alpha x_j^\beta$ where m' is not divisible by either of x_i and x_j. Then since $mx_i \in C_q$ we get $x_i^{\alpha+1}x_j^\beta \in V_{m'}$. We have $x_i^\alpha x_j^{\beta+1} > x_i^{\alpha+1}x_j^\beta$ since $x_j > x_i$. And since $V_{m'}$ is lex we get $x_i^\alpha x_j^{\beta+1} \in V_{m'}$ $\Rightarrow mx_j = m'x_i^\alpha x_j^{\beta+1} \in C_q$. We are done.

(a)($\Rightarrow$) Assume now C_q is a Borel space. Let $A = \{x_i, x_j\}$ be an arbitrary 2 element set where $1 \leq j < i$. We need to prove that C_q is A-multicompressed. It is possible to write $C_q = \bigoplus_{m\in k[A^c]} mV_m$. In this setup it suffices to prove V_m is lex $\forall m$. Fix m and take an arbitrary element $x_i^\alpha x_j^\beta$ from V_m and an arbitrary element $x_i^{\alpha'}x_j^{\beta'}$ from $(k[A])_{q-degm}$ such that $x_i^{\alpha'}x_j^{\beta'} > x_i^\alpha x_j^\beta$. To ensure this last condition on the arbitrary choice done we need $\beta' \geq \beta$. Therefore let $\beta' = \beta + t$ where $t \geq 0$. And since $\alpha' + \beta' = \alpha + \beta$ we get $\alpha = \alpha' + t$. Now since C_q is Borel we get;

$$mx_i^{\alpha'+t}x_j^\beta = mx_i^\alpha x_j^\beta \in C_q \Rightarrow mx_i^{\alpha'+t-1}x_j^{\beta+1} \in C_q.$$

Doing the same operation t times, we get

$$mx_i^{\alpha'}x_j^{\beta+t} = mx_i^{\alpha'}x_j^{\beta'} \in C_q \Rightarrow x_i^{\alpha'}x_j^{\beta'} \in V_m.$$

It proves that V_m is lex since the choice of α, β, α' and β' were arbitrary. We are done.

(b)($\Rightarrow$) Let C_q be lex. Then by 4.0.50 we get it is A-compressed for every set $A \in \{1, 2, ..., n\}$. It clearly implies that C_q is a (3)-compressed monomial space.

(b)($\Leftarrow$) Now let C_q be (3)-multicompressed. Let $u \in C_q$ and $m \in S_q$ such that $m > u$, $deg(m) = deg(u) = q$. We can represent them as follows, $m = x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$ and $u = x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n}$ where $\sum \alpha_l = \sum \beta_l = q$. Let i be the number such that $\alpha_i = \beta_i + t$ for a $t \in Z^+$ and $\alpha_j = \beta_j$ for all j satisfying $1 \leq j < i$. Let $z = x_1^{\beta_1}x_2^{\beta_2}...x_{i-1}^{\beta_{i-1}} = x_1^{\alpha_1}x_2^{\alpha_2}...x_{i-1}^{\alpha_{i-1}}$. Then $m = zx_i^{\alpha_i}x_{i+1}^{\alpha_{i+1}}...x_n^{\alpha_n}$ and $u = zx_i^{\beta_i}x_{i+1}^{\beta_{i+1}}...x_n^{\beta_n}$. Now lets consider the case that $i = n$; it is not possible because $\sum \alpha_l = \sum \beta_l$. Lets analyze the case $i = n - 1$. In this case we have $\alpha_{n-1} = \beta_{n-1}+t$. Then $\beta_n = \alpha_n+t$ follows here since $\sum \alpha_l = \sum \beta_l$. Since C_q is (3)-multicompressed it is also (2)-multicompressed by 4.0.51 . Combining this with

(a) we have that C_q is Borel. Then we get $\frac{u}{x_n}x_{n-1} = zx_{n-1}^{\alpha_{n-1}-(t-1)}x_n^{\alpha_n+t-1} \in C_q$. Doing this last operation t times we get $m \in C_q$. Since the choice of the m and u were arbitrary we get C_q lex which is the desired conclusion. Now it remains to work on the case $n > i+1 > i$. Since C_q is Borel and $u \in C_q$ we have $zx_i^{\beta_i}x_{i+1}^e \in C_q$ where $e = \beta_{i+1} + ... + \beta_n$. Let $A = \{x_i, x_{i+1}, x_n\}$. Since C_q is (3)-multicompressed it is A-multicompressed. Combining this with the fact that $x_i^{\beta_i+t}x_n^{e-t} > x_i^{\beta_i}x_{i+1}^e$ we get $zx_i^{\beta_i+t}x_n^{e-t} \in C_q$. Then using the Borel property of C_q in a good way we can get that $m \in C_q$. So it follows that C_q is lex, since the choice of m and u were arbitrary. □

Lemma 4.0.53. *Let $A \subseteq \{1, 2, ..., n\}$ and A_q be a monomial space. Then there exists a A-multicompressed monomial space T_q s.t. $|A_q| = |T_q|$ and $|S_1T_q| \leq |S_1A_q|$.*

Lemma 4.0.54. *Let A_q be an S_q-monomial space. Then for each fixed $j \in \{1, 2, ..., n-1\}$ $\exists$ a (j)-multicompressed monomial space C_q in S_q s.t. $|C_q| = |A_q|$ and $|S_1C_q| \leq |S_1A_q|$.*

Proof. If A_q is (j)-compressed there is nothing to prove. If not, there exists $A \subset \{1, 2, ..., n\}$ with j elements where A_q is not A-multicompressed. By 4.0.53 there exist an A-multicompressed monomial space T_q^1 s.t $|T_q^1| = |A_q|$ and $|S_1T_q^1| \leq |S_1A_q|$. It is clear that T_q^1 is a lex greater monomial space than A_q. If T_q^1 is (j)-compressed monomial space we are done. If not, apply same procedure to T_q^1 and obtain another monomial space T_q^2 lex greater than T_q^1 and satisfying $|T_q^1| = |T_q^2|$, $|S_1T_q^2| \leq |S_1T_q^1|$. Continuing the same procedure successively we get in each step a lex greater monomial space satisfying the desired inequalities. Since it can not get lex greater forever, at some point we will reach a (j)-multicompressed monomial space T_q with $|T_q| = |A_q|$ and $|S_1T_q| \leq |S_1A_q|$. □

Definition 4.0.55. For a monomial m we denote;

$$max(m) = max\{i \mid x_i \text{ divides } m\}, \;\; min(m) = min\{i \mid x_i \text{ divides } m\}.$$

Also we have some notation to introduce here;

$$\Gamma_{i,j}(A_q) = |\{m \in A_q \mid max(m) \leq i \;,\; x_i^j \nmid m\}|, \;\; t_i(A_q) = |\{m \in A_q \mid max(m) \leq i\}|.$$

Lemma 4.0.56. *Let B_q be a Borel S_q-monomial space. Then we have;*

$$\{S_1 B_q\} = \coprod x_i \{m \in \{B_q\} | max(m) \leq i\}.$$
$$and\ , |S_1 B_q| = \sum_1^n t_j(B_q).$$

Proof. For notational simplicity let $B = \coprod x_i\{m \in \{B_q\}|max(m) \leq i\}$. It is clear that $B \subseteq \{S_1B_q\}$. Now it suffices to prove $\{S_1B_q\} \subseteq B$ is true. Let ux_j be an arbitrary element in $\{S_1B_q\}$ and let $max(u) = t$. If $t > j$ then $\frac{u}{x_t}x_j \in B_q$ since B_q is Borel. Then we get $ux_j = (\frac{u}{x_t}x_j)x_t \in x_t\{m \in B_q|max(m) \leq t\} \Rightarrow ux_j \in B$. For the case $j \geq t$ we have directly $ux_j \in x_j\{m \in B_q|max(m) \leq j\} \Rightarrow ux_j \in B$. So for all cases we proved that an arbitrary monomial taken from S_1B_q falls into B. Which shows that $\{S_1B_q\} = B$ as desired. Now lets prove that the elements of the union are indeed disjoint. Assume that $x_i\{m \in B_q|max(m) \leq i\}$ and $x_j\{m \in B_q|max(m) \leq j\}$ share a common element $x_im = x_jm' \Rightarrow x_i|m'$ and $x_j|m \Rightarrow i \leq j$ and $j \leq i \Rightarrow i = j$. So all elements of the union is disjoint. Then we clearly get $|S_1B_q| = \sum_1^n t_j(B_q)$. □

