ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

THE INVESTIGATION OF DYNAMICS OF THE SPECIFIC NON-LINEAR DIFFERENCE EQUATIONS

DEPARTMENT OF MATHEMATICS

DOCTOR OF PHILOSOPHY THESIS

ERKAN TAŞDEMİR

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ADVISOR: Assoc. Prof. Yüksel SOYKAN

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APPROVAL OF THE THESIS:

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly."

Erkan TAŞDEMİR

ABSTRACT

Doctor of Philosophy Thesis

THE INVESTIGATION OF DYNAMICS OF THE SPECIFIC NON-LINEAR DIFFERENCE EQUATIONS

Erkan TAŞDEMİR

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Yüksel SOYKAN
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In this thesis, we are dealt with the some properties of solutions of some nonlinear difference equations such as periodicity, stability and boundedness. Moreover, in the periodicity sections, we are investigated periodic, eventually periodic and non-periodic cases of solutions of related nonlinear difference equations. This thesis consist of five chapters.

First chapter of the thesis, we present the some important definitions and required theorems.

In chapter 2, we investigate the dynamics of the following nonlinear difference equations $x_{n+1} = x_n x_{n-1} + \alpha$.

In chapter 3, we study the long-term behaviours of a nonlinear difference equation $x_{n+1} = x_{n-1} x_{n-3} - 1$.

ABSTRACT (continued)

In chapter 4, we deal with the periodicity, stability and boundedness character of a nonlinear difference equation
$x_{n+1} = x_{n-2}x_{n-3} - 1.$

In chapter 5, we overcome the dynamical properties of some nonlinear difference equations $x_{n+1} = x_n x_{n-2} + \alpha.$

Keywords: Difference equations, periodicity, boundedness, equilibrium point, stability.

Science Code: 403.03.01

ÖZET

Doktora Tezi

LİNEER OLMAYAN ÖZEL FARK DENKLEMLERİNİN DİNAMİKLERİNİN İNCELENMESİ

Erkan TAŞDEMİR

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Bu tezde, bazı lineer olmayan fark denklemlerinin çözümlerinin periyodiklik, kararlılık ve sınırlılık gibi özellikleri ele alınmaktadır. Ayrıca, tezin periyodiklik bölümlerinde ilgili lineer olmayan fark denklemlerinin çözümlerinin periyodik, eninde sonunda periyodik ve periyodik olmama durumları da incelenmektedir. Bu tez beş bölümden oluşmaktadır.

Tezin ilk bölümünde bazı önemli tanımlar ve gerekli teoremler sunulmuştur.

Ikinci bölümde, lineer olmayan $x_{n+1} = x_n x_{n-1} + \alpha$ fark denklemlerinin dinamikleri incelenmiştir.

Üçüncü bölümde, lineer olmayan $x_{n+1} = x_{n-1} x_{n-3} - 1$ fark denkleminin çözümlerinin davranışları çalışılmıştır.

ÖZET (devam ediyor)

Dördüncü bölümde, lineer olmayan $x_{n+1} = x_{n-2}x_{n-3} - 1$ fark denkleminin periyodiklik, kararlılık ve sınırlılık karakteri ele alınmıştır.

Beşinci bölümde, bazı lineer olmayan $x_{n+1} = x_n x_{n-2} + \alpha$ fark denklemlerinin dinamik özellikleri incelenmiştir.

Anahtar Kelimeler: Fark denklemleri, periyodiklik, sınırlılık, denge noktası, kararlılık.

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TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

Page

LIST OF FIGURES

LIST OF FIGURES (continued)

No Page

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

During the last decades, there has been a significant increase in investigation of the difference equations (or called recursive sequences). One of the most important reason for this increase is undoubtedly applications of mathematical theory to different fields of science as a mathematical model. A mathematical model consist of three basic steps. The first of these steps is to transform the problem into a mathematical problem. Next step is to solve the mathematical problem. The last step is to interpret results of these mathematical problem for a non-mathematical problem. All of these steps are important to the work of different fields of science. Due to its nature, the studies of many fields of science depend on discrete variables. Thanks to these variables, we establish discrete mathematical models by means of difference equations. Some of the techniques applied in difference equations are used in mathematical models of real-life situations such as genetics, ecology, economics, biological systems and so forth. The study of periodicity, stability and boundedness properties of solutions of nonlinear difference equations is beneficial for mathematical models of different applications. Although the linear difference equations a lot studied in the beginning, the nonlinear difference equations have been much more studied for the last years. In particular, some authors have published a number of quality articles on the nonlinear difference equations. Furthermore, difference equations have many applications in applied sciences so there are many papers and books regarding theory and applications of difference equations, (for partial review of the theory of difference equations and their applications see [1] - [28] and the references sited therein).

Our purpose in this thesis is to study investigation of dynamics of both some nonlinear difference equations and specific nonlinear difference equations. Especially, we research periodicity, stability and boundedness properties of related nonlinear difference equations. This thesis include five chapters.

The first chapter consists of general information about the thesis and a summary of the literature. Moreover, in this chapter, there are important definitions and significant theorems used in the study of this thesis.

The second chapter contains some results for periodic and non-periodic solutions of some difference equations

$$x_{n+1} = x_n x_{n-1} + \alpha, \quad n = 0, 1, 2, ...,$$

where the initial conditions x_{-1} and x_0 are real numbers and $\alpha \in \mathbb{R}$. Additionally, this chapter includes some results about the unbounded solutions of the related second order difference equations. At the end of this chapter, graphs are presented to visualize the theoretical results.

The third chapter comprises the periodic, eventually periodic and non-periodic solutions of nonlinear difference equation

$$x_{n+1} = x_{n-1} x_{n-3} - 1, \; n = 0, 1, 2, ...,$$

where the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are real numbers. Morever, this chapter involves some results about the stability character and unbounded solutions of the related nonlinear difference equation. At the end of this chapter, graphs are presented so as to visualize the theoretical results.

The fourth chapter consists of the periodic, eventually periodic and non-periodic solutions of nonlinear difference equation

$$x_{n+1} = x_{n-2} x_{n-3} - 1, \; n = 0, 1, 2, ...,$$

where the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are real numbers. Further, this chapter includes some results about the stability character and unbounded solutions of the related nonlinear difference equation. At the end of this chapter, graphs are presented in order to visualize the theoretical results.

The fifth chapter contains the periodic, eventually periodic and non-periodic solutions of some nonlinear difference equations

$$x_{n+1} = x_n x_{n-2} + \alpha, \; n = 0, 1, 2, ...,$$

where the initial conditions x_{-2}, x_{-1} and x_0 are real numbers and $\alpha \in \mathbb{R}$. Moreover, this chapter consist of some results about the stability character of the related difference equations. At the end of this chapter, graphs are presented to visualize the theoretical results.

Here, we present some important studies on the dynamics of nonlinear difference equations. These studies has an important role during this thesis.

In [15] Kent et al studied the boundedness of solutions, periodicity of solutions, and existence of unbounded solutions and how these behaviors depend on initial conditions of the difference equation

$$x_{n+1} = x_n x_{n-1} - 1.$$

In [13] Kent et al studied the long-term behavior of solutions of the difference equation

$$x_{n+1} = x_{n-1} x_{n-2} - 1.$$

In [14] Kent et al investigated the periodicity of solutions, existence of unbounded solutions and convergence to the negative equilibrium of the difference equation

$$x_{n+1} = x_n x_{n-2} - 1.$$

In [12] Kent and Kosmala studied the periodicity and asymptotic periodicity of solutions, as well as the existence of unbounded solutions of the difference equation

$$x_{n+1} = x_n x_{n-3} - 1.$$

In [22] Stevic and Iričanin, investigated the following difference equation

$$x_{n+1} = x_{n-l} x_{n-k} - 1, n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $k < l$, $\gcd(k, l) = 1$, and the initial values $x_{-l}, ..., x_{-2}, x_{-1}$ are real numbers. They dealt with the behaviour of solutions of the difference equation of general character, by describing the long-term behavior of the solutions of the difference equation for all values of parameters k and l, where the initial values satisfy the following condition $\min\{x_{-l}, ..., x_{-2}, x_{-1}\}$.

In [23] Stevic et al studied the boundedness character of positive solutions of the difference equation

$$x_n = \alpha + \prod_{j=1}^{k} x_{n-j}^{aj}.$$

Now, we present some important definitions and theorems for nonlinear difference equations.

Definition 1.1 (Difference Equation, [5]) *A **difference equation** of order* $(k+1)$ *is an equation of the form*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}),\ n = 0, 1, \ldots \tag{1.1}$$

where F *is a function that maps some set* I^{k+1} *into* I*. The set* I *is usually an interval of real numbers, or a union of intervals, or a discrete set such as the set of integers* $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$.
A solution of Eq.(1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ *that satisfies Eq.(1.1) for all* $n \geq 0$.

Definition 1.2 (Equilibrium, [5]) *A solution of Eq.(1.1) that is constant for all* $n \geq -k$ *is called an **equilibrium solution** of Eq.(1.1). If*

$x_n = \overline{x},$ *for all* $n \geq -k$

is an equilibrium solution of Eq.(1.1), then $\overline{x}$ *is called an **equilibrium point**, or simply an **equilibrium** of Eq.(1.1).*

Definition 1.3 (Stability, [5]) *(i) An equilibrium point* $\overline{x}$ *of Eq.(1.1) is called **locally stable** if, every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $\{x_n\}_{n=-k}^{\infty}$ *is a solution of Eq.(1.1) with*

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$$

then

$|x_n - \bar{x}| < \varepsilon$*, for all* $n \geq 0$.

(ii) An equilibrium point $\overline{x}$ *of Eq.(1.1) is called **locally asymptotically stable** if,* $\bar{x}$ *is locally stable, and if addition there exists* $\gamma > 0$ *such that if* $\{x_n\}_{n=-k}^{\infty}$ *is a solution of Eq.(1.1) with*

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) An equilibrium point $\overline{x}$ of Eq.(1.1) is called a ***global attractor*** *if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1), we have*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) An equilibrium point $\overline{x}$ of Eq.(1.1) is called ***globally asymptotically stable*** *if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq.(1.1).*

(v) An equilibrium point $\overline{x}$ of Eq.(1.1) is called ***unstable*** *if $\bar{x}$ is not locally stable.*

Definition 1.4 (Linearized Equation, [5]) *Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let*

$$q_i = \frac{\partial F}{\partial u_i}\left(\bar{x}, \bar{x}, \cdots, \bar{x}\right), \text{ for } i = 0, 1, 2, \cdots, k$$

denote the partial deriative of $F\left(u_0, u_1, \cdots, u_k\right)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(1.1). The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k},\ n = 0, 1, \cdots, \tag{1.2}$$

is called the ***linearized equation*** *of Eq.(1.1) about the equilibrium point $\bar{x}$.*

Definition 1.5 (Characteristic Equation, [5]) *The equation*

$$\lambda^{k+1} - q_0\lambda^k - q_1\lambda^{k-1} - \cdots - q_{k-1}\lambda - q_k = 0 \tag{1.3}$$

is called the ***characteristic equation*** *of Eq.(1.2) about the equilibrium point $\bar{x}$.*

Theorem 1.1 (The Linearized Stability Theorem, [5]) *Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true:*

(a) When all the roots of Eq.(1.3) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is ***locally asymptotically stable.*** *Moreover, in this here the equilibrium point $\bar{x}$ of Eq.(1.1) is called* ***sink****.*

(b) If at least one root of Eq.(1.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(1.1) is ***unstable.***

(i) The equilibrium point $\bar{x}$ of Eq.(1.1) is called ***hyperbolic*** *if no root of Eq.(1.3) has absolute value equal to one.*

*(ii) If there exists a root of Eq.(1.3) with absolute value equal to one, then the equilibrium $\bar{x}$ is called **nonhyperbolic.***

*(iii) An equilibrium point $\bar{x}$ of Eq.(1.1) is called a **saddle point** if it is hyperbolic and if there exists a root of Eq.(1.3) with absolute value less than one and another root of Eq.(1.3) with absolute value greater than one.*

*(iv) An equilibrium point $\bar{x}$ of Eq.(1.1) is called a **repeller** if all roots of Eq.(1.3) have absolute value greater than one.*

Theorem 1.2 (Clark Theorem, [5]) *Assume that $q_0, q_1, \cdots, q_k$ are real numbers such that*

$$|q_0| + |q_1| + \cdots + |q_k| < 1.$$

Then all roots of Eq.(1.3) lie inside the unit disk.

Definition 1.6 (Periodicity, [5]) *A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called **periodic** with period p if there exists an integer $p \geq 1$ such that*

$$x_{n+p} = x_n, \text{ for all } n \geq -k. \tag{1.4}$$

A solution is called periodic with prime period p if p is the smallest positive integer for which Eq.(1.4) holds.

Theorem 1.3 ([17], p.13) *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:
(a) $f(x, y)$ is non-increasing in each of its arguments;
(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, m) = M \text{ and } f(M, M) = m,$$

then $m = M$.
Then Eq.(1.1) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(1.1) converges to $\overline{x}$.

Theorem 1.4 ([17], p.208) *Let $[a,b]$ be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \times [a,b] \to [a,b]$$

is a continuous function satisfying the following properties:

(a) $f(x,y,z)$ is non-increasing in all three variables x,y,z in $[a,b]$;

(b) If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system

$$M = f(m,m,m) \text{ and } m = f(M,M,M)$$

then

$$m = M.$$

Then Eq(1.1) has a unique equilibrium $\bar{x} \in [a,b]$ and every solution of Eq(1.1) converges to $\bar{x}$.

CHAPTER 2

QUALITATIVE ANALYSIS OF SOME NONLINEAR DIFFERENCE EQUATIONS

In this chapter, we examine the periodic behaviors of solutions and dependence of such behaviors on initial conditions of the some nonlinear difference equations $x_{n+1} = x_n x_{n-1} + \alpha$. Additionally, we study the existence unbounded solutions of the related difference equations.
The results presented in this chapter have been published as article in [25] by us.

2.1 INTRODUCTION

In [15] Kent et al studied the periodicity of solutions, boundedness of solutions, and existence of unbounded solutions of the nonlinear difference equation

$$x_{n+1} = x_n x_{n-1} - 1,$$

where the initial conditions x_{-1} and x_0 are real numbers.
Ladas [1] investigated the stability of the equilibrium points and the long-term behavior of solution of second-order difference equation

$$x_{n+1} = x_n x_{n-1} + \alpha, \quad n = 0, 1, 2, ...,$$

where the initial conditions x_{-1} and x_0 are real numbers and $\alpha \in \mathbb{R}$.

In this chapter we study the periodic character of solutions of the following some nonlinear difference equations

$$x_{n+1} = x_n x_{n-1} + \alpha, \quad n = 0, 1, 2, ..., \tag{2.1}$$

2.2 THE EQUILIBRIUM POINTS OF EQ.(2.1)

This section, we investigate that Eq.(2.1) has exactly two equilibria.

Lemma 2.1 *The equilibrium points of Eq.(2.1) are*

$$\overline{x}_{1,2} = \frac{1 \mp \sqrt{1-4\alpha}}{2}. \tag{2.2}$$

Proof. Let $\overline{x}$ is an equilibrium point of Eq.(2.1). Thus $x_n = \overline{x}$, for all $n \geq -1$. So, we have

$$\overline{x} = \overline{x} \cdot \overline{x} + \alpha.$$

Therefore, we get the equation

$$\overline{x}^2 - \overline{x} + \alpha = 0.$$

Thus, we find that Eq.(2.1) has exactly two equilibrium points, which are $\bar{x}_1$ and $\bar{x}_2$ together:
$\overline{x}_{1,2} = \frac{1 \mp \sqrt{1-4\alpha}}{2}$. ■

remark 2.1 *Eq.(2.2) has three cases for the variable* α*:*
(i) $\overline{x}$ *is complex if* $\alpha > \frac{1}{4}$*,*
(ii) $\bar{x}_1$ *and* $\bar{x}_2$ *are real numbers if* $\alpha < \frac{1}{4}$*,*
(iii) $\overline{x} = \frac{1}{2}$ *is unique equilibrium point if* $\alpha = \frac{1}{4}$*.*

2.3 THE PERIODIC SOLUTIONS OF EQ.(2.1)

In this section, we study the existence of periodic, eventually periodic or non-periodic solutions of Eq.(2.1).

Theorem 2.2 *There are no eventually constant solutions of Eq.(2.1).*

Proof. If $\{x_n\}_{n=-1}^{\infty}$ is eventually constant solutions of Eq.(2.1), then $x_N = x_{N+1} = \overline{x}$ for some $N \geq 0$, where $\bar{x}$ is an equilibrium point. Therefore from $x_{N+1} = x_N x_{N-1} + \alpha$, it follows that

$$x_{N-1} = \frac{x_{N+1} - \alpha}{x_N} = \frac{\overline{x} - \alpha}{\overline{x}} = \overline{x}.$$

Repeating this procedure, we obtain $x_n = \overline{x}$, for $-1 \leq n \leq N+1$ as claimed. ■

Theorem 2.3 *Difference equation (2.1) has no nontrivial period two solutions nor eventually period two solutions.*

Proof. Suppose that $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, for every $k \in \mathbb{N}_0$, and some $N \geq -1$, with $x_N \neq x_{N+1}$. Therefore, we have

$$\begin{aligned}
x_{N+4} &= x_{N+3}x_{N+2} + \alpha \\
x_{N+4} &= x_{N+1}x_{N+2} + \alpha = x_{N+3} \\
x_{N+4} &= x_{N+1}x_N + \alpha = x_{N+2} \\
x_{N+4} &= x_{N-1}x_N + \alpha = x_{N+1}
\end{aligned}$$

From this and since $x_{N+4} = x_N$, we obtain a contradiction which finishes the proof of the result. ∎

The following result shows that there exist exactly three periodic solutions of Eq.(2.1) with minimal period three and gives a description of each.

