Advance Ordering Model with Markov Modulated Demand Information

by

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This is to certify that I have examined this copy of a M.Sc. thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my family

ABSTRACT

Inventory management and control systems are one of the most studied subjects by researchers of industrial engineering, operations research and management. There is a stream in the inventory control literature that focuses on the management of inventories of seasonal products. Inventory managers face a significant amount of uncertainty when they plan the inventories for the seasonal products. One of the methods that inventory managers might apply to reduce the uncertainty is to exploit advance demand information (ADI). The objective of this study is to develop a mathematical model to analyze ordering decisions of the inventory managers dealing with a seasonal product. For this reason we build a dynamic programming model where the inventory manager has multiple ordering opportunities prior to start of the season. Under multiple ordering opportunities he has two decisions to make: the timing of the orders and the order quantity for each ordering opportunity. Our purpose is to investigate the optimal ordering policy in such a situation. In such a context, the inventory manager has to analyze the trade-off between ordering early with discounted prices and observing the economic conditions that are correlated with the demand he faces. A number of works in the literature explores the correlation between demand of certain items and market indices. To take advantage of this demand correlation, we model the exogenous information process as a discrete time Markov process. We develop a dynamic ordering model where the retailer observes the information process. Moreover, we characterize the structure of optimal ordering policy which maximizes the expected profit of the retailer. Furthermore, we analyze three extensions of the advance dynamic ordering model. Finally, we present some numerical examples to compare the dynamic advance ordering policy with existing ordering policies.

ÖZETÇE

Envanter kontrolü ve yönetimi, yöneylem araştırması, endüstri mühendisliği ve işletme alanlarında en çok çalışılan konulardan biridir. Envanter yönetimi literatürünün önemli bir alanı da sezonluk ürünlerin yönetimidir. Sezonluk ürünlerin yönetiminde, yöneticilerin karşı karşıya kaldıkları talep belirsizliği önemli boyuttadır. Bu belirsizliği azaltmanın yöntemlerinden biri de talep hakkındaki ön bilgiden yararlanmaktır. Bu tezde hedeflenen, sezonsal bir ürünle ilgili envanter yöneticisinin sipariş kararlarını ele alan bir matematiksel modeli incelemektir. Bu sebeple envanter yöneticisinin satış sezonu öncesinde birden çok kez sipariş verebildiği bir dinamik program modeli geliştirilmiştir. Birden çok sipariş verme olanağı olduğunda envanter yöneticisinin sipariş zamanları ve miktarları olmak üzere iki temel karar vermesi gerekmektedir. Bu tezin amacı, bu tarz bir durumda eniyi sipariş miktarı ve zamanını araştırmaktır. Bu bağlamda envanter yöneticisi erken sipariş verip indirimli fiyatlardan yararlanma ile siparişi geciktirip ekonomik göstergeleri gözlemlemek arasında dengeleme tahlili yapmalıdır. Ekonomik göstergeler ile belli ürünlerin talepleri arasında korelasyon olduğunu gösteren pek çok çalışma mevcuttur. Bu modelde ekonomik göstergeler ile talep arasındaki korelasyondan faydalanmak adına, ekonomik göstergeler ayrık zamanlı bir Markov süreci ile gösterildiği varsayılmıştır. Ayrıca eniyi sipariş politikasının yapısı karakterize edilmiştir. Buna ek olarak incelenen model üç ayrı açıdan genişletilip, bu durumlardaki eniyi sipariş politikasının yapısı incelenmiştir. Son olarak, incelenen sipariş modeli sayısal örneklerle açıklanmıştır.

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NOMENCLATURE

b	:	Lost sales cost per one unit of inventory
c_n	:	Purchasing cost of one unit of inventory at period n
r	:	Revenue gained from one unit of inventory
s	:	Salvage value for one unit of inventory
α	:	The periodic discounting factor
q_n	:	The order quantity at period n
X_n	:	The state of the information process at period n
P	:	Transition matrix of the information process
$F_D^i(x)$	:	Cumulative distribution function of demand for state i
$f_D^i(x)$	:	Probability density function of demand for state i

Chapter 1

INTRODUCTION

Inventory systems have been thoroughly studied in the operations research and management literature. Various mathematical models regarding inventory control and inventory management exist in the literature. These models vary from deciding on volume and frequency of orders, to the rules that drive these decisions. The impact of cost structure, including inventory holding cost, fixed and variable ordering costs, lost sales or backorder costs on optimal ordering policy are examined in the literature. The nature of the demand structure on inventory decisions has also been investigated by researchers in this realm.

The objective of the study presented in this thesis is to develop a mathematical model to analyze ordering decisions of a retailer dealing with a seasonal product when the retailer has multiple ordering opportunities in her timeline prior to the start of the selling season. When a retailer faces such a situation, she has two fundamental decisions to make: the timing of the orders and the order quantity for each ordering opportunity. Our aim is to investigate the optimal ordering policy of such a retailer. Here the fundamental question is should a retailer put all of her orders at a time or should she orders at different ordering opportunities. The answer to that question is alongside with her cost-information trade-off. Osadchiy et al. (2013), examines the relation with market indices and the demand of discretionary items. Specifically they showed the interrelation of market returns on the retail sales. There is an underlying theory in their work which argues that demand is correlated with financial market indices. Hence, the inventory managers may take advantage of such structure in their decision making processes. Therefore, from the perspective of the retailer it is better to wait as long as possible to gain more information on the uncertain demand prior to the season. On the other hand, from the angle of the supplier receiving the downstream demand earlier creates flexibility and reduces the uncertainty. Thus for many industries putting an order early is cheaper. Not only it helps lessening costs in production environment, in practice where multiple suppliers might be available, early orders allow longer lead-times therefore ordering from overseas suppliers. Tang et al. (2004) analyze this issue and they argue the advantages of offering advance purchase discounts to their downstream for the suppliers. Therefore it is reasonable to model purchasing cost of the products in a non-decreasing way in time. The increasing unit purchasing cost is the first leg of our trade-off argument. The second leg is to reduce uncertainty by exploiting market conditions that give information on demand. One of the ways to model the gain from information is to model an exogenous environment. The factors influence that the "exogenous environment" may include weather, economic conditions, and competition in the market (Song and Zipkin (1993)). They argue that the demand is correlated with financial indices. Therefore they develop a Markov process which modulates the demand. Our second essential assumption in this study is that our demand is modulated by a Markov chain. So we provide a framework to analyze multi-ordering opportunities given the Markov modulated demand structure when the unit purchasing costs are increasing.

To the best of our knowledge, our model is the first model which incorporates Markov modulated demand structure in advance ordering decisions with more than one ordering opportunity available for the retailer.

The main contribution of this thesis to the literature on inventory management is analysis of multiple ordering opportunities before the selling season in a single item single period inventory problem. To carry out the analysis, we extend the related works with multiple preparation epochs in the literature by modeling the exogenous information process as a Markov process. Using Markov modulation in our demand information updates is a crucial part of our work. Besides there are certain implications in the literature on the correlation of demand with market returns especially for products with short-selling season. We characterize the structure of the optimal ordering policy. In the second part of the thesis we study 3 extensions of our model. The second contribution of this work is to examine the Markov multi-ordering framework with fixed ordering costs which is the first extension of the model. In addition to the fixed-ordering costs, we examine the structure of the accumulation of the information. Finally as a special case of the main model, we present a model where the mean of the demand is known and the variance is modulated by a Markov process.

This thesis is organized as follows. In the next chapter, we review the literature on relevant inventory models. Chapter 3 describes the advance ordering model with Markov modulated demand information and analyses structural properties of the model. Chapter 4 is devoted to the extensions of the model described in Chapter 3. In Chapter 5, we illustrate the models discussed in Chapter 3 with numerical experiments. Finally in Chapter 6, we give a summary of the thesis and suggest several directions for future research.

Chapter 2

LITERATURE REVIEW

The literature that includes relevant models and approaches to our study can be listed under two main categories: Inventory control models with advance demand information and models with Markov modulated demand. In his survey, Khouja (1999) summarizes extant literature on the single-period single item (Newsvendor) inventory models and several extensions. These models include multi-product settings (with or without substitution), pricing strategies, vendor-supplier relations, random yield models, different states of information about demand and models with more than one period to prepare for the selling season. Our work falls into the last category. As was mentioned in Khouja (1999), Murray Jr and Silver (1966) is one of the first studies in which the value of information and updating is analysed. In the next part, we consider models that utilize information on demand prior to the season thoroughly.

2.1 Models with Advance Demand Information

In this part we explore previous research on inventory control models that incorporate advance information into their demand estimation processes. In the last decade there has been a growing interest in advance demand information (ADI) models. Karaesmen (2013) presents a review of fundamental ADI models in a recent work. This paper focuses on the effects of inventory and capacity sharing in a supply chain framework where 3 distinct models are considered. The first model is a basic random demand model without taking into account the capacity sharing feature. The second model is a dynamic inventory model with Poisson arrivals and constant lead-times. The last one is a make-stock queue which is also a dynamic inventory model with Poisson arrivals. In the context of models with ADI, make-to-stock queues in an important stream.

Buzacott and Shanthikumar (1994) consider an M/M/1 queue to analyze the trade-off of inventories and lead-time. In their study they assume fixed lead-times. Similarly, Karaesmen et al. (2002) analyze a discrete-time M/M/1 make-to-stock queue system Karaesmen et al. (2003, 2004) explore the performance of make-to-stock queues with fixed lead-times. They consider the characterization of M/M/1 and M/G/1 queues to establish the value of advance demand information. Wijngaard and Karaesmen (2007) present an M/D/1 make-to-stock queue and analyze the impact of ADI on cost reduction.

Milgrom and Roberts (1988) consider a single period random demand model where they analyze getting information on demand by communicating with customers at a cost.Their model allows them to analyze the value of the information on demand through communicating with customers for a cost. Bourland et al. (1996) study the impact of timely demand information in a two-stage supply chain with a component supplier and a final assembly plant who faces the demand. They find that value of information increases with suppliers service level. DeCroix and Mookerjee (1997) characterize a periodic review model where supplier can choose to purchase ADI.