Lemma 4.0.57. *Let A_q be a Borel S_q-monomial space. Then its n-compression is also Borel.*

Lemma 4.0.58. *Let C_q be a n-compressed Borel monomial space and L_q be a lex monomial space with $|L_q| \leq |C_q|$. Then $\Gamma_{i,j}(L_q) \leq \Gamma_{i,j}(C_q)$ for all $1 \leq i \leq n$ and $1 \leq j$.*

Proof. Start with the case $i = n$; by directly from definition of $\Gamma_{i,j}$ we get $\Gamma_{n,q+1}(L_q) = |L_q| \leq |C_q| \leq \Gamma_{n,q+1}(C_q)$. Let it be the first step of the decreasing induction on j. Assume that $\Gamma_{n,j+1}(L_q) \leq \Gamma_{n,j+1}(C_q)$ is true. Based on this induction assumption we will prove that $\Gamma_{n,}(L_q) \leq \Gamma_{n,j}(C_q)$. The definition of $\Gamma_{i,j}$ implies $\Gamma_{n,j}(L_q) \leq \Gamma_{n,j+1}(L_q)$ by free. If C_q does not contain any monomial divisible by x_n^j then $\Gamma_{n,j+1}(C_q) = \Gamma_{n,j}(C_q)$. Combining these together with the induction assumption we get the chain of the inequalities;

$$\Gamma_{n,j}(L_q) \leq \Gamma_{n,j+1}(L_q) \leq \Gamma_{n,j+1}(C_q) = \Gamma_{n,j}(C_q).$$

So we get $\Gamma_{n,j}(L_q) \leq \Gamma_{n,j}(C_q)$ which completes the proof for the case C_q does not contain any monomial divisible by x_n^j.

Now, consider the case where C_q has an element divisible by x_n^j. Let $e = x_1^{e_1} x_2^{e_2} ... x_n^{e_n}$ be the lex last monomial with the property $e_n \geq j$. Lets prove any monomial $m \in S_q$ that is lex greater than e and $x_n^p \| m$ where $1 \leq p \leq j-1$ falls into C_q. The lex last monomial with the mentioned property is clearly $f =: (\frac{e}{x_n^{e_n - p}}) x_{n-1}^{e_n - p}$. f is in C_q because C_q is Borel. Now Since C_q is n-compressed it is written as direct sum of the lex sets along with the multiplication by suitable factors of x_n. Therefore since m and f share same factor of x_n and $m > f$ we can say that $m \in C_q$. That constitutes to the equality;

$$\{m \in C_q \big| x_n^j \nmid m, m > e\} = \{m \in S_q \big| x_n^j \nmid, m > e\}. \tag{1}$$

We will write down a series of results that will be of use;

$$|\{m \in L_q \big| x_n^j \nmid m, m > e\}| \leq |\{m \in S_q \big| x_n^j \nmid m, m > e\}|. \tag{2}$$

$$|\{m \in L_q \big| x_n^j \nmid m, m < e\}| \leq |\{m \in L_q \big| m < e\}|. \tag{3}$$

$$|\{m \in L_q \big| m < e\}| \leq |\{m \in C_q \big| m < e\}|. \tag{4}$$

$$\{m \in C_q \big| m < e\} = \{m \in C_q \big| x_n^j \nmid m, m > e\}. \tag{5}$$

(2) and **(3)** comes directly from enlarging the conditions on m. Lets now prove **(4)** is true. If there is no element in L_q less than e then **(4)** comes true directly. So assume $|\{m \in L_q \mid m < e\}| = h > 0$. Then since L_q is lex, all elements greater or equal to e falls into L_q. Then there are $|L_q| - h$ many elements greater or equal to e in S_q. Then combining this with the fact that $|C_q| \geq |L_q|$ we get $|\{m \in C_q \big| m < e\}| \leq |C_q| - (|L_q| - h) = (|C_q| - |L_q|) + h \geq h = |\{m \in L_q \big| m < e\}|$. So it is done. It is left to prove **(5)**. The sets are same because by definition of e there is no monomial $m \in C_q$ less than e and divisible by x_n^j.

Using (1),(2),(3),(4) and (5) successively we get the following chain of inequalities;

$$\begin{aligned}\Gamma_{n,j}(L_q) &= |\{m \in L_q \big| x_n^j \nmid m, m > e\}| + |\{m \in L_q \big| x_n^j \nmid m, m < e\}| \\ &\leq |\{m \in S_q \big| x_n^j \nmid m, m > e\}| + |\{m \in L_q \big| m < e\}| \\ &\leq |\{m \in C_q \big| x_n^j \nmid m, m > e\}| + |\{m \in C_q \big| m < e\}| \\ &= |\{m \in C_q \big| x_n^j \nmid m, m > e\}| + |\{m \in C_q \big| x_n^j \nmid, m < e\}| = \Gamma_{n,j}(C_q). \quad \textbf{(6)}\end{aligned}$$

Then we get $\Gamma_{n,j}(L_q) \leq \Gamma_{n,j}(C_q)$ which completes the decreasing induction argument on j. Setting $j = 1$ in the proven inequality $\Gamma_{n,1}(L_q) \leq \Gamma_{n,1}(C_q)$ is obtained. We have $\Gamma_{n,1}(L_q) = \{m \in L_q \big| x_n \nmid m\} = L_q/x_n$, similarly $\Gamma_{n,1}(C_q) = C_q/x_n$ is true. Since C_q is n-compressed C_q/x_n is lex. Also L_q/x_n is lex since L_q is lex. Then combining these together it follows that $L_q/x_n \subseteq C_q/x_n$. From this $L_q/\{x_n, x_{n-1}, ..., x_{n-t+1}\} \subseteq C_q/\{x_n, x_{n-1}, ..., x_{n-t+1}\}$ follows. Moreover;

$$\Gamma_{n-t,j}(C_q) = \Gamma_{n-t,j}(C_q/\{x_n, x_{n-1}, ..., x_{n-t+1}\}).$$

and;

$$\Gamma_{n-t,j}(L_q) = \Gamma_{n-t,j}(L_q/\{x_n, x_{n-1}, ..., x_{n-t+1}\}).$$

Then these imply for all t that $\Gamma_{n-t,j}(L_q) \leq \Gamma_{n-t,j}(C_q)$ which ends the proof. $\square$

Lemma 4.0.59. *Let B_q be Borel and L_q be a lex monomial space in S_q with $|L_q| \leq |B_q|$. Then we have $t_i(L_q) \leq t_i(B_q)$ and $\Gamma_{i,j}(L_q) \leq \Gamma_{i,j}(B_q)$.*

Proof. Let C_q be n-compression of B_q. Since B_q is Borel by 4.0.57 C_q is Borel. Moreover by 4.0.58 we have $\Gamma_{i,j}(L_q) \leq \Gamma_{i,j}(C_q)$. Therefore it remains to prove that $\Gamma_{i,j}(C_q) \leq \Gamma_{i,j}(B_q)$. Writing $B_q = \bigoplus\limits_{0 \leq j \leq q} x_n^j K_j$ and $C_q = \bigoplus\limits_{0 \leq j \leq q} x_n^j T_j$ we see by definiton of n-compression that T_j is lexification of K_j. Then $|K_j| = |T_j|$. Using this we get

$$\Gamma_{n,j}(C_q) = \sum_{0 \leq k < j} |T_k| = \sum_{0 \leq k < j} |K_k| = \Gamma_{n,j}(B_q).$$

Now we argue induction on n, the number of variables. Let $i < n$, then clearly $\Gamma_{i,j}(C_q) = \Gamma_{i,j}(C_q/x_n)$ and $\Gamma_{i,j}(B_q) = \Gamma_{i,j}(B_q/x_n)$. Here we get $C_q/x_n = \coprod T_j$

is lex in S_q/x_n and B_q/x_n is Borel in S_q/x_n since B_q is Borel. Since S_q/x_n have $n-1$ variables by induction hypothesis we are done. $\square$

The proof's of 4.0.53 and 4.0.57 will be done in next section along with the proof of macaulay's theorem since they are closely related.

