APPROXIMATION PROPERTIES OF A CLASS OF NONLINEAR SINGULAR INTEGRAL OPERATORS

by

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ABSTRACT

APPROXIMATION PROPERTIES OF A CLASS OF NONLINEAR SINGULAR INTEGRAL OPERATORS

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This thesis is a survey on some approximation properties of nonlinear singular integral operators in the approximation theory. One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. It is to be expected that the better behaved functions are to be constructed from the given f by some smoothing operation on f itself.

This thesis consists of five chapters. The first chapter is devoted to the introduction. The second chapter contains concepts, definitions and also some fundamental informations about Stieltjes integral and bounded variations, which we need further studies and calculations. In the third chapter, we give some informations about the approximation problem and some important inequalities with their proofs. In chapter four, we study the pointwise convergence and the rate of pointwise convergence of nonlinear singular integrals by considering different functional spaces. In the last chapter, we give some examples of kernels of the mentioned the nonlinear singular integral operators.

Keywords: Pointwise Convergence, Carathéodory Kernel, Lipschitz Condition, Convolution, Nonlinear Singular Integrals, Stieltjes Integral, Bounded Variation.

ÖZET

LİNEER OLMAYAN SİNGÜLER İNTEGRAL OPERATORLERİNİN BİR SINIFININ YAKLAŞIM ÖZELLİKLERİ

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Bu tez, yaklaşım teorisindeki lineer olmayan singüler integral operatörlerin bir sınıfının yaklaşım özellikleri üzerine yapılan bir çalışmadır. Analizin temel problemlerinden biri f gibi bazı kötü özelliklere sahip bir fonksiyona, daha iyi özellikleri olan başka bir fonksiyonla yaklaşmaktır.

Bu tez beş bölümden oluşmaktadır. Birinci kısım giriş bölümüne ayrılmıştır. İkinci bölüm kavramlar, tanımlar ve ayrıca ileriki çalışmalar ve hesaplamalar için gerekli olan Stieltjes integrali ve sınırlı salınımlı fonksiyonlar hakkında temel bilgiler içermektedir. Üçüncü bölümde yaklaşım problemi hakkında bilgiler ve ileriki bölümlerde gerekli olan bazı önemli eşitsizliklere ispatları ile birlikte yer verildi. Dördüncü bölümde ise lineer olmayan singüler integrallerin noktasal yakınsaklıkları ve noktasal yakınsaklık hızları farklı fonksiyon uzayları ele alınarak incelendi. Beşinci bölümde ise lineer olmayan singüler integral operatörlerinde değindiğimiz çekirdek örneklerine yer verildi.

Anahtar Kelimeler: Noktasal Yakınsaklık, Carathéodory Çekirdeği, Lipschitz Şartı, Konvolusyon, Lineer Olmayan Singüler İntegral, Stieltjes İntegrali, Sınırlı Salınımlı Fonksiyon.

To my family and my supervisor

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CHAPTER 1

INTRODUCTION

One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. Approximation Theory is that area of analysis which, at its cone, is concerned with the ability to approximate functions by simpler and more easily calculated functions.

This theory takes its origin form at 19th century. At the beginning of the century the functions were viewed by the formulas, such as series, or as a solutions of differential equations. However largely as a consequence of the claims Fourier and result of Dirichlet and the modern concept of a function by its properties were introduced.

The birth of Approximation Theory becomes an unavailable development after a new definitions of the functions. It is in the theory of Fourier series that we find some of the first result of the approximation theory. This include conditions as a functions that ensure the pointwise convergence and uniform convergence of its Fourier Series.

The first question we ask in approximation theory concerns the possibility of approximation. Is the given family of functions from which we plan to approximate dense in the set of functions we wish to approximate? The Weierstrass Approximation Theorems spawned numerous generalization which were applied to other families of functions. They also led to the development of two general methods for determining the density. These are Stone-Weierstrass Theorem generalizing the Weierstrass Theorem to the normed space. Especially in S.W. Theorem the normed space must be compact. A different and more modern approach to density theorem is due to Functional Analysis.

In the theory of Fourier series there appears the problem of approximation of $2\pi-$

periodic functions f, integrable in the interval $[-\pi, \pi]$ by means of linear singular integral operators of the form

$$\int_{-\pi}^{\pi} K_n(t-s)\, f(t)\, dt.$$

For example, in case of the first arithmetic means of partial sums of Fourier series, $K = (K_n)_{n=1}^{\infty}$ is the Fejér kernel

$$K_n(u) = \frac{1}{2n\pi} \left(\frac{\sin(1/2)\, nu}{\sin(1/2)\, u} \right)^2,$$

and the singularity conditions mean that

$$\int_{-\pi}^{-\delta} K_n(u)\, du + \int_{\delta}^{\pi} K_n(u)\, du \to 0,$$

as $n \to \infty$ for every $0 < \delta < \pi$ and

$$\int_{-\pi}^{\pi} K_n(u)\, du = 1 \text{ for } n = 1, 2, \ldots .$$

In [1], these problems are extended in three directions:

a) The interval $[-\pi, \pi]$ is replaced by a compact abelian group G with Haar measure. But, for convenience he considers the following special case; $G = \mathbb{R}$ and the Lebesgue measure on $\mathbb{R}$.

b) The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of indices is replaced by an abstract set W of indices with convergence μ generated by a filter of subset of W,

c) The kernel $K = (K_n)_{n=1}^{\infty}$ is replaced by a kernel $K = (K_w)_{w \in W}$ where $K_w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; the last means that we take nonlinear integral operators T_w;

$$(T_w f)(s) = \int\limits_G K_w(t-s, f(t))\, dt = \int\limits_G K_w(t, f(t+s))\, dt,$$

in place of the above considered linear singular operators. The assumption of linearity of the operators was replaced in [1] by an assumption of a Lipschitz condition for K_w with respect to the second variable. Here, we shall go further and shall assume K_w to satisfy a Lipschitz condition, what will generate a function space of measurable functions. Full proofs of the theorems will be published in [2].

CHAPTER 2

PRELIMINARIES

2.1 Some Important Definitions

Definition 2.1. If A is a subset of the topological space X and if x is a point of X, we say that x is a *limit point* (or *accumulation point*) of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Definition 2.2. We say that $f(x)$ approaches the limit L as x approaches a, and we write

$$\lim_{x\to a} f(x) = L$$

if the following condition is satisfied:
For every number $\varepsilon > 0$ there exists a number $\delta > 0$, depending on ε, such that

$$0 < |x-a| < \delta \quad implies \quad |f(x) - L| < \varepsilon.$$

Definition 2.3. Suppose $\{f_n\}$, $n = 1, 2, 3, ..$, is a sequence of functions defined on set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We shall say that $\{f_n\}$ converges to f *pointwise* on E, if

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad , \ (x \in E),$$

holds.

Definition 2.4. We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, ..$, converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.

Definition 2.5. The sequence f_n uniformly converge to f on $A \Leftrightarrow \forall \varepsilon > 0\ \exists n_0$ such that $n \geq n_0$ and $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$.

Definition 2.6. $(C[a, b])$. As a set X we take the set of all real- valued functions $x, y, \cdots$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$. Choosing the metric defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

where *max* denotes the *maximum*, we obtain a metric space which is denoted by $C[a, b]$.

Definition 2.7. A map $T : X \to Y$ between two vector spaces is called an operator.

Definition 2.8. A linear operator T is an operator such that

- The domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field,
- for each $x, y \in D(T)$ and scalar α,

$$T(x + y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha T x.$$

(Kreyszig, 1989)

Definition 2.9. (Integral Operator). We can define an integral operator $T : C[0,1] \to C[0,1]$ by

$$y = Tx, \qquad where \qquad y(t) = \int_0^1 x(t)k(t,\tau)d\tau.$$

Here k is a given function which is called the kernel of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau - plane$, where $J = [0,1]$. This operator is linear.

(Kreyszig, 1989)

Definition 2.10. (Algebra) Let X be an nonempty set. A collection of subsets of S satisfies the following conditions;

(i) If $A, B \in S$, then $A \cap B \in S$.

(ii) If $A \in S$, then $A^C \in S$.

In this case S is called an *algebra* on X.

Definition 2.11. (σ-algebra) An algebra S of subsets of some set X is called a σ-algebra if every union of a countable collection of members of S is again in S. That is, in addition to S being an algebra, $\bigcup_{n=1}^{\infty} A_n$ belongs to S for every sequence $\{A_n\}$ of S.

Definition 2.12. (Measure) Let S be a σ-algebra of subsets of a set X. A set function $\mu : S \to [0,\infty]$ is called a measure on S if it satisfies the following properties:

1. $\mu(\emptyset) = 0$, and
2. whenever $\{A_n\}$ is a disjoint sequence of S satisfying $\bigcup_{n=1}^{\infty} A_n \in S$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

   holds; that is, μ is $\sigma - additive$.

Note that; A triplet (X, S, μ), where X is a nonempty set, S is a σ-algebra of subsets of X, and μ is a measure on S, is called a measure space.

Definition 2.13. (Measurable Function) Let $f : X \to \mathbb{R}$ be a function. If $f^{-1}(\omega)$ is a measurable set for every open subset ω of $\mathbb{R}$, then f is called a measurable function.

Note that; every constant function is measurable. Indeed, if $f(x) = c$ for all $x \in X$ and ω is an open subset of $\mathbb{R}$, then $f^{-1}(\omega) = \emptyset$ if $c \notin \omega$ and $f^{-1}(\omega) = X$ if $c \in \omega$.

Definition 2.14. (Almost everywhere) Let some property S hold for all the points of a set E, except for the points of a subset E_0 of the set E. If $\mu(E) = 0$, we say that S holds almost everywhere on the set E, or for almost all points of E. In particular, the set of points E_0 excluded may be the void set. We can say, for example, that two functions defined on the set E are equivalent if they are equal almost everywhere on E.

Definition 2.15. (Simple Function) Let assume (X, S, μ) is a measure space. If a measurable function $\phi : X \to \mathbb{R}$ assumes only a finite number of values, then ϕ is called a simple function.

Definition 2.16. (Step Function) A simple function ϕ is called a step function if ϕ has a representation of the form;

$$\phi = \sum_{j=1}^{m} b_j \chi_{B_j},$$

where each B_j is a measurable set of finite measure, i.e., $\mu^*\left(B_j\right) < \infty$.

Clearly; we can say that a simple function is a step function if and only if it vanishes outside of a set of finite measure.

Definition 2.17. (Upper Function) A function $f : X \to \mathbb{R}$ is called an upper function if there exists a sequence $\{\phi_n\}$ of step functions such that;

1. $\phi_n \uparrow f$ (a.e), and

2. $\lim \int \phi_n d\mu < \infty$

   Note that; in particular, every upper function is measurable function and clearly, every step function is an upper function.

Definition 2.18. (Lebesgue Integral) Let f be an upper function, and let $\{\phi_n\}$ be a sequence of step functions such that $\phi_n \uparrow f$ (a.e) holds. Then the *Lebesgue integral* of

f is defined by

$$\int f d\mu = \lim_{n\to\infty} \int \phi_n d\mu.$$

We stress the fact that the value of the Lebesgue integral of an upper function is independent of the generating sequence of step functions.

Alternative definition of Lebesgue integral is like;

Let $f : X \to \mathbb{R}$ be an arbitrary nonnegative bounded measurable function. Then the Lebesgue integral of f is;

$$\int_X f d\mu := \sup \left\{ \int_A f d\mu : A \in \mathrm{A}, \mu(A) < \infty \right\}.$$

2.2 $L_p(D)$ Spaces

Definition 2.19. ($L_p(-\infty, \infty)$ Space). L_p is the set of functions which are Lebesgue integrable to the p^{th} power over $\mathbb{R}$, for $1 \le p < \infty$, and essentially bounded (bounded almost everywhere) on $\mathbb{R}$ if $p = \infty$. For $f \in L_p(-\infty, \infty)$

$$||f||_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p dt \right\}^{\frac{1}{p}},$$

if $1 \le p < \infty$ and in case $p = \infty$

$$||f||_\infty = ess \sup_{x\in\mathbb{R}} |f(t)|.$$

(Butzer and Nessel, 1971)

Definition 2.20. ($L_p[a, b]$ Space). $L_p[a, b]$ is the set of functions which are Lebesgue integrable to the p^{th} power of $[a, b]$. For $f \in L_p[a, b]$

$$||f||_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{1/p}.$$

(Butzer and Nessel, 1971)

Definition 2.21. ($L_1[a,b]$ Space). $L_1[a,b]$ is the set of functions which are Lebesgue integrable to the first power of the period $[a,b]$. For $f \in L_1[a,b]$

$$\|f\|_1 = \int_a^b |f(t)|dt.$$

(Butzer and Nessel, 1971)

Definition 2.22. ($L_p[-\pi,\pi]$ Space). $L_p[-\pi,\pi]$ is the set of functions which are Lebesgue integrable to the p^{th} power of the period $[-\pi,\pi]$. For $f \in L_p[-\pi,\pi]$

$$\|f\|_p = \left\{ \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{1/p}.$$

(Butzer and Nessel, 1971)

Definition 2.23. ($X(\mathbb{R})$ Space). $X(\mathbb{R})$ always denotes one of the spaces C or L_p, $1 \leq p < \infty$. For $f, g \in X(\mathbb{R})$ we write $f(x) = g(x)$ (a.e.) if equality holds for all $x \in X(\mathbb{R})$ in case $X(\mathbb{R}) = C$, and almost everywhere in case $X(\mathbb{R}) = L_p$, $1 \leq p < \infty$, i.e., if $\|f - g\| = 0$. In this event we also write $f = g$ in $X(\mathbb{R})$.

(Butzer and Nessel, 1971)

2.3 Comparing $L_p(D)$ spaces with the situation of D

$L_p(D)$ spaces when D is finite interval

The properties of $L_p(D)$ can change connected with the interval D is finite or infinite. Let $a < b$ are finite numbers and $D = (a,b)$.

In this case, if we use Hölder inequality in $g(x) \equiv 1$;

$$\begin{aligned} \int_D |f(x)|dx &\leq \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} . \left(\int_a^b dx \right)^{\frac{1}{q}} \\ &= \|f\|_{L_p(D)}.(b-a)^{\frac{1}{q}} \end{aligned}$$

is found. So we can write;

$$\|f\|_{L_1(a,b)} \leq \|f\|_{L_p(a,b)}(b-a)^{\frac{1}{q}}.$$

If right hand side of this inequality is finite (i.e. $f \in L_p(a,b)$), then left hand side is also finite (i.e. $f \in L_p(a,b)$). This shows for all $p \geq 1$;

$$L_p(a,b) \subset L_1(a,b). \tag{2.1}$$

Namely;

$$\begin{aligned}\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} &\leq \left(ess \sup_{x\in[a,b]} |f(x)|\right).\left(\int_a^b dx\right)^{\frac{1}{q}} \\ &= (b-a)^{\frac{1}{q}}.\|f\|_{L_\infty(a,b)}\end{aligned}$$

from the inequality for all $p \geq 1$, we see;

$$L_\infty(a,b) \subset L_p(a,b). \tag{2.2}$$

Finally we get from (2.1) and (2.2);

$$L_\infty(a,b) \subset L_p(a,b) \subset L_1(a,b). \tag{2.3}$$

The equation (2.3) shows that when (a,b) is a finite interval; the largest space is $L_1(a,b)$ and the smallest space is $L_\infty(a,b)$ in $L_p(a,b)$ spaces.

Now, let assume $p_1 \geq p_2 \geq 1$. In this case suppose that $\frac{p_1}{p_2} \geq 1$ and $p = \frac{p_1}{p_2}$, we get $\frac{1}{p} = \frac{p_2}{p_1}$ and $\frac{1}{q} = 1 - \frac{p_2}{p_1} = \frac{p_1-p_2}{p_1}$.

Let assume $\varphi \in L_{p_2}(a,b)$.