Chen (2001) analyzes a market segmentation model where a segment of customers may accept delays on their shipments if the retailer gives them a discounted price. They characterize both the optimal schedules and the optimal pricing scheme. As aforementioned, Tang et al. (2004) prove that suppliers are better off by offering advance purchase discount. In their work they focus on a case on sales of "moon cake" with an advance order program they called advance booking discount (ABD). They examine under what conditions ABD is favorable and characterize the optimal discount price. Gallego and Özer (2001, 2003) and Özer and Wei (2004) cover the optimal replenishment and procurement policies for multi-echelon inventory problems under advance demand information. Gurnani and Tang (1999) consider a retailer with a short selling season product where the retailer has exactly 2 opportunities in her timeline prior to the season to order. They present a model where the retailer observes an information process which is correlated to the demand in her second ordering period. Similar to our model they analyze the cost-information trade-off. Moreover their model includes the stochasticity of the the unit cost for the second period. Gavirneni et al. (1999) study value of information sharing in a two-echelon supply chain with a capacitated supplier. They reveal the value of sharing information by considering three levels of information: no, partial and full information sharing models. Gayon et al. (2009a) also consider a make-to-stock capacitated supplier. In their model there are different class of customers. Some classes may put their orders in advance, however they might change or cancel their orders. These classes vary in their demand rates and cancellation probabilities. The authors examine the optimal production policy and show that it is a state-dependent base-stock policy. Gilbert and Ballou (1999) focus on operations of a steel distributer, they consider a pricing strategy to encourage advance ordering by a make-to-stock queue. Hariharan and Zipkin (1995) model an uncapacitated production model with fixed demand and supply lead times. They find that demand lead times affect the performance measures of the system in a way similar to the reduction of supply lead-times. Marklund (2006) analyze evaluating costs and controlling inventories in divergent supply chains where the customers provide perfect advance-order information by considering 3 reservation policies. Koçağa and Şen (2007) investigate a single-echelon spare parts distribution system with two demand classes with advance order information. They present that ADI with efficient inventory allocation leads to cost savings.

One of the earlier studies that exploits more than one ordering opportunity is Lau and Lau (1997). They construct a framework with a re-order opportunity after the selling season starts. The orders realized in early-season is an indicator of whole season demand; therefore, the retailer has a chance to exploit. In their analysis they compare one-order and two-order cases under normal and beta demand distributions. Although their model may not be classified under advance demand information models, they show that observation of demand is valuable. Tan (2008) present a forecasting framework for imperfect ADI. Tan et al. (2009) also consider imperfect ADI in a single-stage inventory allocation system with 2 demand classes. Mostard et al. (2011)

consider ways of forecast the demand if there exist pre-order data from a group of customers. They use 5 different forecast methods and they perform an empirical study with the data they had been provided by a private company. Their study is one of the most recent work in the literature related to forecasting of the demand in the textile industry. In the next section we consider the models with quick response strategy.

Models based on Quick Response

Quick response (QR) policy is a well-known inventory management strategy which is used successfully by a number of major retailers. Its idea is to respond quickly to market changes and cut ordering lead times (Hammond (1990)). It was first established in the 1980s in the American clothing industry. QR is essential for managing fast-moving products with short serviceable life (also known as fast-fashion products). Fast-moving products have a long lead time relative to their selling season length and they are not usually carried over into future selling periods without major loss in profit. Typical industries where QR practice is popular include the fashion clothing and toys industries. Fisher and Raman (1996) developed a case study which is an implementation of quick response approach at Sport Obermeyer, a leading fashion ski wear designer and manufacturer. The case focuses on comparison of two production locations in Hong Kong and Mainland China. Ordering from Hong Kong is more expensive; however, it is faster, more flexible and allows smaller lot sizes. That work includes the analysis for approximately 800 stock keeping units (SKU). They also observe that quick response systems allow for adjusting the inventory of a retail item as closely as possible to the optimal level by updating the demand forecast with market data .

Donohue (2000) studies efficiency of the supply contract for the fashion industry. They consider 2 modes of production which differ in production unit costs: slow and fast modes. Additionally they analyze decentralized case where the situation is a Stackelberg game. In addition to Donohue (2000), Chen et al. (2006, 2010) works on the coordination issues with different price structures.

Iyer and Bergen (1997) study the benefits of accurate response practices on the profitability of suppliers and manufacturers. It is the first study on the QR strategy that have both a single-manufacturer and a single-retailer in the supply chain. In their two stage model market information is realized and then the retailer updates her estimation on the mean of the demand with known variance for the second stage in a Bayesian fashion. Choi et al. (2006) develop another forecast updating model that uses Bayesian updates. Their model goes beyond the model of Iyer and Bergen (1997) by allowing dual-updating where both mean and variance are unknown and subject to updates. Eppen and Iyer (1997) also study flexibility through backup agreement contracts when information updating is available.

Quick response is an essential strategy for the "Fast Fashion" industry. Cachon and Swinney (2011) question the importance of the QR for a fast fashion system. They investigate value of QR by comparing three arrangements: QR, both QR and enhanced product design, and traditional system with neither QR nor enhanced product design. They also analyze the effect of existence of strategic customers.

Caro and Martınez-de Albéniz (2014) conduct an extensive review regarding "Fast Fashion" industry and analyze future opportunities in this stream of research. They have data of ten fast fashion retailers which they analyzed. Serel (2009) simultaneously analyses inventory management and pricing in a quick response system. Choi and Sethi (2010) established another recent review on innovative quick response programs. Their work includes single and multiple delivery modes, exogenous on endogenous pricing and the value of information in recent technologies such as RFID systems.

Martingale Model of Forecast Evolution

In this section we present the martingale model of forecast evolution (MMFE) method and its use in the inventory control literature. As developed by Graves (1999) and Heath and Jackson (1994), MMFE has been used in the literature to model the gain from forecast updating of the demand. MMFE is a descriptive model that characterizes the sequence of forecast update vectors where these update terms (or error terms) are independent and identically distributed with the normal distribution. Graves (1999) develop an inventory model where demand is nonstationary and has an autoregressive integrated moving average (ARIMA) structure. By that time the models in that context had stationary demand structure where, in practice, stationary demand is not applicable.

Güllü (1996) studies an additive evolution MMFE model in a capacitated structure and compares a forecast evolution model and the standard single product multi-period inventory model. Under certain distributional assumptions, the author proves that forecast evolution model has lower expected cost compared to the standard model.

Toktay and Wein (2001) focus on forecast updating in a capacitated single item make-to-stock production system with stationary demand. Their results indicate optimality of a state-dependent modified base-stock policy and they set the setup cost in the model to 0.

As Milgrom and Roberts (1988) initiates Zhu and Thonemann (2004) also analyze a situation in which a retailer could get information from customers on their demand in a decentralized supply chain. Additionally, they observe the imperfect information and utilize MMFE framework to have correlated demand. Oh and Özer (2013) also utilize MMFE in their demand structure. They generalized the forecast evolution framework for the existence of multiple forecasters. Moreover they consider the imperfect information case among parties which they referred to as Martingale Model of Asymmetric Forecast Evolutions. Their framework allows them to analyze multi-agent mechanism design problems by incorporating a fluctuating demand environment.

As we mentioned earlier, Wang et al. (2012) consider a newsvendor model with multiple ordering opportunities similar to ours. However, they directly adopt MMFE in their demand structure while our model consider a more generalized case where the demand is modulated by a Markov chain. They characterize the optimal base-stock levels analytically and they showed the optimality of the base-stock policy. Their analysis is significance in the sense that they analyze increasing cost and value of information trade-off.

2.2 Models with Markov Modulated Demand

We model the fluctuating information on demand in a Markovian demand structure. Iglehart and Karlin (1962) were one of the first who consider periodic-review model where demand is modulated by a Markov chain over a discrete-time. Song and Zipkin (1993) present the first model which considers Markov modulated fluctuating demand as a generalization of Poisson process. Their work is strongly related to Iglehart and Karlin (1962). In comparison to the previous study, Song and Zipkin (1993) has a more tractable model. They allow fixed ordering costs, stochastic lead times and countably-infinite state space.

Modeling the demand with Markov modulated structure is well-established in inventory control literature. Bensoussan (2012) discusses several implications of the Markovian demand in their chapter including both backlogging and loss-sale situations in a multiperiod inventory model.

Özekici and Parlar (1999) consider a multi-period dynamic inventory model where they focus on the unreability of the supplier. In their analysis, the Markovian structure that modulates the demand process also affects the supply and cost parameters. They showed that in the absence of the fixed order cost, the optimal ordering policy is an environment-dependent base-stock policy. When the fixed ordering cost enters to the picture, the optimal policy is a state-dependent (s, S) policy. Erdem and Özekici (2002) also focuses on random supply with fluctuating demand as Özekici and Parlar (1999). They both analyze inventory models in random environments with random yields. However, their supply structure is different. The optimal policy in their model is a base-stock policy where optimal base-stock levels are state-dependent.

Gayon et al. (2009b) investigates a multi-class production system where they allow cancellation of orders. Besides the lead times are random. They investigate the structure of the production control policy and the inventory allocation policy. The optimal production policy is a state-dependent base-stock policy.

Yin and Rajaram (2007) also analyze a periodic review model with Markov modulated demand structure. Their model jointly considers inventory control and pricing in a periodic review problem over a finite horizon. In their model, the authors adapt state-dependent variable, fixed and surplus costs. In addition to being dependent on the state for the cost structure, they showed benefits of dynamic pricing in their numerical instances.

Gallego and Hu (2004) model both their demand and supply processes as Markov processes. They have two independent Markov chains that separately derives the evolution of the demand and supply processes. The authors analyze the model with structural results and they provide numerical illustrations. Özkan et al. (2013) also model their demand process as Markov modulated in a resource capacity control revenue management problem.

The last work we consider in this section belongs to Song and Zipkin (2012). They study a framework where there is sequentially revealed information on demand. In their model the retailer orders once and she may put cancellation on orders observing Markov process signals. Their angle of modeling an exogenous process as Markov is quite similar to the way we structure our model.

