PSEUDOSPECTRA OF LINEAR OPERATORS

by

FATMA KAYILI

THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE ABANT İZZET BAYSAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS

APRIL 2009

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Nihat Çelebi

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Zafer Ercan

Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Yusuf Cesur

Supervisor

Examining Committee Members

Assoc. Prof. Dr. Cabir Terzioğlu

Asst. Prof. Dr. Ali Öztürk

Asst. Prof. Dr. Yusuf Cesur

ABSTRACT

PSEUDOSPECTRA OF LINEAR OPERATORS

Kayılı, Fatma

M.Sc., Department of Mathematics

Supervisor: Asst. Prof. Dr. Yusuf Cesur

April 2009, 47 pages

In this study, we investigate the eigenvalues of a non-hermitian Toeplitz matrix A. These are usually highly sensitive to perturbations, having condition numbers that increase exponentially with the dimension N. An equivalent statement is that the resolvent $(zI - A)^{-1}$ of a Toeplitz matrix may be much larger in norm than the eigenvalues alone would suggest-exponentially large as a function of N, even when z is far from the spectrum. Because of these facts, the meaningfulness of the eigenvalues of non-hermitian Toeplitz matrices for any but the most theoretical purposes should be considered suspect. In many applications it is more meaningful to investigate the $\varepsilon - pseudoeigenvalues$: the complex numbers z with $\|(zI - A)^{-1}\| \geq \varepsilon^{-1}$.

In the second part of study we investigate the pseudospectra of Linear Operators and analyzes the pseudospectra of Toeplitz matrices, and in particular relates them to the symbols of the matrices.

In the third part of study we investigate our results for reasonably complete in block Toeplitz matrices with smoothly varying coefficients.

This study presents computed examples of pseudospectrum for the varying coefficients block Toeplitz matrices with using Mathematica and Matlab programming, and applications in numerical analysis.

Keywords: Eigenvalue, eigenvector, pseudoeigenvector, spectrum, pseudospectrum, Block Toeplitz matrix, symbol.

ÖZET

LİNEER OPERATÖRLERİN PSÖDOSPEKTRUMU

Kayılı, Fatma

Master Tezi, Matematik Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Yusuf Cesur

Nisan 2009, 47 sayfa

Bu çalışmada, hermitian olmayan bir A Toeplitz matrisinin özdeğerlerini araştırıyoruz. Genellikle bunlar küçük ötelemelere karşı aşırı duyarlıdırlar ve koşul sayıları N boyutuna bağlı olarak üstel olarak artmaktadır. Buna eşdeğer bir ifade ise bir Toeplitz matrisine ait özdeğerleri $(zI - A)^{-1}$ rezolvent fonksiyonu ile incelenebilmesidir. Rezolventin, spektruma ait z noktalarına çok yakın noktalarda N'e bağlı olarak üstel olarak hızla norm değerleri artar. Bu nedenle hermitian olmayan Toeplitz matrislerine ait birçok teorik çalışmada rezolventten yararlanılarak çalışılır. Bu bakış altında birçok uygulamada ε-ötelenmiş özdeğerleri araştırma işlemlerinde rezolventten yararlanılır. Kısaca bir z kompleks sayısında $\|(zI - A)^{-1}\| \geq \varepsilon^{-1}$ yardımıyla ε-ötelenmiş A matrisinin özdeğerleri incelenebilmektedir.

Çalışmanın ikinci kısmında ise Lineer Operatörlerin pseudospektrumu araştırılmaktadır. Burada ayrıca Toeplitz matrislerin spektrumlarına ait özellikleri semboller yardımıyla da açıklanmaktadır.

Çalışmanın üçüncü bölümünde, elde ettiğimiz sonuçlar Blok Toeplitz matrislerin değişken katsayılı durumları için de araştırıldı.

Bu çalışmada Blok Toeplitz matrislerinin pseudospektrumu ile ilgili örneklerin araştırılmasında Mathematica ve Matlab programlarından ve nümerik analiz tekniklerinden yararlanılmıştır.

Anahtar Kelimeler: Özdeğer, özvektör, ötelenmiş özdeğer, spectrum, psödospektrum, Block Toeplitz matris, sembol.

ACKNOWLEDGEMENTS

I would like to acknowledge my indebtedness to my supervisor Assistant Prof. Dr. Yusuf Cesur who has guided, encouraged and believed in me throughout this thesis. I am grateful to him for a critical reading of my studies and valuable suggestions in correcting my errors. I also like to thank to the other members of my Committee, Associate Prof. Dr. Cabir Terzioğlu and Assistant Prof. Dr. Ali Öztürk for their assistance. It is also a pleasure to acknowledge a special dept of my gratitude to all the professors who gave me help in my master studies.

Finally, I would like to thank to my family for their un-vanishing support and encouragement.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 History of Pseudospectra

In this section we attempt to survey the history of pseudospectra. Many of nonhermitian matrices that arise in applications have eigenvalues that are highly sensitive to perturbations. Indeed, for a family of matrices with variable dimension N, the sensitivity of the eigenvalues often increases exponentially as $N \to \infty$. Highly sensitive eigenvalues are a reflection of the fact that the basis of eigenvectors is highly ill conditioned, and when this is the case, it is unlikely that there is good reason for working with the basis. Yet any statement about eigenvalues depends ultimately on the properties of the associated eigenvectors, whether or not they are mentioned explicitly.

We believe that in problems like this, a fruitful alternative is the analysis of *pseudo-eigenvalues*. In particular, the purpose of this thesis is to investigate the pseudospectra of various kinds of Toeplitz matrices, a special family of matrices with broad applications of integral equations, finite-difference equations, matrix iterations, spline approximation, signal processing, and other problems. Let A be a real or complex square matrix of dimension N, and let $\|.\|$ denote the 2-norm. The definition of pseudo-eigenvalues is as follows:

Definition 1.1 *Given $\varepsilon > 0$, the number $\lambda \in \mathbb{C}$ is an ε* -pseudo-eigenvalue *of A if any of the following equivalent conditions is satisfied:*

(i) λ *is an eigenvalue of* $A+E$ *for some* $E \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ *with* $\|E\| \leqslant \varepsilon$ *;*

(ii) $\exists u \in \mathbb{C}^{\mathbb{N}}$*,* $\|u\| = 1$ *such that* $\|(A-\lambda I)u\| \leq \varepsilon$*;*

(iii) $\|(A-\lambda I)^{-1}\| \geq \varepsilon^{-1}$*;*

(iv) $\sigma_N(\lambda I - A) \leq \varepsilon$*.*

The set of all ε -pseudo-eigenvalues of A, the $\varepsilon - pseudospectrum$, is denoted by $\sigma_\varepsilon(A)$ or simply σ_ε.

A pseudo-eigenvalue, in other words, need not be near to any exact eigenvalue, but it is an exact eigenvalue of some nearby matrix.

In the context of Hermittian and near- Hermittian systems, mathematical physicists have for years spoken of *quasimodes*, which are the same as what we would call *pseudomodes* or *pseudoeigenvectors*.

In the nonnormal context, the earliest definition of pseudospecta is given in J.M. Varah's 1967 thesis at Stanford University, *The Computation of Bounds for the Invariant Subspaces of a General Matrix Operator* [1].

Varah introduced the notion of an ε- pseudoeigenvalue under the name r-approximate eigenvalue. Motivated by his analysis of the accuracy of eigenpairs produced by computer implementations of the inverse iteration algorithm, Varah incorporated a parameter, η_1, in his definition to describe floating - point precision.

Definition 1.2 λ *is an* ε*-approximate eigenvalue of* A_r *if there exists* $\varphi \in P(rQ)$*, with* $\|\varphi\| = 1$*, such that* $\|A_r\varphi - \lambda\varphi\| \leq \varepsilon$ *. We call* φ *an* ε *-approximate eigenfunction corresponding to* λ *.*

Definition of pseudospectra from Landau, 1975 [2].

In Novosibirsk, S. K. Godunov and his colleagues conducted research related to pseudospectra throughout the 1980s. Those involved included A. G. Antonov, A.

Y. Bulgakov (later Haydar Bulgak, O. P. Kirilyuk, V. I Kostin, A. N. Malyshev, and S. I Razzakov. Gudunov, together with Ryabenkii and others, had made significant contributions in the 1960s to the study of how nonnormality affects the numerical stability of discretized differential equations. In fact, in their 1962 monograph, Godunov and Ryabenkii introduce the spectrum of the family of operators $\{R_h\}$, indexed by a mesh parameter h [3]. A point $z \in \mathbb{C}$ is in this set if, for every $\varepsilon > 0$, z is an ε-pseudoeigenvalue of R_h for all sufficiently small h.

The Novosibirsk group defined the *ε-spectrum* by relative rather than absolute perturbations :

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \left\|(z-A)^{-1}\right\| \geq (\varepsilon \|A\|)^{-1}\}$$

In [4] and [5], computed plots of pseudospecra were presented and called 'spectral portraits of matrices' containing various 'patches of spectrum', and computations based on both contour plotting and curve tracing were discussed. A sketch of two pseudospectra had appeared earlier in a 1985 paper by Kostin and Razzakov [6].

Pseudospectra were investigated in several papers by J. W. Demmel in the mid-1980s, appearing with the labels $S(A, \varepsilon)$ in [7] and $\sigma(\varepsilon, A)$ in [8]. The former paper contains the first published computer plot.

Beginning in the mid-1980s, D. Hinrichsen and A. J. Pritchard wrote a number of papers on the *stability radius* of a matrix, i.e., the distance to the set of unstable matrices. In a 1992 paper [9] they introduced the term *spectral value* set and notation $\sigma(A, \rho)$ to denote the real structured ε- pseudospectrum of a nonnormal matrix. Another paper by Hinrichsen and Kelb in 1993 extended these ideas to complex perturbations [10].

Trefethen's first publications that mentioned pseudospectra were [11] and [12] in 1990 (the first uses the term ε-approximate eigenvalues). These were an outgrowth of a 1987 paper by Trefethen and Trummer, who found eigenvalues that were extraordinary sensitive to perturbations but failed fully to appreciate their significance [13]. A crucial collaborator in this work with pseudospectra in 1988 and made many contributions after that date. Reddy's early work on these topics was summarized in his 1991 thesis at MIT [14].

In 1992 Trefethen's paper 'Pseudospectra of Matrices' appeared, which presented the idea of pseudospectra and exhibited thirteen examples [15]. It was after this point that the idea began to be widely known. A later companion paper, ' Pseudospectra of Linear Operators', presented ten examples involving operators on infinite-dimensional spaces [16].

The papers by Demmel and Wilkinson mentioned above cite each other, and they both cite the paper of Varah [17]. None of the papers discussed here published before 1990 cite any of the others. These data suggest that pseudospectra have been invented at least five times:

J. M. Varah 1967 r-approximate eigenvalues, 1979 ε-spectrum

H. J. Landau 1975 ε-approximate eigenvalues

S. K. Godunov et al 1982 spectral portraid

L. N. Trefethen 1990 ε-pseudospectrum

D. Hinrichsen and A. J. Pritchard 1992 spectral value set.

1.2 Basic Definitions

Definition 1.3 Eigenvalue, Eigenvectors, Spectrum and Resolvent set of a matrix in finite dimensional spaces:

An eigenvalue of a square matrix $A = (a_{ij})$ is a number λ such that

$$Av = \lambda v$$

has a solution $v \neq 0$. This v is called an eigenvector of A corresponding to that eigenvalue λ. The eigenvectors corresponding to that eigenvalue λ and the zero vector form a vector subspace of X which is called the eigenspace of A corresponding to that eigenvalue λ.