If we use Hölder inequality;

$$|f(x)| = |\varphi(x)|^{p_2},$$

$$g(x) \equiv 1,$$

$$p = \frac{p_1}{p_2} \quad \text{and} \quad \frac{1}{q} = 1 - \frac{p_2}{p_1},$$

$$\begin{aligned}
\int_a^b |\varphi(x)|^{p_2}\, dx &\leq \left(\int_a^b |\varphi(x)|^{p_2 \frac{p_1}{p_2}}\, dx\right)^{\frac{p_2}{p_1}} (b-a)^{\frac{p_1-p_2}{p_1}} \\
&= \left(\int_a^b |\varphi(x)|^{p_1}\, dx\right)^{\frac{p_2}{p_1}} (b-a)^{\frac{p_1-p_2}{p_1}}
\end{aligned}$$

is found and then we see;

$$\left(\int_a^b |\varphi(x)|^{p_2} dx\right)^{\frac{1}{p_2}} \leq (b-a)^{\frac{p_1-p_2}{p_1 p_2}} \left(\int_a^b |\varphi(x)|^{p_1} dx\right)^{\frac{1}{p_1}}.$$

Namely if;

$$p_1 \geq p_2 \geq 1$$

$$\|\varphi\|_{L_{p_2}(a,b)} \leq \|\varphi\|_{L_{p_1}(a,b)}.(b-a)^{\frac{p_1-p_2}{p_1 p_2}}$$

with another term if;

$$p_1 \geq p_2 \geq 1,$$

$$L_{p_1(a,b)} \subset L_{p_2(a,b)}.$$

This shows that if (a, b) is a finite interval, then $L_p(a, b)$ spaces extend with p' s be reduced from the equation (2.3). As;

$$L_\infty(a,b) \subset L_{p_1}(a,b) \subset L_{p_2}(a,b) \subset L_1(a,b), \qquad p_1 > p_2. \tag{2.4}$$

$L_p(D)$ spaces when $D = (-\infty, \infty)$

We researched that there is the relation (2.4) with $L_p(D)$ spaces in finite interval $D(a, b)$. Namely, $L_1(D)$ space contains all properties of $L_p(D)$, $1 < p \leq \infty$. But unfortunately, there is not any relation as if (2.4), when D is an infinite interval. This shows the properties of $L_p(D)$ spaces can be change with respect to p in infinite intervals. For example it is obvious that, for the function $f_1(x) \equiv 1$ is member of $L_\infty(-\infty, \infty)$, but it is not the member of $L_p(-\infty, \infty)$, $L_\infty(a, \infty)$ or $L_\infty(-\infty, b)$. That is to say;

$$L_\infty(-\infty, \infty) \not\subset L_p(-\infty, \infty).$$

Also for $\alpha > 0$ is a number;

$$f_2(x) = \left\{ \begin{array}{ll} 0 \le x \le \alpha, & \frac{1}{x^{\frac{1}{p+1}}}; \\ otherwise, & 0; \end{array} \right\},$$

$f_2(x)$ is member of $L_p(-\infty, \infty)$, but the neighborhood of $f_2(x)$ at $x = 0$ is infinite and increasing, so $f_2 \notin L_\infty(-\infty, \infty)$, namely;

$$L_p(-\infty, \infty) \not\subset L_\infty(-\infty, \infty).$$

Let find some examples for the other cases, if $p_1 > p_2$, take $f_3(x)$ as;

$$f_3(x) = \left\{ \begin{array}{ll} 0 \le x \le \alpha, & \frac{1}{\sqrt[p_1]{x}}; \\ otherwise, & 0; \end{array} \right\}.$$

In this case $\frac{p_2}{p_1} < 1$, so;

$$\|f_3\|_{L_{p_2}(-\infty,\infty)} = \left(\int_0^\alpha \frac{1}{|x|^{\frac{p_2}{p_1}}} dx \right)^{\frac{1}{p_2}} < \infty.$$

Namely, $f_3 \in L_{p_2}(-\infty, \infty)$. Besides;

$$\|f_3\|_{L_{p_1}(-\infty,\infty)} = \left(\int_0^\alpha \frac{1}{x} dx \right)^{\frac{1}{p_1}} = \infty$$

and $f_3 \notin L_{p_1}(-\infty, \infty)$. This shows that, when $p_1 > p_2$;

$$L_{p_1}(-\infty, \infty) \not\subset L_{p_2}(-\infty, \infty)$$

Also , again take $f_4(x)$, if $\alpha > 0$ and $p_1 > p_2$,

$$f_4(x) = \left\{ \begin{array}{ll} \alpha \le x \le \infty, & \frac{1}{\sqrt[p_2]{x}}; \\ otherwise, & 0; \end{array} \right\}.$$

We know $\frac{p_1}{p_2} > 1$, so;

$$\|f_4\|_{L_{p_1}(-\infty,\infty)} = \int_{\alpha}^{\infty} \left(\frac{1}{x^{\frac{p_1}{p_2}}} \right)^{\frac{1}{p_1}} < \infty.$$

Namely $f_4 \in L_{p_1}(-\infty, \infty)$, but;

$$\left(\int_{-\infty}^{\infty} |f_4(x)|^{p_2} dx \right)^{\frac{1}{p_2}} = \left(\int_{\alpha}^{\infty} \frac{1}{x} dx \right)^{\frac{1}{p_2}} = \infty$$

inequality shows that;

$$f_4 \notin L_{p_2}(-\infty, \infty).$$

Because of $p_1 > p_2$,

$$L_{p_2}(-\infty, \infty) \not\subset L_{p_1}(-\infty, \infty).$$

This examples shows that; there is not a relation in $L_p(-\infty, \infty)$ spaces like equation (2.4). Similar examples are valid for $L_p(\alpha, \infty)$ and $L_p(-\infty, b)$ spaces.

Definition 2.24. (Minkowski' s Inequality). Let $f, g \in X(\mathbb{R})$. Then $(f + g) \in X(\mathbb{R})$ and

$$\|f + g\|_{X(\mathbb{R})} \le \|f\|_{X(\mathbb{R})} + \|g\|_{X(\mathbb{R})}.$$

If p is such that $1 \le p \le \infty$, the conjugate number q is defined through $(1/p)+(1/q) = 1$ in case $1 < p < \infty$; $q = \infty$ if, $p = 1$, and $q = 1$ if $p = \infty$.

Definition 2.25. (Hölder' s Inequality). Let $f \in L_p$, $1 \le p \le \infty$, and $g \in L_q$. Then $fg \in L_1$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$. Namely;

$$\left| \int fg \right| \le \left(\int |f|^p \right)^{1/p} . \left(\int |g|^q \right)^{1/q} \tag{2.5}$$

Proof. We first note that if $\int |f|^p = 0$, then $|f|^p = 0$ almost everywhere; so $f = 0$, and therefore $fg = 0$, almost everywhere. Then fg is integrable, $\int fg = 0$, and (2.5) holds trivially, as it does also in the case where $\int |g|^q = 0$. Thus we may assume that $\int |f|^p > 0$ and $\int |g|^q > 0$. We then have, almost everywhere,

$$\begin{aligned}\frac{|fg|}{(\int |f|^p)^{1/p}(\int |g|^q)^{1/q}} &= \left(\frac{|f|^p}{\int |f|^p}\right)^{1/p}\left(\frac{|g|^q}{\int |g|^q}\right)^{1/q} \\ &\leq \frac{|f|^p}{p\int |f|^p} + \frac{|g|^q}{q\int |g|^q}\end{aligned}$$

So;

$$|fg| \leq \left(\int |f|^p\right)^{1/p} \cdot \left(\int |g|^q\right)^{1/q} \cdot \frac{|f|^p}{p\int |f|^p} + \frac{|g|^q}{q\int |g|^q} \tag{2.6}$$

Now, fg is measurable and the right-hand side of (2.6) is integrable. Hence, fg is integrable and

$$\left|\int fg\right| \leq \int |fg| \leq \left(\int |f|^p\right)^{\frac{1}{p}} \cdot \left(\int |g|^q\right)^{\frac{1}{q}} \cdot \left(\frac{1}{p} + \frac{1}{q}\right),$$

from which (2.5) follows. □

Definition 2.26. (Hölder- Minkowski Inequality). Let $f(x,y)$ be defined and measurable on $\mathbb{R}^2$. If $\|f(x,y)\|_{X(\mathbb{R})} \in L_1$, then

$$\left\|\int_{-\infty}^{\infty} f(x,y)dy\right\|_{X(\mathbb{R})} \leq \int_{-\infty}^{\infty} \|f(x,y)\|_{X(\mathbb{R})}dy.$$

This is also known as the generalized Minkowski inequality.

Definition 2.27. (Lipschitz condition) A finite function $f(x)$ defined on $[a,b]$ is said to satisfy a Lipschitz condition if there exists a constant K such that for any two points x and y in $[a,b]$,

$$|f(x) - f(y)| \leq K|x-y|$$

holds true.

(Natanson, Volume I, 1964)

Definition 2.28. (Convolution) Let consider the product $f(x+t)\,g(t)$. For this product almost all x is a summable function of t in $[-\pi, \pi]$, and supposing

$$\rho(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t)\,g(t)\,dt,$$

we have $\rho(x) \in L[0, 2\pi]$. This function $\rho(x)$ is known as the convolution for $f(x)$ and $g(x)$.

The definition of convolution on real-line is;

Let two real-valued functions f and g are defined and measurable on $\mathbb{R}$. The expression

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)\,g(u)\,du$$

is called *the convolution* of f and g.

Definition 2.29. (Kernel) A kernel is a function $\phi_n(t, x)$ which $(n = 1, 2, 3, ...)$ defined in the square $(a \leq t \leq b, a < x < b)$ and such that;

$$\lim_{n\to\infty} \int_{\beta}^{\theta} \phi_n(t, x)\,dt = 1,$$

provided where, $a \leq \beta < x < \theta \leq b$. It is self-evident that $\phi_n(t, x)$ is assumed summable with respect to t for every fixed x.

And also note that; an integral of the form;

$$f_n(x) = \int_{a}^{b} \phi_n(t, x)\,f(t)\,dt$$

where $\phi_n(t, x)$ is a kernel, is called a *singular integral*.

(Natanson Volume II, 1960)

Definition 2.30. Let Λ be an index set and λ_0 be an accumulation point of the set Λ. When the kernel function $K_\lambda(t)$ satisfies the $\lim_{\lambda\to\lambda_0} K_\lambda(t_0) = \infty$ at the point t_0, then we say the kernel function $K_\lambda(t)$ is "Singular Kernel", and

$$L_\lambda(f;x) = \int_D f(t)K_\lambda(t-x)dt, \qquad x \in D$$

are called the family of convolution type singular integral operators depends on parameters (x,λ).

Definition 2.31. If the singular kernel function $K_\lambda(t)$ satisfies the condition;

$$\lim_{\lambda\to\lambda_0}\left[\sup_{\delta\leq|t|}|K_\lambda(t)|\right] = 0 \qquad \forall\delta > 0,$$

then the function is called "Approximation Identity".

Definition 2.32. (Carathéodory kernel) The function $K : G\times\mathbb{R} \to \mathbb{R}$ is *a Carathéodory kernel function*, i.e. it is $\sum$-measurable in for every $u \in \mathbb{R}$, and it is continuous in $\mathbb{R}$ for every $t \in G$, with $K(t,0) = 0$.

2.4 Some Characteristic Points

Definition 2.33. (D- Point). A point $x \in \mathbb{R}$ is called a $D-$ *point* of the function $f \in L_1$ if

$$\lim_{h\to0}\frac{1}{h}\int_x^{x+h}[f(t) - f(x)]dt = 0. \tag{2.7}$$

D-points stand for differentiability; for the D-points of an integrable function f are precisely those points at which the indefinite integral of f is differentiable to the value $f(x)$.

Proof. If we substitute $h = -h$ in the equation (2.7),

$$\lim_{h\to0}\frac{1}{h}\int_{x-h}^{x} f(t)dt = f(x). \tag{2.8}$$

Then sum equations (2.7) and (2.8)

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h}[f(t)-f(x)]dt=0. \tag{2.9}$$

(2.9) is valid (a. e). Namely, if (2.7) is exists, then (2.8) and (2.9) exist.

We can write (2.7) and (2.8) as;

$$\lim_{h\to 0}\frac{1}{h}\int_{0}^{h}[f(x+t)-f(x)]dx=0. \tag{2.10}$$

$$\lim_{h\to 0}\frac{1}{h}\int_{0}^{h}[f(x-t)-f(x)]dx=0. \tag{2.11}$$

and sum the equations (2.10) and (2.11);

$$\lim_{h\to 0}\frac{1}{h}\int_{0}^{h}[f(x+t)-f(x-t)-2f(x)]dx=0. \tag{2.12}$$

Namely, if x is D- point of function f, at this point (2.10), (2.11) and (2.12) are valid.

□

(Bary, 1964)

Definition 2.34. (Lebesgue Point) A point $x\in\mathbb{R}$ is called a Lebesgue point of the function $f\in L_1$ if

$$\lim_{h\to 0}\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt=0.$$

(Bary, 1964)

Evidently, any Lebesgue point of f is a D-point of f but not conversely, and any point of continuity of an integrable function f is a Lebesgue point. Indeed; let $L(f)$ be the set of all Lebesgue points of $f\in L_1$ and $D(f)$ be the set of all points of all D-points of $f\in L_1$. Then

$$\left|\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt\right|\leq\frac{1}{h}\int_{x}^{x+h}|f(t)-f(x)|dt.$$

The inequality shows that every Lebesgue point is also D-point. Thus,

$$L(f) \subset D(f).$$

Now, it is clear that any point, where $f(x)$ is continuous, is a Lebesgue point; in fact, if there exists any $\varepsilon > 0$ and the function $f(x)$ is continuous at the point x, it can be found a $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \qquad for \qquad |h| \leq \delta$$

and then

$$\frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt < \varepsilon,$$

or we can write

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0.$$

This shows that any continuity point of the function $f \in L_1$ is also a Lebesgue point. Let $C(f)$ be the set of all points of continuity of $f \in L_1$. Then we can write

$$C(f) \subset L(f) \subset D(f).$$

(Bary, 1964)

Definition 2.35. (μ *– generalized Lebesgue point*) A point $x \in \mathbb{R}$, is called a μ *– generalized Lebesgue point* of the function $f \in L_1$, if

$$\lim_{h \to 0^+} \frac{1}{\mu(h)} \int_0^h |f(x_0 + t) - f(x_0)| \, dt = 0,$$

where $\mu(t)$ be an absolutely continuous, increasing function defined on $[0, b - a]$ with $\mu(0) = 0$.

2.5 Functions of Bounded (Finite) Variation

Let a function $f(x)$ be defined on $[a, b]$. Subdivide $[a, b]$ into parts by means of the points;

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b,$$

and if we form the sum;

$$\bigvee = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

So, we can give following definition.

Definition 2.36. (Finite Variation) The least upper bound of the set of all possible sums $\bigvee$ is called the total variation of $f(x)$ on $[a, b]$ and is denoted by;

$$\bigvee_a^b f.$$

If,

$$\bigvee_a^b f < +\infty,$$

then $f(x)$ is said to be a function of finite variation on $[a, b]$. We also say that $f(x)$ is bounded variation on $[a, b]$. This kind of functions define a set which is denoted by;

$$BV[a, b].$$

Theorem 2.1. *A monotonic function on* $[a, b]$ *is bounded variation on* $[a, b]$.

Proof. Let f be a monotone increasing on $[a, b]$. Then consider the partition and define the following sum:

$$
\begin{aligned}
\bigvee &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\
&= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \ldots + (f(x_n) - f(x_{n-1})) \qquad (2.13) \\
&= f(x_n) - f(x_0) \qquad (2.14)
\end{aligned}
$$

Since we know that;

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b,$$

the above equation takes the form;

$$
\begin{aligned}
\bigvee &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\
&= f(b) - f(a)
\end{aligned}
$$

So;

$$\bigvee_a^b f = f(b) - f(a) < +\infty.$$

Therefore; $f \in BV[a, b]$. □

Definition 2.37. A finite function $f(x)$ is defined on $[a, b]$, is said to satisfy a Hölder condition if there exists a constant K such that for any two points x and y in $[a, b]$,

$$|f(x) - f(y)| \le K.|x - y|^{\alpha}$$

where $0 < \alpha \le 1$. Then the functions satisfying Hölder condition define the following set of functions;

$$Lip_K\alpha = \{f : [a, b] \to \mathbb{R}, |f(x) - f(y)| \le K|x - y|^{\alpha}\},$$

for all $x, y \in [a, b]$ and $0 < \alpha \le 1$.

Note that;

$Lip_K(1)$ is called Lipshitz Class and;

$$Lip_K(1) \subset Lip_K\alpha.$$

If the function $f(x)$ has a derivative $f'(x)$ at every point of $[a,b]$ and if $f'(x)$ is bounded, then clearly from The Mean Value Theorem one has,

$$f(x) - f(y) = f'(z)\,|(x-y)|$$

for $x < z < y$,

So;

$$|f(x) - f(y)| = |f'(z)|\,|(x-y)| \leq K.\,|(x-y)|\,.$$

Therefore; If a function has a derivative than it has an element of Lip-class.