Our Model

As we mentioned in the first chapter, we believe that our model is the first model which incorporates Markov modulated demand structure in advance ordering decisions with multiple ordering periods prior to the selling season. As Chapter 2 demonstrates ADI is well-acknowledged in the production and inventory control literature. Furthermore using Markov processes to model the economic uncertainty is effective and prevalent. Thus in its nature our model shares common features with the existing models on inventory management literature. For the comparison purposes, we want to highlight two studies here. Wang et al. (2012) examines a multi-period advance ordering model, similar to our model. They adopt a special case of MMFE from Oh and Özer (2013). In their study they have N ordering periods and at each of these periods the retailer observes an error factor. They use two different demand structures. Their first structure is the addictive structure where the error terms observed in each period accumulates and constitutes the total demand. Second, they provide the same approach to a multiplicative model which evolves in multiplication of the error terms observed during the planning season. Even though the structure of the unit purchasing cost vectors look alike, our demand modeled as Markov modulated can be considered as a generalized version of MMFE. Secondly, Song and Zipkin (2012) consider the situation that demand is driven by an exogenous Markov process similar to ours. However in their model the retailer is allowed to order one specific time. They utilize the environmental information for cancellations. In that sense, their model can be seen as the dual of our model.

Chapter 3

ADVANCE ORDERING MODEL WITH MARKOV MODULATED DEMAND INFORMATION

In this chapter we consider an ordering methodology to a single-period inventory model with multiple ordering opportunities in advance. In Section (3.1) we introduce our mathematical model and in Section (3.2) of this chapter we analyze the mathematical structure of the model.

3.1 Model Definition

In this section we formulate a finite horizon, discrete time control model for a single-period inventory problem where information is available on demand for N ordering periods prior to the selling season. To analyze the value of the information, we need to define the information process. The information process $X = \{X_n; n = 1, 2, \cdots, N\}$ is a discrete time Markov chain with state space E and probability transition matrix P. The state of X at any period gives information on demand during the selling season. The retailer observes X_n at the beginning of period n for N decision periods. In each period, the retailer decides the ordering quantity after observing X_n in the beginning of period n. We denote the order quantity of period n with q_n and $Q = (q_1, q_2, \cdots, q_N)$ denotes the vector of order quantities for all periods until the selling season.

The last period to give an order is period N. At the beginning of period N, state X_N is observed and q_N is ordered. Subsequent to period N, the selling season starts and demand D occurs during the selling season. Without loss of generality we suppose that D is a non-negative continuous random variable with cumulative probability distribution function $F_D^i(x) = P(D \leq x | X_N = i)$ and probability density function $f_D^i(x)$.

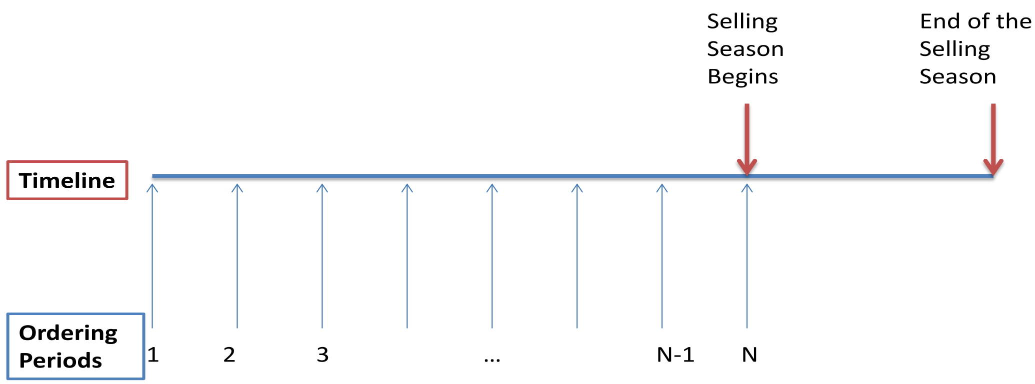

Figure 3.1: Ordering Periods Illustration

The timeline of the ordering periods is given in the Figure 3.1. Orders arrive at the beginning of the selling season. Thus, the retailer does not hold any type of inventory during the ordering periods.

In this setting the unit procurement cost of items at period n is c_n which is assumed to be non-decreasing in n. Total procurement cost in a period is linear in ordering quantity in that period. Our key goal here is to analyze gains from information on demand D versus increasing unit procurement costs on the time line as time elapses.

At period N, the problem turns into a classical single-period inventory problem. Revenue from each unit sold is shown with r. The stock-out cost due to unit lost-sale is b, s is the salvage of each item left in the inventory and α is the periodic discounting factor of future profit. We impose certain conditions on the economic parameters of the model for nontrivial results. First, $r > c_n > s$ for all $n = 1, \cdots, N$. Second, the stock-out cost $b \geq 0$. Additionally, $c_n < \alpha c_{n+1}$.

We let $V_n(i, x)$ denote the maximum expected profit from period n on when information process is in state i and the x is the total number of orders that has been placed in the first $n-1$ periods. y is the number of orders placed so far, including period n. The optimal ordering policy is obtained by solving the Bellman equation

$$V_n(i,x) = \max_{y \geq x} G_n(i,y) + c_n x \tag{3.1}$$

where

$$G_n(i,y) = -c_n y + \alpha \sum_{k \in E} P_{ik} V_{n+1}(k,y). \tag{3.2}$$

Moreover, the boundary condition for the last period is

$$V_N(i,x) = \max_{y \geq x} L(i,y) + c_N x \tag{3.3}$$

where

$$L(i,y) = -c_N y + E\left[r \min\{D,y\} + s \max\{0, y-D\} - b \max\{0, D-y\} | X_N = i\right]. \tag{3.4}$$

3.2 Structural Properties

In this section we explore the structural properties of the optimal ordering policy. To discuss the structure we need to consider the profit-to-go function $V_n(i,x)$ in the dynamic programming model. Thus we first examine the mathematical structure of $V_n(i,x)$. We show that $V_n(i,x)$ is concave in x for all i and n. Therefore, the optimal ordering policy is a base-stock policy. We demonstrate concavity of $V_n(i,x)$ by mathematical induction in Lemma [3.2.1].

Lemma 3.2.1. *$V_N(i,x)$ is concave in x for all $i \in E$.*

Proof. When we look at the structure of $V_N(i,x)$ shown in (3.3), we observe that $V_N(i,x)$ is simply summation of maximization of $L(i,y)$ and $c_N x$. Since $c_N x$ is a linear function of x, it is concave in x. To demonstrate $V_N(i,x)$ is concave in x, we need to show that the first term of right-hand side of the (3.3) is concave in x.

First we show concavity of $L(i,y)$ in y. In (3.4) we substitute $\min\{D,y\} = y - \max\{0, y-D\}$ and $\max(0, D-y) = D - y + \max(0, y-D)$. Then, (3.4) becomes

$$L(i,y) = -c_N y + E\left[r(y - \max\{0, y-D\}) + s\max\{0, y-D\} - b(D-y) - b\max\{0, y-D\} | X_N = i\right]. \quad (3.5)$$

Then, we rearrange the terms to obtain

$$L(i,y) = (r+b-c_N)y - (r+b-s)E[\max\{0, y-D\}|X_N = i] - bE[D|X_N = i]. \quad (3.6)$$

When the expectation is calculated $L(i,y)$ becomes

$$L(i,y) = (r+b-c_N)y - (r+b-s)\int_0^y (y-x)f_D^i(x)dx - bE[D|X_N = i]. \quad (3.7)$$

Since $L(i,y)$ is differentiable in y, it allows us to track concavity by taking derivatives. If the first derivative of a function is monotonically decreasing then it is concave. Besides, the maximizer of the function is where the first derivative equal to 0. The first and second derivatives of $L(i,y)$ are

$$\frac{dL(i,y)}{dy} = (r+b-c_N) - (r+b-s)F_D^i(y) \quad (3.8)$$

and

$$\frac{d^2L(i,y)}{dy^2} = -(r+b-s)f_D^i(y). \quad (3.9)$$

For concavity, the second derivative $-(r+b-s)f_D^i(y)$ should be less than or equal to 0. The nontriviality restriction on the economic parameters of the model $r \geq s$ and $b \geq 0$ so that $(r+b-s)$ is always non-negative, leads to non-positivity of the second derivative. It is clear that first derivative of $L(i, y)$ is decreasing in y and $L(i, y)$ is concave in y for all i.

According to Heyman and Sobel (2003) convexity is preserved under minimization. Applying their theorem to our model we say that concavity is preserved under maximization. The maximization operation to $L(i, y)$ over the convex set of $y \geq x$ maintains concavity. Therefore, $V_N(i, x)$ is concave in x . □

As a preparation step to the following proposition, in Lemma [3.2.1] we prove the concavity of $V_N(i, x)$, next we consider concavity of $V_n(i, x)$ for all n.

Proposition 3.2.2. *$V_n(i, x)$ is concave in x for all i and n.*

Proof. By mathematical induction method;

a) From Lemma [1], $V_N(i, x)$ is concave which constitutes the boundary argument for induction,

b) Assuming $V_{n+1}(i, x)$ is concave in x, we will show that $V_n(i, x)$ is also concave in x for all i.

We consider the structure of the function $V_n(i, x)$ in (3.1) where $G_n(i, y)$ is given in (3.2). Since P_{ij} is the probability of progressing from state i to state j for the information process, the function $\sum_{j \in E} P_{ij} V_{n+1}(j, y)$ is a positive linear combination of concave functions since $V_{n+1}(i, x)$ is concave in x by the induction assumption. According to Zangwill (1969) positive linear combinations of convex functions are convex. It is also true for combination of concave functions. It is seen in (3.1) that $G_n(i, y)$ is summation of a linear and a concave function in y, therefore it is also concave.

As we mention in the proof of Lemma [1], concavity is preserved under maximization. Since $G_n(i, y)$ is concave in y, $V_n(i, x)$ is concave in x because of the induction assumption and maximization operation. □

Until this part, we focus on the mathematical structure of profit-to-go function. From now on using the concavity property of $V_n(i, x)$, we present the characterization of the optimal ordering policy starting with from Proposition [3.2.3]

Proposition 3.2.3. *Optimal ordering policy is a base-stock policy which is dependent on the information state i.*

Proof. From Proposition [3.2.2] we know that $V_n(i,x)$ is concave. Therefore, $V_n(i,x)$ has a global maximizer for every i. Let $S_n(i)$ be the maximizer of $G_n(i,x)$. For every period n, if starting ordering quantity $x \leq S_n(i)$, then the order quantity for this period becomes $q_n^*(i) = S_n(i) - x$; if $x > S_n(i)$ then $G_n(i,x)$ is nonincreasing to the right of x (by concavity) so $q_n^*(i) = 0$. Thus, the optimal policy is a base-stock policy. □

So far we did not need to compare the information states in E, we have not stated a rule to order information states. Therefore we need the ordering of information states. The ordering is as follows: An information state i is better than state j $(i \geq j)$ if and only if D_i, the demand that will be observed at the selling season when $X_N = i$, is first-order stochastically larger than D_j. Equivalently, from the definition of the first-order stochastic dominance in terms of the cumulative distribution function of the D_i and D_j; state i is better than state j if $F_D^i(x) \leq F_D^j(x)$.