The set $\sigma(A)$ of all eigenvalues of A is called the spectrum of A. Its complement $\rho(A) = \mathbb{C} - \sigma(A)$ in the complex plane is called resolvent set of A.

Definition 1.4 Regular value, Resolvent set and Spectrum of a matrix in infinite dimensional spaces:

Let $X \neq \{0\}$ be a complex normed space and $A : D(A) \to X$ a linear operator with domain $D(A) \subset X$. A regular value z of A is a complex number such that;

(R1): $R_z(A) = (zI - A)^{-1}$ exists.

(R2): $R_z(A)$ is bounded.

(R3): $R_z(A)$ is defined on a set which is dense in X.

The resolvent set $\rho(A)$ of A is the set of all regular values z of A.

It's complement $\sigma(A) = \mathbb{C} - \rho(A)$ in the complex plane is called the spectrum of operator A, and a $z \in \sigma(A)$ is called spectral value of A.

Point Spectrum (σ_p) : *The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_z(A)$ doesn't exist. A $z \in \sigma_p(A)$ is called an eigenvalue of A.*

Continuous Spectrum (σ_c) : *The continuous spectrum $\sigma_c(A)$ is a set such that $R_z(A)$ exists and satisfies (R3) but are not (R2), that is , $R_z(A)$ is bounded.*

Residual Spectrum (σ_r) : *The residual spectrum $\sigma_r(A)$ is the set such that $R_z(A)$ exists (and may be bounded or not) but does not satisfy (R3) that is the domain of $R_z(A)$ is not dense on X.*

If X is infinite dimensional, then A can have spectral values which are not eigenvalues.

Definition 1.5 *The Euclidean norm of a vector $x = [x_1, x_2, ..., x_n]$ is :*

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Definition 1.6 *The Euclidean norm of a matrix A_{mxn} is :*

$$\|\mathrm{A}\|_2 = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

where a_{ij} is the ith row and jth column of the matrix A.

Definition 1.7 Normed Space, Banach Space :

A normed space X is a vector space with a norm defined on it. A Banach space is a complete normed space. Here a norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties;

$(N1)$ $\|x\| \geqslant 0$

$(N2)$ $\|x\| = 0 \Leftrightarrow x = 0$

$(N3)$ $\|\alpha x\| = |\alpha| \, \|x\|$

$(N4)$ $\|x + y\| \leqslant \|x\| + \|y\|$ *(Triangle Inequality);*

here x and y are arbitrary vectors in X and α is any scalar.

A norm on X defines a metric d on X which is given by

$$d(x,y) = \|x-y\| \qquad\qquad (\mathrm{x, y \in X})$$

and is called the metric induced by the norm. The normed space just defined is denoted by $(X, \|.\|)$ or simply by X.

Definition 1.8 Direct Product and Direct Sum:

The direct product $\prod_{i\in I} V_i$ of a family of vector spaces V_i, where i runs through some index set I, consists of the set of all tuples $(v_i)_{i\in I}$, which specify for each index i an element v_i of V_i. Addition and scalar multiplication is performed componentwise. A variant of this construction is the direct sum $\oplus_{i\in I} V_i$, where only tuples with finitely many nonzero vectors are allowed. If the index set I is finite, the two constructions agree, but differ otherwise.

Definition 1.9 Tensor Product:

The tensor product $V \otimes W$ of two vector spaces V and W over a field K can be defined by the method of generators and relations.

To construct $V \otimes W$, one begins with the set of ordered pairs in the Cartesian product $V \times W$. For the purposes of this construction, regard this Cartesian product as a set rather than a vector space. The free vector space on $V \times W$ is defined by taking the vector space in which the elements of $V \times W$ are a basis. Symbolically,

$$F(V \times W) = \left\{ \sum_{i=1}^{n} a_i e_{v_i \times w_i} \,|n \in \mathbb{N}, a_i \in K, (v_i \times w_i) \in V \times W \right\}$$

where we have used the symbol $e_{(v\times w)}$ to emphasize that these are taken to be linearly independent for distinct $(v \times w) \in V \times W$.

Definition 1.10 Linear Operator:

A linear operator A is an operator such that

(i) the domain $D(A)$ of A is a vector space and the range $R(A)$ lies in a vector space over the same field,

(ii) for all $x, y \in D(A)$ and scalars α,

$$A(x + y) = Ax + Ay$$

$$A(\alpha x) = \alpha Ax.$$

Definition 1.11 Bounded Linear Operator:

Let X and Y be normed spaces and $A : D(A) \to Y$ a linear operator,where $D(A) \subset X$. The operator A is said to be bounded if there is a real number c such that for all $x \in D(A)$.

$$\|Ax\| \leq c. \|x\|$$

Definition 1.12 Orthonormal Basis:

An orthonormal system $\{\varphi_1, \varphi_2, ...\}$ is called an orthonormal basis for H if for every $v \in H$, $v = \sum_k \alpha_k \varphi_k$ for some $\alpha_1, \alpha_2, ...$ in $\mathbb{C}$.

Definition 1.13 Subharmonic Function :

One can show that a real-valued, continuous function φ of a complex variable (that is, of two real variables) defined on a set $G \subset \mathbb{C}$ is subharmonic if and only if for any closed disc $D(z, r) \subset G$ of center z and radius r one has

$$\varphi(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \varphi\left(z + re^{i\theta}\right) d\theta.$$

Definition 1.14 l_2 *: The vector space l_2 consists of the set of those sequences $(\xi_1, \xi_2, ...)$ of complex numbers for which $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$. The complex valued function $\langle ., . \rangle$ defined on $l_2 \times l_2$ by $\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \bar{\eta}_i$ where $x = (\xi_i)$, $y = (\eta_i)$, is called the inner product on l_2. The norm, $\|x\|$, on l_2 is defined by*

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{\infty} |\xi_i|^2 \right)^{1/2}$$

The distance between the vectors $x = (\xi_1, \xi_2, ...)$ and $y = (\eta_1, \eta_2, ...)$ is $\|x - y\| = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2 \right)^{1/2}$.

Definition 1.15 Seperable Hilbert Space:

A Hilbert space H is called seperable if there exist vectors $v_1, v_2, ...$ which span a subspace dense in H.

Definition 1.16 Lebesgue-measurable function:

A Lebesgue-measurable function $f : \mathbb{R} \to \mathbb{R}$ is a real function such that for every real number a, the set

$\{x \in \mathbb{R} : f(x) > a\}$ is a Lebesgue-measurable set.

Definition 1.17 First definition of Pseudospectra:

Let $A \in \mathbb{C}^{N \times N}$ and $\varepsilon > 0$ be arbitrary . The ε-pseudospectrum $\sigma_\varepsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\left\|(z - A)^{-1}\right\| > \varepsilon^{-1}$$

Definition 1.18 Second definition of Pseudospectra:

$\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that

$$z \in \sigma(A+E)$$

for some $E \in \mathbb{C}^{NxN}$ with $\|E\| < \varepsilon$.

The ε-pseudospectrum is the set of numbers that are eigenvalues of some perturbed matrix $A+E$ with $\|E\| < \varepsilon$.

Definition 1.19 Third definition of Pseudospectra:

$\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that

$$\|(z-A)\,v\| < \varepsilon$$

for some $v \in \mathbb{C}^N$ with $\|v\| = 1$.

Definition 1.20 Jordan Block Matrix:

A matrix, also called a canonical box matrix, having zeros everywhere except along the diagonal and superdiagonal, with each element of the diagonal consisting of a single number λ and each element of the superdiagonal consisting of a 1.

For example,

$$\begin{bmatrix} \lambda & 1 & 0 & & 0 & 0 \\ 0 & \lambda & 1 & & 0 & 0 \\ 0 & 0 & \lambda & & & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

Definition 1.21 Toeplitz Matrix: *Any $n \times n$ matrix A of the form*

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & \ddots & \vdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2-n} & \vdots & \ddots & a_0 & a_1 \\ a_{1-n} & a_{2-n} & \cdots & a_{-1} & a_0 \end{bmatrix}$$

is a Toeplitz Matrix. If the i , j element of A is denoted A_{ij} , then we have $A_{ij} = a_{j-i}$.

Let $f(z)$ denote the symbol

$$f(z) = \sum_{k=-\infty}^{\infty} a_k.z^k = ... + a_{-1}.z^{-1} + a_0.z^0 + a_1.z^1 + ...$$

with

$$\sum_{k=-\infty}^{\infty} |a_k| < \infty.$$

Definition 1.22 Kronecker Delta: *The set ϕ_{ij} is orthonormal since*

$$\langle \phi_{jk}, \phi_{mn} \rangle = \int_a^b \int_a^b \varphi_j(s)\varphi_k(t)\overline{\varphi_m(s)\varphi_n(t)}dsdt$$

$$= \int_a^b \varphi_j(s)\overline{\varphi_m(s)}ds \int_a^b \varphi_k(t)\overline{\varphi_n(t)}dt$$

$$= \delta_{jm}\delta_{kn}$$

where δ_{jm}, called the Kronecker Delta, is defined to be 1 if $j = m$ and 0 if $j \neq m$.

CHAPTER 2

PSEUDOSPECTRA

2.1 Eigenvalues

In the simplest context of matrices, the definitions are as follows. Let A be an $N \times N$ matrix with real or complex coefficients; we write $A \in \mathbb{C}^{N\times N}$. Let v be a nonzero real or complex column vector of length N, and let λ be a real or complex scalar; we write $v \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$. Then v is an eigenvector of A, and $\lambda \in \mathbb{C}$ is its corresponding eigenvalue, if

$$Av = \lambda v.$$

(Even if A is real, its eigenvalues are in general complex unless A is selfadjoint.) The set of all eigenvalues of A is the spectrum of A, a nonempty subset of the complex plane $\mathbb{C}$ that we denote by $\sigma(A)$. The spectrum can also be defined as the set of points $z \in \mathbb{C}$ where the resolvent matrix,

$$(z - A)^{-1},$$

does not exist. Where $(z - A)$ is shorthand for $zI - A$, where I is the identity matrix.

2.2 Pseudospecta of Lineer Operators

In this section we set down the fundamentals of the theory of pseudospectra for linear operators in Banach space. Fortunately, the technical details do not matter for many applications, and we use a language closer to linear algebra than functional analysis.

Let X be a complex Banach space, that is, a complete normed vector space over the complex field $\mathbb{C}$ with norm $\|.\|$. We shall consider linear operators mapping X into itself. Such an operators A has a domain denoted by $D(A) \subseteq X$, which may or may not be all of X. We denote by $B(X)$ the set of bounded operators on X ; for $A \in B(X)$, we assume without loss of generality $D(A) = X$. We denote by $C(X)$ the set of closed operators on X. (An operators A is closed provided that if $\{u_k\}$ is a sequence in $D(A)$ converging to a limit $u \in X$ and if $\{Au_k\}$ converges to a limit $v \in X$, then $u \in D(A)$ and $Au = v$.) An unbounded closed operator will necessarily have $D(A) \neq X$, although many such operators are densely defined, meaning that the closure of $D(A)$ is X.

For $A \in C(X)$ and $E \in B(X)$, $A+E$ is also in $C(X)$, with domain $D(A+E) = D(A)$. A special case is the situation where E is a multiple of the identity. Thus perturbations of closed operators by bounded operators or shifts introduce no technical difficulties.

Given $A \in C(X)$, a bounded inverse is an operator $A^{-1} \in B(X)$ such that AA^{-1} is the identity on X and $A^{-1}A$ is the identity on $D(A)$. This is the only kind of inverse we shall be concerned with, and when an expression like A^{-1} or $(z - A)^{-1}$ appears, it should always be interpreted as a bounded inverse, defined on all of X.

