DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MATHEMATICAL PROBLEMS OF ELECTROMAGNETIC WAVES IN LAYERED MEDIA

by

Şengül ERSOY

February, 2015

İZMİR

MATHEMATICAL PROBLEMS OF ELECTROMAGNETIC WAVES IN LAYERED MEDIA

A Thesis Submitted to the
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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"MATHEMATICAL PROBLEMS OF ELECTROMAGNETIC WAVES IN LAYERED MEDIA"** completed by **ŞENGÜL ERSOY** under supervision of **PROF.DR. VALERY YAKHNO** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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Şengül ERSOY

MATHEMATICAL PROBLEMS OF ELECTROMAGNETIC WAVES IN LAYERED MEDIA

ABSTRACT

In the first part of the thesis, the method of the approximate computation of the magnetic and electric matrix Green's functions in the rectangular parallelepiped with the perfect conducting boundary and the multi-layered isotropic structures has been suggested and investigated. This method based on the Fourier series expansion meta-approach. The method has been implemented in MATLAB and the error analysis of this method has been done by MATLAB tools. The computational experiments confirmed the robustness of the method. Moreover it has been shown that the method is applicable for the numerical computation of the electromagnetic wave propagation in rectangular parallelepiped arising from the pulse point sources. The modern computer tools allow us to observe the electromagnetic wave propagation.

In the second part of the thesis the analytical method of solving the initial value problem for the time-dependent Maxwell's equations in the general electrically and magnetically anisotropic vertical inhomogeneous media has been described. This method is based on the following. The Maxwell's equations are written as symmetric hyperbolic system of the first order. This system is written in terms of the Fourier images with respect to lateral variables. The coefficients of this system are functions depending on vertical variable only. Applying method of integration along the characteristics of the symmetric hyperbolic system reduces to a system of Volterra integral equations. Method of successive approximations is applied for solving the obtained system of integral equations. Finally, the inverse Fourier transform is applied to find a solution of the initial value problem for the time-dependent Maxwell's equations.

The methods and techniques of the generalized functions and partial differential equations of hyperbolic and elliptic types as well as the theory and application of the mathematical and functional analysis, and computational methods and computational

tools are actively applied and used in the thesis.

Keywords: Maxwell's equations, Green's function, analytical method, simulations, initial value problem, initial boundary value problem, perfect conducting boundary conditions, interface conditions.

KATMANLI ORTAMDAKİ ELEKTROMANYETİK DALGALARIN MATEMATİKSEL PROBLEMLERİ

ÖZ

Tezin ilk bölümünde çok katmanlı izotropik yapıda ve üzerinde mükemmel sınır koşulları tanımlanmış dikdörtgensel paralel yüzlü içindeki manyetik ve elektrik matris Green fonksiyonlarının yaklaşık olarak hesaplanması için bir yöntem önerilmiş ve araştırılmıştır. Bu yöntem Fourier serileri açılımı yaklaşımına dayanmaktadır. Yöntem MATLAB'ta uygulanmış ve MATLAB araçları kullanılarak yöntemin hata analizi yapılmıştır. Sayısal deneyimler bu yöntemin doğruluğunu göstermektedir. Ayrıca bu yöntemin dikdörtgensel paralel yüzlü içindeki noktasal darbe kaynağının oluşturduğu elektromanyetik dalga yayılımının sayısal olarak hesaplanmasında uygulanılabilirliği gösterilmiştir.

Tezin ikinci bölümünde genel elektriksel ve manyetiksel anizotropik dik homojen olmayan ortamdaki zamana bağlı Maxwell denklemlerinin başlangıç değer probleminin çözümü için bir analitik yöntem ele alınmıştır. Bu yöntem aşağıda belirtilenlere dayanmaktadır. Maxwell denklemleri birinci dereceden simetrik hiperbolik sistem olarak yazılır. Elde edilen sistemin yatay değişkenler cinsinden Fourier görüntüleri yazılır. Bu sistemin katsayıları dik değişkene bağlı fonksiyonlardan oluşmaktadır. Karakteristikler üzerinde integrasyon yöntemi uygulanarak simetrik hiperbolik sistem Volterra integral denklemleri sistemine indirgenir. Elde edilen integral denklem sistemini çözmek için ardışık yaklaşım yöntemi uygulanır. Son olarak ters Fourier dönüşümü uygulanarak zamana bağlı Maxwell denklemlerinin başlangıç değer probleminin çözümü elde edilir.

Tezde genelleştirilmiş fonksiyonların ve hiperbolik ve eliptik tip kısmi diferansiyel denklemlerin yöntem ve teknikleri, matematiksel ve fonksiyonel analiz teorisi ve uygulamaları, hesaplama metotları ve araçları aktif bir şekilde kullanılmıştır.

Anahtar Kelimeler : Maxwell denklemleri, Green fonksiyonu, analitik yöntem,

simülasyon, başlangıç değer problemi, başlangıç sınır değer problemi, mükemmel iletken sınır koşulları, arayüz koşulları.

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CHAPTER ONE
INTRODUCTION

1.1 Some Results of Past Research Related to the Thesis

The fundamental equations of the electromagnetic theory which were established by James Clerk Maxwell in 1873, are called Maxwell's equations. In 1888, they were experimentally verified by Heinrich Rudolf Hertz. Since then the theory of electromagnetic waves has played a magnificent role in the development of radio, television, optical communications, radar, microwave heating, remote sensing, and in many other practical applications.

The study of the Maxwell's equations in bounded domains with the natural boundary conditions is an important issue of the interdisciplinary science which has many applications in wireless telecommunications and microwave engineering (Eom, 2004, Russenschuck, 2010). Mobile communication systems, equipment and environment conditions are studied systematically in papers Fujimoto (1991) and Dolmans (1996). Nowadays general models of electromagnetic wave propagation in closed areas are of great interest because distribution of the electromagnetic field in these environments is very complicated. The interaction of electromagnetic waves inside a real indoor environment (cabinets, desks, etc) has been studied in (Dolmans, 1995), where the walls, ceiling and floors of an indoor environment have been modeled as perfectly conducting and the geometry of this model is a rectangular cavity.

Although the mathematical background of the theory of the initial boundary and boundary value problems for the Maxwell's equations has been described in Dautray & Lions (1993) nevertheless different aspects of correctness, such as existence, uniqueness, stability estimates, of the classical and generalized statements of the initial and boundary value problems, are of great interest at present (Belgacem & Bernardi, 1999, Boffi & Gastaldi, 2003, Bulmezaoud & El Rhabi, 2003, Buffa & Perugia, 2006, Krigman & Wayne, 2007, Izsák et al., 2008, Denisenko, 2011, Gui & Li, 2011). The results of these papers have been used for constructing analytical and numerical methods (Yee, 1966, Monk & Süli, 1994, Bulmezaoud & El Rhabi, 2005, Sazonov et al., 2006, Izsák et al., 2008, Petropavlovsky & Tsynkov, 2012, Gunel & Zoral, 2010, 2015).

Let us mention some of them. An implicit a posteriori error estimation technique for the time-harmonic Maxwell's equations has been developed in (Izsák et al., 2008). Boulmezaoud and Rhabi in (Bulmezaoud & El Rhabi, 2005) have used the mortar spectral element method for numerically solving the Maxwell's equations in a bounded domain of $\mathbf{R}^3$ with a perfect conducting boundary. In (Sazonov et al., 2006), Yee's scheme has been generalized to unstructured meshes, and the co-volume method has been used for the integration of the Maxwell's equations. The scattering of electromagnetic waves by perfectly conducting objects has been computed numerically and simulation of the electromagnetic wave propagation has been given.

Analytical methods also play an important role for the construction of solutions of the Maxwell's equations. The Green's function technique has attracted considerable attention and is widely used for solving electromagnetic boundary value problems (Yaghjian, 1980, Tai & Yaghjian, 1981, Collin, 1990, Tai, 1993, Chew, 1995, Dolmans, 1995, Lukic & Yakoylev, 2002, Narayanaswamy & Chen, 2010, Fallahi & B., 2011, Sevgi, 2006). Most of the studies related to the Green's functions in the electromagnetic theory have been done for the frequency domain. In (Dolmans, 1995) the frequency dependent dyadic electric and magnetic Green's functions corresponding to the elementary electric and magnetic current point sources have been presented inside the cavity with perfect conducting boundary by the eigenvalues and eigenmodes. This presentation contains solenoidal and non-divergenceless terms. In (Tai, 1993) the vector wave functions have been used to derive the expressions of the dyadic Green's functions. These functions were first introduced in (Hansen, 1937). Then Tai (1993) has applied the sets of vector wave functions corresponding to TE (transverse electric) and TM (transverse magnetic) modes for the construction of frequency dependent dyadic Green's functions for the rectangular waveguide and then, using the method of scattering superposition, for the construction of frequency dependent Green's function in the rectangular cavity. The main results have been presented by quite complicated formulae and procedures for the derivation of frequency dependent Green's functions Tai (1993). The computation of their values has not been made.

Some approaches for the computation of time-dependent electric and magnetic Green's functions have been developed for unbounded domain (Yakhno, 2010, 2011, Yakhno et al., 2012, Yakhno & Yakhno, 2012). The computation of time-dependent electric and magnetic Green's functions for the bounded domain with perfect conducting conditions has not been achieved so far.

The electromagnetic wave propagations in multi-layered media are described by the Maxwell's equations with piece-wise constant coefficients. For solving the problems of electromagnetic wave propagation in multi-layered media, analytical and numerical methods have been applied (Dmitriev et al., 2002, Okhmatovski & Cangellaris, 2003, Lukic & Yakoylev, 2002, Li et al., 2004, Oğuzer et al., 2011, Chen et al., 2012, Stumpf et al., 2003).

1.2 The Goal of the Thesis

The goal of the thesis is to work out a method for the approximate computation of magnetic and electric Green's functions in rectangular parallelepiped with the multi-layered structure and the perfect conducting boundary and to develop a method of solving the initial value problem for time-dependent Maxwell's equations in the anisotropic inhomogeneous three dimensional space.

1.3 Methods of the Thesis

The methods of the generalized functions and partial differential equations of hyperbolic and elliptic types as well as the theory and application of the mathematical and functional analysis, and computational methods and computational tools are actively applied and used in the thesis.

1.4 The Descriptions of the Main Problems and Results of the Thesis

The first problem of the thesis consists of an approximate computation of frequency dependent electric and magnetic Green's functions in a rectangular parallelepiped with the perfect conducting boundary.

Let b_1, b_2, b_3 be positive reals and let

$$V = \left\{ x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1,\ \ 0 < x_2 < b_2,\ \ 0 < x_3 < b_3 \right\}$$

be a rectangular parallelepiped; Γ be the boundary of V; $\omega \in \mathbf{R}$ (frequency, $\omega > 0$), let $x^0 = (x_1^0, x_2^0, x_3^0) \in V$ be 1D and 3D parameters, respectively. The 3×3 matrices

$$\mathbf{G}^H(x,\omega,x^0) = \begin{pmatrix} H_1^1(x,\omega,x^0) & H_1^2(x,\omega,x^0) & H_1^3(x,\omega,x^0) \\ H_2^1(x,\omega,x^0) & H_2^2(x,\omega,x^0) & H_2^3(x,\omega,x^0) \\ H_3^1(x,\omega,x^0) & H_3^2(x,\omega,x^0) & H_3^3(x,\omega,x^0) \end{pmatrix},$$

$$\mathbf{G}^E(x,\omega,x^0) = \begin{pmatrix} E_1^1(x,\omega,x^0) & E_1^2(x,\omega,x^0) & E_1^3(x,\omega,x^0) \\ E_2^1(x,\omega,x^0) & E_2^2(x,\omega,x^0) & E_2^3(x,\omega,x^0) \\ E_3^1(x,\omega,x^0) & E_3^2(x,\omega,x^0) & E_3^3(x,\omega,x^0) \end{pmatrix},$$

whose columns $\mathbf{H}^s(x,\omega,x^0)$, $\mathbf{E}^s(x,\omega,x^0)$, and $s = 1,2,3$ satisfy

$$curl_x \mathbf{H}^s(x,\omega,x^0) = -i\omega\epsilon \mathbf{E}^s(x,\omega,x^0) + \delta(x - x^0)\vec{\mathbf{e}^s}, \quad x \in V, \tag{1.1}$$

$$curl_x \mathbf{E}^s(x,\omega,x^0) = i\mu\omega \mathbf{H}^s(x,\omega,x^0), \tag{1.2}$$

$$(\mathbf{E}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0,\ \ (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0,\ \ s = 1,2,3, \tag{1.3}$$

are called frequency-dependent magnetic and electric Green's functions in V, respectively. Here i is the imaginary unit ($i^2 = -1$); $\epsilon > 0$, $\mu > 0$ are given constants characterizing the permittivity and permeability of the electromagnetic medium, which is located inside V; $\vec{\mathbf{e}^s}$ are the basis vectors from Euclidean space $\mathbf{R}^3$ ($\vec{\mathbf{e}^1} = (1,0,0)^T$, $\vec{\mathbf{e}^2} = (0,1,0)^T$, $\vec{\mathbf{e}^3} = (0,0,1)^T$); $x^0 = (x_1^0, x_2^0, x_3^0) \in V$ is the 3D parameter; $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, and $\delta(x_j - x_j^0)$ is the Dirac delta distribution concentrated at $x_j = x_j^0$, $j = 1,2,3$; $\vec{\mathbf{n}}$ is an outward unit normal vector on Γ. The relations in (1.3) are called the perfect conducting boundary conditions (Eom (2004)).

Remark 1.4.1. We note that $div_x \mathbf{H}^s(x, \omega, x^0) = 0$ for all $x \in V$ is a consequence of (1.2).

A method of the approximate computation of frequency dependent Green's functions is described in Chapter 2 of this thesis. This method consists of the following. The equations for magnetic Green's function are written in special form which does not contain elements of electric Green's function. These equations are elliptic partial differential equations and contain the components of the generalized vector functions (distributions) in their right hand sides. We approximate components of these generalized vector functions by the Fourier series with a finite number of terms and then applied standard Fourier series expansion meta-approach for solving the boundary value problem for the elliptic partial differential equations in the bounded region. After calculation of all columns of magnetic matrix Green's function the approximate computation of the columns of electric Green's function is given by explicit calculation of the differential operator $curl_x$ from each column of magnetic matrix Green's function with respect to 3D space variable x. This method is implemented in MATLAB. The error analysis and computational experiments are described in Chapter 2 and published in Yakhno & Ersoy (2014).

We note that the Fourier transform and Laplace transform are applied very often for solving initial value or initial boundary value problems of the differential equations. In this case there are three steps of solving. In the first step the problem for the considered differential equation can be written in terms of the Fourier (or Laplace) images. After solving the obtained problem, on the last step, we apply the inverse Fourier (or Laplace) transform to the found solution. This last step is more complicated because the application of the inverse Fourier transform can not always be realized easily in the class of the functions where we try to find a solution of the original problem. For partial differential equations, having the time and space variables, the frequency dependent formulation of the problem can be considered as the result of the application the Fourier transform to the time-dependent formulation. The computation of frequency dependent electric and magnetic Green's functions in a rectangular parallelepiped with perfect conducting boundary by the Fourier series meta-approach has been worked in Yakhno & Ersoy (2014) and described in Chapter 2. But the computation of time-dependent electric and magnetic Green's functions has not been achieved so far for the bounded domain with perfect conducting conditions.

The second problem of the thesis is an approximate computation of time dependent magnetic and electric Green's functions in a rectangular parallelepiped with the perfect conducting boundary.

Let $b_1, b_2, b_3 \in \mathbf{R}^+$ and $x = (x_1, x_2, x_3)^T \in \mathbf{R}^3$;

$$V = \left\{ x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \;\; 0 < x_2 < b_2, \;\; 0 < x_3 < b_3 \right\} \tag{1.4}$$

be a rectangular parallelepiped; $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \left\{ x = (x_1, x_2, x_3)^T : \quad x_1 = 0 \vee x_1 = b_1, \quad 0 < x_2 < b_2, \quad 0 < x_3 < b_3 \right\},$$
$$\Gamma_2 = \left\{ x = (x_1, x_2, x_3)^T : \quad x_2 = 0 \vee x_2 = b_2, \quad 0 < x_1 < b_1, \quad 0 < x_3 < b_3 \right\},$$
$$\Gamma_3 = \left\{ x = (x_1, x_2, x_3)^T : \quad x_3 = 0 \vee x_3 = b_3, \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2 \right\}.$$

The 3×3 matrices

$$\mathbf{G}^H(x,t,x^0) = \begin{pmatrix} H_1^1(x,t,x^0) & H_1^2(x,t,x^0) & H_1^3(x,t,x^0) \\ H_2^1(x,t,x^0) & H_2^2(x,t,x^0) & H_2^3(x,t,x^0) \\ H_3^1(x,t,x^0) & H_3^2(x,t,x^0) & H_3^3(x,t,x^0) \end{pmatrix},$$
$$\mathbf{G}^E(x,t,x^0) = \begin{pmatrix} E_1^1(x,t,x^0) & E_1^2(x,t,x^0) & E_1^3(x,t,x^0) \\ E_2^1(x,t,x^0) & E_2^2(x,t,x^0) & E_2^3(x,t,x^0) \\ E_3^1(x,t,x^0) & E_3^2(x,t,x^0) & E_3^3(x,t,x^0) \end{pmatrix},$$

whose columns $\mathbf{H}^s(x,t,x^0)$, $\mathbf{E}^s(x,t,x^0)$, $s = 1,2,3$ satisfy

$$curl_x \mathbf{H}^s = \epsilon \frac{\partial \mathbf{E}^s}{\partial t} + \delta(x - x^0)\delta(t)\vec{\mathbf{e}^s}, \quad x \in V, \quad t \in \mathbf{R}, \tag{1.5}$$
$$curl_x \mathbf{E}^s = -\mu \frac{\partial \mathbf{H}^s}{\partial t}, \tag{1.6}$$
$$\mathbf{E}^s\big|_{t<0} = 0, \quad \mathbf{H}^s\big|_{t<0} = 0, \tag{1.7}$$
$$(\mathbf{E}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \quad (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0, \quad s = 1,2,3, \tag{1.8}$$

are called magnetic and the electric Green's functions, respectively, in V. Here $\epsilon > 0$, $\mu > 0$ are given constants characterizing the permittivity and permeability; $\vec{\mathbf{e}^s}$ are the basis vectors from Euclidean space $\mathbf{R}^3$ ($\vec{\mathbf{e}^1} = (1,0,0)^T,\ \vec{\mathbf{e}^2} = (0,1,0)^T,\ \vec{\mathbf{e}^3} = (0,0,1)^T$); $x^0 = (x_1^0, x_2^0, x_3^0)^T \in V$ is the 3D parameter; t is the time variable; $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, and $\delta(x_j - x_j^0)$ is the Dirac delta distribution concentrated at $x_j = x_j^0$, $j = 1,2,3$; $\delta(t)$ is the Dirac delta distribution at $t = 0$; $\vec{\mathbf{n}}$ is an outward unit normal vector on Γ. We note that the relations in (1.8) are called the perfect conducting boundary conditions (see, Eom (2004)).

Remark 1.4.2. We note $div_x \mathbf{H}^s = 0$ for all $x \in V$ is a consequence of (1.6),(1.7).

We note that there exists a connection between time dependent Green's functions and frequency dependent Green's functions which can be expressed by the Fourier transform with respect to the time variable. Moreover time-dependent Green's function, for example, of the wave equation in a whole space can be derived explicitly by the application of the Fourier transform to the Green's function of the Helmholtz equation. The Fourier transform and inverse Fourier transform are defined here in the class of the generalized functions (tempered distributions) and for some scalar hyperbolic differential equations the application of the Fourier transform can be done explicitly. Unfortunately the explicit or approximate computation of the inverse Fourier transform, applied to frequency dependent electric and magnetic Green's functions in a bounded domain of the three dimensional space, is unknown. This computation requires knowledge of frequency dependent Green's functions for all frequencies that is difficult to realize in practice. Moreover time-dependent electric and magnetic Green's functions are singular generalized functions and numerical methods in the space of generalized functions are not developed yet. To overcome these difficulties we suggest the adaptation of the Fourier series meta-approach for the direct computation an approximate (regularized) time dependent electric and magnetic Green's functions in a rectangular parallelepiped with a perfect conducting boundary and do not use frequency dependent Green's functions. This method consists of the following. The equations for magnetic Green's function are written in a special form which does not contain elements of electric Green's function. These equations are hyperbolic partial differential equations and our method is based on the standard Fourier series expansion meta-approach for solving the initial boundary value problem for hyperbolic partial differential equations in the bounded region. The elements of the approximate Green's function for magnetic field are found by the explicit formulae. These formulae partial sums of the Fourier series. Using the partial sums we derive explicitly the approximate elements of electric Green's matrix by integration with respect to time variable. The simple implementation of our method for computing Green's functions in a rectangular parallelepiped is based on the obtained presentations and does not contain any type of discretization. The implementation of the method has been done in MATLAB. The error analysis and computational experiments are described in Chapter 3.

Time-dependent magnetic and electric Green's functions for the Maxwell's equations in multi-layered parallelepiped with the perfect conducting boundary are considered in Chapter 6. The columns of these Green's functions are solution of Maxwell's equations satisfying the boundary and matching (interface) conditions. The coefficients of the Maxwell's equations are the piece-wise constant functions. Let us describe these equations, conditions and definition of the Green's functions.

Let b_1, b_2, b_3, r_i, $i = 1,2,...,N$ be positive reals with $0 = r_0 < r_1 < ... < r_N = b_2$, $x = (x_1, x_2, x_3)^T \in \mathbf{R}^3$;

$$D = \left\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2, \quad 0 < x_3 < b_3\right\}$$

be a rectangular parallelepiped, Γ be the boundary of D; and D_i, $i = 1,2,...,N$ be the parallelepipeds

$$D_i = \left\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad r_{i-1} < x_2 < r_i, \quad 0 < x_3 < b_3\right\};$$

$D^0 = \bigcup\limits_{i=1}^{N} D_i$.

Let μ_i, ϵ_i, $i = 1,2,\ldots,N$ be given positive numbers; $\mu(x_2)$, $\epsilon(x_2)$ be two piecewise constant functions defined on $[0, b_2]$ such that

$$\mu(x_2) = \mu_i, \quad \epsilon(x_2) = \epsilon_i, \text{ for } x_2 \in (r_{i-1}, r_i), \ i = 1,2,\ldots,N.$$

Find the 3×3 matrices

$$\mathbf{G}^H(x,t,x^0) = \begin{pmatrix} H_1^1(x,t,x^0) & H_1^2(x,t,x^0) & H_1^3(x,t,x^0) \\ H_2^1(x,t,x^0) & H_2^2(x,t,x^0) & H_2^3(x,t,x^0) \\ H_3^1(x,t,x^0) & H_3^2(x,t,x^0) & H_3^3(x,t,x^0) \end{pmatrix},$$

$$\mathbf{G}^E(x,t,x^0) = \begin{pmatrix} E_1^1(x,t,x^0) & E_1^2(x,t,x^0) & E_1^3(x,t,x^0) \\ E_2^1(x,t,x^0) & E_2^2(x,t,x^0) & E_2^3(x,t,x^0) \\ E_3^1(x,t,x^0) & E_3^2(x,t,x^0) & E_3^3(x,t,x^0) \end{pmatrix},$$

whose columns $\mathbf{H}^s(x,t,x^0)$, $\mathbf{E}^s(x,t,x^0)$, and $s = 1,2,3$ satisfy

$$curl_x\mathbf{H}^s = \epsilon(x_2)\frac{\partial \mathbf{E}^s}{\partial t} + \delta(x - x^0)\delta(t)\vec{\mathbf{e}^s}, \;\; x \in D^0, \;\; t \in \mathbf{R}^+, \tag{1.9}$$

$$curl_x\mathbf{E}^s = -\mu(x_2)\frac{\partial \mathbf{H}^s}{\partial t}, \tag{1.10}$$

the initial data

$$\mathbf{H}^s\big|_{t<0} = 0, \tag{1.11}$$

$$\mathbf{E}^s\big|_{t<0} = 0, \tag{1.12}$$

the perfect conducting boundary conditions

$$(\mathbf{E}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \;\; (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0, \tag{1.13}$$

and matching (interface) conditions

$$\left(\mathbf{H}^s\big|_{x_2=r_i-0} - \mathbf{H}^s\big|_{x_2=r_i+0}\right) \times \vec{\nu} = 0, \quad \left(\mu_i\, \mathbf{H}^s\big|_{x_2=r_i-0} - \mu_{i+1}\, \mathbf{H}^s\big|_{x_2=r_i+0}\right) \cdot \vec{\nu} = 0, \tag{1.14}$$

$$\left(\mathbf{E}^s\big|_{x_2=r_i-0} - \mathbf{E}^s\big|_{x_2=r_i+0}\right) \times \vec{\nu} = 0, \quad \left(\epsilon_i\, \mathbf{E}^s\big|_{x_2=r_i-0} - \epsilon_{i+1}\, \mathbf{E}^s\big|_{x_2=r_i+0}\right) \cdot \vec{\nu} = 0. \tag{1.15}$$

Here $\vec{\mathbf{e}^s}$ are the basis vectors from Euclidean space $\mathbf{R}^3$ ($\vec{\mathbf{e}^1} = (1,0,0)^T$, $\vec{\mathbf{e}^2} = (0,1,0)^T$, $\vec{\mathbf{e}^3} = (0,0,1)^T$); $x^0 = (x_1^0, x_2^0, x_3^0)^T \in D$, t is the time variable; $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, $\delta(x_j - x_j^0)$ is the Dirac delta distribution concentrated at $x_j = x_j^0$, $j = 1,2,3$; $\delta(t)$ is the Dirac delta distribution at $t = 0$; $\vec{\mathbf{n}}$ is an outward unit normal vector on Γ; $\vec{\nu} = (0,1,0)$. The matrices $\mathbf{G}^H(x,t,x^0)$, $\mathbf{G}^E(x,t,x^0)$ are called time-dependent magnetic and electric matrix Green's functions for $x \in D^0$, $t \in \mathbf{R}^+$.