If $f(x)$ satisfies a Lipschitz condition, then;

$$\begin{aligned} \bigvee &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &\leq \sum_{k=0}^{n-1} K.\,|x_{k+1} - x_k| \\ &= K \sum_{k=0}^{n-1} (x_{k+1} - x_k) = K.\,[b-a]\,. \end{aligned}$$

Therefore;

$$\bigvee_a^b (f) = K.\,[b-a] < \infty,$$

so, if $f \in Lip_k(1)$ then $f \in BV[a,b]$.

Theorem 2.2. *Every function of bounded variation on* $[a,b]$ *is bounded in* $[a,b]$.

Proof. We choose any arbitrary point $x \in [a, b]$, then;

$$\bigvee = |f(x) - f(a)| + |f(b) - f(x)| \leq \bigvee_a^b f.$$

This implies that;

$$|f(x) - f(a)| \leq \bigvee_a^b f,$$

and,

$$|f(b) - f(x)| \leq \bigvee_a^b f,$$

If we choose first part, then

$$\begin{aligned} |f(x) - f(a)| &\leq \bigvee_a^b f \\ &\Rightarrow \quad |f(x)| \leq \bigvee_a^b f + |f(a)| \end{aligned}$$

for all $x \in [a, b]$,

Hence, f is bounded on $[a, b]$.

This finishes the proof. □

Theorem 2.3. *The sum, difference and product of two functions of bounded variation are also functions of bounded variations.*

Proof. Let $f(x)$ and $g(x)$ belong to $BV[a, b]$. And let $S(x)$ is the their sum namely;

$$S(x) = f(x) + g(x).$$

So;

$$\begin{aligned}
\bigvee_a^b (S) &= \sum_{k=0}^{n-1} |S(x_{k+1}) - S(x_k)| \\
&= \sum_{k=0}^{n-1} |f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)| \\
&\leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|.
\end{aligned}$$

This implies;

$$\bigvee_a^b (S) \leq \bigvee_a^b (f) + \bigvee_a^b (g) < +\infty.$$

Since $f, g \in BV[a,b]$;

$$\bigvee_a^b f(x) < +\infty$$

and

$$\bigvee_a^b g(x) < +\infty.$$

Therefore; $S(x)$, which denotes the sum of $f(x)$ and $g(x)$, is an element of $BV[a,b]$.

Now, we will have a look for their product.

Assume that $S(x)$ denotes their product;

$$S(x) = f(x).g(x).$$

Then;

$$\begin{aligned}
\bigvee(S) &= \sum_{k=0}^{n-1} |S(x_{k+1}) - S(x_k)| \\
&= \sum_{k=0}^{n-1} |f(x_{k+1}).g(x_{k+1}) - f(x_k).g(x_k)| \\
&= \sum_{k=0}^{n-1} |f(x_{k+1}).g(x_{k+1}) - f(x_k).g(x_{k+1}) + f(x_k).g(x_{k+1}) - f(x_k).g(x_k)| \\
&= \sum_{k=0}^{n-1} \left| g(x_{k+1})[f(x_{k+1}) - f(x_k)] + f(x_k)[g(x_{k+1}) - g(x_k)] \right| \\
&\leq |g(x_{k+1})| \bigvee_a^b (f) + |f(x_k)| \bigvee_a^b (g) \\
&\leq M \bigvee_a^b (f) + N \bigvee_a^b (g).
\end{aligned}$$

Since $f, g \in BV[a,b]$, we can say that $|f(x)| \leq N$, and $|g(x)| \leq M$, for some $M, N \in \mathbb{R}$. This finishes the proof. □

Theorem 2.4. *Let $f \in BV[a,b]$ and let $a < c < b$, then;*

$$\bigvee_a^b (f) = \bigvee_a^c (f) + \bigvee_c^b (f).$$

Proof. First of all we need to observe that we have two intervals, $[a,c]$ and $[c,b]$. If we subdivide each of the intervals by means of the points;

Let $[a,c]$;

$$a = y_0 < y_1 < y_2 < \ldots < y_m = c,$$

and $[c,b]$;

$$c = z_0 < z_1 < z_2 < \ldots < z_n = b.$$

Then, if we form the sums;

$$\bigvee_1 = \sum_{k=0}^{n-1} |f(y_{k+1}) - f(y_k)|$$

and the other interval is;

$$\bigvee_2 = \sum_{k=0}^{n-1} |f(z_{k+1}) - f(z_k)|.$$

So;

$$\bigvee = \bigvee_1 + \bigvee_2,$$

where $\bigvee$ is the variation on $[a,b]$. It follows that,

$$\bigvee_1 + \bigvee_2 \leq \bigvee_a^b (f)$$

or

$$\bigvee_a^c (f) + \bigvee_c^b (f) \leq \bigvee_a^b (f).$$

If we show;

$$\bigvee_a^b (f) \leq \bigvee_a^c (f) + \bigvee_c^b (f),$$

then the proof will be completed. Now subdivide $[a,b]$ by means of the points;

$$a = x_0 < x_1 < x_2 < ... < x_n = b.$$

If $c = x_m$, then we can express;

$$\begin{aligned}\bigvee &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &= \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=m}^{n-1} |f(x_{k+1}) - f(x_k)|.\end{aligned}$$

Consequently;

$$\bigvee = \bigvee_1 + \bigvee_2 \leq \bigvee_a^c (f) + \bigvee_c^b (f),$$

and hence,

$$\bigvee_a^b (f) \leq \bigvee_a^c (f) + \bigvee_c^b (f).$$

If c does not belong to the partition, then we define a new partition which includes the point c. Then our partition contains (n+2) points. The other parts are the same. The proof of this theorem is completed. □

Corollary 2.1. *Let $a < c < b$. If the function $f(x)$ has finite (bounded) variation on $[a,b]$, it has finite variation on each of the intervals $[a,c]$ and $[c,b]$, and conversely.*

Theorem 2.5. *A function $f(x)$ defined and finite on $[a,b]$, is a function of finite variation if and only if; it is representable as the difference of two increasing function.*

Proof. First of all, we need to choose a function, which has to be increasing, namely;

$$\psi(x) = \bigvee_a^x (f),$$

where, $\psi(a) = 0$ and $a < x \leq b$. We know that our function is an increasing function on $[a,b]$. From the increasing function's property, we can set;

$$\vartheta(x) = \psi(x) - f(x).$$

Then, we obtain another increasing function. In fact; if $a \leq x < y \leq b$, then;

$$\begin{aligned} \vartheta(y) &= \psi(y) - f(y) \\ &= \psi(x) + \bigvee_x^y (f) - f(y) \end{aligned}$$

and,

$$\begin{aligned} \vartheta(y) - \vartheta(x) &= \bigvee_x^y (f) + f(x) - f(y) \\ &= \bigvee_x^y (f) - [f(y) - f(x)]. \end{aligned}$$

However, from the definition of total variation, it is clear that,

$$f(y) - f(x) \leq \bigvee_x^y (f).$$

So that;

$$\vartheta(y) - \vartheta(x) \geq 0.$$

Therefore, $\vartheta(x)$ is an increasing function. So we can easily write

$$f(x) = \psi(x) - \vartheta(x).$$

Finally note that; we know and prove that monotone functions are bounded variations and we also know that we can use sum and difference with bounded variations functions. We can easily prove the other using these knowledge. □

Corollary 2.2. *If a function $f(x)$ has finite variation on $[a,b]$, then at almost every point of $[a,b]$ the derivative $f'(x)$ exists, and is finite. Furthermore, $f'(x)$ is a summable function on $[a,b]$.*

2.6 The Stieltjes Integral

We now take up a very important generalization of the Riemann integral which is called the Stieltjes Integral.

Definition 2.38. Let $f(x)$ and $g(x)$ be finite functions defined on the closed interval $[a,b]$. Subdivide $[a,b]$ into parts by means of the points;

$$a = x_0 < x_1 < ... < x_n = b,$$

choose a point φ_k in $[x_k, x_{k+1}]$ for $k = 0, ..., n-1$, and form the sum;

$$\sigma = \sum_{k=0}^{n-1} f(\varphi_k)\left[g(x_{k+1}) - g(x_k)\right].$$

If the sum σ tends to a finite limit I as;

$$\lambda = \max(x_{k+1} - x_k) \to 0,$$

independently of both the method of subdivision and the choice of the points φ_k, this limit is called the Stieltjes Integral of the function $f(x)$ with respect to the function $g(x)$ and designated by;

$$\int_a^b f(x)\, dg(x),$$

or

$$(S)\int_a^b f(x)\, dg(x).$$

The exact meaning of the definition in this: The number I is the Stieltjes Integral of the function $f(x)$ with respect to the function $g(x)$ if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for an arbitrary method of subdivision for which $\lambda < \delta$, the inequality

$$|\sigma - I| < \varepsilon,$$

holds, for all choices of the points φ_k.

Note that; it is clear that the Riemann integral is a special case of the Stieltjes integral, we can easily obtain by setting $g(x) = x$.

Properties of Stieltjes Integral

Now, we will have a look at the list of obvious properties of the Stieltjes Integral.

1.
$$\int_a^b [f_1(x) + f_2(x)]\, dg(x) = \int_a^b f_1(x)\, dg(x) + \int_a^b f_2(x)\, dg(x).$$

We can prove this property easily;

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=0}^{n-1} [f_1(\xi_k) + f_2(\xi_k)]\, \Delta g &= \lim_{n\to\infty} \sum_{k=0}^{n-1} [f_1(\xi_k)\, \Delta g + f_2(\xi_k)\, \Delta g] \\ &= \lim_{n\to\infty} \left[\sum_{k=0}^{n-1} f_1(\xi_k)\, \Delta g + \sum_{k=0}^{n-1} f_2(\xi_k)\, \Delta g\right] \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} f_1(\xi_k)\, \Delta g + \lim_{n\to\infty} \sum_{k=0}^{n-1} f_2(\xi_k)\, \Delta g \\ &= \int f_1(x)\, dg(x) + \int f_2(x)\, dg(x). \end{aligned}$$

2.
$$\int_a^b f(x)\, d\left[g_1(x) + g_2(x)\right] = \int_a^b f(x)\, dg_1(x) + \int_a^b f(x)\, dg_2(x)$$

3. If k and m are constants, then

$$\int_a^b k.f(x)\, dmg(x) = k.m \int_a^b f(x)\, dg(x).$$

Note that; in all three properties of Stieltjes Integral, the existence of right-hand member implies the existence of the left-hand member.

4. If $a < c < b$, and all three integrals involved in the equality,

$$\int_a^b f(x)\, dg(x) = \int_a^c f(x)\, dg(x) + \int_c^b f(x)\, dg(x) \tag{2.15}$$

exists, then the equality (2.15) holds.

Notice that, in order to prove this property of the integral, it is necessary only to see that the point c is included in the points of subdivision of $[a, b]$, when we form the sum σ for the integral

$$\int_a^b f(x)\, dg(x). \tag{2.16}$$

It is not difficult to prove that the existence of the integral (2.16) implies the existence of both the integrals (2.15), but we will not discuss it now. We now only notice that the converse statement is not true. Let us give an example.

Example 2.1. Let the function $f(x)$ and $g(x)$ be defined on the segment $[-1, +1]$, where

$$f(x) = \begin{cases} 0, & if \quad -1 \le x \le 0 \\ 1, & if \quad 0 < x \le 1 \end{cases}$$

and,

$$g(x) = \begin{cases} 0, & if \quad -1 \le x \le 0 \\ 1, & if \quad 0 \le x \le 1 \end{cases} .$$

It is easy to see that the integrals

$$\int_{-1}^{0} f(x)\, dg(x),$$

and

$$\int_{0}^{1} f(x)\, dg(x)$$

are exist. Because all of the sums σ are equal to zero. But at the same time, the integral

$$\int_{-1}^{1} f(x)\, dg(x),$$

does not exist. In fact, subdivide the segment $[-1, +1]$, into parts so that the point 0, is not a point of subdivision, and if we form the sum;

$$\sigma = \sum_{k=0}^{n-1} f(\varphi_k)\,[g(x_{k+1}) - g(x_k)] .$$

It is easy to see that if $x_i < 0 < x_{i+1}$, only the $i - th$ term remains in the sum σ, because if the points x_k and x_{k+1} lie on the same side of the points 0, then

$$g(x_k) = g(x_{k+1}) .$$

This implies that

$$\begin{aligned} \sigma &= f(\varphi_i)\,[g(x_{i+1}) - g(x_i)] \\ &= f(\varphi_i) . \end{aligned}$$

Depending upon whether $\varphi_i < 0$ or $\varphi_i > 0$, we have

$$\sigma = 0$$

or

$$\sigma = 1,$$

so that σ has no limit.

5. The existence of one of the integrals

$$\int_a^b f(x)\,dg(x)$$

and

$$\int_a^b g(x)\,df(x),$$

implies the existence of the other. In this case, the equality

$$\int_a^b f(x)\,dg(x) + \int_a^b g(x)\,df(x) = [f(x)\,g(x)]_a^b \tag{2.17}$$

holds, where

$$[f(x)\,g(x)]_a^b = f(b)\,g(b) - f(a)\,g(a)\,. \tag{2.18}$$

The relation (2.17) is called the *formula for integration by parts*. Let us prove this formula. Suppose that the integral

$$\int_a^b g(x)\,df(x)$$

exists. Subdivide $[a, b]$ into parts and form the sum,

$$\sigma = \sum_{k=0}^{n-1} f(\varphi_k)\,[g(x_{k+1}) - g(x_k)]\,.$$

The sum σ can also be written as;

$$\sigma = \sum_{k=0}^{n-1} f(\varphi_k)\, g(x_{k+1}) - \sum_{k=0}^{n-1} f(\varphi_k)\, g(x_k)$$

and this expression is clearly equal to

$$\sigma = -\sum_{k=0}^{n-1} g(x_k)\left[f(\varphi_k) - f(\varphi_{k-1})\right] + f(\varphi_{n-1})\, g(x_n) - f(\varphi_0)\, g(x_0)\,.$$

Adding and subtracting expression (2.18) on the right side of the last equality, we find;

$$\begin{aligned}\sigma &= \left[f(x)\, g(x)\right]_a^b \\ &\quad -\left\{g(a)\left[f(\varphi_k) - f(a)\right] + \sum_{k=1}^{n-1} g(x_k)\left[f(\varphi_k) - f(\varphi_{k-1})\right] + g(b)\left[f(b) - f(\varphi_{n-1})\right]\right\}.\end{aligned}$$

The expression in the curly brackets is nothing but the sum formed for the integral

$$\int_a^b g(x)\, d f(x)\,,$$

where the points of subdivision of $[a, b]$ are

$$a \le \varphi_0 \le \varphi_1 \le \varphi_2 \le \ldots \le \varphi_{n-1} \le b,$$

and the points $a, x_1, x_2, \ldots, x_{n-1}, b$ are points of the closed intervals $[a, \varphi_0]\,, [\varphi_0, \varphi_1]\,, \ldots, [\varphi_{n-1}, b]$. As $\max(x_{k+1} - x_k)$ approaches zero,

$$\max(\varphi_{k+1} - \varphi_k)$$

also approaches zero, so that the sum in the curly brackets tends to the integral

$$\int_a^b g(x)\, d f(x)\,.$$

And this completes the proof.

It is natural to ask the question under what conditions the Stieltjes integral exists. We will answer this question to giving following theorem.

Theorem 2.6. *The integral*

$$\int_a^b f(x)\,dg(x),$$

exists if the function $f(x)$ *is continuous on* $[a,b]$ *and* $g(x)$ *is of finite variation on* $[a,b]$.

Proof. We may obviously suppose that the function $g(x)$ is increasing, because every function of finite variation is the difference of two functions. Subdivide $[a,b]$ by means of the points

$$x_0 = a < x_1 < \ldots < x_n = b$$

and designate by m_k and M_k, the least and greatest values, respectively, of the function $f(x)$ on $[x_k, x_{k+1}]$. Let

$$s = \sum_{k=0}^{n-1} m_k\,[g(x_{k+1}) - g(x_k)],$$

and

$$S = \sum_{k=0}^{n-1} M_k\,[g(x_{k+1}) - g(x_k)].$$

It is clear that

$$s \le \sigma \le S,$$

for all choices of the points in the closed intervals $[x_k, x_{k+1}]$. It is also easy to verify that the sum s does not decrease and S does not increase when new points of subdivision are added. From this, it follows that none of the sums s surpasses any of the sum S. In fact, having two methods, I and II, of subdividing the segment $[a,b]$, with corresponding s_1, S_1 and s_2, S_2, respectively, we can form a method III by combining the points of subdivision of both of the methods I and II. If the sums s_3 and S_3 correspond to the method III, we have

$$s_1 \le s_3 \le S_3 \le S_2,$$

so that, in fact, $s_1 \le S_2$.