Matrix Representations of Bounded Linear Operators

In this part it is shown how to associate a matrix with a given bounded linear operator on a separable Hilbert space.

Suppose $A \in L(H)$, where H is a Hilbert space with orthonormal basis $\varphi_1, \varphi_2, \ldots$. Then for $x \in H$, $x = \sum_j \langle x, \varphi_j \rangle \varphi_j$. It follows from the linearity and continuity of A, applied to the sequence of partial sums of this series, that

$$Ax = \sum_j \langle x, \varphi_j \rangle A\varphi_j. \tag{2.1}$$

Now

$$A\varphi_j = \sum_k \langle A\varphi_j, \varphi_k \rangle \varphi_k. \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$Ax = \sum_j \sum_k \langle x, \varphi_j \rangle \langle A\varphi_j, \varphi_k \rangle \varphi_k = \sum_k \sum_j \langle x, \varphi_j \rangle \langle A\varphi_j, \varphi_k \rangle \varphi_k. \tag{2.3}$$

Thus if $Ax = y$, then (2.3) implies

$$\begin{pmatrix} \langle A\varphi_1, \varphi_1 \rangle & \langle A\varphi_2, \varphi_1 \rangle & \cdots \\ \langle A\varphi_1, \varphi_2 \rangle & \langle A\varphi_2, \varphi_2 \rangle & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \langle x, \varphi_1 \rangle \\ \langle x, \varphi_2 \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle y, \varphi_1 \rangle \\ \langle y, \varphi_2 \rangle \\ \vdots \end{pmatrix}.$$

This matrix equation leads to the following definition.

Definition 2.1 *Let $\varphi_1, \varphi_2, \ldots$ be an orthonormal basis for a Hilbert space H and let A be in $L(H)$. The matrix (a_{ij}) corresponding to A and $\varphi_1, \varphi_2, \ldots$ is defined by $a_{ij} = \langle A\varphi_j, \varphi_j \rangle$.*

Example 2.2 *Let $S_r \in L(\ell_2)$ be the forward shift given by*

$$S_r(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...).$$

If $e_1, e_2, ...$ is the standard orthonormal basis for ℓ_2, then

$$\langle S_r e_j, e_i \rangle = \langle e_{j+1}, e_i \rangle = \delta_{j+1,i}.$$

Thus the matrix corresponding to S_r and $e_1, e_2, ...$ is the lower diagonal matrix

$$\begin{pmatrix} 0 & 0 & 0 & ... \\ 1 & 0 & 0 & ... \\ 0 & 1 & 0 & ... \\ . & . & . & . \end{pmatrix}$$

Example 2.3 *Let $H = L_2([-\pi, \pi])$ and let $a(t)$ be a bounded complex valued Lebesgue measurable function on $[-\pi, \pi]$.*

Define $A \in L(H)$ by

$$(Af)(t) = a(t) f(t).$$

The "doubly infinite" matrix $\{a_{jk}\}_{j,k=-\infty}^{\infty}$ corresponding to A and the orthonormal basis $\varphi_n(t) = \frac{1}{\sqrt{2\pi}} e^{\text{int}}$, $n = 0, \pm 1, \pm 2, ...$ is obtained as follows. Let

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(t) e^{-\text{int}} dt.$$

Then

$$a_{jk} = \langle A\varphi_k, \varphi_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(t)\, e^{\mathrm{i}(k-j)t} dt = a_{j-k}.$$

Thus the matrix is

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & & \\ \cdots & a_1 & a_0 & a_{-1} & \cdots & & & \\ & \cdots & a_1 & a_0 & a_{-1} & \cdots & & \\ & & \cdots & a_1 & a_0 & a_{-1} & \cdots & \\ & & & \ddots & \ddots & \ddots & & \end{pmatrix}$$

which is called a Toeplitz matrix. The entry a_0 is located in row zero column zero.

Example 2.4 *Let K be the integral operator on $L_2([a,b])$ with kernel function $k \in L_2([a,b] \times [a,b])$. Given an orthonormal basis $\{\varphi_j\}$ for $L_2([a,b])$, then $\Phi_{ij}(s,t) = \varphi_i(s)\varphi_j(t)$, $\quad 1 \leqslant i, j$, forms an orthonormal basis for $L_2([a,b] \times [a,b])$. Now the matrix (a_{ij}) corresponding to K and $\{\varphi_i\}$ is given by*

$$a_{ij} = \langle K\varphi_j, \varphi_i \rangle = \int_a^b \int_a^b k(t,s)\varphi_j(s)\,\bar{\varphi}_i(t)\, ds dt = \langle k, \Phi_{ji} \rangle.$$

Thus we see that each a_{ij} is a Fourier coefficient of k with respect to $\{\varphi_{ij}\}$. Consequently, Parseval's equality gives

$$\sum_{i,j} |a_{ij}|^2 = \sum_{i,j} |\langle k, \Phi_{ij} \rangle|^2 = \|k\|^2.$$

So far we started with a bounded linear operator and considered its matrix representation relative to an orthonormal basis. In some problems the reverse direction is important: an infinite matrix is given and one looks for a bounded *linear operator for which the given matrix is the matrix of the operator relative to some orthonormal basis. In the sequel we shall use the following definition.*

Let $(a_{jk})_{j,k=1}^{\infty}$ *be an infinite matrix. We say that this matrix induces a bounded linear operator* A *on* ℓ_2 *if* $(a_{jk})_{j,k=1}^{\infty}$ *is the matrix of* A *with respect to the standard orthonormal basis* $e_1, e_2, ...$ *of* ℓ_2*, that is,*

$$a_{jk} = \langle Ae_k, e_i \rangle \,, \quad j, k = 1, 2, ...$$

In this case we simply write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

As a first example consider a diagonal matrix

$$diag\,(\omega_1, \omega_2, \omega_3, ...) = (\delta_{jk}\omega_k)_{j,k=1}^{\infty} \,.$$

Here δ_{jk} *is the Kronecker delta and the diagonal entries are* $\omega_1, \omega_2, \omega_3, ...$ *This matrix induces a bounded linear operator* D *on* ℓ_2 *if and only if* $\sup_{j\geqslant 1} |\omega_j| < \infty$*, and in this case*

$$\|D\| = \sup_{j\geqslant 1} |\omega_j| \,. \tag{2.4}$$

Indeed, if the matrix $diag\,(\omega_j)_{j=1}^{\infty}$ *induces a bounded linear operator* D *on* ℓ_2*,then*

$$|\omega_j| = |\langle De_j, e_j\rangle| \leqslant \|D\| . \quad j = 1, 2, ...,$$

and hence $\sup_{j\geqslant 1}|\omega_j| \leqslant \|D\| < \infty$*. Conversely, if* $\sup_{j\geqslant 1}|\omega_j| < \infty$ *, then*

$$D\,(x_1, x_2, ...) = (\omega_1 x_1, \omega_2 x_2, ...)$$

is a bounded linear operator on ℓ_2 *and the matrix of* D *with respect to the standard orthonormal basis of* ℓ_2 *is the diagonal matrix diag* $(\omega_j)_{j=1}^{\infty}$*. Moreover,*

$$\|Dx\| = \left(\sum_{j=1}^{\infty} |\omega_j x_j|^2\right)^{1/2} \leqslant \left(\sup_{j\geqslant 1}|\omega_j|\right) . \|x\| ,$$

and hence (2.4) holds. If

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty,$$

then the matrix with respect to the standard orthonormal basis of ℓ_2 *of the operator* A *defined is precisely equal to* $(a_{ij})_{ij=1}^{\infty}$*.*

Norm of the resolvent

Theorem 2.5 *Given* $A \in C(X)$*, and with* $\|(z-A)^{-1}\|$ *defined as* ∞ *for* $z \in \sigma(A)$*, the norm of resolvent* $\|(z-A)^{-1}\|$ *is a function from* $z \in \mathbb{C}$ *to* $(0, \infty]$ *with the following properties. It is continuous and unbounded and takes the value* ∞ *precisely on* $\sigma(A)$*. For* $z \notin \sigma(A)$ *it is subharmonic and satisfies the maximum principle as well as the bound*

$$\left\|(z-A)^{-1}\right\| \geq \frac{1}{dist(z,\sigma(A))} \tag{2.5}$$

If $z \notin \sigma(A)$, then $z \notin \sigma(A+E)$ for any $E \in B(X)$ that satisfies

$$\|E\| \leq \frac{1}{\|(z-A)^{-1}\|};$$

conversely, for any $\mu > \|(z-A)^{-1}\|^{-1}$, there exists $E \in B(X)$ with $\|E\| < \mu$ such that $(A+E)u = zu$ for some nonzero $u \in X$.

According to Theorem 2.5 every z that is not in $\sigma(A)$ is also not in $\sigma(A+E)$ for sufficiently small $\|E\|$.

We defined de ε-pseudospectrum of a matrix A in three equivalent ways. The same three definitions apply for linear operators in Banach space. Statements of aspects of this equivalence in various contexts can be found in a number of publications, including [18-21].

Three equivalent definitions of pseudospectra

Let $A \in C(X)$ and $\varepsilon > 0$ be arbitrary. The ε-pseudospectrum $\sigma_\varepsilon(A)$ of A is the set of $z \in \mathbb{C}$ defined equivalently by any of the conditions

$$\left\|(z-A)^{-1}\right\| > \varepsilon^{-1}, \tag{2.6}$$

$$z \in \sigma(A+E) \text{ for some } E \in B(X) \text{ with } \|E\| < \varepsilon, \tag{2.7}$$

$$\|(z-A)\,v\| < \varepsilon \text{ for some } v \in \mathbb{C}^N \text{with } \|v\| = 1. \tag{2.8}$$

Theorem 2.6 *For any matrix $A \in \mathbb{C}^{N\times N}$ the three definitions above are equivalent.*

Proof. For $z \in \sigma(A)$ the equivalence is trivial, so assume $z \notin \sigma(A)$, implying the existence of $(z-A)^{-1}$. To prove (2.7) $\Rightarrow$ (2.8), suppose that $(A+E)\,v = zv$ for some $E \in \mathbb{C}^{N\times N}$ with $\|E\| < \varepsilon$ and some nonzero $v \in \mathbb{C}^N$, which we may take to be normalized, $\|v\| = 1$. Then $\|(z-A)\,v\| = \|Ev\| < \varepsilon$, as required. To prove (2.8) $\Rightarrow$ (2.6), suppose $(z-A)\,v = su$ for some $v, u \in \mathbb{C}^N$ with $\|v\| = \|u\| = 1$ and $s < \varepsilon$. Then $(z-A)^{-1} u = s^{-1} v$, so $\left\|(z-A)^{-1}\right\| \geqslant s^{-1} > \varepsilon^{-1}$. Finally, to prove (2.6) $\Rightarrow$ (2.7), suppose $\left\|(z-A)^{-1}\right\| > \varepsilon^{-1}$. Then $(z-A)^{-1} u = s^{-1} v$ and consequently $zv - Av = su$ for some $v, u \in \mathbb{C}^N$ with $\|v\| = \|u\| = 1$ and $s < \varepsilon$. To establish (2.7), it is enough to show that there exists a matrix $E \in \mathbb{C}^{N\times N}$ with $\|E\| = s$ and $Ev = su$, for then v will be an eigenvector of $A+E$ with eigenvalue z. In fact, E can be taken to be a rank-1 matrix of the form $E = suw^*$ for some $w \in \mathbb{C}^N$ with $w^* v = 1$. If $\|.\|$ is the 2-norm, this is evident simply by taking $w = v$. In the case of an arbitrary norm $\|.\|$, the existence of a vector w satisfying the required conditions can be interpreted as the existence of a linear functional L on $\mathbb{C}^N$ with $\|Lv\| = 1$ and $\|L\| = 1$, which is guaranteed by the Hahn-Banach theorem. A less highbrow version of the same proof can be carried out by the method of dual norms [22].