Theorem 2.4 *There exists exactly three periodic solution of Eq.(2.1) with minimal period three. They are given by three pairs of initial conditions*

$x_{-1} = -1, x_0 = -1;$

$x_{-1} = -1, x_0 = \alpha + 1$

and

$x_{-1} = \alpha + 1, x_0 = -1$

where $\alpha \neq 0$.

Proof. The case $\alpha = -1$ was investigated in [15]. Now, we can write terms of a period-three solution of Eq.(2.1) as

$x_{-1} = a,$

$x_0 = b,$

$x_1 = ab + \alpha,$

and

$$\begin{aligned}
x_2 &= (ab + \alpha) b + \alpha = a, \\
x_3 &= a (ab + \alpha) + \alpha = b.
\end{aligned}$$

So

$$\begin{aligned}
ab^2 + \alpha b + \alpha - a &= 0, \\
ba^2 + \alpha a + \alpha - b &= 0.
\end{aligned}$$

Thus, this is indeed a solution of period three if the system below is satisfied.

$$
\begin{aligned}
(b+1)(ab-a+\alpha) &= 0 \qquad (2.3)\\
(a+1)(ab-b+\alpha) &= 0 \qquad (2.4)
\end{aligned}
$$

Therefore, we obtain three cases when after solving (2.3) and (2.4),

(i) Suppose that $b+1=0$. From (2.4) then $(a+1)(ab-b+\alpha)=0$ implies that $ab-b+\alpha=0$ or $1+a=0$ and hence $a=-1$ or $a=\alpha+1$.

(ii) Suppose that $a+1=0$. From (2.3) then $(b+1)(ab-a+\alpha)=0$ implies that $ab-a+\alpha=0$ or $1+b=0$ and hence $b=-1$ or $b=\alpha+1$.

(iii) Suppose that $a+1\neq 0$ and $b+1\neq 0$ such that

$$
\begin{aligned}
ab-a+\alpha &= 0 \qquad (2.5)\\
ab-b+\alpha &= 0 \qquad (2.6)
\end{aligned}
$$

Therefore, after solving (2.5)–(2.6) we find that

$a=\bar{x}_1$ and $b=\bar{x}_1$

or

$a=\bar{x}_2$ and $b=\bar{x}_2$.

In this case, there are no periodic solutions of Eq.(2.1) with prime period three, because it's trivial solutions.

Hence, there exists exactly three periodic solutions with minimal period three of Eq.(2.1) given by

$x_{-1}=-1, x_0=-1, x_1=\alpha+1, ...$

$x_{-1}=\alpha+1, x_0=-1, x_1=-1, ...$

$x_{-1}=-1, x_0=\alpha+1, x_1=-1, ...$

as claimed. ■

In the sequel, we will refer to any one of these three periodic solutions of Eq.(2.1) as

$$..., \alpha+1, -1, -1, \alpha+1, -1, -1, ... \qquad (2.7)$$

Theorem 2.5 *Difference equation (2.1) has no nontrivial period-four solutions.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a period-four solution of Eq.(2.1). Then $x_{4n-1} = a$, $x_{4n} = b$, $x_{4n+1} = c$ and $x_{4n+2} = d$ for every $n \in \mathbb{N}_0$ and some a, b, c, $d \in \mathbb{R}$ such that at least two of them are different. Therefore by direct calculation we have

$$x_1 = x_0x_{-1} + \alpha = ab + \alpha = c \tag{2.8}$$

$$x_2 = x_1x_0 + \alpha = b(ab + \alpha) + \alpha = d \tag{2.9}$$

$$x_3 = x_2x_1 + \alpha = cd + \alpha = a \tag{2.10}$$

$$x_4 = x_3x_2 + \alpha = ad + \alpha = b. \tag{2.11}$$

Thus, from (2.8)–(2.11), we have two cases,

(i) If $\alpha \neq 0$, then $a = b = c = d = \bar{x}_1$ and $a = b = c = d = \bar{x}_2$,

(ii) If $\alpha = 0$, then $a = b = c = d = 0$ and $a = b = c = d = 1$.

Consequently, there are no periodic solutions of Eq.(2.1) with period-four, because these are trivial solutions or non-periodic solutions. ■

Theorem 2.6 *Difference equation (2.1) has no nontrivial period-five solutions.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a period-five solution of Eq.(2.1). Then $x_{5n-1} = a$, $x_{5n} = b$, $x_{5n+1} = c$, $x_{5n+2} = d$ and $x_{5n+3} = e$ for every $n \in \mathbb{N}_0$ and some a, b, c, d, $e \in \mathbb{R}$ such that at least two of them are different. Therefore by direct calculation we have

$$x_1 = x_0x_{-1} + \alpha = ab + \alpha = c$$

$$x_2 = x_1x_0 + \alpha = bc + \alpha = d$$

$$x_3 = x_2x_1 + \alpha = cd + \alpha = e$$

$$x_4 = x_3x_2 + \alpha = de + \alpha = a$$

$$x_5 = x_4x_3 + \alpha = ae + \alpha = b.$$

Then $a = b = c = d = e = \bar{x}_1$ and $a = b = c = d = e = \bar{x}_2$. So, there are no periodic solutions of Eq.(2.1) with period-five, because it is a trivial solution. ■

2.4 LONG TERM BEHAVIOUR OF EQ.(2.1)

In this section, we find sets of initial conditions of Eq.(2.1) for which unbounded solutions exist. The following theorem takes care of long-term behaviour of Eq.(2.1).

Theorem 2.7 *Let $\alpha > \frac{1}{4}$ and x_{-1}, $x_0 < -1$. Then, the following statements hold true:*
(i)
$0 < x_1 < x_4 < x_7 < ...$
$... < x_8 < x_6 < x_5 < x_3 < x_2 < x_0 < -1.$
(ii) The subsequences $\{x_{3n}\}_{n=0}^{\infty}$, $\{x_{3n-1}\}_{n=0}^{\infty}$ tend to $-\infty$, $\{x_{3n+1}\}_{n=0}^{\infty}$ tends to $+\infty$.

Proof. We first have, $x_1 = x_0 x_{-1} + \alpha > 0$. We will prove that, $x_2 < x_0 < -1$. Since $(x_0+1)^2 > 0$, we have that $x_0^2 + 2x_0 + 1 > 0$. Thus, $1 + \frac{2}{x_0} + \frac{1}{x_0^2} > 0$ so $\frac{2}{x_0} + \frac{1}{x_0^2} > -1$. Since $x_{-1} < -1$, we must have, $x_{-1} < \frac{2}{x_0} + \frac{1}{x_0^2}$. Thus $x_{-1}x_0 > 2 + \frac{1}{x_0}$ and $x_{-1}x_0 + \alpha > 2 + \alpha + \frac{1}{x_0}$. Therefore $x_1 > 2 + \alpha + \frac{1}{x_0}$. It follows that,
$x_1 x_0 < 2x_0 + \alpha x_0 + 1$
$x_1 x_0 + \alpha < 2x_0 + \alpha x_0 + 1 + \alpha < x_0$
$x_2 < x_0 < -1.$
Next, we show that x_3 is not only less than -1, but it is also less then x_2. To show this, we note that since $x_2 < x_0$ and $x_1 = x_0 x_{-1} + \alpha > 0$ we have $x_2 x_1 < x_0 x_1$. This gives $x_2 x_1 + \alpha < x_0 x_1 + \alpha$, and hence $x_3 < x_2 < x_0 < -1$. We want to show $x_4 > x_1$. To show this, we start by observing that $x_3 < -1$ and so $x_3 < x_1$. Therefore, since $x_3 = x_2 x_1 + \alpha < x_1$, we obtain with $x_2 + 1 < 0$,
$x_2 x_1 - x_1 + \alpha < 0$
$x_1 (x_2 - 1) + \alpha < 0$
$x_1 (x_2 - 1) < -\alpha$
$x_1 (x_2 - 1)(x_2 + 1) > -\alpha (x_2 + 1)$
$x_1 x_2^2 - x_1 > -\alpha x_2 - \alpha$
$x_1 x_2^2 + \alpha x_2 > x_1 - \alpha$
$x_2 (x_1 x_2 + \alpha) > x_1 - \alpha$
$x_2 x_3 > x_1 - \alpha$
$x_2 x_3 + \alpha > x_1$
$x_4 > x_1.$
By induction, it can be proved that,
$0 < x_1 < x_4 < x_7 < ...$
$... < x_8 < x_6 < x_5 < x_3 < x_2 < x_0 < -1.$
In other words, $\{x_{3n+1}\}_{n=0}^{\infty}$ is a positive increasing subsequence and $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are negative decreasing subsequences.

(ii) Now we verify that these subsequences are unbounded and thus our solution is unbounded. Suppose that two decreasing sequences, $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are bounded from below. Then they each must converge to a finite limit (which is the same finite limit and less then $\overline{x}_2$). But by Eq.(2.1), the third increasing subsequence $\{x_{3n+1}\}_{n=0}^{\infty}$ must also converge to a finite limit (which is positive), where,

$x_{3n+1} = x_{3n}x_{3n-1} + \alpha = |x_{3n}|\left|x_{3(n-1)+2}\right| + \alpha$.

This is impossible because there are no periodic solution with minimal period two (see section 2.2, theorem 2.3). So $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are unbounded (They tend to $-\infty$) and thus $\{x_{3n+1}\}_{n=0}^{\infty}$ is also unbounded (it tends to $+\infty$) by Eq.(2.1) again. ■

2.5 NUMERICAL EXAMPLES

In this section, we present some numerical examples about theoretical results for Eq.(2.1).

Example 2.1 *Consider the Eq.(2.1) with initial conditions* $x_{-1} = -1$, $x_0 = -1$ *and* $\alpha = 3$, *then there are periodic solutions with period three as (2.7). The following figure verifies our theoretical results. (See Figure 2.1)*

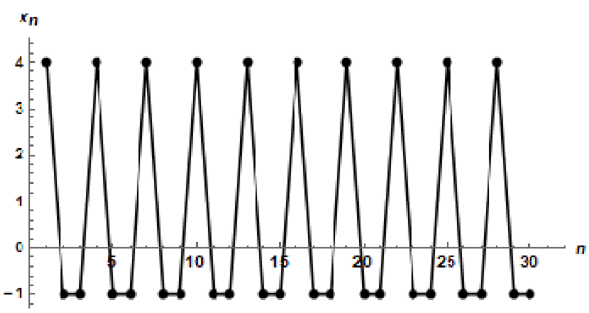


Figure 2.1: Plot of the difference equation $x_{n+1} = x_n x_{n-1} + 3$.

Example 2.2 *Consider the Eq.(2.1) with initial conditions* $x_{-1} = \alpha + 1 = \frac{3}{2}$, $x_0 = -1$ *and* $\alpha = \frac{1}{2}$, *then there are periodic solutions with period three as (2.7). The following figure verifies our theoretical results. (See Figure 2.2)*

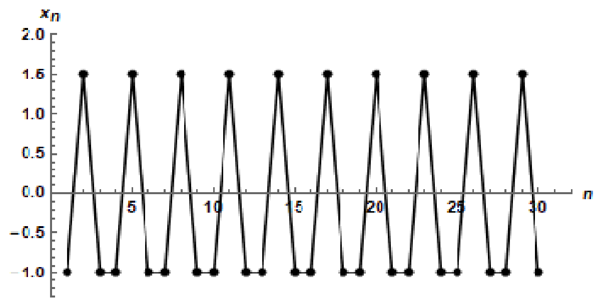


Figure 2.2: Plot of the difference equation $x_{n+1} = x_n x_{n-1} + \frac{1}{2}$.

Example 2.3 *Consider the Eq.(2.1) with initial conditions $x_{-1} = -1$, $x_0 = \frac{3}{5}$ and $\alpha = -\frac{2}{5}$, then there are periodic solutions with period three as (2.7). The following figure verifies our theoretical results. (See Figure 2.3)*

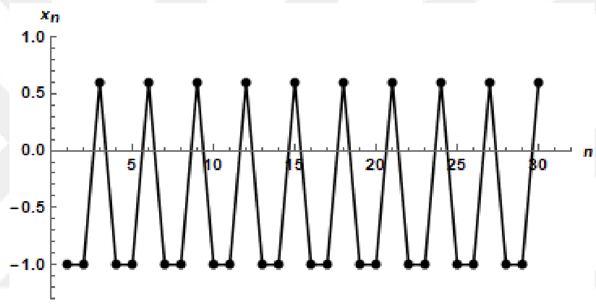


Figure 2.3: Plot of the difference equation $x_{n+1} = x_n x_{n-1} - \frac{2}{5}$.

Example 2.4 *Consider the Eq.(2.1) with initial conditions $x_{-1} = -1.0003$, $x_0 = -1.0002$ and $\alpha = 0.28$, then there are unbounded solutions as Theorem 2.7. The following figure verifies our theoretical results. (See Figure 2.4)*

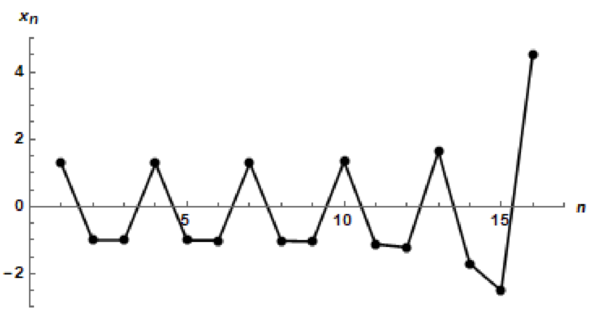


Figure 2.4: Plot of the difference equation $x_{n+1} = x_n x_{n-1} + 0.28$.

CHAPTER 3

LONG-TERM BEHAVIOR OF SOLUTIONS OF A NONLINEAR DIFFERENCE EQUATION

In this chapter we study the solution of the nonlinear difference equation

$x_{n+1} = x_{n-1}x_{n-3} - 1.$

Especially, we examine the periodic, eventually periodic and non-periodic solutions of related difference equation. Additionally, we study both the existence of unbounded solutions of related difference equation and convergence behaviour of equilibrium points of related difference equation.

The results presented in this chapter have been published as articles in [26] and [27] by us.

3.1 INTRODUCTION

In [15] Kent et al studied the boundedness of solutions, periodicity of solutions, and existence of unbounded solutions and how these behaviors depend on initial conditions of the difference equation

$x_{n+1} = x_n x_{n-1} - 1.$

In [13] Kent et al studied the long-term behavior of solutions of the difference equation

$x_{n+1} = x_{n-1}x_{n-2} - 1.$

In [14] Kent et al investigated the periodicity of solutions, existence of unbounded solutions and converging to the negative equilibrium of the difference equation

$x_{n+1} = x_n x_{n-2} - 1.$

In [12] Kent and Kosmala studied the periodicity and asymptotic periodicity of solutions, as well as the existence of unbounded solutions of the difference equation

$x_{n+1} = x_n x_{n-3} - 1.$

In [22], Stevic and Iričanin, investigated the following difference equation

$$x_{n+1} = x_{n-l}x_{n-k} - 1, n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $k < l$, $\gcd(k, l) = 1$, and the initial values $x_{-l}, ..., x_{-2}, x_{-1}$ are real numbers. They dealt with the behaviour of solutions of the difference equation of general character, by describing the long-term behavior of the solutions of the difference equation for all values of parameters k and l, where the initial values satisfy the following condition $\min\{x_{-l}, ..., x_{-2}, x_{-1}\}$.

In this chapter we study the solution of the following difference equation

$$x_{n+1} = x_{n-1}x_{n-3} - 1, \quad n = 0, 1, 2, ... \tag{3.1}$$

The difference equation (3.1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, \ n \in \mathbb{N}_0, \tag{3.2}$$

with particular choices of k and l, where $k, l \in \mathbb{N}_0$.

This chapter can be regarded as a continuation of systematic investigation of Eq.(3.2).

3.2 THE EQUILIBRIUM POINTS OF EQ.(3.1)

In this section, we investigate the equilibrium points of Eq.(3.1).

Lemma 3.1 *The equilibrium points of Eq.(3.1) are*

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2}, \ \bar{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{3.3}$$

Proof. Let $\overline{x}$ is an equilibrium point of Eq.(3.1). Thus $x_n = \overline{x}$, for all $n \geq -3$. So, we have

$$\overline{x} = \overline{x} \cdot \overline{x} - 1.$$

Therefore, we get the equation

$$\overline{x}^2 - \overline{x} - 1 = 0.$$

Thus, we find that Eq.(3.1) has exactly two equilibrium points, which $\bar{x}_1$ is negative and $\bar{x}_2$ is positive:

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2}, \ \bar{x}_2 = \frac{1+\sqrt{5}}{2}.$$

$\bar{x}_2$ is called as Golden Number and $\bar{x}_1$ is its conjugate. ■

3.3 PERIODIC SOLUTIONS OF EQ.(3.1)

In this section we prove some results regarding periodicity of solutions of Eq.(3.1). Eq.(3.1) has constant solutions, namely $x_n = \bar{x}_1$ for all $n \geq -3$, and $x_n = \bar{x}_2$ for all $n \geq -3$. But we have

Theorem 3.2 *There are no eventually constant solutions of Eq.(3.1) unless we take initial conditions as equilibrium points.*

Proof. If $\{x_n\}_{n=-3}^{\infty}$ is eventually constant solutions of Eq.(3.1), then $x_N = x_{N+1} = x_{N+2} = x_{N+3} = \bar{x}$, for some $N \in \mathbb{N}_0$, where $\bar{x}$ is an equilibrium point. However, Eq.(3.1) gives $x_{N+3} = x_{N+1}x_{N-1} - 1$, which implies

$$x_{N-1} = \frac{x_{N+3}+1}{x_{N+1}} = \frac{\bar{x}+1}{\bar{x}} = \bar{x}.$$

Repeating the procedure, we obtain that $x_n = \bar{x}$ for $-3 \leq n \leq N+3$. Hence, there are no eventually constant solutions. ■

The following theorem demonstrates that Eq.(3.1) has solutions that are periodic with prime period two.