Now we show that information states and base-stock levels for a period have the same ordering. To do that we impose the following IFR property of Markov chains which is the main assumption of the following proposition. We suppose that $E = \{1, 2, \cdots, S\}$ consists of ordered states.

Definition 3.2.4. *A Markov chain with state space E and transition matrix P is said to be increasing failure rate (IFR) if*

$$f(i) = \sum_{j=k}^{S} P_{ij} \tag{3.10}$$

is nondecreasing in i for all $k = 1, 2, \cdots, S$.

IFR Markov chains is one of the class of Markov chains where rows of its transition probability matrix are in increasing stochastic order. As may be observed from this definition, the IFR property provides a structure such that the information is more likely to improve if transition probability matrix is IFR. Base-stock level of period N increases with the increase in observed information state. In order to see non-decreasing trend in the base-stock levels at better information states, we use the first derivative of $V_n(i,x)$ which we denote with $V_n'(i,x)$.

Throughout the remainder of this thesis we suppose that X is IFR unless stated otherwise. In the next proposition, we present that the ordering of profit-to-go functions is in the same direction with the ordering of the information states for a given period n.

Proposition 3.2.5. *$V_n(i,x) \leq V_n(j,x)$ when b=0, for all $i \leq j$ and $n = 1, \cdots, N$.*

Proof. By mathematical induction:

First we show the statement for period N, remember that $V_N(i,x)$ in (3.3) is

$$V_N(i,x) = \max_{y \geq x} L(i,y) + c_N x. \tag{3.11}$$

To show that it is increasing in i,we need to show that $\max_{y\geq x} L(i,y)$ is increasing in i. If we show that $L(i,y)$ is increasing in i, then the induction hypothesis holds for period N. Therefore we first rewrite $L(i,y)$ using (3.6) as

$$L(i,y) = (r - c_N)y - (r - s)E[\max\{0, y - D\}|X_N = i] \tag{3.12}$$

by setting $b = 0$. Here it is clear that to compare the values of $L(i,y)$ for different states, comparing the expectation term $E[\max\{0, y - D\}|X_N = i]$ is sufficient. We demonstrate the comparison of $E[\max\{0, y - D\}|X_N = i]$ for different states by constructing a coupling argument. Let D_i and D_j be random variables with first-order stochastic ordering

$$D_i \succeq D_j \tag{3.13}$$

for $i \geq j$.

Then, we can write

$$y - D_i \preceq y - D_j \tag{3.14}$$

which implies

$$\max\{0, y - D_i\} \preceq \max\{0, y - D_j\}. \tag{3.15}$$

for $i \geq j$. Finally, we can conclude the decreasing behavior of $E[\max\{0, y-D\}|X_N = i]$ using (3.15) and the definition of expected value of random variables. Therefore $V_N(i,x)$ is increasing in i. Note that, for the cases with $b > 0$, it is not possible to show the increase of $V_N(i,x)$ at better states without certain assumptions on the problem parameters.

We use mathematical induction to complete the proposition for all ordering periods. The last period optimal expected profit function $V_N(i,x)$ forms the boundary condition of the induction argument that $V_n(i,x)$ is increasing in x. Using $V_n(i,x) = \underset{y\geq x}{max} G_n(i,y) + c_n x$, if we show $G_n(i,y)$ is increasing in i; then we are done. By simply looking at the components of $G_n(i,y)$ in (3.2), we observe that

$$G_n(i,y) = -c_n y + \alpha \sum_{k\in E} P_{ik} V_{n+1}(k,y)$$

Since the transition matrix P is IFR, using the property of IFR matrices if $V_{n+1}(i,y)$ is increasing in i, then $\sum_{k\in E} P_{ik} V_{n+1}(k,y)$ is also increasing in i. We have already shown that last period function is increasing in i, which constitutes the boundary of our induction argument. Thus, $V_n(i,x)$ is increasing in i. This completes the proof. □

Proposition 3.2.6. *a)* $V_n'(i,x) \leq V_n'(j,x)$ *for all* $i \leq j$ *and* $n = 1, \cdots, N$,

b) $S_n(i) \leq S_n(j)$ *for all* $i \leq j$ *and* $n = 1, \cdots, N$.

Proof. By mathematical induction: We show statements a) and b) synchronously. First we show b) for period N. In Lemma [3.2.1], concavity of $L(i,x)$ was shown; in addition, Proposition [3.2.3] demonstrates that the optimal ordering policy is base-stock. From the definition of base-stock policy, $S_N(i)$ is the maximizer of the $L(i,x)$. Therefore base-stock levels is where the first derivative of $L(i,x)$ equals to 0. This implies

$$L'(i, S_N(i)) = 0 \tag{3.16}$$

or

$$L'(i, S_N(i)) = (r + b - c_N) - (r + b - s)F_D^i(S_N(i)) = 0. \tag{3.17}$$

Rearranging the terms, we obtain the explicit solution

$$F_D^i(S_N(i)) = (r + b - c_N)/(r + b - s). \tag{3.18}$$

Now to compare base-stock level $S_N(i)$ for information state i with base-stock level $S_N(j)$ for any other information state j, we use (3.18). It is known that $F_D^i(x)$ is the cumulative probability function of D when information state is observed as i. It is monotonically non-decreasing in x. Since right-hand side of (3.18) is independent of information state

$$F_D^i(S_N(i)) = F_D^j(S_N(j)). \tag{3.19}$$

We already know that $F_D^i(x) \geq F_D^j(x)$ for any x whenever $i \leq j$ by our assumption. Replacing x with $S_N(i)$, $F_D^i(S_N(i)) \geq F_D^j(S_N(i))$. Also using (3.19)

$$F_D^i(S_N(i)) = F_D^j(S_N(j)) \geq F_D^j(S_N(i)). \tag{3.20}$$

Since $F_D^j(x)$ is a monotonically non-decreasing function of x,

$$S_N(j) \geq S_N(i). \tag{3.21}$$

Now we will show b) for period N.

Remember that $V_N(i, x)$ in (3.3) has the form

$$V_N(i, x) = \max_{y \geq x} L(i, y) + c_N x \tag{3.22}$$

which can be written as

$$V_N(i,x) = \begin{cases} L(i, S_N(i)) + c_N x & x \le S_N(i) \\ L(i,x) + c_N x & x > S_N(i) \end{cases} \tag{3.23}$$

Therefore, the derivative of $V_N(i,x)$ is

$$V_N'(i,x) = \begin{cases} c_N & x \le S_N(i) \\ L'(i,x) + c_N & x > S_N(i) \end{cases}. \tag{3.24}$$

As shown shortly before, $S_N(j) \ge S_N(i)$. This information allows us determine the ordering of $V_N'(i,x)$ in i. To compare $V_N'(i,x)$ and $V_N'(j,x)$ for all x and show that

$$V_N'(i,x) \le V_N'(j,x) \tag{3.25}$$

we need to consider 3 different regions.

1-) $x < S_N(i) \le S_N(j)$

In this region

$$V_N'(i,x) = V_N'(j,x) = c_N \tag{3.26}$$

so that statement (3.25) holds.

2-) $S_N(i) \le x \le S_N(j)$

In this region $V_N'(i,x) = L'(i,x) + c_N$ and $V_N'(j,x) = c_N$. Since $L'(i,x)$ is negative for $x > S_N(i)$,

$$V_N'(i,x) = L'(i,x) + c_N < c_N = V_N'(j,x). \tag{3.27}$$

Hence statement (3.25) holds in this region.

3-) $S_N(i) \le S_N(j) < x$

For the last region, $V_N'(i,x) = L'(i,x) + c_N$ and $V_N'(j,x) = L'(j,x) + c_N$. To demonstrate the ordering, we need to compare $L'(i,x)$ and $L'(j,x)$ in this region. Recall that

$$L'(i,x) = (r+b-c_N) - (r+b-s)F_D^i(x).$$

To complete the reasoning here, consider the main assumption between state i and state j which states that $i \leq j$ if and only if $F_D^j(x) \leq F_D^i(x)$. This clearly implies that

$$L'(i,x) \leq L'(j,x) \tag{3.28}$$

so that statement (3.25) is true.

We have shown both statements of Proposition (3.2.6) for period N. The boundary conditions of the induction argument are demonstrated. Now, assuming statements for periods $n+1$ to N are true we show them for period n.

Similar to the proof of period N, to show ordering of order-up-to levels; we use the definition of base-stock level of period n. It is the maximizer of the $G_n(i,x)$. Hence base-stock level $S_n(i)$ is where the first derivative of $G_n(i,x)$ equals to 0. This implies

$$G_n{}'(i, S_n(i)) = 0 \tag{3.29}$$

or

$$G_n'(i, S_n(i)) = -c_n + \sum_{k\in E} \alpha P_{ik} V_{n+1}'(k, S_n(i)) = 0 \tag{3.30}$$

for any state i. The induction hypothesis now supposes that $S_{n+1}(j) \geq S_{n+1}(i)$ and $V_{n+1}'(i,x) \leq V_{n+1}'(j,x)$. Then, for period n to show the ordering of $S_n(i)$ and $S_n(j)$, we use the IFR property of transition matrix P, and say that $\sum_{k\in E} P_{ik} V_{n+1}'(k,x)$ is increasing in i. Then (2) implies that

$$G_n'(i,x) \leq G_n'(j,x). \tag{3.31}$$

Replacing x with $S_n(i)$ in equation (3.31), $G_n'(i, S_n(i)) \leq G_n'(j, S_n(i))$. Also, using (3.29)

$$G'_n(i, S_n(i)) = G'_n(j, S_n(j)) \leq G'_n(j, S_n(i)).$$

Since $G'_n(i,x)$ is a monotonically non-increasing function of x,

$$S_n(j) \geq S_n(i). \tag{3.32}$$

We have now completed b) for every period n. The proof will be completed by demonstrating

$$V'_n(i,x) \leq V'_n(j,x). \tag{3.33}$$

Recall the mathematical structure of $V_n(i,x)$

$$V_n(i,x) = \begin{cases} G_n(i, S_n(i)) + c_n x & x \leq S_n(i) \\ G_n(i,x) + c_n x & x > S_n(i) \end{cases} \tag{3.34}$$

for n=$1, 2, \cdots, N-1$. Then the first derivative $V'_n(i,x)$ is

$$V'_n(i,x) = \begin{cases} c_n & x \leq S_n(i) \\ G'_n(i,x) + c_n & x > S_n(i) \end{cases} \tag{3.35}$$

for n=$1, 2, \cdots, N-1$.