With this in mind, we denote the least upper bound of the set $\{s\}$ of all lower sums by the symbol I:

$$I = \sup\{s\}.$$

For every method of subdivision, it is clear that

$$s \leq I \leq S,$$

and consequently, by (3),

$$|\sigma - I| < S - s.$$

If we choose and arbitrary $\varepsilon > 0$ and find a $\delta > 0$ such that $|f(x'') - f(x')| < \varepsilon$ whenever $|x'' - x'| < \delta$, then we have

$$M_k - m_k < \varepsilon$$

for $k = 0, 1, ..., n-1$, and $\lambda < \delta$, and accordingly

$$S - s < \varepsilon[g(b) - g(a)].$$

Clearly,

$$|\sigma - I| < \varepsilon[g(b) - g(a)],$$

for $\lambda < \delta$. In other words,

$$\lim_{\lambda \to 0} = I,$$

so that I is the integral

$$\int_a^b f(x)\, dg(x).$$

It follows from the theorem just proved that every function of finite variation is integrable with respect to every continuous function.

Hence; the proof is complete. $\square$

Theorem 2.7. *If the function $f(x)$ is continuous on $[a,b]$ and if the function $g(x)$ has a Riemann integrable derivative $g'(x)$ at every point $[a,b]$, then*

$$(S)\int_a^b f(x)\,dg(x) = (R)\int_a^b f(x)\,g'(x)\,dx. \tag{2.19}$$

Proof. It follows from the conditions of the theorem that $g(x)$ satisfies the Lipschitz condition and therefore is of finite variation. Hence the integral on the left side of (2.19) exists. On the other hand, the function $g'(x)$, and the product $f(x)g'(x)$ with it, are continuous almost everywhere, so that the right member of (2.19) also exists. It remains to show that the two sides of (2.19) are equal. For this purpose, subdivide $[a,b]$ by means of the points

$$x_0 = a < x_1 < ... < x_n = b,$$

and, to every difference $g(x_{k+1}) - g(x_k)$, apply the mean value theorem we get:

$$g(x_{k+1}) - g(x_k) = g'(\bar{x}_k)(x_{k+1} - x_k),$$

where, $x_k < \bar{x}_k < x_{k+1}$. In forming the sum σ for the integral

$$\int_a^b f(x)\,dg(x),$$

we may take the point x_k which appears in the mean value theorem for the point φ_k. The sum σ then becomes

$$\sigma = \sum_{k=0}^{n-1} f(\bar{x}_k)\,g'(\bar{x}_k)(x_{k+1} - x_k).$$

This is a Riemann sum for the function $f(x)g'(x)$. Refining the subdivision and taking the limit, we obtain the equation (2.19). □

Theorem 2.8. *Let $f(x)$ be a function continuous on $[a,b]$ and let $g(x)$ be constant in* *each of the intervals* $(a, c_1), (c_1, c_2), ..., (c_m, b)$, *where,*

$$a < c_1 < c_2 < ... < c_m < b.$$

Then,

$$\begin{aligned}\int_a^b f(x)\,dg(x) &= f(a)[g(a+0) - g(a)] + \sum_{k=1}^{m} f(c_k)[g(c_k+0) - g(c_k-0)] \\ &\quad + f(b)[g(b) - g(b-0)]. \qquad (2.20)\end{aligned}$$

Proof. We can easily see that

$$\begin{aligned}\bigvee_a^b(g) &= |g(a+0) - g(a)| + |g(b) - g(b-0)| \\ &\quad + \sum_{k=1}^{m}\{|g(c_k) - g(c_k-0)| + |g(c_k+0) - g(c_k)|\},\end{aligned}$$

so that the function $g(x)$ has finite variation on $[a, b]$. Therefore $g(x)$ has finite variation on every subinterval of $[a, b]$. Therefore;

$$\int_a^b f(x)\,dg(x) = \sum_{k=1}^{m}\int_{c_k}^{c_{k+1}} f(x)\,dg(x), \qquad (2.21)$$

where we write $c_0 = a$, $c_{m+1} = b$. It remains to compute the integral

$$\int_{c_k}^{c_{k+1}} f(x)\,dg(x).$$

Subdividing $[c_k, c_{k+1}]$ and forming the sum σ for the interval $[c_k, c_{k+1}]$ we obviously have

$$\sigma = f(\varphi_0)[g(c_k+0) - g(c_k)] + f(\varphi_{k+1})[g(c_{k+1}) - g(c_{k+1}-0)],$$

because all other terms vanish. In the limit, therefore,

$$\int_{c_k}^{c_{k+1}} f(x)\,dg(x) = f(c_k)\left[g(c_k+0) - g(c_k)\right] + f(c_{k+1})\left[g(c_{k+1}) - g(c_{k+1}-0)\right]. \quad (2.22)$$

The equality (2.20) follows from (2.21) and (2.22). Hence the proof is complete with this. □

CHAPTER 3

APPROXIMATION PROBLEM AND IMPORTANT INEQUALITIES

3.1 The Approximation Problem

We mentioned that one of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have ”better” properties than f. Suppose that a given function f can be shown that the limit of set of functions $\varphi_n(x)$ at the point which has good properties at x_0. Namely,

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0)$$

be as described above. In this case the series of $\varphi_n(x)$ approximates or converges to f.

The second fundamental problem of the approaching theorem is finding the rate of approach to the problem.

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0)$$

while $n \to \infty$, then the series $(\alpha_n) = (f(x_0) - \varphi_n(x_0))$ is a zero sequence. If we find an another zero sequence β_n with $\alpha_n = o(\beta_n)$, then one has

$$f(x_0) - \varphi_n(x_0) = o(\beta_n).$$

This Landau representation shows that the $f(x_0) - \varphi_n(x_0)$ is approaches to zero more rapidly than β_n, when $n \to \infty$.

3.2 Some Important Inequalities

Lemma 3.1. *(I. P. NATANSON)[3] Let a summable function $f(t)$ possessing the property,*

$$M = \sup_{0<h\leq b-a}\left\{\frac{1}{h}\left|\int_a^{a+h} f(t)\,dt\right|\right\} < \infty \tag{3.1}$$

be defined on the closed interval $[a,b]$. For any nonnegative decreasing function $g(t)$, defined and summable on $[a,b]$, the integral

$$\int_a^b f(t)\,g(t)\,dt \tag{3.2}$$

exists. Here we can specify that it might be improper for $t = a$ and then, the inequality

$$\left|\int_a^b f(t)\,g(t)\,dt\right| \leq M\int_a^b g(t)\,dt \tag{3.3}$$

is valid.

To clarify the conditions of the lemma, we note that the case where $g(a) = \infty$ is not excluded. If, however, $g(a) < +\infty$, the function $g(t)$ is bounded and the integral (3.2) exists as an ordinary Lebesgue integral.

Proof. Before starting to prove, we note that, without loss of generality, we may assume $g(b) = 0$. In fact, if this were not so, we would simply introduce the new function $g^*(t)$ defined by the equalities;

$$g^*(t) = \left\{\begin{array}{lll} g(t) & ,if & a \leq t < b \\ 0 & ,if & t = b \end{array}\right\}$$

to replace $g(t)$.

Having proved the theorem for $g^*(t)$, we could then replace $g^*(t)$ by $g(t)$ everywhere because this would have no effect on the value of the integrals involved. Thus, we shall assume $g(b) = 0$.

Let $a < \omega < b$. On the closed interval $[\omega, b]$, the function $g(t)$ is bounded and the integral

$$\int_{\omega}^{b} f(t)\, g(t)\, dt \tag{3.4}$$

certainly exists. If we set

$$F(t) = \int_{\omega}^{t} f(u)\, du,$$

integral (3.4) can be expressed in the form of a Stieltjes integral

$$\int_{\omega}^{b} f(t)\, g(t)\, dt = \int_{\omega}^{b} g(t)\, dF(t),$$

for which, after integrating by parts, we find

$$\int_{\omega}^{b} f(t)\, g(t)\, dt = -F(\omega)\, g(\omega) + \int_{\omega}^{b} F(t)\, d[-g(t)].$$

However, by (3.1), we have that

$$F(t) \leq M(t-a). \tag{3.5}$$

Indeed, we can see;

$$\frac{F(t)}{(t-a)} = \frac{1}{t-a}\left|\int_{\omega}^{t} f(u)\, du\right| \leq \sup_{t}\left\{\frac{1}{t-a}\left|\int_{\omega}^{t} f(u)\, du\right|\right\} = M.$$

Equation (3.5) is satisfies. And since $g(t)$ decreases,

$$g(\omega)(\omega-a) \leq \int_{a}^{\omega} g(t)\, dt. \tag{3.6}$$

This means that,

$$|F(\omega)\, g(\omega)| \leq M \int_{a}^{\omega} g(t)\, dt. \tag{3.7}$$

On the other hand, the function $-g(t)$ increases this and (3.5) imply that

$$\left| \int_{\omega}^{b} F(t)\, d[-g(t)] \right| \leq M \int_{\omega}^{b} (t-a)\, d[-g(t)]\,.$$

We transform the integral in the right member using the formula for integration by parts:

$$\int_{\omega}^{b} (t-a)\, d[-g(t)] = g(\omega)(\omega - a) + \int_{\omega}^{b} g(t)\, dt.$$

From this equation, and with the equation (3.6), it follows that;

$$\int_{\omega}^{b} (t-a)\, d[-g(t)] \leq \int_{a}^{\omega} g(t)\, dt + \int_{\omega}^{b} g(t)\, dt = \int_{a}^{b} g(t)\, dt.$$

Combining the above equation, we obtain

$$\left| \int_{a}^{b} f(t)\, g(t)\, dt \right| \leq M \left\{ \int_{a}^{\omega} g(t)\, dt + \int_{a}^{b} g(t)\, dt \right\}. \tag{3.8}$$

Although this inequality was established under the assumption that $g(b) = 0$, it also remains valid without this hypothesis as was already explained. This means we can replace the limit b by β, where $\omega < \beta < b$. But then, letting ω and β tend to a, we see that

$$\lim \int_{\omega}^{\beta} f(t)\, g(t)\, dt = 0,$$

which proves the existence of integral (3.2). Finally, we pass to the limit as $\omega \to a$ in (3.8), then we obtain

$$\left| \int_{a}^{b} f(t)\, g(t)\, dt \right| \leq M \int_{a}^{b} g(t)\, dt.$$

This completes the proof of the lemma. $\square$

Now we will give Roman Taberski's generalization of Natanson's Lemma.

Lemma 3.2. *Let* $\varphi(t) \in BV[a+\eta, b]$*, where* $0 < \eta < b-a$*, such that;*

$$\upsilon(s) = var_{s \le t \le b}\varphi(t)$$

since $a \le s < b$ *and* $\upsilon(b) = 0$*, then*

$$\int_a^b \upsilon(s)\,ds < \infty$$

is valid. If;

$$M = \sup_{0<h\le b-a}\left|\frac{1}{h}\int_a^{a+h} f(t)\,dt\right| < \infty, \tag{3.9}$$

where $f \in L_1[a,b]$*, then*

$$I = \int_{a^+}^b f(t)\varphi(t)\,dt, \tag{3.10}$$

indefinite Lebesgue integral is exists and,

$$|I| \le M\int_a^b [\upsilon(s) + |\varphi(b)|]\,ds \tag{3.11}$$

is valid.

Proof. Let define new functions as

$$F(t) = \int_a^t f(u)\,du,$$

where, $a \le t \le b$, and

$$I_{\gamma,\beta} = \int_\gamma^\beta f(t)\varphi(t)\,dt,$$

where, $a < \gamma < \beta \leq b$.

If we apply the formula of integration by parts in the integral $I_{\gamma,\beta}$, we obtain

$$\begin{aligned} I_{\gamma,\beta} &= \int_{\gamma}^{\beta} \varphi(t)\, dF(t) \\ &= [\varphi(t) F(t)]_{\gamma}^{\beta} - \int_{\gamma}^{\beta} F(t)\, d\varphi(t) \end{aligned}$$

and if we use our condition (3.9), we get

$$\begin{aligned} \left| \int_{\gamma}^{\beta} F(t)\, d\varphi(t) \right| &\leq M \int_{\gamma}^{\beta} (t-a)\, d[-\upsilon(t)] \\ &= M \left[[(a-t)\,\upsilon(t)]_{\gamma}^{\beta} + \int_{\gamma}^{\beta} \upsilon(t)\, dt \right]. \end{aligned}$$

Now, if we consider

$$(\gamma - a)\, |\varphi(\gamma) - \varphi(b)| \leq (\gamma - a)\, \upsilon(a) \leq \int_{\gamma}^{a} \upsilon(t)\, dt,$$

then,

$$I = \int_{a^{+}}^{b} F(t)\, d\varphi(t)$$

is exists and,

$$\left| \int_{a^{+}}^{b} F(t)\, d\varphi(t) \right| \leq M \int_{a}^{b} \upsilon(t)\, dt$$

is valid. This completes the proof. □

Lemma 3.3. *(A.D. GADJIEV)[4] Let $\mu(t)$ be an increasing and absolutely continuous function, defined on $[0, b-a]$, moreover $\mu(0) = 0$. Besides, let $\varphi(t)$ be a function of* *bounded variation in each interval of* $[a+\eta, b]$ *where* $0 < \eta < b - a$,

$$\int_a^b [\mu(t-a)]_t' \, \upsilon(t)\, dt < \infty, \tag{3.12}$$

where,

$$\upsilon(t) = var_{t \le s \le b} \varphi(s), \qquad (a \le t < b).$$

If

$$M = \sup_{0<h\le b-a} \frac{1}{\mu(h)} \left| \int_a^{a+h} f(t)\, dt \right| < \infty, \qquad f \in L_1[a,b], \tag{3.13}$$

then,

$$\left| \int_{a^+}^{b} f(t)\varphi(t)\, dt \right| \le M \int_a^b [\upsilon(t) + |\varphi(b)|]\, [\mu(t-a)]_t' \, dt. \tag{3.14}$$

Proof. Denote

$$F(t) = \int_a^t f(u)\, du, \qquad (a \le t \le b)$$

and

$$I_{\alpha,\beta} = \int_\alpha^\beta f(t)\varphi(t)\, dt, \qquad (a < \alpha < \beta \le b).$$

If we apply the formula of integration by parts, we will obtain

$$I_{\alpha,\beta} = \int_\alpha^\beta f(t)\, dF(t)\, dt = \varphi(t) F(t)|_\alpha^\beta - \int_\alpha^\beta F(t)\, d\varphi(t)\, dt.$$

If we apply (3.13), it is easy to show that

$$\int_\alpha^\beta F(t)\, d\varphi(t)\, dt \le M \left\{ -\mu(t-a)\, \upsilon(t)|_\alpha^\beta + \int_\alpha^\beta \upsilon(t)\, [\mu(t-a)]_t' \, dt \right\}.$$

In the view of equation (3.12) and since $\mu(t)$ is an absolutely continuous function

$$\mu(\alpha - a)\,\upsilon(a) \le \int_a^{\alpha} \upsilon(t)\,[\mu(t-a)]_t'\,dt$$

is true, then the integral

$$\int_{a^+}^{b} F(t)\,d\varphi(t)$$

is exist and does not exceed absolute value the expression

$$\left|\int_{a^+}^{b} F(t)\,d\varphi(t)\right| \le M \int_a^{b} \upsilon(t)\,[\mu(t-a)]_t'\,dt$$

holds. The rest is clear.

Hence, this completes the proof. $\square$

The following lemma is proved quite analogously.

Lemma 3.4. *[4] If $\psi(t)$ is function of bounded variation in each interval $\langle a, b-\eta\rangle$ such that*

$$\int_a^{b} [\mu(b-t)]_t'\,W(t)\,dt < \infty,$$

where $W(t) = var_{a\le s\le t}\psi(s)$, $(a < s \le b)$, then

$$\left|\int_a^{b^-} f(t)\,\psi(t)\,dt\right| \le -M_1 \int_a^{b} [W(t) + |\psi(a)|]\,[\mu(b-t)]_t'\,dt,$$

where

$$M_1 = \sup_{0<h\le b-a} \frac{1}{\mu(h)} \left|\int_{b-h}^{b} f(t)\,dt\right| < +\infty.$$

Remark 3.1. *In the case $\mu(t) = t$ lemmas* (3.3) *and* (3.4) *are proved by R.Taberski [3]. If $\mu(t) = t$ and $\varphi(t)$ is monotonically decreasing on (a, b) function, then the well-known I.P. Natanson's lemma* (3.1) *follows from lemma* (3.3).