Chapter 5

Proof of the Macaulay's Theorem

Proof. We argue induction on t, the number of variables. It is clear for $t = 1$ that $|S_1L_q| = |S_1A_q| = 1$. For the case $t = 2$ we have $S = k[x_1, x_2]$. Ordering of the monomials in S_q is as $x_1^q > x_1^{q-1}x_2 > x_1^{q-2}x_2^2 > ... > x_1x_2^{q-1} > x_2^q$. Let $z = |A_q| = |L_q|$. Then the lex last monomial in L_q is clearly $x_1^{q-z+1}x_2^{z+1}$. Therefore the lex last element of S_1L_q becomes $x_1^{q-z+1}x_2^z$. Then using this with the fact that S_1L_q is lex by 3.0.37 we get $|S_1L_q| = z + 1$. Now it is left to prove that $|S_1A_q| \geq z + 1$. Let $m_1 > m_2 > ... > m_z$ be elements of A_q. Then we get $m_1x_1 > m_2x_1 > ... > m_zx_1 > m_zx_2$. All of these are different and in S_1A_q. Then we proved that $\{S_1A_q\}$ has at least $z - 1$ elements. So we are done with the case $t = 2$ and $t = 1$.

Let those be the base step of the induction. Assume for $t \leq n-1$ the Macaulay's theorem is true. We nned to prove it is true for $t = n$. We will do so by proving the following assertions one by one. The notation 3.0.44($n-1$) means the theorem 3.0.44 is true for all $t \leq n-1$. Same notation is used for everything below.

$$3.0.44(n-1) \Rightarrow 4.0.57(n) \Rightarrow 4.0.59(n). \qquad \textbf{(1)}$$

$$3.0.44(n-1) \Rightarrow 4.0.53(n) \Rightarrow 4.0.54(n). \qquad \textbf{(2)}$$

$$3.0.44(n-1) \Rightarrow 4(n). \qquad \textbf{(3)}$$

$$4(n) \Rightarrow 3.0.44(n). \qquad \textbf{(4)}$$

For (1); 4.0.57(n) $\Rightarrow$ 4.0.59(n) was proven in the previous section. Therefore lets focus on proving the first implication. Let $A_q = \coprod_{0\leq j\leq q} x_n^{q-j}K_j$ and $C_q = \coprod_{0\leq j\leq q} x_n^{q-j}L_j$. From the notation L_j is lexification of K_j. Let $mx_i \in C_q$ then we are to prove that $mx_l \in C_q$ for all $1 \leq l \leq i$. Write $mx_i = ux_ix_n^{\alpha}$, $\alpha \geq o$ and $x_n \nmid u$. Clearly $ux_i < ux_l$, then since $ux_i \in L_{q-\alpha}$ and $L_{q-\alpha}$ is lex we get $ux_l \in L_{q-\alpha} \Rightarrow mx_l \in C_q$. These all work for the case $i \neq n$. Now consider the case $i = n$; write $mx_n = yx_n^{\beta}$ where $x_n \nmid y$. To show C_q is Borel we need to prove that $(yx_n^{\beta-1})x_l = mx_l \in C_q$ for all $l < n$. We have both K_j and L_j lying in S_1/x_n. So we can apply 3.0.44(n-1) here. Applying it, we get $|S_1/x_nL_j| \leq |S_1/x_nK_j|$. Moreover since A_q is Borel it follows that $(S_1/x_n)K_j \subseteq K_{j+1}$. Combining them together we get $|(S_1/x_n)L_j| \leq |(S_1/x_n)K_j| \leq |K_{j+1}| = |L_{j+1}|$. From here we get $(S_1/x_n)L_j \subseteq L_{j+1}$.(It is because both of them are lex) $y \in L_{q-\beta}$, then by the previous achievement it is seen $yx_l \in L_{q-\beta+1}$ where $l < n \Rightarrow yx_lx_n^{\beta-1} \in C_q$. So it is done.

For (2); 4.0.53(n) $\Rightarrow$ 4.0.54(n) was again proven in previous section. So it suffices to prove the first implication only. We have for every $t < n$ Macaulay's theorem true. let $A_q = \bigoplus_{m\in k[A^c]} mV_m$ where V_m is a monomial space in $S/k[A^c] = k[A]$ of degree $q - deg(m)$. Now let $T_q = \bigoplus_{m\in k[A^c]} mL_m$ where L_m's are lexifications of V_m's. Since $|L_m| = |V_m|$ we clearly have $|A_q| = |T_q|$. Now lets prove $|S_1A_q| \geq |S_1T_q|$. It is easy to see that $S_1A_q = \bigoplus_{m\in k[A^c]} m\{\coprod_{x_j\in A^c} L_{m/x_j} + AV_m\}$ and $S_1T_q = \bigoplus_{m\in k[A^c]} m\{\coprod_{x_j\in A^c} L_{m/x_j} + AL_m\}$. According to that it suffices to prove $|\coprod L_{m/x_j} + AL_m| \leq |\coprod V_{m/x_j} + AV_m|$. L_{m/x_j} is lex and of degree $q - deg(m) + 1$. Also AL_m is lex in k[A] by 3.0.37 and it is of degree $q - deg(m) + 1$. Therefore it follows that $|\coprod L_{m/x_j} + AL_m| = max\{|AL_m|, |L_{m/x_j}| | x_j \in A^c\}$. On the other hand we have $|L_{m/x_j}| = |V_{m/x_j}|$ and by 3.0.44(n-1) $|AL_m| \leq |AV_m|$.(Here keep in mind that $|A| \leq n-1$ by definition of A-compression) Using these and combining the previous observations we get the following chain of inequalities;

$$|\coprod L_{m/x_j} + AL_m| = max\{|AL_m|, |L_{m/x_j}||x_j \in A^c\}$$
$$\leq max\{|AV_m|, |V_{m/x_j}||x_j \in A^c\} \leq |\coprod V_{m/x_j} + AV_m|.$$

We are done.

For (3); this is added to stress on the point that almost whole $4(n)$ is now standing on the body of 3.0.44(n-1). Therefore in **(4)** we can use $4(n)$.

For (4); this is the main part of the induction proof that uses $4(n)$. Since we have $4.0.54(n)$ and $n > 2$ it follows that there exist a (2)-multicompressed monomial space C_q s.t $|C_q| = |A_q|$, and $|S_1 C_q| \leq |S_1 A_q|$. C_q is Borel by 4.0.52. Let L_q be lexification of C_q. Clearly it is also lexification of A_q. Lets prove $|S_1 L_q| \leq |S_1 C_q|$. By using 4.0.56 $|S_1 L_q| = \sum_{i=1}^{n} t_i(L_q)$, and $|S_1 C_q| = \sum_{i=1}^{n} t_i(C_q)$. Since by 4.0.59(n), $t_i(L_q) \leq t_i(C_q)$ $\forall i$ it follows that $|S_1 L_q| \leq |S_1 C_q|$. So we are done. $\square$

Here it is worthy to mention that almost all results given in chapter 4 are turned true in the process of proving Macaulay's theorem. If it was not true, almost all of the chapter 4 would collapse. In the following chapters we will make use of some of these results to prove a continuation of Macaulay's theorem.

Chapter 6

Statement of the Main Theorem

For every graded ideal J we know by Macaulay's Theorem that there is a lex ideal L with the same Hilbert function as J. Moreover for that ideal we have $|S_1L_q| \leq |S_1J_q|$ by 3.0.42. Lets now present a corollary which establish a relation with Macaulay's Theorem and the the betti numbers of the graded ideals.

Definition 6.0.60. Let M be a monomial ideal. Then we denote the set of degree q minimal monomial generators of M as $G_q(M)$.

Corollary 6.0.61. *Let J be a graded ideal and L is a lex ideal with the same Hilbert function as J. Then $\beta_{0,q}(J) \leq \beta_{0,q}(L)$ for all q.*

Proof. By 3.0.40 we have $\beta^S_{0,q}(J) = |J_q| - |S_1J_{q-1}|$ and $\beta^S_{0,q}(L) = |L_q| - |S_1L_{q-1}|$. Since L and J has the same Hilbert function $|L_q| = |J_q|$. And by 3.0.42 $|S_1L_q| \leq |S_1J_q|$. These all together constitutes to $\beta^S_{0,q}(J) = |J_q| - |S_1J_{q-1}| \leq |L_q| - |S_1L_{q-1}| = \beta^S_{0,q}(L)$ which completes the proof. □

This corollary determine an upper bound for the zeroth betti numbers of graded ideals. Now it is time to present an extension of this which is the main theorem of this paper. The proof will be given in section 8.