Remark 1.4.3. $div_x\mathbf{H}^s = \mathbf{0}$, $x \in D^0$ follows from (1.10), (1.11). This is the necessary condition that $\mathbf{H}^s$ satisfies (1.9)-(1.15) $x \in D^0$.

A method for an approximate computation of columns of time-dependent magnetic and electric Green's functions in the multi-layered rectangular parallelepiped with the perfect conducting boundary is described in Chapter 6. This method is a generalization of the method described in Chapter 3. We have shown that the adaption of the Fourier series meta-approach can be done for the approximate computation of columns of time-dependent magnetic Green's function in multi-layered parallelepiped and then electric Green's function can be computed approximately. In Chapters 4 and 5 we solve the auxiliary problems which help to adapt the method of Chapter 3 for the approximate computation of time-dependent magnetic Green's functions in multi-layered parallelepiped with perfect conducting boundary.

In Chapter 7 we consider $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}$ as the space and time variables, $\mu_{ij}(x_3) > 0$ and $\epsilon_{ij}(x_3) > 0$ as the given functions depending on x_3, $\mathcal{M}(x_3) = (\mu_{ij}(x_3))_{3\times3}$ and $\mathcal{E}(x_3) = (\epsilon_{ij}(x_3))_{3\times3}$ as the given 3×3 matrices, $\mathbf{J}(x,t) = (J_1(x,t), J_2(x,t), J_3(x,t))$ as a given vector function, $\mathbf{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))$ and $\mathbf{H}(x,t) = (H_1(x,t), H_2(x,t), H_3(x,t))$ as the unknown vector functions satisfying the following Maxwell's equations

$$curl_x\mathbf{H}(x,t) = \mathcal{E}(x_3)\frac{\partial \mathbf{E}(x,t)}{\partial t} + \mathbf{J}(x,t), \tag{1.16}$$

$$curl_x\mathbf{E}(x,t) = -\mathcal{M}(x_3)\frac{\partial \mathbf{H}(x,t)}{\partial t}, \tag{1.17}$$

with the initial conditions

$$\mathbf{E}(x,t)\Big|_{t=0} = 0, \quad \mathbf{H}(x,t)\Big|_{t=0} = 0. \tag{1.18}$$

Here, $\mathbf{E}(x,t)$, $\mathbf{H}(x,t)$ denote the electric and magnetic fields, respectively; $\mathbf{J}(x,t)$ is the electric current density; $\mathcal{E}(x_3)$ is the electric permittivity and $\mathcal{M}(x_3)$ is the magnetic permeability.

The problem of finding $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ satisfying (1.16)-(1.18) is called the initial value problem for the Maxwell's equations in vertically inhomogeneous, electrically and magnetically anisotropic media. The main object of the study in this chapter is an analytical method for solving this problem under the following assumptions. We assume that $\mathcal{E}(x_3)$, $\mathcal{M}(x_3)$ are such symmetric positive definite matrices that there exist positive constants $\bar{\epsilon_0}$, $\bar{\mu_0}$ satisfying the following inequality

$$\langle \mathcal{E}(x_3)\cdot\xi,\xi\rangle \geq \bar{\epsilon_0}\cdot|\xi|^2,\ \ \forall\xi\in\mathbf{R}^3,\ \ \forall x_3\in\mathbf{R},$$
$$\langle \mathcal{M}(x_3)\cdot\xi,\xi\rangle \geq \bar{\mu_0}\cdot|\xi|^2,\ \ \forall\xi\in\mathbf{R}^3,\ \ \forall x_3\in\mathbf{R},$$

for an arbitrary vector $\xi\in\mathbf{R}^3$ and $x_3\in\mathbf{R}$.

Let T be a positive number, $a_0 = \max\{\bar{\epsilon_0},\bar{\mu_0}\}$, $M = \dfrac{1}{a_0}$. The triangle $\Delta(M,T)$ is defined by

$$\Delta(M,T) = \{(x_3,t):\ \ 0\leq t\leq T,\ \ -M(T-t)\leq x_3\leq M(T-t)\}. \tag{1.19}$$

We assume also that the elements of the matrices $\mathcal{E}(x_3)$, $\mathcal{M}(x_3)$ are twice continuously differentiable functions over $[-MT,MT]$. Moreover there exists the Fourier transform with respect to x_1 and x_2 of the components of $\mathbf{J}(x,t) = (J_1(x,t),J_2(x,t),J_3(x,t))$. The Fourier transformed image of the functions $J_k(x,t)$ denoted by $\widetilde{J_k}(\nu,x_3,t)$, are assumed to be from $C(C^1(\Delta(M,T));C_0(\mathbf{R}^2))$, $k = 1,2,3$. Here, $\nu = (\nu_1,\nu_2)$ is a parameter from $\mathbf{R}^2$. The class $C(C^1(\Delta(M,T);C_0(\mathbf{R}^2)))$ consists of all continuous functions, depending on (x_3,t,ν) with $\Big((x_3,t)\in\Delta(M,T),\nu\in\mathbf{R}^2\Big)$ and are differentiable with respect to x_3,t and have finite supports with respect to ν.

The main result of Chapter 7 is an analytical method for solving (1.16)-(1.18). The main steps of this methods are the following. The problem (1.16)-(1.18) is presented in the form of the initial value problem for the symmetric hyperbolic system of the first order. The initial value problem is written in terms of the Fourier images of the Fourier transform with respect to x_1 and x_2 variables. Applying integration along characteristics to the obtained symmetric hyperbolic system we reduce the initial value problem for this system to the system of Volterra integral equations. Applying method of successive approximations, a solution of this integral equation system is found in the space $C(C^1(\Delta(M;T));C_0(\mathbf{R}^2))$. Finally, applying the inverse Fourier transform, a solution of the original initial value problem for the Maxwell's equations is determined in the space $C^1(\Delta(M;T);PW(\mathbf{R}^2))$. We denote $C^1(\Delta(M,T);\ PW(\mathbf{R}^2))$ the class of all continuous functions of $(x_3,t)\in\Delta(M,T)$ and $(x_1,x_2)\in\mathbf{R}^2$, that is, differentiable with respect to x_3 and t over $\Delta(M,T)$ and belongs to $PW(\mathbf{R}^2)$ for $(x_1,x_2)\in\mathbf{R}^2$. Here $PW(\mathbf{R}^2)$ denotes the Paley Weiner Space of $\mathbf{R}^2$ (see, Appendix A9).

We note that the analytical method for the study of the initial value problem for the Maxwell's equations with smooth coefficients depending on one variable for a particular case of anisotropy (uniaxial and bi-axial crystals) has been studied by Yakhno & Sevimlican (2007, 2010, 2011).

The method of the works (Yakhno & Sevimlican, 2007; 2010; 2011) is based on the reduction of the Maxwell's equations to hyperbolic equations of the second order, the application of the D'Alamberd method and successive approximations. The application of the approach of the works (Yakhno & Sevimlican, 2007; 2010; 2011) for solving IVP for Maxwell's equations with the smooth function coefficients in anisotropic media with the general structures of the anisotropy is questionable. The result of Chapter 7 has shown that the initial value problem for the Maxwell's equations with the smooth function coefficients depending on one space variable in the general electrically and magnetically anisotropic media can be solved by the analytical method based on the application of the theory and methods of the symmetric hyperbolic systems of the first order and the functional analysis.

CHAPTER TWO
COMPUTING THE ELECTRIC AND MAGNETIC MATRIX GREEN'S FUNCTIONS IN A RECTANGULAR PARALLELEPIPED WITH A PERFECT CONDUCTING BOUNDARY

In this chapter a boundary value problem (BVP) for frequency dependent Maxwell's equations in a rectangular parallelepiped with perfect conducting boundary is studied. The method of solving the BVP, that is, approximate computation of magnetic and electric matrix Green's functions is described. The column elements of magnetic and electric Green's functions are found in the form of Fourier series with a finite number of terms where the basis functions are the eigenfunctions of the 3D elliptic operator Δ_x. Using these eigenfunctions regularizations (approximations) of the 3D Dirac delta function and its derivatives are given. The method is implemented in MATLAB and computational experiments are given to show applicability of the suggested method.

2.1 Frequency-Dependent Magnetic and Electric Green's Functions

Let b_1, b_2, b_3 be positive reals and let

$$V = \left\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1,\ \ 0 < x_2 < b_2,\ \ 0 < x_3 < b_3\right\}$$

be a rectangular parallelepiped; Γ be the boundary of V; $\omega \in \mathbf{R}$ (frequency, $\omega > 0$), let $x^0 = (x_1^0, x_2^0, x_3^0) \in V$ be 1D and 3D parameters, respectively. The 3×3 matrices

$$\mathbf{G}^H(x,\omega,x^0) = \begin{pmatrix} H_1^1(x,\omega,x^0) & H_1^2(x,\omega,x^0) & H_1^3(x,\omega,x^0) \\ H_2^1(x,\omega,x^0) & H_2^2(x,\omega,x^0) & H_2^3(x,\omega,x^0) \\ H_3^1(x,\omega,x^0) & H_3^2(x,\omega,x^0) & H_3^3(x,\omega,x^0) \end{pmatrix},$$

$$\mathbf{G}^E(x,\omega,x^0) = \begin{pmatrix} E_1^1(x,\omega,x^0) & E_1^2(x,\omega,x^0) & E_1^3(x,\omega,x^0) \\ E_2^1(x,\omega,x^0) & E_2^2(x,\omega,x^0) & E_2^3(x,\omega,x^0) \\ E_3^1(x,\omega,x^0) & E_3^2(x,\omega,x^0) & E_3^3(x,\omega,x^0) \end{pmatrix},$$

whose columns $\mathbf{H}^s(x,\omega,x^0)$, $\mathbf{E}^s(x,\omega,x^0)$, and $s=1,2,3$ satisfy

$$curl_x\mathbf{H}^s(x,\omega,x^0) = -i\omega\epsilon\mathbf{E}^s(x,\omega,x^0) + \delta(x-x^0)\vec{\mathbf{e}^s}, \quad x\in V, \tag{2.1}$$

$$curl_x\mathbf{E}^s(x,\omega,x^0) = i\mu\omega\mathbf{H}^s(x,\omega,x^0), \tag{2.2}$$

$$(\mathbf{E}^s\times\vec{\mathbf{n}})\big|_\Gamma = 0, \ \ (\mathbf{H}^s\cdot\vec{\mathbf{n}})\big|_\Gamma = 0, \ \ s=1,2,3, \tag{2.3}$$

are called frequency-dependent magnetic and electric Green's functions in V, respectively. Here i is the imaginary unit ($i^2=-1$); $\epsilon>0$, $\mu>0$ are given constants characterizing the permittivity and permeability of the electromagnetic medium, which is located inside V; $\vec{\mathbf{e}^s}$ are the basis vectors from Euclidean space $\mathbf{R}^3$, ($\vec{\mathbf{e}^1}=(1,0,0)^T$, $\vec{\mathbf{e}^2}=(0,1,0)^T$, $\vec{\mathbf{e}^3}=(0,0,1)^T$); $\delta(x-x^0)=\delta(x_1-x_1^0)\delta(x_2-x_2^0)\delta(x_3-x_3^0)$, where $\delta(x_j-x_j^0)$ is the Dirac delta distribution concentrated at $x_j=x_j^0$, $j=1,2,3$; $\vec{\mathbf{n}}$ is an outward unit normal vector on Γ. The relations in (2.3) are called the perfect conducting boundary conditions.

Remark 2.1.1. We note that $div_x\mathbf{H}^s(x,\omega,x^0)=0$ for all $x\in V$ is a consequence of (2.2).

Applying $curl_x$ to (2.1) and using (2.2), we find

$$\omega^2\epsilon\mu\mathbf{H}^s - curl_x curl_x\mathbf{H}^s = -curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right], \ \ x\in V. \tag{2.4}$$

Taking the cross product of both sides of (2.1) and $\vec{\mathbf{n}}$, and using (2.3) we get

$$(curl_x\mathbf{H}^s\times\vec{\mathbf{n}})\big|_\Gamma = 0, \quad (\mathbf{H}^s\cdot\vec{\mathbf{n}})\big|_\Gamma = 0. \tag{2.5}$$

It follows from (2.1) that

$$\mathbf{E}(x,\omega,x^0) = \frac{i}{\omega\epsilon}\left[curl_x\mathbf{H}^s(x,\omega,x^0) - \delta(x-x^0)\vec{\mathbf{e}^s}\right]. \tag{2.6}$$

Remark 2.1.2. If $\mathbf{H}^s$ is a vector function satisfying (2.4)-(2.5), $\mathbf{E}^s$ is a vector function satisfying (2.6) for a fixed value of $s=1,2,3$, then $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (2.1)-(2.3), that is, $\mathbf{H}^s$ and $\mathbf{E}^s$ are columns of frequency-dependent magnetic and electric Green's functions. From (2.6), we find $curl_x\mathbf{H}^s(x,\omega,x^0) = -i\omega\epsilon\mathbf{E}^s(x,\omega,x^0)+\delta(x-x^0)\vec{\mathbf{e}^s}$. Substituting this expression into (2.4) we have $curl_x\mathbf{E}^s(x,\omega,x^0) = i\mu\omega\mathbf{H}^s(x,\omega,x^0)$. Moreover, taking the cross product of (2.6) and $\vec{\mathbf{n}}$, letting $x\in\Gamma$ and using (2.5) we find that $\mathbf{E}^s$ satisfies the first condition of (2.3). Therefore, we obtain that if $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (2.4),(2.5) and (2.6) then $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy equations (2.1)-(2.3) also, that is, $\mathbf{H}^s$ and $\mathbf{E}^s$ are s-columns of frequency-dependent electric and magnetic Green's functions.

Remark 2.1.3. Let $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (2.4) and (2.6). Then $div_x\mathbf{H}^s = 0,\ x \in V$.

2.2 The Eigenvalue-Eigenfunction Problem for the Operator $\Delta_x\mathbf{I}_3$ in Rectangular Parallelepiped

Let us consider the problem of finding all values of λ (eigenvalues) for which there exists a nonzero vector-valued function $\mathbf{U}(x) = (U_1(x), U_2(x), U_3(x))^T$ (vector-valued eigenfunction) satisfying

$$\Delta_x\mathbf{I}_3\mathbf{U}(x) + \lambda\mathbf{U}(x) = 0, \quad x \in V, \tag{2.7}$$

$$(curl_x\mathbf{U}(x)\times\vec{\mathbf{n}})\Big|_\Gamma = 0, \quad (\mathbf{U}\cdot\vec{\mathbf{n}})\Big|_\Gamma = 0, \tag{2.8}$$

where $\mathbf{I}_3$ is the 3×3 identity matrix.

In this section we show that the eigenvalue-eigenfunction problem (2.7),(2.8) can be decomposed into three eigenvalue-eigenfunction problems for the scalar-valued Laplace operator in the rectangular parallelepiped with the mixed boundary conditions. These subproblems can be solved by a standard technique Weinberger (1995).

Lemma 2.2.1. *Let the boundary Γ of the parallelepiped V be presented as $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3$, where Γ_1, Γ_2 and Γ_3 are the sets defined by*

$$\Gamma_1 = \{x = (x_1, x_2, x_3): \quad x_1 = 0 \vee x_1 = b_1, \quad 0 < x_2 < b_2, \quad 0 < x_3 < b_3\},$$

$$\Gamma_2 = \{x = (x_1, x_2, x_3): \quad x_2 = 0 \vee x_2 = b_2, \quad 0 < x_1 < b_1, \quad 0 < x_3 < b_3\},$$

$$\Gamma_3 = \{x = (x_1, x_2, x_3): \quad x_3 = 0 \vee x_3 = b_3, \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2\}.$$

The boundary condition in (2.8) can be written as

$$\left(\alpha_j(x)\frac{\partial U_j(x)}{\partial\vec{n}} + \beta_j(x)U_j(x)\right)\Bigg|_\Gamma = 0, \tag{2.9}$$

where the functions $\alpha_j(x)$ and $\beta_j(x)$, $j = 1,2,3$ are defined by

$$\alpha_j(x) = \begin{cases} 0, & x\in\Gamma_j, \\ 1, & x\in\Gamma\setminus\Gamma_j; \end{cases} \qquad \beta_j(x) = \begin{cases} 1, & x\in\Gamma_j, \\ 0, & x\in\Gamma\setminus\Gamma_j. \end{cases}$$

Proof. Let us prove this for $j = 1$ and consider the part of the boundary where $x_1 = b_1$. Thus, the outward normal unit vector is $\vec{\mathbf{e}^1} = (1,0,0)^T$. The second condition of (2.8) reads

$$U_1(b_1,x_2,x_3) = 0, \ \ for \ \ 0 < x_2 < b_2, \ \ 0 < x_3 < b_3.$$

The first condition of (2.8) takes the form

$$\frac{\partial U_2}{\partial x_1}(b_1,x_2,x_3) - \frac{\partial}{\partial x_2}[U_1(b_1,x_2,x_3)] = 0,$$
$$\frac{\partial U_3}{\partial x_1}(b_1,x_2,x_3) - \frac{\partial}{\partial x_3}[U_1(b_1,x_2,x_3)] = 0.$$

Combining the last three equalities we can write the boundary condition (2.8) in the form

$$\left.\left(\alpha_j(x)\frac{\partial U_j(x)}{\partial \vec{\mathbf{n}}} + \beta_j(x)U_j(x)\right)\right|_{x_1=b_1} = 0, \ \ j = 1,2,3,$$

where $\alpha_1 = 0, \beta_1 = 1$; $\alpha_2 = 1, \beta_2 = 0$; $\alpha_3 = 1, \beta_3 = 0$. Similar arguments hold for the lower part of Γ_1 and for the parts of the boundaries Γ_2 and Γ_3 leading (2.9). □

Remark 2.2.2. All eigenvalues of the problem (2.7)-(2.8) are real and nonnegative (see, Appendix A1).

Let λ_{kmn} denote the eigenvalues and $\mathbf{U}^{kmn}(x)$ the corresponding vector-valued eigenfunctions of (2.7)-(2.8). The eigenfunctions $\mathbf{U}^{kmn}(x)$ can be written in the form

$$\mathbf{U}^{kmn}(x) = U_1^{kmn}(x)\vec{\mathbf{e}^1} + U_2^{kmn}(x)\vec{\mathbf{e}^2} + U_3^{kmn}(x)\vec{\mathbf{e}^3},$$

and λ_{kmn}, $U_j^{kmn}(x)$, $j = 1,2,3$ satisfy

$$\Delta_x U_j^{kmn}(x) + \lambda_{kmn} U_j^{kmn}(x) = 0, \ \ x \in V; \tag{2.10}$$
$$\left.\left(\alpha_j(x)\frac{\partial U_j^{kmn}(x)}{\partial \vec{\mathbf{n}}} + \beta_j(x)U_j^{kmn}(x)\right)\right|_{\Gamma} = 0. \tag{2.11}$$

Applying the standard technique of the spectral theory (see, for example, Weinberger (1995)) we find all eigenvalues and corresponding eigenfunctions of the problem (2.10),(2.11) for each j as:

For $j = 1$,

$$\lambda_{kmn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2,$$

$$U_1^{kmn}(x) = A_{kmn} \sin\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{m\pi}{b_2}x_2\right)\cos\left(\frac{n\pi}{b_3}x_3\right), \tag{2.12}$$

$k = 1,2,...;\quad n,m = 0,1,...$ are eigenvalues and corresponding eigenfunctions.

For $j = 2$,

$$\lambda_{kmn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2,$$

$$U_2^{kmn}(x) = B_{kmn} \cos\left(\frac{k\pi}{b_1}x_1\right)\sin\left(\frac{m\pi}{b_2}x_2\right)\cos\left(\frac{n\pi}{b_3}x_3\right), \tag{2.13}$$

$m = 1,2,...; k,n = 0,1,...$ are eigenvalues and corresponding eigenfunctions.

For $j = 3$,

$$\lambda_{kmn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2,$$

$$U_3^{kmn}(x) = C_{kmn} \cos\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{m\pi}{b_2}x_2\right)\sin\left(\frac{n\pi}{b_3}x_3\right),$$

$n = 1,2,...; k,m = 0,1,...$ are eigenvalues and corresponding eigenfunctions.

Here A_{kmn}, B_{kmn}, C_{kmn} are constants, such that

$$\int_0^{b_3}\int_0^{b_2}\int_0^{b_1} U_j^{kmn}(x)\cdot U_j^{k'm'n'}(x)\,dx = \begin{cases} 0, & if\ (k,m,n) \neq (k',m',n'), \\ 1, & if\ k = k',\ m = m',\ n = n'\ . \end{cases}$$

$k,m,n = 0,1,...;\ j = 1,2,3$. That is, $A_{kmn} = B_{kmn} = C_{kmn} = \sqrt{\frac{8}{b_1b_2b_3}}$, $A_{k0n} = A_{km0} = B_{0mn} = B_{km0} = C_{0mn} = C_{k0n} = \sqrt{\frac{4}{b_1b_2b_3}}$ $A_{k00} = B_{0m0} = C_{00n} = \sqrt{\frac{2}{b_1b_2b_3}}$, $k = 1,2,...;\ m = 1,2,...;\ n = 1,2,....$

Remark 2.2.3. Let $U_j^{kmn}(x)$, $j = 1,2,3$, be eigenfunctions of (2.10)-(2.11), defined above. By the direct computation we can show that $\mathbf{U}^{kmn}(x)$ satisfies the boundary conditions (2.8) for the rectangular parallelepiped V.

Remark 2.2.4. For each $j = 1,2,3$ the system of eigenfunctions $\left\{U_j^{kmn}(x)\right\}$ is orthogonal and complete in the space of the square integrable functions over V. That is, any scalar-valued function $h(x) \in C_0^\infty(V)$ can be written in the form of uniformly convergent series over $\overline{V}$ (the closure of V) (see, for example, Weinberger (1995))

$$h(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_j^{kmn} U_j^{kmn}(x), \tag{2.14}$$

where

$$h_j^{kmn} = \iiint_V h(x) U_j^{kmn}(x)\,dx. \tag{2.15}$$

The constants h_j^{kmn} defined in (2.15) are called the Fourier coefficients of $h(x)$ relative to the basis functions $\left\{U_j^{kmn}(x)\right\}$.

Lemma 2.2.5. *(Weinberger, 1995) Let $h(x) \in C_0^\infty(V)$. Then*

$$\int\int_V\int |h(x)|^2 dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[h_j^{kmn}\right]^2 < \infty \tag{2.16}$$

holds. Here h_j^{kmn} are the Fourier coefficients (2.15). The equality (2.16) is known as Parseval's equality.

The proof of this lemma can be found, for example, in (Weinberger, 1995).

As a result, any vector function $\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x))^T$ with components from $C_0^\infty(V)$ can be presented in the form

$$\Psi(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\psi_1^{kmn}U_1^{kmn}(x)\vec{\mathbf{e}^1} + \psi_2^{kmn}U_2^{kmn}(x)\vec{\mathbf{e}^2} + \psi_3^{kmn}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right), \forall x \in \overline{V}, \tag{2.17}$$

where

$$\psi_j^{kmn} = \int\int_V\int \psi_j(x)U_j^{kmn}(x)\, dx, \quad j = 1,2,3. \tag{2.18}$$

We note that if $\psi_j(x) \in C_0^\infty(V)$ then $\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_j^{kmn}U_j^{kmn}(x)$, for each $j = 1,2,3$, is uniformly convergent series over $\overline{V}$.