Theorem 3.3 *Eq.(3.1) has prime period-two solutions if and only if the initial conditions are*

$$\begin{aligned} x_{-3} &= \bar{x}_1,\ x_{-2} = \bar{x}_2,\ x_{-1} = \bar{x}_1,\ x_0 = \bar{x}_2 \\ x_{-3} &= \bar{x}_2,\ x_{-2} = \bar{x}_1,\ x_{-1} = \bar{x}_2,\ x_0 = \bar{x}_1. \end{aligned}$$

Proof. If the initial conditions are as given, then by some calculations we can see easily that these solutions are of period two. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-two solution of Eq.(3.1). Then $x_{2n-3} = a$, $x_{2n-2} = b$, $x_{2n-1} = a$ and $x_{2n} = b$ for all $n \in \mathbb{N}_0$ and for some a, $b \in \mathbb{R}$ such that $a \neq b$. We have $x_1 = x_{-1}x_{-3} - 1 = a^2 - 1 = a$ and $x_2 = x_0x_{-2} - 1 = b^2 - 1 = b$. From these two equations we obtain that

$$a^2 - a - 1 = 0$$

and

$$b^2 - b - 1 = 0.$$

If either $a = \bar{x}_1$ and $b = \bar{x}_2$ or $a = \bar{x}_2$ and $b = \bar{x}_1$, we obtain the prime period two solution. On the other hand, either $a = \bar{x}_1$ and $b = \bar{x}_1$ or $a = \bar{x}_2$ and $b = \bar{x}_2$, we obtain the trivial solution which is an equilibrium solution. ■

The following theorem shows that Eq.(3.1) has prime period three solutions and this theorem has an important role for Theorem 3.5.

Theorem 3.4 *Eq.(3.1) has prime period-three solutions if and only if the initial conditions are*

$$\begin{aligned}
x_{-3} &= -1,\ x_{-2} = -1,\ x_{-1} = 0,\ x_0 = -1 \\
x_{-3} &= -1,\ x_{-2} = 0,\ x_{-1} = -1,\ x_0 = -1 \\
x_{-3} &= 0,\ x_{-2} = -1,\ x_{-1} = -1,\ x_0 = 0.
\end{aligned}$$

Proof. If the initial conditions are as given, then by some calculations it is clear to see that these solutions are of period three. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-three solution of Eq.(3.1). Then $x_{3n-3} = a$, $x_{3n-2} = b$, $x_{3n-1} = c$ and $x_{3n} = a$ for all $n \in \mathbb{N}_0$ and for some a, b, $c \in \mathbb{R}$ such that at least two of them are different. We have

$$x_1 = x_{-1}x_{-3} - 1 = ac - 1 = b \tag{3.4}$$

$$x_2 = x_0 x_{-2} - 1 = ab - 1 = c \tag{3.5}$$

$$x_3 = x_1 x_{-1} - 1 = bc - 1 = a. \tag{3.6}$$

From (3.4) – (3.6) we have three cases to be considered.

(i) If $a = 0$ and $b = -1$, then $c = -1$, we obtain the first prime period-three solution.

(ii) If $a = -1$ and $b = 0$, then $c = -1$, we obtain the second prime period-three solution.

(iii) If $a = -1$ and $b = -1$, then $c = 0$, we obtain the last prime period-three solution. ■

Throughout this chapter, we will refer to any one of these three periodic solutions of Eq.(3.1) as

$$..., 0, -1, -1, 0, -1, -1, ... \tag{3.7}$$

Theorem 3.5 *All eventually periodic solutions with prime period-three are of the form*

$$(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, -1, -1, 0, -1, -1, ...)$$

where, $N \geq -3$, $x_{N+2}x_N = 1$, $x_{N+3} = -1, x_{N+1} = 0$, *and, if* $N \neq -3$, $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$ *for* $0 \leq n \leq N$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1) that is eventually periodic with prime period-three, then by Theorem 3.4, there is an $N \geq -3$ such that $x_{N+4} = 0$, $x_{N+5} = -1$, $x_{N+6} = -1$ and $x_{N+7} = 0$. Hence, $0 = x_{N+4} = x_{N+2}x_N - 1$ and thus $x_{N+2}x_N = 1$. Hence, $-1 = x_{N+5} = x_{N+3}x_{N+1} - 1$ and then $x_{N+3}x_{N+1} = 0$. Therefore, $-1 = x_{N+6} = x_{N+4}x_{N+2} - 1$ and then $x_{N+4}x_{N+2} = 0$. It's true from $x_{N+4} = 0$. Thus, $0 = x_{N+7} = x_{N+5}x_{N+3} - 1$ and thus $x_{N+5}x_{N+3} = 1$. Hence, $x_{N+3} = -1$ because $x_{N+5} = -1$. So, $x_{N+1} = 0$. Consequently, while $x_{N+3} = -1$, $x_{N+1} = 0$ and $x_{N+2}x_N = 1$, all eventually periodic solutions prime period-three are of the form (3.7). From Eq.(3.1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$ for $0 \leq n \leq N$, as desired. ∎

Theorem 3.6 *Eq.(3.1) has no prime period-five solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-five solution of Eq.(3.1). Then, $x_{5n-3} = a$, $x_{5n-2} = b$, $x_{5n-1} = c$, $x_{5n} = d$ and $x_{5n+1} = e$ for all $n \in \mathbb{N}_0$ and a, b, c, d, $e \in \mathbb{R}$ such that at least two of them are different. From Eq.(3.1)

$$x_1 = x_{-1}x_{-3} - 1 = ac - 1 = e \tag{3.8}$$

$$x_2 = x_0x_{-2} - 1 = bd - 1 = a \tag{3.9}$$

$$x_3 = x_1x_{-1} - 1 = ec - 1 = b \tag{3.10}$$

$$x_4 = x_2x_0 - 1 = ad - 1 = c \tag{3.11}$$

$$x_5 = x_3x_1 - 1 = be - 1 = d \tag{3.12}$$

From (3.8) – (3.12) we obtain

$$a = b = c = d = e = \bar{x}_1$$

$$a = b = c = d = e = \bar{x}_2.$$

as desired. ∎

The next theorem shows that Eq.(3.1) has prime period-six for certain initial conditions.

Theorem 3.7 *Eq.(3.1) has prime period-six solutions if and only if the initial conditions are*

Case 1: $x_{-3} = 0,\ x_{-2} = \bar{x}_1,\ x_{-1} = -1,\ x_0 = \bar{x}_1;$

Case 2: $x_{-3} = -1,\ x_{-2} = \bar{x}_1,\ x_{-1} = 0,\ x_0 = \bar{x}_1;$

Case 3: $x_{-3} = -1,\ x_{-2} = \bar{x}_1,\ x_{-1} = -1,\ x_0 = \bar{x}_1;$

Case 4: $x_{-3} = \bar{x}_1,\ x_{-2} = 0,\ x_{-1} = \bar{x}_1,\ x_0 = -1;$

Case 5: $x_{-3} = \bar{x}_1,\ x_{-2} = -1,\ x_{-1} = \bar{x}_1,\ x_0 = 0;$

Case 6: $x_{-3} = \bar{x}_1,\ x_{-2} = -1,\ x_{-1} = \bar{x}_1,\ x_0 = -1;$

Case 7: $x_{-3} = 0,\ x_{-2} = \bar{x}_2,\ x_{-1} = -1,\ x_0 = \bar{x}_2;$

Case 8: $x_{-3} = -1,\ x_{-2} = \bar{x}_2,\ x_{-1} = 0,\ x_0 = \bar{x}_2;$

Case 9: $x_{-3} = -1,\ x_{-2} = \bar{x}_2,\ x_{-1} = -1,\ x_0 = \bar{x}_2;$

Case 10: $x_{-3} = \bar{x}_2,\ x_{-2} = 0,\ x_{-1} = \bar{x}_2,\ x_0 = -1;$

Case 11: $x_{-3} = \bar{x}_2,\ x_{-2} = -1,\ x_{-1} = \bar{x}_2,\ x_0 = 0;$

Case 12: $x_{-3} = \bar{x}_2,\ x_{-2} = -1,\ x_{-1} = \bar{x}_2,\ x_0 = -1;$

Case 13: $x_{-3} = -1,\ x_{-2} = 0,\ x_{-1} = 0,\ x_0 = -1;$

Case 14: $x_{-3} = 0,\ x_{-2} = 0,\ x_{-1} = -1,\ x_0 = -1;$

Case 15: $x_{-3} = -1,\ x_{-2} = -1,\ x_{-1} = 0,\ x_0 = 0;$

Case 16: $x_{-3} = 0,\ x_{-2} = -1,\ x_{-1} = -1,\ x_0 = -1;$

Case 17: $x_{-3} = -1,\ x_{-2} = -1,\ x_{-1} = -1,\ x_0 = -1.$

Proof. If the initial conditions are as given, then by some calculations it is easy to see that these solutions are of period six. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-six solution of Eq.(3.1). Then, $x_{6n-3} = a$, $x_{6n-2} = b$, $x_{6n-1} = c$, $x_{6n} = d$, $x_{6n+1} = ac - 1$ and $x_{6n+2} = bd - 1$ for all $n \in \mathbb{N}_0$ and a, b, c, d, e, $f \in \mathbb{R}$ such that at least two of them are different. We have

$$x_1 = x_{-1}x_{-3} - 1 = ac - 1 \tag{3.13}$$

$$x_2 = x_0x_{-2} - 1 = bd - 1 \tag{3.14}$$

$$x_3 = x_1x_{-1} - 1 = (ac-1)c - 1 = a \tag{3.15}$$

$$x_4 = x_2x_0 - 1 = d(bd-1) - 1 = b \tag{3.16}$$

$$x_5 = x_3x_1 - 1 = a(ac-1) - 1 = c \tag{3.17}$$

$$x_6 = x_4x_2 - 1 = b(bd-1) - 1 = d. \tag{3.18}$$

From (3.13) – (3.18), we obtain that 25 cases.

Case 1: $a = 0,\ b = \bar{x}_1,\ c = -1,\ d = \bar{x}_1$

Case 2: $a = -1,\ b = \bar{x}_1,\ c = 0,\ d = \bar{x}_1$

Case 3: $a = -1,\ b = \bar{x}_1,\ c = -1,\ d = \bar{x}_1$

Case 4: $a = \bar{x}_1,\ b = 0,\ c = \bar{x}_1,\ d = -1$

Case 5: $a = \bar{x}_1,\ b = -1,\ c = \bar{x}_1,\ d = 0$

Case 6: $a = \bar{x}_1,\ b = -1,\ c = \bar{x}_1,\ d = -1$

Case 7: $a = 0,\ b = \bar{x}_2,\ c = -1,\ d = \bar{x}_2$

Case 8: $a = -1,\ b = \bar{x}_2,\ c = 0,\ d = \bar{x}_2$

Case 9: $a = -1,\ b = \bar{x}_2,\ c = -1,\ d = \bar{x}_2$

Case 10: $a = \bar{x}_2,\ b = 0,\ c = \bar{x}_2,\ d = -1$

Case 11: $a = \bar{x}_2,\ b = -1,\ c = \bar{x}_2,\ d = 0$

Case 12: $a = \bar{x}_2,\ b = -1,\ c = \bar{x}_2,\ d = -1$

Case 13: $a = -1,\ b = 0,\ c = 0,\ d = -1$

Case 14: $a = 0,\ b = 0,\ c = -1,\ d = -1$

Case 15: $a = -1,\ b = -1,\ c = 0,\ d = 0$

Case 16: $a = 0,\ b = -1,\ c = -1,\ d = -1$

Case 17: $a = -1,\ b = -1,\ c = -1,\ d = -1$

Case 18: $a = 0, b = -1, c = -1, d = 0$

Case 19: $a = -1, b = -1, c = 0, d = -1$

Case 20: $a = -1, b = 0, c = -1, d = -1$

Case 21: $a = -1, b = -1, c = -1, d = 0$

Case 22: $a = \bar{x}_1, b = \bar{x}_2, c = \bar{x}_1, d = \bar{x}_2$

Case 23: $a = \bar{x}_2, b = \bar{x}_1, c = \bar{x}_2, d = \bar{x}_1$

Case 24: $a = \bar{x}_2, b = \bar{x}_2, c = \bar{x}_2, d = \bar{x}_2$

Case 25: $a = \bar{x}_1, b = \bar{x}_1, c = \bar{x}_1, d = \bar{x}_1$.

Then, there are period-six solution of Eq.(3.1) for 25 cases. Although Case 1 – Case 17 have prime period-six, Case 18 – Case 21 have prime period-three, Case 22 – Case 23 have prime period-two and Case 24 – Case 25 are trivial solutions. Thereafter, Case 1 – Case 6 have period-six cycle as

$..., 0, \bar{x}_1, -1, \bar{x}_1, -1, \bar{x}_1, ...,$

and Case 7 – Case 12 have period-six cycle as

$..., 0, \bar{x}_2, -1, \bar{x}_2, -1, \bar{x}_2, ...,$

and Case 13 – Case 17 have period-six cycle as

$..., 0, 0, -1, -1, -1, -1, ...$

Likewise, prime period-three and prime period-two investigated in Theorem 3.4 and Theorem 3.3 respectively. ■

The following theorem demonstrates that Eq.(3.1) has eventually periodic solutions with prime period-six and that their forms.

Theorem 3.8 *All eventually periodic solutions with prime period-six are of the forms*

Form 1: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, \bar{x}_2, -1, \bar{x}_2, -1, \bar{x}_2, 0, \bar{x}_2, -1, ...)$

(Case 1 – Case 6) where, $N \geq -3$, $x_{N+1} = \bar{x}_2$, $x_{N+3} = \bar{x}_2$, $x_N x_{N+2} = 1$, *and, if* $N \neq -3$, $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$ *for* $0 \leq n \leq N$.

Form 2: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, \bar{x}_1, -1, \bar{x}_1, -1, \bar{x}_1, 0, \bar{x}_1, -1, ...)$

(Case 7 – Case 12) where, $N \geq -3$, $x_{N+1} = \bar{x}_1$, $x_{N+3} = \bar{x}_1$, $x_N x_{N+2} = 1$, *and, if* $N \neq -3$, $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$ *for* $0 \leq n \leq N$.

Form 3: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, 0, -1, -1, -1, -1, 0, 0, -1, ...)$

(Case 13 – Case 17) where, $N \geq -3$, $x_{N+1} x_{N+3} = 1$, $x_N x_{N+2} = 1$, *and, if* $N \neq -3$, $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$ *for* $0 \leq n \leq N$.

Proof.

Form 1: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1) that is eventually periodic with prime period-six. Then by Theorem 3.7, there is an $N \geq -3$ such that $x_{N+4} = 0$ and $x_{N+5} = \bar{x}_2$. Hence, $0 = x_{N+4} = x_N x_{N+2} - 1$ and consequently $x_N x_{N+2} = 1$. Hence, $\bar{x}_2 = x_{N+5} = x_{N+3} x_{N+1} - 1$ and then $x_{N+3} x_{N+1} = 1 + \bar{x}_2$. Hence, $x_{N+3} x_{N+1} = 1 + \bar{x}_2 = \frac{3+\sqrt{5}}{2} = (\bar{x}_2)^2$ and so $x_{N+3} = x_{N+1} = \bar{x}_2$. Therefore,

$$\begin{aligned}
x_{N+6} &= x_{N+4} x_{N+2} - 1 = -1 \\
x_{N+7} &= x_{N+5} x_{N+3} - 1 = \bar{x}_2 \\
x_{N+8} &= x_{N+6} x_{N+4} - 1 = -1 \\
x_{N+9} &= x_{N+7} x_{N+5} - 1 = \bar{x}_2 \\
x_{N+10} &= x_{N+8} x_{N+6} - 1 = 0 \\
x_{N+11} &= x_{N+9} x_{N+7} - 1 = \bar{x}_2.
\end{aligned}$$

From Eq.(3.1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$, for $0 \leq n \leq N$, as desired.

Form 2: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1) that is eventually periodic with prime period-six. Then by Theorem 3.7, there is an $N \geq -3$ such that $x_{N+4} = 0$ and $x_{N+5} = \bar{x}_1$.

Hence, $0 = x_{N+4} = x_N x_{N+2} - 1$ and consequently $x_N x_{N+2} = 1$. Hence, $\bar{x}_1 = x_{N+5} = x_{N+3}x_{N+1} - 1$ and then $x_{N+3}x_{N+1} = 1 + \bar{x}_1$. Hence, $x_{N+3}x_{N+1} = 1 + \bar{x}_1 = \frac{3-\sqrt{5}}{2} = (\bar{x}_1)^2$ and so $x_{N+3} = x_{N+1} = \bar{x}_1$. Therefore,

$$\begin{aligned}
x_{N+6} &= x_{N+4}x_{N+2} - 1 = -1 \\
x_{N+7} &= x_{N+5}x_{N+3} - 1 = \bar{x}_1 \\
x_{N+8} &= x_{N+6}x_{N+4} - 1 = -1 \\
x_{N+9} &= x_{N+7}x_{N+5} - 1 = \bar{x}_1 \\
x_{N+10} &= x_{N+8}x_{N+6} - 1 = 0 \\
x_{N+11} &= x_{N+9}x_{N+7} - 1 = \bar{x}_1.
\end{aligned}$$

From Eq.(3.1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1} + 1)/x_{n-3}$, for $0 \leq n \leq N$, as desired.