Properties of pseudospectra:

Theorem 2.7 *Given $A \in C(X)$, the pseudospectra $\{\sigma_\varepsilon(A)\}_{\varepsilon>0}$ have the following properties. They can be defined equivalently by any of the conditions (2.6)-(2.8). Each $\sigma_\varepsilon(A)$ is a nonempty open subset of $\mathbb{C}$, and any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$. The pseudospectra are strictly nested supersets of the spectrum: $\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A)$, and conversely, for any $\delta > 0$, $\sigma_{\varepsilon+\delta}(A) \supseteq \sigma_\varepsilon(A) + \Delta_\delta$, where Δ_δ is the open disk of radius δ.*

Theorem 2.8 *Let A be an arbitrary Toeplitz operator with absolutely summable entries, and let A_N denote its $N \times N$ Toeplitz matrix section. Then*

$$\lim_{N \to \infty} \delta_\varepsilon (A_N) = \delta_\varepsilon (A) .$$

for each $\varepsilon > 0$, and therefore

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \delta_\varepsilon (A_N) = \delta (A) .$$

Theorem 2.9 *Let $A \in \mathbb{C}^{N \times N}$ and $\varepsilon > 0$ be arbitrary.*

(i) $\sigma_\varepsilon (A)$ is nonempty, open, and bounded, with at most N connected components, each containing one or more eigenvalues of A.

(ii) If $\|.\| = \|.\|_2$, then $\sigma_\varepsilon (A^*) = \overline{\sigma_\varepsilon (A)}$.

(iii) If $\|.\| = \|.\|_2$, then $\sigma_\varepsilon (A_1 \oplus A_2) = \sigma_\varepsilon (A_1) \cup \sigma_\varepsilon (A_2)$.

(iv) For any $c \in \mathbb{C}$, $\sigma_\varepsilon (A + c) = c + \sigma_\varepsilon (A)$.

(v) For any nonzero $c \in \mathbb{C}$, $\sigma_{|c|\varepsilon} (cA) = c\sigma_\varepsilon (A)$.

In part (iii), $A_1 \oplus A_2$ denotes the direct sum of two square matrices A_1 and A_2, whose dimensions need not be equal; in other words it is the block diagonal matrix

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} .$$

To prove the assertion about connected components in (i), one can use the fact that $\log \|(z - A^{-1})\|$ is a subharmonic function and hence satisfies the maximum principle except at the eigenvalues of A. Part (v) is also worth a comment, for although elementary, it is surprising at first: the ε-pseudospectrum of $2A$ is not twice the ε-pseudospectrum of A, but twice the $\varepsilon/2$-pseudospectrum of A.

If A is normal, then the spectrum $\sigma(A)$ is a satisfactory answer to that question for almost every purpose. The family of pseudospectra $\{\sigma_\varepsilon(A)\}$ is an attempt to provide a satisfactory answer in the case where A is not normal, perhaps far from normal. It is not a perfect answer; pseudospectra lack the simplicity of spectra, yet despite their complexity, they do not provide exact answers to the questions one would like to ask about the behavior of A.

Through the equivalence of definitions (2.6) and (2.7) they provide a reminder that eigenvalues that are sensitive to perturbations may be of limited significance in determining the behavior of A. Finally, they provide an appealing geometric interpretation of nonnormality. The fact is, nonnormal matrices and operator do not live in the complex plane, but one can get a good start in predicting their behavior if, in addition to the usual calculation of eigenvalues, one plots a few contour lines of the resolvent norm or the eigenvalues of a few randomly perturbed matrices. Beautiful plots can be obtained in seconds for matrices of dimensions in the hundreds with the MATLAB system EigTool [23].

CHAPTER 3

PSEUDO-EIGENVALUES OF TOEPLITZ MATRICES

3.1 Block Toeplitz Matrices

In this chapter we try to plot the pseudo-eigenvalues of the $N \times N$ bidiagonal matrix

$$A = \begin{pmatrix} 1 & \gamma & & & & \\ & -1 & \gamma & & & \\ & & 1 & \gamma & & \\ & & & -1 & \gamma & \\ & & & & 1 & \gamma \\ & & & & & -1 \end{pmatrix}, \quad \gamma \in \mathbb{C} \tag{3.1}$$

whose eigenvalues are obviously $\{-1, 1\}$. This matrix looks like, a Jordan block, except that the diagonal elements are not constant but cycle repetitively through a fixed sequence. Figure (3.1) shows computed eigenvalues corresponding to $\gamma = 1$, $N = 77$, in two dimensional complex plane for ten randomly perturbation $A + E$ with $\|E\| \approx \varepsilon = 10^{-2}$ and Figure (3.2) shows computed eigenvalues corresponding to $\gamma = 1$, $N = 77$, in three dimension for ten randomly perturbation $A + E$ with $\|E\| \approx \varepsilon = 10^{-2}$.

The pseudo-eigenvalues in each of these pictures lie approximately along a lemniscate: the curve or union of curves in the complex plane along which a polynomial $p(\lambda)$ has constant modulus $|p(\lambda)| = C$. For the matrix A of (3.1), the polynomial is $|p(\lambda)| = \lambda^2 - 1$, and C depends on the choice of γ . By varying γ and the sequence of elements on the diagonal, one can obtain pseudospectra that approximate arbitrary lemniscates of arbitrary polynomials.

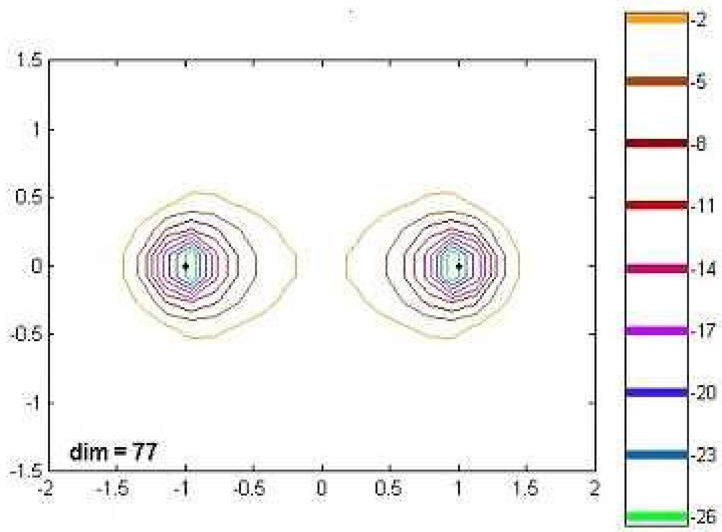


Figure 3.1: The plot shows the eigenvalues of matrix A with MatLab, where A bidiagonal matrix (3.1) of dimension $N = 77$. The eigenvalues of A are ± 1. An approximation to the boundary of the ε-pseudospectrum, $\varepsilon = 10^{-2}$, 10^{-5}, 10^{-8},...,10^{-26}.

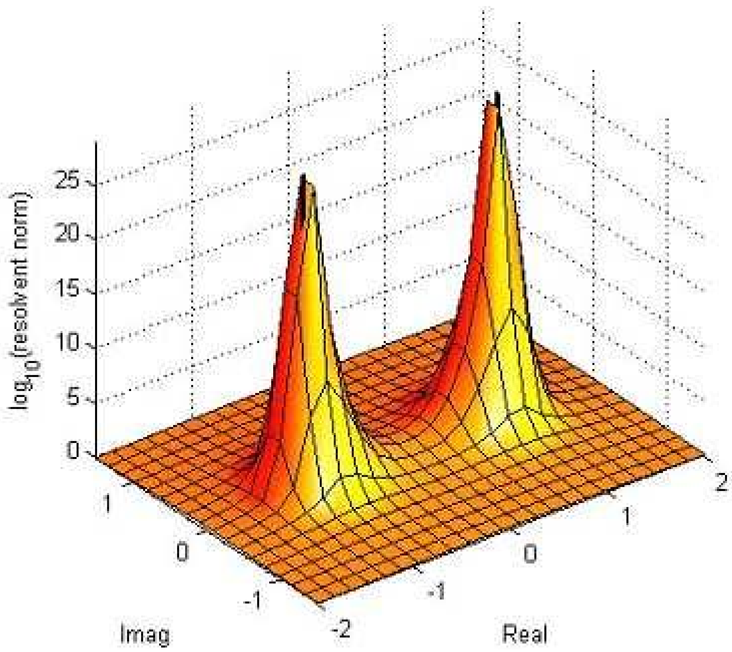


Figure 3.2: The resolvent norm as a function of $z \in \mathbb{C}$ for the matrix of A. The spikes occur at eigenvalues of A; in principle they extend to infinity.

The left plots in the Figure (3.3) shows two dimensional plotting for perturbed matrix A. And on the right side of this figure shows three dimensional plotting of ε-pseudospectral points of A.

Figure (3.4) and figure (3.5) shows computed eigenvalues corresponding to $\gamma = 1$, $N = 1000$ in two and three dimensional complex plane for ten randomly perturbation $A + E$ with $\|E\| \approx \varepsilon = 10^{-1}, 10^{-2}, ..., 10^{-10}$.