Similar to the relation in the last period, to show the ordering of $V'_n(i,x)$ we need to examine 3 different regions.

1-) $x < S_n(i) \leq S_n(j)$

Very similar to the last period, in this region

$$V'_n(i,x) = V'_n(j,x) = c_n. \tag{3.36}$$

Hence statement (3.33) holds in this region.

2-) $S_n(i) \leq x \leq S_n(j)$

In this region $V'_n(i,x)$=$G'_n(i,x) + c_n$; and $V'_n(j,x) = c_n$. Since $G'_n(i,x)$ is negative for the $x > S_n(i)$,

$$V_N'(i,x) = G_n'(i,x) + c_n < c_n = V_n'(j,x). \quad (3.37)$$

So statement (3.33) also holds.

3-) $S_n(i) \leq S_n(j) < x$

For the remaining region, $V_n'(i,x) = G_n'(i,x) + c_n$ and $V_n'(j,x) = G_n'(j,x) + c_n$. To demonstrate the ordering, we need to compare $G_n'(i,x)$ and $G_n'(j,x)$ in this region. Recall that

$$G_n'(i,x) = -c_n + \sum_{k \in E} \alpha P_{ik} V_{n+1}'(k,x). \quad (3.38)$$

From the IFR property of the transition matrix P, $\sum_{k \in E} P_{ik} V_{n+1}'(k,x)$ is increasing in i. Therefore,

$$G_n'(i,x) \leq G_n'(j,x) \text{ and } V_n'(i,x) \leq V_n'(j,x)$$

So the proof is completed for both statements and for all periods n. □

We define the term $p-$stochastic dominance as a generalized version of the first-order stochastic dominance. It differs from the classical first-order stochastic dominance in the region of the ordering of cumulative distribution functions.

Definition 3.2.7. *A random variable X has $p-$stochastic dominance over another random variable Y if $F_X(x) \leq F_Y(x)$ for $x \geq F_Y^{-1}(p)$.*

In this context, classical first order stochastic dominance may be referred as 0-stochastic dominance. Because for $p = 0$, $x \geq F_Y^{-1}(p)$ becomes $x \geq 0$, which is simply the first-order stochastic dominance. So far, we have imposed 0-stochastic dominance on the demands observed in different states. However, under $p-$stochastic dominance of the demands most of the structural properties of the model still holds.

Proposition 3.2.8. *If the demands observed in ordered states have $p-$stochastic ordering for all information states with $p \leq (r + b - c_N)/(r + b - s)$, then Proposition [3.2.6] holds.*

Proof. We will follow an approach similar to that of Proposition [3.2.6]. First we need to show that the base-stock levels are ordered in terms of i for period N. In the proof of Proposition [3.2.6], we stated that the optimal ordering policy is base-stock policy based on the concavity of $V_N(i,x)$. Here we observe that for the $p-$stochastic dominance assumption, the concavity of $V_N(i,x)$ is still preserved. As shown in the proof of Lemma [3.2.1], $V_N(i,x)$ is concave for every i regardless of the stochastic ordering between the information states. Next, using the definition of the base-stock levels, we demonstrate the ordering of the base-stock levels for different states as in the proof of Proposition [3.2.6]. Remember when x equals to $S_N(i)$, $L(i,x)$ attains its maximum value. So the first derivative of $L(i,x)$ is equal to 0 at that point such that

$$L'(i, S_N(i)) = 0 \tag{3.39}$$

or

$$L'(i, S_N(i)) = (r + b - c_N) - (r + b - s)F_D^i(S_N(i)) = 0. \tag{3.40}$$

Then

$$S_N(i) = (F_D^i)^{-1}((r + b - c_N)/(r + b - s)).$$

Since

$$p \leq (r + b - c_N)/(r + b - s))$$

we obtain

$$F_D^j(S_N(i)) \leq F_D^i(S_N(i)) = (r + b - c_N)/(r + b - s)$$

and this implies that

$$S_N(j) \geq S_N(i)$$

for all $i \leq j$.

Next we show that $V_N'(i,x)$ is ordered in i. Remember that

$$V_N(i,x) = \begin{cases} L(i, S_N(i)) + c_N x & x \leq S_N(i) \\ L(i,x) + c_N x & x > S_N(i) \end{cases} \tag{3.41}$$

and its derivative is

$$V_N'(i,x) = \begin{cases} c_N & x \leq S_N(i) \\ L'(i,x) + c_N & x > S_N(i) \end{cases} . \tag{3.42}$$

As in the proof of Proposition [3.2.6], we use the relation $S_N(i) \leq S_N(j)$ to compare values of $V_N'(i,x)$ and $V_N'(j,x)$ for all x. Here we consider 3 different regions.

1-) $x < S_N(i) \leq S_N(j)$

Very similar to previous results, in this region,

$$V_N'(i,x) = V_N'(j,x) = c_N \tag{3.43}$$

so that $V_N'(i,x) \leq V_N'(j,x)$ for all x values.

2-) $S_N(i) \leq x \leq S_N(j)$

For this region $V_N'(i,x)$=$L'(i,x) + c_N$, and $V_N'(j,x) = c_N$. Since $L'(i,x)$ is negative for $x > S_N(i)$,

$$V_N'(i,x) = L'(i,x) + c_N < c_N = V_N'(j,x). \tag{3.44}$$

Hence statement (3.25) holds in this region.

3-)$S_N(i) \leq S_N(j) < x$

In this region, we need to be more careful to show (3.25). Here $V_N'(i,x) = L'(i,x) + c_N$ and $V_N'(j,x) = L'(j,x) + c_N$. To analyze the ordering of those values, we need to determine the ordering of $L'(i,x)$ and $L'(j,x)$ in this region. Remember that

$$L'(i,x) = (r + b - c_N) - (r + b - s)F_D^i(x).$$

According to $p-$stochastic dominance, $i \leq j$ if and only if $F_D^j(x) \leq F_D^i(x)$ for $x \geq (F_D^i)^{-1}(p) = S_N(i)$. Hence in this region of x there is an ordering between cumulative distribution functions of the demands to be observed in state i and j. Now,

$$F_D^j(x) \leq F_D^i(x) \tag{3.45}$$

implies that

$$L'(i,x) \le L'(j,x) \tag{3.46}$$

so that

$$V_N'(i,x) \le V_N'(j,x). \tag{3.47}$$

Here it should be noted that when the number of states is greater than 2; statement (3.47) holds for all i and j supposing $p-$stochastic dominance ordering among all states.

The boundary conditions of the induction argument are demonstrated for all x values is period N. Now, by induction, to show ordering of order-up-to levels, definition of base-stock levels is utilized. $S_n(i)$ is the maximizer of the $G_n(i,x)$ and

$$G_n{}'(i, S_n(i)) = 0 \tag{3.48}$$

or

$$G_n'(i, S_n(i)) = -c_n + \sum_{k \in E} \alpha P_{ik} V_{n+1}'(k, S_n(i)) = 0 \tag{3.49}$$

for any state i. Since the boundary condition of the induction hypothesis is satisfied, assuming $S_{n+1}(i) \le S_{n+1}(j)$ and $V_{n+1}'(i,x) \le V_{n+1}'(j,x)$, we will show $S_n(i) \le S_n(j)$ and $V_n'(i,x) \le V_n'(j,x)$ for period n. As before, since P is IFR and $V_{n+1}'(k,x)$ is increasing in i, $\sum_{k \in E} P_{ik} V_{n+1}'(k,x)$ is increasing in i as well. Then,

$$G_n'(i,x) \le G_n'(j,x). \tag{3.50}$$

Replacing x with $S_n(i)$ in equation (3.50), $G_n'(i, S_n(i)) \le G_n'(j, S_n(i))$. Also, using (3.48)

$$G_n'(i, S_n(i)) = G_n'(j, S_n(j)) \le G_n'(j, S_n(i)).$$

Since $G_n'(j,x)$ is a monotonically non-increasing function of x,

$$S_n(j) \ge S_n(i). \tag{3.51}$$

Next, we demonstrate

$$V_n'(i,x) \leq V_n'(j,x) \tag{3.52}$$

for period n. Recall the mathematical structure of $V_n(i,x)$ given by

$$V_n(i,x) = \begin{cases} G_n(i,S_n(i)) + c_n x & x \leq S_n(i) \\ G_n(i,x) + c_n x & x > S_n(i) \end{cases} \tag{3.53}$$

for n=$1, 2,\cdots,N-1$. Then, $V_n'(i,x)$ is

$$V_n'(i,x) = \begin{cases} c_n & x \leq S_n(i) \\ G_n'(i,x) + c_n & x > S_n(i) \end{cases} \tag{3.54}$$

for n=$1, 2,\cdots,N-1$.

Seeing that $V_n'(i,x)$ changes at the base-stock level points. We analyze the ordering of $V_n'(i,x)$ in 3 different regions in this regard.

1-) $x < S_n(i) \leq S_n(j)$

In this region

$$V_n'(i,x) = V_n'(j,x) = c_n. \tag{3.55}$$

Hence (3.52) holds.

2-) $S_n(i) \leq x \leq S_n(j)$

In this region $V_n'(i,x)$=$G_n'(i,x) + c_n$ and $V_n'(j,x) = c_n$. Since $G_n'(i,x)$ is negative for the x values $x > S_n(i)$,

$$V_N'(i,x) = G_n'(i,x) + c_n < c_n = V_n'(j,x) \tag{3.56}$$

so that statement (3.52) also holds.