Form 3: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1) that is eventually periodic with prime period-six. Then by Theorem 3.7, there is an $N \geq -3$ such that $x_{N+4} = 0$ and $x_{N+5} = 0$. Hence, $0 = x_{N+4} = x_N x_{N+2} - 1$ and consequently $x_N x_{N+2} = 1$. Hence, $0 = x_{N+5} = x_{N+3}x_{N+1} - 1$ and then $x_{N+3}x_{N+1} = 1$. Therefore,

$$\begin{aligned}
x_{N+6} &= x_{N+4}x_{N+2} - 1 = -1 \\
x_{N+7} &= x_{N+5}x_{N+3} - 1 = -1 \\
x_{N+8} &= x_{N+6}x_{N+4} - 1 = -1 \\
x_{N+9} &= x_{N+7}x_{N+5} - 1 = -1 \\
x_{N+10} &= x_{N+8}x_{N+6} - 1 = 0 \\
x_{N+11} &= x_{N+9}x_{N+7} - 1 = 0.
\end{aligned}$$

From Eq.(3.1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1} + 1)/x_{n-3}$, for $0 \leq n \leq N$, as desired. ■

Also, in [12], the authors investigated Eq.(3.1) where the interval $(-1, 0)$ is invariant.

Theorem 3.9 ([12], Theorem 4.1) *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3.2). If*

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (-1, 0),$$

then $x_n \in (-1, 0)$ for all $n \geq -k$.

Therefore, Eq.(3.1), (where $k = 3$ and $l = 1$ in Eq.(3.2)), has an invariant interval under the initial conditions in $(-1, 0)$.

3.4 STABILITY OF SOLUTIONS OF EQ.(3.1)

In this section, we determine the stability nature of the two equilibria of Eq.(3.1).

Theorem 3.10 *The positive equilibrium of Eq.(3.1), $\bar{x}_2$, is unstable.*

Proof. The characteristic equation of equilibria of Eq.(3.1) is the following:

$$\lambda^4 - \bar{x}_2\lambda^2 - \bar{x}_2 = 0 \tag{3.19}$$

with eigenvalues

$$\begin{aligned}
\lambda_1, \lambda_2 &= \pm\frac{1}{2}\sqrt{1+\sqrt{5}-2\sqrt{\frac{5}{2}\sqrt{5}+\frac{7}{2}}} \\
\lambda_3, \lambda_4 &= \pm\frac{1}{2}\sqrt{1+\sqrt{5}+2\sqrt{\frac{5}{2}\sqrt{5}+\frac{7}{2}}}.
\end{aligned}$$

Therein, $|\lambda_1|, |\lambda_2| < 1$ and $|\lambda_3|, |\lambda_4| > 1$. Herewith, $\bar{x}_2$ is unstable, which is a saddle point. ∎

Lemma 3.11 *The negative equilibrium of Eq.(3.1), $\bar{x}_1$, is local asymptotically stable.*

Proof. The characteristic equation of equilibria of Eq.(3.1) is the following:

$$\lambda^4 - \bar{x}_1\lambda^2 - \bar{x}_1 = 0 \tag{3.20}$$

with eigenvalues

$$\begin{aligned}
\lambda_1, \lambda_2 &= \pm\frac{1}{2}\sqrt{1-\sqrt{5}-2\sqrt{\frac{7}{2}-\frac{5}{2}\sqrt{5}}} \\
\lambda_3, \lambda_4 &= \pm\frac{1}{2}\sqrt{1-\sqrt{5}+2\sqrt{\frac{7}{2}-\frac{5}{2}\sqrt{5}}}.
\end{aligned}$$

Therefore, $|\lambda_1|, |\lambda_2|, |\lambda_3|$ and $|\lambda_4| < 1$ as desired. ∎

3.5 CONVERGENCE OF NEGATIVE EQUILIBRIUM OF EQ.(3.1)

In this section, we focus on convergence of negative equilibrium of Eq.(3.1). During the last years, although many authors have focused on convergence of positive equilibrium point of solutions of difference equations, some authors prefer to study negative equilibrium of difference equations. (See [1], [17]) Eq.(3.1) is equivalent to the following difference equation

$$y_{n+1} = 1 - y_{n-1}y_{n-3}, \; n = 1, 2, ... \tag{3.21}$$

with change to variable $y_n = -x_n$. Thus, the initial conditions of (3.21) are $y_{-3} = -x_{-3}$, $y_{-2} = -x_{-2}$, $y_{-1} = -x_{-1}$ and $y_0 = -x_0$.

Besides, the interval $(0, 1)$ is invariant for Eq.(3.21) as in [12] as Theorem 4.1. Now, we take two subsequences $t_n = y_{2n-1}$ and $w_n = y_{2n}$ for $n \geq -1$. Therefore, we obtain that

$$t_{n+1} = 1 - t_n t_{n-1}; t_{-1}, t_0 \in I \tag{3.22}$$

and

$$w_{n+1} = 1 - w_n w_{n-1}; \; w_{-1}, w_0 \in I \tag{3.23}$$

with the positive equilibrium point $\overline{y} = \overline{t} = \overline{w} = \frac{\sqrt{5}-1}{2}$.

Theorem 3.12 *Every solution of Eq.(3.21) with the initial conditions in* $(0, 1)$ *converges to the positive equilibrium* $\bar{y} = (\sqrt{5} - 1)/2$.

Now, we will give the proof of Theorem 3.12 after giving two lemmas which we need their proof.

Lemma 3.13 *Every solutions of Eq.(3.22) converges to* $\bar{t} = \frac{\sqrt{5}-1}{2}$.

Proof. We apply Theorem 1.3. Let $g : [a, b] \times [a, b] \to [a, b]$ is a function defined by

$$g(u, v) = 1 - u \cdot v.$$

Then

$$\begin{aligned} \frac{\partial g(u,v)}{\partial u} &= -v < 0, \\ \frac{\partial g(u,v)}{\partial v} &= -u < 0. \end{aligned}$$

Therefore g is non-increasing function in each of its arguments.

Now, we consider that $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, m) \text{ and } m = g(M, M).$$

Hence, from Eq.(3.22)

$$\begin{aligned} M &= 1 - m \cdot m, \\ m &= 1 - M \cdot M. \end{aligned}$$

Thus, we have

$$M = m.$$

Then, the proof is completed. ■

Now, we investigate convergence behaviour of Eq.(3.23).

Lemma 3.14 *Every solutions of Eq.(3.23) converges to* $\overline{w} = \frac{\sqrt{5}-1}{2}$.

Proof. We apply Theorem 1.3. Let $g : [a, b] \times [a, b] \to [a, b]$ is a function defined by

$$g(u, v) = 1 - u \cdot v.$$

Then

$$\begin{aligned} \frac{\partial g(u, v)}{\partial u} &= -v < 0, \\ \frac{\partial g(u, v)}{\partial v} &= -u < 0. \end{aligned}$$

Therefore g is non-increasing function in each of its arguments.

Now, we consider that $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, m) \text{ and } m = g(M, M).$$

Hence, from Eq.(3.23)

$$\begin{aligned} M &= 1 - m \cdot m, \\ m &= 1 - M \cdot M. \end{aligned}$$

Thus, we have

$$M = m.$$

So, the proof is completed. ■

Proof of Theorem 3.12. We consider Lemma 3.13 and Lemma 3.14. We also know that sequences t_n and w_n are subsequences of y_n. So, y_n converges to the positive equilibrium $\bar{y}$. Due to the fact that $\bar{y}$ is positive equilibrium of Eq.(3.21), Eq.(3.1) converges to negative equilibrium point. So, the proof is completed. ■

3.6 EXISTENCE OF UNBOUNDED SOLUTIONS OF EQ.(3.1)

In this section, we find sets of initial conditions of Eq.(3.1) for which unbounded solutions exist.

Our study is a special case of Eq.(3.2) where $k = 1$ and $l = 3$. In [22] and [12] the authors obtain some results for general condition Eq.(3.2). We now give some results for this special case which has not been studied so far as far as we examined.

Lemma 3.15 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1). If $x_{-3},\ x_{-2},\ x_{-1},\ x_0 < -1$ then the following statements are true:*

(i) $x_1,\ x_2 > 0$

(ii) $x_3,\ x_4,\ x_5,\ x_6 < -1$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1) and $x_{-3},\ x_{-2},\ x_{-1},\ x_0 < -1$. Then,

$x_1 = x_{-1}x_{-3} - 1 > 0$

$x_2 = x_0 x_{-2} - 1 > 0$

$x_3 = x_1 x_{-1} - 1 < -1$

$x_4 = x_2 x_{-2} - 1 < -1$

$x_5 = x_3 x_1 - 1 < -1$

$x_6 = x_4 x_2 - 1 < -1.$ ■

The following theorem has an important role which existence of unbounded solutions of Eq.(3.1).

Theorem 3.16 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1), then*

$$x_{n+6} - x_n = (x_{n+2} + 1)(x_{n+4} - x_n). \tag{3.24}$$

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1), hence,

$x_{n+6} - x_n = (x_{n+4}x_{n+2} - 1) - x_n$

$x_{n+6} - x_n = ((x_{n+2}x_n - 1)\, x_{n+2} - 1) - x_n$

$x_{n+6} - x_n = x_{n+2}^2 x_n - x_{n+2} - 1 - x_n$

$x_{n+6} - x_n = x_n \left(x_{n+2}^2 - 1\right) - (x_{n+2} + 1)$

$x_{n+6} - x_n = (x_{n+2} + 1)(x_n (x_{n+2} - 1) - 1)$

$x_{n+6} - x_n = (x_{n+2} + 1)(x_n x_{n+2} - x_n - 1)$

$x_{n+6} - x_n = (x_{n+2} + 1)(x_{n+4} - x_n)$. ■

Theorem 3.17 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1). If $x_{-3}, x_{-2}, x_{-1}, x_0 < -1$, then,*

$$
\begin{aligned}
0 &< x_1 < x_7 < x_{13} < ..., \\
0 &< x_2 < x_8 < x_{14} < ..., \\
-1 &> x_3 > x_9 > x_{15} > ..., \\
-1 &> x_4 > x_{10} > x_{16} > ..., \\
-1 &> x_5 > x_{11} > x_{17} > ..., \\
-1 &> x_6 > x_{12} > x_{18} > ....
\end{aligned}
$$

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3.1). From Lemma 3.15 and Theorem 3.16, for $n = 1$,

$$x_7 - x_1 = (x_3 + 1)(x_5 - x_1)$$

therefore, since $x_1 > 0$, $x_3 < -1$ and $x_5 < -1$,

$$x_7 - x_1 > 0$$

hence

$$x_7 > x_1 > 0.$$

From Theorem 3.16, for $n = 2$,

$$x_8 - x_2 = (x_4 + 1)(x_6 - x_2)$$

thus, for $x_2 > 0$, $x_4 < -1$ and $x_6 < -1$,

$$x_8 - x_2 > 0$$

then

$$x_8 > x_2 > 0.$$

From Theorem 3.16, for $n = 3$,

$$x_9 - x_3 = (x_5 + 1)(x_7 - x_3)$$

hence, since $x_3 < -1$, $x_5 < -1$ and $x_7 > 0$,

$$x_9 - x_3 < 0$$

thus

$x_9 < x_3 < -1.$

From Theorem 3.16, for $n = 4$,

$x_{10} - x_4 = (x_6 + 1)(x_8 - x_4)$

then, since $x_4 < -1$, $x_6 < -1$ and $x_8 > 0$,

$x_{10} - x_4 < 0$

hence

$x_{10} < x_4 < -1.$

From Theorem 3.16, for $n = 5$,

$x_{11} - x_5 = (x_7 + 1)(x_9 - x_5)$

and from Eq.(3.1)

$x_{11} - x_5 = (x_7 + 1)\left(x_5^2 x_3 - 2x_5 - 1\right)$

hence, for $x_3 < -1$ and $x_7 > 0$,

$x_{11} - x_5 < 0$

thus

$x_{11} < x_5 < -1.$

From Theorem 3.16, for $n = 6$,

$x_{12} - x_6 = (x_8 + 1)(x_{10} - x_6)$

and from Eq.(3.1)

$x_{12} - x_6 = (x_8 + 1)\left(x_6^2 x_4 - 2x_6 - 1\right)$

so, since $x_4 < -1$ and $x_8 > 0$,

$x_{12} - x_6 < 0$

consequently

$x_{12} < x_6 < -1.$

From Theorem 3.16, for $n = 7$,

$x_{13} - x_7 = (x_9 + 1)(x_{11} - x_7)$

consequently, for $x_7 > 0$, $x_9 < -1$ and $x_{11} < -1$,

$x_{13} - x_7 > 0$

therefore

$x_{13} > x_7 > x_1 > 0.$

By induction, it can be proved that

$$\begin{aligned}
0 &< x_1 < x_7 < x_{13} < ..., \\
0 &< x_2 < x_8 < x_{14} < ..., \\
-1 &> x_3 > x_9 > x_{15} > ..., \\
-1 &> x_4 > x_{10} > x_{16} > ..., \\
-1 &> x_5 > x_{11} > x_{17} > ..., \\
-1 &> x_6 > x_{12} > x_{18} > ....
\end{aligned}$$

The proof is completed. ■

3.7 NUMERICAL EXAMPLES

In this section, we present some numerical examples about theoretical results for Eq.(3.1).

Example 3.1 *Consider the Eq.(3.1) with the initial conditions $x_{-3} = x_{-1} = \bar{x}_2$ and $x_{-2} = x_0 = \bar{x}_1$, then there are periodic solutions with period two as Theorem 3.3. The following figure verifies our theoretical results. (See Figure 3.1)*

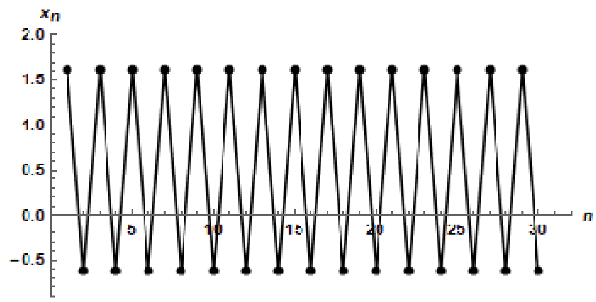


Figure 3.1: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = x_{-1} = \bar{x}_2$, $x_{-2} = x_0 = \bar{x}_1$.

Example 3.2 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = 0$ *and* $x_0 = -1$, *then there are periodic solutions with period three as Theorem 3.4. The following figure verifies our theoretical results. (See Figure 3.2)*

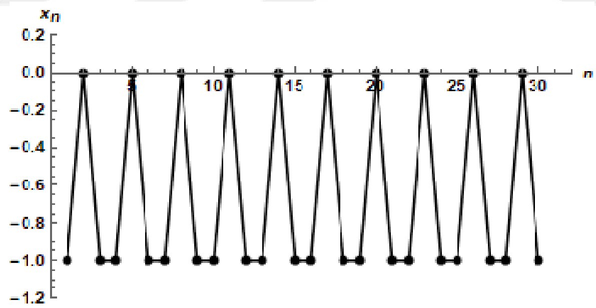


Figure 3.2: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = 0$ and $x_0 = -1$.

Example 3.3 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = \frac{9}{4}$, $x_{-2} = \frac{7}{16}$, $x_{-1} = \frac{10}{9}$ *and* $x_0 = 4$, *then there are eventually periodic solutions with period three as Theorem 3.5. The following figure verifies our theoretical results. (See Figure 3.3)*

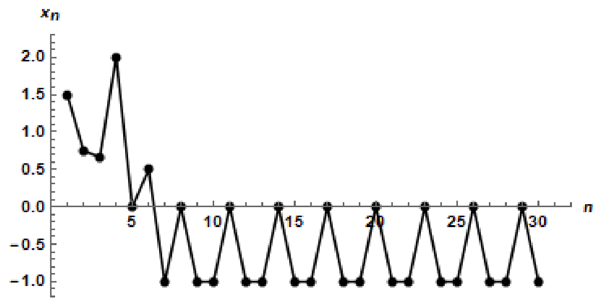


Figure 3.3: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = \frac{9}{4}$, $x_{-2} = \frac{7}{16}$, $x_{-1} = \frac{10}{9}$ and $x_0 = 4$.

Example 3.4 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = -1$, $x_{-2} = \frac{1-\sqrt{5}}{2}$, $x_{-1} = 0$ *and* $x_0 = \frac{1-\sqrt{5}}{2}$, *then there are periodic solutions with period six as Case 1 – Case 6 in Theorem 3.7. The following figure verifies our theoretical results. (See Figure 3.4)*

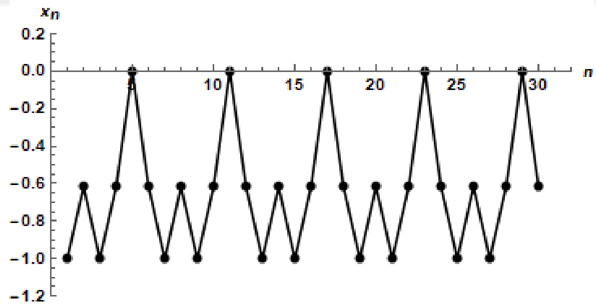


Figure 3.4: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = \frac{1-\sqrt{5}}{2}$, $x_{-1} = 0$ and $x_0 = \frac{1-\sqrt{5}}{2}$.