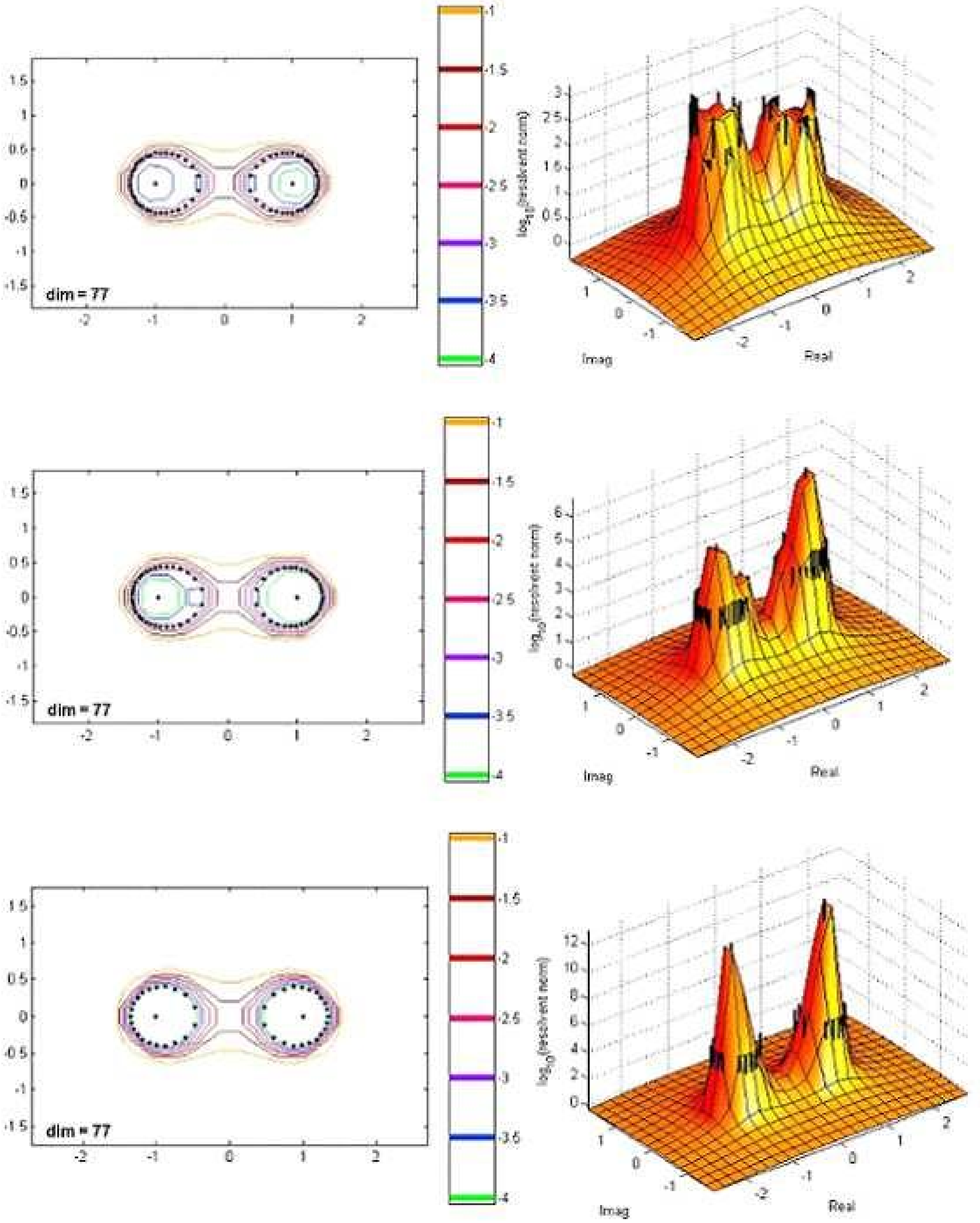


Figure 3.3: Six plots corresponding to $N = 77$ and $\|E\| = 10^{-2}$ show eigenvalues of 3 randomly perturbed matrices $A + E$.

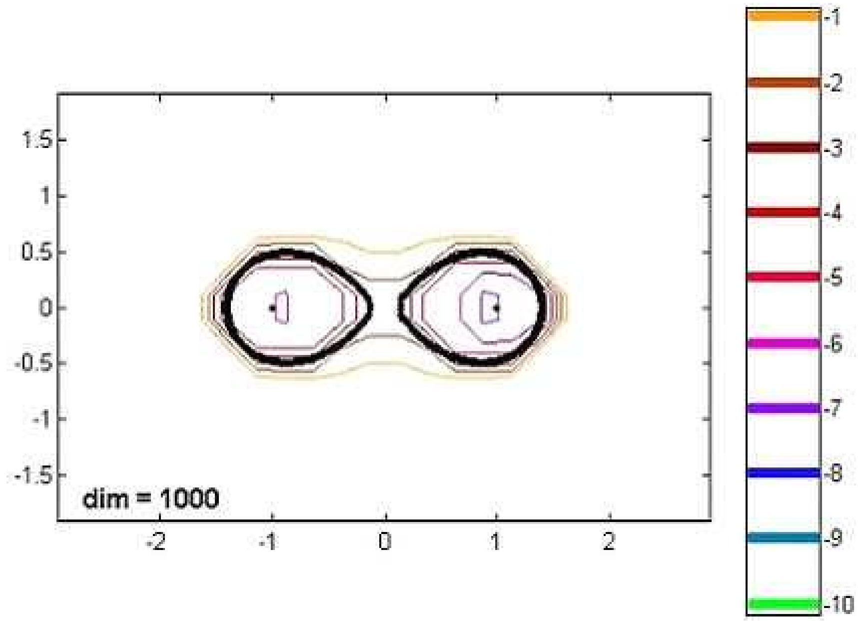


Figure 3.4: Boundaries of pseudospectra $\sigma_\varepsilon(A)$, $\varepsilon = 10^{-1}, 10^{-2}, ..., 10^{-10}$, for the matrix (3.1) of dimension N = 1000.

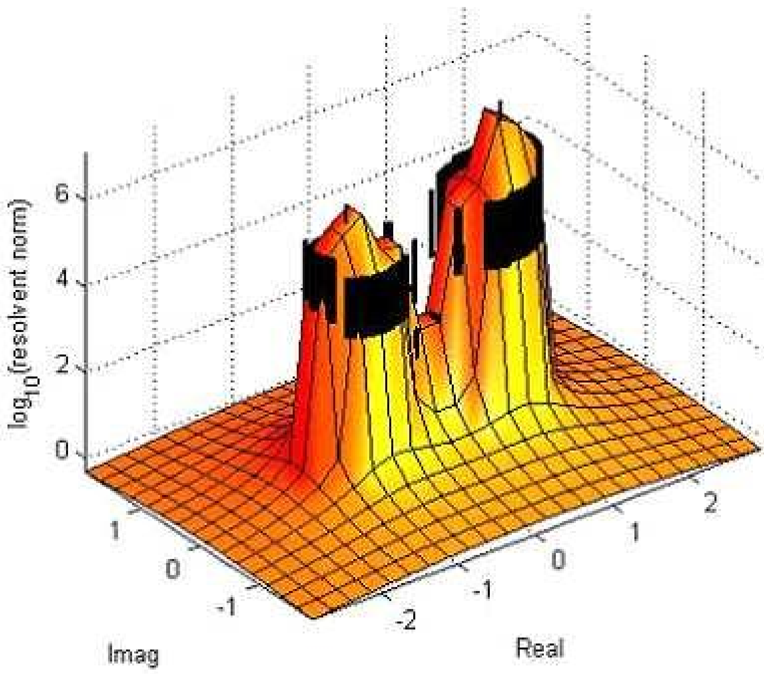


Figure 3.5: The resolvent norm as a function of $z \in \sigma(A)$ for the matrix of figure 3.4. The spikes occur at eigenvalues of A; in principle they extend to infinity.

The explanation for this behavior is as follows. The matrix A of (3.1) can be viewed as an upper bidiagonal block Toeplitz matrix of dimension N/2,

$$A = \begin{pmatrix} D & E & \\ & D & E \\ & & D \end{pmatrix}, \tag{3.2}$$

where D and E are the 2 X 2 matrices

$$D = \begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}. \tag{3.3}$$

A will accordingly have approximate pseudo-eigenvectors of the form u $= \mathrm{U} \otimes \left(1, z, z^2, ..., z^{(N-1)/2}\right)^T$, where U is a 2-vector and $\otimes$ denotes the tensor product. This 2-vector will be either of the eigenvectors of the symbol of A, which is now the 2×2 matrix function

$$f(z) = D + zE = \begin{pmatrix} 1 & \gamma \\ \gamma z & -1 \end{pmatrix}. \tag{3.4}$$

The ε-pseudo-eigenvalues of A will be the corresponding eigenvalues λ of $f(z)$, namely the roots of the equation

$$\lambda^2 - 1 = \gamma^2 z. \tag{3.5}$$

If z ranges over the disk D_r , these values of λ fill the region bounded approximately by the lemniscate $|\lambda^2 - 1| = \gamma^2 r$. Since the highest power of z present is $z^{(N-1)/2}$, the appropriate value of r will be

$$r \approx \varepsilon^{2/N} \tag{3.6}$$

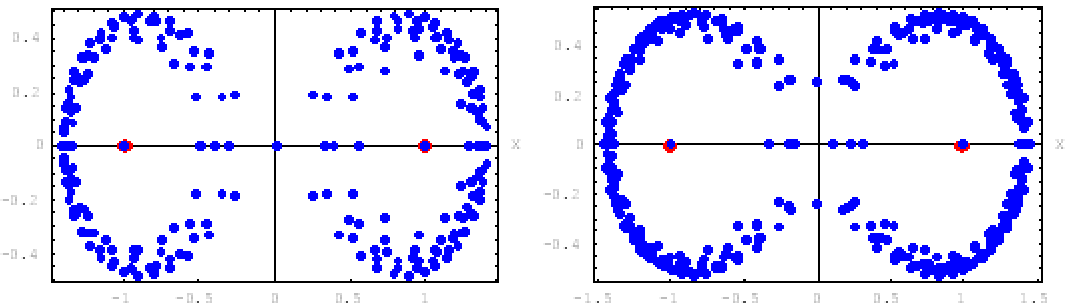
If this explanation of Figure 3.1 is correct, then a pseudospectrum bounded approximately by the critical lemniscate $|\lambda^2 - 1| = 1$ should be obtained if we pick γ and N to satisfy

$$1 = \gamma^2 r = \gamma^2 \varepsilon^{2/N},$$

that is,

$$\gamma = \varepsilon^{-1/N}, \tag{3.7}$$

Figure 3.6 shows results of an experiment with this value of γ for N = 75 and 100. As predicted, the eigenvalues fall roughly on the critical lemniscate.



$N = 75, \quad \gamma = \varepsilon^{-1/N} \approx 1.063$ $\qquad$ $N = 100, \quad \gamma = \varepsilon^{-1/N} \approx 1.096$

Figure 3.6: The eigenvalues of A are ploted by Mathematica and they are marked by red points. Eigenvalues of $A + E$ (3.1) are marked by blue points, $\|E\| \approx \varepsilon = 10^{-2}$, for γ given by (3.7); result from ten random perturbations are superimposed.

These examples give just a hint of the phenomena that may arise in pseudospectra of block Toeplitz matrices. For block Toeplitz matrices with larger blocks and more than two nonzero diagonals, the variety of pseudospectra that can be obtained goes far beyond the class of lemniscates. For the spectra of block Toeplitz operators, results can be found in [24] and [25].

Program (3.1) compute the spectral points (eigenvalues) of A for dimension $N = 150$, and Program (3.2) compute the ε-pseudospectral points for this matrix with Mathematica.

Figure (3.7) shows compute eigenvalues of A and $A + E$ for $N = 150, 350$, with $\|E\| = 10^{-2}, 10^{-4}, 10^{-6}$.

Figure (3.8) and Figure (3.9) shows computed eigenvalues corresponding to $\gamma = 1$, $N = 500$ and $N = 600$, for ten and five randomly perturbation $A + E$ with $\|E\| \approx \varepsilon = 10^{-2}$.

```
n = 150;
A = DiagonalMatrix[Table[0, {n}]];
For[i = 1, i ≤ n - 1, i ++, A[[i,i+1]] = 1 ;];
For[i = 1, i ≤ n, i ++, A[[i,i]] = (-1) ^ (i + 1)  ;];
Print[MatrixForm[A]];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i ++,
  λi  = solset[[i,1,2]];] ;
Print["       A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i ++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
Print["{(xk,yk)} = ", nokta = Table[{ComplexExpand[Re[Chop[N[λi]]]],
    ComplexExpand[Im[Chop[N[λi]]] ]}, {i, n}]]
Needs["Graphics`Colors`"];
dots = ListPlot[nokta, PlotStyle → { Red , PointSize[0.015]},
   DisplayFunction → Identity];
Show[dots, PlotRange → All, Axes → False, AxesLabel → {"X", "Y"},
  DisplayFunction → $DisplayFunction];
```

Program 3.1: Program to compute the spectral points of A for dimension $N = 150$ with Mathematica.

```
n = 150;
A = DiagonalMatrix[Table[0, {n}]];
For[i = 1, i ≤ n - 1, i++, A[[i,i+1]] = 1 ;];
For[i = 1, i ≤ n, i++, A[[i,i]] = (-1)^(i + 1) ;];
Print[A[[75,50]] = 0.01];
Print[MatrixForm[A]];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["       A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
Print["{(xk,yk)} = ", nokta = Table[{ComplexExpand[Re[Chop[N[λi]]]],
    ComplexExpand[Im[Chop[N[λi]]] ]}, {i, n}]]
Needs["Graphics`Colors`"];
dots1 = ListPlot[nokta, PlotStyle → {Blue , PointSize[0.025]},
   DisplayFunction → Identity];
Show[dots1, PlotRange → All, Frame → True, AxesLabel → {"X", "Y" },
  DisplayFunction → $DisplayFunction];
```

Program 3.2: Program to compute the spectral points of $A + E$ for dimension $N = 150$ with Mathematica.