2.3 Computation of Frequency-Dependent Magnetic Green's Function

2.3.1 Regularizations of the Dirac Delta Function and Its Derivatives

Let, $s = 1$, $x \in \mathbf{R}^3$, $x^0 \in \mathbf{R}^3$, $\delta(x-x^0) = \delta(x_1-x_1^0)\delta(x_2-x_2^0)\delta(x_3-x_3^0)$ be the Dirac delta distribution. The vector distribution $curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right] = -curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right]$ (for the proof, see Appendix A2) can be written as follows

$$-curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right] = curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right] = 0\cdot\vec{\mathbf{e}^1} + \frac{\partial\delta(x-x^0)}{\partial x_3^0}\vec{\mathbf{e}^2} - \frac{\partial\delta(x-x^0)}{\partial x_2^0}\vec{\mathbf{e}^3} \tag{2.19}$$

Using formulas (2.16), (2.18) we find the Fourier series expansion of $curl_{x^0}\left[\delta(x-x^0)\mathbf{e}^{\vec{1}}\right]$ in the form of (2.17), where

$$\psi_1^{kmn} = 0, \ \ \psi_2^{kmn} = \frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}, \ \ \psi_3^{kmn} = -\frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}. \tag{2.20}$$

(See, for example Vladimirov (1971, 2002), Schwartz (1966)).

Lemma 2.3.1. *Let $x^0 \in V$ be fixed, and $h(x) \in C_0^\infty(V)$. Then the series*

$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}h_2^{kmn}, \tag{2.21}$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}h_3^{kmn}, \tag{2.22}$$

are absolutely convergent.

Here h_2^{kmn} and h_3^{kmn} are defined by (2.15) for the given function $h(x)$.

The proof of Lemma 2.3.1 is given in Appendix A3.

Using Lemma 2.3.1 the distributions $\frac{\partial\delta(x-x^0)}{\partial x_3^0}$ and $\frac{\partial\delta(x-x^0)}{\partial x_2^0}$ can be defined by

$$\left\langle\frac{\partial\delta(x-x^0)}{\partial x_3^0}, h(x)\right\rangle = \frac{\partial h(x^0)}{\partial x_3^0} = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}h_2^{kmn}, \tag{2.23}$$

$$\left\langle\frac{\partial\delta(x-x^0)}{\partial x_2^0}, h(x)\right\rangle = \frac{\partial h(x^0)}{\partial x_2^0} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}h_3^{kmn}, \tag{2.24}$$

for any function $h(x) \in C_0^\infty(V)$.

Let us consider the infinitely differentiable functions $\psi_2^{1,N}(x, x^0)$, $\psi_3^{1,N}(x, x^0)$ defined by

$$\psi_2^{1,N}(x, x^0) = \sum_{k=0}^{N}\sum_{m=1}^{N}\sum_{n=0}^{N}\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}U_2^{kmn}(x), \tag{2.25}$$

$$\psi_3^{1,N}(x, x^0) = -\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=1}^{N}\frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}U_3^{kmn}(x). \tag{2.26}$$

Lemma 2.3.2. *Let $x^0 \in V$ be fixed, and functions $\psi_2^{1,N}(x, x^0)$, $\psi_3^{1,N}(x, x^0)$ be defined by*

(2.25), (2.26). Then for any $h(x) \in C_0^\infty(V)$

$$\lim_{N\to\infty} \int\int_V\int \psi_2^{1,N}(x,x^0)h(x)dx = \left\langle \frac{\partial\delta(x-x^0)}{\partial x_3^0}, h(x) \right\rangle, \tag{2.27}$$

$$\lim_{N\to\infty} \int\int_V\int \psi_3^{1,N}(x,x^0)h(x)dx = -\left\langle \frac{\partial\delta(x-x^0)}{\partial x_2^0}, h(x) \right\rangle. \tag{2.28}$$

The proof of Lemma 2.3.2 follows from Lemma 2.3.1 and standard arguments of the theory of distributions (see, for example, Vladimirov (1971, 2002)).

Remark 2.3.3. Equalities (2.27) and (2.28) mean that

$$\lim_{N\to\infty} \psi_2^{1,N}(x,x^0) = \frac{\partial\delta(x-x^0)}{\partial x_3^0} \quad \textit{and} \quad \lim_{N\to\infty} \psi_3^{1,N}(x,x^0) = -\frac{\partial\delta(x-x^0)}{\partial x_2^0}$$

in the space of generalized functions $\mathcal{D}'(V)$ (see, for example, (Vladimirov, 1971, 2002)).

Remark 2.3.4. $\psi_2^{1,N}(x,x^0)$, $\psi_3^{1,N}(x,x^0)$ can be taken as regularizations (approximations) of $\frac{\partial\delta(x-x^0)}{\partial x_3^0}$ and $-\frac{\partial\delta(x-x^0)}{\partial x_2^0}$, respectively. Here N is a parameter of the regularization (see, for example, Vladimirov (1971, 2002)).

Remark 2.3.5. Let

$$\begin{aligned} \Psi^{1,N}(x,x^0) &= (0, \psi_2^{1,N}, \psi_3^{1,N})^T \\ &= \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0} U_2^{kmn}(x)\vec{\mathbf{e}^2} - \frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0} U_3^{kmn}(x)\vec{\mathbf{e}^3} \right). \end{aligned} \tag{2.29}$$

Using Lemma 2.3.2 and Remarks 2.3.3 and 2.3.4 we find that

$$\lim_{N\to\infty} \Psi^{1,N}(x,x^0) = curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right]$$

in the space of distributions $\mathcal{D}'(V)$ and therefore the vector function $\Psi^{1,N}(x,x^0)$, with infinitely differentiable components, can be taken as a regularization of the generalized vector function $curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right]$.

Similarly, we find that the vector functions $\Psi^{2,N}(x,x^0)$ and $\Psi^{3,N}(x,x^0)$ defined by

$$\Psi^{2,N}(x,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(-\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0} U_1^{kmn}(x)\vec{\mathbf{e}^1} + \frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0} U_3^{kmn}(x)\vec{\mathbf{e}^3} \right), \tag{2.30}$$

$$\Psi^{3,N}(x,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(\frac{\partial U_1^{kmn}(x^0)}{\partial x_2^0} U_1^{kmn}(x)\vec{\mathbf{e}^1} - \frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0} U_2^{kmn}(x)\vec{\mathbf{e}^2} \right), \tag{2.31}$$

can be taken as the regularizations of $curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^2}\right]$ and $curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^3}\right]$.

2.3.2 Solving (2.4)-(2.5) with the Regularized Inhomogeneous Term

Let us consider the equations (2.4)-(2.5) for the frequency-dependent magnetic Green's function. We replace $curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]$ by their regularizations defined by (2.29), (2.30), (2.31) for $s = 1,2,3$, respectively. Moreover, equations (2.4)-(2.5) are replaced by the following equations

$$\Delta_x I_3 \mathbf{H}^{s,N} + \omega^2 \epsilon\mu \mathbf{H}^{s,N} = \Psi^{s,N}(x,x^0), \tag{2.32}$$

$$(curl_x \mathbf{H}^{s,N} \times \vec{\mathbf{n}})\big|_\Gamma = 0, \quad (\mathbf{H}^{s,N}\cdot\vec{\mathbf{n}})\big|_\Gamma = 0. \tag{2.33}$$

We will find solutions of the problem (2.32)-(2.33) in the form

$$\mathbf{H}^{s,N}(x,\omega,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\Big(Y_1^{s,kmn}(\omega,x^0)U_1^{kmn}(x)\vec{\mathbf{e}^1} + Y_2^{s,kmn}(\omega,x^0)U_2^{kmn}(x)\vec{\mathbf{e}^2} + Y_3^{s,kmn}(\omega,x^0)U_3^{kmn}(x)\vec{\mathbf{e}^3}\Big), \tag{2.34}$$

where $x \in V$; $U_j^{kmn}(x)$ are eigenfunctions of the eigenvalue-eigenfunction problem (2.10),(2.11); $Y_j^{s,kmn}(\omega,x^0)$, $j = 1,2,3$, are the unknown Fourier coefficients.

Substituting (2.34) into (2.32) and using orthogonality of $U_j^{kmn}(x)$, we find

$$\left(\omega^2\epsilon\mu - \lambda_{kmn}\right)Y_j^{s,kmn}(\omega,x^0) = \psi_j^{s,kmn}(x^0),$$

for $j = 1,2,3$; $s = 1,2,3$; $k = 0,1,...$, $m = 0,1,...$, $n = 0,1,....$
Let $\omega^2 \neq \dfrac{\lambda_{kmn}}{\epsilon\mu}$, then $Y_j^{s,kmn}(\omega,x^0)$ is uniquely determined by

$$Y_j^{s,kmn}(\omega,x^0) = \frac{\psi_j^{s,kmn}(x^0)}{\omega^2\epsilon\mu - \lambda_{kmn}}, \tag{2.35}$$

where

$$\psi_j^{s,kmn}(x^0) = \iiint_V \Psi_j^s(x,x^0)U_j^{kmn}(x)\,dx,$$

$k,m,n = 0,1,\ldots$. As a result we obtain the following explicit formula

$$\mathbf{H}^{1,N}(x,\omega,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\left(\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}U_2^{kmn}(x)\vec{\mathbf{e}^2} - \frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right), \tag{2.36}$$

for all $x \in V$, $x^0 \in V$ and all parameter ω satisfying $\omega^2\epsilon\mu \neq \lambda_{kmn}$ ($k = 0,1,...;\ m = 0,1,...;\ n = 0,1,...$).

Similarly, the solutions of (2.32)-(2.33) for $s = 2$ and $s = 3$ are obtained in the form

$$\mathbf{H}^{2,N}(x,\omega,x^0) = -\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\left(\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}U_1^{kmn}(x)\vec{\mathbf{e}^1} - \frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right), \tag{2.37}$$

$$\mathbf{H}^{3,N}(x,\omega,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\left(\frac{\partial U_1^{kmn}(x^0)}{\partial x_2^0}U_1^{kmn}(x)\vec{\mathbf{e}^1} - \frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0}U_2^{kmn}(x)\vec{\mathbf{e}^2}\right). \tag{2.38}$$

Using the explicit formulas for $\mathbf{H}^{s,N}(x,\omega,x^0)$ and $U_j^{kmn}(x)$ we can check directly that $\mathbf{H}^{s,N}(x,\omega,x^0)$ satisfy the boundary condition (2.33).

Remark 2.3.6. By direct computation we can show that $\mathbf{H}^{s,N}(x,\omega,x^0)$, defined by (2.36)-(2.38), satisfy $div_x\mathbf{H}^{s,N}(x,\omega,x^0) = 0,\ s = 1,2,3.$

2.3.3 Frequency-Dependent Magnetic Green's Function and Its Regularization

In this section we show that $\lim_{N\to\infty} H_j^{s,N}(x,\omega,x^0)$, ($s = 1,2,3;\ j = 1,2,3$) exists in the space of distributions $\mathcal{D}'(V)$, and the distributions $H_j^s = \lim_{N\to\infty} H_j^{s,N}$ are the entries of the matrix-valued Green's function $\mathbf{G}^H(x,\omega,x^0)$.

Let us consider for example, $s = 1,\ j = 2$ (the other eight cases are verbatim) and an arbitrary function $h(x) \in C_0^\infty(V)$ of the form (2.14), (2.15). The integrable over V function $H_j^{s,N}(x,t,x^0)$ defines a regular functional (distribution) by

$$\left\langle H_2^{1,N}(x,\omega,x^0), h(x)\right\rangle = \sum_{k=0}^{N}\sum_{m=1}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}h_2^{kmn}.$$

Using Lemma 2.3.1 we find that $\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}h_2^{kmn}$ is absolutely convergent for any $h(x) \in C_0^\infty(V)$. This means that $\lim_{N\to\infty}\left\langle H_2^{1,N}(x,\omega,x^0), h(x)\right\rangle$ exists for any $h(x) \in C_0^\infty(V)$ and defines the linear continuous functional $H_2^{1,N}(x,\omega,x^0)$ over $C_0^\infty(V)$.

Similarly, we can find $H_j^s(x,\omega,x^0)$ as $\lim_{N\to\infty} H_j^{s,N}(x,\omega,x^0)$ in the complete space $\mathcal{D}'(V)$ (see, for example, Vladimirov (1971, 2002)) for $s = 1,2,3$ and $j = 1,2,3$. Moreover, since $\mathbf{H}^{s,N}(x,\omega,x^0) = \left(H_1^{s,N}(x,\omega,x^0), H_2^{s,N}(x,\omega,x^0), H_3^{s,N}(x,\omega,x^0)\right)^T$ satisfies (2.32)-(2.33) then $\mathbf{H}^s(x,\omega,x^0) = \left(H_1^s(x,\omega,x^0), H_2^s(x,\omega,x^0), H_3^s(x,\omega,x^0)\right)^T$ satisfies the following equalities

$$\omega^2\epsilon\mu\mathbf{H}^s - curl_x curl_x \mathbf{H}^s = -curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right],$$
$$(curl_x\mathbf{H}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \quad (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0,$$

that is, equalities (2.4)-(2.5). This means that $\mathbf{H}^s(x,\omega,x^0)$ is the $s-column$ of frequency-dependent magnetic matrix Green's function.

2.4 Approximate Computation of the Electric Green's Function

Let columns of the magnetic Green's function be derived approximately, that is, solutions $\mathbf{H}^{s,N}$, $s = 1,2,3$ of the problem (2.32)-(2.33) be computed by (2.36)-(2.38). For the approximate derivation of $\mathbf{E}^s(x,\omega,x^0)$, satisfying (2.6), we replace $curl_x\mathbf{H}^s$ by $curl_x\mathbf{H}^{s,N}$ and $\delta(x-x^0)\vec{\mathbf{e}^s}$ by its regularization $\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} U_s^{kmn}(x^0)U_s^{kmn}(x)\vec{\mathbf{e}^s}$ in (2.6) for $s = 1,2,3$, respectively and find an explicit formula for $\mathbf{E}^{s,N}(x,\omega,x^0)$ as follows

$$\mathbf{E}^{s,N}(x,\omega,x^0) = \frac{i}{\omega\epsilon}\left[curl_x\mathbf{H}^{s,N}(x,\omega,x^0) - \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} U_s^{kmn}(x^0)U_s^{kmn}(x)\vec{\mathbf{e}^s}\right], \quad (2.39)$$

for all $x \in V$, $x^0 \in V$ and all parameter ω satisfying $\omega^2\epsilon\mu \neq \lambda_{kmn}$.

2.4.1 The Regularization of the Electric Green's Function

Let us consider (2.39) for $s = 2$, that is, $\mathbf{E}^{2,N}$. The differentiable function $E_2^{2,N}(x,\omega,x^0)$ is the second component $\mathbf{E}^{2,N}$. This function defines a linear continuous regular functional over $C_0^\infty(V)$ by the following relation (see, for example, Vladimirov

(1971, 2002))

$$\left\langle E_2^{2,N}, h(x)\right\rangle = -\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\Big[\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}\int\int_V\int U_1^{kmn}(x)\frac{\partial h(x)}{\partial x_3}dx$$
$$+\frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0}\int\int_V\int U_3^{kmn}(x)\frac{\partial h(x)}{\partial x_1}dx\Big]+\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}U_2^{kmn}(x^0)h_2^{kmn}, \tag{2.40}$$

for an arbitrary $h(x) \in C_0^\infty(V)$. The series in the right-hand side of the last equality is absolutely convergent for any $h(x) \in C_0^\infty(V)$ (see, Appendix A4). This means that the limit of the inner product $\lim_{N\to\infty}\langle E_2^{2,N}(x,\omega,x^0), h(x)\rangle$ exists for any $h(x) \in C_0^\infty(V)$ and defines a linear continuous regular functional over $C_0^\infty(V)$. We denote this functional as $E_2^2(x,\omega,x^0)$, that is, $E_2^2(x,\omega,x^0) = \lim_{N\to\infty} E_2^{2,N}(x,\omega,x^0)$ in the complete space $\mathcal{D}'(V)$. Similarly, we can find $E_j^s(x,\omega,x^0)$ as $\lim_{N\to\infty} E_j^{s,N}(x,\omega,x^0)$ in the space $\mathcal{D}'(V)$ for $s = 1,2,3$ and $j = 1,2,3$.

The s-column (sth column) of the electric Green's function is

$$\mathbf{E}^s(x,\omega,x^0) = \frac{i}{\omega\epsilon}\left[curl_x\mathbf{H}^s(x,\omega,x^0) - \delta(x-x^0)\vec{\mathbf{e}}^s\right].$$

Now, let $\mathbf{E}^s(x,\omega,x^0) = \left(E_1^s(x,\omega,x^0), E_2^s(x,\omega,x^0), E_3^s(x,\omega,x^0)\right)^T$ be the generalized vector function, whose components are distributions defined as $E_j^s = \lim_{N\to\infty} E_j^{s,N}$, $j = 1,2,3$. Since $\mathbf{E}^{s,N}(x,\omega,x^0) = \left(E_1^{s,N}(x,\omega,x^0), E_2^{s,N}(x,\omega,x^0), E_3^{s,N}(x,\omega,x^0)\right)^T$ satisfies (2.39), then the generalized vector function $\mathbf{E}^s(x,\omega,x^0)$ satisfies (2.2).

Remark 2.4.1. We note that the vector of generalized functions $\mathbf{H}^s(x,\omega,x^0)$, $\mathbf{E}^s(x,\omega,x^0)$, defined in Sections 2.3.3 and 2.4.1, satisfy (2.4)-(2.5) and (2.6). It follows that $\mathbf{H}^s(x,\omega,x^0)$ and $\mathbf{E}^s(x,\omega,x^0)$ satisfy (2.1)-(2.3), and therefore, $\mathbf{H}^s(x,\omega,x^0)$, $\mathbf{E}^s(x,\omega,x^0)$ are s-columns of frequency-dependent electric and magnetic Green's functions $\mathbf{G}^E$ and $\mathbf{G}^H$, respectively.

2.5 Computational Experiments

In this section, we present computational experiments to show applicability of the approximate computation of frequency-dependent magnetic and electric matrix Green's functions in the rectangular parallelepiped. Here we consider the parallelepiped

$$V = \left\{x \in \mathbf{R}^3 : \quad 0 < x_1 < 3, \quad 0 < x_2 < 4, \quad 0 < x_3 < 2.5\right\}$$

which contains a dielectric material characterized by the electric permittivity ϵ and the magnetic permeability μ. We choose ϵ and μ as $c = 1/\sqrt{\epsilon\mu}$ ($c \approx 3 \cdot 10^8\, m/s$ is the speed of light) and ω is equal to the value of c. The boundary of V, denoted as Γ, is a perfect conducting boundary on which the tangential components of the electric field component and normal component of the magnetic field component vanish. There is an electric current $\mathbf{J}(x, \omega)$ inside V which produces the electric and magnetic fields. In our experiments we take $\mathbf{J}^s(x, \omega)$, $s = 1, 2, 3$ as

$$\mathbf{J}^s(x, \omega) = \delta(x - x^0)\vec{\mathbf{e}^s},$$

$\vec{\mathbf{e}^s}$ are the standard basis vectors of $\mathbf{R}^3$; $x^0 = (x_1^0, x_2^0, x_3^0)$ is the 3D parameter from V and chosen as $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$; $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, and $\delta(x_j - x_j^0)$ is the Dirac delta distribution at $x_j = x_j^0$, for each $j = 1, 2, 3$.

Two types of computational experiments will be presented. The first one is related to convergence and error analysis of our method; the second one is the computation of the elements of the matrix magnetic and electric Green's functions in the rectangular parallelepiped V and the visualization of the computed magnetic and electric fields components by MATLAB graphic tools.

2.5.1 Convergence Analysis

The Green's function of the magnetic field is a 3×3 matrix whose elements are distributions. Generally speaking, distributions do not have values at fixed points (Vladimirov, 1971, 2002). The elements of the regularized (approximate) matrix Green's function, which we have computed by our method, are classical differentiable and integrable functions over V having the values at fixed points. These classical functions define regular distributions by the standard rule (see, Vladimirov (1971, 2002)). The comparison of the distributions can be made by comparing their integral characteristics. Some integral characteristics of the elements of the matrix Green's function of the magnetic field can be found explicitly. For example, let $H_2^1(x, \omega, x^0)$ be a solution of (2.4)-(2.5) for $s = 1$, $j = 2$ and $h_2^1(x_3,\omega,x^0)$ be defined by

$$h_2^1(x_3,\omega,x^0) = \int_0^{b_1}\int_0^{b_2} H_2^1(x,\omega,x^0)\cos\left(\frac{\pi x_1}{b_1}\right)\sin\left(\frac{\pi x_2}{b_2}\right)dx_2dx_1.$$

We can show that $h_2^1(x_3,\omega,x^0)$ satisfies

$$\frac{\partial^2 h_2^1}{\partial x_3^2} + k^2 h_2^1 = -\frac{\partial}{\partial x_3}\delta(x_3 - x_3^0)\cos\left(\frac{\pi x_1^0}{b_1}\right)\sin\left(\frac{\pi x_2^0}{b_2}\right), \quad 0 < x_3 < b_3, \tag{2.41}$$

$$\left.\frac{dh_2^1}{dx_3}\right|_{x_3=0} = 0, \quad \left.\frac{dh_2^1}{dx_3}\right|_{x_3=b_3} = 0, \tag{2.42}$$

where $k^2 = \omega^2\epsilon\mu - \left(\frac{\pi}{b_1}\right)^2 - \left(\frac{\pi}{b_2}\right)^2$. A solution of (2.41)-(2.42) can be written in the form

$$h_2^1(x_3,\omega,x^0) = \begin{cases} \frac{\sin\left(k(b_3-x_3^0)\right)}{\sin(kb_3)}\cos(kx_3)\cos\left(\frac{\pi x_1^0}{b_1}\right)\sin\left(\frac{\pi x_2^0}{b_2}\right), & 0 < x_3 < x_3^0, \\ -\frac{\sin(kx_3^0)}{\sin(kb_3)}\cos(k(b_3 - x_3))\cos\left(\frac{\pi x_1^0}{b_1}\right)\sin\left(\frac{\pi x_2^0}{b_2}\right), & x_3^0 < x_3 < b_3. \end{cases} \tag{2.43}$$

We can compare the values of $h_2^1(x_3,\omega,x^0)$, as the values of the integral characteristics of H_2^1, with the values of the integral characteristic

$$\begin{aligned} h_2^{1,N}(x_3,\omega,x^0) &= \int_0^{b_1}\int_0^{b_2} H_2^{1,N}(x,\omega,x^0)\cos\left(\frac{\pi x_1}{b_1}\right)\sin\left(\frac{\pi x_2}{b_2}\right)dx_2dx_1 \\ &= \sum_{n=1}^{N}\frac{-1}{k^2-\left(\frac{n\pi}{b_3}\right)^2}\frac{2n\pi}{(b_3)^2}\cos\left(\frac{\pi x_1^0}{b_1}\right)\sin\left(\frac{\pi x_2^0}{b_2}\right)\sin\left(\frac{n\pi x_3^0}{b_3}\right)\cos\left(\frac{n\pi x_3}{b_3}\right) \end{aligned} \tag{2.44}$$

of $H_2^{1,N}$. Here $H_2^{1,N}$ is the element of the approximate matrix Green's function computed by our method and N is the parameter of the approximation corresponding to the number of terms in each series of the right hand side of (2.44).