CHAPTER 4

NONLINEAR SINGULAR INTEGRALS DEPENDING ON TWO PARAMETERS

4.1 Convergence in $L_p(2\pi)$ and $L_p(\mathbb{R})$

The theory of approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [5] and widely developed in [5]. He considers nonlinear integral operators, replacing linearity assumption by Lipschitz condition for kernel functions generating the operators and satisfying suitable singularity assumptions. After this discovery, several mathematicians have undertaken the program of extending approximation by nonlinear integral operators in many ways and to several settings, including modular function spaces, pointwise and uniform convergence of operators, Korovkin type theorems, abstract function spaces, sampling series and so on. Especially, this kind of operators were extensively studied by C. Bardaro, J. Musielak and G. Vinti [6]-[7] in connection with the theory of Modular space. Operators of type (1) and its special cases were studied by Swiderski and Wachnicki [8], Karsli [9]-[10] and Karsli-Ibikli [11] in some Lebesgue spaces. Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators. We also refer the reader to [14]-[17], [20]-[24] as well as the monographs [5] and [18] where other kinds of convergence results concerning the pointwise convergence and rate of pointwise convergence of linear and nonlinear singular integral operators in the Lebesgue and Musielak-Orlicz spaces.

We talked about the theory of Fourier series, in the theory of Fourier series there appears the problem of approximation of $2\pi-$ *periodic* functions f, integrable in the interval $[-\pi, \pi]$ by means of linear, singular integral operators K_n of the form

$$\int_{-\pi}^{\pi} K_n(t-s) f(t)\, dt.$$

In papers [3] Taberski has shown that the pointwise convergence at which the points are Lebesgue point of integrable functions in $L_1(-\pi, \pi)$, by the family of convolution type singular integral operators depending on two parameters of the form

$$U_\lambda(f;x) = \int_{-\pi}^{\pi} f(t) K_\lambda(t-x)\, dt \quad , \quad x \in (-\pi, \pi) \tag{4.1}$$

where $K(t, \lambda)$ is a kernel which satisfies suitable assumptions. Moreover, Taberski has shown also that in [3] approximation properties of the derivatives of this operators.

After this fundamental study of Taberski [3], these operators were studied by some other authors. Gadjiev [4] and Rydzewska [25] obtained the pointwise convergence of these operators at which the points are respectively, generalized Lebesgue point and $\mu-$generalized Lebesgue point of integrable functions in $L_1(-\pi, \pi)$. In 1984, Bardaro and Gori Cocchieri [26] estimated the degree of pointwise approximation of Féjer-type singular integrals, which are the special case of (4.1), in the class of functions $f \in L_1(R)$ at which the points are generalized Lebesgue point. Very recently Karslı and İbikli [12], [13] studied both pointwise convergence and pointwise convergence of the *rth* derivatives of the operators $T_\lambda(f;x)$, which are defined as

$$T_\lambda(f;x) = \int_{a}^{b} f(t)\, K_\lambda(t-x)\, dt \quad , \quad x \in (a, b).$$

In approximation theory, however, applications we limited to linear integral operators because the notion of singularity of an integral operators was closely connected with its linearity. In 1981, Musielak [27] used the nonlinear integral operators, which were defined as

$$(T_w f)(s) = \int\limits_G K_w(t-s, f(t))\,dt \quad , \quad s \in G. \tag{4.2}$$

We were mentioned about by the assumption of linearity of the operators was replaced by an assumption of a Lipschitz condition for K_w with respect to the second variable. This study allows us that using the classical method for linear integral operators [18] to obtain the convergence of the nonlinear integral operators, although the notion of singularity of an integral operators was closely connected with its linearity [5]. We surveyed the Swiderski's and Wachnicki's studies in this thesis and for further paper on the subject which we are examining in this thesis, were written by Karsli [28].

Now, here as we said that before in the chapter of introduction these problems are extended in three directions but in this survey, we will consider the nonlinear integral operators. That, the kernel

$$K = (K_n)_{n=1}^{\infty}$$

is replaced by a kernel

$$K = (K_w)_{w \in W}$$

where $K_w : G \times \mathbb{R} \to \mathbb{R}$; the last means that we take nonlinear integral operators T_w;

$$(T_w f)(s) = \int\limits_G K_w(t-s, f(t))\,dt = \int\limits_G K_w(t, f(t+s))\,dt,$$

in place of the considered linear operators. Now denote the family of integral operators T_w converge to the function f on the space $L_p[a, b]$.

It is important to specify that, in this chapter we consider the case $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with addition mod 2π. And we denote $B(\theta)$ the family of all neighborhoods of the neutral element θ of G. Let W be a nonempty set of indices and w_0 be an accumulation point of W. We take a family $K = (K_w)_{w \in W}$ of functions $K_w : G \times \mathbb{R} \to \mathbb{R}$, where $K_w(t, 0) = 0$ for $t \in G$, such that K_w for every $w \in W$ are integrable functions with respect to the first variable for all values of the second variable.The family K will be called kernel. In 2000, T. Swiderski and E. Wachnicki investigated the pointwise convergence of the operator $(T_w)(f)$ defined as;

$$(T_w f)(s) = \int_G K_w(s-t, f(t))\, dt \tag{4.3}$$

as $(w, s) \to (w_{0,} s_0)$, where w_0 is an accumulation point of the set of indices and s_0 an accumulation point G. If we write $t \to (s-t)$ the equation (4.3) turns into;

$$(T_w f)(s) = \int_G K_w(s-t, f(t))\, dt = \int_G K_w(t, f(s-t))\, dt. \tag{4.4}$$

In papers [1]-[2] and [29]-[30] the convergence of the integral (4.4) in some normed spaces have been studied. In the present paper we investigate the pointwise convergence of the integral (4.4) as $(w, s) \to (w_{0,} s_0)$ in the sense of the product convergence, where s_0 is an accumulation point of G. Here f belongs to $L_p(G)$, $1 \le p \le +\infty$. First we shall define a new class, which provide the existence of the integral $T_w(f)$.

Definition 4.1. We shall say that a function $K_w : G \times \mathbb{R} \to \mathbb{R}$ belongs to $Class\ A$ if the following conditions hold:

(A) There exists an integrable function $L_w : G \to \mathbb{R}$ such that;

$$|K_w(t, u) - K_w(t, v)| \le L_w(t)\, |u - v| \tag{4.5}$$

for any $w \in W, t \in G, u, v \in \mathbb{R}$.

(B) This condition is known as singularity condition and it can be written two way;

$$\lim_{w \to w_0} \frac{1}{u} \int_G K_w(t, u)\, dt = 1.$$

We can also show like this way;

$$|u| \left| \int_G L_w(t)\, dt - 1 \right| \to 0$$

for every $u \in \mathbb{R} \setminus \{0\}$.

(C) There exist a number $M > 0$ such that

$$\int_G L_w(t)\, dt \leq M$$

for all $w \in W$.

(D) For every $U \in B(\theta)$

$$\lim_{w \to w_0} \int_{G \setminus U} L_w(t)\, dt = 0.$$

.

According to equation (4.5) it is easy to see that *Class A* is a kernel and since $K_w(t, 0) = 0$, we can say that

$$|K_w(t, u)| = L_w(t)\, |u| \tag{4.6}$$

for $t \in G$, $u, v \in \mathbb{R}$.

This condition also known as assumption of linearity of the operators. Notice that we mention that the assumption of linearity for operators being replaced by an assumption of a Lipschitz condition for K_w with respect to the second variable. We need to show that the function $T_w(f)$ is exist on the same space or not? Hence, by the property of convolution [31], the following theorem holds.

Theorem 4.1. *If* $f \in L_p(G)$, $1 \leq p \leq \infty$, *then* $T_w(f) \in L_p(G)$ *and*

$$\|T_w f\|_p \leq \|L_w\|_1 \, \|f\|_p \tag{4.7}$$

for every $w \in W$.

Proof. First we need to observe that

$$
\begin{aligned}
\|T_w f\|_p &= \left\| \int_G K_w(t-x, f(t))\, dt \right\|_p \\
&= \left(\int_G \left| \int_G K_w(t-x, f(t))\, dt \right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_G \left(\int_G |K_w(t-x, f(t))|^p\, dx \right)^{\frac{1}{p}} dt \right) \\
&\leq \left(\int_G \left(\int_G (L_w(t-x)\,|f(t)|)^p\, dx \right)^{\frac{1}{p}} dt \right).
\end{aligned}
$$

Since by condition (A) and if we substitute $(t-x) \to t$, we get

$$
\begin{aligned}
&\leq \left(\int_G \left(\int_G (L_w(t)\,|f(t+x)|)^p\, dx \right)^{\frac{1}{p}} dt \right) \\
&\leq \left(\int_G \left((L_w(t))^p \int_G |f(t+x)|^p\, dx \right)^{\frac{1}{p}} dt \right) \\
&\leq \left(\int_G L_w(t) \left(\int_G |f(t+x)|^p\, dx \right)^{\frac{1}{p}} dt \right).
\end{aligned}
$$

Now if we write $t \to (t-x)$,

$$
\begin{aligned}
&\leq \left(\int_G L_w(t-x) \left(\int_G |f(t)|^p\, dx \right)^{\frac{1}{p}} dt \right) \\
&= \int_G L_w(t-x)\, \|f\|_p\, dt \\
&= \|f\|_p \int_G L_w(t-x)\, dt \\
&\leq \|f\|_p \,.\, \|L_w\|_1 \,.
\end{aligned}
$$

Therefore the proof is complete. $\square$

We guaranteed the existence of the integral $T_w f$. Now we need to prove the following theorem which contains additional condition to our *Class A*.

Theorem 4.2. *Let $U \in B(\theta)$. Denote*

$$R_q^U(w) = \begin{cases} \left(\int_{G\setminus U} (L_w(t))^q dt\right)^{\frac{1}{q}} & as \quad 1 \le q \le +\infty \\ \sup\limits_{G\setminus U} ess\, L_w(t) & as \quad q = +\infty \end{cases}$$

and let $f \in L_p(G)$, $1 \le p \le +\infty$. Assume that a kernel belongs to Class A and satisfies the following condition;

(E)

$$\lim_{w \to w_0} R_q^U(w) = 0,$$

for every $U \in B(\theta)$, where $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(T_w f)(s) \to f(s_0)$ as $(w, s) \to (w_0, s_0)$, when s_0 is an accumulation point of G at which function f is continuous.

Proof. First we know the convolution type integral operator as;

$$(T_w f)(s) = \int_G K_w(s - t, f(t))\, dt$$

If we write $t \to (t + s_0)$, it takes the form

$$(T_w f)(s) = \int_G K_w(s - t, f(t))\, dt = \int_G K_w(s - s_0 - t, f(t + s_0))\, dt$$

and

$$\int_G K_w(s - s_0 - t, f(s_0))\, dt = \int_G K_w(t, f(s_0))\, dt$$

for every $s, s_0 \in G$.

Therefore;

$$\begin{aligned}
|(T_w f)(s) - f(s_0)| &= \left| \int_G K_w(s-t, f(t))\, dt - f(s_0) \right| \\
&= \left| \begin{array}{c} \int_G K_w(s-s_0-t, f(t+s_0))\, dt - \int_G K_w(s-s_0-t, f(s_0))\, dt \\ + \int_G K_w(s-s_0-t, f(s_0))\, dt - f(s_0) \end{array} \right| \\
&\leq \left| \int_G K_w(s-s_0-t, f(t+s_0)) - K_w(s-s_0-t, f(s_0))\, dt \right| \\
&\quad + \left| \int_G K_w(t, f(s_0))\, dt - f(s_0) \right| \\
&\leq \int_G |K_w(s-s_0-t, f(t+s_0)) - K_w(s-s_0-t, f(s_0))|\, dt \\
&\quad + \left| \int_G K_w(t, f(s_0))\, dt - f(s_0) \right| \\
&= I_1(w,s) + I_2(w).
\end{aligned}$$

If we use the condition (B) for the second part of the integral, we get;

$$I_2(w) = \left| \int_G K_w(t, f(s_0))\, dt - f(s_0) \right| \to 0$$

as $w \to w_0$. Since,

$$\lim_{w \to w_0} \int_G K_w(t, f(s_0))\, dt = f(s_0).$$

Now we will prove $I_1(w,s) \to 0$ as $(w,s) \to (w_0, s_0)$.

Let $\varepsilon > 0$. Since the function f is continuous at s_0, we can find $U \in B(\theta)$ such that;

$$|f(t+s_0) - f(s_0)| < \frac{\varepsilon}{3M} \tag{4.8}$$

for $t \in U$, where M is constant occurring in assumption (C). We may obviously assume

$U = -U$. Hence by (A) we obtain;

$$
\begin{aligned}
I_1(w,s) &= \int_G |K_w(s-s_0-t, f(t+s_0) - K_w(s-s_0-t, f(s_0))|\,dt \\
&\leq \int_G L_w(s-s_0-t)\,|f(t+s_0) - f(s_0)|\,dt \\
&\leq \int_U L_w(s-s_0-t)\,|f(t+s_0) - f(s_0)|\,dt \\
&\quad + \int_{G\setminus U} L_w(s-s_0-t)\,|f(t+s_0) - f(s_0)|\,dt.
\end{aligned}
$$

Using the condition (A) and triangle inequality the integral takes the form;

$$
\begin{aligned}
&\leq \int_U L_w(s-s_0-t)\,|f(t+s_0) - f(s_0)|\,dt + \int_{G\setminus U} L_w(s-s_0-t)\,|f(t+s_0)|\,dt \\
&\quad + \int_{G\setminus U} L_w(s-s_0-t)\,|f(s_0)|\,dt \\
&= J_1(w,s) + J_2(w,s) + J_3(w,s).
\end{aligned}
$$

From using the condition (C) and the property (4.8) we can write J_1 as;

$$
\begin{aligned}
J_1(w,s) &= \int_U L_w(s-s_0-t)\,|f(t+s_0) - f(s_0)|\,dt \\
&\leq \frac{\varepsilon}{3M}\int_U L_w(s-s_0-t)dt \\
&\leq \frac{\varepsilon}{3M}\int_G L_w(t)dt \\
&\leq \frac{\varepsilon}{3}.
\end{aligned}
$$

We can choose $U \in B(\theta)$ for which there exists $V \in B(\theta)$ such that $2V \subset U$, where $2V = \{x+y \ ; \ x,y \in V\}$. As before we take $V = -V$. Let $s \in G$ and $s - s_0 \in V$, then $V + (s - s_0) \subset U$ and

$$
\begin{aligned}
J_2(w,s) &= \int_{G\setminus U} L_w(s-s_0-t)\,|f(t+s_0)|\,dt \\
&\leq \int\limits_{G\setminus V+(s-s_0)} L_w(s-s_0-t)\,|f(t+s_0)|\,dt
\end{aligned}
$$

Since ($s_0 \to s$) and $t = -t$, then;

$$\leq \int_{G\setminus V} L_w(t)\,|f\,(s-t)|\,dt.$$

And by using Hölder inequality;

$$\begin{aligned}
&\leq \left(\int_{G\setminus V} L_w^q\,(t)\,dt\right)^{\frac{1}{q}} . \left(\int_{G\setminus V} |f\,(s-t)|^p\,dt\right)^{\frac{1}{p}} \\
&\leq \left(\int_{G\setminus V} L_w^q\,(t)\,dt\right)^{\frac{1}{q}} . \left(\int_{G} |f\,(s-t)|^p\,dt\right)^{\frac{1}{p}} .
\end{aligned}$$

Since; $\left(\int_{G\setminus V} |f\,(s-t)|^p\,dt\right)^{\frac{1}{p}} \leq \left(\int_{G} |f\,(s-t)|^p\,dt\right)^{\frac{1}{p}}$ and $t \to (s-t)$,

$$\leq R_q^U\,(w)\,.\,\|f\|_p$$

Hence by (E) there exists a neighborhood of W_1 of w_0 such that

$$J_2\,(w,s) \leq \frac{\varepsilon}{3}$$

for $w \in W_1$ and $s \in V + s_0$.