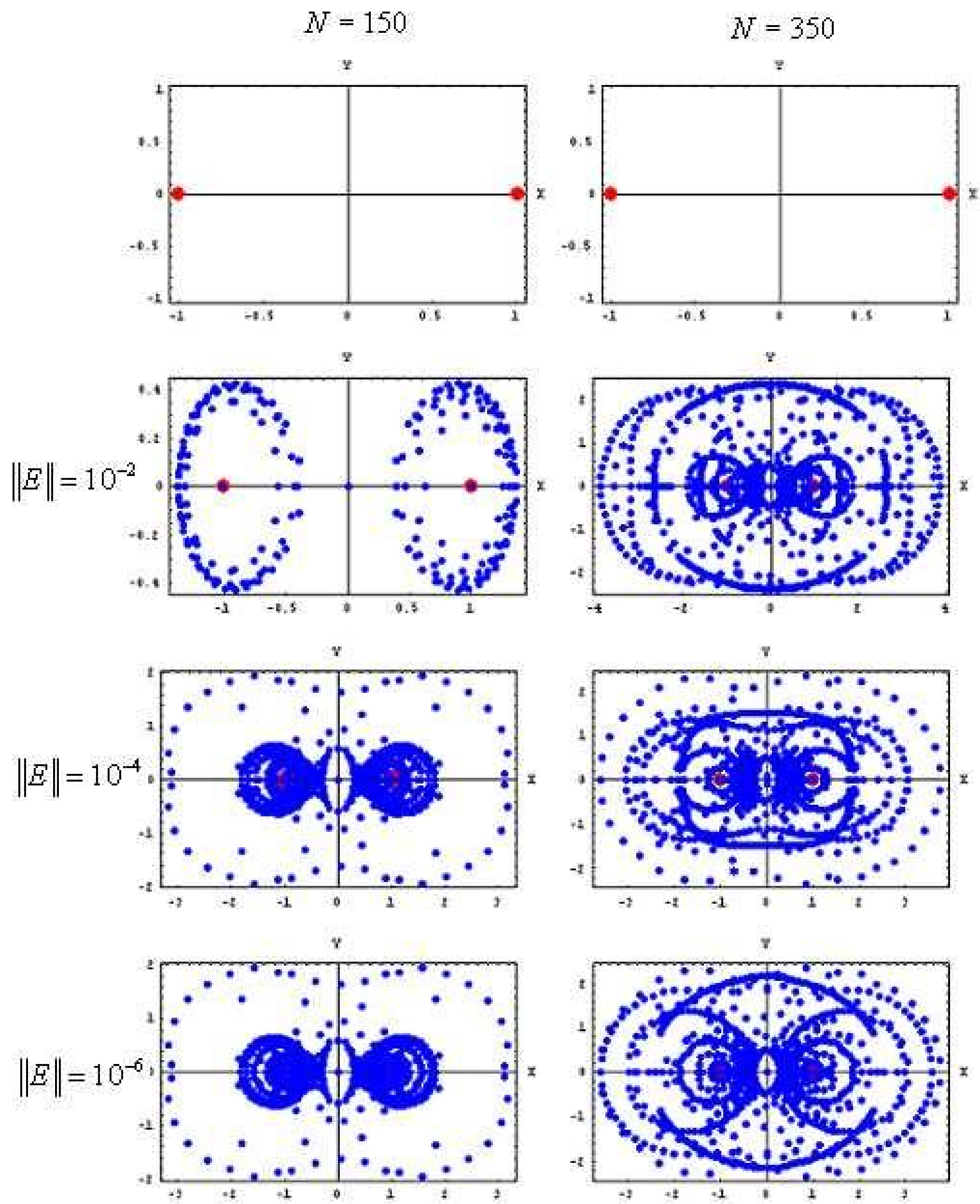


Figure 3.7: Eight plots corresponding to $N = 150, 350$, the eigenvalues of A are marked by red points and $\|E\| = 10^{-2}, 10^{-4}, 10^{-6}$. Each plot shows eigenvalues of 10 randomly perturbed matrices $A + E$ and this eigenvalues are marked by blue points plotted by Mathematica.

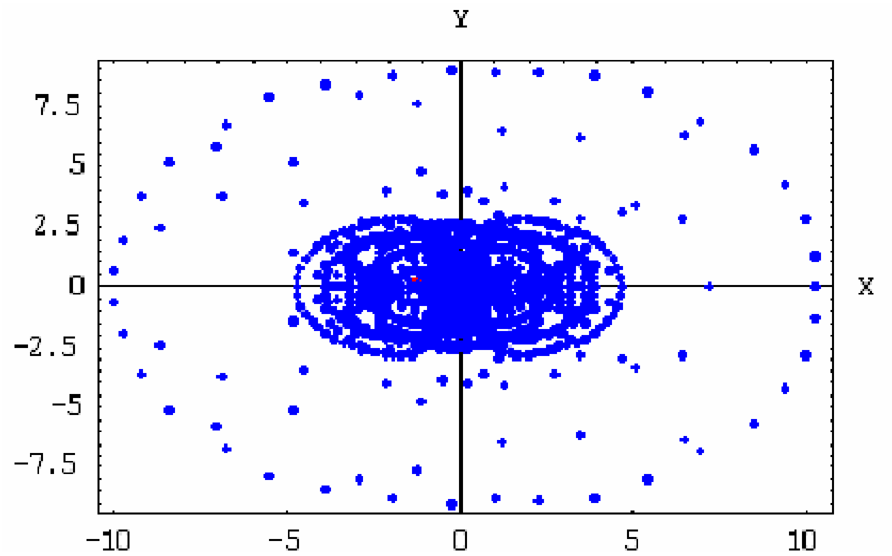


Figure 3.8: The plot corresponding to $N = 500$, $\|E\| = 10^{-2}$ and shows eigenvalues of 10 randomly perturbed matrices $A + E$.

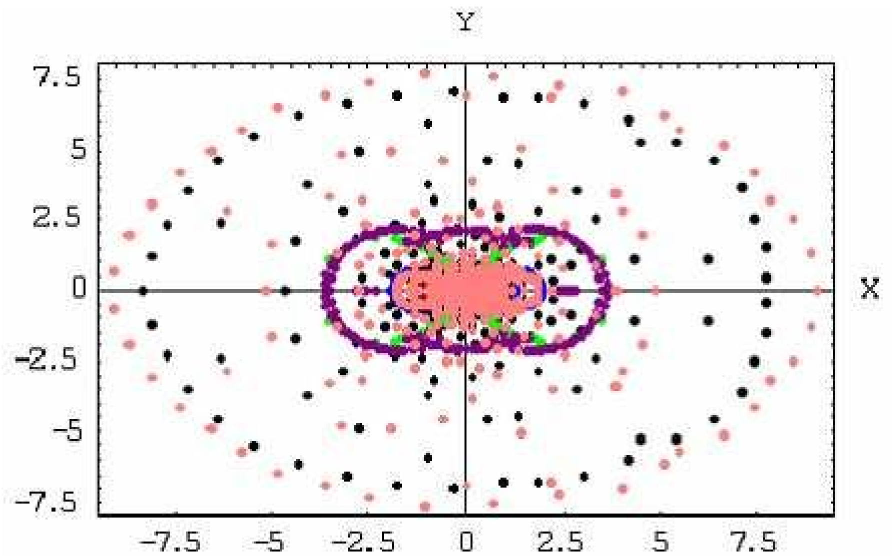


Figure 3.9: The plot corresponding to $N = 600$, $\|E\| = 10^{-2}$ and shows eigenvalues of 5 randomly perturbed matrices $A + E$.

3.2 Variable Coefficients

Another variation on theme Toeplitz matrices is the coefficients varyusually smoothly. Such problems are related to integral equations with variable kernels, and it is natural to expect the resulting spectra and pseudospectra to be associated with the symbols $f(z)$ obtained by freezing the coefficients at various points. Roughly speaking, the pseudospectra of matrices obtained in this fashion approximate superpositions of pseudospectra of the associated Toeplitz matrices defined by frozen coefficients (though not for the same ε). We shall not attempt to make this statement precise. In the hermitian case for exact spectra, theorems to this effect have been given in a well-known paper of Kac, Murdock, and Szegö [27].

We shall give two examples. First, Figure (3.10) shows computed pseudo-eigenvalues of the bidiagonal matrix

$$A = \begin{pmatrix} \omega & 1 & & & \\ & \omega^2 & 1 & & \\ & & \omega^3 & \ddots & \\ & & & \ddots & 1 \\ & & & & \omega^N \end{pmatrix}, \qquad \omega = e^{2\pi i/N}, \tag{3.8}$$

with N = 100. For large N, the pseudospectra of A approximate a superposition of pseudospectra of bidiagonal Toeplitz matrices with elements $\varepsilon^{i\theta}$ on the diagonal and 1 on the superdiagonal. This is a superposition of disks about the points $\varepsilon^{i\theta}$- in other words, a region in the shape of a ring sausage, as in the figure.

The fact that the sausage has ends may be viewed as accidental. Figure (3.11) shows computed pseudo-eigenvalues of the bidiagonal matrix in three dimensional case for dimension $N = 100$ with MatLab. The left plots in the Figure (3.12) shows two dimensional plotting for perturbed matrix A. And on the right side of this figure shows three dimensional plotting of ε-pseudospectral points of A.

Program (3.3) and Program (3.4) plotting the spectral points (eigenvalues) and ε-pseudospectral of A in two and three dimensional complex plane for dimension $N = 100$ with MatLab.

```
clc;
clear;
n=100;
w=cos(2*pi/n)+i*sin(2*pi/n);
A=zeros(n);
for i=1:n
A(i,i)=w^i;
end
for i=1:1:n-1
A(i,i+1)=1;
end
```

Program 3.3 : Program for plotting $\sigma(A)$ with using MatLab and EigTool for $N = 100$.

```
clc;
clear;
n=100;
w=cos(2*pi/n)+i*sin(2*pi/n);
A=zeros(n);
for i=1:n
A(i,i)=w^i;
end
for i=1:1:n-1
A(i,i+1)=1;
end
A(90,15)=0.01;
```

Program 3.4 : Program for plotting $\sigma(A + E)$ with using MatLab and EigTool for $N = 100$.

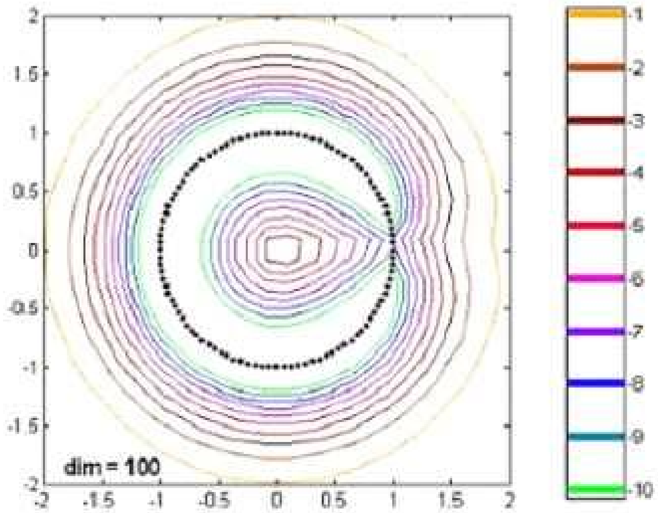


Figure 3.10: Boundaries of pseudospectra $\sigma_\varepsilon(A)$, $\varepsilon = 10^{-1}, 10^{-2}, ..., 10^{-10}$, for the matrix (3.8) of dimension N = 100.