Theorem 6.0.62 (Main Theorem). *Assume* $char(k) = 0$ *and let* J *be a graded ideal in* S. *Let* L *be a lex ideal with the same Hilbert function with* J. *Then* $\beta_{i,j}(J) \leq \beta_{i,j}(L)$ *for all* i, j.

Chapter 7

Eliahou-Kervaire Resolution

Definition 7.0.63. For a monomial m denote $max(m) = max\{i \mid x_i \text{ divides } m\}$ and $min(m) = min\{i \mid \text{ divides } m\}$.

Proposition 7.0.64. *A Borel ideal M is given. If $w \in M$ is a monomial, then there is a unique decomposition $w = rl$ s.t r is a minimal monomial generator of M and $max(r) \leq min(l)$.*

Proof. We first prove that there exist $m \in M$ s.t $w = m.v$ and $max(m) \leq min(v)$. The case w is a minimal monomial generator is trivial. Now assume w is not a minimal monomial generator. Then $\exists u_0 \in M$ s.t $w = u_0 v_0$. If $max(u_0) \leq min(v_0)$ we are done. Otherwise let $max(u_0) = j_0$, $min(v_0) = i_0$ and $j_0 > i_0$. Then since M is Borel and $u_0 \in M$ we have $u_1 := \frac{u_0}{x_{j_0}} x_{i_0} \in M$. And $w = u_0 v_0 = \frac{u_0}{x_{j_0}} x_{i_0} \frac{v_0}{x_{i_0}} x_{j_0}$. Clearly $j_1 := max(u_1) = max(\frac{u_0}{x_{j_0}} x_{i_0}) < max(u_0) = j_0$. Also let $v_1 = \frac{v_0}{x_{i_0}} x_{j_0}$. Again clearly $i_1 := min(v_1) = min(\frac{v_0}{x_{i_0}} x_{j_0} < min(v_0) = i_0$. To sum up, $w = u_1 v_1$, $u_1 \in M$ and $i_1 \geq i_0$, $j_0 \geq j_1$. Continuing the same process we generate an increasing sequence $(i_k)_0^\infty$ and a decreasing sequence $(j_k)_0^\infty$ with the monomials $u_k \in M$ and v_k. Therefore for a suitable k we will get $j_k \leq i_k$, at this point set $m = u_k$ and $v = v_k$.

Now using this lets prove the existence part of the statement. By previous result we have $m \in M$ s.t $w = m.v$ and $max(m) \leq min(v)$. If m is a minimal monomial generator we are done. Otherwise applying same thing on m we get $\exists m_0 \in M$ s.t $m = m_0 h_0$ where $max(m_0) \leq min(h_0)$. Clearly $max(m_0) \leq max(m_0 h_0) = max(m) \leq min(v)$. To sum up all of them, $min(vh_0) = min(min(v), min(h_0)) \geq max(m_0)$. So we get $max(m_0) \leq min(vh_0)$. Lets apply the process that we followed for m to m_0. Then we get $\exists m_1 \in M$ s.t $m_0 = m_1 h_1$, $max(m_1) \leq min(vh_0 h_1)$. Continuing successively we get $w = m_k h_k h_{k-1} ... h_1 h_0 v$ s.t $m_k \in M$ and $max(m_k) \leq min(h_k h_{k-1} ... h_1 h_0 v)$. Since m_k is getting smaller at each step it will soon reach to a m.m.g, say at the step N. Then set $m_N = r$ and $h_N h_{N-1} ... h_1 h_0 v = l$ where we clearly proved that $max(r) \leq min(l)$. So the existence part is done. For the uniqueness, let $w = r_1 l_1 = r_2 l_2$ where r_1 and r_2 are minimal monomial generators with $max(r_1) \leq min(l_1)$ and $max(r_2) \leq min(l_2)$. Now let $w = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ and $r_1 = x_1^{\alpha_1} x_2^{\alpha_2} ... x_s^{\theta_s}$, $r_2 = x_1^{\alpha_1} x_2^{\alpha_2} ... x_t^{\beta_t}$. Now without loss of generality let $(\sum_{i=1}^{s-1} \alpha_i) + \theta_s \leq (\sum_{i=1}^{t-1} \alpha_i) + \beta_t \Rightarrow r_1 \mid r_2 \Rightarrow r_1 = r_2$ since both of them are m.m.g. □

Definition 7.0.65. In the notation of we define the beginning and end of a monomial w in M as $b(w) := r$ and $e(w) := l$ respectively.

Remark 7.0.66. Let M be a Borel ideal and A is a monomial in it. If $B = A.C$ s.t $max(A) \leq min(C)$ we have $b(B) = b(A)$.

Lemma 7.0.67. *Let M be a Borel ideal and $m \in M$. Then we have;*

$$b(b(mx_t)x_q) = b(b(mx_q)x_t) \ \ \forall t, q \ \ s.t. \ \ t < q < max(m).$$

Proof. Let $m = x_1^{\alpha_1} x_2^{\alpha_2} ... x_t^{\alpha_t} ... x_q^{\alpha_q} ... x_n^{\alpha_n}$ and let $b(mx_t) = x_1^{\alpha_1} ... x_s^{\theta_s}$, where here $\alpha_t := \alpha_t + 1$, $b(mx_q) = x_1^{\alpha_1} ... x_l^{\theta_l}$ where here $\alpha_q := \alpha_q + 1$.

Lets first consider the case $s \leq l$. If $b(mx_t) = b(mx_q)$ then clearly $s = l$. Moreover we have $s = l \leq t$, because otherwise $x_t^{\alpha_t + 1}$ would divide $b(mx_t)$ but not $b(mx_q)$. So $max(b(mx_t)) = max(b(mx_q)) \leq t < q$, then by using 7.0.66 we get the following;

$$b(b(mx_t)x_q) = b(mx_t) = b(mx_q) = b(b(mx_q)x_t).$$

We are done. Combining this result with the fact that $b(mx_t)$ and $b(mx_q)$ are m.m.g we see that it is left to analyze the subcase $b(mx_t) \nmid b(mx_q)$. Only cases making this possible are obviously $s \geq t$ or $s = t$ and $\theta_s = \alpha_t + 1$. Then we have $l \geq s \geq t$ which implies $x_t^{\alpha_t} \| b(mx_q)$. Therefore $(mx_t) \mid b(mx_q)x_t$. Then for the case $s \leq q$ we get by 7.0.64 that $b(b(mx_q)x_t) = b(mx_t)$. Also if $s \leq q$ we get $max(b(mx_t)) \leq q$ implying $b(b(mx_t)x_q) = b(mx_t)$ which gives us what we want. Now for the case $s > q$ we get $l \geq s > q > t$. Which implies $x_t^{\alpha_t+1} \mid b(mx_q)x_t$ and $x_q^{\alpha_q+1} \mid b(mx_t)x_q$. Then using 7.0.66 we conclude that $b(b(mx_q)x_t) = b(b(mx_t)x_q)$.