3-) $S_n(i) \leq S_n(j) < x$

Here, $V_n'(i,x) = G_n'(i,x) + c_n$ and $V_n'(j,x) = G_n'(j,x) + c_n$. Ordering should be determined by comparing values of $G'_n(i,x)$ and $G'_n(j,x)$. Remember

$$G'_n(i,x) = -c_n + \sum_{k\in E} \alpha P_{ik} V'_{n+1}(k,x). \tag{3.57}$$

From the IFR property of the transition matrix P, $\sum_{k\in E} P_{ik} V'_{n+1}(k,x)$ is increasing in i. Therefore

$$G'_n(i,x) \leq G'_n(j,x)$$

and

$$V'_n(i,x) \leq V'_n(j,x).$$

So the proof is completed for all periods under $p-$stochastic dominance assumption. □

The main findings in this chapter are as follows:

- First, we introduce our main dynamic programming model in Section [3.1].

- Second, we explore the mathematical structure of the profit-to-go function and the optimal ordering policy of the retailer. The optimal ordering policy is a state-dependent base-stock policy.

- Moreover we describe IFR property for the matrices and we make IFR assumption for the probability transition matrix of the information process.

- We consider the characterization of the base-stock levels in Proposition [3.2.6].

- As a relaxation to the main stochastic dominance assumption in Proposition [3.2.6], $p-$stochastic dominance is defined. Proposition [3.2.8] represents the results under this relaxation.

Chapter 4

EXTENSIONS

4.1 Dynamic Ordering Model with Markov Modulated Demand Information and Fixed Ordering Cost

In this section we introduce fixed ordering costs in the model studied in the previous chapter. In the beginning of each period n, the retailer observes information state X_n, and decides ordering quantity q_n for that period as the in the base model. The extension here is the retailer faces a constant fixed ordering cost K if he decides to order. The unit procurement cost of items at time n is c_n which is again non-decreasing in n. Here, the total procurement cost in a period is not linear in ordering quantity in that period.

Optimal ordering policy is obtained by solving the Bellman equation

$$V_n(i,x) = \max_{y \geq x} \{K\delta(y-x) + G_n(i,y)\} + c_n x \tag{4.1}$$

where the function for last period is

$$V_N(i,x) = \max_{y \geq x} \{K\delta(y-x) + L(i,y)\} + c_N x. \tag{4.2}$$

Here, $\delta(\tau)$ is binary indicator function which is 1 only if $\tau > 0$.

Next we characterize the structure of the fixed ordering costs model.

Structural Properties

Due to the fixed ordering cost K, mathematical structure of the $V_n(i,x)$ changes. We show that $V_n(i,x)$ is not concave in x, however it is $K-$concave in x for all i and n.

Lemma 4.1.1. *$V_N(i,x)$ is $K-$concave in x for all i.*

Proof. We first rewrite $V_N(i,x)$ in (4.2) as

$$V_N(i,x) = \max_{y\geq x}\{K\delta(y-x) + L(i,y)\} + c_N x.$$

In Sethi et al. (2006), they showed that a function $m(x)$ in the form of

$$m(x) = \min_{y\geq x}\{K\delta(y-x) + h(y)\} \tag{4.3}$$

is $K-$convex if $h(y)$ is $K-$convex in y. This result holds for $K-$concave functions by changing minimum in (4.3) with maximum. In our model, $V_N(i,x)$ is in the form of the designated function which can be written as

$$V_N(i,x) = \max_{y\geq x}\{K\delta(y-x) + L(i,y)\} + c_n x. \tag{4.4}$$

In Chapter 3, in the proof of Lemma [3.2.1], we showed that $L(i,y)$ is concave in y. In other words, $L(i,y)$ is 0-concave. All concave functions are $K-$concave for all $K \geq 0$. Thus, $L(i,y)$ is $K-$concave. Clearly this completes our proof, $V_N(i,x)$ which is summation of a linear function of x and a function in the form of $m(x)$ in (4.3) is $K-$concave in x.

Proposition 4.1.2. *$V_n(i,x)$ is $K-$concave in x for all i and n.*

Proof. By mathematical induction, in Lemma [4.1.1], we show that $V_N(i,x)$ is $K-$concave in x. Assuming $V_{n+1}(i,x)$ is $K-$concave, we will prove $K-$*concavity* of $V_n(i,x)$ using a very similar approach. Recall that $V_n(i,x)$ in (4.1) satisfies

$$V_n(i,x) = \max_{y\geq x}\{K\delta(y-x) + G_n(i,y)\} + c_n x$$

and

$$G_n(i,y) = -c_n y + \alpha\sum_{k\in E} P_{ik} V_{n+1}(k,y)$$

where $V_{n+1}(k, y)$ is $K-$concave from our induction hypothesis.

Note that $\alpha \sum_{k \in E} P_{ik} V_{n+1}(k, y)$ is $\alpha K-$concave due to the fact that the summation of the P_{ik}'s equals to 1. Also $-c_n y$ is a linear function of y which indicates that it is 0-concave. Summation of a 0-concave and an $\alpha K-$concave function is $\alpha K-$concave. Since α is between 0 and 1 by definition, $\alpha K \leq K$ and an $\alpha K-$concave function is also $K-$concave. So, $G_n(i, y)$ is $K-$concave. Therefore with the same reasoning as in Lemma [4.1.1], $V_n(i, x)$ is the summation of $c_n x$, which is linear and 0-concave in x and a function in the form of $m(x)$ in Sethi et al. (2006), therefore it is $K-$concave. □

Corollary 4.1.3. *Optimal ordering policy is a state-dependent (s, S) policy.*

Proof. This follows from the K-concavity of $G_n(i, y)$ for all i and n. □

Note that an (s, S) policy is defined an ordering policy with an optimal order up to level S and a re-order point s in multi-period inventory models. In (s, S) policy the retailer orders if her inventory level is under the re-order point s and she orders to fill her inventory up-to level S. In Porteus (2002), they have shown that when the profit function of the inventory model is $K-$concave, optimal ordering policy is an (s, S) policy. In fixed cost version of our model, we have $K-$concave profit functions which also depend on the current information state. Therefore our optimal ordering policy with fixed-cost model is a state-dependent (s, S) policy.

Proposition [4.1.2] indicates $K-$concavity of $G_n(i, y)$ and $V_n(i, x)$ so an (s, S) policy is optimal for the advance ordering model with fixed ordering costs. Here S is the order up-to level as it is defined in base-stock policies. We denote S as $S_n(i)$ which is the maximizer of $G_n(i, y)$ for every fixed cost $K > 0$. It is optimal to order if the current inventory level is under the re-order point $s_N(i)$ which is defined by

$$G_n(i, s_n(i)) = G_n(i, S_n(i)) + K.$$

We will show that order-up-to levels are also ordered for this version of the model for the last period N, as we showed in Section 3.2 for base-stock levels. For this reason, we need the ordering of the first derivatives of $V_n(i, x)$ with respect to states.

Proposition 4.1.4. *$S_N(i) \leq S_N(j)$ for all $i \leq j$.*

Proof. In Lemma [4.1.1] concavity of $L(i,x)$ was shown. In addition, Corollary [4.1.3] demonstrates that the optimal ordering policy is an (s,S) policy. From the definition of (s,S) policies, $S_N(i)$ is the maximizer of $L(i,x)$. Therefore, the first derivative of $L(i,x)$ equals to 0 at $S_N(i)$. So that

$$L'(i, S_N(i)) = 0$$

or

$$L'(i, S_N(i)) = (r + b - c_N) - (r + b - s)F_D^i(S_N(i)) = 0$$

Rearranging the terms,

$$F_D^i(S_N(i)) = (r + b - c_N)/(r + b - s). \tag{4.5}$$

As we demonstrated for the main version of the model in Section 3.2, we compare order-up-to level $S_N(i)$ for information state i with order-up-to level $S_N(j)$ any other information state $j \geq i$ using (4.5). We know that $F_D^i(x)$ is a monotonically increasing function of x and

$$F_D^i(S_N(i)) = F_D^j(S_N(j)). \tag{4.6}$$

Then to show that the order-up-to level of period N is lower when observed state is i, than the better state $j \geq i$, from the stochastic ordering of state i and j that $F_D^i(x) \geq F_D^j(x)$ for all x. Replacing x with $S_N(i)$, $F_D^i(S_N(i)) \geq F_D^j(S_N(i))$. Also, using (4.6)

$$F_D^i(S_N(i)) = F_D^j(S_N(j)) \geq F_D^j(S_N(i)). \tag{4.7}$$

Since $F_D^j(x)$ is a monotonically non-decreasing function of x,

$$S_N(j) \geq S_N(i) \tag{4.8}$$

and this completes our proof. □

4.2 Dynamic Order Model with Accumulating Demand Information

In this section we expand our model with more dimensional information state transition matrices. In this setting, for every period n retailer is able to have all information state sequences from period 1 to n in the beginning of period n. Now the transition matrix of information process X expands in dimension. This setting increases the complexity of the control problem; however, it is realistic to consider the path of the information process. Here we denote $\bar{i}_n$ as the collection of information states thus far, which can be defined as $\bar{i}_n = \{i_1, i_2, \cdots, i_n\}$. So, states of the X process up-to period n is shown as $\overline{X}_n = \{X_1, X_2, \cdots, X_n\}$ rather than just the information of the current period. Here $\bar{i}_n$ constitutes the state of $\bar{X}_n$. Definition of the transition matrix should now be changed. The new probability transition matrix of the process is a $|E|^{n-1} \times |E|$ matrix for period n which grows in dimension each period. We now have the transition matrix

$$P(X_{n+1} = k | \overline{X}_n = \bar{i}_n) = P_{\bar{i}_n k}.$$

The maximum expected profit function becomes

$$V_n(\bar{i}_n, x) = \max_{y \geq x} G_n(\bar{i}_n, y) + c_n x \tag{4.9}$$

where

$$G_n(\bar{i}_n, y) = -c_n y + \alpha \sum_{k \in E} P_{\bar{i}_n k} V_{n+1}((i_1, i_2, \cdots, i_n, k), y). \tag{4.10}$$

The boundary condition is

$$V_N(\bar{i}_N, x) = \max_{y \geq x} L(\bar{i}_N, y) + c_N x \tag{4.11}$$

Here, demand D for period N is only affected by the information observed in the period N. Thus $P(D \leq x | \overline{X}_N = \bar{\imath}_N) = P(D \leq x | X_N = i_N) = F_D^{i_N}(x)$. The path of the information affects the state transition due to the new form of P compared to the model discussed in Chapter 3. The concavity of the $V_n(\bar{i}_n, x)$ is shown in the following proposition.