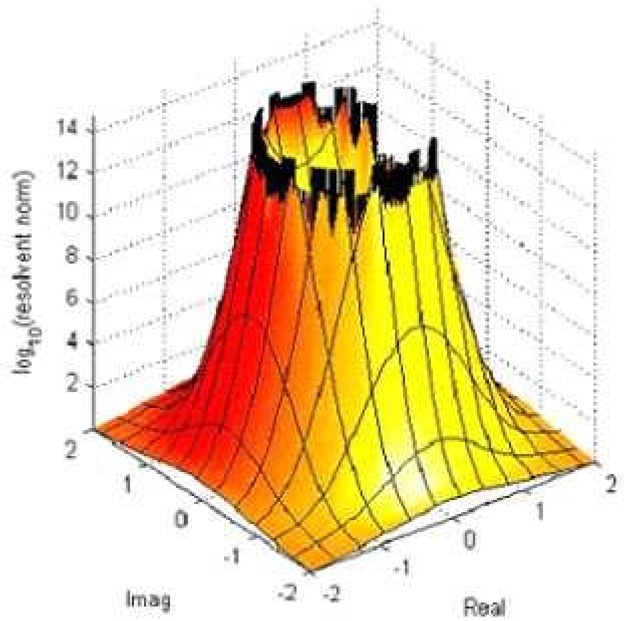


Figure 3.11: The resolvent norm as a function of $z \in \sigma(A)$ for the matrix (3.8). The spikes occur at eigenvalues of A; in principle they extend to infinity.

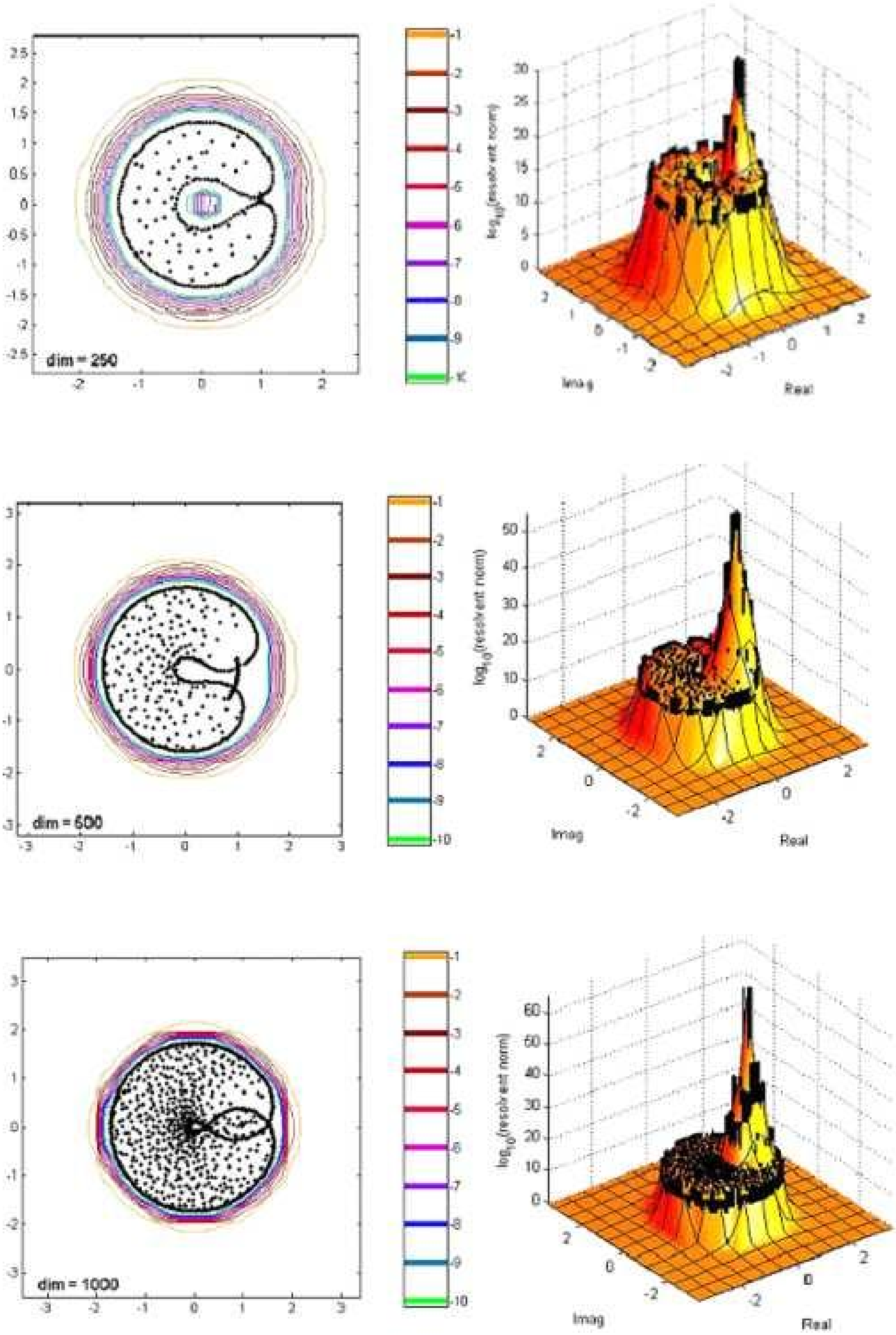


Figure 3.12: Eigenvalues of $(A+E)$ (3.8), $\|E\| = 10^{-4}$, $N = 250, 500, 1000$. The eigenvalues of A lie on the unit circle and are marked by asterisks.

In our second example is also bidiagonal, we vary the superdiagonal rather than the diagonal elements (Figure 3.13). The matrix is

$$B = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \frac{1}{2} & & & \\ & & 0 & \frac{1}{3} & & \\ & & & 0 & \ddots & \\ & & & & \ddots & \frac{1}{N-1} \\ & & & & & 0 \end{pmatrix}, \tag{3.9}$$

again of dimension N = 100. If the dimension were $N = \infty$, the transpose of this matrix could be interpreted as the map

$$f(z) \mapsto \int_0^z f(w) dw \tag{3.10}$$

described in the basis of monomials. Very loosely speaking, for each $r \in [0.01, 1]$, it is as if B had a Jordan block with r on the superdiagonal, with the dimension of the block increasing as r decreases. Superimposing the resulting pseudospectra gives a radially symmetric dependence on ε that remains nontrivial even in the limit $N = \infty$. The basis of our definition of pseudospectra, that to fully understand the behavior of matrices and operators one must sometimes be prepared to consider finite ε. Program (3.5) and Program (3.6) plotting the spectral points (eigenvalues) and ε-pseudospectral of B in two and three dimensional complex plane for dimension $N = 100$ with MatLab and Eigentool.

```
clc;
clear;
n=100;
B=zeros(n);
for i=1:n-1
B(i,i+1)=1/i;
end
```

Program 3.5 : Program for plotting $\sigma(B)$ with using MatLab and EigTool.

```
clc;
clear;
n=100;
B=zeros(n);
for i=1:n-1
B(i,i+1)=1/i;
end
B(56,98)=0.01;
```

Program 3.6 : Program for plotting $\sigma(B+E)$ with using MatLab and Eigentool.

Figure (3.14) shows computed pseudo-eigenvalues of the bidiagonal matrix B (3.9) in three dimensional case for dimension $N = 100$ with MatLab. The left plots in the Figure (3.15) shows two dimensional plotting for perturbed matrix B. And on the right side of this figure shows three dimensional plotting of ε-pseudospectral points of B.

Figure (3.16) and Figure (3.17) shows computed eigenvalues corresponding to $N = 100$ and $N = 500$, for ten randomly perturbation $B + E$ with $\|E\| = 10^{-2}$.

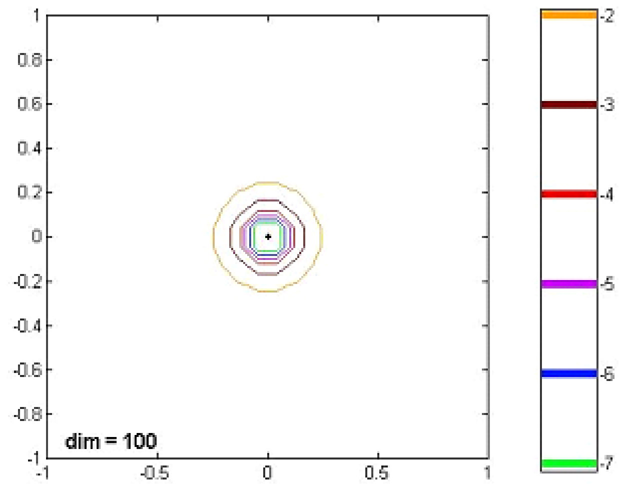


Figure 3.13: The plots shows the eigenvalues of matrix B with MatLab, where B bidiagonal matrix (3.9) of dimension $N = 100$. The only eigenvalue of B is 0. An approximation to the boundary of the ε-pseudospectrum, $\varepsilon = 10^{-2}$, 10^{-3},...,10^{-7}.

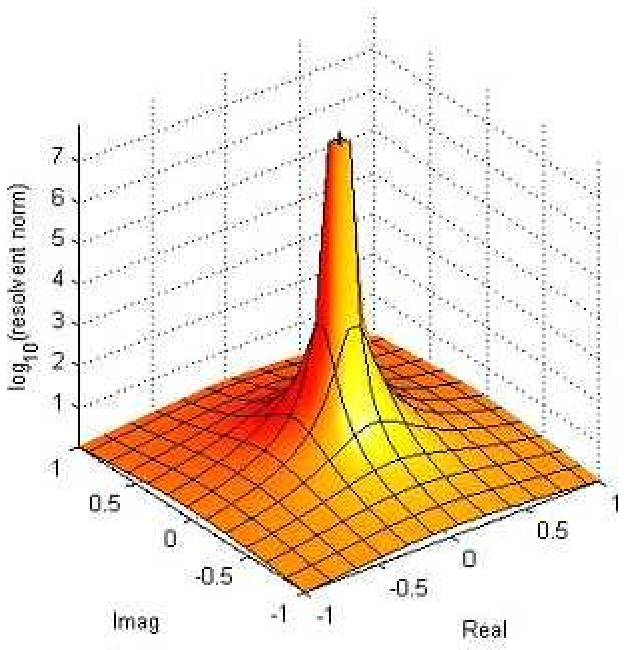


Figure 3.14: The resolvent norm as a function of $z \in \sigma(B)$ for the matrix (3.9). The spikes occur at eigenvalues of B; in principle they extend to infinity.