It is left to consider the case $s > l$. Using the same argumentation above $b(mx_q) \nmid b(mx_t)$ is the only subcase left to consider. This is only possible if $x_q^{\alpha_q+1} \mid b(mx_q)$ since $s > l$. That means $s > l \geq q > t$. Therefore $x_t^{\alpha_t} \mid b(mx_q)$ also. It implies that $x_t^{\alpha_t+1} \mid b(mx_q)x_t$ and $x_q^{\alpha_q+1} \mid b(mx_t)x_q$. Then by 7.0.66 we get $b(b(mx_q)x_t) = b(b(mx_t)x_q)$. $\square$

construction 7.0.68. Let M be a Borel ideal. Denote $\{m_1, m_2, ...m_r\}$ the set of minimal monomial generators of M. For each m_i and for each sequence $1 \leq j_1 < ... < j_p < max(m_i)$ we consider the free S-module $S(m_i x_{j_1} ... x_{j_p})$ with one generator $(m_i; j_1, ..., j_p)$ of multi degree $m_i x_{j_1} ... x_{j_p}$. We set up a sequence of homomorphisms of S-modules;

$$E : ... \xrightarrow{d_{p+1}} E_p \xrightarrow{d_p} E_{p-1} \xrightarrow{d_{p-1}} ... \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 = S \xrightarrow{d_0} S/M \longrightarrow 0.$$

$$E_{p+1} = \bigoplus_{\text{p is fixed}} S(m_i x_{j_1} ... x_{j_p}).$$

Where $(m_i; j_1, ..., j_p)$ are the basis of the modules in the sum. Now we define $d = \delta - \mu$ as

$$\delta(m_i; j_1, ...j_p) = \sum_{1 \leq q \leq p} (-1)^q x_{j_q} (m_i, j_1 ..., \tilde{j_q}, ..., j_p).$$

and

$$\mu(m_i; j_1, ..., j_p) = \sum_{1 \leq q < p} (-1)^q \frac{m_i x_{j_q}}{b(m_i) x_{j_q}} (b(m_i x_{j_q}); j_1, ..., \tilde{j_q}, ..., j_p).$$

Here $\tilde{j_q}$ means j_q is missing in the given base element. Also for the sake of being well defined we count $(b(m_i x_{j_q}); j_1, ..., \tilde{j_q}, ..., j_p)$ zero if $j_p \geq max(b(m_i x_{j_q}))$.

Proposition 7.0.69. *The sequence of the homomorphisms E of the S-modules constructed in 7.0.68 is indeed a complex.*

Proof. We need to prove $d_p d_{p+1} = 0$ to show E is a complex. Following chain of the implications are by definition of d;

$$\begin{aligned} d_p d_{p+1} = 0 &\iff (\delta_p - \mu_p)(\delta_{p+1} - \mu_{p+1}) = 0 \\ &\iff \delta_p \delta_{p+1} + (\mu_p \mu_{p+1} - (\delta_p \mu_{p+1} - \mu_p \delta_{p+1})) = 0. \end{aligned}$$

The desired equation is satisfied if the following statements are true;

$$\delta_p \delta_{p+1} = 0. \tag{1}$$

$$\delta_p \mu_{p+1} + \mu_p \delta_{p+1} = 0. \tag{2}$$

$$\mu_p \mu_{p+1} = 0. \tag{3}$$

For (1)

$$\begin{aligned} \delta_p \delta_{p+1}(m_i; j_1, ..., j_p) &= \sum_{1 \le q \le p} (-1)^q x_{j_q} d_p(m_i; j_1, ..., \tilde{j_q}, ..., j_p) \\ &= \Big(\sum_{1 \le q < t \le p} (-1)^{q+t-1} x_{j_q} x_{j_t} (m_i; j_1, ..., \tilde{j_q}, ..., \tilde{j_t}, ..., j_p)\Big) \\ &+ \Big(\sum_{1 \le r < q \le p} (-1)^{q+r} x_{j_q} x_{j_r} (m_i; j_1, ..., \tilde{j_r}, ..., \tilde{j_q}, ..., j_p)\Big). \end{aligned}$$

Let $s > l$, now consider the coefficient of a generic element $x_{j_s} x_{j_l} (m_i; j_1, ..., \tilde{j_l}, ..., \tilde{j_s}, ..., j_p)$ of the sum. In the first sum of the last equation setting $q = s$, $t = l$ we get the coefficient $(-1)^{l+s-1}$. Again setting $q = s$, $t = l$ in the second sum of the last equation $(-1)^{l+s}$ contributes to coefficient of the generic element. Therefore the coefficient of the generic element becomes $(-1)^{l+s-1} + (-1)^{l+s}$ which equals to 0. So it is done.

For (2)

$$\begin{aligned} \delta_p \mu_{p+1}(m_i; j_1, ..., j_p) &= \sum_{1 \le q \le p-1} \frac{(-1)^q m_i x_{j_q}}{b(m_i x_{j_q})} \delta_p(b(m_i x_{j_q}); j_1, ..., \tilde{j_q}, ..., j_p) \\ &= \sum_{1 \le q < t \le p} \frac{(-1)^{q+t-1} m_i x_{j_q} x_{j_t}}{b(m_i x_{j_q})} (b(m_i x_{j_q}); j_1, ..., \tilde{j_q}, ..., \tilde{j_t} ..., j_p) \\ &+ \sum_{1 \le r < q \le p-1} \frac{(-1)^{q+r} m_i x_{j_q} x_{j_r}}{b(m_i x_{j_q})} (b(m_i x_{j_q}); j_1, ..., \tilde{j_r}, ..., \tilde{j_q}, ..., j_p). \end{aligned}$$

Let $s > l$. To determine the generic elements of the sum put $q = l$, $s = t$ in the first sum and put $r = l$, $s = q$ in the second sum of the equation. Then we get two forms of the generic elements of the sum;

$$\frac{(-1)^{s+l-1} m_i x_{j_l} x_{j_s}}{b(m_i x_{j_l})} (b(m_i x_{j_l}); j_1, ..., \tilde{j}_l, ..., \tilde{j}_s ..., j_p). \tag{4}$$

$$\frac{(-1)^{s+l} m_i x_{j_s} x_{j_l}}{b(m_i x_{j_s})} (b(m_i x_{j_s}); j_1, ..., \tilde{j}_l, ..., \tilde{j}_s ..., j_p). \tag{5}$$

$$\begin{aligned} \mu_p \delta_{p+1}(m_i; j_1, ..., j_p) &= \sum_{1 \le q \le p} (-1)^q m_i x_{j_q} \mu_p(m_i; j_1, ..., \tilde{j}_q, ..., j_p) \\ &= \sum_{1 \le q < t \le p-1} \frac{(-1)_i^{q+t-1} x_{j_q} x_{j_t}}{b(m_i x_{j_t})} (b(m_i x_{j_t}); j_1, ..., \tilde{j}_q, ..., \tilde{j}_t ..., j_p) \\ &+ \sum_{1 \le r < q \le p} \frac{(-1)^{q+r} m_i x_{j_q} x_{j_r}}{b(m_i x_{j_r})} (b(m_i x_{j_r}); j_1, ..., \tilde{j}_r, ..., \tilde{j}_q, ..., j_p). \end{aligned}$$

Again putting $q = l$, $s = t$ in the first sum and putting $r = l$, $s = q$ in the second sum we get the two forms of generic elements of the sum;

$$\frac{(-1)^{s+l-1} m_i x_{j_l} x_{j_s}}{b(m_i x_{j_s})} (b(m_i x_{j_s}); j_1, ..., \tilde{j}_l, ..., \tilde{j}_s ..., j_p). \tag{6}$$

$$\frac{(-1)^{s+l} m_i x_{j_s} x_{j_l}}{b(m_i x_{j_l})} (b(m_i x_{j_l}); j_1, ..., \tilde{j}_l, ..., \tilde{j}_s ..., j_p). \tag{7}$$

It is apparently seen that the sum of generic elements in **(4)**, **(5)**, **(6)** and **(7)** becomes zero. Which proves **(2)**.

For (3)

$$\begin{aligned} \mu_p \mu_{p+1}(m_i; j_1, ..., j_p) &= \sum_{1 \le q \le p-1} \frac{(-1)^q m_i x_{j_q}}{b(m_i x_{j_q})} \mu_p(b(m_i x_{j_q}); j_1, ..., \tilde{j}_q, ..., j_p) \\ &= \sum_{1 \le q < t \le p-1} \frac{(-1)^{q+t-1} m_i x_{j_q} x_{j_t}}{b(b(m_i x_{j_q}) x_{j_t})} (b(b(m_i x_{j_q}) x_{j_t}); j_1, ..., \tilde{j}_q, ..., \tilde{j}_t, ..., j_p) \\ &+ \sum_{1 \le r < q \le p-1} \frac{(-1)^{q+r} m_i x_{j_q} x_{j_r}}{b(b(m_i x_{j_q}) x_{j_r})} (b(b(m_i x_{j_q}) x_{j_r}); j_1, ..., \tilde{j}_r, ..., \tilde{j}_q, ..., j_p). \end{aligned}$$

Again let $s > l$. Setting $q = l$, $t = s$ in the first sum of the equation we get the generic element;

$$\frac{(-1)^{s+l-1} m_i x_{j_l} x_{j_s}}{b(b(m_i x_{j_l}) x_{j_s})} (b(b(m_i x_{j_l}) x_{j_s}); j_1, ..., \tilde{j}_l, ..., \tilde{j}_s, ..., j_p).$$

and putting $r = l$, $q = s$ in the second sum we get the generic element;

$$\frac{(-1)^{s+l} m_i x_{j_s} x_{j_l}}{b(b(m_i x_{j_s}) x_{j_l})} (b(b(m_i x_{j_s}) x_{j_l}); j_1, ..., \tilde{j_l}, ..., \tilde{j_s}, ..., j_p).$$

By 7.0.67 we have $b(b(m_i x_{j_s}) x_{j_l}) = b(b(m_i x_{j_l}) x_{j_s})$. It implies that the sum of the generic elements represented above is zero. Which proves $\mu_p \mu_{p+1} = 0$. We are done.