Proposition 4.2.1. *$V_n(\bar{i}_n, x)$ is concave in x for all $\bar{i}_n$.*

Proof. We show this by mathematical induction. Similar to the proof of Proposition [3.2.2], we first show the induction hypothesis for period N. Here note that since path of the information observed in period N has no effect on the demand, only the last observation affects the demand and $L(\bar{i}_n, y) = L(i_N, y)$. Hence $V_N(\bar{\imath}_N, x)$ is equal to $V_N(i_N, x)$. So from this equivalency we conclude the concavity of $V_N(\bar{\imath}_N, x) = V_N(i_N, x)$ using Lemma [3.2.1]. This forms the boundary argument for induction. However, it should be shown for periods $n = 1$ to $N - 1$ in more detail.

When the structure of the function $V_n(\bar{\imath}_n, x)$ is considered in (4.9), the concavity of $V_n(\bar{\imath}_n, x)$ is contingent upon the concavity of $G_n(\bar{\imath}_n, x)$, whereas concavity is preserved under maximization. Next, remember $G_n(\bar{\imath}_n, x)$ in (4.10) it is simply the summation of $-c_n x$ and $\alpha \sum_{k \in E} P_{\bar{i}_n k} V_{n+1}((i_1, i_2, \cdots, i_n, k), x)$. It is clear that $-c_n x$ is concave as it is a linear function of x. Moreover, $\alpha \sum P_{\bar{i}_n k} V_{n+1}((i_1, i_2, \cdots, i_n, k), x)$ is concave because it is positive linear combination of $V_{n+1}((i_1, i_2, \cdots, i_n, k), x)$. Utilizing concavity for the period N and knowing that positive linear combinations of concave functions are concave, we conclude that $V_n(\bar{\imath}_n, x)$ is also concave □

Proposition 4.2.2. *Optimal ordering policy is a state-dependent base-stock policy.*

Proof. As we stated in the Section [3.2] and showed for our main model, if profit function of an inventory control problem is concave then the optimal ordering policy is base-stock policy. We demonstrated in Proposition [4.2.1] that the profit function of the dynamic order model with accumulating demand information is concave. Therefore, optimal ordering policy is a state-dependent base-stock policy as we showed in Proposition [3.2.6]. □

4.3 Dynamic Order Model with Demand Information by Reducing Variance

Most of the managerial decision makers are risk-averse as March and Shapira (1987) argued. Also in his detailed review, Tang (2006) discussed several models and approaches to coping with risk-inducing uncertainty in supply chains. Looking from the risk minimization perspective, we extend our model by considering that information states represent the volatility on the demand.

In this version of the model, demand D is a normal random variable with mean μ and variance σ^2. Here observing the information process provides knowledge on σ^2. Better information state signals low variance of the D. Namely, if $i \leq j$ then $\sigma_i^2 \geq \sigma_j^2$ and we suppose that $\mu_i = \mu$ for all i.

Under normal demand assumption with ordered variances, $V_n(i, x)$ is still concave. However, for the normal distributions, stochastic ordering of D for different information states does not exist. As mentioned in Chapter [2], there are many probabilistic distributions that ensure first-order stochastic dominance. However, for normal distribution, first-order stochastic dominance can only be attained with variables which have same variances. For our modification, there is no first-order stochastic ordering among demands for different states. On the other hand, $p-$stochastic ordering enables us to draw some conclusions.

Proposition 4.3.1. *$S_n(i) \geq S_n(j)$ if $p = (r + b - c_N)/(r + b - s) \geq 0.5$ for all $i \leq j$*

Proof. If we are able to show existence of $p-$stochastic ordering here, with the advantage of Proposition [3.2.8], we will show the ordering of base-stock levels with variance.

As in the definition of $p-$stochastic dominance, for 2 random variables to be $p-$stochastically ordered, the cumulative distribution functions should be ordered after the p percentile of one of them.

If D has the normal distribution with mean μ and variance σ^2, then

$$P(D \leq x) = \Phi(\frac{x-\mu}{\sigma}) = \frac{1}{2}(1 + erf(\frac{x-\mu}{\sigma\sqrt{2}}))$$

where $erf(x)$ is

$$erf(x) = \frac{1}{\sqrt{\pi}}{}_{-x}^{\,x} e^{-t^2} dt$$

which is an increasing function of x.

Looking at the cumulative distribution functions of normal distributed random variables with the same mean μ, we see that all the cumulative distribution functions intercept at the mean such that

$$F_D^i(\mu) = 0.5$$

for all i. So for the values of $x \geq \mu \geq 0$

$$erf(\frac{x-\mu}{\sigma_i\sqrt{2}}) \geq erf(\frac{x-\mu}{\sigma_j\sqrt{2}})$$

for all $\sigma_i \leq \sigma_j$. Therefore, $F_D^i(x) \leq F_D^j(x)$ for $x \geq \mu$.

On the other hand for the values of $x \leq \mu$, the inequalities become

$$erf(\frac{x-\mu}{\sigma_i\sqrt{2}}) \leq erf(\frac{x-\mu}{\sigma_j\sqrt{2}})$$

and

$$F_D^i(x) \geq F_D^j(x).$$

As a result, there is no $p-$stochastic dominance among ordered-variances for same-mean normal random variables in this region.

Thus, normal random variables with same mean and different variances are $p-$stochastically ordered for $p \geq 0.5$. Proposition[3.2.8] shows that base-stock levels are ordered in i when there is $p-$stochastic dominance. It is exactly the case here in this version of the model and $S_n(i) \geq S_n(j)$ if $p = (r+b-c_N)/(r+b-s) \geq 0.5$. $\square$

In this chapter we analyzed the the structure of certain extensions to our main model. In the next chapter we illustrate the results we obtained thus far with examples.

Chapter 5

NUMERICAL ANALYSIS AND ILLUSTRATIONS

In this chapter, we will provide numerical illustrations for the theoretical results which we obtained and we will do sensitivity analysis on the problem parameters.

A Comparison: Ordering without Observing Advance Information

Most of the time, retailers do not receive information on demand prior to selling season. They order by taking into account the current market condition and commit their orders accordingly. In this incident the retailer is only able to exploit the current information she has. Thus to determine the order quantity for that product, she follows the current state information as an indicator of the demand and orders accordingly. More explicitly the optimal ordering quantity is the optimal quantity of the single period inventory problem with known probability density function F_D^i.

Furthermore the retailer may estimate the probabilities that the current state i changes to the state j, namely the probability transition matrix P of the states. In that case she includes P with F_D^i.

On the other hand, if a retailer has the insight that evolution of the demand follows a Markovian structure, she may exploit that information and she uses P^n instead of P in her calculations.

In this section we compare the ordering schemes and the expected profit of these two types of retailers with the advance information taker retailer who utilizes the model presented in Chapter 3. In the remainder of this work, we compare these 3 types of retailers. We refer to the retailer who orders according to current state of the information as the standard retailer. The second type of retailer who consider P^n while putting her orders is the benchmark retailer. Finally, the retailer who uses the dynamic ordering model is called as the dynamic ordering retailer. We compare the orders and the expected profit of these three types of retailers.

We consider 3 information states; namely low, medium and high. For illustration purposes, we assume that demand has a Poisson distribution with rates $50, 150$ and 250 from low to high. There are 8 ordering periods prior to the season. Our economic parameters are as follows: we set the unit revenue $r = 45$, the purchasing cost vector $c = 15, 15.1, 15.2, 15.3, 15.4, 15.5, 15.6, 15.7$. We set the unit penalty cost from the lost demand b and the unit salvage value s that gained from the unsold items to 0.

Another important parameter in our model is the transition probability matrix of the information process given as

$$\text{P}= \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.05 & 0.95 \end{bmatrix}.$$

Under these parameters the order quantities of the standard retailer for the initial information states low, medium and high are $157, 239$ and 254 respectively. The benchmark retailer orders 253 units for all of the information states observed. The dynamic ordering retailer orders $157, 158$ and 159 for the initial information states low, medium and high respectively at the beginning. The optimal ordering policy of the retailer is given in Figure [5.1]. Under these parameters, it is optimal to not order for low and medium state but for the high state he orders extra 95 units at the period 8 which is the last period.

Moreover the expected profit of each of the 3 types of retailers we consider is shown in the Table [5.1]

Table 5.1: The Expected Profit Comparison of Three Types of Retailers

	Low	Medium	High
Standard Retailer	4724.4	4742.9	4768.8
Benchmark Retailer	6732.6	6750.5	6775.5
Dynamic Ordering Retailer	9174.0	9199.0	9249.2

In Table [5.2], we show the change in profit margins compared to the standard retailer for the benchmark and the dynamic ordering retailers. It is clear that there is substantial gain in using the dynamic model.

Table 5.2: The Expected Profit Comparison of Three Types of Retailers in Percentages

	Low	Medium	High
Standard Retailer	100	100	100
Benchmark Retailer	142.50	142.32	142.08
Dynamic Ordering Retailer	194.18	193.95	193.95

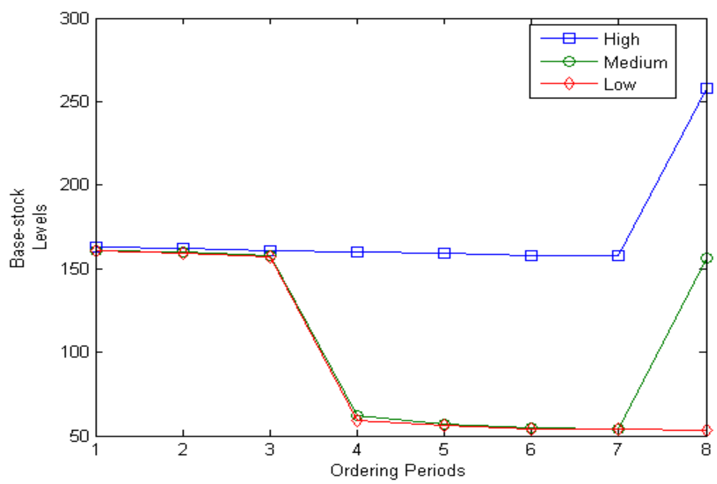

Figure 5.1: Optimal Base-stock Levels for Illustration 1

Figure [5.1] shows that, as we have proved in Proposition [3.2.6] the base-stock levels are ordered alongside with the information vector.