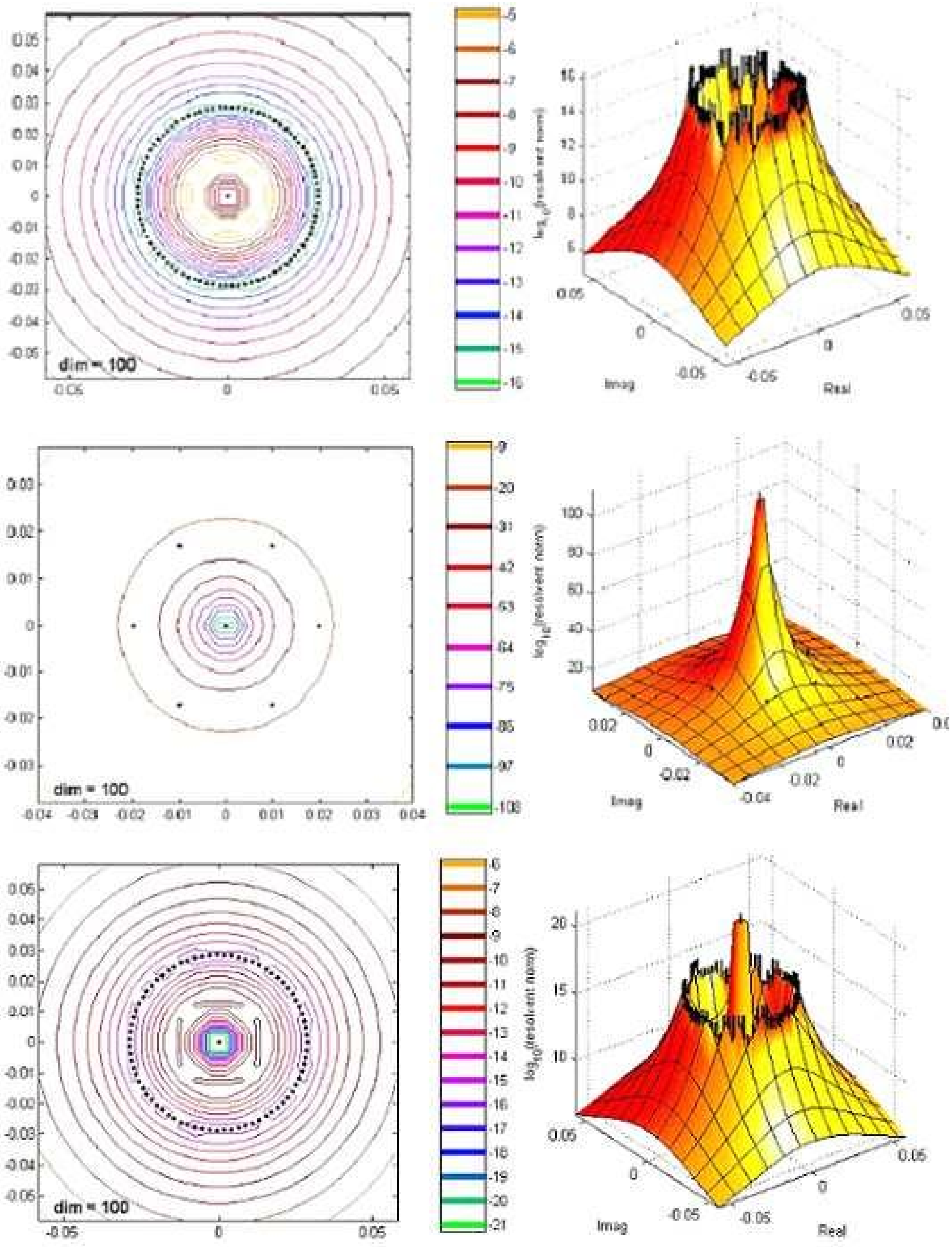


Figure 3.15: Six plots corresponding to $N = 100$ and $\|E\| = 10^{-2}$ show eigenvalues of 3 randomly perturbed matrices $B + E$ (3.9) with using MatLab and Eigtool.

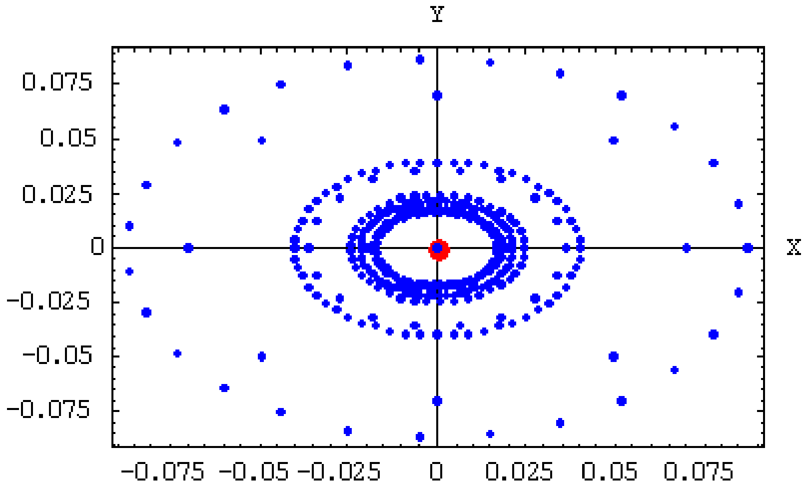


Figure 3.16: The plot corresponding to $N = 100$, $\|E\| = 10^{-2}$ and shows eigenvalues of 10 randomly perturbed matrices $B + E$ ploted by Mathematica.

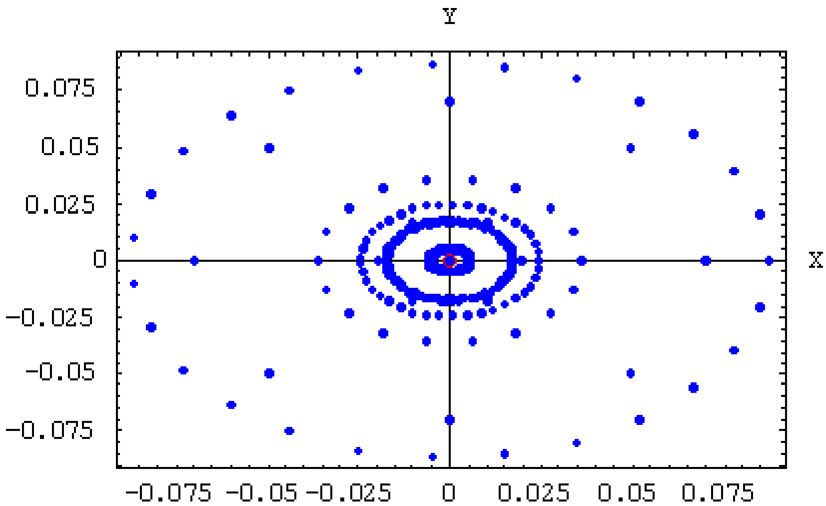


Figure 3.17: The plot corresponding to $N = 500$, $\|E\| = 10^{-2}$ and shows eigenvalues of 10 randomly perturbed matrices $B + E$ ploted by Mathematica.

CONCLUSIONS

In this study we investigate the ε-pseudospectrum and ε-pseudoeigenvalues for some Toeplitz matrices. We investigate and analyse our results for the pseudospectra of block Toeplitz matrices and for the varying coefficients block Toeplitz matrices with using Mathematica and MatLab programming. In this study we write Mathematica programs and use MatLab with Eigentool to compute and plot the ε-pseudospectral region for higher dimension Toeplitz matrices. We plot ε-pseudospectral region for $N = 500, 600, 1000$. We don't plot ε-pseudospectrum for $N = 5000, 10000, 20000$ dimensional Toeplitz matrices because we don't have a computer with delicate CPU to compute our problems.

REFERENCES

[1] J. M. Varah. The compitation of bounds for the invariants subspaces of a general matrix operators. Technical Report CS 66, Computer Sicience Department, Stanford University, 1967.

[2] H. J. Landau. On Szegeö's eigenvalue distribution theorem and non-Hermitian kernels. *J. d'Analyse Math.,* 28:335-357, 1975.

[3] S. K. Godunov and V. S. Ryabenkii. *Theory of Difference Schemes:* An Introduction. North-Holland, Amsterdam, 1964. Russian original published by Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962.

[4] S. K. Godunov, O. P. Kirilyuk, and V. I. Kostin. Spectral portraits of matrices. Technical Report Preprint 3, Inst. of Math., Sib. Branch of USSR Acad. Sci., 1990. In Russian.

[5] V. I. Kostin. On definition of matrices' spectra. In M. Durand and F. E. Dabaghi, editors, *High Performance Computing II.* North-Holland, msterdam, 1991.

[6] V. I. Kostin and S. I. Razzakov. On convergence of the power orthogonal method of spectrum computing. *Trans. Inst. Math. Sib. Branch Acad. Sci.,* 6:55-84, 1985. In Russian.

[7] J. W. Demmel. A counterexample for two conjectures about stability. *IEEE Trans. Auto. Control,* AC-32:340-343, 1987.

[8] J. W. Demmel. Nearest defective matrices and the geometry of ill-conditioning. In M. G. Cox. and S. Hammarling, editors, *Reliable Numerical Computatiton,* pages 35-55. Oxford University Press, Oxford, 1990.

[9] D. Hinrichsen and A. J. Pritchard. On spectral variations under bounded real matrix perturbations. *Numer. Math.,* 60:500-524, 1992.

[10] D. Hinrichsen and B. Kelb. Spectral value set: A graphical tool for robustness analysis. *Sys. Control Lett.,* 21:127-136, 1993.

[11] L. N. Trefethen. Approximation theory and numerical linear algebra. In J. C. Mason and M. G. Cox, editors, *Algorithms for Approximation II.* Chapman and Hall, London, 1990.

[12] S. C. Reddy and L. N. Trefethen. Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues. *Comp. Methods Appl. Mech. Eng.,* 80:147-164, 1990.

[13] L. N. Trefethen and M. R. Trummer. An instability pheonomenon in spectral methods. *SIAM J. Numer. Anal.,* 24:1008-1023, 1987.

[14] S. C. Reddy. *Pseudospectra of Operators and Discretization Matrices and an Application to Stability of the Method of Lines.* Ph.D. thesis, MIT, Cambridge, MA. 1991.

[15] L. N. Trefethen. Pseudospectra of matrices. In D. F. Griffiths and G. A. Watson, editors, *Numerical Analysis 1991,* pages 234-266. Longman Scientific and Technical, Harlow, Essex, UK, 1992.

[16] L. N. Trefethen. Pseudospectra of linear operators. *SIAM Review,* 39:383-406, 1997.

[17] J. M. Varah. On the separation of two matrices. *SIAM J. Numer. Anal.,* 16:216-222, 1979.

[18] A. Böttcher. Pseudospectra and singular values of large convolution operators. *J. Int. Eqs. Appl.,* 6:267-301, 1994.

[19] A. Böttcher and B. Silbermann. *Introduction to Large Truncated Toeplitz Matrices.* Springer-Verlag, New York, 1999.

[20] F. Chaitin-Chatelin and A. Harrabi. About definitions of pseudospectra of closed operators in Banach spaces. Tecnichal Report TR/PA/98/08, CERFACS, August 1998.

[21] E. Gallestey, D. Hinrichsen, and A. J. Pritchard Spectral value ses of closed linear operators. Proc. Roy. Soc. Londan A,456:1397-1418,2000.

[22] J. H. Wilkinson. Sensitivity of eigenvalues II. *Utilitas Math.*, 30:243-286,1986.(16,43)

[23] T. G. Wright. EigTool, 2002. Software avaible at http://www.comlab.ox.ac.uk/pseudospectra/eigtool. (21,140,144,160,371,375,390,397,416,434)

[24] I. C. Gohberg and I. A. Fel'dman *Convolution Equations and Projection Methots for Their Solution,* Amer. Math Soc., Providence,1974.

[25] L. E. Lerer. On the asymptotic distributation of the spectra of finite truncations of Wiener-Hopf operetors, Dokl. Akad. Nauk SSSR 207:1651-1655(1972).

[26] M. Kac, W. L. Murdock, and G. Szegö, On the eigen-values of certain hermitian forms, *J. Rational Mech. Anal.*2:767-800 (1953).