Example 3.5 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = -1$, $x_{-2} = \frac{1+\sqrt{5}}{2}$, $x_{-1} = 0$ *and* $x_0 = \frac{1+\sqrt{5}}{2}$, *then there are periodic solutions with period six as Case 7 – Case 12 in Theorem 3.7. The following figure verifies our theoretical results. (See Figure 3.5)*

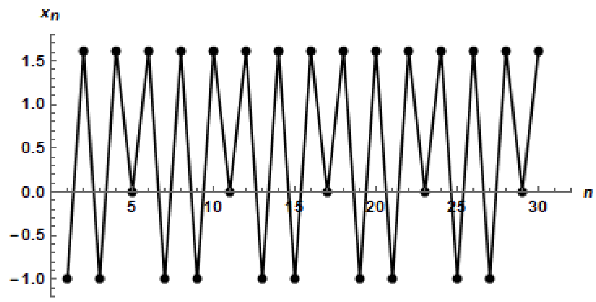


Figure 3.5: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = \frac{1+\sqrt{5}}{2}$, $x_{-1} = 0$ and $x_0 = \frac{1+\sqrt{5}}{2}$.

Example 3.6 *Consider the Eq.(3.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = 0$, $x_{-1} = 0$ and $x_0 = -1$, then there are periodic solutions with period six as Case 13 – Case 17 in Theorem 3.7. The following figure verifies our theoretical results. (See Figure 3.6)*

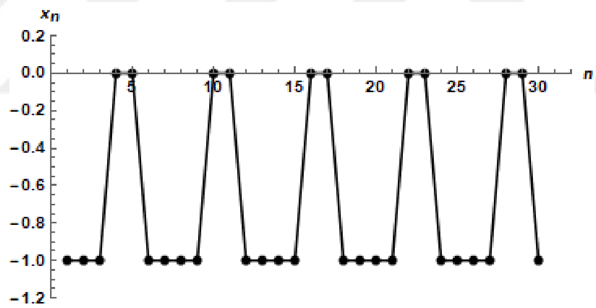


Figure 3.6: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = 0$, $x_{-1} = 0$ and $x_0 = -1$.

Example 3.7 *Consider the Eq.(3.1) with the initial conditions $x_{-3} = \bar{x}_2$, $x_{-2} = \frac{1}{4}$, $x_{-1} = \bar{x}_2$ and $x_0 = 6$, then there are eventually periodic solutions with period six as form 1 in Theorem 3.8. The following figure verifies our theoretical results. (See Figure 3.7)*

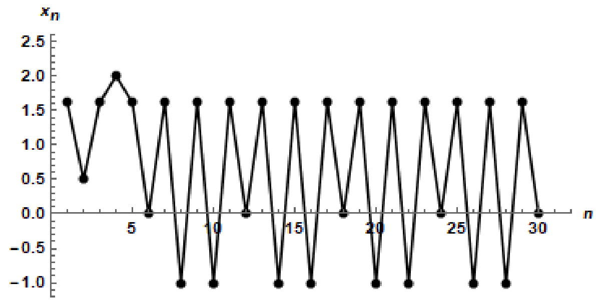


Figure 3.7: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = \bar{x}_2$, $x_{-2} = \frac{1}{4}$, $x_{-1} = \bar{x}_2$ and $x_0 = 6$.

Example 3.8 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = \bar{x}_1$, $x_{-2} = \frac{1}{4}$, $x_{-1} = \bar{x}_1$ *and* $x_0 = 6$, *then there are eventually periodic solutions with period six as form 2 in Theorem 3.8. The following figure verifies our theoretical results. (See Figure 3.8)*

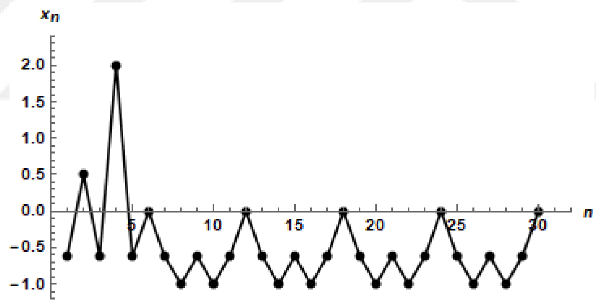


Figure 3.8: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = \bar{x}_1$, $x_{-2} = \frac{1}{4}$, $x_{-1} = \bar{x}_1$ and $x_0 = 6$.

Example 3.9 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = \frac{9}{4}$, $x_{-2} = \frac{16}{9}$, $x_{-1} = \frac{10}{9}$ *and* $x_0 = \frac{21}{16}$, *then there are eventually periodic solutions with period six as form 3 in Theorem 3.8. The following figure verifies our theoretical results. (See Figure 3.9)*

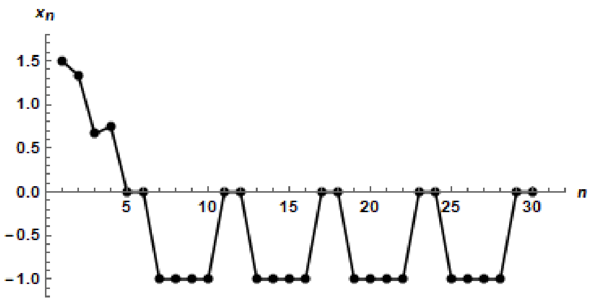


Figure 3.9: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = \frac{9}{4}$, $x_{-2} = \frac{16}{9}$, $x_{-1} = \frac{10}{9}$ and $x_0 = \frac{21}{16}$.

Example 3.10 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = -0.1$, $x_{-2} = -0.3$, $x_{-1} = -0.5$ *and* $x_0 = -0.9$, *then Eq.(3.1) converges to negative equilibrium* $\bar{x}_1 = \frac{1-\sqrt{5}}{2}$ *as Theorem 3.12. The following figure verifies our theoretical results. (See Figure 3.10)*

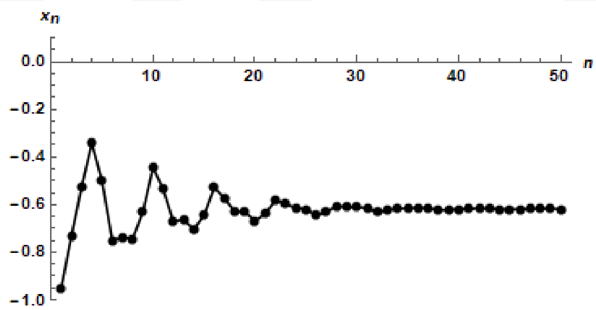


Figure 3.10: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -0.1$, $x_{-2} = -0.3$, $x_{-1} = -0.5$ and $x_0 = -0.9$.

Example 3.11 *Consider the Eq.(3.1) with the initial conditions* $x_{-3} = -0.3$, $x_{-2} = -0.8$, $x_{-1} = -0.6$ *and* $x_0 = -0.2$, *then Eq.(3.1) converges to negative equilibrium* $\bar{x}_1 = \frac{1-\sqrt{5}}{2}$ *as Theorem 3.12. The following figure verifies our theoretical results. (See Figure 3.11)*

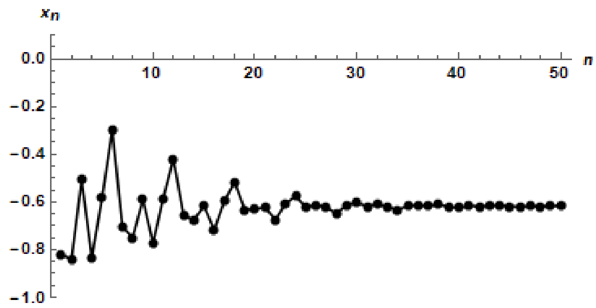


Figure 3.11: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -0.3$, $x_{-2} = -0.8$, $x_{-1} = -0.6$ and $x_0 = -0.2$.

Example 3.12 *Consider the Eq.(3.1) with the initial conditions $x_{-3} = -1.002$, $x_{-2} = -1.003$, $x_{-1} = -1.02$ and $x_0 = -1.002$, then the terms of Eq.(3.1) behave as Theorem 3.17. The following figure verifies our theoretical results. (See Figure 3.12)*

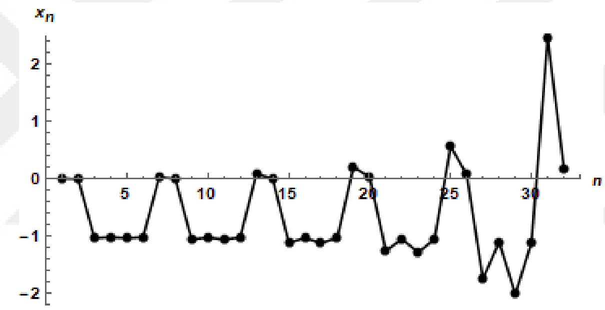


Figure 3.12: Plot of the Eq.(3.1) with the initial conditions $x_{-3} = -1.002$, $x_{-2} = -1.003$, $x_{-1} = -1.02$ and $x_0 = -1.002$.

CHAPTER 4

DYNAMICAL ANALYSIS OF A NONLINEAR DIFFERENCE EQUATION

In this chapter, we investigate the dynamics of the solutions of the following nonlinear difference equation $x_{n+1} = x_{n-2}x_{n-3} - 1$. Besides, we study the periodic behaviours of related difference equation especially asymptotic periodicity and eventually periodicity. Then, we examine the unbounded solutions of related difference equation.
The results presented in this chapter have been published as article in [28] by us.

4.1 INTRODUCTION

Up to the present, many authors investigated to dynamics of various forms of difference equation $x_{n+1} = x_{n-k}x_{n-l} - 1$, $n \in \mathbb{N}_0$ such as $k = 0$, $l = 1$ in [15]; $k = 0$, $l = 2$ in [14]; $k = 1$, $l = 2$ in [13]; $k = 0$, $l = 3$ in [12]. Besides, Stević and Iričanin have obtained some results regarding the general form of the related difference equation in [22].
In this chapter, we investigate the dynamics of the solutions of the following nonlinear difference equation

$$x_{n+1} = x_{n-2}x_{n-3} - 1, \quad n = 0, 1, 2, ..., \tag{4.1}$$

with arbitrary initial conditions x_{-3}, $x_{-2,}$ $x_{-1,}$ x_0. The Eq.(4.1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, \; n \in \mathbb{N}_0, \tag{4.2}$$

with specific selection of k and l, where $k, l \in \mathbb{N}_0$.
This chapter can be considered as a continuance of our systematic analysis of Eq.(4.2).

4.2 THE EQUILIBRIUM POINTS OF EQ.(4.1)

In this section we study the equilibrium points of Eq.(4.1).

Lemma 4.1 *The equilibrium points of Eq.(4.1) are*

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2}, \ \bar{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{4.3}$$

Proof. Let $\overline{x}$ is an equilibrium point of Eq.(4.1). Thus $x_n = \overline{x}$, for all $n \geq -3$. So, we have

$$\overline{x} = \overline{x} \cdot \overline{x} - 1.$$

Therefore, we get the equation

$$\overline{x}^2 - \overline{x} - 1 = 0.$$

Thus, we find that Eq.(4.1) has exactly two equilibrium points, which $\bar{x}_1$ is negative and $\bar{x}_2$ is positive:

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2}, \ \bar{x}_2 = \frac{1+\sqrt{5}}{2}.$$

Note that these are the Golden Number and its conjugate. ■

4.3 EXISTENCE OF PERIODICITY OF EQ.(4.1)

In this section, we show that Eq.(4.1) has minimal prime periodic solutions with period seven. Also Eq.(4.1) has eventually periodic solutions with period seven.

Theorem 4.2 *Eq.(4.1) has no eventually constant solutions.*

Proof. *If $\{x_n\}_{n=-3}^{\infty}$ is eventually constant solutions of Eq.(4.1), hence $x_N = x_{N+1} = x_{N+2} = x_{N+3} = \bar{x}$, for some $N \in \mathbb{N}_0$, where $\bar{x}$ is an equilibrium point. However, Eq.(4.1) gives $x_{N+3} = x_N x_{N-1} - 1$, which implies*

$$x_{N-1} = \frac{x_{N+3}+1}{x_N} = \frac{\bar{x}+1}{\bar{x}} = \bar{x}.$$

Repetition the procedure, we get that $x_n = \bar{x}$ for $-3 \leq n \leq N+3$. Then, the proof is completed. ■

Theorem 4.3 *There are no nontrivial nor eventually period-two solutions of Eq.(4.1).*

Proof. Suppose that $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, for all $k \in \mathbb{N}_0$, and some $N \geq -1$, with $x_N \neq x_{N+1}$. Therefore, we have

$$x_{N+4} = x_{N+1}x_N - 1 \tag{4.4}$$

$$x_{N+4} = x_{N-1}x_N - 1 = x_{N+3} \tag{4.5}$$

$$x_{N+4} = x_{N-1}x_{N-2} - 1 = x_{N+2} \tag{4.6}$$

$$x_{N+4} = x_{N-3}x_{N-2} - 1 = x_{N+1} \tag{4.7}$$

From (4.5) – (4.7) and since $x_{N+4} = x_N$ we arrive a contradiction, as desired. ■

Theorem 4.4 *Eq.(4.1) has no minimal prime period-three solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-three solution of Eq.(4.1). Then, $x_{3n-3} = a$, $x_{3n-2} = b$, $x_{3n-1} = c$ and $x_{3n} = a$ for all $n \in \mathbb{N}_0$ and a, b and $c \in \mathbb{R}$ such that at least two are different from each other. From Eq.(4.1), we have

$$x_1 = x_{-2}x_{-3} - 1 = ba - 1 = b \tag{4.8}$$

$$x_2 = x_{-1}x_{-2} - 1 = cb - 1 = c \tag{4.9}$$

$$x_3 = x_0x_{-1} - 1 = ac - 1 = a \tag{4.10}$$

From (4.8) – (4.10) we obtain that

$$a = b = c = \bar{x}_1$$

or

$$a = b = c = \bar{x}_2.$$

Thus, the proof is completed. ■

Theorem 4.5 *Eq.(4.1) has no minimal prime period-four solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-four solution of Eq.(4.1). Then, $x_{4n-3} = a$, $x_{4n-2} = b$, $x_{4n-1} = c$ and $x_{4n} = d$ for all $n \in \mathbb{N}_0$ and a, b, c and $d \in \mathbb{R}$ such that at least two of them are different. From Eq.(4.1), we have

$$x_1 = x_{-2}x_{-3} - 1 = ba - 1 = a \tag{4.11}$$

$$x_2 = x_{-1}x_{-2} - 1 = cb - 1 = b \tag{4.12}$$

$$x_3 = x_0x_{-1} - 1 = dc - 1 = c \tag{4.13}$$

$$x_4 = x_1x_0 - 1 = ad - 1 = d. \tag{4.14}$$

From (4.11) – (4.14) we obtain that

$a = b = c = d = \bar{x}_1$

or

$a = b = c = d = \bar{x}_2$

as desired. ■

Theorem 4.6 *Eq.(4.1) has no minimal prime period-five solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a periodic solution of Eq.(4.1) with minimal prime period-five. Then, $x_{5n-3} = a$, $x_{5n-2} = b$, $x_{5n-1} = c$, $x_{5n} = d$ and $x_{5n+1} = e$ for all $n \in \mathbb{N}_0$ and a, b, c, d and $e \in \mathbb{R}$ such that at least two of them are different. From Eq.(4.1), we obtain

$$x_1 = x_{-2}x_{-3} - 1 = ba - 1 = e \tag{4.15}$$

$$x_2 = x_{-1}x_{-2} - 1 = cb - 1 = a \tag{4.16}$$

$$x_3 = x_0x_{-1} - 1 = dc - 1 = b \tag{4.17}$$

$$x_4 = x_1x_0 - 1 = ed - 1 = c \tag{4.18}$$

$$x_5 = x_2x_1 - 1 = ae - 1 = d. \tag{4.19}$$

From (4.15) – (4.19) we have

$a = b = c = d = e = \bar{x}_1$

or

$a = b = c = d = e = \bar{x}_2$

as desired. ■

Theorem 4.7 *Eq.(4.1) has no period solutions with minimal prime period-six.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-six solution of Eq.(4.1). Then, $x_{6n-3} = a$, $x_{6n-2} = b$, $x_{6n-1} = c$, $x_{6n} = d$, $x_{6n+1} = e = ac - 1$ and $x_{6n+2} = f = bd - 1$ for all $n \in \mathbb{N}_0$

and a, b, c, d, e and $f \in \mathbb{R}$ such that at least two of them are different. We have

$$\begin{aligned}
x_1 &= x_{-2}x_{-3} - 1 = ab - 1 = e && (4.20)\\
x_2 &= x_{-1}x_{-2} - 1 = bc - 1 = f && (4.21)\\
x_3 &= x_0x_{-1} - 1 = cd - 1 = a && (4.22)\\
x_4 &= x_1x_0 - 1 = de - 1 = b && (4.23)\\
x_5 &= x_2x_1 - 1 = ef - 1 = c && (4.24)\\
x_6 &= x_3x_2 - 1 = fa - 1 = d. && (4.25)
\end{aligned}$$