In Figures 2.1-2.4 and Table 2.1 the values of $h_2^{1,N}$ are compared with the values of h_2^1 for fixed $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$, and ϵ and μ such that $c = 1/\sqrt{\epsilon\mu}$, and $\omega = c$, and different numbers N. The graph of h_2^1 has jumps at two points and graphs of $h_2^{1,N}$ do not have the small oscillations near the jumps of h_3^2 (see Figures 2.1-2.4). This behavior is known as the Gibbs phenomenon, which manifests the peculiar manner in which the Fourier series of a piecewise continuously differentiable function behaves at jumps discontinuity. In all cases of the Gibbs phenomenon and regardless of the number of harmonics (terms of the Fourier series) it is observed that overshoot of the ripples has

Table 2.1 The accuracy of the calculation of the magnetic field for $x_1^0 = 2$; $x_2^0 = 3$; $x_3^0 = 1.5$; $r_N = \frac{|h_2^1 - h_2^{1,N}|}{|h_2^1|}$; $N = 15, 50, 100, 150$.

x_3	r_{15}	r_{50}	r_{100}	r_{150}
0.5	0.0705	0.0213	0.0107	0.0071
0.7	0.0597	0.0232	0.0116	0.0078
0.9	0.0208	0.0271	0.0136	0.0090
1.1	0.0492	0.0356	0.0179	0.0119
1.3	0.1806	0.0625	0.0316	0.0211
1.7	0.1525	0.0531	0.0268	0.0179
1.9	0.0417	0.0317	0.0159	0.0106
2.1	0.0213	0.0251	0.0126	0.0084
2.2	0.0645	0.0235	0.0118	0.0078
2.4	0.0233	0.0220	0.0110	0.0073

a constant magnitude around 18%. The maximum of relative error $r_N(x_3) = \frac{|h_2^1 - h_2^{1,N}|}{|h_2^1|}$ at the point $x_3 = 1.3$ of the interval $[0, 2.5]$ for $N = 15, 50, 100, 150$ are $r_{15} = 0.1806$, $r_{50} = 0.0625$, $r_{100} = 0.0316$, $r_{150} = 0.0211$, respectively. The error is monotonically decreasing for each fixed x_3. Making the error analysis we have concluded that the biggest N corresponds to the smallest error between values of $h_2^{1,N}$ and h_2^1. We have taken $N = 100$ as a reasonable value between accuracy and the time for our computation in MATLAB.

2.5.2 Approximate Computation and Visualization of the Elements of Matrix Magnetic and Electric Green's Functions

The method described in Sections 2.3 and 2.4 has been applied for the approximate computation of frequency-dependent magnetic and electric Green's matrices in the rectangular parallelepiped

$$V = \left\{ x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \; 0 < x_2 < b_2, \; 0 < x_3 < b_3 \right\},$$

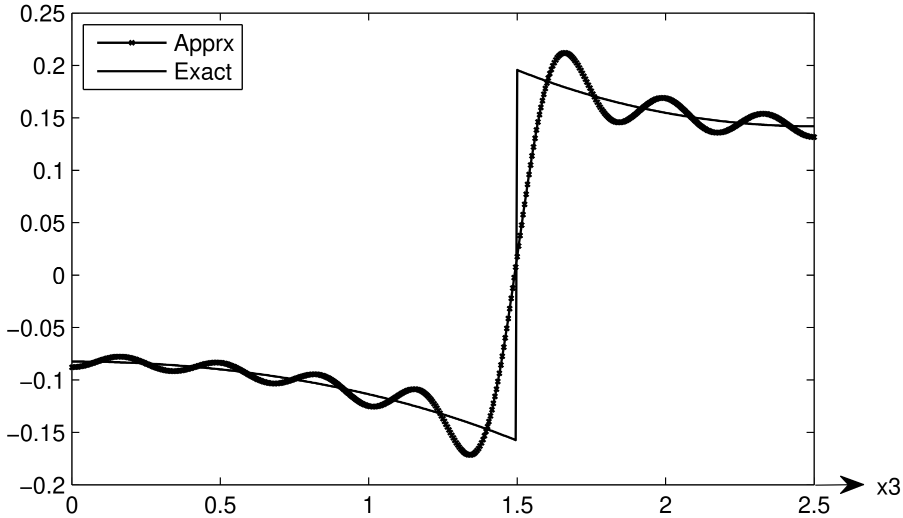

Figure 2.1 The graphs of h_2^1 and $h_2^{1,N}$ for $N = 15$, $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$.

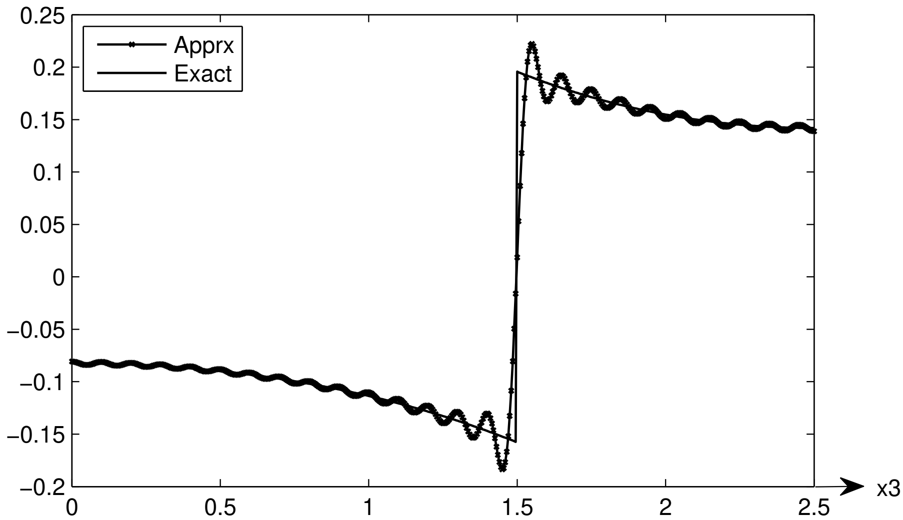

Figure 2.2 The graphs of h_2^1 and $h_2^{1,N}$ for $N = 50$, $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$.

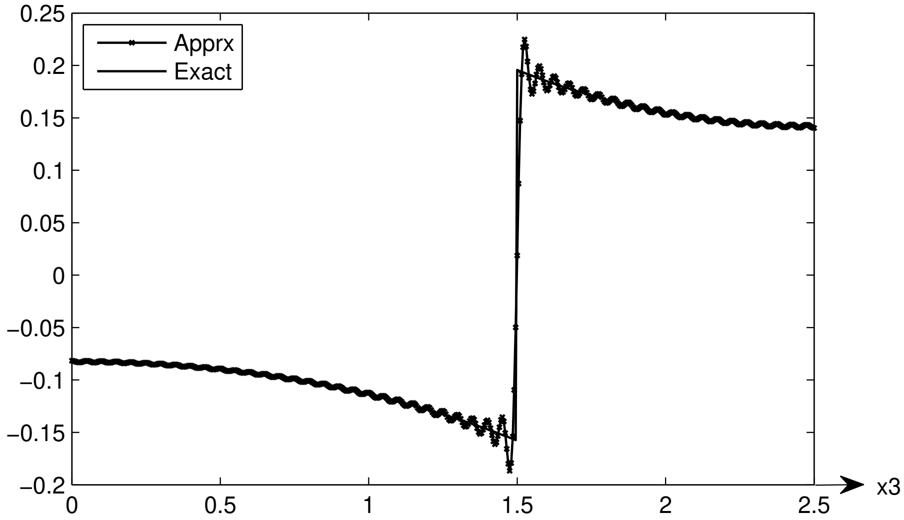

Figure 2.3 The graphs of h_2^1 and $h_2^{1,N}$ for $N = 100$, $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$.

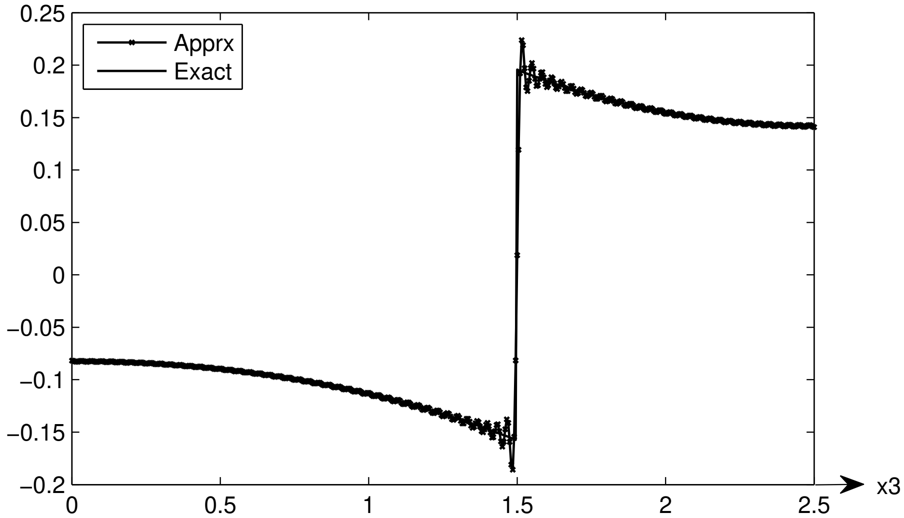

Figure 2.4 The graphs of h_2^1 and $h_2^{1,N}$ for $N = 150$, $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$.

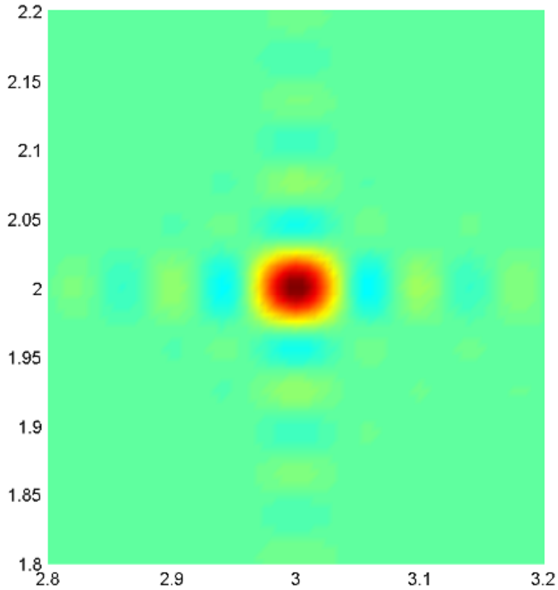

Figure 2.5 The second component of the first column of the regularized magnetic Green's function, $H_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

respectively. We note that each column of magnetic and electric frequency-dependent Green's matrices are the magnetic and electric fields arising from the pulse dipole polarized in the direction of the standard basis vector of $\mathbf{R}^3$. For computation we take $b_1 = 3$, $b_2 = 4$, $b_3 = 2.5$; ε and μ such that $c = 1/\sqrt{\varepsilon\mu}$; $\omega = c$ (c is the value of the light velocity); $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$. Computation of $H_j^{s,N}$ and $E_j^{s,N}$ have been imple- mented in MATLAB and their graphs are presented. Figures 2.5-2.6 and 2.7-2.8 present the second and third components of the first column of the regularized magnetic matrix, respectively, $H_2^{1,100}(x, \omega, x^0)$ and $H_3^{1,100}(x, \omega, x^0)$. Figs. 2.9-2.10, 2.11-2.12, 2.13-2.14 present each component of the first column of the regularized electric matrix, respectively, $E_1^{1,100}(x, \omega, x^0)$, $E_2^{1,100}(x, \omega, x^0)$ and $E_3^{1,100}(x, \omega, x^0)$. The views of the surfaces $z = H_2^{1,N}$, $z = H_3^{1,N}$, $z = E_1^{1,N}$, $z = E_2^{1,N}$, $z = E_3^{1,N}$ (N=100) from the top of z-axis are presented in Figs. 2.5, 2.7, 2.9, 2.11, 2.13, where x_1-axis is vertical and x_2-axis is horizontal one. Figures 2.6, 2.8, 2.10, 2.12, 2.14 are 3D images of the surfaces $H_2^{1,100}(x, \omega, x^0)$, $H_3^{1,100}(x, \omega, x^0)$, $E_1^{1,100}(x, \omega, x^0)$, $E_2^{1,100}(x, \omega, x^0)$ and $E_3^{1,100}(x, \omega, x^0)$ for $x_3 = 1.5$, respectively. Here, x_1 and x_2 are horizontal axes and z-axis is the magnitude.

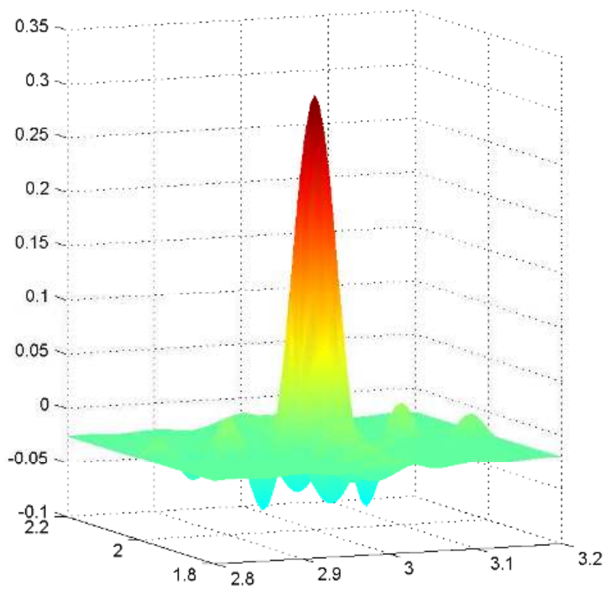

Figure 2.6 The second component of the first column of the regularized magnetic Green's function, $H_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

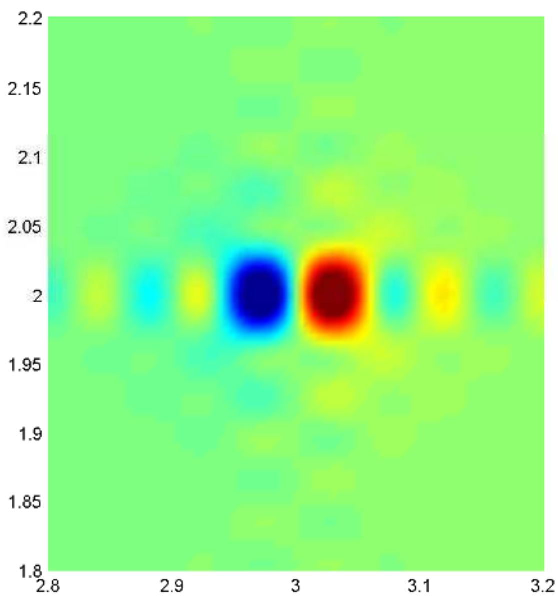

Figure 2.7 The third component of the first column of the regularized magnetic Green's function, $H_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

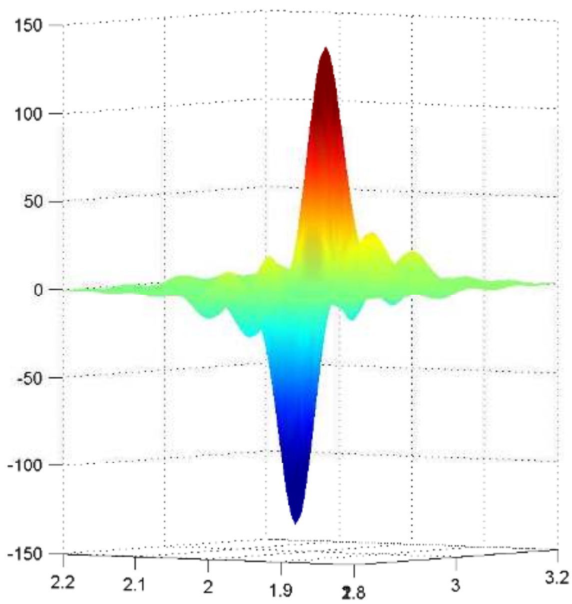

Figure 2.8 The third component of the first column of the regularized magnetic Green's function, $H_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

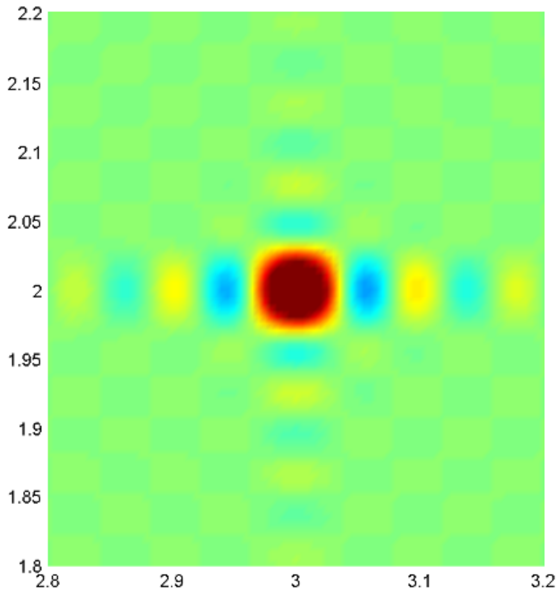

Figure 2.9 The first component of the first column of the regularized electric Green's function, $E_1^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

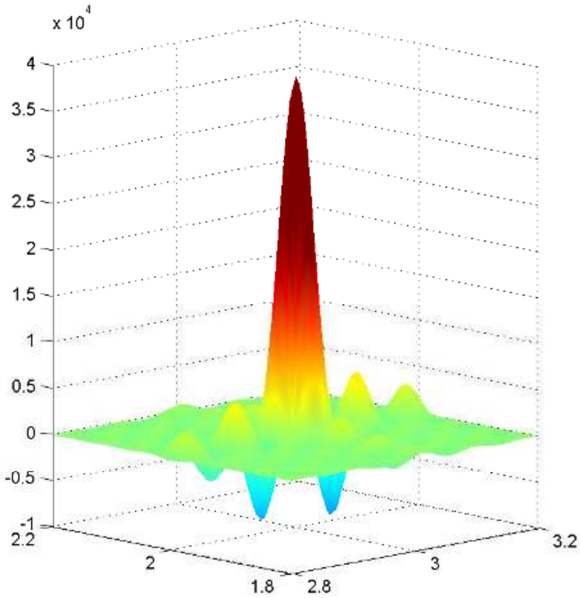

Figure 2.10 The first component of the first column of the regularized electric Green's function, $E_1^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

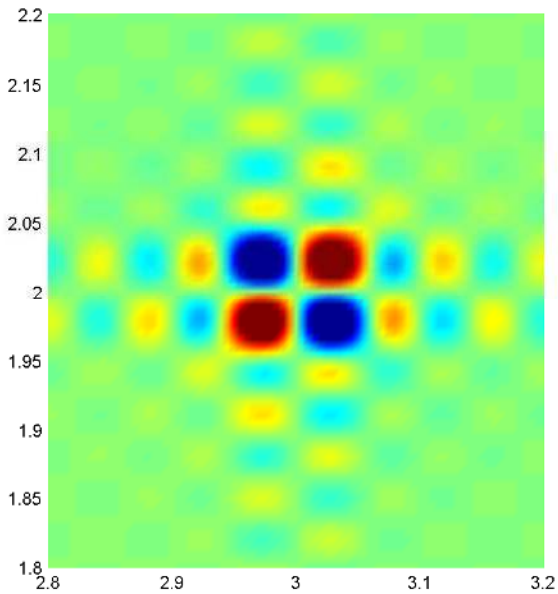

Figure 2.11 The second component of the first column of the regularized electric Green's function, $E_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

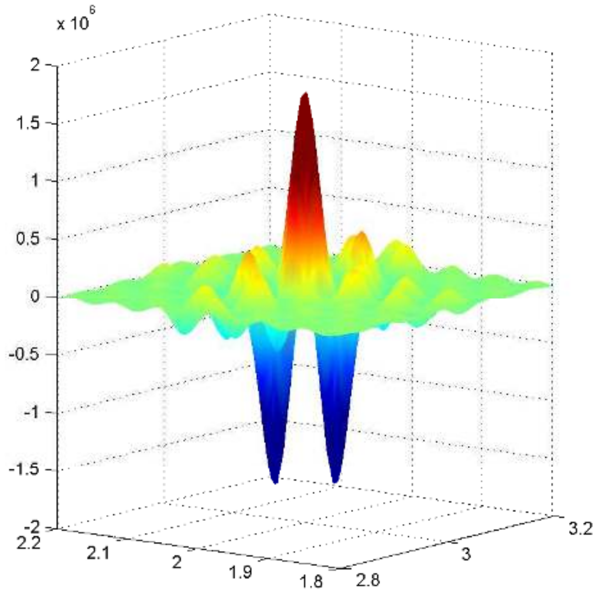

Figure 2.12 The second component of the first column of the regularized electric Green's function, $E_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

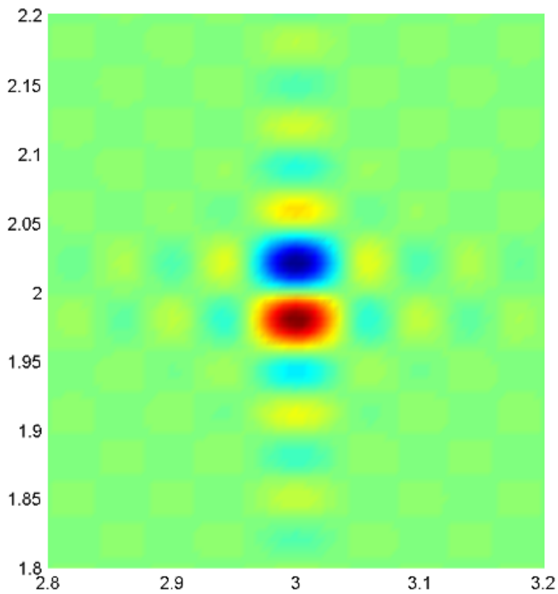

Figure 2.13 The third component of the first column of the regularized electric Green's function, $E_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

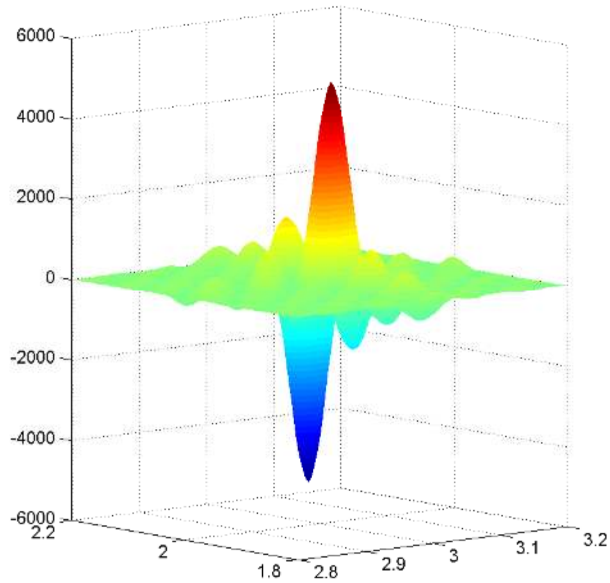

Figure 2.14 The third component of the first column of the regularized electric Green's function, $E_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$.

2.6 Summary of Chapter Two

A new method for the approximate computation of frequency-dependent magnetic and electric Green's matrices in the rectangular parallelepiped with a perfect conducting boundary has been developed. The columns of the magnetic and electric matrix Green's function in the space of generalized functions (distributions) and their regularizations (approximations) have been obtained . The method has been implement-ed in MATLAB. The computational experiments have been confirmed its robustness.

CHAPTER THREE
THE APPROXIMATE COMPUTATION OF THE TIME DEPENDENT MAGNETIC AND ELECTRIC MATRIX GREEN'S FUNCTIONS IN A RECTANGULAR PARALLELEPIPED

In this chapter an initial boundary value problem (IBVP) for the Maxwell's equations in a rectangular parallelepiped with perfect conducting boundary is studied. A method based on the Fourier series meta-approach is suggested. This method consists of the following: the construction of the eigenvalues and corresponding orthogonal set of the eigenfunctions of the operator $\Delta_x I_3$; regularizations (approximations) of the Dirac delta function and its derivatives using the found eigenfunctions as basis functions; approximate computation of the elements of the magnetic Green's function using the Fourier series meta approach; the derivation of approximate elements of the electric Green's function by direct integration with respect to time variable. The method is implemented in MATLAB and computational experiments are given to show applicability of the suggested method.