Now we show the base-stock levels of the model for a different purchasing unit cost vector. The new cost vector is $15.199, 15.2, 15.203, 15.208, 15.215, 15.3, 20, 25$ which is convexly increasing. The remaining parameters are the same. Figure [5.2] illustrates the ordering of optimal base-stock levels.

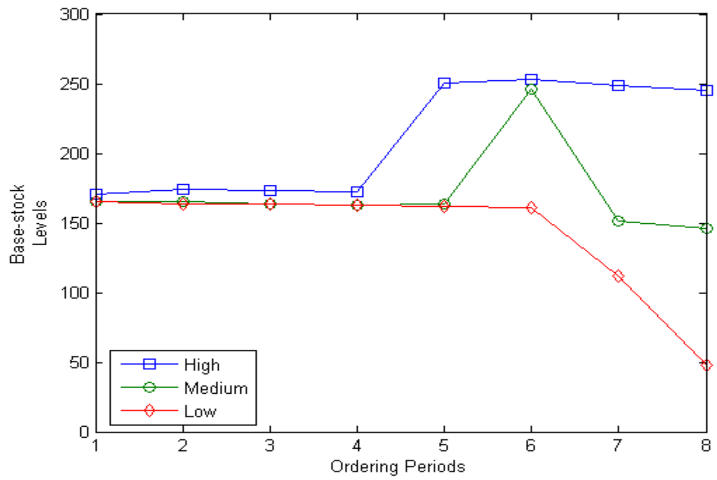

Figure 5.2: Optimal Base-stock Levels for Illustration 2

Figure [5.1] and Figure [5.2] show how optimal base-stock levels, hence the optimal ordering policy, depends on the structure of unit purchasing cost vector. Next we show the profits of the retailers for the convexly increasing purchasing cost setting in Table [5.3].

Table 5.3: The Expected Profit Comparison of Three Types of Retailers under Convex Cost Increments

	Low	Medium	High
Standard Retailer	4644.8	4663.3	4689.2
Benchmark Retailer	6681.9	6699.7	6724.8
Dynamic Ordering Retailer	9171.7	9204.8	9342.3

For the convex increasing cost case, when the values are compared to the standard retailer in percentages we obtain Table [5.4].

Table 5.4: The Expected Profit Comparison of Three Types of Retailers under Convex Cost Increments in Percentages

	Low	Medium	High
Standard Retailer	100	100	100
Benchmark Retailer	143.86	143.67	143.41
Dynamic Ordering Retailer	197.46	197.39	199.23

When investigating the advantages of the dynamic ordering model compared to standard model, as shown in Table [5.2] and Table [5.4], we observe that for the second cost structure, the benefit of using the dynamic ordering model is higher. In the next section, we conduct sensitivity analysis on some economic parameters of the model.

Analysis on Problem Parameters

Although we have significant findings of the structure of the optimal policy and the profit-to-go function, it is not easy to show the effect of marginal changes for the problem parameters. For this reason, to understand our model thoroughly, we experiment on some parameters numerically for certain values.

Keeping the economic parameters the same with the first illustration in the previous section, we observe the effect of the cost increments on the profits. The unit purchasing vector is linear in the illustration; however, we analyze the profits for 5 different unit cost increments. Here, the profits of the first and second types of retailers do not change due to the fact that they order only at the first period. We still present their profits in Table [5.5] for comparison purposes.

In addition to the values of the profits, we show the ratio of the profits of the benchmark and dynamic ordering retailer to the standard retailer in Table [5.6].

Furthermore, we observe the effect of changing the unit revenues and the critical ratio of the newvendor solution in Table [5.7] and the percentage change in Table [5.8].

Table 5.5: The Expected Profit Comparison of Threes Types of Retailers under Different Unit Cost Increments

	Cost Increment	0.5	1	1.5	2	2.5
Low	Standard Retailer	4724.4	4724.4	4724.4	4724.4	4724.4
	Benchmark Retailer	6732.6	6732.6	6732.6	6732.6	6732.6
	Dynamic Ordering Retailer	9159.7	9174.0	9189.2	9220.6	9314.0
Medium	Standard Retailer	4742.9	4742.9	4742.9	4742.9	4742.9
	Benchmark Retailer	6750.5	6750.5	6750.5	6750.5	6750.5
	Dynamic Ordering Retailer	9184.9	9199.0	9229.2	9275.8	9550.1
High	Standard Retailer	4768.8	4768.8	4768.8	4768.8	4768.8
	Benchmark Retailer	6775.5	6775.5	6775.5	6775.5	6775.5
	Dynamic Ordering Retailer	9235.2	9249.2	9279.4	9356.4	10603.0

Table 5.6: The Expected Profit Comparison of Three Types of Retailers under Different Unit Cost Increments in Percentages

	Cost Increment	0.5	1	1.5	2	2.5
Low	Standard Retailer	100	100	100	100	100
	Benchmark Retailer	142.51	142.51	142.51	142.51	142.51
	Dynamic Ordering Retailer	193.88	194.18	194.51	195.17	197.15
Medium	Standard Retailer	100	100	100	100	100
	Benchmark Retailer	142.33	142.33	142.33	142.33	142.33
	Dynamic Ordering Retailer	193.66	193.95	194.59	195.57	201.36
High	Standard Retailer	100	100	100	100	100
	Benchmark Retailer	142.08	142.08	142.08	142.08	142.08
	Dynamic Ordering Retailer	193.66	193.95	194.59	196.2	222.34

Table 5.7: The Expected Profit Comparison of Three Types of Retailers under Different Unit Revenue

	Unit Revenue	40	45	50	55	60	65
	Critical Ratio	0.375	0.33	0.3	0.27	0.25	0.23
Low	Standard Retailer	3532.8	4724.4	5916	7107.6	8299.2	9490.8
	Benchmark Retailer	5561.1	6732.6	7906.9	9083.6	10262	11442
	Dynamic Ordering Retailer	7975.8	9174	10360	11533	12723	13914
Medium	Standard Retailer	3549.2	4742.9	5936.5	7130.2	8323.9	9517.5
	Benchmark Retailer	5576.9	6750.5	7926.9	9105.6	10286	11468
	Dynamic Ordering Retailer	8011.9	9199	10389	11581	12774	13954
High	Standard Retailer	3572.3	4768.8	5965.4	7161.9	8358.5	9555
	Benchmark Retailer	5599	6775.5	7954.9	9136.5	10320	11505
	Dynamic Ordering Retailer	8056.6	9249.2	10444	11642	12826	14026

Table 5.8: The Expected Profit Comparison of Three Types of Retailers under Different Unit Revenue in Percentages

	Unit Revenue	40	45	50	55	60	65
	Critical Ratio	0.375	0.33	0.3	0.27	0.25	0.23
Low	Standard Retailer	100	100	100	100	100	100
	Benchmark Retailer	157.41	142.51	133.65	127.8	123.65	120.56
	Dynamic Ordering Retailer	225.76	194.18	175.12	162.26	153.3	146.61
Medium	Standard Retailer	100	100	100	100	100	100
	Benchmark Retailer	157.13	142.33	133.53	127.7	123.57	120.49
	Dynamic Ordering Retailer	225.74	193.95	175	162.42	153.46	146.61
High	Standard Retailer	100	100	100	100	100	100
	Benchmark Retailer	156.73	142.08	133.35	127.57	123.47	120.41
	Dynamic Ordering Retailer	225.53	193.95	175.08	162.55	153.45	146.79

In addition to the effect of the change in the unit cost vector and the unit revenues, we also investigate the effect of demand variability. For this purpose, we use the same demand rates for low and medium states (50 and 100 respectively). We analyze the results as changing the high demand rate as 230, 260, 290, 320 and 350 in our 5 illustrations. The profits of the 3 different retailers are given in Table [5.9] and the proportional comparison is given in Table [5.10].

Table 5.9: The Expected Profit Comparison of Three Types of Retailers under Different High State Demand Rate

	Demand Rate of the High State	230	260	290	320	350
Low	Standard Retailer	3918.3	5127.5	6336.7	7546	8754.1
	Benchmark Retailer	6236.3	6980.9	7726.3	8472.4	9219.2
	Dynamic Ordering Retailer	8642.2	9418.6	10196	10974	11753
Medium	Standard Retailer	3933.7	5147.5	6361.2	7574.9	8787.5
	Benchmark Retailer	6251.2	7000.2	7750	8500.7	9251.9
	Dynamic Ordering Retailer	8663.6	9445.6	10229	11012	11796
High	Standard Retailer	3955.5	5175.5	6395.5	7615.5	8834.5
	Benchmark Retailer	6272.1	7027.4	7783.5	8540.4	9297.8
	Dynamic Ordering Retailer	8708.7	9498.5	10289	11081	11873

Table 5.10: The Expected Profit Comparison of Three Types of Retailers under Different High State Demand Rate in Percentages

	Demand Rate of the High State	230	260	290	320	350
Low	Standard Retailer	100	100	100	100	100
	Benchmark Retailer	159.16	136.15	121.93	112.28	105.31
	Dynamic Ordering Retailer	220.56	183.69	160.90	145.43	134.26
Medium	Standard Retailer	100	100	100	100	100
	Benchmark Retailer	158.91	135.99	121.83	112.22	105.28
	Dynamic Ordering Retailer	220.24	183.50	160.80	145.37	134.24
High	Standard Retailer	100	100	100	100	100
	Benchmark Retailer	158.57	135.78	121.70	112.14	105.24
	Dynamic Ordering Retailer	220.17	183.53	160.88	145.51	134.39

Chapter 6

CONCLUSIONS

In this thesis, we discuss a single-period, single-item inventory problem when the inventory manager has multiple ordering opportunities prior to the selling season. In the first part of the thesis, we characterize a dynamic ordering model with Markovian demand evolution and examine the mathematical structure of the optimal ordering policy. In the second part of the thesis we study 3 extensions of the dynamic ordering model. One of the fundamental purposes of this study is to build a mathematical model which maximizes the profit of the retailer via information gained on the economic condition of the market which projects the state of the demand. A natural extension to this model would be applying financial hedging to the profit to go function utilizing the state of the economic market by constructing a mean-variance portfolio. This portfolio consists of the options and derivatives which hedges the net revenue from down-side risks. The analysis of the structure of the cost vector might be a relevant extension to our study. Another opportunity to make an extension to this study is exploring different contracting strategies between the retailer and her suppliers. In the literature, supply chain contracting issues is well-studied, however the number of works that study Markov modulation is limited.

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