From (4.20) – (4.25) we obtain

$$a = b = c = d = e = f = \bar{x}_1$$

or

$$a = b = c = d = e = f = \bar{x}_2$$

as desired. ■

Theorem 4.8 *There are periodic solutions of Eq.(4.1) with minimal prime period-seven if and only if*
(i) $x_{-3} = 0$, $x_{-2} = m$, $x_{-1} = -1$, $x_0 = -1$;
(ii) $x_{-3} = -1$, $x_{-2} = m$, $x_{-1} = 0$, $x_0 = 0$;
(iii) $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = -1$, $x_0 = m$;
(iv) $x_{-3} = m$, $x_{-2} = -1$, $x_{-1} = -1$, $x_0 = -1$;
(v) $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = m$, $x_0 = 0$;
where m is arbitrary.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a periodic solution of Eq.(4.1) with minimal prime period-seven. Then, $x_{7n-3} = a$, $x_{7n-2} = b$, $x_{7n-1} = c$, $x_{7n} = d$, $x_{7n+1} = e = ac - 1$, $x_{7n+2} = f = bc - 1$ and $x_{7n+3} = g = cd - 1$ for all $n \in \mathbb{N}_0$ and a, b, c, d, e, f and $g \in \mathbb{R}$ such that at least two are different from each other. We have

$$\begin{aligned}
x_1 &= x_{-2}x_{-3} - 1 = ab - 1 = e \\
x_2 &= x_{-1}x_{-2} - 1 = bc - 1 = f \\
x_3 &= x_0x_{-1} - 1 = cd - 1 = g \\
x_4 &= x_1x_0 - 1 = de - 1 = a \\
x_5 &= x_2x_1 - 1 = ef - 1 = b \\
x_6 &= x_3x_2 - 1 = fg - 1 = c \\
x_7 &= x_4x_3 - 1 = ga - 1 = d.
\end{aligned}$$

Thus, the following equalities are obtained:

$$x_4 = d(ab-1) - 1 = a \tag{4.26}$$

$$x_5 = (ab-1)(bc-1) - 1 = b \tag{4.27}$$

$$x_6 = (bc-1)(cd-1) - 1 = c \tag{4.28}$$

$$x_7 = (cd-1)a - 1 = d. \tag{4.29}$$

From (4.26) – (4.29), then by direct calculation we have

(i) $a = 0,\ c = -1,\ d = -1$*;*

(ii) $a = -1,\ c = 0,\ d = 0$*;*

(iii) $a = -1,\ b = -1,\ c = -1$*;*

(iv) $b = -1,\ c = -1,\ d = -1$*;*

(v) $a = -1,\ b = -1,\ d = 0$*;*

and so,

$$x_{-3} = 0, x_{-2} = m, x_{-1} = -1, x_0 = -1$$

$$x_{-3} = -1, x_{-2} = m, x_{-1} = 0, x_0 = 0$$

$$x_{-3} = -1, x_{-2} = -1, x_{-1} = -1, x_0 = m$$

$$x_{-3} = m, x_{-2} = -1, x_{-1} = -1, x_0 = -1$$

$$x_{-3} = -1, x_{-2} = -1, x_{-1} = m, x_0 = 0$$

where m *is arbitrary, as desired.* ■

remark 4.1 *Consequently, all minimal prime period-seven solutions are of the forms;*
(i) If $x_{-3} = 0, x_{-2} = m, x_{-1} = -1, x_0 = -1,$ *then* $(-1, -m-1, 0, 0, m, -1, -1, ...)$,
(ii) If $x_{-3} = -1, x_{-2} = m, x_{-1} = 0, x_0 = 0,$ *then* $(-m-1, -1, -1, -1, m, 0, 0, ...)$,
(iii) If $x_{-3} = -1, x_{-2} = -1, x_{-1} = -1, x_0 = m,$ *then* $(0, 0, -m-1, -1, -1, -1, m, ...)$,
(iv) If $x_{-3} = m, x_{-2} = -1, x_{-1} = -1, x_0 = -1,$ *then* $(-m-1, 0, 0, m, -1, -1, -1, ...)$,
(v) If $x_{-3} = -1, x_{-2} = -1, x_{-1} = m, x_0 = 0,$ *then* $(0, -m-1, -1, -1, -1, m, 0, ...)$.

From now on, we will refer to any one of these seven periodic solutions of Eq.(4.1) as

$$..., -1, -1, -1, m, 0, 0, -m-1, ... \tag{4.30}$$

where m is arbitrary.

Theorem 4.9 *There are eventually periodic solutions with minimal period-seven and they have two forms, respectively:*

Form 1: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, -1, -1, -1, m, 0, 0, -m-1, ...)$

where, $N \geq -3$, $x_{N+1}x_N = 0, x_{N+2}x_{N+1} = 0, x_{N+3}x_{N+2} = 0$, *and, if* $N \neq -3$, $x_{n-2} = (x_{n+1} + 1)/x_{n-3}$ *for* $0 \leq n \leq N$.

Form 2: $(x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_N, x_{N+1}, x_{N+2}, x_{N+3}, 0, 0, -m-1, -1, -1, -1, m, ...)$

where, $N \geq -3$, $x_{N+1}x_N = 1, x_{N+2}x_{N+1} = 1$, *and, if* $N \neq -3$, $x_{n-2} = (x_{n+1} + 1)/x_{n-3}$ *for* $0 \leq n \leq N$.

Proof.
Form 1: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4.1) that is eventually periodic with prime period-seven. Then by Theorem 4.8, there is an $N \geq -3$ such that $x_{N+4} = -1, x_{N+5} = -1$ and $x_{N+6} = -1$. Then, $-1 = x_{N+4} = x_N x_{N+1} - 1$ and consequently $x_N x_{N+1} = 0$. Hence, $-1 = x_{N+5} = x_{N+2}x_{N+1} - 1$ and then $x_{N+2}x_{N+1} = 0$. Hence, $-1 = x_{N+6} = x_{N+3}x_{N+2} - 1$ and so $x_{N+3}x_{N+2} = 0$. Therefore,

$$\begin{aligned}
x_{N+7} &= x_{N+4}x_{N+3} - 1 = m \\
x_{N+8} &= x_{N+5}x_{N+4} - 1 = 0 \\
x_{N+9} &= x_{N+6}x_{N+5} - 1 = 0 \\
x_{N+10} &= x_{N+7}x_{N+6} - 1 = -m-1 \\
x_{N+11} &= x_{N+8}x_{N+7} - 1 = -1.
\end{aligned}$$

From Eq.(4.1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$, for $0 \leq n \leq N$, as desired.

Form 2: Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4.1) that is eventually periodic with prime period-seven. Then by Theorem 4.8, there is an $N \geq -3$ such that $x_{N+4} = 0, x_{N+5} = 0$ and $x_{N+6} = -m - 1$. Then, $0 = x_{N+4} = x_N x_{N+1} - 1$ and consequently $x_N x_{N+1} = 1$. Hence, $0 = x_{N+5} = x_{N+2} x_{N+1} - 1$ and then $x_{N+2} x_{N+1} = 1$. Hence, $-m - 1 = x_{N+6} = x_{N+3} x_{N+2} - 1$ and so $x_{N+3} x_{N+2} = -m$. Therefore,

$$\begin{aligned}
x_{N+7} &= x_{N+4} x_{N+3} - 1 = -1 \\
x_{N+8} &= x_{N+5} x_{N+4} - 1 = -1 \\
x_{N+9} &= x_{N+6} x_{N+5} - 1 = -1 \\
x_{N+10} &= x_{N+7} x_{N+6} - 1 = m \\
x_{N+11} &= x_{N+8} x_{N+7} - 1 = 0.
\end{aligned}$$

From Eq.(4.1), if $N \neq -3$, we get $x_{n-1} = (x_{n+1} + 1) / x_{n-3}$, for $0 \leq n \leq N$, as desired. ■

remark 4.2 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4.1). If $x_{-3} x_{-2} = 1$ and $x_{-1} x_0 = 1$, then x_n converges to period-seven cycle as*

$$\cdots, 0, 0, 0, -1, -1, -1, -1, \cdots. \tag{4.31}$$

Proof. Let $x_{-3} = a, x_{-2} = 1/a, x_{-1} = b$ and $x_0 = 1/b$ for $a \neq 0$ and $b \neq 0$. From Eq.(4.1),

$$\begin{aligned}
x_1 &= x_{-2} x_{-3} - 1 = 0 \\
x_2 &= x_{-1} x_{-2} - 1 = \frac{b}{a} - 1 \\
x_3 &= x_0 x_{-1} - 1 = 0 \\
x_4 &= x_1 x_0 - 1 = -1.
\end{aligned}$$

Hence, by induction Eq.(4.1) converges to period-seven cycle as (4.31). The proof is completed. ■

4.4 ASYMPTOTICALLY PERIODIC SOLUTIONS OF EQ.(4.1)

In this section, we study the existence of asymptotic periodic solutions of Eq.(4.1).

Theorem 4.10 *Eq.(4.1) has the asymptotically seven-periodic solutions as (4.30) for the initial conditions x_{-3}, x_{-2}, x_{-1}, $x_0 \in (-1, 0)$.*

Proof. Focus on the asymptotically seven-periodic solutions, we get $u_k^{(0)} = x_{n+7k}$, $u_k^{(1)} = x_{n+7k-1}$, $u_k^{(2)} = x_{n+7k-2}$, $u_k^{(3)} = x_{n+7k-3}$, $u_k^{(4)} = x_{n+7k-4}$, $u_k^{(5)} = x_{n+7k-5}$ and $u_k^{(6)} = x_{n+7k-6}$. Now, we make the ansatz as in [3]:

$$u_k^{(0)} = \sum_{v=0}^{\infty} a_v p^v t^{vk}, \ a_0 = m; \tag{4.32}$$

$$u_k^{(1)} = \sum_{v=0}^{\infty} b_v p^v t^{vk}, \ b_0 = 0; \tag{4.33}$$

$$u_k^{(2)} = \sum_{v=0}^{\infty} c_v p^v t^{vk}, \ c_0 = 0; \tag{4.34}$$

$$u_k^{(3)} = \sum_{v=0}^{\infty} d_v p^v t^{vk}, \ d_0 = -m-1; \tag{4.35}$$

$$u_k^{(4)} = \sum_{v=0}^{\infty} e_v p^v t^{vk}, \ e_0 = -1; \tag{4.36}$$

$$u_k^{(5)} = \sum_{v=0}^{\infty} f_v p^v t^{vk}, \ f_0 = -1; \tag{4.37}$$

$$u_k^{(6)} = \sum_{v=0}^{\infty} g_v p^v t^{vk}, \ g_0 = -1; \tag{4.38}$$

with arbitrary p and $m \in (-1, 0)$. We choose $p > 0$ and from Eq.(4.1), it follows that:

$$\begin{aligned}
u_k^{(0)} &= u_k^{(3)} u_k^{(4)} - 1 \\
u_{k+1}^{(6)} &= u_k^{(2)} u_k^{(3)} - 1 \\
u_{k+1}^{(5)} &= u_k^{(1)} u_k^{(2)} - 1 \\
u_{k+1}^{(4)} &= u_k^{(0)} u_k^{(1)} - 1 \\
u_{k+1}^{(3)} &= u_{k+1}^{(6)} u_k^{(0)} - 1 \\
u_{k+1}^{(2)} &= u_{k+1}^{(5)} u_{k+1}^{(6)} - 1 \\
u_{k+1}^{(1)} &= u_{k+1}^{(4)} u_{k+1}^{(5)} - 1.
\end{aligned}$$

Substitution of (4.32) – (4.38) into these equations. Hence, when we compare the coefficients, we obtain that

$$\begin{aligned}
a_1 &= b_1 = c_1 = d_1 = e_1 = f_1 = g_1 = 0 \\
a_2 &= b_2 = c_2 = d_2 = e_2 = f_2 = g_2 = 0
\end{aligned}$$

and by induction,

$a_n = b_n = c_n = d_n = e_n = f_n = g_n = 0,$ for all $n > 0$.

Therefore $x_{n+7k} = u_k^{(0)} = \sum_{v=0}^{\infty} a_v p^v t^{vk} = m + 0 + 0 + ...,$ so x_{n+7k} converges to m. Similarly, x_{n+7k-1} converges to 0, x_{n+7k-2} converges to 0, x_{n+7k-3} converges to $-m-1$, x_{n+7k-4} converges to -1, x_{n+7k-5} converges to -1 and x_{n+7k-6} converges to -1. Hence, the proof is complete. ■

4.5 STABILITY OF EQ.(4.1)

In this section, we examine the stability of the two equilibria of Eq.(4.1).

Theorem 4.11 *The positive equilibrium point of Eq.(4.1), $\bar{x}_2$, is unstable.*

Proof. The characteristic equation of equilibria of Eq.(4.1) is the following:

$$\lambda^4 - \bar{x}_2\lambda - \bar{x}_2 = 0$$

with eigenvalues
$\lambda_1 \approx -0.7756$
$\lambda_2 \approx 1.4044$
$\lambda_3, \lambda_4 \approx -0.3142 \pm 1.1773i$.
Therefore, $|\lambda_1| < 1$ and $|\lambda_2|, |\lambda_3|, |\lambda_4| > 1$. Therefore, $\bar{x}_2$ is unstable, which is a saddle point. ■

Theorem 4.12 *The negative equilibrium point of Eq.(4.1), $\bar{x}_1$, is unstable.*

Proof. The characteristic equation of equilibria of Eq.(4.1) is the following:

$$\lambda^4 - \bar{x}_1\lambda - \bar{x}_1 = 0$$

with eigenvalues
$\lambda_1, \lambda_2 \approx -0.6412 \pm 0.4125i$
$\lambda_3, \lambda_4 \approx -0.6412 \pm 0.8075i$.
Therefore, $|\lambda_1|, |\lambda_2| < 1$ and $|\lambda_3|, |\lambda_4| > 1$. So, $\bar{x}_1$ is unstable and which is a saddle point.
■

4.6 EXISTENCE OF UNBOUNDED SOLUTIONS OF EQ.(4.1)

In this section, we work the existence of unbounded solutions of Eq.(4.1).

Theorem 4.13 *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4.1). If $x_{-3}, x_{-2}, x_{-1}, x_0 > \bar{x}_2 = \frac{1+\sqrt{5}}{2}$, the following statements hold true:*

(i) $x_{-2} < x_1 < x_4 < \cdots$, $x_{-1} < x_2 < x_5 < \cdots$ and $x_0 < x_3 < x_6 < \cdots$;

(ii) the solutions tend to $+\infty$.

Proof. *(i)* Since $x_{-2} > \frac{1+\sqrt{5}}{2}$, we obtain $\frac{1}{x_{-2}} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_{-2}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-3}.$$

Then, $x_{-3} > 1 + \frac{1}{x_{-2}}$. Hence, $x_{-3}x_{-2} > x_{-2} + 1$. Therefore, $x_{-3}x_{-2} - 1 > x_{-2}$. Thus, $x_{-2} < x_1$. Since $x_{-1} > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_{-1}} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_{-1}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}.$$

Then, $x_{-2} > 1 + \frac{1}{x_{-1}}$. Hence, $x_{-1}x_{-2} > x_{-1} + 1$. Therefore, $x_{-1}x_{-2} - 1 > x_{-1}$. Thus, $x_{-1} < x_2$. Since $x_0 > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_0} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_0} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}.$$

Then, $x_{-1} > 1 + \frac{1}{x_0}$. Hence, $x_0x_{-1} > x_0 + 1$. Therefore, $x_0x_{-1} - 1 > x_0$. Thus, $x_0 < x_3$. Since $x_1 > x_{-2} > \frac{1+\sqrt{5}}{2}$, we have $\frac{1}{x_1} < \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$. Hence,

$$1 + \frac{1}{x_1} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_0.$$

Then, $x_0 > 1 + \frac{1}{x_1}$. Hence, $x_1x_0 > x_1 + 1$. Therefore, $x_1x_0 - 1 > x_1$. Thus, $x_1 < x_4$. Hence, by induction it easily follows that

$$x_{-2} < x_1 < x_4 < \cdots \tag{4.39}$$

$$x_{-1} < x_2 < x_5 < \cdots \tag{4.40}$$

$$x_0 < x_3 < x_6 < \cdots. \tag{4.41}$$

(ii) Suppose one of (4.39) – (4.41) subsequences given in (i) is bounded. Hence, from Eq.(4.1), we obtain

$$x_{n-3} = \frac{1 + x_{n+1}}{x_{n-2}},\ n \in \mathbb{N}_0.$$

Therefore, the subsequences $(x_{3n})_{n=0}^{\infty}$, $(x_{3n-1})_{n=0}^{\infty}$ and $(x_{3n-2})_{n=0}^{\infty}$ must be convergent. Thereby, there are two situations for whole solution of Eq.(4.1). Then, in the first case, all solutions of Eq.(4.1) converges to a periodic solution with period three. But this is not possible. Because, there are not nontrivial period three solution of Eq.(4.1). In the other case, all solutions of Eq.(4.1) converge to an equilibria. Unfortunately, this is impossible. Because the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are bigger then the largest equilibria. This is a contradiction, as desired. ■

4.7 NUMERICAL EXAMPLES

In this section, we present some numerical examples about theoretical results for Eq.(4.1).

Example 4.1 *Consider the Eq.(4.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = -1$, $x_0 = m$ and $m = 2$, then Eq.(4.1) has periodic solutions with minimal prime period-seven as (4.30). The following figure verifies our theoretical results. (See Figure 4.1)*

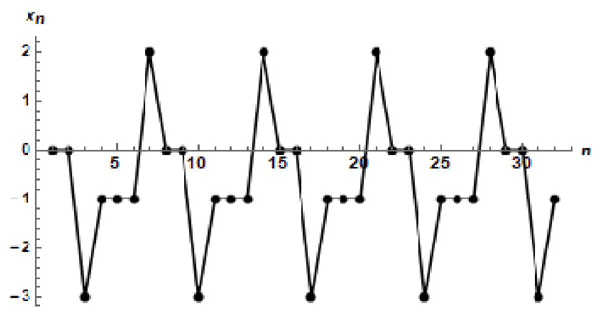


Figure 4.1: Plot of the Eq.(4.1) with the initial conditions $x_{-3} = -1$, $x_{-2} = -1$, $x_{-1} = -1$, $x_0 = m$ and $m = 2$.