□

Definition 7.0.70. The sequence of homomorphisms E that is proven to be complex in 7.0.69 is called Eliahou-Kervaire complex.

Now we represent a lemma that has great importance in the upcoming results.

Lemma 7.0.71. *Let M be a Borel ideal and suppose $m_1 \underset{rlex}{<} m_2 \underset{rlex}{<} ... \underset{rlex}{<} m_r$ are m.m.g of M in revlex order. Then for $i \geq 0$ we have $(m_1, m_2, ..., m_i) : m_{i+1} = (x_1, x_2, ... x_{max(m_{i+1})})$.*

Proof. To prove this we should first prove that $B_i = (m_1, m_2, ..., m_i)$ is a Borel ideal for all $i \leq r$. Let $x_t m \in B_i$ be fixed. To prove B_i is Borel, we should prove $x_j m \in B_i$ $\forall j \leq t$. Let q be the maximum element of the set $\{l \leq i \mid m_l$ divides $x_t m\}$. Since $(m_1, m_2, ..., m_q) \subseteq B_i$ it suffices to prove $x_j \in (m_1, m_2, ..., m_q)$ $\forall\ j < t$. Since m_q divides $x_t m$ we write $x_t m = m_q v$. If $x_t \nmid m_q$ then $m_q \mid m$ which also implies $m_q \mid x_j m \quad \forall\ j < t$. So done. For the case $x_t \mid m_q$ from the Borel property of M it follows that $\frac{m_q}{x_t} x_j \in M$. Since $j < t$ we have $\frac{m_q}{x_t} x_j \underset{rlex}{<} m_q$. Then if $\frac{m_q}{x_t} x_j$ is a m.m.g it must be in $\{m_1, ..., m_q\}$. It implies $x_j m = \frac{m_q}{x_t} x_j v \in (m_1, m_2, ..., m_q)$ which was the desired conclusion. If $\frac{m_q}{x_t} x_j$ is not a m.m.g then it must have a m.m.g as a divisor. Let it be m'. Then $m' < \frac{m_q}{x_t} x_j < m_q$. Therefore $m' \in \{m_1, ..., m_q\}$. Since $x_j m$ is divided by m' we get $x_j m \in (m_1, ..., m_q)$. So we are done.

For $\supseteq$; since $m_{i+1} \in M$ we have $A = \frac{m_{i+1}}{x_{max(m_{i+1})}} x_{max(m_{i+1})-1} \in M$ by Borel property. Also clearly $m_{i+1} \underset{rlex}{>} A$. Then we get A is divisible by one of $m_1, m_2, ..., m_i$.

Then we get

$$A \subset (m_1, m_2, ..., m_i) \Longrightarrow Ax_{max(m_{i+1})} \in (m_1, ..., m_i).$$

So we have $m_{i+1}x_{max(m_{i+1})-1} \in (m_1, ..., m_i)$. Then by Borel property of $(m_1, m_2, ..., m_i)$ we get $m_{i+1}x_k \in (m_1, m_2, ..., m_i)$ $\forall$ k s.t $k \leq x_{max(m_{i+1})-1}$. This proves what we want.

For $\subseteq$; assume not. ie; there exists a monomial w such that $wm_{i+1} \in (m_1, ..., m_i)$ but $w \notin (x_1, x_2, ..., x_{max(m_{i+1})-1})$. The last condition gives that $x_k \nmid w$ $\forall$ $k < max(m_{i+1})$. Therefore $min(w) \geq max(m_{i+1})$. Which constitutes to $b(wm_{i+1}) = m_{i+1}$. Also since $wm_{i+1} \in (m_1, ..., m_i)$ $\exists t$ s.t $1 \leq t \leq i$ and $m_r \mid wm_{i+1}$. Setting $u_0 = m_t$, $v_0 = \frac{wm_{i+1}}{m_t}$ and applying the process followed in the proof of 7.0.64 we reach $b(wm_{i+1}) = m_{i+1}$ with reductions on $u_0 = m_t$ respect to revlex order. So it is seen that $m_t \underset{rlex}{\geq} m_{i+1}$ which contradicts with the revlex ordering of the minimal monomial generators. We are done. $\square$

Definition 7.0.72. Let (U, d) and (U', d') be complexes of finitely generated R-modules. And let $\varphi : (U, d) \to (U', d')$ be a homomorphism of complexes. Then (W, d'') is defined to be a mapping cone of U and U' where $W = U_{i-1} \bigoplus U'_i$ and $d''_i(a + b) = -d_{i-1}(a) + \varphi_{i-1}(a) + d'_i(b)$ where $a \in U_{i-1}$ and $b \in U'_i$. The map φ is called the comparison map.

Remark 7.0.73. (W, d'') defined in 7.0.71 is a complex of finitely generated R-modules.

Lemma 7.0.74. *Let (U, d) and (U', d') be free resolutions on the finitely generated R-modules V and V' respectively. And define $\varphi : (U, d) \to (U', d')$ as lifting of an injective homomorphism $\varphi_0 : V \to V'$. Then the mapping cone W of U and U' is the free resolution of $V'/\varphi(V)$.*

Proof. Set up the following exact sequence in the obvious way;

$$0 \to U' \overset{incl.}{\to} W \overset{surj.}{\to} U[-1] \to 0. \qquad \mathbf{(1)}$$

This yields the long exact sequences of homologies.

$$... \to H_i(U') \to H_i(W) \to H_{i-1}(U) \to H_{i-1}(U') \to H_{i-1}(W) \to H_{i-2}(U) \to .... \tag{\textbf{2}}$$

Now since U and U' are free resolutions of V and V' respectively, we have $H_i(U) = H_i(U') = 0$ and $H_0(U) = V$, $H_0(U') = V'$. Then from **(2)** we get $H_i(W) = 0$ for $i \geq 2$ and also the only part of **(2)** that is nonzero and in nonnegative homological degree becomes the following;

$$0 \to H_1(W) \overset{\sigma}{\to} H_0(U) = V \overset{\varphi=\varphi_0}{\to} H_0(U') = V' \overset{\lambda}{\to} H_0(W) \to 0. \tag{\textbf{3}}$$

Since the sequence above is a part of **(2)** it must be an exact sequence. Since it is given that φ is injective $im\sigma = \ker\varphi = \varnothing$. It implies that $H_1(W) = 0$ since σ is injective. Moreover by 1st isomorphism theorem we obtain $V'/ker(\lambda) \cong im(\lambda)$. Since **(3)** is exact we have λ is surjective and $\ker(\lambda) = im(\varphi)$. Therefore we get $V'/im(\varphi) = H_0(W)$. To sum up, it is proven that $H_i(W) = 0 \ \forall i > 0$ and $H_0(W) = V'/im(\varphi)$ which clearly constitutes to the result that W is a free resolution of the module $V'/\varphi(V)$. □

Proposition 7.0.75. *Let m be a fixed minimal monomial generator of the module M in 7.0.68. Then construct a sequence of R-modules;*

$$K : ... \xrightarrow{d_{p+1}} K_p \xrightarrow{d_p} K_{p-1} \xrightarrow{d_{p-1}} ... \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \longrightarrow 0.$$

where K_i is formed by the basis set $\{(m; j_1, ..., j_i) \mid 1 \leq j_1 \leq ... \leq j_i \leq r\}$. And d_p is set to $-\delta_{p+1}$ where δ is induced by the one in 7.0.68. Then K is a free resolution of the module $S/(x_1, x_2, ...x_r)$.