3.1 Time-Dependent Magnetic and Electric Green's Functions

Let $b_1, b_2, b_3 \in \mathbf{R}^+$, $x = (x_1, x_2, x_3)^T$ be a 3D variable from $\mathbf{R}^3$;

$$V = \left\{ x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2, \quad 0 < x_3 < b_3 \right\} \tag{3.1}$$

be a rectangular parallelepiped; Γ be the boundary of V.
The 3×3 matrices

$$\mathbf{G}^H(x,t,x^0) = \begin{pmatrix} H_1^1(x,t,x^0) & H_1^2(x,t,x^0) & H_1^3(x,t,x^0) \\ H_2^1(x,t,x^0) & H_2^2(x,t,x^0) & H_2^3(x,t,x^0) \\ H_3^1(x,t,x^0) & H_3^2(x,t,x^0) & H_3^3(x,t,x^0) \end{pmatrix},$$

$$\mathbf{G}^E(x,t,x^0) = \begin{pmatrix} E_1^1(x,t,x^0) & E_1^2(x,t,x^0) & E_1^3(x,t,x^0) \\ E_2^1(x,t,x^0) & E_2^2(x,t,x^0) & E_2^3(x,t,x^0) \\ E_3^1(x,t,x^0) & E_3^2(x,t,x^0) & E_3^3(x,t,x^0) \end{pmatrix},$$

whose s-columns $\mathbf{H}^s(x,t,x^0)$, $\mathbf{E}^s(x,t,x^0)$, $s = 1,2,3$ satisfy

$$curl_x \mathbf{H}^s = \epsilon \frac{\partial \mathbf{E}^s}{\partial t} + \delta(x - x^0)\delta(t)\vec{\mathbf{e}^s}, \quad x \in V, \quad t \in \mathbf{R}, \tag{3.2}$$

$$curl_x \mathbf{E}^s = -\mu \frac{\partial \mathbf{H}^s}{\partial t}, \tag{3.3}$$

$$\mathbf{E}^s\big|_{t<0} = 0, \quad \mathbf{H}^s\big|_{t<0} = 0, \tag{3.4}$$

$$(\mathbf{E}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \ (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0, \ s = 1,2,3, \tag{3.5}$$

are called the magnetic Green's function (matrix) and the electric Green's function (matrix), respectively, in V. Here ϵ and μ are given positive constants characterizing the electric permittivity and magnetic permeability, respectively; $\vec{\mathbf{e}^s}$ are the basis vectors of $\mathbf{R}^3$; x^0 is a 3D parameter from V; t is the time variable; $\delta(t)$ is the Dirac delta function at $t = 0$ (see, for example Vladimirov (1971)); $\vec{\mathbf{n}}$ is the unit normal vector to the boundary Γ of the parallelepiped V.

Remark 3.1.1. We note that $div_x \mathbf{H}^s = 0$ for all $x \in V$ is a consequence of (3.3) and (3.4).

Remark 3.1.2. The boundary conditions (3.5) are called the perfect conducting conditions (see, for example, Eom (2004)).

3.1.1 Equations for the Time-Dependent Magnetic Green's Function

Applying $curl_x$ to (3.2) and using (3.3) with Remark3.1.1, we find

$$\frac{\partial^2 \mathbf{H}^s}{\partial t^2} + c^2 curl_x curl_x \mathbf{H}^s = c^2 curl_x \left[\delta(x - x^0)\vec{\mathbf{e}^s}\right]\delta(t), \tag{3.6}$$

where $c = 1/\sqrt{\epsilon\mu}$.

It follows from equation (3.3) and the second condition (3.4) that

$$\mathbf{H}^s\big|_{t<0} = 0, \quad \frac{\partial \mathbf{H}^s}{\partial t}\bigg|_{t<0} = 0. \tag{3.7}$$

Taking the cross product of both sides of the equation (3.2) and $\vec{\mathbf{n}}$, and using the conditions in (3.5) we get

$$(curl_x \mathbf{H}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \ \ (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0. \tag{3.8}$$

The equations (3.6)-(3.8) will be used for the computation of the time-dependent magnetic Green's function.

3.1.2 *Equations for Time-Dependent Electric Green's Function*

Let $\mathbf{H}^s(x,t,x^0)$ be the s-column of the magnetic Green's function, which we assume to be computed. Then it follows from (3.2) and (3.4) that

$$\frac{\partial \mathcal{E}^s}{\partial t} = \frac{1}{\epsilon} curl_x \mathbf{H}^s, \tag{3.9}$$

$$\mathcal{E}^s \mid_{t<0} = 0, \tag{3.10}$$

where

$$\mathcal{E}^s = \mathbf{E}^s + \frac{1}{\epsilon}\delta(x-x^0)\Theta(t)\vec{\mathbf{e}^s}, \tag{3.11}$$

and $\Theta(t)$ is the Heaviside step function, that is, $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t < 0$.

The equations (3.9)-(3.10) will be used for the computation of time-dependent electric Green's function if the magnetic Green's function is known.

Remark 3.1.3. If $\mathbf{H}^s$ is a vector function satisfying (3.6)-(3.8), $\mathcal{E}^s$ is a vector function satisfying (3.9)-(3.10) and $\mathbf{E}^s = \mathcal{E}^s - \frac{1}{\epsilon}\delta(x-x^0)\Theta(t)\vec{\mathbf{e}^s}$ for a fixed value of $s = 1,2,3$. Then $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (3.2)-(3.5), that is, $\mathbf{H}^s$ and $\mathbf{E}^s$ are s-columns of time-dependent magnetic and electric Green's functions. Really, from (3.9), we find $curl_x\mathbf{H}^s = \epsilon\frac{\partial \mathbf{E}^s}{\partial t} + \vec{\mathbf{e}^s}\delta(x-x^0)\delta(t)$. Substituting the expression for $curl_x\mathbf{H}^s$ into (3.8) we have $\frac{\partial}{\partial t}\left(\mu\frac{\partial \mathbf{H}^s}{\partial t} + curl_x\mathbf{E}^s\right) = 0$. Integrating this equality and using (3.7),(3.10) we find that $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy equation (3.3). Moreover, taking the cross product of the equation (3.9) and $\vec{\mathbf{n}}$, letting $x \in \Gamma$ and using (3.8) we find $\frac{\partial}{\partial t}(\mathbf{E}^s \times \vec{\mathbf{n}})|_\Gamma = 0$. Integrating the last equality with respect to t from $-\infty$ to t and using (3.10) we find that $\mathbf{E}^s$ satisfies the first condition of (3.5). Therefore we obtain that if $\mathbf{H}^s$ and $\mathcal{E}^s$ satisfy (3.6)-(3.8), and (3.9)-(3.11) then $\mathbf{H}^s$, $\mathbf{E}^s$ satisfy equations (3.2)-(3.5). That is, $\mathbf{H}^s$, and $\mathbf{E}^s$ are s-columns of the time-dependent magnetic and electric Green's functions.

Remark 3.1.4. We note that if $\mathbf{H}^s$ satisfies (3.6)-(3.8) then $div_x\mathbf{H}^s = 0$, $x \in V$, that is, $div_x\mathbf{H}^s = 0$, $x \in V$ is the necessary condition for (3.6)-(3.8).

We use equations (3.6)-(3.8) for computation of the time-dependent magnetic Green's function and equations (3.9)-(3.11) for finding the time-dependent electric

Green's function when the magnetic Green's function is known. The method for the computation of the magnetic Green's function is based on the Fourier series expansion and describes as follows: The first step is to find eigenvalues and corresponding to them vector-eigenfunctions of the vector differential operator $\Delta_x \mathbf{I}_3$ ($\mathbf{I}_3$ is the identity 3×3 matrix). We note that the set of all vector-eigenfunctions are orthogonal and complete in the space of the vector functions with the square integrable components over the considered parallelepiped. This set is used as a basis for the Fourier series expansion. The second step of the method is a regularization (approximation) of the generalized function $curl_x \left[\delta(x - x^0)\vec{\mathbf{e}^s}\right]$ in the form of the Fourier series according to the obtained system of vector-eigenfunctions with a finite number of terms. The third step is solving the initial boundary value problem (3.6)-(3.8) when $curl_x \left[\delta(x - x^0)\vec{\mathbf{e}^s}\right]$ is replaced by the regularized vector-function. We note that the solution of such problem depends on the parameter of the regularization. We show that the solution of (3.6)-(3.8) is a limit of the regularized solution in the space of the generalized function $\mathcal{D}'(V)$ (see, Vladimirov (2002)) when the parameter of the regularization tends to infinity. The fourth step is finding a solution of (3.9),(3.10) when the solution of (3.6)-(3.8) is replaced by a regularized solution found in Step 3. We prove that the solution of (3.9),(3.10) is a limit in $\mathcal{D}'(V)$ of the regularized solution obtained in Step 4 when the parameter of the regularization tends to infinity. The vector functions constructed in Steps 3 and 4 are s-columns of the regularized (approximate) magnetic and electric Green's functions.

3.2 The Eigenvalue-Eigenfunction Problem for the Operator $\Delta_x \mathbf{I}_3$ in Rectangular Parallelepiped

Let us consider the problem of finding all values of λ (eigenvalues) for which there exists a nonzero vector-valued function $\mathbf{U}(x) = (U_1(x), U_2(x), U_3(x))$ with $x \in V$ satisfying

$$\Delta_x \mathbf{I}_3 \mathbf{U}(x) + \lambda \mathbf{U}(x) = 0, \tag{3.12}$$

$$\left(curl_x \mathbf{U}(x) \times \vec{\mathbf{n}}\right)\Big|_{\Gamma} = 0, \quad \left(\mathbf{U} \cdot \vec{\mathbf{n}}\right)\Big|_{\Gamma} = 0. \tag{3.13}$$

Let λ_{kmn} denote the eigenvalues and $\mathbf{U}^{kmn}(x)$ be the corresponding vector-valued eigenfunctions of (3.12)-(3.13). The eigenfunctions $\mathbf{U}^{kmn}(x)$ can be written in the form

$$\mathbf{U}^{kmn}(x) = U_1^{kmn}(x)\vec{\mathbf{e}^1} + U_2^{kmn}(x)\vec{\mathbf{e}^2} + U_3^{kmn}(x)\vec{\mathbf{e}^3},$$

and λ_{kmn}, $U_i^{kmn}(x)$, $i = 1,2,3$ satisfy

$$\Delta_x U_i^{kmn}(x) + \lambda_{kmn} U_i^{kmn}(x) = 0, \quad x \in V; \tag{3.14}$$

$$\left.\left(\alpha_i(x)\frac{\partial U_i^{kmn}(x)}{\partial \vec{\mathbf{n}}} + \beta_i(x) U_i^{kmn}(x)\right)\right|_{\Gamma} = 0. \tag{3.15}$$

This eigenvalue-eigenfunction problem is solved in Section 2.2, and all eigenvalues and corresponding eigenfunctions of the problem (3.14),(3.15) are found for each i as follows:

For $i = 1$,

$$\lambda_{kmn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2,$$

$$U_1^{kmn}(x) = A_{kmn}\sin\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{m\pi}{b_2}x_2\right)\cos\left(\frac{n\pi}{b_3}x_3\right), \tag{3.16}$$

$k = 1,2,...; \quad n,m = 0,1,...$ are eigenvalues and corresponding eigenfunctions.

For $i = 2$,

$$\lambda_{kmn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2,$$

$$U_2^{kmn}(x) = B_{kmn}\cos\left(\frac{k\pi}{b_1}x_1\right)\sin\left(\frac{m\pi}{b_2}x_2\right)\cos\left(\frac{n\pi}{b_3}x_3\right), \tag{3.17}$$

$m = 1,2,...; k,n = 0,1,...$ are eigenvalues and corresponding eigenfunctions.

For $i = 3$,

$$\lambda_{kmn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2,$$

$$U_3^{kmn}(x) = C_{kmn}\cos\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{m\pi}{b_2}x_2\right)\sin\left(\frac{n\pi}{b_3}x_3\right), \tag{3.18}$$

$n = 1,2,...; k,m = 0,1,...$ are eigenvalues and corresponding eigenfunctions.

Here A_{kmn}, B_{kmn}, C_{kmn} are constants, such that

$$\int_0^{b_3}\int_0^{b_2}\int_0^{b_1} U_i^{kmn}(x)\cdot U_i^{k'm'n'}(x)\,dx = \begin{cases} 0, & if\ (k,m,n) \neq (k',m',n'), \\ 1, & if\ k = k',\ m = m',\ n = n'. \end{cases}$$

$k,m,n = 0,1,...;\ j = 1,2,3$. That is, $A_{kmn} = B_{kmn} = C_{kmn} = \sqrt{\frac{8}{b_1 b_2 b_3}}$, $A_{k0n} = A_{km0} = B_{0mn} = B_{km0} = C_{0mn} = C_{k0n} = \sqrt{\frac{4}{b_1 b_2 b_3}}$, $A_{k00} = B_{0m0} = C_{00n} = \sqrt{\frac{2}{b_1 b_2 b_3}}$, $k = 1,2,...;$ $m = 1,2,...; n = 1,2,....$

3.3 Computation of Time-Dependent Magnetic Matrix Green's Function

3.3.1 Regularizations of the Dirac Delta Function and Its Derivatives

The vector functions defined by

$$\Psi^{1,N}(x,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}U_2^{kmn}(x)\vec{\mathbf{e}^2} - \frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right), \tag{3.19}$$

$$\Psi^{2,N}(x,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(-\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}U_1^{kmn}(x)\vec{\mathbf{e}^1} + \frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right), \tag{3.20}$$

$$\Psi^{3,N}(x,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(\frac{\partial U_1^{kmn}(x^0)}{\partial x_2^0}U_1^{kmn}(x)\vec{\mathbf{e}^1} - \frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0}U_2^{kmn}(x)\vec{\mathbf{e}^2}\right), \tag{3.21}$$

can be taken as regularizations of $curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^1}\right]$, $curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^2}\right]$, and $curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}^3}\right]$ (for detailed explanation see Section 2.3.1).

3.3.2 Solving (3.6)-(3.8) with Regularized Inhomogeneous Term

Let us consider the equations (3.6)-(3.8) for the time-dependent magnetic Green's function. We replace $curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]$ by their regularizations defined by (3.19), (3.20), (3.21) for $s = 1,2,3$, respectively. Moreover, (3.6)-(3.8) are replaced by the following equations

$$\frac{\partial^2 \mathbf{H}^{s,N}}{\partial t^2} - c^2\Delta_x \mathbf{H}^{s,N} = -c^2\Psi^{s,N}(x,x^0)\delta(t), \tag{3.22}$$

$$\mathbf{H}^{s,N}\big|_{t<0} = 0, \quad \frac{\partial \mathbf{H}^{s,N}}{\partial t}\bigg|_{t<0} = 0, \tag{3.23}$$

$$(curl_x\mathbf{H}^{s,N}\times\vec{\mathbf{n}})\big|_{\Gamma} = 0, \quad (\mathbf{H}^{s,N}\cdot\vec{\mathbf{n}})\big|_{\Gamma} = 0. \tag{3.24}$$

We will find solutions of the problem (3.22)-(3.24) in the form

$$\mathbf{H}^{s,N}(x,t,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\Big(T_1^{s,kmn}(t,x^0)U_1^{kmn}(x)\vec{\mathbf{e}^1} + T_2^{s,kmn}(t,x^0)U_2^{kmn}(x)\vec{\mathbf{e}^2} + T_3^{s,kmn}(t,x^0)U_3^{kmn}(x)\vec{\mathbf{e}^3}\Big), \tag{3.25}$$

where $x \in V,\ t > 0;\ U_i^{kmn}(x)$ are eigenfunctions of the eigenvalue-eigenfunction problem (3.14),(3.15); $T_i^{s,kmn}(t,x^0)$ are the unknown Fourier coefficients for $i = 1,2,3$.

By substituting (3.25) into (3.22), (3.23) and using orthogonality of $U_i^{kmn}(x)$, we find for $s = 1$:

$$\frac{d^2}{dt^2}T_1^{1,kmn}(t,x^0) + c^2\lambda_{kmn}T_1^{1,kmn}(t,x^0) = 0, \quad t > 0, \tag{3.26}$$

$$T_1^{1,kmn}(0,x^0) = 0, \quad \frac{d}{dt}T_1^{1,kmn}(0,x^0) = 0; \tag{3.27}$$

$$\frac{d^2}{dt^2}T_2^{1,kmn}(t,x^0) + c^2\lambda_{kmn}T_2^{1,kmn}(t,x^0) = -c^2\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}\delta(t), \quad t > 0, \tag{3.28}$$

$$T_2^{1,kmn}(0,x^0) = 0, \quad \frac{d}{dt}T_2^{1,kmn}(0,x^0) = 0; \tag{3.29}$$

$$\frac{d^2}{dt^2}T_3^{1,kmn}(t,x^0) + c^2\lambda_{kmn}T_3^{1,kmn}(t,x^0) = c^2\frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}\delta(t), \quad t > 0, \tag{3.30}$$

$$T_3^{1,kmn}(0,x^0) = 0, \quad \frac{d}{dt}T_3^{1,kmn}(0,x^0) = 0, \tag{3.31}$$

$k, m, n = 0, 1, \ldots$. The solutions of (3.26)-(3.27), (3.28)-(3.29) and (3.30)-(3.31) can be written in explicit form. As a result we obtain with (3.25)

$$\begin{aligned}\mathbf{H}^{1,N}(x,t,x^0) = -\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{c\sin(c\sqrt{\lambda_{kmn}}t)}{\sqrt{\lambda_{kmn}}}\Bigg(&\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}U_2^{kmn}(x)\mathbf{e}^{\vec{2}} \\ &- \frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}U_3^{kmn}(x)\mathbf{e}^{\vec{3}}\Bigg).\end{aligned} \tag{3.32}$$

Similarly, the solutions of (3.22)-(3.24) for $s = 2$ and $s = 3$ are obtained in the form

$$\begin{aligned}\mathbf{H}^{2,N}(x,t,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{c\sin(c\sqrt{\lambda_{kmn}}t)}{\sqrt{\lambda_{kmn}}}\Bigg(&\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}U_1^{kmn}(x)\mathbf{e}^{\vec{1}} \\ &- \frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0}U_3^{kmn}(x)\mathbf{e}^{\vec{3}}\Bigg),\end{aligned} \tag{3.33}$$

$$\begin{aligned}\mathbf{H}^{3,N}(x,t,x^0) = -\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{c\sin(c\sqrt{\lambda_{kmn}}t)}{\sqrt{\lambda_{kmn}}}\Bigg(&\frac{\partial U_1^{kmn}(x^0)}{\partial x_2^0}U_1^{kmn}(x)\mathbf{e}^{\vec{1}} \\ &- \frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0}U_2^{kmn}(x)\mathbf{e}^{\vec{2}}\Bigg).\end{aligned} \tag{3.34}$$

Using the explicit formulas for $\mathbf{H}^{s,N}(x,t,x^0)$ and $U_i^{kmn}(x)$ we can check directly that $\mathbf{H}^{s,N}(x,t,x^0)$ satisfy boundary condition (3.24).

Remark 3.3.1. The vector functions $\mathbf{H}^{s,N}(x,t,x^0)$, defined by, (3.32)-(3.34), satisfy

$$div_x\mathbf{H}^{s,N}(x,t,x^0) = 0, \quad s = 1,2,3.$$

3.3.3 *Time-Dependent Magnetic Green's Function and Its Regularization*

In this section we show that $\lim_{N\to\infty} H_i^{s,N}(x,t,x^0)$, $(s = 1,2,3; \quad i = 1,2,3)$ exists in the space of distributions $\mathcal{D}'(V)$, and the distributions $H_i^s = \lim_{N\to\infty} H_i^{s,N}$ are the entries of the matrix-valued Green's function $\mathbf{G}^H(x,t,x^0)$.

Let us consider for example, $s = 1$, $i = 2$ and an arbitrary function $h \in C_0^\infty(V)$ of the form (2.14), (2.15). The integrable over V function $H_i^{s,N}(x,t,x^0)$ defines a regular functional (distribution) by

$$\left\langle H_2^{1,N}(x,t,x^0), h(x)\right\rangle = -\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} \frac{c}{\sqrt{\lambda_{kmn}}} \frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0} \sin(c\sqrt{\lambda_{kmn}}t) h_2^{kmn}.$$

Using Lemma 2.3.1 we find that $\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(c/\sqrt{\lambda_{kmn}})\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}\sin(c\sqrt{\lambda_{kmn}}t)h_2^{kmn}$ is absolutely convergent for any $h(x) \in C_0^\infty(V)$. This means that $\lim_{N\to\infty}\left\langle H_2^{1,N}, h\right\rangle$ exists for any $h(x) \in C_0^\infty(V)$ and defines linear continuous functional $H_2^{1,N}(x,t,x^0)$ over $C_0^\infty(V)$.

Similarly, we can find $H_i^s(x,t,x^0)$ as $\lim_{N\to\infty} H_i^{s,N}(x,t,x^0)$ in the complete space $\mathcal{D}'(V)$ (see, for example, Vladimirov (2002)) for $s = 1,2,3$ and $i = 1,2,3$.

Moreover, since $\mathbf{H}^{s,N}(x,t,x^0) = \left(H_1^{s,N}(x,t,x^0), H_2^{s,N}(x,t,x^0), H_3^{s,N}(x,t,x^0)\right)^T$ satisfies equalities (3.22),(3.23),(3.24) then $\mathbf{H}^s(x,t,x^0) = \left(H_1^s(x,t,x^0), H_2^s(x,t,x^0), H_3^s(x,t,x^0)\right)^T$ is the solution of (IBVP)

$$\frac{\partial^2 \mathbf{H}^s}{\partial t^2} - c^2\Delta_x \mathbf{H}^s = c^2 curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]\delta(t),$$

$$\mathbf{H}^s\big|_{t<0} = 0, \qquad \frac{\partial \mathbf{H}^s}{\partial t}\bigg|_{t<0} = 0,$$

$$(curl_x \mathbf{H}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \quad (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0.$$

This means that $\mathbf{H}^s(x,t,x^0)$ is the $s-column$ of time-dependent magnetic matrix Green's function.

3.4 Approximate Solution of (3.9),(3.10)

Assume that columns of the magnetic Green's function be derived approximately. That is, solutions $\mathbf{H}^{s,N}$, $s = 1,2,3$ of the problem (3.22)-(3.24) be computed by formulae (3.32), (3.33), (3.34). For the approximate derivation of $\mathcal{E}^s(x,t,x^0)$, satisfying (3.9)-(3.10), we replace $curl_x\mathbf{H}^s(x,t,x^0)$ by $curl_x\mathbf{H}^{s,N}(x,t,x^0)$ in (3.9), (3.10) for $s = 1,2,3$, respectively and consider the following initial value problem

$$\frac{\partial \mathcal{E}^{s,N}(x,t,x^0)}{\partial t} = \frac{1}{\epsilon} curl_x\mathbf{H}^{s,N}(x,t,x^0), \tag{3.35}$$

$$\mathcal{E}^{s,N}(x,t,x^0)\Big|_{t<0} = 0. \tag{3.36}$$

A solution of (3.35), (3.36) is given by

$$\mathcal{E}^{s,N}(x,t,x^0) = \frac{\Theta(t)}{\epsilon}\int_0^t curl_x\mathbf{H}^{s,N}(x,\tau,x^0)\, d\tau. \tag{3.37}$$

Using (3.11) we find from (3.37) an explicit formula for $\mathbf{E}^{s,N}(x,t,x^0)$ as follows

$$\mathbf{E}^{s,N}(x,t,x^0) = \frac{\Theta(t)}{\epsilon}\int_0^t curl_x\mathbf{H}^{s,N}(x,\tau,x^0)\, d\tau - \frac{1}{\epsilon}\Phi^{s,N}(x,x^0)\Theta(t)\vec{\mathbf{e}}^s, \tag{3.38}$$

or in component form

$$\mathbf{E}^{1,N}(x,t,x^0) = \frac{\Theta(t)}{\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\cos(c\sqrt{\lambda_{kmn}}t)-1}{\lambda_{kmn}} curl_x\left[\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}U_2^{kmn}(x)\vec{\mathbf{e}^2}\right.$$
$$\left.-\frac{\partial U_3^{kmn}(x^0)}{\partial x_2^0}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right] - \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\epsilon}U_1^{kmn}(x^0)U_1^{kmn}(x)\Theta(t)\vec{\mathbf{e}^1},$$

$$\mathbf{E}^{2,N}(x,t,x^0) = -\frac{\Theta(t)}{\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\cos(c\sqrt{\lambda_{kmn}}t)-1}{\lambda_{kmn}} curl_x\left[\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}U_1^{kmn}(x)\vec{\mathbf{e}^1}\right.$$
$$\left.-\frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0}U_3^{kmn}(x)\vec{\mathbf{e}^3}\right] - \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\epsilon}U_2^{kmn}(x^0)U_2^{kmn}(x)\Theta(t)\vec{\mathbf{e}^2},$$

$$\mathbf{E}^{3,N}(x,t,x^0) = \frac{\Theta(t)}{\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\cos(c\sqrt{\lambda_{kmn}}t)-1}{\lambda_{kmn}} curl_x\left[\frac{\partial U_1^{kmn}(x^0)}{\partial x_2^0}U_1^{kmn}(x)\vec{\mathbf{e}^1}\right.$$
$$\left.-\frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0}U_2^{kmn}(x)\vec{\mathbf{e}^2}\right] - \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\epsilon}U_3^{kmn}(x^0)U_3^{kmn}(x)\Theta(t)\vec{\mathbf{e}^3}.$$

The vector functions defined by

$$\Phi^{s,N}(x,x^0) = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}U_s^{kmn}(x^0)U_s^{kmn}(x)\vec{e^s} \tag{3.39}$$

are regularizations of the generalized vector functions $\delta(x-x^0)\vec{\mathbf{e}^s}$, $s = 1,2,3$.