Example 4.2 *Consider the Eq.(4.1) with the initial conditions $x_{-3} = \frac{122625}{1376256}$, $x_{-2} = \frac{21504}{1125}$, $x_{-1} = \frac{225}{1024}$ and $x_0 = \frac{64}{9}$, then Eq.(4.1) has eventually seven-periodic solutions as Theorem 4.9. The following figure verifies our theoretical results. (See Figure 4.2)*

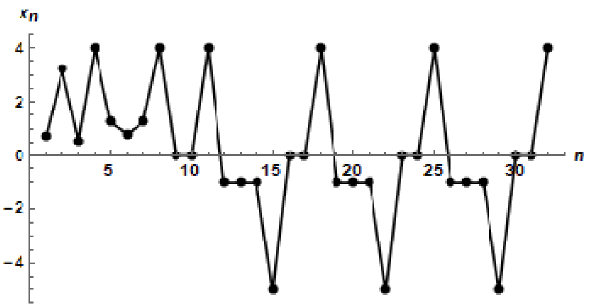


Figure 4.2: Plot of the Eq.(4.1) with the initial conditions $x_{-3} = \frac{122625}{1376256}$, $x_{-2} = \frac{21504}{1125}$, $x_{-1} = \frac{225}{1024}$ and $x_0 = \frac{64}{9}$.

Example 4.3 *Consider the Eq.(4.1) with the initial conditions* $x_{-3} = -\frac{2}{3}$, $x_{-2} = -\frac{3}{2}$, $x_{-1} = \frac{1}{5}$ *and* $x_0 = 5$, *then Eq.(4.1) converges to seven-periodic solutions as Remark 4.2. The following figure verifies our theoretical results. (See Figure 4.3)*

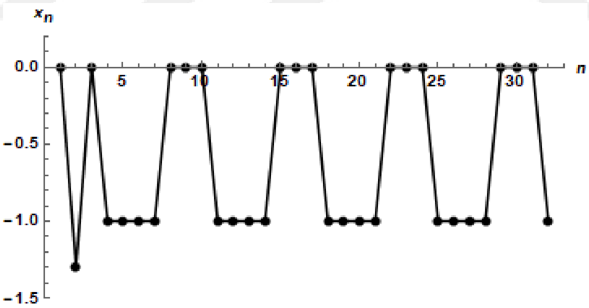


Figure 4.3: Plot of the Eq.(4.1) with the initial conditions $x_{-3} = -\frac{2}{3}$, $x_{-2} = -\frac{3}{2}$, $x_{-1} = \frac{1}{5}$ and $x_0 = 5$.

Example 4.4 *Consider the Eq.(4.1) with the initial conditions* $x_{-3} = -0.45$, $x_{-2} = -0.55$, $x_{-1} = -0.7$ *and* $x_0 = -0.75$, *then Eq.(4.1) has asymptotically seven-periodic solutions as Theorem 4.10. The following figure verifies our theoretical results. (See Figure 4.4)*

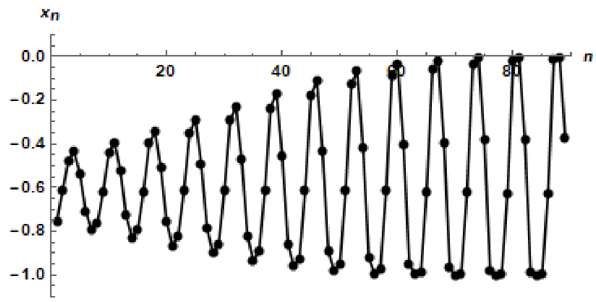


Figure 4.4: Plot of the Eq.(4.1) with the initial conditions $x_{-3} = -0.45$, $x_{-2} = -0.55$, $x_{-1} = -0.7$ and $x_0 = -0.75$.

Example 4.5 *Consider the Eq.(4.1) with the initial conditions* $x_{-3} = 1.63$, $x_{-2} = 1.64$, $x_{-1} = 1.62$ *and* $x_0 = 1.63$, *then Eq.(4.1) has unbounded solutions as Theorem 4.13. The following figure verifies our theoretical results. (See Figure 4.5)*

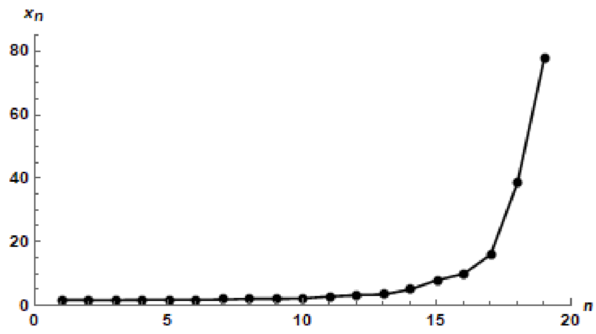


Figure 4.5: Plot of the Eq.(4.1) with the initial conditions $x_{-3} = 1.63$, $x_{-2} = 1.64$, $x_{-1} = 1.62$ and $x_0 = 1.63$.

CHAPTER 5

STABILITY AND PERIODIC SOLUTIONS OF SOME NONLINEAR DIFFERENCE EQUATIONS

In this chapter we deal with the behaviour of solutions of the following difference equation $x_{n+1} = x_n x_{n-2} + \alpha$. Specifically, we investigate that there are periodic or non-periodic solutions of related difference equations under which conditions hold. Moreover, we study convergence of the solutions of related difference equations.

5.1 INTRODUCTION

In [14] Kent et al investigated that the periodicity and the boundedness of solutions of the nonlinear difference equation of $x_{n+1} = x_n x_{n-2} - 1$. Besides, in [25] we investigated the periodicity and stability of some nonlinear difference equations $x_{n+1} = x_n x_{n-1} + \alpha$.

In this chapter we study the dynamics of solutions of the following nonlinear difference equations

$$x_{n+1} = x_n x_{n-2} + \alpha, \ n \in \mathbb{N}_0; \ \alpha, x_{-2}, x_{-1}, x_0 \in \mathbb{R}. \tag{5.1}$$

5.2 THE EQUILIBRIUM POINTS OF EQ.(5.1)

In this section, we estimate the equilibrium points of Eq.(5.1).

Lemma 5.1 *The equilibrium points of Eq.(5.1) are*

$$\overline{x}_{1,2} = \frac{1 \mp \sqrt{1-4\alpha}}{2}. \tag{5.2}$$

Proof. Let $\overline{x}$ is an equilibrium point of Eq.(5.1). Thus $x_n = \overline{x}$, for all $n \geq -2$. So, we have

$$\overline{x} = \overline{x} \cdot \overline{x} + \alpha.$$

Therefore, we get the equation

$\overline{x}^2 - \overline{x} + \alpha = 0.$

Thus, we find that Eq.(5.1) has exactly two equilibrium points, which $\bar{x}_1$ and $\bar{x}_2$ together: $\overline{x}_{1,2} = \frac{1 \mp \sqrt{1-4\alpha}}{2}$. ■

remark 5.1 *Eq.(5.2) has three cases for the variable α:*
(i) If $\alpha > \frac{1}{4}$ then $\overline{x}$ is complex,
(ii) If $\alpha < \frac{1}{4}$ then $\bar{x}_1$ and $\bar{x}_2$ are real numbers,
(iii) If $\alpha = \frac{1}{4}$ then $\overline{x} = \frac{1}{2}$ is unique equilibrium point.

5.3 PERIODIC BEHAVIOURS OF EQ.(5.1)

In this section, we investigate that there are periodic or non-periodic solutions of Eq.(5.1) with certain initial conditions.

Theorem 5.2 *Eq.(5.1) has not eventually constant solutions.*
Proof. *If $\{x_n\}_{n=-2}^{\infty}$ is eventually constant solutions of Eq.(5.1), then $x_N = x_{N+1} = x_{N+2} = \bar{x}$, for some $N \in \mathbb{N}_0$, where $\bar{x}$ is an equilibrium point. However, Eq.(5.1) gives $x_{N+2} = x_{N+1}x_{N-1} + \alpha$, which implies*

$$x_{N-1} = \frac{x_{N+3} - \alpha}{x_{N+1}} = \frac{\bar{x} - \alpha}{\bar{x}} = \bar{x}.$$

Repeating the process, we get that $x_n = \bar{x}$ for $-3 \leq n \leq N+2$. Thus, Eq.(5.1) has not eventually constant solutions. ■

The following theorem demonstrates that Eq.(5.1) has solutions that are periodic with prime period two.

Theorem 5.3 *There are periodic solutions of Eq.(5.1) with prime period two if and only if the initial conditions are*

$$x_{-2} = -\frac{1+\sqrt{-4\alpha-3}}{2}, x_{-1} = \frac{-1+\sqrt{-4\alpha-3}}{2}, x_0 = -\frac{1+\sqrt{-4\alpha-3}}{2}$$

or

$$x_{-2} = \frac{-1+\sqrt{-4\alpha-3}}{2}, x_{-1} = -\frac{1+\sqrt{-4\alpha-3}}{2}, x_0 = \frac{-1+\sqrt{-4\alpha-3}}{2}$$

where $\alpha \leq -\frac{3}{4}$.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a prime period-two solution of Eq.(5.1). Then, $x_{2n-1} = a$ and $x_{2n} = b$ for all $n \in \mathbb{N}_0$ and a, $b \in \mathbb{R}$ such that $a \neq b$. From Eq.(5.1), we have

$$x_1 = x_0 x_{-2} + \alpha = b^2 + \alpha = a \tag{5.3}$$

$$x_2 = x_1 x_{-1} + \alpha = a^2 + \alpha = b. \tag{5.4}$$

From (5.3) and (5.4), we obtain that

$$\begin{aligned} a &= -\frac{1+\sqrt{-4\alpha-3}}{2} \text{ and } b = \frac{-1+\sqrt{-4\alpha-3}}{2} \\ a &= \frac{-1+\sqrt{-4\alpha-3}}{2} \text{ and } b = -\frac{1+\sqrt{-4\alpha-3}}{2} \\ a &= b = \frac{1-\sqrt{1-4\alpha}}{2} = \bar{x}_1 \end{aligned}$$

or

$$a = b = \frac{1+\sqrt{1-4\alpha}}{2} = \bar{x}_2.$$

Therefore, Eq.(5.1) has nontrivial periodic solutions with prime period two as

$$x_{-2} = -\frac{1+\sqrt{-4\alpha-3}}{2}, x_{-1} = \frac{-1+\sqrt{-4\alpha-3}}{2}, x_0 = -\frac{1+\sqrt{-4\alpha-3}}{2}$$

or

$$x_{-2} = \frac{-1+\sqrt{-4\alpha-3}}{2}, x_{-1} = -\frac{1+\sqrt{-4\alpha-3}}{2}, x_0 = \frac{-1+\sqrt{-4\alpha-3}}{2}.$$

Consequently, the proof is completed. ■

remark 5.2 *For $\alpha \leq -\frac{3}{4}$, Eq.(5.1) has prime period-two solutions as*

$$..., \frac{-1+\sqrt{-4\alpha-3}}{2}, -\frac{1+\sqrt{-4\alpha-3}}{2}, \frac{-1+\sqrt{-4\alpha-3}}{2}, ... \tag{5.5}$$

The next theorem shows that Eq.(5.1) has no period-three solutions for $\alpha \neq 1$.

Theorem 5.4 *Eq.(5.1) has nontrivial period-three solutions if and only if*

$$\alpha = 1 \text{ and } x_{-2} = 1 - \frac{1}{x_{-1}}, x_0 = \frac{1}{1-x_{-1}}$$

where $0 \neq x_{-1} \neq 1$.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a prime period-three solution of Eq.(5.1). Then, $x_{3n-2} = a$, $x_{3n-1} = b$ and $x_{3n} = c$ for all $n \in \mathbb{N}_0$ and a, b, $c \in \mathbb{R}$ such that at least two of them are distinct from the other. From Eq.(5.1), we have

$$x_1 = x_0 x_{-2} + \alpha = ca + \alpha = a \tag{5.6}$$

$$x_2 = x_1 x_{-1} + \alpha = ab + \alpha = b \tag{5.7}$$

$$x_3 = x_2 x_0 + \alpha = bc + \alpha = c. \tag{5.8}$$

From (5.6) – (5.8) we obtain that if $\alpha \neq 1$, then

$a = b = c = \bar{x}_1$ or $a = b = c = \bar{x}_2$;

and if $\alpha = 1$, then

$a = \dfrac{b-1}{b}, c = \dfrac{1}{1-b}$

as desired. ■

remark 5.3 *Eq.(5.1) has three periodic solutions for $\alpha = 1$ as*

$$..., \frac{-1+x_{-1}}{x_{-1}}, x_{-1}, \frac{1}{1-x_{-1}}, ... \tag{5.9}$$

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a three periodic solution of Eq.(5.1) and $\alpha = 1$. Let the initial conditions as Theorem 5.4. Hence,

$$\begin{aligned}
x_1 &= x_0 x_{-2} + 1 = \frac{1}{1-x_{-1}}.\frac{x_{-1}-1}{x_{-1}} + 1 = \frac{-1+x_{-1}}{x_{-1}}, \\
x_2 &= x_1 x_{-1} + 1 = \frac{-1+x_{-1}}{x_{-1}}.x_{-1} + 1 = x_{-1}, \\
x_3 &= x_2 x_0 + 1 = x_{-1}.\frac{1}{1-x_{-1}} + 1 = \frac{1}{1-x_{-1}}, \\
x_4 &= x_3 x_1 + 1 = \frac{1}{1-x_{-1}}.\frac{x_{-1}-1}{x_{-1}} + 1 = \frac{-1+x_{-1}}{x_{-1}}.
\end{aligned}$$

The proof is completed. ■

Theorem 5.5 *Eq.(5.1) has no nontrivial periodic solutions with minimal period four.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a minimal period-four solution of Eq.(5.1). Then, $x_{4n-3} = a$, $x_{4n-2} = b$, $x_{4n-1} = c$ and $x_{4n} = d$ for all $n \in \mathbb{N}_0$ and a, b, c, $d \in \mathbb{R}$ such that at least two of them are distinct from the other. From Eq.(5.1), we obtain

$$x_1 = x_0 x_{-2} + \alpha = db + \alpha = a \tag{5.10}$$

$$x_2 = x_1 x_{-1} + \alpha = ac + \alpha = b \tag{5.11}$$

$$x_3 = x_2 x_0 + \alpha = bd + \alpha = c \tag{5.12}$$

$$x_4 = x_3 x_1 + \alpha = ca + \alpha = d. \tag{5.13}$$

From (5.10) – (5.13) we get that

$$\textit{Case1}: \begin{array}{l} a = -\frac{1+\sqrt{-4\alpha-3}}{2}, \\ b = \frac{-1+\sqrt{-4\alpha-3}}{2}, \\ c = -\frac{1+\sqrt{-4\alpha-3}}{2}, \\ d = \frac{-1+\sqrt{-4\alpha-3}}{2}; \end{array}$$

$$\textit{Case2}: \begin{array}{l} a = \frac{-1+\sqrt{-4\alpha-3}}{2}, \\ b = -\frac{1+\sqrt{-4\alpha-3}}{2}, \\ c = \frac{-1+\sqrt{-4\alpha-3}}{2}, \\ d = -\frac{1+\sqrt{-4\alpha-3}}{2}; \end{array}$$

Case3: $a = b = c = d = \bar{x}_1$;

Case4: $a = b = c = d = \bar{x}_2$.

Note that *Case 1* and *Case 2* are periodic solutions of Eq.(5.1) with minimal period-two. The other cases are trivial solutions of Eq.(5.1) but they are non-periodic solutions. ■

Theorem 5.6 *There are no nontrivial periodic solutions of Eq.(5.1) with minimal period five.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a minimal period-five solution of Eq.(5.1). Then, $x_{5n-4} = a, x_{5n-3} = b, x_{5n-2} = c, x_{5n-1} = d$ and $x_{5n} = e$ for all $n \in \mathbb{N}_0$ and a, b, c, d, $e \in \mathbb{R}$ such that at least two of them are different. From Eq.(5.1), we have

$$x_1 = x_0 x_{-2} + \alpha = ec + \alpha = a \tag{5.14}$$

$$x_2 = x_1 x_{-1} + \alpha = ad + \alpha = b \tag{5.15}$$

$$x_3 = x_2 x_0 + \alpha = be + \alpha = c \tag{5.16}$$

$$x_4 = x_3 x_1 + \alpha = ca + \alpha = d \tag{5.17}$$

$$x_5 = x_4 x_2 + \alpha = db + \alpha = e \tag{5.18}$$

From (5.14) – (5.18) we obtain that

$a = b = c = d = e = \bar{x}_1$ or $a = b = c = d = e = \bar{x}_2$.