Similarly, there exists a neighborhood W_2 of w_0 such that;

$$\begin{aligned}
J_3\,(w,s) &= \int_{G\setminus U} L_w(s-s_0-t)\,|f\,(s_0)|\,dt \\
&\leq \int\limits_{G\setminus V+(s-s_0)} L_w(s-s_0-t)\,|f\,(s_0)|\,dt \\
&\leq \int_{G\setminus V} L_w(t)\,|f\,(s)|\,dt \\
&\leq \left(\int_{G\setminus V} L_w^q\,(t)\,dt\right)^{\frac{1}{q}} . \left(\int_{G\setminus V} |f\,(t)|^p\,dt\right)^{\frac{1}{p}} \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

And since;

$$J_3(w,s) \leq \frac{\varepsilon}{3}$$

for $w \in W_2$ and $s \in V + s_0$.

Consequently;

$$\begin{aligned} I_1(w,s) &= J_1(w,s) + J_2(w,s) + J_3(w,s) \\ I_1(w,s) &< \varepsilon \end{aligned}$$

for $w \in W_1 \cap W_2$ and $s \in V + s_{0.}$

Therefore this finishes the proof of Theorem 2. □

Now we will expand our space again, we take $G = \mathbb{R}$ then following theorem holds.

Theorem 4.3. *Let* $f \in L_p(G)$, $1 \leq p \leq +\infty$. *Assume that a kernel K satisfies the conditions* $(A), (B), (C), (D), (E)$. *Moreover we assume that there exists* $\delta_0 > 0$ *that the function* L_w *is increasing in* $[-\delta_0, 0]$ *and decreasing* $[0, \delta_0]$ *for any* $w \in W$. *Let* $s_0 \in G$. *We assume that;*

$$\lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} |f(t+s_0) - f(s_0)| \, dt = 0. \tag{4.9}$$

Let $C > 0$. *Denote*

$$Z_C = \{(w,s) \in W \times G : |s - s_0| \; L_w(0) \leq C\}.$$

Then $T_w f(s) \to f(s_0)$ *as* $(w,s) \to (w_0, s_0)$ *and* $(w,s) \in Z_C$.

Proof. If we consider the theorem we will recognize that it will sufficient to show that;

$$\int_G L_w(s - s_0 - t) \, |f(t+s_0) - f(s_0)| \, dt \to 0$$

as $(w,s) \to (w_0, s_0)$ and $(w,s) \in Z_C$.

Let $\varepsilon > 0$. By (4.9) there exists a $\delta \in \left(0, \frac{\delta_0}{2}\right)$ such that

$$\begin{aligned} \frac{1}{h}\int_{-h}^{0} |f(t+s_0) - f(s_0)|\, dt &= \frac{\varepsilon}{M+2C}, \\ \frac{1}{h}\int_{0}^{h} |f(t+s_0) - f(s_0)|\, dt &= \frac{\varepsilon}{M+2C} \end{aligned}$$

for $0 < h \leq \delta$, where M is the constant occurring in condition (C). We have;

$$\begin{aligned} & \int_G L_w(s - s_0 - t)\,|f(t+s_0) - f(s_0)|\, dt \\ = & \int\limits_{\substack{|t|>\delta \\ t\in G}} L_w(s - s_0 - t)\,|f(t+s_0) - f(s_0)|\, dt \\ & + \int\limits_{\substack{|t|\leq\delta \\ t\in G}} L_w(s - s_0 - t)\,|f(t+s_0) - f(s_0)|\, dt \\ = & \; I_1(w,s) + I_2(w,s). \end{aligned}$$

Analogously to the proof of the Theorem (3.1), we can show that $I_1(w,s) \to 0$ as $(w,s) \to (w_0, s_0)$. We have to show $I_2(w,s)$.

$$\begin{aligned} I_2(w,s) &= \int\limits_{\substack{|t|\leq\delta \\ t\in G}} L_w(s - s_0 - t)\,|f(t+s_0) - f(s_0)|\, dt \\ &= \int_{-\delta}^{0} L_w(s - s_0 - t)\,|f(t+s_0) - f(s_0)|\, dt \\ &\quad + \int_{0}^{\delta} L_w(s - s_0 - t)\,|f(t+s_0) - f(s_0)|\, dt. \end{aligned}$$

Hence if we use Roman Taberski's generalization of the Natanson's Lemma, $I_2(w,s)$

takes the form;

$$
\begin{aligned}
I_2(w,s) &\le \frac{\varepsilon}{M+2C}\int_{-\delta}^{0}\left[\bigvee_{-\delta\le t\le z} L_w(s-s_0-t)+L_w(s-s_0-(-\delta))\right]dz \\
&\quad +\frac{\varepsilon}{M+2C}\int_{0}^{\delta}\left[\bigvee_{z\le t\le \delta} L_w(s-s_0-t)+L_w(s-s_0-(\delta))\right]dz \\
&\le \frac{\varepsilon}{M+2C}\int_{-\delta}^{0}\left[\bigvee_{-\delta\le t\le z} L_w(s-s_0-t)+L_w(s-s_0+\delta)\right]dz \\
&\quad +\frac{\varepsilon}{M+2C}\int_{0}^{\delta}\left[\bigvee_{z\le t\le \delta} L_w(s-s_0-t)+L_w(s-s_0-\delta)\right]dz.
\end{aligned}
$$

Since we know $L_w(t)$ function's properties, we need to transform $L_w(s-s_0-t)$ to the $L_w(t)$. So we need to change $t\to(s-s_0-t)$, then

$$
\begin{aligned}
&= \frac{\varepsilon}{M+2C}\left(\begin{array}{l}\int\limits_{-\delta}^{0}\left[\bigvee\limits_{s-s_0-z\le t\le s-s_0+\delta} L_w(t)+L_w(s-s_0+\delta)\right]dz \\ +\int\limits_{0}^{\delta}\left[\bigvee\limits_{s-s_0-\delta\le t\le s-s_0-z} L_w(t)+L_w(s-s_0+\delta)\right]dz\end{array}\right) \\
&= \frac{\varepsilon}{M+2C}\left(\begin{array}{l}\int\limits_{s-s_0}^{s-s_0+\delta}\left[\bigvee\limits_{z\le t\le s-s_0+\delta} L_w(t)+L_w(s-s_0+\delta)\right]dz \\ +\int\limits_{s-s_0-\delta}^{s-s_0}\left[\bigvee\limits_{s-s_0-\delta\le t\le z} L_w(t)+L_w(s-s_0+\delta)\right]dz\end{array}\right) \\
&= \frac{\varepsilon}{M+2C}\left(U_1(s,t)+U_2(s,t)\right)
\end{aligned}
$$

where

$$\bigvee_{\alpha\le t\le\beta} L_w(t),$$

denotes the total variation of L_w in the interval $[\alpha,\beta]$.

Now since we do not know that $(s-s_0)$ positive or negative we need to assume that

$0 < s - s_0 < \delta$. Then $s - s_0 - \delta > -\delta > -\delta_0$ and $s - s_0 + \delta < 2\delta < \delta_0$. Hence;

$$\begin{aligned}
U_1(s,t) &= \int_{s-s_0}^{s-s_0+\delta} \left[\bigvee_{z \le t \le s-s_0+\delta} L_w(t) + L_w(s - s_0 + \delta) \right] dz \\
&= \int_{s-s_0}^{s-s_0+\delta} \left[\left(\bigvee_{z \le t \le s-s_0+\delta} L_w(t) \right) + L_w(s - s_0 + \delta) \right] dz \\
&= \int_{s-s_0}^{s-s_0+\delta} \left(L_w(z) - L_w(s - s_0 + \delta) + L_w(s - s_0 + \delta) \right) dz \\
&= \int_{s-s_0}^{s-s_0+\delta} L_w(z)\, dz.
\end{aligned}$$

Now we will consider the second part of the integral. We know the function is decreasing on the interval $[0, \delta_0]$, so it is obvious that $L_w(z) > L_w(s - s_0 + \delta)$, and since $s - s_0 > 0$ and $(s - s_0 - \delta) < 0$ we need to split the integral into two parts;

$$\begin{aligned}
U_2(s,t) &= \int_{s-s_0-\delta}^{s-s_0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w(t) + L_w(s - s_0 + \delta) \right] dz \\
&= \int_{s-s_0-\delta}^{0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w(t) + L_w(s - s_0 + \delta) \right] dz \\
&\quad + \int_{0}^{s-s_0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w(t) + L_w(s - s_0 + \delta) \right] dz \\
&= \int_{s-s_0-\delta}^{0} \left[\left(\bigvee_{s-s_0-\delta \le t \le z} L_w(t) \right) + L_w(s - s_0 + \delta) \right] dz \\
&\quad + \int_{0}^{s-s_0} \left[\left(\bigvee_{s-s_0-\delta \le t \le z} L_w(t) \right) + L_w(s - s_0 + \delta) \right] dz.
\end{aligned}$$

Here, we can easily see that our function $L_w(s - s_0 + \delta)$ independent of variables t and

z we can split it;

$$\begin{aligned}
&= (0 - s + s_0 + \delta) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w (t) \right] dz \\
&\quad + (s - s_0 - 0) L_w (s - s_0 + \delta) + \int_{0}^{s-s_0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w (t) \right] dz \\
&= (-s + s_0 + \delta + s - s_0) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w (t) \right] dz \\
&\quad + \int_{0}^{s-s_0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w (t) \right] dz \\
&= (\delta) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w (t) \right] dz + \int_{0}^{s-s_0} \left[\bigvee_{s-s_0-\delta \le t \le z} L_w (t) \right] dz \\
&= (\delta) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} (L_w (z) - L_w (s - s_0 + \delta)) \, dz \\
&\quad + \int_{0}^{s-s_0} \left[\bigvee_{s-s_0-\delta \le t \le 0} L_w (t) + \bigvee_{0 \le t \le z} L_w (t) \right] dz \\
&= (\delta) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} (L_w (z) - L_w (s - s_0 + \delta)) \, dz \\
&\quad + \int_{0}^{s-s_0} [(L_w (0) - L_w (s - s_0 + \delta)) + (L_w (0) - L_w (z))] \, dz \\
&= (\delta) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} (L_w (z) - L_w (s - s_0 + \delta)) \, dz \\
&\quad + \int_{0}^{s-s_0} [2L_w (0) - L_w (s - s_0 + \delta) - L_w (z)] \, dz \\
&= (\delta) L_w (s - s_0 + \delta) - (s_0 - s + \delta) L_w (s - s_0 + \delta) + \int_{s-s_0-\delta}^{0} L_w (z) \, dz \\
&\quad + (s - s_0) 2L_w (0) - (s - s_0) L_w (s - s_0 + \delta) - \int_{0}^{s-s_0} L_w (z) \, dz
\end{aligned}$$

$$
\begin{aligned}
&= (\delta) L_w (s - s_0 + \delta) + (s - s_0) 2L_w (0) - (\delta) L_w (s - s_0 + \delta) \\
&\quad + \int\limits_{s-s_0-\delta}^{0} L_w (z) \, dz - \int\limits_{0}^{s-s_0} L_w (z) \, dz \\
&= 2 (s - s_0) L_w (0) + \int\limits_{s-s_0-\delta}^{0} L_w (z) \, dz - \int\limits_{0}^{s-s_0} L_w (z) \, dz \\
&\leq \int\limits_{s-s_0-\delta}^{0} L_w (z) \, dz + 2 (s - s_0) L_w (0) .
\end{aligned}
$$

Consequently by (C), we get;

$$
\begin{aligned}
I_2 (w, s) &= \int\limits_{-\delta}^{0} L_w (s - s_0 - t) \, | f (t + s_0) - f (s_0)| \, dt \\
&\quad + \int\limits_{0}^{\delta} L_w (s - s_0 - t) \, | f (t + s_0) - f (s_0)| \, dt \\
&\leq \frac{\varepsilon}{M + 2C} \left(\int\limits_{s-s_0}^{s-s_0+\delta} L_w (z) \, dz + \int\limits_{s-s_0-\delta}^{0} L_w (z) \, dz + 2 (s - s_0) L_w (0) \right) \\
&\leq \frac{\varepsilon}{M + 2C} \left(\int\limits_{s-s_0-\delta}^{s-s_0+\delta} L_w (z) \, dz + 2C \right) \\
&\leq \frac{\varepsilon}{M + 2C} \left(\int_G L_w (z) \, dz + 2C \right) \\
&\leq \frac{\varepsilon}{M + 2C} (M + 2C) = \varepsilon
\end{aligned}
$$

for $0 < s - s_0 < \delta$ and $(w, s) \in Z_C$.

We can also obtain the same inequality $-\delta < s - s_0 < 0$.

Therefore $I_2 (w, s) \to 0$ as $(w, s) \to (w_0, s_0)$, where $(w, s) \in Z_C$.

Remark 4.1. *The last theorem relates to earlier results of R. Taberski [3] and S. Siudult [19] for some linear kernels which we are defined in chapter 5, example 1.*

In the next section we study both pointwise convergence and rate of pointwise convergence on the arbitrary interval $\langle a, b \rangle$.

4.2 Convergence and Rate of Convergence in $L_1(\langle a,b\rangle)$

The purpose of this section is to study both the pointwise approximation and the rate of pointwise approximation of the operators

$$T_\lambda(f;x) = \int_a^b K_\lambda(t-x, f(t))\,dt \quad , \quad x \in < a, b >, \tag{4.10}$$

on a $\mu-$generalized Lebesgue point of $f \in L_1(\langle a,b\rangle)$ as $(x,\lambda) \to (x_0,\lambda_0)$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$.

Let $L_1(\langle a,b\rangle)$ be the class of functions Lebesgue integrable in the interval $\langle a,b\rangle$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$. Now we will define a new class of function.

Definition 4.2. We will call the function $K_\lambda : G \times \mathbb{R} \to \mathbb{R}$ belongs to *Class* β, if it satisfies the following conditions:

a) Let $L_\lambda(t)$ be an integrable function with respect to t such that;

$$|K_\lambda(t,u) - K_\lambda(t,v)| \leq L_\lambda(t)\,|u-v|\,,$$

for every $t \in G, u, v \in \mathbb{R}$, and for any $\lambda \in \Lambda$. Here while we think about $G = \mathbb{R}$ we can easily see that $t, u, v \in \mathbb{R}$.

b) $\lim\limits_{\lambda\to\lambda_0} \int\limits_{\mathbb{R}\setminus U} L_\lambda(t)dt = 0$, for every $U \in \mathcal{U}(0)$. By $\mathcal{U}(0)$ we denote the family of all neighborhoods of zero in $\mathbb{R}$.

c) $\lim_{\lambda\to\lambda_0}\left[\sup_{|t|\geq\delta} L_\lambda(t)\right] = 0$, for every $\delta > 0$.

d) $\lim_{\lambda\to\lambda_0} \int_G K_\lambda(t,u)\,dt = u$, for every $u \in \mathbb{R}$.

e)There exists a $\delta_0 > 0$, such that $L_\lambda(t)$ is non-decreasing on $(-\delta_0, 0]$ and non-increasing on $[0, \delta_0)$ as a function of t, for each $\lambda \in \Lambda$.

The condition (a) implies that the class of all functions $K_\lambda(t,u)$ is a kernel. Now we will give the following existence theorem.

Theorem 4.4. *Let $1 \leq p < \infty$ and assume that a function $K_\lambda(t,u)$ satisfies condition (a). If $f \in L_p(\langle a,b\rangle)$, then $T(f) \in L_p(\langle a,b\rangle)$ and*

$$\|T(f)\|_{L_p(\langle a,b\rangle)} \leq H(\lambda)\,\|f\|_{L_p(\langle a,b\rangle)}$$

for each $\lambda \in \Lambda$.

Firstly, we assume that $\langle a,b\rangle = [a,b]$ and let the function μ be an absolutely continuous, increasing function defined on $[0, b-a]$ with $\mu(0) = 0$.

Theorem 4.5. *Suppose that a kernel function $K_\lambda(t,u)$ belongs the Class β. Then at each point x_0 for which*

$$\lim_{h\to 0^+} \frac{1}{\mu(h)} \int_0^h |f(x_0+t) - f(x_0)|\,dt = 0, \quad 0 \leq \alpha < 1 \tag{4.11}$$

holds we have

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T_\lambda(f;x) - f(x_0)| = 0,$$

on any planar set Z on which the function

$$\int_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|t-x_0|)\,dt + 2L_\lambda(0)\mu(|x-x_0|) \tag{4.12}$$

is bounded, where $0 < \delta < \delta_0$.