3.4.1 Regularization of the Electric Green's Function

Let us consider the differentiable function $\mathcal{E}_2^{2,N}(x,t,x^0)$, given by (3.37). This function defines a linear continuous regular functional over $C_0^\infty(V)$ by the following relation (see, for example, Vladimirov (1971))

$$\left\langle \mathcal{E}_2^{2,N}, h(x) \right\rangle = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} \frac{\cos(c\sqrt{\lambda_{kmn}}t)-1}{\lambda_{kmn}} \Big[\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0} \int\int_V\int U_1^{kmn}(x)\frac{\partial h(x)}{\partial x_3}dx$$

$$+\frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0} \int\int_V\int U_3^{kmn}(x)\frac{\partial h(x)}{\partial x_1}dx\Big]\frac{\Theta(t)}{\epsilon}, \tag{3.40}$$

for all $h(x) \in C_0^\infty(V)$. The series in the right-hand side of the last equality is absolutely convergent for any $h(x) \in C_0^\infty(V)$ (see, Appendix A4), that is, $\lim_{N\to\infty} \langle \mathcal{E}_2^{2,N}(x,t,x^0), h(x)\rangle$ exists for any $h(x) \in C_0^\infty(V)$ and defines a linear continuous regular functional over $C_0^\infty(V)$. We denote this functional as $\mathcal{E}_2^2(x,t,x^0)$, that is, $\mathcal{E}_2^2(x,t,x^0) = \lim_{N\to\infty} \mathcal{E}_2^{2,N}(x,t,x^0)$ in the complete space $\mathcal{D}'$. Similarly, we can find $\mathcal{E}_i^s(x,t,x^0)$ as $\lim_{N\to\infty} \mathcal{E}_i^{s,N}(x,t,x^0)$ in $\mathcal{D}'(V)$ for $s = 1,2,3$ and $i = 1,2,3$.

The s-column of the electric Green's function is

$$\mathbf{E}^s(x,t,x^0) = \frac{\Theta(t)}{\epsilon}\int_0^t curl_x\mathbf{H}^s(x,\tau,x^0)\,d\tau - \frac{1}{\epsilon}\delta(x-x^0)\Theta(t)\vec{\mathbf{e}}^s.$$

Let $\mathcal{E}^s(x,t,x^0) = \left(\mathcal{E}_1^s(x,t,x^0), \mathcal{E}_2^s(x,t,x^0), \mathcal{E}_3^s(x,t,x^0)\right)^T$ be the generalized vector function, whose components are distributions defined as $\mathcal{E}_i^s = \lim_{N\to\infty}\mathcal{E}_i^{s,N}$, $i = 1,2,3$. Since $\mathcal{E}^{s,N}(x,t,x^0) = \left(\mathcal{E}_1^{s,N}(x,t,x^0), \mathcal{E}_2^{s,N}(x,t,x^0), \mathcal{E}_3^{s,N}(x,t,x^0)\right)^T$ satisfies (3.35), (3.36) then the generalized vector function $\mathcal{E}^s(x,t,x^0)$ satisfies (3.9),(3.10) and the generalized vector function $\mathbf{E}^s(x,t,x^0)$, defined by

$$\mathbf{E}^s(x,t,x^0) = \mathcal{E}^s(x,t,x^0) - \frac{1}{\epsilon}\delta(x-x^0)\Theta(t)\vec{\mathbf{e}}^s, \tag{3.41}$$

satisfies (3.2).

Remark 3.4.1. We note that the generalized vector functions $\mathbf{H}^s(x,t,x^0)$, $\mathcal{E}^s(x,t,x^0)$, defined in Sections 3.3 and 3.4, satisfy (3.6)-(3.8) and (3.9),(3.10). The generalized vector function $\mathbf{E}^s(x,t,x^0)$, defined by (3.41), satisfy (3.2). It follows that $\mathbf{H}^s(x,t,x^0)$ and $\mathbf{E}^s(x,t,x^0)$ satisfy (3.2)-(3.5), and therefore, $\mathbf{H}^s(x,t,x^0)$, $\mathbf{E}^s(x,t,x^0)$ are s-columns of time-dependent electric and magnetic Green's functions $\mathbf{G}^E$ and $\mathbf{G}^H$, respectively.

3.5 Computational Experiments

In this section, we present computational experiments to show applicability of the method for approximate computation of time-dependent magnetic and electric matrix Green's functions in the parallelepiped V with $b_1 = 3$, $b_2 = b_3 = 2$, that is,

$$V = \Big\{ x \in \mathbf{R}^3 : \quad 0 < x_1 < 3, \;\; 0 < x_2 < 2, \;\; 0 < x_3 < 2 \Big\}.$$

Two types of computational experiments have been done. The first one is related to convergence analysis of the method; the second one is the computation of the elements of the magnetic and electric Green's functions in V and visualization of the computed magnetic and electric fields components by MATLAB graphic tools.

3.5.1 Convergence and Error Analysis

To obtain a reasonable accuracy and time for the computation of the approximate Green's functions for electric and magnetic fields in MATLAB by the personal computer it is necessary to determine a value of the parameter of the regularization N. We note that the Green's functions of the magnetic and electric fields are 3×3 matrices whose elements are distributions. Generally speaking, distributions do not have values at fixed points (Vladimirov, 1971, 2002, Antosik & Mikusinski, 1973) (see also Appendix A5). The elements of the regularized (approximate) matrix Green's function for the magnetic field (computed by our method) are classical differentiable and integrable functions over V having values at the fixed points. These classical functions define the regular distributions by the standard rule (see, Vladimirov (1971, 2002), Antosik & Mikusinski (1973) and Appendix A5). The comparison of distributions can be made by the comparison of their integral characteristics. Some integral characteristics of the elements of the matrix Green's function of the magnetic field can be found explicitly. These characteristics can be compared with integral characteristics of the approximate Green's function for different parameter of the approximation N to select reasonable one. Let us consider the second column $\mathbf{H}^2$ of the matrix Green's function of the magnetic field. The vector function $\mathbf{H}^2$ is a solution of (3.6)-(3.8) for

$s = 2$. Integrating (3.6) with respect to x_1^0 and x_2^0 and defining $\vec{h}^2(x,t,x_3^0)$ by

$$\vec{h}^2(x,t,x_3^0) = \int_0^{b_1}\int_0^{b_2} \mathbf{H}^2(x,t,x^0)\sin\left(\frac{\pi x_1^0}{b_1}\right)dx_2^0 dx_1^0,$$

we find that $\vec{h}^2 = (h_1^2, h_2^2, h_3^2)$ satisfies

$$\frac{\partial^2 h_1^2}{\partial t^2} - c^2\Delta_x h_1^2 = c^2 \sin\left(\frac{\pi x_1}{b_1}\right)\frac{\partial \delta(x_3 - x_3^0)}{\partial x_3^0}\delta(t), \tag{3.42}$$

$$\frac{\partial^2 h_2^2}{\partial t^2} - c^2\Delta_x h_2^2 = 0, \tag{3.43}$$

$$\frac{\partial^2 h_3^2}{\partial t^2} - c^2\Delta_x h_3^2 = c^2\frac{\pi}{b_1}\cos\left(\frac{\pi x_1}{b_1}\right)\delta(x_3 - x_3^0)\delta(t), \tag{3.44}$$

$$\vec{h}^2\Big|_{t\le 0} = 0, \quad \frac{\partial \vec{h}^2}{\partial t}\Big|_{t\le 0} = 0, \tag{3.45}$$

$$(curl_x \vec{h}^2 \times \vec{\mathbf{n}})\Big|_\Gamma = 0, \quad (\vec{h}^2 \cdot \vec{\mathbf{n}})\Big|_\Gamma = 0. \tag{3.46}$$

A solution of (3.42)-(3.46) can be written in the form

$$h_1^2 = c^2 \sin\left(\frac{\pi x_1}{b_1}\right)\frac{\partial h(x_3,t,x_3^0)}{\partial x_3^0},$$

$$h_2^2 = 0,$$

$$h_3^2 = c^2\frac{\pi}{b_1}\cos\left(\frac{\pi x_1}{b_1}\right)h(x_3,t,x_3^0), \tag{3.47}$$

where $h(x_3,t,x_3^0)$ satisfies

$$\frac{\partial^2 h}{\partial t^2} - c^2\frac{\partial^2 h}{\partial x_3^2} + c^2\left(\frac{\pi}{b_1}\right)^2 h = \delta(x_3 - x_3^0)\delta(t), \tag{3.48}$$

$$h|_{t\le 0} = 0, \quad \frac{\partial h}{\partial t}\Big|_{t\le 0} = 0, \tag{3.49}$$

$$h(x_3,t,x_3^0)|_{x_3=0} = 0, \quad h(x_3,t,x_3^0)|_{x_3=b_3} = 0. \tag{3.50}$$

It is well known that there is no influence of the boundary conditions (3.50) on the solution $h(x_3,t,x_3^0)$ of the initial boundary value problem (3.48)-(3.50) for all times when the wave front $|x_3 - x_3^0| = ct$ does not reach the boundaries $x_3 = 0$ and $x_3 = b_3$. This means that the solution of the initial value problem and the solution of the boundary value problem are equal for these times.

The function $h(x_3,t,x_3^0)$ as a solution of the initial value problem (3.48)-(3.49) for $x_3 \in (0,b_3)$, $0 < t < \min\left\{\dfrac{x_3^0}{c}, \dfrac{b_3 - x_3^0}{c}\right\}$ reads as follows (see, Vladimirov (1971))

$$h(x_3,t,x_3^0) = \frac{\Theta(ct - |x_3 - x_3^0|)}{2c} J_0\left(\frac{\pi}{b_1}\sqrt{c^2t^2 - (x_3 - x_3^0)^2}\right),$$

where $J_0(t)$ is the Bessel function of the first kind of the real variable t. As a result of it the integral characteristic $h_3^2(x,t,x_3^0)$ of the third component of the second column of the matrix Green's function is found in the explicit form for $x_3 \in (0,b_3)$, $0 < t < \min\left\{\dfrac{x_3^0}{c}, \dfrac{b_3 - x_3^0}{c}\right\}$. Let $\mathbf{H}^{2,N}$ be the second column of the matrix Green's function of the magnetic field computed by our method and N be the parameter of the approximation corresponding to the number of the terms of the right hand side of (3.33). Let us consider the third component $H_3^{2,N}$ of $\mathbf{H}^{2,N}$ and its integral characteristic $h_3^{2,N}(x,t,x_3^0)$, defined by similar rule as $h_3^2(x,t,x_3^0)$, that is by

$$h_3^{2,N}(x,t,x_3^0) = \int_0^{b_1}\int_0^{b_2} H_3^{2,N}(x,t,x^0)\sin\left(\frac{\pi x_1^0}{b_1}\right)dx_2^0 dx_1^0. \tag{3.51}$$

Remark 3.5.1. Let h_3^2 and $h_3^{2,N}$ be defined by (3.47)-(3.50) and (3.51), respectively. We note that $h_3^{2,N}$ can be written in the form of the Fourier series. Moreover, using (3.18), (3.33), (3.51) and the reasoning of Appendix A5 we can show that $h_3^2 = \lim_{N\to\infty} h_3^{2,N}$ in $\mathcal{D}'(V\times\mathbf{R})$. This means that $h_3^{2,N}$ is an approximation of h_3^2 for a fixed N.

We compare the values of $h_3^2(x,t,x_3^0)$ and $h_3^{2,N}(x,t,x_3^0)$. The comparison of the values $h_3^{2,N}$ and h_3^2 is presented in Figures 3.1-3.4 and Table 3.1. The values of $h_3^{2,N}$ are compared with the values of h_3^2 for fixed $t = 0.4/c$, $x_3^0 = 0.5$, $x_1 = 1$ and different numbers N. The graph of h_3^2 has jumps at two points and graphs of $h_3^{2,N}$ do not have the small oscillations near the jumps of h_3^2 (see Figures 3.1-3.4). This behavior is known as the Gibbs phenomenon, which manifests the peculiar manner in which the Fourier series of a piecewise continuously differentiable function behaves at jumps discontinuity. In all cases of the Gibbs phenomenon and regardless of the number of harmonics (terms of the Fourier series) it is observed that overshoot of the ripples has a constant magnitude around 18%. The maximum of a relative error $r_N(x_3) = \dfrac{|h_3^2 - h_3^{2,N}|}{|h_3^2|}$

Table 3.1 The accuracy of the calculation of the integral characteristics h_3^2 of the magnetic Green's function computed by our method for $t = 0.4/c$; $x_3^0 = 0.5$; $x_1 = 1$; $r_N = \dfrac{|h_3^2 - h_3^{2,N}|}{|h_3^2|}$; $N = 5, 10, 25, 50, 100, 150$.

x_1	r_5	r_{10}	r_{25}	r_{50}	r_{100}	r_{150}
0.2	0.2561	0.0491	0.0587	0.0041	0.0285	0.0005
0.3	0.0144	0.0864	0.0112	0.0193	0.0175	0.0065
0.4	0.1671	0.0024	0.0385	0.0002	0.0139	0.0000
0.5	0.1908	0.0740	0.0411	0.0156	0.0127	0.0052
0.6	0.0969	0.0096	0.0241	0.0005	0.0129	0.0001
0.7	0.0852	0.1059	0.0112	0.0236	0.0153	0.0079
0.8	0.3155	0.0624	0.0765	0.0047	0.0246	0.0006

at the points of the interval $[0.2, 0.8]$ for $N = 5, 10, 25, 50, 100, 150$ are $r_5 = 0.3155$, $r_{10} = 0.1059$, $r_{25} = 0.0765$, $r_{50} = 0.0236$, $r_{100} = 0.0285$, $r_{150} = 0.0079$, respectively. Making the error analysis we have concluded that the biggest N corresponds to the smallest error between values of $h_3^{2,N}$ and h_3^2 at the points of the interval $[0.2, 0.8]$. We have taken $N = 100$ as a reasonable value between accuracy and the time for our computation in MATLAB by the personal computer.

3.5.2 *Approximate Computation and Visualization of Elements of Magnetic and Electric Green's Matrices*

The suggested method has been applied for the computation of the time-dependent magnetic and electric Green's matrices in the parallelepiped V. More precisely, we have derived elements $H_i^{s,N}$ and $\mathcal{E}_i^{s,N}$ of the regularized time-dependent magnetic and electric matrices, respectively. The parameter of the regularization N has been chosen as 100. We note that each column of the magnetic and electric time-dependent Green's matrices is a magnetic or electric field, respectively, arising from the pulse dipole polarized in the direction of the basis vector of $\mathbf{R}^3$. Computation of $H_i^{s,N}$ and $\mathcal{E}_i^{s,N}$ has been implemented in MATLAB and their graphs are presented.

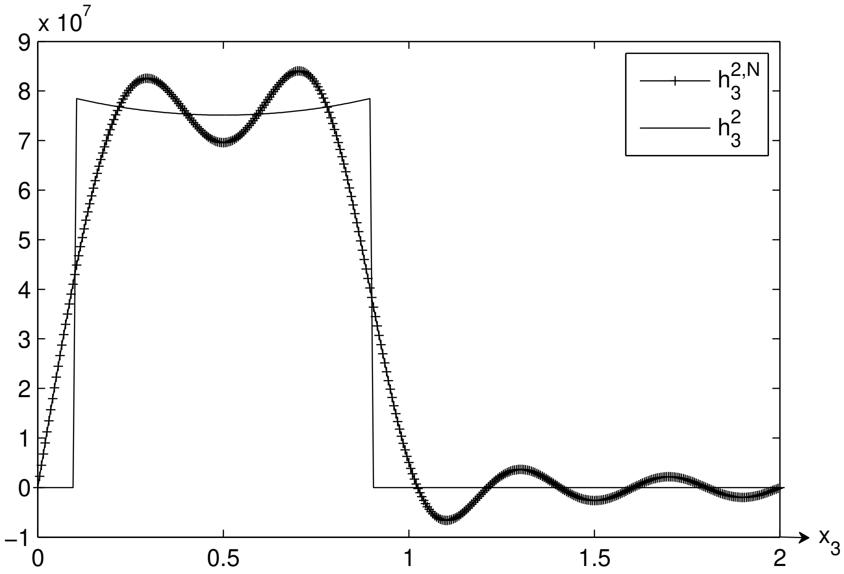

Figure 3.1 The graphs of h_3^2 and $h_3^{2,N}$ for $N = 10$, $t = \dfrac{0.4}{c}$, $x_3^0 = 0.5$, $x_1 = 1$; the horizontal axis is x_3-axis.

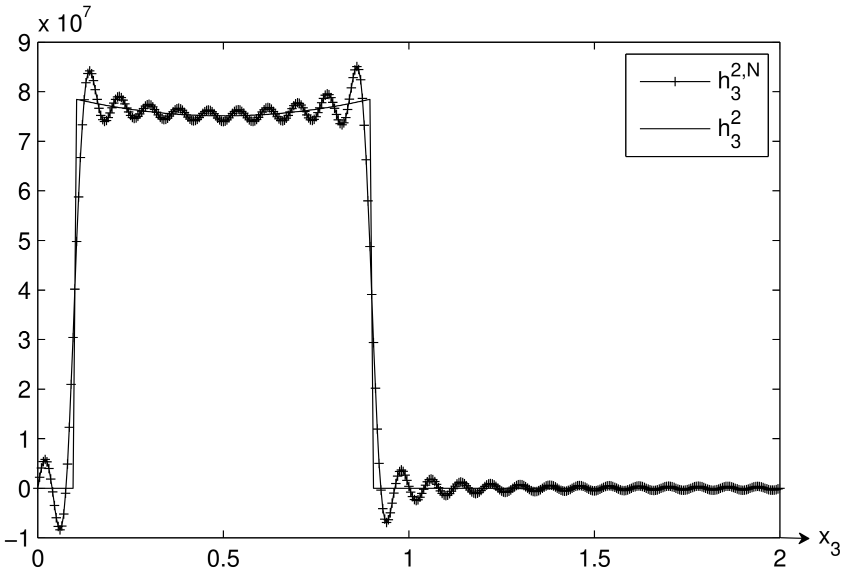

Figure 3.2 The graphs of h_3^2 and $h_3^{2,N}$ for $N = 50$, $t = \dfrac{0.4}{c}$, $x_3^0 = 0.5$, $x_1 = 1$; the horizontal axis is x_3-axis.

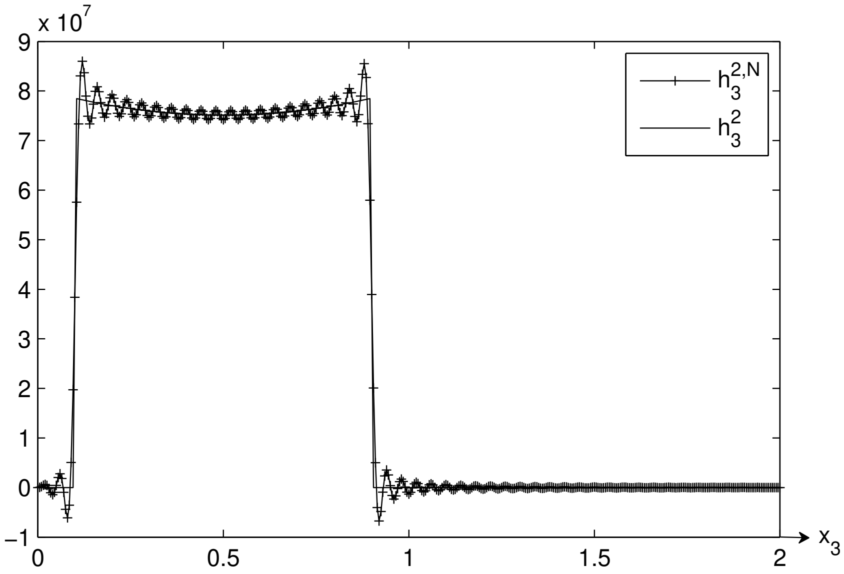

Figure 3.3 The graphs of h_3^2 and $h_3^{2,N}$ for $N = 100$, $t = \dfrac{0.4}{c}$, $x_3^0 = 0.5$, $x_1 = 1$; the horizontal axis is x_3-axis.

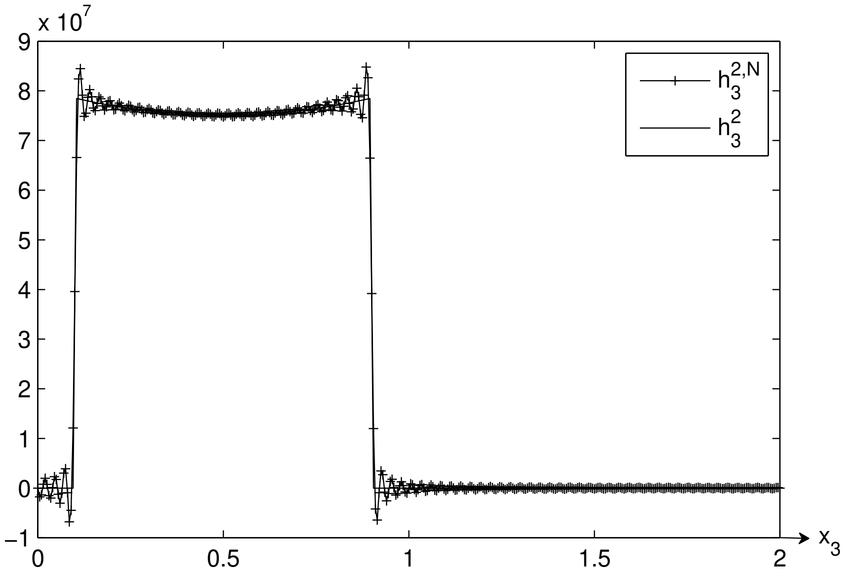

Figure 3.4 The graphs of h_3^2 and $h_3^{2,N}$ for $N = 150$, $t = \dfrac{0.4}{c}$, $x_3^0 = 0.5$, $x_1 = 1$; the horizontal axis is x_3-axis.

Figures 3.5-3.8 depict the first component of the second column of the approximate magnetic Green's matrix, whereas Figures 3.9-3.12 present the second component of the second column of the approximate electric Green's matrix. Figures 3.6, 3.10 are 3D images of the surfaces $z = H_1^{2,N}(x_1, x_2, 0.5, 0.1/c; x^0)$ and $z = \mathcal{E}_2^{2,N}(x_1, x_2, 0.5, 0.1/c; x^0)$, respectively, where x_1 and x_2 are the horizontal axes and z-axis is the magnitude of $H_1^{2,N}$ or $\mathcal{E}_2^{2,N}$ for $N = 100$. The view of the surfaces $z = H_1^{2,N}(x_1, x_2, 0.5, t; x^0)$ and $z = \mathcal{E}_2^{2,N}(x_1, x_2, 0.5, t; x^0)$ from the top of z-axis is presented in Figures 3.5, 3.7, 3.8, and 3.9, 3.11, 3.12, where x_1-axis, x_2-axis are the horizontal and vertical axes, respectively. The first component of the magnetic field and the second component of the electric field, arising from the pulse dipole $\mathbf{J}^2(x, t, x^0) = \delta(x - x^0, t)\vec{\mathbf{e}}^2$ at time $t = 0.1/c$, are presented in Figures 3.5, 3.6, 3.9, 3.10, respectively. Figures 3.5, 3.9 represent the map of the surfaces $z = H_1^{2,N}(x_1, x_2, 0.5, t, x^0)$, $z = \mathcal{E}_2^{2,N}(x_1, x_2, 0.5, t, x^0)$ for $N = 100$ at $t = 0.1/c$. The 3D images of these surfaces for $t = 0.1/c$ are given in Figures 3.6, 3.10.

The dynamic of the magnetic and electric fields is presented in Figures 3.5-3.12. We observe propagations of the electric and magnetic waves. The forward and reflected wave fronts are presented by the essential fluctuations. The fronts of the magnetic and electric waves, presented in Figures 3.5,3.6 and 3.9,3.10, did not reach the boundary of the parallelepiped. In Figure 3.11 the front of the electric wave touches the boundary Γ. In Figures 3.7, 3.8, 3.12 we see further propagation of the magnetic and electric waves and the reflections of these waves from the different parts of the boundary. Moreover Figures 3.5-3.12 present the total magnetic and electric fields as a superposition of the forward and reflected waves from the conducting boundary of the parallelepiped. In Figures 3.5-3.12. there is also a visible ripple which is related to the errors of approximations (see Appendices A5 and A6).