Hence, there are no periodic solutions of Eq.(5.1) with period-five, because it's trivial solutions or non-periodic solutions. ■

Theorem 5.7 *There are no nontrivial periodic solutions of Eq.(5.1) with minimal period six.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a minimal period-five solution of Eq.(5.1). Then, $x_{6n-2} = a$, $x_{6n-1} = b$, $x_{6n} = c$, $x_{6n+1} = d$, $x_{6n+2} = e$ and $x_{6n+3} = f$ for all $n \in \mathbb{N}_0$ and a, b, c, d, e, $f \in \mathbb{R}$ such that at least two of them are different. From Eq.(5.1), we have

$$x_1 = x_0 x_{-2} + \alpha = ac + \alpha = d \tag{5.19}$$

$$x_2 = x_1 x_{-1} + \alpha = bd + \alpha = e \tag{5.20}$$

$$x_3 = x_2 x_0 + \alpha = ec + \alpha = f \tag{5.21}$$

$$x_4 = x_3 x_1 + \alpha = fd + \alpha = a \tag{5.22}$$

$$x_5 = x_4 x_2 + \alpha = ae + \alpha = b \tag{5.23}$$

$$x_6 = x_5 x_3 + \alpha = bf + \alpha = c \tag{5.24}$$

From (5.19) – (5.24) we obtain that

$a = b = c = d = e = f = \bar{x}_1$ or $a = b = c = d = e = f = \bar{x}_2$.

Hence, there are no periodic solutions of Eq.(5.1) with period-six, because it's trivial solutions or non-periodic solutions. Therefore, the proof is completed. ■

Theorem 5.8 *There are no nontrivial periodic solutions of Eq.(5.1) with minimal period seven.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a minimal period-five solution of Eq.(5.1). Then, $x_{7n-2} = a$, $x_{7n-1} = b$, $x_{7n} = c$, $x_{7n+1} = d$, $x_{7n+2} = e$, $x_{7n+3} = f$ and $x_{7n+4} = g$ for all $n \in \mathbb{N}_0$ and a,

b, c, d, e, f, $g \in \mathbb{R}$ such that at least two of them are different. From Eq.(5.1), we have

$$x_1 = x_0x_{-2} + \alpha = ac + \alpha = d \tag{5.25}$$

$$x_2 = x_1x_{-1} + \alpha = bd + \alpha = e \tag{5.26}$$

$$x_3 = x_2x_0 + \alpha = ec + \alpha = f \tag{5.27}$$

$$x_4 = x_3x_1 + \alpha = fd + \alpha = g \tag{5.28}$$

$$x_5 = x_4x_2 + \alpha = ge + \alpha = a \tag{5.29}$$

$$x_6 = x_5x_3 + \alpha = af + \alpha = b \tag{5.30}$$

$$x_7 = x_6x_4 + \alpha = bg + \alpha = c \tag{5.31}$$

From (5.25) – (5.31) we obtain that

$a = b = c = d = e = f = g = \bar{x}_1$ or $a = b = c = d = e = f = g = \bar{x}_2$.

Hence, there are no periodic solutions of Eq.(5.1) with period-seven, because it's trivial solutions or non-periodic solutions. ■

Up to now we have investigated the periodic character of solutions of Eq.(5.1). From now on we will examine that the local stability of Eq.(5.1).

5.4 LOCAL STABILITY ANALYSIS OF EQ.(5.1)

In this section, we investigate the local stability of Eq.(5.1) at the equilibrium points $\bar{x}_1$ and $\bar{x}_2$.

Lemma 5.9 *The linearized equation of Eq.(5.1) about its equilibrium point $\overline{x}$ is*

$$z_{n+1} - \overline{x} \cdot z_n - \overline{x} \cdot z_{n-2} = 0 \tag{5.32}$$

Proof. Let I be some interval of real numbers and let

$$f : I \times I \times I \to I$$

be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = x_nx_{n-2} + \alpha.$$

Therefore, we attain that

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = [x_{n-2}](\overline{x}, \overline{x}, \overline{x}) = \overline{x}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}} (\overline{x}, \overline{x}, \overline{x}) = [0] (\overline{x}, \overline{x}, \overline{x}) = 0$$

and

$$q_2 = \frac{\partial f}{\partial x_{n-2}} (\overline{x}, \overline{x}, \overline{x}) = [x_n] (\overline{x}, \overline{x}, \overline{x}) = \overline{x}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(5.1), then the linearized equation associated with Eq.(5.1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 \cdot z_n + q_1 \cdot z_{n-1} + q_2 \cdot z_{n-2}$$

then

$$z_{n+1} - \overline{x} \cdot z_n - \overline{x} \cdot z_{n-2} = 0.$$

Therefore, the proof is completed. ■

The next lemma shows that the positive equilibrium $\bar{x} = \frac{1}{2}$ of Eq.(5.1) is non-hyperbolic equilibrium point.

Lemma 5.10 *The positive equilibrium $\bar{x} = \frac{1}{2}$ of Eq.(5.1) is unstable. Because it is nonhyperbolic equilibrium point.*

Proof. The characteristic equation of linearized equation (5.32) is following:

$$\lambda^3 - \frac{1}{2}\lambda^2 - \frac{1}{2} = 0$$

with eigenvalues

$$\lambda_1 = 1,\ \lambda_{2,3} = \frac{-1 \pm \sqrt{7}i}{4}.$$

Hence, $|\lambda_1| = 1$, $|\lambda_{2,3}| < 1$. Therefore, the positive equilibrium $\bar{x} = \frac{1}{2}$ of Eq.(5.1) is nonhyperbolic equilibrium point. Thus, it is unstable. ■

Lemma 5.11 *Let $\alpha < \frac{1}{4}$. The equilibrium $\bar{x}_1 = \frac{1+\sqrt{1-4\alpha}}{2}$ of Eq.(5.1) is unstable.*

Proof. Let $\alpha < \frac{1}{4}$ and $\bar{x} = \frac{1+\sqrt{1-4\alpha}}{2}$. Thus, the characteristic equation of linearized equation (5.32) is following:

$$\lambda^3 - \frac{1 + \sqrt{1 - 4\alpha}}{2}\lambda^2 - \frac{1 + \sqrt{1 - 4\alpha}}{2} = 0.$$

So,

$$2\lambda^3 - \left(1+\sqrt{1-4\alpha}\right)\lambda^2 - \left(1+\sqrt{1-4\alpha}\right) = 0.$$

Now we take

$$P(t) = 2t^3 - \left(1+\sqrt{1-4\alpha}\right)t^2 - \left(1+\sqrt{1-4\alpha}\right).$$

Therefore,

$$P(1) = -2\sqrt{1-4\alpha} < 0.$$

Otherwise,

$$P(1+\sqrt{1-4\alpha}) > 0.$$

So,

$$P(1) < 0 < P(1+\sqrt{1-4\alpha}).$$

Thus, there is $P(\lambda) = 0$ such that $1 < \lambda < 1+\sqrt{1-4\alpha}$. Therefore, one root of $P(t) = 0$ has absolute value grater then one. So, the equilibrium point $\bar{x}_1 = \frac{1+\sqrt{1-4\alpha}}{2}$ of Eq.(5.1) is unstable. ■

The next lemma demonstrate that the equilibrium $\bar{x}_2$ of Eq.(5.1) is local asymptotic stable.

Lemma 5.12 *If $\alpha \in \left(-\frac{3}{4}, \frac{1}{4}\right)$, then the equilibrium $\bar{x}_2 = \frac{1-\sqrt{1-4\alpha}}{2}$ of Eq.(5.1) is local asymptotic stable.*

Proof. Let $\alpha < \frac{1}{4}$ and $\bar{x} = \frac{1-\sqrt{1-4\alpha}}{2}$. Thus, the characteristic equation of linearized equation (5.32) is the following:

$$\lambda^3 - \frac{1-\sqrt{1-4\alpha}}{2}\lambda^2 - \frac{1-\sqrt{1-4\alpha}}{2} = 0.$$

If

$$|q_0| + |q_1| + |q_2| < 1$$

then Clark Theorem, it follows from that all roots of the characteristic equation lie in an open disk $|\lambda| < 1$. Therefore,

$$\left|\frac{1-\sqrt{1-4\alpha}}{2}\right| + |0| + \left|\frac{1-\sqrt{1-4\alpha}}{2}\right| < 1.$$

Then

$$\left|1-\sqrt{1-4\alpha}\right| < 1.$$

So

$$-\frac{3}{4} < \alpha < \frac{1}{4}.$$

Hence, if $\alpha \in \left(-\frac{3}{4}, \frac{1}{4}\right)$ then $\bar{x}_2$ is local asymptotic stable. ■

5.5 CONVERGENCE BEHAVIOUR OF EQ.(5.1)

Recall that Eq.(5.1) has an equilibrium $\bar{x}_2$, which is locally asymptotically stable if $\alpha \in \left(-\frac{3}{4}, \frac{1}{4}\right)$ by Lemma 5.12. In this section, we investigate convergence behaviour of Eq.(5.1).

Theorem 5.13 *Every solution of Eq.(5.1) with the initial conditions in* $(-1, 0)$ *converges to the equilibrium* $\bar{x}_2 = \frac{1-\sqrt{1-4\alpha}}{2}$.

Proof. We apply Theorem 1.4. Let $g : [a, b] \times [a, b] \times [a, b] \to [a, b]$ is a function defined by $g(u, v, t) = ut + \alpha$. Then

$$\frac{\partial g(u, v, t)}{\partial u} = t$$

$$\frac{\partial g(u, v, t)}{\partial v} = 0$$

$$\frac{\partial g(u, v, t)}{\partial t} = u.$$

Therefore g is non-increasing function in each of its arguments. Now, we consider that $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $M = f(m, m, m)$ and $m = f(M, M, M)$. Hence, from Eq.(5.1)

$$M = m \cdot m + \alpha$$

$$m = M \cdot M + \alpha.$$

Thus we have

$$M = m.$$

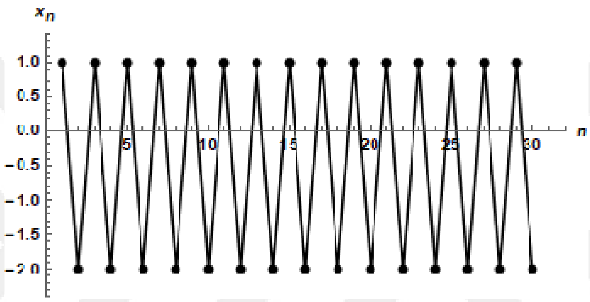
Then, according to Theorem 1.4, every solution of Eq.(5.1) with the initial conditions in $(-1, 0)$ converges to the equilibrium $\bar{x}_2 = \frac{1-\sqrt{1-4\alpha}}{2}$. Hence, proof is completed. ■

5.6 NUMERICAL EXAMPLES

Here we will present some numerical examples about theoretical results in previous sections.

Example 5.1 *Consider the Eq.(5.1) with the initial conditions* $x_{-2} = -2$, $x_{-1} = 1$, $x_0 = -2$ *and* $\alpha = -3$, *then Eq.(5.1) has minimal prime two periodic solutions as (5.5). The following figure verifies our theoretical results. (See Figure 5.1)*

Figure 5.1:
Plot of the Eq.(5.1) with the initial conditions $x_{-2} = -2$, $x_{-1} = 1$, $x_0 = -2$ *and* $\alpha = -3$.



Plot of the Eq.(5.1) with the initial conditions $x_{-2} = -2$, $x_{-1} = 1$, $x_0 = -2$ and $\alpha = -3$.

Example 5.2 *Consider the Eq.(5.1) with the initial conditions* $x_{-2} = \frac{-1+\sqrt{17}}{2}$, $x_{-1} = -\frac{1+\sqrt{17}}{2}$, $x_0 = \frac{-1+\sqrt{17}}{2}$ *and* $\alpha = -5$, *then Eq.(5.1) has minimal prime two periodic solutions as (5.5). The following figure verifies our theoretical results. (See Figure 5.2)*

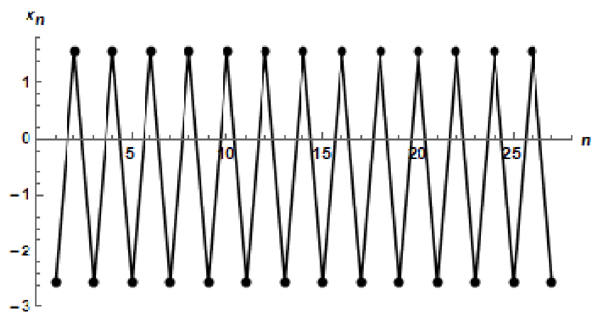


Figure 5.2: Plot of the Eq.(5.1) with the initial conditions $x_{-2} = \frac{-1+\sqrt{17}}{2}$, $x_{-1} = -\frac{1+\sqrt{17}}{2}$, $x_0 = \frac{-1+\sqrt{17}}{2}$ and $\alpha = -5$.

Example 5.3 *Consider the Eq.(5.1) with the initial conditions* $x_{-2} = 3$, $x_{-1} = -\frac{1}{2}$, $x_0 = \frac{2}{3}$ *and* $\alpha = 1$, *then Eq.(5.1) has three periodic solutions as (5.9). The following figure verifies our theoretical results. (See Figure 5.3)*

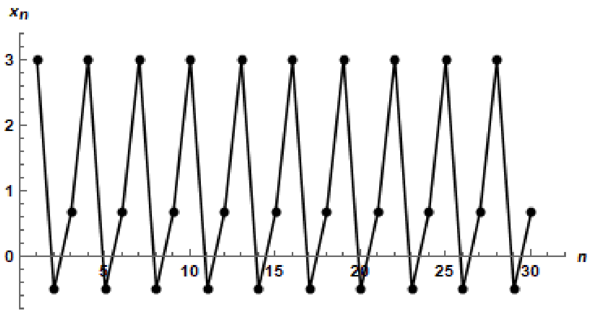


Figure 5.3: Plot of the Eq.(5.1) with the initial conditions $x_{-2} = 3$, $x_{-1} = -\frac{1}{2}$, $x_0 = \frac{2}{3}$ and $\alpha = 1$.

Example 5.4 *Consider the Eq.(5.1) with the initial conditions* $x_{-2} = -\frac{2}{5}$, $x_{-1} = \frac{5}{7}$, $x_0 = \frac{7}{2}$ *and* $\alpha = 1$, *then Eq.(5.1) has three periodic solutions as (5.9). The following figure verifies our theoretical results. (See Figure 5.4)*

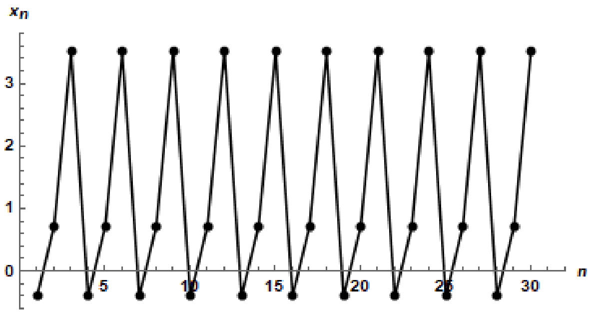
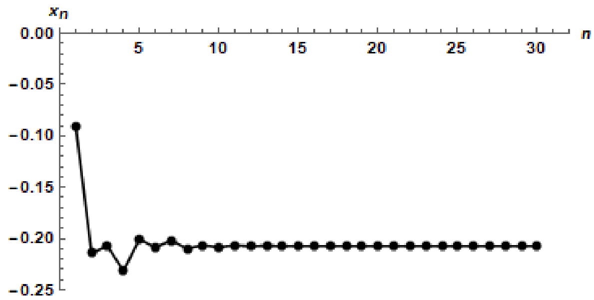


Figure 5.4: Plot of the Eq.(5.1) with the initial conditions $x_{-2} = -\frac{2}{5}$, $x_{-1} = \frac{5}{7}$, $x_0 = \frac{7}{2}$ and $\alpha = 1$.

Example 5.5 *Consider the Eq.(5.1) with the initial conditions* $x_{-2} = -0.8$, $x_{-1} = -0.4$, $x_0 = -0.2$ *and* $\alpha = -0.25$, *then Eq.(5.1) converges to* $\bar{x}_2 = \frac{1-\sqrt{1-4\alpha}}{2} = \frac{1-\sqrt{2}}{2} \approx -0.207$ *as Theorem 5.13. The following figure verifies our theoretical results. (See Figure 5.5)*

Figure 5.5:
Plot of the Eq.(5.1) with the initial conditions $x_{-2} = -0.8$, $x_{-1} = -0.4$, $x_0 = -0.2$ *and* $\alpha = -0.25$.



Plot of the Eq.(5.1) with the initial conditions $x_{-2} = -0.8$, $x_{-1} = -0.4$, $x_0 = -0.2$ and $\alpha = -0.25$.

Example 5.6 *Consider the Eq.(5.1) with the initial conditions* $x_{-2} = -0.15$, $x_{-1} = -0.95$, $x_0 = -0.45$ *and* $\alpha = 0.2$*, then Eq.(5.1) converges to* $\bar{x}_2 = \frac{1-\sqrt{1-4\alpha}}{2} = \frac{1-\sqrt{0.2}}{2} \approx 0.27$ *as Theorem 5.13. The following figure verifies our theoretical results. (See Figure 5.6)*

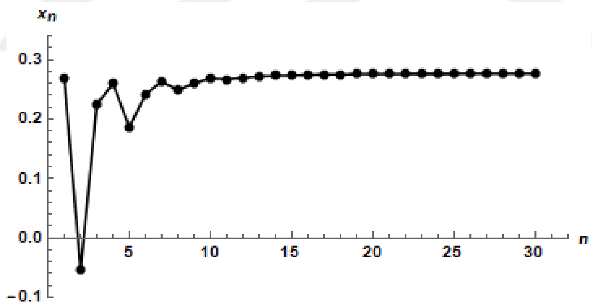


Figure 5.6: Plot of the Eq.(5.1) with the initial conditions $x_{-2} = -0.15$, $x_{-1} = -0.95$, $x_0 = -0.45$ and $\alpha = 0.2$.

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