Proof. Let us before to begin proof, firstly we introduce a new function $\tilde{f} \in L_1(\mathbb{R})$ as

$$\tilde{f}(t) := \begin{cases} f(t) & , \quad t \in <a,b> \\ 0 & , \quad t \notin <a,b> \end{cases}, \tag{4.13}$$

where $\langle a, b\rangle$ is an arbitrary interval in $\mathbb{R}$. We assume that

$$x_0 + \delta < b, \quad x_0 - \delta > a \text{ and } 0 < x_0 - x < \frac{\delta}{2}, \tag{4.14}$$

for any $0 < \delta < \delta_0$. Let

$$|I(x, \lambda)| := |T_\lambda(f; x) - f(x_0)|,$$

from (4.13) and the condition (d), we can write;

$$\begin{aligned} |I(x,\lambda)| &= \left| \int_a^b K_\lambda(t-x, f(t))\, dt - f(x_0) \right| \\ &= \left| \int_{\mathbb{R}} K_\lambda\left(t-x, \tilde{f}(t)\right) dt - f(x_0) \right| \\ &= \left| \begin{array}{c} \int_{\mathbb{R}} K_\lambda\left(t-x, \tilde{f}(t)\right) dt - \int_{\mathbb{R}} K_\lambda\left(t-x, \tilde{f}(x_0)\right) dt \\ + \int_{\mathbb{R}} K_\lambda\left(t-x, \tilde{f}(x_0)\right) dt - f(x_0) \end{array} \right|. \end{aligned}$$

By the triangle inequality the above equality takes the following form:

$$\begin{aligned} |I(x,\lambda)| \leq & \int_{\mathbb{R}} \left| K_\lambda(t-x, \tilde{f}(t)) - K_\lambda(t-x, \tilde{f}(x_0)) \right| dt \\ & + \left| \int_{\mathbb{R}} K_\lambda(t-x, \tilde{f}(x_0)) dt - f(x_0) \right|. \end{aligned} \tag{4.15}$$

According to the condition a),

$$\left| K_\lambda(t-x, \tilde{f}(t)) - K_\lambda(t-x, \tilde{f}(x_0)) \right| \leq L_\lambda(t-x) \left| \tilde{f}(t) - \tilde{f}(x_0) \right|. \tag{4.16}$$

holds for any $\lambda \in \Lambda$. According to (4.16) we can rewrite (4.15) as follows;

$$|I(x,\lambda)| \leq \int_{\mathbb{R}} L_\lambda(t-x)\left|\tilde{f}(t)-\tilde{f}(x_0)\right|dt + \left|\int_{\mathbb{R}} K_\lambda(t-x,\tilde{f}(x_0))dt - f(x_0)\right|.$$

For using the property of the $\mu -$ *generalized Lebesque point*, we need to split the above inequality as terms like;

$$|I(x,\lambda)| \leq \left\{ \int_{t\notin\langle a,b\rangle} + \int_{\substack{|t-x_0|>\delta,\\ t\in\langle a,b\rangle}} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} \right\} L_\lambda(t-x)\left|\tilde{f}(t)-\tilde{f}(x_0)\right|dt \tag{4.17}$$

$$+\left|\int_{\mathbb{R}} K_\lambda\left(t-x,\tilde{f}(x_0)\right)dt - f(x_0)\right|. \tag{4.18}$$

Here, we have five integrals that we can show them as;

$$|I(x,\lambda)| := I_0(x,\lambda) + I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda)\ . \tag{4.19}$$

Thus, it is sufficient to show that the first five terms on the right hand side of (4.19) tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Firstly we consider $I_4(x,\lambda)$.

From the condition (d) and using the new function $\tilde{f}(t)$, it is easy to see that;

$$I_4(x,\lambda) = \left|\int_{\mathbb{R}} K_\lambda(t-x,\tilde{f}(x_0))dt - f(x_0)\right| \to 0 \tag{4.20}$$

as $(x,\lambda) \to (x_0,\lambda_0)$.

Now we consider $I_2(x,\lambda)$.

Before going on our proof we need to define a new function.

Denote

$$F(t) := \int_{t}^{x_0} |f(y) - f(x_0)|\,dy. \tag{4.21}$$

According to (4.11), to each $\varepsilon > 0$ there exists a $\delta > 0$ such that,

$$F(t) \le \varepsilon\,\mu\,(x_0 - t), \tag{4.22}$$

for all $0 < x_0 - t \le \delta$. We now fix this δ and if we take $t = x_0 - \delta$, then the equation (4.21), takes the form

$$F(x_0 - \delta) := \int_{x_0-\delta}^{x_0} |f(y) - f(x_0)|\,dy. \tag{4.23}$$

And it is also seen that;

$$F(x_0 - \delta) \le \varepsilon\,\mu\,(\delta). \tag{4.24}$$

Now, evaluation of $I_2(x, \lambda)$ is done by the following way:

$$|I_2\,(x, \lambda)| = \int_{x_0-\delta}^{x_0} L_\lambda\,(t - x)\left|\tilde{f}\,(t) - \tilde{f}\,(x_0)\right| dt \tag{4.25}$$

We use the equation (4.23), then the equation (4.25) takes the form;

$$|I_2\,(x, \lambda)| = \int_{x_0-\delta}^{x_0} L_\lambda\,(t - x)\,d\,(-F\,(x_0 - \delta))\,.$$

Here, using the method of integration by parts, it gives;

$$\begin{aligned} |I_2\,(x, \lambda)| &= \left| -F\,(x_0 - \delta)\,L_\lambda\,(x_0 - \delta - x) + \int_{x_0-\delta}^{x_0} F\,(t)\,\frac{\partial}{\partial t} L_\lambda\,(t - x)\,dt \right| \\ &\le |-F\,(x_0 - \delta)|\,L_\lambda\,(x_0 - \delta - x) + \int_{x_0-\delta}^{x_0} |F\,(t)|\,\frac{\partial}{\partial t} L_\lambda\,(t - x)\,dt. \end{aligned}$$

Then, according to the equation (4.24), we get;

$$|I_2(x,\lambda)| \leq \varepsilon\mu(\delta) L_\lambda(x_0-\delta-x) + \varepsilon\int_{x_0-\delta}^{x_0} \mu(x_0-t)\frac{\partial}{\partial t}L_\lambda(t-x)\,dt.$$

Again using integration by parts;

$$\begin{aligned}
|I_2(x,\lambda)| &\leq \varepsilon\mu(\delta) L_\lambda(x_0-\delta-x) \\
&\quad +\varepsilon\left\{-\mu(\delta) L_\lambda(x_0-\delta-x) + \int_{x_0-\delta}^{x_0} \mu'(x_0-t) L_\lambda(t-x)\,dt\right\} \\
&= \varepsilon\mu(\delta) L_\lambda(x_0-\delta-x) - \varepsilon\mu(\delta) L_\lambda(x_0-\delta-x) \\
&\quad + \int_{x_0-\delta}^{x_0} \mu'(x_0-t) L_\lambda(t-x)\,dt \\
&= \varepsilon\int_{x_0-\delta}^{x_0} \mu'(x_0-t) L_\lambda(t-x)\,dt.
\end{aligned}$$

Then, if we substitute t with $(t+x)$, the above expression takes the form;

$$= \varepsilon\int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-t-x) L_\lambda(t)\,dt.$$

Setting,

$$I_{2,1}(x,\lambda) := \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-t-x) L_\lambda(t)\,dt,$$

according to condition e), and by (4.14) we can now obtain the following estimate:

$$\begin{aligned}
I_{2,1}(x,\lambda) &= \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-t-x) L_\lambda(t)\,dt \\
&= \int_{x_0-x-\delta}^{0} \mu'(x_0-t-x) L_\lambda(t)\,dt + \int_{0}^{x_0-x} \mu'(x_0-t-x) L_\lambda(t)\,dt.
\end{aligned}$$

Then, we can write above equation as;

$$
\begin{aligned}
I_{2,1}(x,\lambda) &= \int_{x_0-x-\delta}^{0} \left(\bigvee_{x_0-x-\delta\le s\le t} L_\lambda(s)\,\mu'(x_0-t-x) \right) dt + \int_{0}^{x_0-x} \left(\bigvee_{x_0-x-\delta\le s\le t} L_\lambda(s)\,\mu'(x_0-t-x) \right) dt \\
&\quad + L_\lambda(x_0-\delta-x) \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-t-x)\,dt \\
&= \int_{x_0-x-\delta}^{0} (L_\lambda(t) - L_\lambda(x_0-\delta-x))\mu'(x_0-x-t)\,dt \\
&\quad + \int_{0}^{x_0-x} \left\{ \left(\bigvee_{x_0-x-\delta\le s\le 0} + \bigvee_{0\le s\le t} \right) L_\lambda(s)\,\mu'(x_0-t-x) \right\} dt \\
&\quad + L_\lambda(x_0-\delta-x) \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-t-x)\,dt \\
&= \int_{x_0-x-\delta}^{0} (L_\lambda(t) - L_\lambda(x_0-\delta-x))\mu'(x_0-x-t)\,dt \\
&\quad + \int_{0}^{x_0-x} \left\{ \left(\bigvee_{x_0-x-\delta\le s\le 0} + \bigvee_{0\le s\le t} \right) L_\lambda(s)\,\mu'(x_0-t-x) \right\} dt \\
&\quad + L_\lambda(x_0-\delta-x) \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-t-x)\,dt \\
&= \int_{x_0-x-\delta}^{0} (L_\lambda(t) - L_\lambda(x_0-\delta-x))\mu'(x_0-x-t)\,dt \\
&\quad + \int_{0}^{x_0-x} \left[(L_\lambda(0) - L_\lambda(x_0-\delta-x)) + (L_\lambda(0) - L_\lambda(t)) \right] \mu'(x_0-t-x)\,dt \\
&\quad + L_\lambda(x_0-\delta-x) \int_{x_0-x}^{x_0} \mu'(x_0-t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_{x_0-x-\delta}^{0} L_\lambda(t)\,\mu'(x_0-x-t)\,dt + L_\lambda(x_0-\delta-x)\int\limits_{x_0-x}^{x_0}\mu'(x_0-t)\,dt \\
&\quad -L_\lambda(x_0-\delta-x)\int\limits_{0}^{x_0-x}\mu'(x_0-x-t)\,dt - \int\limits_{0}^{x_0-x} L_\lambda(t)\,\mu'(x_0-x-t)\,dt \\
&\quad -L_\lambda(x_0-\delta-x)\int\limits_{x_0-x-\delta}^{0}\mu'(x_0-x-t)\,dt + 2L_\lambda(0)\int\limits_{0}^{x_0-x}\mu'(x_0-x-t)\,dt \\
&= \int\limits_{x_0-x-\delta}^{0} L_\lambda(t)\,\mu'(x_0-x-t)\,dt + 2L_\lambda(0)\int\limits_{0}^{x_0-x}\mu'(x_0-x-t)\,dt \\
&\quad + \int\limits_{0}^{x_0-x} L_\lambda(t)\,\mu'(x_0-x-t)\,dt - 2\int\limits_{0}^{x_0-x} L_\lambda(t)\,\mu'(x_0-x-t)\,dt \\
&\leq \int\limits_{x_0-x-\delta}^{x_0-x} L_\lambda(t)\,\mu'(x_0-x-t)\,dt + 2L_\lambda(0)\int\limits_{0}^{x_0-x}\mu'(x_0-x-t)\,dt \\
&= \int\limits_{x_0-x-\delta}^{x_0-x} L_\lambda(t)\,\mu'(x_0-x-t)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|)\,dt.
\end{aligned}
$$

If we substitute $t \to (t-x)$, then above expression takes the form

$$
I_{2,1}(x,\lambda) = \int\limits_{x_0-\delta}^{x_0} L_\lambda(t-x)\,\mu'(x_0-t)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|)\,dt
$$

Consequently,

$$
|I_2(x,\lambda)| \leq \varepsilon\left(\int\limits_{x_0-\delta}^{x_0} L_\lambda(t-x)\,\mu'(x_0-t)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|)\right). \tag{4.26}
$$

We can use a very similar method for $I_3(x,\lambda)$. Then , we find the following inequality:

$$
|I_3(x,\lambda)| \leq \varepsilon\int_{x_0}^{x_0+\delta} L_\lambda(t-x)\,\mu'(t-x_0)\,dt. \tag{4.27}
$$

Combining (4.26) and (4.27) we obtain

$$
\begin{aligned}
|I_2(x,\lambda)| + |I_3(x,\lambda)| \quad &\leq \quad \varepsilon \int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\mu'\left(|x_0-t|\right) dt \\
&\quad +\varepsilon 2L_\lambda(0)\mu\left(|x-x_0|\right), \qquad (4.28)
\end{aligned}
$$

which in view of (4.12) tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Moreover, let us consider about $I_1(x,\lambda)$, if we use (4.13),we can write $I_1(x,\lambda)$ as follows;

$$
I_1(x,\lambda) = \int_{|t-x_0|>\delta, t\in\langle a,b\rangle} L_\lambda(t-x)\left|\tilde{f}(t) - \tilde{f}(x_0)\right| dt.
$$

Since $|t-x_0| > \delta$ is equal to $t > x_0+\delta$ and $t < x_0-\delta$ and $\tilde{f}(t)$ turns into $f(t)$ because $t \in \langle a,b\rangle$ and we can split $I_1(x,\lambda)$ into two parts as:

$$
\begin{aligned}
I_1(x,\lambda) \quad &= \quad \int\limits_{a}^{x_0-\delta} L_\lambda(t-x)\left|f(t) - f(x_0)\right| dt + \int\limits_{x_0+\delta}^{b} L_\lambda(t-x)\left|f(t) - f(x_0)\right| dt \\
: \quad &= I_{1,1}(x,\lambda) + I_{1,2}(x,\lambda).
\end{aligned}
$$

By (4.14), it is easy to see that the first part of the integral;

$$
\begin{aligned}
I_{1,1}(x,\lambda) \quad &= \quad \int\limits_{a}^{x_0-\delta} L_\lambda(t-x)\left|f(t) - f(x_0)\right| dt \\
&\leq \quad \sup_{t\in\langle a,x_0-\delta]} L_\lambda(t-x) \int\limits_{a}^{x_0-\delta} \left|f(t) - f(x_0)\right| dt.
\end{aligned}
$$

When $t \in \langle a, x_0-\delta]$, we have $t \leq x_0-\delta$ and $t \geq a$. According to (4.14), if $t \leq x_0-\delta$ then,

$$
t - x \leq x_0 - x - \delta < -\frac{\delta}{2}
$$

and hence we get $|t-x| > \frac{\delta}{2}$ and thus, we have the following inequality;

$$
I_{1,1}(x,\lambda) \leq \sup_{|u|>\frac{\delta}{2}} L_\lambda(u)\left(\|f\|_{L_1\langle a,b\rangle} + |f(x_0)|(b-a)\right).
$$

In the same way , we obtain ;

$$I_{1,2}(x,\lambda) \leq \sup_{|u|>\frac{\delta}{2}} L_\lambda(u)\left(\|f\|_{L_1\langle a,b\rangle} + |f(x_0)|(b-a)\right).$$

Thus, if we combine last two inequality we get;

$$I_1(x,\lambda) \leq 2\sup_{|u|>\frac{\delta}{2}} L_\lambda(u)\left(\|f\|_{L_1\langle a,b\rangle} + |f(x_0)|(b-a)\right) \tag{4.29}$$

which in view of the condition (c) tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Finally, let us consider the integral $I_0(x,\lambda)$. According to (4.13) we write;

$$I_0(x,\lambda) = \int\limits_{t\notin\langle a,b\rangle} L_\lambda(t-x)\left|\tilde{f}(t) - \tilde{f}(x_0)\right| dt.$$

Since $\tilde{f}(t) = 0$ when $t \notin \langle a,b\rangle$,

$$\begin{aligned} I_0(x,\lambda) &= \int_{t\notin\langle a,b\rangle} L_\lambda(t-x)|f(x_0)|\,dt \\ &= |f(x_0)|\int_{t\notin\langle a,b\rangle} L_\lambda(t-x)\,dt. \end{aligned}$$

Thus, according to (4.14), when $t \notin \langle a,b\rangle$, we have $t<a$ or $t>b$. If $t<a$, then

$$t-x < a-x < x_0 - x - \delta < -\frac{\delta}{2} < 0,$$

while if $t>b$, then

$$t-x > b-x < x_0 - x + \delta < \delta > 0.$$

This implies that, for any $\delta > 0$ there exists a neighborhood U of zero such that;

$$\begin{aligned} \int\limits_{t\notin\langle a,b\rangle} L_\lambda(t-x)\,dt &\leq \int\limits_{|t-x|>\frac{\delta}{2}} L_\lambda(t-x)\,dt \\ &= \int\limits_{\mathbb{R}\setminus U} L_\lambda(u)\,du \end{aligned}$$

Thus, we obtain the result of

$$I_0(x,\lambda) \leq |f(x_0)| \int\limits_{\mathbb{R}\setminus U} L_\lambda(u)\,du \tag{4.30}$$

which in view of the condition b) tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Therefore, if we combine our part of integral's result as (4.28), (4.29) and (4.30) we get the following inequality:

$$\begin{aligned} I(x,\lambda) \leq\ & |f(x_0)| \int\limits_{\mathbb{R}\setminus U} L_\lambda(u)\,du + 2 \sup_{|u|>\frac{\delta}{2}} L_\lambda(u)\left(\|f\|_{L_1\langle a,b\rangle} + |f(x_0)|(b-a)\right) \\ & +\varepsilon\left(\int\limits_{x_0-\delta}^{x_0} L_\lambda(t-x)\,\mu'(x_0-t)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|)\,dt\right) \\ & +\varepsilon\int\limits_{x_0}^{x_0+\delta} L_\lambda(t-x)\,\mu'(t-x_0)\,dt + \left|\int\limits_{\mathbb{R}} K_\lambda\left(t-x,\tilde{f}(x_0)\right)dt - f(x_0)\right| \end{aligned}$$

and hence,

$$\begin{aligned} I(x,\lambda) \leq\ & |f(x_0)| \int\limits_{\mathbb{R}\setminus U} L_\lambda(u)\,du + 2 \sup_{|u|>\frac{\delta}{2}} L_\lambda(u)\left(\|f\|_{L_1\langle a,b\rangle} + |f(x_0)|(b-a)\right) \\ & +\varepsilon\left(\int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|x_0-t|)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|)\,dt\right) \\ & +\left|\int\limits_{\mathbb{R}} K_\lambda\left(t-x,\tilde{f}(x_0)\right)dt - f(x_0)\right|. \end{aligned}$$

Finally, our theorem now follows, in view of conditions of (b), (c), (4.13) and (4.14) i.e,

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} T_\lambda(f;x) = f(x_0).$$

Hence, this completes the proof of our theorem.