Proof. We get by using 7.0.69 that $d_{p+1}d_p = -\delta_p(-\delta_{p-1}) = \delta_p\delta_{p-1} = 0$. To prove K is the free resolution of the module $S/(x_1, x_2, ..., x_r)$ it suffice to prove the following list orderly.

$$H_0(K) = S/(x_1, x_2, ..., x_r). \tag{\textbf{1}}$$

$$H_1(K) = 0. \tag{\textbf{2}}$$

$$H_i(K) = 0 \ \ \forall \ \ i > 0. \qquad \textbf{(3)}$$

For (1); By definition of homology $H_0(K) = K_0/im(d_1)$. K_0 is generated by $(m, \varnothing)$, so $K_0 \cong S$. On the other hand $im(d_1)$ is generated by the set $\{d_1(m,j) \big| j \leq r\} = \{x_j(m, \varnothing) \big| j \leq r\}$. So clearly $im(d_1) = (x_1, x_2, ..., x_r)$. These constitutes to $H_0(K) = S/(x_1, x_2, ..., x_r)$.

For (2); K_1 is generated by $\{(m,1), (m,2), ..., (m,r)\}$. Now let $w \in ker(d_1)$. Then it can be written as $w = \sum\limits_{i=1}^{r} s_i(m,i)$ where $s_i \in S$. Since $d_1(w) = 0$ we have $\sum\limits_{i=1}^{r} s_i x_i = 0$. It is not hard to see that the $ker(d_1)$ is generated by the elements of the form $\frac{u}{x_j}(m,j) - \frac{u}{x_i}(m,i)$ where u is a monomial divisible by both x_i and x_j. And clearly $d_2(\frac{u}{x_i x_j}(m,i,j)) = \frac{u}{x_j}(m,j) - \frac{u}{x_i}(m,i)$. Therefore $\frac{u}{x_j}(m,j) - \frac{u}{x_i}(m,i) \in im(d_2)$ which yields $ker(d_1) \subseteq im(d_2)$. Since K is complex we have $im(d_2) \subseteq ker(d_1)$ is known by free. Then $ker(d_1) = im(d_2)$ which constitutes to $H_1(K) = 0$.

For (3); The complex where $r = 1$ becomes, $0 \to K_1 \xrightarrow{x_1} K_0 \to 0$ and by **(2)** we have $H_1(K) = 0$. Then for r=1 **(3)** is true also. Let it be the first step of induction on r. Assume that **(3)** is true for $r = t-1$. Let K^{t-1} be the complex when r is set to $t-1$ and K^t like so. We need to prove $H_i K^t = 0$ for all $i > 0$. By **(2)** $H_1 K^t = 0$. So it suffices to prove $H_i K^t$ for $i > 1$. The basis set of K_i^t equals to following;

$$\begin{aligned}
& \{(m; j_1, j_2, ..., j_i) \big| 1 \leq j_1 ... < j_i \leq t\} \\
= \ & \{(m; j_1, j_2, ..., j_i) \big| 1 \leq j_1 ... < j_i \leq t-1\} \cup \{(m; j_1, ..., j_{i-1}, t) \big| 1 \leq j_1 ... < j_{i-1} < t\} \\
\cong \ & \{(m; j_1, j_2, ..., j_i) \big| 1 \leq j_1 ... < j_i < t\} \cup \{(m; j_1, ..., j_{i-1}) \big| 1 \leq ... < j_{i-1} \leq t\}.
\end{aligned}$$

From here we clearly get $K_i^t \cong K_i^{t-1} \oplus K_{i-1}^{t-1} = K_i^{t-1} \oplus (K^{t-1}[-1])_i$. So we can set up the following exact sequence of complexes with the obvious inclusion and surjection maps;

$$0 \longrightarrow K^{t-1} \longrightarrow K^t \longrightarrow K^{t-1}[-1] \longrightarrow 0.$$

This yields the long exact sequence of homologies;

$$... \longrightarrow H_i(K^{t-1}) \longrightarrow H_i(K^t) \longrightarrow H_i(K^{t-1}[-1]) = H_{i-1}(K^{t-1}) \longrightarrow ...$$

by induction assumption we have $H_i(K^t) = H_{i-1}(K^{t-1}) = 0$ for $i > 1$. Then we get by exactness of the sequence that $H_i(K^t) = 0$ for $i > 1$. So we are done. **Then we proved that K is a free resolution of $S/(x_1, x_2, ..., x_r)$.** $\square$

Proposition 7.0.76. *Let M be a Borel where its minimal monomial generators are $m_1, m_2, ..., m_r$ ordered in revlex order. Also let $M_i = \{m_1, ..., m_i\}$. Then the following sequence is exact*

$$0 \longrightarrow S/(M_i : m_{i+1}) \underset{\sigma}{\xrightarrow{m_{i+1}}} S/M_i \underset{\gamma}{\xrightarrow{surj.}} S/M_{i+1} \longrightarrow 0.$$

and $S/M_{i+1} \cong \frac{S/M_i}{\sigma(S/(M_i:m_{i+1}))}$.

Proof. Lets prove it is exact first. Let $a = s_1 + M_i : m_{i+1}$, $b = s_2 + M_i : m_{i+1}$ s.t $am_{i+1} = bm_{i+1}$ ie $s_1 m_{i+1} - s_2 m_{i+1} \in M_i \Rightarrow s_1 - s_2 \in M_i : m_{i+1} \Rightarrow a = b$. Then γ is injective. Now lets prove that $ker(\gamma) = im(\sigma)$. Let $a = s_1 + M_i \in ker(\gamma)$ ie this clearly points that $m_{i+1} \mid s_1 \Rightarrow a \in im(\sigma)$. Since it is known by free that $im(\sigma) \subseteq ker(\gamma)$ and γ is surjective we are done with the exactness.
So by 1st isomorphism theorem we get $\frac{S/M_i}{ker(\gamma)} = im(\gamma)$. Then by surjective property of γ and $ker(\gamma) = im(\sigma)$ we get $S/M_{i+1} \cong \frac{S/M_i}{\sigma(S/(M_i:m_{i+1}))}$. We are done. $\square$

After all the preparation done now we are ready to state and prove the following hardcore theorem which will enable us to calculate Betti numbers of a Borel module in a precise way.

Theorem 7.0.77. *Eliahou-Kervaire complex of the Borel module M is the minimal free resolution of S/M.*

Proof. We will continue with the notation used in 7.0.76. We argue induction on i; assume Eliahou-Kervaire complex E' of M_i is a free resolution of S/M_i. By using 7.0.71 we know $S/(M_i : m_{i+1}) = S/(x_1, x_2, ..., x_l)$ where $l = max(m_{i+1}) - 1$. Set $V = S/(M_i : m_{i+1})$ and $V' = S/M_i$. Blending with this notation in 7.0.76 we proved that the map $\sigma : V \underset{m_{i+1}}{\longrightarrow} V'$ is an injective map and $S/M_{i+1} \cong V'/\sigma(V)$. Putting $m = m_{i+1}$ and $r = max(m_{i+1}) - 1$ in 7.0.75 we get a free resolution K of the module $V = S/(M_i : m_{i+1})$ and by the induction assumption we have the Eliahou-Kervaire resolution E' of $V' = S/M_i$. Import the map μ from 7.0.68. Let

$-\mu : K \to E'$ where we define $\mu(m_{i+1}, \varnothing) = -m_{i+1}$. Lets prove it is indeed the lifting of the map σ between V and V'. There is two cases needed to be proven separately;

$$\mu_{t-1}(\delta_t(m_{i+1}; j_1, ..., j_p)) = -d_t(\mu_t(m_{i+1}; j_1, ..., j_p)) \forall t \geq 1. \qquad \textbf{(1)}$$

$$\sigma\delta_0(m_{i+1}; \varnothing) = -d_0\mu_0(m_{i+1}, \varnothing). \qquad \textbf{(2)}$$

$$\begin{array}{ccccccccc}
... & K_t & \xrightarrow{-\delta_t} & K_{t-1} & ... & K_0 & \xrightarrow{\delta_0} & V & \longrightarrow & 0 \\
 & \downarrow & \circlearrowleft & \downarrow & ... & \downarrow & \circlearrowleft & \downarrow & & . \\
... & E'_t & \xrightarrow{d_t} & E_{t-1} & ... & E'_0 & \xrightarrow{d_0} & V' & \longrightarrow & 0
\end{array}$$