3.6 Summary of Chapter Three

A new method for the approximate computation of time-dependent magnetic and electric Green's matrices in the rectangular parallelepiped with a perfect conducting boundary has been developed. The method is based on the adaption of the Fourier

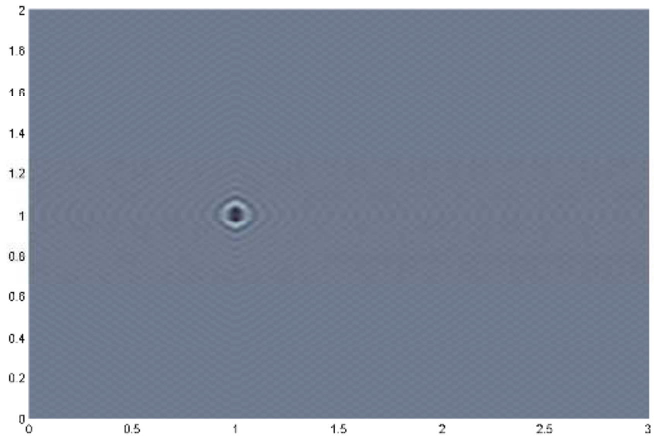

Figure 3.5 The first component of the second column of the approximate magnetic Green's function for $N = 100$, $x^0 = (1,1,0.5)$, $x_3 = 0.5$ at times $t = 0.1/c$.

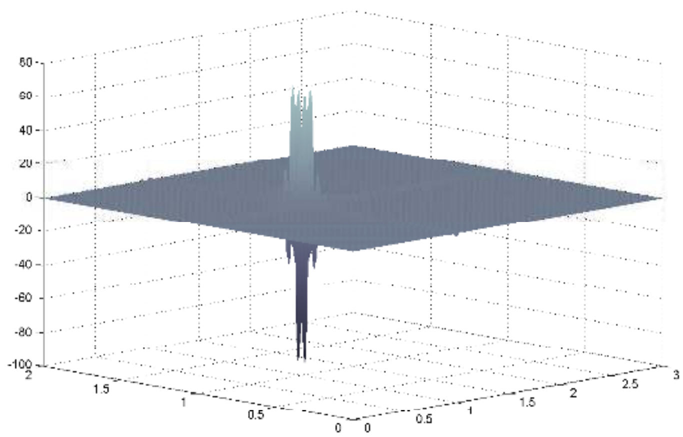

Figure 3.6 The first component of the second column of the approximate magnetic Green's function for $N = 100$, $x^0 = (1,1,0.5)$, $x_3 = 0.5$ at times $t = 0.1/c$.

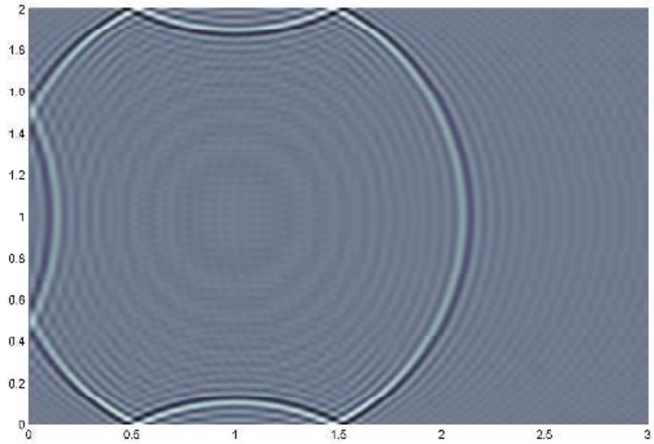

Figure 3.7 The first component of the second column of the approximate magnetic Green's function for $N = 100$, $x^0 = (1, 1, 0.5)$, $x_3 = 0.5$ at times $t = 1.15/c$.

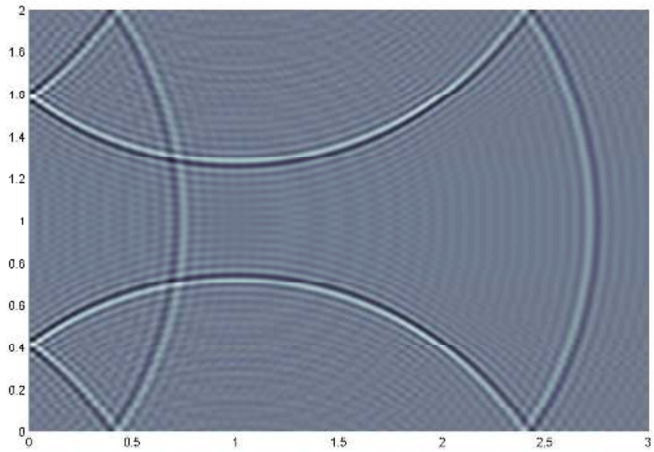

Figure 3.8 The first component of the second column of the approximate magnetic Green's function for $N = 100$, $x^0 = (1, 1, 0.5)$, $x_3 = 0.5$ at times $t = 1.75/c$.

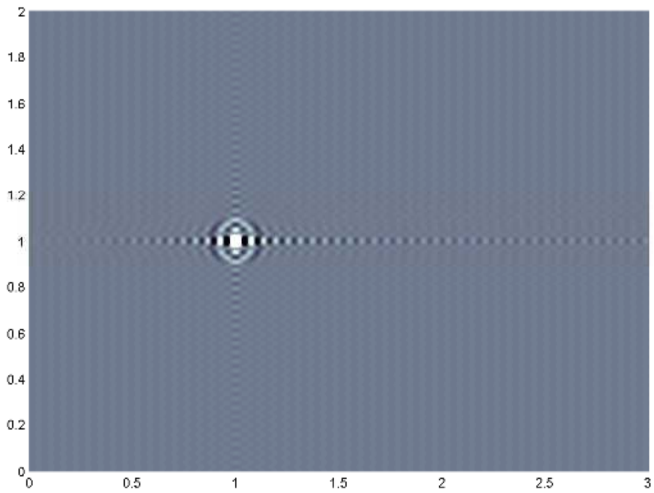

Figure 3.9 The second component of the second column of the approximate electric Green's function for $N = 100$, $x^0 = (1, 1, 0.5)$, $x_3 = 0.5$ at times $t = 0.1/c$.

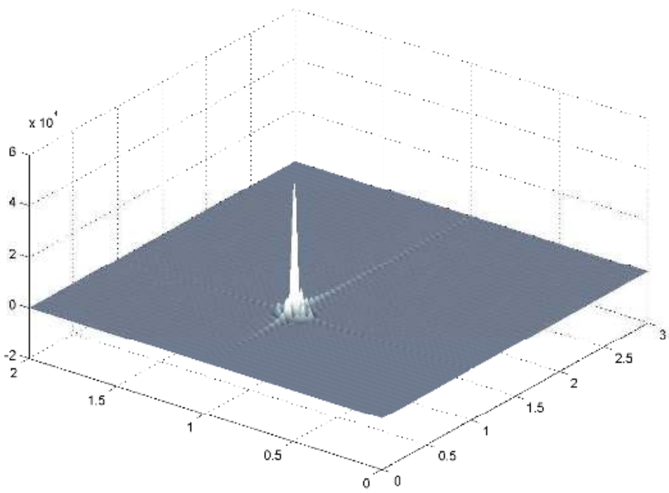

Figure 3.10 The second component of the second column of the approximate electric Green's function for $N = 100$, $x^0 = (1, 1, 0.5)$, $x_3 = 0.5$ at times $t = 0.1/c$.

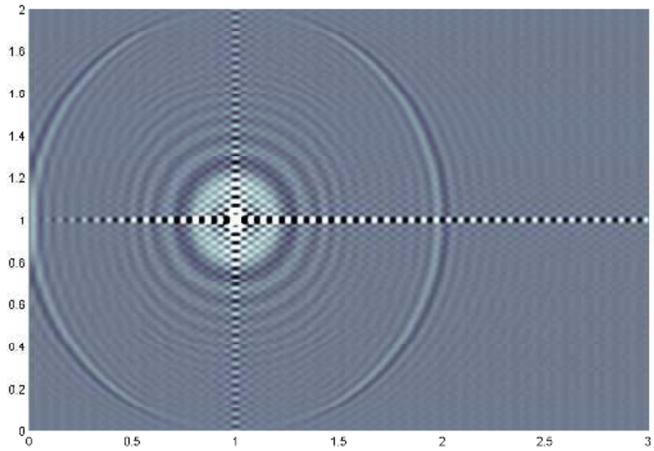

Figure 3.11 The second component of the second column of the approximate electric Green's function for $N = 100$, $x^0 = (1, 1, 0.5)$, $x_3 = 0.5$ at times $t = 1/c$.

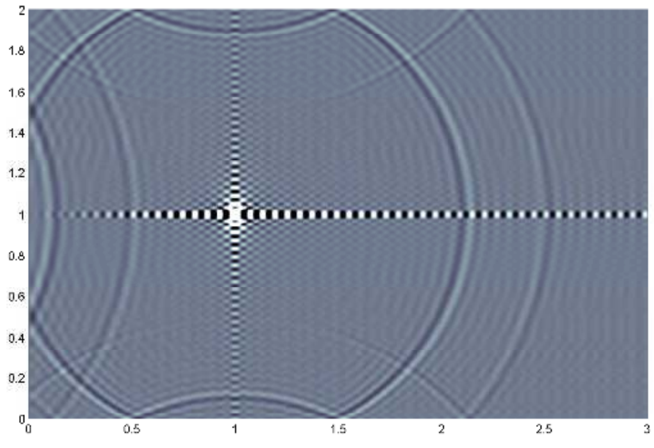

Figure 3.12 The second component of the second column of the approximate electric Green's function for $N = 100$, $x^0 = (1, 1, 0.5)$, $x_3 = 0.5$ at times $t = 1.5/c$.

series meta-approach for solving the initial boundary value problem of the hyperbolic equations. We note that the generalized vector functions $curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]\delta(t),\ s = 1,2,3,$ appear in the differential equations for the columns of the magnetic Green's matrix. We have suggested an approximation of these generalized vector functions for the computation of the considered Green's functions. The elements of the approximate Green's function for the magnetic field have been found in the form of the partial sums of the Fourier series with a finite number of terms. Using them we have derived explicitly the approximate elements of the electric Green's matrix by integration with respect to the time variable. Moreover we have proved that if all Fourier coefficients derived, then, using the obtained Fourier series, the time dependent magnetic and electric Green's functions are defined in the space of the generalized functions (distributions.) Using the method, a visualization of the behavior of the magnetic and electric fields inside a given parallelepiped with perfect conducting boundary has been made.

The columns of the magnetic and electric matrix Green's function in the space of generalized functions and their regularizations has been obtained . The method has been implemented in MATLAB. The computational experiments have been confirmed its robustness.

CHAPTER FOUR

AN INITIAL BOUNDARY VALUE PROBLEM FOR 1D WAVE EQUATION WITH PIECEWISE CONSTANT COEFFICIENT

In this chapter we study an IBVP for the 1D wave equation with piecewise constant coefficient. This problem and its solution will be used in Chapter 6. The method of characteristics is applied for solving this IBVP.

4.1 Statement of the IBV Problem for the One-dimensional Wave Equation with Piecewise Constant Coefficient

Let $y \in \mathbf{R}$ be the space variable; $t \in \mathbf{R}$ be the time variable; c_k, ϵ_k, y_k, $k = 1, 2, \ldots, n$, be given positive constants and $y_0 = 0$, $y_n = b > 0$; $f_k(t)$, $k = 1, 2, \ldots, n-1$, be given continuous functions. Find $u(y,t)$ satisfying the wave equation for $(y,t) \in (y_{k-1}, y_k) \times \mathbf{R}^+$ (that is, $y \in (y_{k-1}, y_k)$, $t \in \mathbf{R}^+$), $k = 1, 2, \ldots, n$,

$$u_{tt} - c_k^2 u_{yy} = 0, \tag{4.1}$$

the initial data

$$u(y,0) = 0, \;\; u_t(y,0) = 0, \tag{4.2}$$

boundary conditions

$$u_y\Big|_{y=y_0} = 0, \;\; u_y\Big|_{y=y_n} = 0, \tag{4.3}$$

and matching (interface) conditions:

$$u|_{y=y_k+0} = u|_{y=y_k-0}, \;\; k = 1, 2, \ldots, n-1, \tag{4.4}$$

$$u_y\Big|_{y=y_k+0} = \frac{\epsilon_{k+1}}{\epsilon_k}\, u_y\Big|_{y=y_k-0} - \frac{\epsilon_{k+1}}{\epsilon_k} f_k(t), \;\; k = 1, 2, \ldots, n-1. \tag{4.5}$$

This problem is called the initial boundary value problem (IBVP).

4.2 Reduction of IBVP for The Wave Equation to IBVP for a System

Let us consider a new variable z and a sequence of points z_k, $k = 1,2,\ldots,n$ which are defined by formula

$$z = \begin{cases} \frac{y}{c_1},\ y \in (y_0, y_1), & \text{if } z_1 = \frac{y_1}{c_1}, \\ z_k + \frac{y - y_k}{c_{k+1}},\ y \in (y_k, y_{k+1}), & \text{if } z_{k+1} = z_k + \frac{y_{k+1} - y_k}{c_{k+1}},\ k = 1,2,\ldots,n-1, \end{cases} \tag{4.6}$$

with $z_0 = 0$, $z_n = B$.

Let $y = y(z)$ be the inverse of the function $z = z(y)$ defined by (4.6).

$$y = \begin{cases} c_1 z, & z \in (z_0, z_1) \\ y_k + c_{k+1}(z - z_k), & z \in (z_k, z_{k+1}),\ k = 1,2,\ldots,n-1. \end{cases}$$

Introduce a new function $v(z,t) = u(y(z),t)$. Equations (4.1)-(4.5) can be written in terms of $v(z,t)$ as follows:

$$v_{tt} - v_{zz} = 0,\ (z,t) \in (z_{k-1}, z_k) \times \mathbf{R}^+,\ k = 1,2,\ldots,n, \tag{4.7}$$

$$v(z,0) = 0,\ v_t(z,0) = 0, \tag{4.8}$$

$$v_z|_{z=0} = 0,\ v_z|_{z=z_n} = 0, \tag{4.9}$$

$$v|_{z=z_k+0} = v|_{z=z_k-0},\ 1,2,\ldots,n-1, \tag{4.10}$$

$$v_z|_{z=z_k+0} = d_{k,k+1}\, v_z|_{z=z_k-0} - \frac{\epsilon_{k+1}}{\epsilon_k} c_{k+1} f_k(t),\ 1,2,\ldots,n-1, \tag{4.11}$$

where $d_{k,k+1} = \frac{c_{k+1}}{c_k}\frac{\epsilon_{k+1}}{\epsilon_k}$.

Letting $V = \frac{\partial v}{\partial t}$, $\sigma = \frac{\partial v}{\partial z}$, (4.7) can be written for $(z,t) \in (z_{k-1}, z_k) \times \mathbf{R}^+$ as follows

$$\frac{\partial V}{\partial t} = \frac{\partial \sigma}{\partial z},\ \frac{\partial \sigma}{\partial t} = \frac{\partial V}{\partial z},$$

or equivalently

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} V \\ \sigma \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} V \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.12}$$

Let

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \mathbf{S}^T \begin{pmatrix} V \\ \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V - \sigma \\ V + \sigma \end{pmatrix},\ \text{where } \mathbf{S}^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

here $*$ denotes the transposition. Then (4.7)-(4.11) can be written in terms of the vector function $\mathbf{Y}(z,t)$ in the following form

$$\frac{\partial}{\partial t}\mathbf{Y}+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial z}\mathbf{Y}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ z\in(z_{k-1},z_k),\ k=1,2,\ldots,n, \tag{4.13}$$

$$\mathbf{Y}(z,0)=0, \tag{4.14}$$

$$-Y_1(0,t)+Y_2(0,t)=0,\ -Y_1(B,t)+Y_2(B,t)=0, \tag{4.15}$$

$$Y_1(z_k+0,t)+Y_2(z_k+0,t)=Y_1(z_k-0,t)+Y_2(z_k-0,t),\ k=1,2,\ldots,n-1, \tag{4.16}$$

$$\begin{aligned}-Y_1(z_k+0,t)+Y_2(z_k+0,t)&=-d_{k,k+1}Y_1(z_k-0,t)+d_{k,k+1}Y_2(z_k-0,t)\\ &-\sqrt{2}\frac{\epsilon_{k+1}}{\epsilon_k}c_{k+1}f_k(t),\ k=1,2,\ldots,n-1.\end{aligned} \tag{4.17}$$

The problem of finding $\mathbf{Y}(z,t)$ satisfying (4.13)-(4.17) we call the initial boundary value problem for the system (4.13). If $\mathbf{Y}(z,t)$ is a solution of (4.13)-(4.17) then $(V,\sigma)^T$ can be found by

$$\begin{pmatrix} V \\ \sigma \end{pmatrix}=\mathbf{S}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} Y_1+Y_2 \\ -Y_1+Y_2 \end{pmatrix},\ \text{where } \mathbf{S}=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

and the function $v(z,t)$ can be found by integrating $V(z,\tau)$ with respect to τ from 0 to t

$$v(z,t)=\int_0^t V(z,\tau)\,d\tau,$$

and finally, we find

$$u(y,t)=v(z(y),t).$$

4.3 Solution of the IBVP (4.13)-(4.17)

4.3.1 *Construction of Functions* $Y_1(z_k-0,t)$, $Y_1(z_k+0,t)$, $Y_2(z_k-0,t)$, $Y_2(z_k+0,t)$

The system of equation (4.13) has the following form

$$\frac{\partial}{\partial t}Y_1+\frac{\partial}{\partial z}Y_1=0,\quad \frac{\partial}{\partial t}Y_2-\frac{\partial}{\partial z}Y_2=0, \tag{4.18}$$

for $(z,t)\in(z_{k-1},z_k)\times\mathbf{R}^+$, $k=1,2,\ldots,n$, and can be written as

$$\frac{d}{dz}[Y_1(z,z)]=0,\ \frac{d}{dz}[Y_2(-z,z)]=0.$$

Let $t_k = z_k - z_{k-1}$, using the initial condition (4.14) and the boundary condition (4.15) we find

$$Y_1(z_k - 0, t) = \begin{cases} Y_1(z_k - t, 0), & t \le t_k, \\ Y_1(z_{k-1} + 0, t - t_k), \quad t > t_k, & k = 1, 2, \ldots, n, \end{cases}$$

$$Y_2(z_k + 0, t) = \begin{cases} Y_2(z_k + t, 0), & t \le t_{k+1}, \\ Y_2(z_{k+1} - 0, t - t_{k+1}), \quad t > t_{k+1}, & k = 0, 1, \ldots, n-1. \end{cases}$$

Finding $Y_1(z_k - 0, t)$, $Y_2(B, t)$ **for** $t \in (0, t_k),\ k = 1, 2, \ldots, n.$

Equation (4.18) may be written in terms of ξ and τ as

$$\frac{\partial}{\partial \tau} Y_1(\xi, \tau) + \frac{\partial}{\partial \xi} Y_1(\xi, \tau) = 0.$$

The characteristics of this equation are given by

$$\frac{d\tau}{ds} = 1,\ \frac{d\xi}{ds} = 1.$$

The characteristic passing through the point (z_k, t), $k = 1, 2, \ldots, n$ is defined by

$$\tau = \xi + t - z_k.$$

(4.18) along this characteristic may be written in the following form

$$\frac{d}{d\xi} [Y_1(\xi, \xi + t - z_k)] = 0.$$

Integrating the last equation along the curve $\tau = \xi + t - z_k$, from $(z_k - t, 0)$ to (z_k, t) for $t \in (0, t_k)$, we find

$$\int_{z_k - t}^{z_k} \frac{d}{d\xi} [Y_1(\xi, \xi + t - z_k)] = 0,$$

that is,

$$Y_1(z_k - 0, t) = Y_1(z_k - t, 0),\ t \in (0, t_k).$$

Using the initial condition (4.14) we find

$$Y_1(z_k - 0, t) = 0,\ t \in (0, t_k),\ k = 1, 2 \ldots, n. \tag{4.19}$$

For the particular case where $k = n$ we have

$$Y_1(B, t) = 0,\ t \in (0, t_n). \tag{4.20}$$

Therefore, using (4.15), we find

$$Y_2(B,t) = 0, \ \ t \in (0,t_n). \tag{4.21}$$

Finding $Y_2(z_k+0,t)$, $Y_1(0,t)$ **for** $t \in (0,t_{k+1})$, $k = 0,1,\ldots,n-1$.

Equation (4.18) may be written in terms of ξ and τ as

$$\frac{\partial}{\partial \tau} Y_2(\xi,\tau) - \frac{\partial}{\partial \xi} Y_2(\xi,\tau) = 0.$$

The characteristic of this equation passing through the point (z_k,t) is given by

$$\tau = -\xi + t + z_k, \ \ k = 0,1,\ldots,n-1.$$

Equation (4.18) along this characteristic may be written as follows

$$\frac{d}{d\xi}\left[Y_2(\xi,-\xi+t+z_k)\right] = 0.$$

Integrating the last equation along the curve $\tau = -\xi + t + z_k$, from (z_k,t) to $(z_k+t,0)$ for $t \in (0,t_{k+1})$ we find

$$\int_{z_k}^{z_k+t} \frac{d}{d\xi}\left[Y_2(\xi,-\xi+t+z_k)\right] = 0,$$

that is,

$$Y_2(z_k+0,t) = Y_2(z_k+t,0), \ \ t \in (0,t_{k+1}).$$

Using the initial condition (4.14) we find

$$Y_2(z_k+0,t) = 0, \ \ t \in (0,t_{k+1}), \ \ k = 0,1\ldots,n-1. \tag{4.22}$$

For the particular case where $k = 0$ we have

$$Y_2(0,t) = 0, \ \ t \in (0,t_1). \tag{4.23}$$

Therefore, using (4.15), we find

$$Y_1(0,t) = 0, \ \ t \in (0,t_1). \tag{4.24}$$

Finding $Y_1(z_k+0,t)$, $Y_2(z_k-0,t)$ **for** $t \in (0,t^*)$, $k = 1,2,\ldots,n-1$.

The functions $Y_1(z_k-0,t)$, $Y_2(z_k+0,t)$, $k=1,2,\dots,n-1$, are found in the previous steps. We use these functions for finding the unknowns $Y_1(z_k+0,t)$, $Y_2(z_k-0,t)$, $k=1,2,\dots,n-1$, for $t\in(0,t^*)$, $t^*=\min\{t_1,t_2,\dots,t_n\}$. Using the matching conditions (4.16), (4.17) we get the following algebraic system

$$\underbrace{\begin{pmatrix} 1 & -1 \\ 1 & d_{k,k+1} \end{pmatrix}}_{\mathbf{A}_{k,k+1}} \begin{pmatrix} Y_1(z_k+0,t) \\ Y_2(z_k-0,t) \end{pmatrix} = \begin{pmatrix} Y_1(z_k-0,t)-Y_2(z_k+0,t) \\ d_{k,k+1}Y_1(z_k-0,t)+Y_2(z_k+0,t)+\sqrt{2}\frac{\epsilon_{k+1}}{\epsilon_k}c_{k+1}f_k(t) \end{pmatrix}. \tag{4.25}$$

Since $\det\mathbf{A}_{k,k+1}=d_{k,k+1}+1>0$, using the Cramer's rule, we get unknown functions $Y_1(z_k+0,t)$ and $Y_2(z_k-0,t)$ as

$$Y_1(z_k+0,t)=\frac{1}{1+d_{k,k+1}}\begin{vmatrix} G_{k,k+1}(t) & -1 \\ F_{k,k+1}(t) & d_{k,k+1} \end{vmatrix}=\frac{d_{k,k+1}G_{k,k+1}(t)+F_{k,k+1}(t)}{1+d_{k,k+1}}, \tag{4.26}$$

$$Y_2(z_k-0,t)=\frac{1}{1+d_{k,k+1}}\begin{vmatrix} 1 & G_{k,k+1}(t) \\ 1 & F_{k,k+1}(t) \end{vmatrix}=\frac{F_{k,k+1}(t)-G_{k,k+1}(t)}{1+d_{k,k+1}}, \tag{4.27}$$

and the functions $G_{k,k+1}(t)$, $F_{k,k+1}(t)$ are defined as

$$G_{k,k+1}(t)=Y_1(z_k-0,t)-Y_2(z_k+0,t),$$
$$F_{k,k+1}(t)=d_{k,k+1}Y_1(z_k-0,t)+Y_2(z_k+0,t)+\sqrt{2}\frac{\epsilon_{k+1}}{\epsilon_k}c_{k+1}f_k(t),$$

for $t\in(0,t^*)$, $k=1,2,\dots,n-1$.