□

If we consider the case $\langle a,b\rangle = \mathbb{R}$, then we will have the following theorem which we prove special case in above theorem.

Theorem 4.6. *Suppose that the hypothesis of Theorem (4.5) is satisfied. Then at each point x_0 for which*

$$\lim_{h\to 0^+}\frac{1}{\mu(h)}\int_0^h |f(x_0+t)-f(x_0)|\,dt=0,\quad 0\le\alpha<1 \tag{4.31}$$

holds, we have for $f\in L_1(R)$,

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T_\lambda(f;x)-f(x_0)|=0.$$

Proof. The proof of this theorem contains analogously the same methods and inequalities for Theorem (4.5). □

Now, we will estimate the rate of pointwise convergence.

Theorem 4.7. *Suppose that the hypothesis of Theorem (4.5) are satisfied. Setting*

$$\Delta(\mu,x,\lambda,\delta):=\int_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|t-x_0|)\,dt,$$

where $0<\delta<\delta_0$, let the following assumptions be satisfied:

- *For every $|u|>0$*

$$L_\lambda(u)=o(\Delta(\mu,x,\lambda,\delta))$$

  as $(x,\lambda)\to(x_0,\lambda_0)$,

- *As $(x,\lambda)\to(x_0,\lambda_0)$,*

$$\left|\int_{\mathbb{R}} K_\lambda\left(t-x,\tilde{f}(x_0)\right)dt-f(x_0)\right|=o(\Delta(\mu,x,\lambda,\delta)).$$

- *As* $(x,\lambda)\to(x_0,\lambda_0)$,

$$\int\limits_{\mathbb{R}\setminus U} L_\lambda(u)\,du = o\left(\Delta(\mu,x,\lambda,\delta)\right).$$

Then at each point x_0 for which is called μ – generalized Lebesgue point holds, we have for $f\in L_1\langle a,b\rangle$ as $(x,\lambda)\to(x_0,\lambda_0)$,

$$|T_\lambda(f;x)-f(x_0)| = o\left(\Delta(\mu,x,\lambda,\delta)\right).$$

Proof. Suppose that (4.14) is satisfied for any $0<\delta<\delta_0$. Setting;

$$|I(x,\lambda)| := |T_\lambda(f;x)-f(x_0)|.$$

According to Theorem (4.5), we get the following inequality:

$$\begin{aligned} I(x,\lambda) \quad \le \quad & |f(x_0)|\int\limits_{\mathbb{R}\setminus U} L_\lambda(u)\,du + 2\sup_{|u|>\frac{\delta}{2}} L_\lambda(u)\left(\|f\|_{L_1\langle a,b\rangle}+|f(x_0)|(b-a)\right) \\ & +\varepsilon\left(\int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|x_0-t|)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|)\,dt\right) \\ & +\left|\int\limits_{\mathbb{R}} K_\lambda\left(t-x,\tilde{f}(x_0)\right)dt - f(x_0)\right|. \end{aligned}$$

From our $Class\,\beta$ conditions (a) to (c) we obtained the desired result, i.e,

$$|T_\lambda(f;x)-f(x_0)| = o\left(\Delta(\mu,x,\lambda,\delta)\right),$$

as $(x,\lambda)\to(x_0,\lambda_0)$.

Since the function μ is an absolutely continuous function, then it is clear that

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\left(\int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|t-x_0|)\,dt\right) = 0.$$

Thus, the proof of the theorem is complete. □

CHAPTER 5

EXAMPLES

In this chapter we inspect some examples, which we can be use in the previous chapters.

Example 5.1. A well-known example of a Kernel satisfying the *Class A* conditions, is the linear kernel with respect to the second variable, i.e.,

$$K_w(t,u) = L_w(t)\, u \ ,$$

where L_w satisfies the conditions $(C), (D)$ and moreover;

$$\lim_{w\to w_0} \int_G L_w(t)\, dt = 1.$$

This special case leads to known singular integrals of the convolution type, widely used in the Approximation Theory [18], such as Poisson, Fejer etc.

Example 5.2. Let us take $G = [-\pi, \pi)$ with the operation of addition mod 2π. In G we consider the natural topology and the Lebesque Integral. Let $u \in \mathbb{R}$. We define;

$$K_n(t,u) = \begin{cases} \frac{nu}{2} + \sin\frac{nu}{2} & \text{if} \quad t \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & \text{if} \quad t \in [-\pi,\pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases} .$$

for $n \in \mathbb{N}$. In this case the set of indices W is equal to $\mathbb{N}$ and $w_0 = +\infty$.

Let $u, v \in \mathbb{R}$. While $t \in \left[-\frac{1}{n}, \frac{1}{n}\right]$, we observe that;

$$\begin{aligned} |K_n(t,u) - K_n(t,v)| &= \left|(\frac{nu}{2} + \sin\frac{nu}{2}) - (\frac{nv}{2} + \sin\frac{nv}{2})\right| \\ &= \left|(\frac{nu}{2} - \frac{nv}{2}) + (\sin\frac{nu}{2} - \sin\frac{nv}{2})\right| \end{aligned}$$

By using the Mean Value Theorem;

$$\begin{aligned} &= \left|\frac{nu}{2} - \frac{nv}{2} + |\cos c| \,. \left|\frac{nu}{2} - \frac{nv}{2}\right|\right| \\ &\leq \left|\frac{n(u-v)}{2} + \frac{n(u-v)}{2}\right| \\ &\leq n\,|u-v| \end{aligned}$$

where $c \in [-\frac{1}{n}, \frac{1}{n}]$ and since $|\cos c| \leq 1$,

and; if $t \in [-\pi, \pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]$,

$$|K_n(t,u) - K_n(t,v)| = 0,$$

since $K_n\,(t,u) = 0$ while $t \in [-\pi, \pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]$.

Hence,

$$|K_n(t,u) - K_n(t,v)| \leq L_w(t)\,|u-v|\,.$$

So we can shown L_w as;

$$L_n(t) = \begin{cases} n & \text{if} \quad t \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & \text{if} \quad t \in [-\pi, \pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}$$

We already seen that condition (A) is satisfied and we can easily check that the conditions (B), (C) and (D) are satisfied, too.

Let's begin to check the condition (B):

$$\begin{aligned} \frac{1}{u}\int_G K_n(t,u)dt &= \frac{1}{u}\int_G (\frac{nu}{2} + \sin\frac{nu}{2})dt \\ &= 1 + \frac{2}{un}.\sin\frac{nu}{2} \to 1 \end{aligned}$$

while $n \to \infty$.

The condition (C):

$$
\begin{aligned}
\int_G L_n(t)dt &= \int_{-\pi}^{\pi} L_n(t)dt \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} ndt + \int_{t\in[-\pi,\pi)\setminus\left[-\frac{1}{n},\frac{1}{n}\right]} 0dt \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} ndt + 0 \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} ndt = n.\left[\frac{1}{n} - \left(-\frac{1}{n}\right)\right] \\
&= n.\frac{2}{n} = 2 < \varepsilon
\end{aligned}
$$

So, we can see that it is bounded.

The condition (D):

This condition is obvious to see that the condition tell us,

$$
\lim_{w\to w_0} \int_{G\setminus U} L_w(t)\,dt = 0.
$$

And our function L is already $L_n(t) = 0$ while $t \in [-\pi, \pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]$.

Example 5.3. The other example is a kernel K_w, which belongs to *Class A* the kernel defined by

$$
K_n(t,u) = \begin{cases} n^2 \sin\frac{u}{2n} & if \quad t \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & if \quad t \in [-\pi,\pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}.
$$

Here $G = [-\pi, \pi)$ is considered with the operation of addition mod 2π and the set of indices W is equal to $\mathbb{N}$, $w_0 = +\infty$.

Let $u, v \in \mathbb{R}$. Then

$$
\begin{aligned}
|K_n(t,u) - K_n(t,v)| &= \left| n^2 \sin\frac{u}{2n} - n^2 \sin\frac{v}{2n} \right| \\
&= \left| n^2 \left(\sin\frac{u}{2n} - \sin\frac{v}{2n} \right) \right|.
\end{aligned}
$$

By using the Mean Value Theorem,

$$\begin{aligned} &\leq n^2.\,|\cos c|\,.\left|\frac{u}{2n}-\frac{v}{2n}\right| \\ &\leq n^2.\frac{|u-v|}{2n} \\ &= \frac{n}{2}\,|u-v|\,, \end{aligned}$$

$c \in \left[-\frac{1}{n}, \frac{1}{n}\right]$ So;

$$\begin{aligned} |K_n\,(t,u) - K_n(t,v)| &\leq \frac{n}{2}\,|u-v| \\ &= L_n\,(t)\,|u-v| \end{aligned}$$

while $t \in \left[-\frac{1}{n}, \frac{1}{n}\right]$.

It is clear that;

$$|K_n(t,u) - K_n(t,v)| = 0.$$

Because $K_n(t,u) = 0$ while $t \in [-\pi, \pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]$.

Hence, we can write $L_n(t)$ as;

$$L_n(t) = \begin{cases} \frac{n}{2} & if \quad t \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & if \quad t \in [-\pi, \pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}$$

Moreover we can see that the conditions are satisfied:

Let's begin to check the condition (B):

$$\begin{aligned} \frac{1}{u}\int_G K_n(t,u)dt &= \frac{1}{u}\int_{-\frac{1}{n}}^{\frac{1}{n}} n^2.\sin\frac{u}{2n}dt \\ &= \frac{1}{u}.n^2\sin\frac{u}{2n}.\frac{2}{n} \\ &= \frac{2n}{u}.\sin\frac{u}{2n} \to 1 \end{aligned}$$

and the condition (C):

$$\begin{aligned}
\int_G L_n(t)dt &= \int_{-\pi}^{\pi} L_n(t)dt \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}dt + \int_{t\in[-\pi,\pi)\setminus\left[-\frac{1}{n},\frac{1}{n}\right]} 0dt \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}dt = \frac{n}{2}.\left[\frac{1}{n} - \left(-\frac{1}{n}\right)\right] \\
&= \frac{n}{2}.\frac{2}{n} = 1 < \varepsilon
\end{aligned}$$

and finally, we can easily see that the condition (D) is obviously true that we know, $L_n(t) = 0$ where $t \in [-\pi,\pi) \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]$.

Example 5.4. In this example we think $G = \mathbb{R}$. We introduce the function

$$K_\lambda(t,u) = \begin{cases} \lambda u + \sin\frac{\lambda u}{2}, & t \in \left[0, \frac{1}{\lambda}\right] \\ 0, & t \notin \left[0, \frac{1}{\lambda}\right] \end{cases} . \tag{5.1}$$

In the view of our new function (5.1), it is seen that for every $u, v \in R$, and $t \in \left[0, \frac{1}{\lambda}\right]$,

$$|K_\lambda(t,u) - K_\lambda(t,v)| \le \lambda\,|u - v|\,. \tag{5.2}$$

Since we know $K_\lambda(t,0) = 0$. It is easy to see for every $u, v \in R$, and $t \notin \left[0, \frac{1}{\lambda}\right]$,

$$|K_\lambda(t,u) - K_\lambda(t,v)| = 0. \tag{5.3}$$

Combining (5.2) and (5.3) we find

$$L_\lambda(t) = \begin{cases} \lambda, & t \in \left[0, \frac{1}{\lambda}\right] \\ 0, & t \notin \left[0, \frac{1}{\lambda}\right] \end{cases} ,$$

and moreover we have

$$\int_{\mathbb{R}} L_\lambda(t)\,dt = \int_{[0,\frac{1}{\lambda}]} \lambda dt \tag{5.4}$$
$$= \lambda.\left(\frac{1}{\lambda}-0\right) = 1 < \infty. \tag{5.5}$$

In addition, by (5.5), $L_\lambda(t)$ is non-decreasing on $(-\infty,0]$, and non-increasing on $[0,\infty)$, as a function of t, for each λ. Furthermore,

$$\begin{aligned}\int_G K_\lambda(t-x,u)\,dt &= \int_{[0,\frac{1}{\lambda}]} \left[\lambda u + \sin\frac{\lambda u}{2}\right] dt \\ &= \left[\lambda u + \sin\frac{\lambda u}{2}\right].\frac{1}{\lambda} \\ &= u + \frac{1}{\lambda}\sin\frac{\lambda u}{2} \to u\end{aligned}$$

as $(x,\lambda)\to(x_0,\infty)$. This implies that the kernel function $K_\lambda(t,u)$ satisfies the whole conditions which we gave in Theorem (4.5).

If we use (5.1) in Theorem (4.5) and Theorem (4.7) , we find;

$$\begin{aligned}&\int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|t-x_0|)\,dt + 2L_\lambda(0)\,\mu(|x-x_0|) \\ = &\int\limits_{[0,1/\lambda]} \lambda dt + 2\lambda\,|x-x_0|\,.\end{aligned}$$

The first part of the right-hand side of this equality is finite. Besides;

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \lambda\,|x-x_0| = M < \infty,$$

if and only if the rates of convergence of $\lambda\to\infty$ and $x\to x_0$ are equivalent.

If we use (5.1) in Theorem (4.5) and Theorem (4.7), we obtain

$$\Delta(\mu,x,\lambda,\delta) = \int_{x_0-\delta}^{x_0+\delta} L_\lambda(t-x)\,\mu'(|t-x_0|)\,dt.$$

From the previous expression we obtain

$$\lim_{(x,\lambda)\to(x_0,\infty)} \Delta(\mu, x, \lambda, \delta) = 0$$

and the rate of convergence of $\Delta(\mu, x, \lambda, \delta) \to 0$ is equivalent to $\frac{1}{\lambda} \to 0$.

From the property of $L_\lambda(t)$ function, it is seen that $L_\lambda(u) = o\left(\frac{1}{\lambda}\right)$. Since

$$\int_G K_\lambda(t-x,u)\,dt = u + \frac{1}{\lambda}\sin\frac{\lambda u}{2} \to u \text{ as } (x,\lambda) \to (x_0,\infty).$$

It is easy to observe that;

$$\frac{1}{\lambda}\sin\frac{\lambda u}{2} = o\left(\frac{1}{\lambda^a}\right) \qquad (0 < a < 1),$$

In this case, we obviously have

$$|T_\lambda(f;x) - f(x_0)| = o(\Delta(\mu, x, \lambda, \delta)) = o\left(\frac{1}{\lambda}\right).$$

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