For **(1)**; we are to prove $\mu\delta + d\mu = 0$. We know by definition that $d = \delta - \mu$. And by the proof of 7.0.69 that $\delta^2 = \mu^2 = \delta\mu + \mu\delta = 0$. Using them we get the following chain of equations;

$$\mu\delta + d\mu = \mu\delta + (\delta - \mu)\mu = \underbrace{(\mu\delta + \delta\mu)}_{0} - \underbrace{\mu^2}_{0} = 0.$$

For **(2)**; by using the definition of δ_0, σ and μ we get;

$$\sigma\delta_0(m_{i+1}; \varnothing) = \sigma(1+(M_i : m_{i+1})) = m_{i+1}+(M_i) = -d_0(-m_{i+1}) = -d_0\mu_0(m_{i+1}; \varnothing).$$

Let W be the mapping cone of K and E' where the comparison map between them is $-\mu$. Now all requirements of 7.0.76 are set. Applying it we get that W is a free resolution of $V'/\sigma(V) = S/M_{i+1}$. If we prove W is isomorphic to Eliahou-Kervaire complex, say E, of M_{i+1} we are done. To show it lets start with proving $W_t = E_t$. The basis set of K_t, E'_t and E_t are written below orderly;

$$B_1 = \{(m_{i+1}; j_1, ..., j_{t-1}) \mid j_1 \leq j_2 \leq ... < max(m_{i+1})\}.$$

$$B_2 = \{(m_j; j_1, ..., j_{t-1}) \mid 1 \leq j \leq i,\ j_1 \leq j_2 \leq ... < max(m_j)\}.$$

$$B_3 = \{(m_j; j_1, ..., j_{t-1}) \mid 1 \leq j \leq i+1,\ j_1 \leq j_2 \leq ... < max(m_j)\}.$$

It is easy to see that $B_3 = B_1 \cup B_2$. Also since $W_t = K_{t-1} \oplus E'_t$ we get $W_t \cong E_t$. It is left to prove the map $\varphi_t : W_t = K_{t-1} \oplus E'_t \longrightarrow W_{t-1} = K_{t-2} \oplus E'_{t-1}$ is same with d_t. Importing from 7.0.72 with essential arrangements we get that;

$$\varphi_t(a+b) = \delta_{t-1}(a) - \mu_{t-1}(a) + d_t(b) = d_t(a) + d_t(b) = d_t(a+b)$$

$$\forall a \in K_{t-1} \text{ and } \forall b \in E_t.$$

Therefore we get $\varphi = d$. As a result it is shown that the Eliahou-Kervarire complex E of M_{i+1} is a free resolution of S/M_{i+1}. As a last thing, it is not hard to see that $d(E) \subseteq (x_1, x_2, ..., x_n)E$ since in the definition of d every input element gives the output with some multiplication by a x_z. So the resolution is minimal. □

Now we can calculate the betti numbers of a Borel ideal as follows;

Corollary 7.0.78. *Let M be a Borel ideal minimally generated by monomials $m_1, m_2, ..., m_r$. Then we have;*

$$b^S_{p,p+q}(M) = \sum_{deg(m_i)=q,\ 1\leq\ i\leq r} \binom{max(m_i)-1}{p} \quad \forall\ p,\ q.$$

Proof. $b^S_{p,p+q}(M)$ means the number of summands in the homological degree p and of the form $S(-(p+q))$. Eliahou-Kervaire minimal free resolution has the basis;

$$\{(m_i; j_1, ..., j_p) \mid 1 \leq j_1 < ... < j_p < max(m_i),\ 1 \leq i \leq r\}.$$

in homological degree p by 7.0.68. For a fixed m_i there are clearly $\binom{max(m_i)-1}{p}$ ways to choose the sequence $j_1, ..., j_p$. Where $S(m_i j_1 j_2 ... j_p)$ is in the form $S(-(p+q))$. We are done. □

Chapter 8

Proof of the Main Theorem

In this section we present the proof of the main theorem of this paper. Before giving it we prove a lemma that will be of use.

Lemma 8.0.79. *If M is a Borel ideal in S, then we have;*

$$b^S_{i,i+j}(M) = |M_j| \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(M_j) \binom{p-1}{i-1} - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i}.$$

Proof. By 7.0.78 we get,

$$b^S_{i,i+j}(M) = \sum_{deg(m_t)=j} \binom{max(m_t)-1}{i} = \sum_{p=1}^{n} |\{m \in G_j(M) \big| \ max(m) = p\}| \binom{p-1}{i}.$$

Also by 3.0.40 we know that $G_j(M) = M_j - S_1 M_{j-1}$, applying it to our equation we get,

$$\sum_{p=1}^{n} |\{m \in M_j \big| max(m) = p\}| \binom{p-1}{i} - \sum_{p=1}^{n} |\{m \in S_1 M_{j-1} \big| max(m) = p\}| \binom{p-1}{i}.$$

Moreover using 4.0.56 $S_1 M_{j-1} = \bigsqcup x_p\{m \in M_{j-1} \big| max(m) \leq p\}$ is obtained. Then only monomials in $S_1 M_{j-1}$ with the maximum indices p are in the $x_p\{m \in M_j - 1 \big| max(m) \leq p\}$ summand clearly. Therefore we get;

$$|\{m \in S_1 M_{j-1} \big| max(m) = p\}| = |\{m \in M_j - 1 \big| max(m) \leq p\}| = t_p(M_{j-1}).$$

Here we recalled the definition of $t_p(M_{j-1})$ from chapter 4. Combining all these we get;

$$= \sum_{p=1}^{n} |\{m \in M_j \big| max(m) = p\}| \binom{p-1}{i} - \sum_{p=1}^{n} t_p M_{j-1} \binom{p-1}{i}.$$

Also we have $|\{m \in M_j \big| max(m) = p\}| = t_p(M_j) - t_{p-1}(M_j)$. Applying this to our equation,

$$\begin{aligned}
&= \sum_{p=1}^{n} (t_p(M_j) - t_{p-1}(M_j)) \binom{p-1}{i} - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i} \\
&= \sum_{p=1}^{n} t_p(M_j) \binom{p-1}{i} - \sum_{p=1}^{n} t_{p-1}(M_{j-1}) \binom{p-1}{i} - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i} \\
&= t_n(M_j) \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(M_j) \left[\binom{p}{i} - \binom{p-1}{i} \right] - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i} \\
&= |M_j| \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(M_j) \binom{p-1}{i-1} - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i}.
\end{aligned}$$

So we are done. □

Proof of The Main Theorem

We stick to the notation in 6.0.62. Let M be the generic initial ideal of J. By 2.0.31 we have M is a Borel ideal and its Hilbert function is same with the Hilbert function of J. In addition to this we have;

$$b^S_{i,i+j}(J) \leq b^S_{i,i+j}(M) \text{ for all i, j } \mathit{by} \text{ 3.0.43.}$$

Now since both M and L are Borel we can apply the work done in chapter 7;

$$b^S_{i,i+j}(M) = |M_j| \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(M_j) \binom{p-1}{i-1} - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i}. \quad \mathbf{(1)}$$

$$b^S_{i,i+j}(L) = |L_j| \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(L_j) \binom{p-1}{i-1} - \sum_{p=1}^{n} t_p(L_{j-1}) \binom{p-1}{i}. \quad \mathbf{(2)}$$

We have Hilbert function of L, J and M are same. Combining this with L is lex and M is Borel we can apply 4.0.59 here to get;

$$t_p(L_j) \leq t_p(M_j) \ \ , t_p(L_{j-1}) \leq t_p(M_{j-1}).$$

Also we have $|M_j| = |L_j|$ since their Hilbert functions are same. So all these together with **(1)** and **(2)** implies that;

$$b^S_{i,i+j}(J) \leq b^S_{i,i+j}(M) \text{ for all i, j.}$$

We are done.

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