Finding $Y_1(z_k-0,t)$, $Y_2(B,t)$ for $t\in(t_k,t_k+t^*)$, $k=1,2,\dots,n$.

Here, we need to find the functions $Y_1(z_k-0,t)$, for $t\in(t_k,t_k+t^*)$, $k=1,2,\dots,n$. Integrating the equation (4.18) along the curve $\tau=\xi+t-z_k$, from $(z_{k-1},t-t_k)$ to (z_k,t) for $t\in(t_k,t_k+t^*)$ we find

$$\int_{z_{k-1}}^{z_k}\frac{d}{d\xi}\left[Y_1(\xi,\xi+t-z_k)\right]=0,$$

that is,

$$Y_1(z_k-0,t)=Y_1(z_{k-1}+0,t-t_k),\ \ t\in(t_k,t_k+t^*),\ \ k=1,2,\dots,n.$$

In the previous steps, the functions $Y_1(z_k+0,t)$ have been constructed for $t\in(0,t^*)$, $k=1,2,\dots,n-1$. By shifting t variable t_k units down we get

$$Y_1(z_{k-1}+0,t-t_k)=\frac{d_{k-1,k}G_{k-1,k}(t-t_k)+F_{k-1,k}(t-t_k)}{1+d_{k-1,k}},\ \ t\in(t_k,t_k+t^*),\ \ k=2,3,\dots,n.$$

Hence we find

$$Y_1(z_k-0,t)=\frac{d_{k-1,k}G_{k-1,k}(t-t_k)+F_{k-1,k}(t-t_k)}{1+d_{k-1,k}},\ t\in(t_k,t_k+t^*),\ k=2,3,\dots,n. \quad (4.28)$$

Let us consider the particular case $k = 1$

$$Y_1(z_1-0,t)=Y_1(0,t-t_1).$$

In the previous steps we find $Y_1(0,t)=0$ for $t\in(0,t_1)$. By shifting the t variable t_1 units down we get

$$Y_1(0,t-t_1)=0,\ t\in(t_1,2t_1).$$

Hence, we find

$$Y_1(z_1-0,t)=0,\ t\in(t_1,2t_1). \quad (4.29)$$

For finding the function $Y_2(B,t)$, we consider the case $k=n$ of the result (4.28) and using (4.15) we find

$$Y_2(B,t)=Y_1(B,t)=\frac{d_{n-1,n}G_{n-1,n}(t-t_n)+F_{n-1,n}(t-t_n)}{1+d_{n-1,n}},\ t\in(t_n,t_n+t^*). \quad (4.30)$$

Finding $Y_2(z_k+0,t)$, $Y_1(0,t)$ for $t\in(t_{k+1},t_{k+1}+t^*)$, $k=0,1,\dots,n-1$.

Here, we need to find $Y_2(z_k+0,t)$ for $t\in(t_{k+1},t_{k+1}+t^*)$, $k=0,1,\dots,n-1$. Integrating (4.18) along the curve $\tau=-\xi+t+z_k$ from (z_k,t) to $(z_{k+1},t-t_k)$ we find

$$Y_2(z_k+0,t)=Y_2(z_{k+1}-0,t-t_{k+1}),\ t\in(t_{k+1},t_{k+1}+t^*).$$

In the previous steps the function $Y_2(z_k-0,t)$, $k=1,2,\dots,n-1$ has been constructed for $t\in(0,t^*)$. By shifting t variable t_{k+1} units down we get

$$Y_2(z_{k+1}-0,t-t_{k+1})=\frac{F_{k+1,k+2}(t-t_{k+1})-G_{k+1,k+2}(t-t_{k+1})}{1+d_{k+1,k+2}},\ t\in(t_{k+1},t_{k+1}+t^*),$$

$k=0,1\dots,n-2$.

Hence, we find

$$Y_2(z_k+0,t)=\frac{F_{k+1,k+2}(t-t_{k+1})-G_{k+1,k+2}(t-t_{k+1})}{1+d_{k+1,k+2}},\ t\in(t_{k+1},t_{k+1}+t^*), \quad (4.31)$$

$k = 0, 1 \ldots, n-2$.

Similarly, for the case $k = n-1$ we find

$$Y_2(z_{n-1}+0,t) = Y_2(B, t-t_n) = 0, \;\; t_n < t < 2t_n.$$

For finding the function $Y_1(0,t)$, we consider the case $k = 0$ and use (4.15), and find

$$Y_1(0,t) = Y_2(0,t) = \frac{F_{1,2}(t-t_1) - G_{1,2}(t-t_1)}{1+d_{1,2}}, \;\; t \in (t_1, t_1+t^*). \tag{4.32}$$

Finding $Y_1(z_k+0,t)$, $Y_2(z_k-0,t)$ **for** $t^* < t < 2t^*$ $k = 1, 2, \ldots, n-1$.

Similar to the previous steps the functions $Y_1(z_k-0,t)$, $Y_2(z_k+0,t)$ can be found as

$$Y_1(z_k+0,t) = \frac{d_{k,k+1}G_{k,k+1}(t) + F_{k,k+1}(t)}{1+d_{k,k+1}}, \tag{4.33}$$

$$Y_2(z_k-0,t) = \frac{F_{k,k+1}(t) - G_{k,k+1}(t)}{1+d_{k,k+1}}, \tag{4.34}$$

for $t \in (t^*, 2t^*)$, $k = 1, 2, \ldots, n-1$.

Finding $Y_1(z_k-0,t)$, $Y_2(B,t)$ **for** $t \in (t_k+t^*, t_k+2t^*)$, $k = 1, 2, \ldots, n$.

Here we need to find the functions $Y_1(z_k-0,t)$, for $t \in (t_k+t^*, t_k+2t^*)$, $k = 1, 2, \ldots, n$. Integrating the equation (4.18) along the curve $\tau = \xi + t - z_k$, from $(z_{k-1}, t-t_k)$ to (z_k, t)

$$Y_1(z_k-0,t) = Y_1(z_{k-1}+0, t-t_k), \;\; k = 1, 2, \ldots, n.$$

In the previous steps the function $Y_1(z_k+0,t)$, $k = 1, 2, \ldots, n-1$, has been constructed for $t \in (0,t^*) \cup (t^*, 2t^*)$. Hence, we find

$$Y_1(z_k-0,t) = \frac{d_{k-1,k}G_{k-1,k}(t-t_k) + F_{k-1,k}(t-t_k)}{1+d_{k-1,k}}, \;\; t \in (t_k+t^*, t_k+2t^*), \tag{4.35}$$

$k = 2, \ldots, n$.

Using the constructed function $Y_1(0,t)$, $t \in (0,t_1) \cup (t_1, t_1+t^*)$, we find the function $Y_1(z_1-0,t)$ for $t \in (2t_1, 2t_1+t^*)$ as

$$Y_1(z_1-0,t) = \frac{F_{1,2(t-2t_1)} - G_{1,2}(t-2t_1)}{1+d_{1,2}}. \tag{4.36}$$

For finding the function $Y_2(B,t)$, we consider the case $k = n$ of the result (4.35) and using (4.15) we find

$$Y_2(B,t) = \frac{d_{n-1,n}G_{n-1,n}(t-t_n) + F_{n-1,n}(t-t_n)}{1+d_{n-1,n}}, \;\; t \in (t_n+t^*, t_n+2t^*). \tag{4.37}$$

Finding $Y_2(z_k+0,t)$, $Y_1(0,t)$ **for** $t \in (t_{k+1}+t^*, t_{k+1}+2t^*)$, $k = 0,1,\ldots,n-1$.

Similar to the previous steps the functions $Y_2(z_k+0,t)$, $k = 0,1,\ldots,n-1$, can be found using the following equality

$$Y_2(z_k+0,t) = Y_2(z_{k+1}-0,t-t_{k+1}).$$

In the previous steps the functions $Y_2(z_k-0,t)$ $k = 1,2,\ldots,n-1$, have been constructed for $t \in (0,t^*) \cup (t^*,2t^*)$. Using this result we find

$$Y_2(z_k+0,t) = \frac{F_{k+1,k+2}(t-t_{k+1}) - G_{k+1,k+2}(t-t_{k+1})}{1+d_{k+1,k+2}}, \; t \in (t_{k+1}+t^*, t_{k+1}+2t^*), \quad (4.38)$$

$k = 0,1\ldots,n-2$. Similarly for the case $k = n-1$ we find

$$Y_2(z_{n-1}+0,t) = \frac{d_{n-1,n}G_{n-1,n}(t-2t_n) + F_{n-1,n}(t-2t_n)}{1+d_{n-1,n}}, \; t \in (2t_n, 2t_n+t^*).$$

For finding the function $Y_1(0,t)$, we consider the case $k = 0$ and use (4.15), and find

$$Y_1(0,t) = Y_2(0,t) = \frac{F_{1,2}(t-t_1) - G_{1,2}(t-t_1)}{1+d_{1,2}}, \; t \in (t_1+t^*, t_1+2t^*). \quad (4.39)$$

Finding $Y_1(z_k+0,t)$, $Y_2(z_k-0,t)$, $k = 1,2,\ldots,n-1$.

Similar to the previous steps the functions $Y_1(z_k-0,t)$, $Y_2(z_k+0,t)$ can be found as

$$Y_1(z_k+0,t) = \frac{d_{k,k+1}G_{k,k+1}(t) + F_{k,k+1}(t)}{1+d_{k,k+1}}, \quad (4.40)$$

$$Y_2(z_k-0,t) = \frac{F_{k,k+1}(t) - G_{k,k+1}(t)}{1+d_{k,k+1}}, \quad (4.41)$$

for $t \in (2t^*, 3t^*)$, $k = 1,2,\ldots,n-1$.

Similar we can do in general case. Really, let $p \in \mathbb{Z}^+$ denote the order of the step. Then for $p > 2$, the functions $Y_1(z_k-0,t)$, $Y_1(z_k+0,t)$, $Y_2(z_k+0,t)$, $Y_2(z_k-0,t)$ can be find by the formulas

$$Y_1(0,t) = Y_2(0,t) = \frac{F_{1,2}(t-t_1) - G_{1,2}(t-t_1)}{1+d_{1,2}}, \; t \in (t_1+(p-1)t^*, t_1+pt^*).$$

$$Y_1(B,t) = Y_2(B,t) = \frac{d_{n-1,n}G_{n-1,n}(t-t_n) + F_{n-1,n}(t-t_n)}{1+d_{n-1,n}}, \; t \in (t_n+(p-1)t^*, t_n+pt^*).$$

$$Y_1(z_1-0,t) = \frac{F_{1,2}(t-2t_1) - G_{1,2}(t-2t_1)}{1+d_{1,2}}, \; t \in (2t_1+(p-2)t^*, 2t_1+(p-1)t^*),$$

$$Y_1(z_k-0,t) = \frac{d_{k-1,k}G_{k-1,k}(t-t_k) + F_{k-1,k}(t-t_k)}{1+d_{k-1,k}}, \; t \in (t_k+(p-1)t^*, t_k+pt^*),$$

where $k = 2,\ldots,n-1$.

$$Y_2(z_k+0,t) = \frac{F_{k+1,k+2}(t-t_{k+1}) - G_{k+1,k+2}(t-t_{k+1})}{1+d_{k+1,k+2}},\ t \in (t_{k+1}+(p-1)t^*, t_{k+1}+pt^*),$$

where $k = 1,2,\ldots,n-2$,

$$Y_2(z_{n-1}+0,t) = \frac{d_{n-1,n}G_{n-1,n}(t-2t_n) + F_{n-1,n}(t-2t_n)}{1+d_{n-1,n}},$$

for $t \in (2t_n+(p-2)t^*, 2t_n+(p-1)t^*)$.

$$Y_1(z_k+0,t) = \frac{d_{k,k+1}G_{k,k+1}(t) + F_{k,k+1}(t)}{1+d_{k,k+1}},\ \text{for } t \in (pt^*, (p+1)t^*),\ k = 1,2,\ldots,n-1.$$

$$Y_2(z_k-0,t) = \frac{F_{k,k+1}(t) - G_{k,k+1}(t)}{1+d_{k,k+1}},\ \text{for } t \in (pt^*, (p+1)t^*),\ k = 1,2,\ldots,n-1.$$

4.3.2 Solution of IBVP (4.13)-(4.17) by Method of Characteristics

Firstly, let us consider the IBVP (4.13)-(4.17) and choose an arbitrary point in $(z_0,z_1)\cup(z_2,z_3)\cup\ldots\cup(z_{n-1},z_n)\times\mathbf{R}^+$, such that, $(z,t)\in(z_{k-1},z_k)\times\mathbf{R}^+$. Note that in the previous section, we have shown that the system (4.13) can be rewritten in component form as in equation (4.18). Then these equations have been rewritten in terms of ξ and τ. Now, let us consider these equations. The characteristics passing through the point $(z,t)\in(z_{k-1},z_k)\times\mathbf{R}^+$ can be found as $\tau = \xi+t-z$ and $\tau = -\xi+t+z$. Equation (4.18) along these characteristics can be written as

$$\frac{d}{d\xi}Y_1(\xi,\xi+t-z) = 0, \quad \frac{d}{d\xi}Y_2(\xi,-\xi+t+z) = 0.$$

Integrating the equations above with respect to ξ from z_{k-1} to z and z to z_k, respectively, we find

$$Y_1(z,t) = Y_1(z_{k-1}+0, t-(z-z_{k-1})),\ (z,t)\in(z_{k-1},z_k)\times\mathbf{R}^+, \tag{4.42}$$

$$Y_2(z,t) = Y_2(z_k-0, t+(z-z_k)),\ (z,t)\in(z_{k-1},z_k)\times\mathbf{R}^+, \tag{4.43}$$

where $k = 1,2,\ldots,n$. Note that the functions $Y_1(z_{k-1}+0, t-(z-z_{k-1}))$, and $Y_2(z_k-0, t+(z-z_k))$ have been constructed in the previous section. Then using this result we find solution of the IBVP for the system (4.13)-(4.17) as

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Y_1(z_{k-1}+0, t-(z-z_{k-1})) \\ Y_2(z_k-0, t+(z-z_k)) \end{pmatrix},\ (z,t)\in(z_{k-1},z_k)\times\mathbf{R}^+,\ k = 1,2,\ldots,n. \tag{4.44}$$

4.4 Solution of IBVP (4.1)-(4.5)

Now, let us find solution of the IBVP (4.7)-(4.11). Firstly, substituting Y_1 and Y_2 into the formulas for V and σ we get

$$V = \frac{1}{\sqrt{2}}\left[Y_1(z_{k-1}+0, t-(z-z_{k-1})) + Y_2(z_k-0, t+(z-z_k))\right], \tag{4.45}$$

$$\sigma = \frac{1}{\sqrt{2}}\left[-Y_1(z_{k-1}+0, t-(z-z_{k-1})) + Y_2(z_k-0, t+(z-z_k))\right], \tag{4.46}$$

for $(z,t) \in (z_{k-1}, z_k) \times \mathbf{R}^+,\ k = 1,2,\ldots,n$. Integrating the equality $V = \frac{\partial v}{\partial \tau}$ with respect to τ from 0 to t we find solution of the IBVP (4.7)-(4.11) as

$$v(z,t) = \int_0^t \frac{1}{\sqrt{2}}\left[Y_1(z_{k-1}+0, \tau-(z-z_{k-1})) + Y_2(z_k-0, \tau+(z-z_k))\right]\, d\tau, \tag{4.47}$$

for $(z,t) \in (z_{k-1}, z_k) \times \mathbf{R}^+,\ k = 1,2,\ldots,n$. As a result we find solution of the problem (4.1)-(4.5)

$$u(y,t) = v(z(y),t),\ (y,t) \in (y_{k-1}, y_k) \times \mathbf{R}^+,\ k = 1,2,\ldots,n. \tag{4.48}$$

4.5 Summary of Chapter Four

Initial boundary value problem for the 1D wave equation with piecewise constant coefficient has been solved. The solution of this problem has the following steps. The IBVP for 1D wave equation is reduced to an initial value problem for symmetric hyperbolic equations with two independent variables and piecewise constant coefficients. Applying the method of characteristics the IBVP for the obtained symmetric hyperbolic equations has been solved explicitly. Finally, the solution of the original initial value problem has been written by explicit formula. The solution of this problem will be used in the Chapter 6.

CHAPTER FIVE
THE EIGENVALUE-EIGENFUNCTION PROBLEM IN LAYERED RECTANGULAR PARALLELEPIPED

Three eigenvalue-eigenfunction problems for the scalar differential operator $c^2(x_2)\Delta_x$ with piecewise constant function $c(x_2)$ in N-layered rectangular parallelepiped in $\mathbf{R}^3$ are studied in this chapter. The formulation of each problem contains the boundary and matching (interface) conditions.

The procedure of finding eigenvalues of the considered problems are described in this chapter. This procedure is implemented in MATLAB. Eigenfunctions are constructed in explicit form. The regularizations of 1D, 2D and 3D Dirac delta functions are given in terms of the Fourier series with finite number of terms of the basis functions. The results and explicit formulae, and regularization of the Dirac delta function will be used in Chapter 6.

5.1 Solving the Eigenvalue-Eigenfunction Problems in Layered Parallelepiped

Let b_1, b_2, b_3, r_i, $i = 1,2,...,N$ be given real positive numbers with $0 = r_0 < r_1 < ... < r_N = b_2$, $x = (x_1, x_2, x_3)^T \in \mathbf{R}^3$;

$$D = \Big\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2, \quad 0 < x_3 < b_3\Big\}$$

be a rectangular parallelepiped, Γ be the boundary of D; and D_i, $i = 1,2,...,N$ be the parallelepipeds

$$D_i = \Big\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad r_{i-1} < x_2 < r_i, \quad 0 < x_3 < b_3\Big\}$$

and $D^0 = \bigcup\limits_{i=1}^{N} D_i$.

Let μ_i, ϵ_i, for $i = 1,2,\ldots,N$ given real positive numbers; $\mu(x_2)$, $\epsilon(x_2)$ be two piecewise constant functions defined on $[0, b_2]$ such that

$$\mu(x_2) = \mu_i, \quad \epsilon(x_2) = \epsilon_i, \text{ for } x_2 \in (r_{i-1}, r_i), \ i = 1,2,\ldots,N.$$

We define the function $c(x_2)$ as

$$c(x_2) = c_i = \frac{1}{\sqrt{\mu_i \epsilon_i}}, \text{ for } x_2 \in (r_{i-1}, r_i), \; i = 1,2,\ldots,N. \tag{5.1}$$

Let us consider the problem of finding all values of λ (eigenvalues) for which there exists a nonzero scalar function $U_j(x)$, $j = 1,2,3$ satisfying

$$\Delta_x U_1(x) + \frac{1}{c^2(x_2)}\lambda U_1(x) = 0, \quad x \in D^0, \tag{5.2}$$

$$U_1(x)|_{x_1=0} = 0, \quad U_1(x)|_{x_1=b_1} = 0, \tag{5.3}$$

$$\left.\frac{\partial U_1}{\partial x_2}\right|_{x_2=0} = 0, \quad \left.\frac{\partial U_1}{\partial x_2}\right|_{x_2=b_2} = 0, \tag{5.4}$$

$$\left.\frac{\partial U_1}{\partial x_3}\right|_{x_3=0} = 0, \quad \left.\frac{\partial U_1}{\partial x_3}\right|_{x_3=b_3} = 0, \tag{5.5}$$

$$U_1|_{x_2=r_i-0} = U_1|_{x_2=r_i+0}, \tag{5.6}$$

$$\left.\frac{\partial U_1}{\partial x_2}\right|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\left.\frac{\partial U_1}{\partial x_2}\right|_{x_2=r_i-0}, \tag{5.7}$$

$$\Delta_x U_2(x) + \frac{1}{c^2(x_2)}\lambda U_2(x) = 0, \quad x \in D^0, \tag{5.8}$$

$$\left.\frac{\partial U_2}{\partial x_1}\right|_{x_1=0} = 0, \quad \left.\frac{\partial U_2}{\partial x_1}\right|_{x_1=b_1} = 0, \tag{5.9}$$

$$U_2(x)|_{x_2=0} = 0, \quad U_2(x)|_{x_2=b_2} = 0, \tag{5.10}$$

$$\left.\frac{\partial U_2}{\partial x_3}\right|_{x_3=0} = 0, \quad \left.\frac{\partial U_2}{\partial x_3}\right|_{x_3=b_3} = 0, \tag{5.11}$$

$$U_2|_{x_2=r_i-0} = \frac{\mu_{i+1}}{\mu_i} U_2|_{x_2=r_i+0}, \tag{5.12}$$

$$\left.\frac{\partial U_2}{\partial x_2}\right|_{x_2=r_i+0} = \left.\frac{\partial U_2}{\partial x_2}\right|_{x_2=r_i-0}, \tag{5.13}$$

$$\Delta_x U_3(x) + \frac{1}{c^2(x_2)}\lambda U_3(x) = 0, \quad x \in D^0, \tag{5.14}$$

$$\left.\frac{\partial U_3}{\partial x_1}\right|_{x_1=0} = 0, \quad \left.\frac{\partial U_3}{\partial x_1}\right|_{x_1=b_1} = 0, \tag{5.15}$$

$$\left.\frac{\partial U_3}{\partial x_2}\right|_{x_2=0} = 0, \quad \left.\frac{\partial U_3}{\partial x_2}\right|_{x_2=b_2} = 0, \tag{5.16}$$

$$U_3(x)|_{x_3=0} = 0, \quad U_3(x)|_{x_3=b_3} = 0, \tag{5.17}$$

$$U_3|_{x_2=r_i-0} = U_3|_{x_2=r_i+0}, \tag{5.18}$$

$$\left.\frac{\partial U_3}{\partial x_2}\right|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\left.\frac{\partial U_3}{\partial x_2}\right|_{x_2=r_i-0}, \tag{5.19}$$

where $i = 1,2,...,N-1$.

We find the eigenfunctions of the problems (5.2)-(5.7), (5.8)-(5.13), (5.14)-(5.19) in the form

$$U_j^{kn}(x_1,x_2,x_3) = Z_j^{kn}(x_2)W_j^{kn}(x_1,x_3), \quad j = 1,2,3. \tag{5.20}$$

The functions $W_j^{kn}(x_1,x_3)$ are defined explicitly for each $j = 1,2,3$ as:
For $j = 1$,

$$W_1^{kn}(x_1,x_3) = a_{kn}\sin\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{n\pi}{b_3}x_3\right), \quad k = 1,2,\ldots;\ n = 0,1,\ldots, \tag{5.21}$$

$a_{k0} = \sqrt{\frac{2}{b_1 b_3}}$, $a_{kn} = \dfrac{2}{\sqrt{b_1 b_3}}$, $k = 1,2,\ldots,$ $n = 1,2,\ldots$.
For $j = 2$,

$$W_2^{kn}(x_1,x_3) = b_{kn}\cos\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{n\pi}{b_3}x_3\right), \quad k = 0,1,\ldots;\ n = 0,1,\ldots,$$

$b_{k0} = b_{0n} = \sqrt{\frac{2}{b_1 b_3}}$, $b_{kn} = \dfrac{2}{\sqrt{b_1 b_3}}$, $b_{00} = \sqrt{\frac{1}{b_1 b_3}}$, $k = 1,2,\ldots,$ $n = 1,2,\ldots$.
For $j = 3$,

$$W_3^{kn}(x_1,x_3) = c_{kn}\cos\left(\frac{k\pi}{b_1}x_1\right)\sin\left(\frac{n\pi}{b_3}x_3\right), \quad k = 0,1,\ldots;\ n = 1,2,\ldots, \tag{5.22}$$

$c_{0k} = \sqrt{\frac{2}{b_1 b_3}}$, $c_{kn} = \dfrac{2}{\sqrt{b_1 b_3}}$, $k = 1,2,\ldots,$ $n = 1,2,\ldots$.

Substituting (5.20) into (5.2)-(5.7), (5.8)-(5.13), (5.14)-(5.19), we find the following equations

$$c^2(x_2)\frac{d^2 Z_1^{kn}(x_2)}{dx_2^2} + \left(\lambda - c^2(x_2)\xi_{kn}\right)Z_1^{kn}(x_2) = 0, \quad x_2 \in (0,r_1)\cup\ldots\cup(r_{N-1},b_2), \tag{5.23}$$

$$\left.\frac{dZ_1^{kn}}{dx_2}\right|_{x_2=0} = 0, \quad \left.\frac{dZ_1^{kn}}{dx_2}\right|_{x_2=b_2} = 0, \tag{5.24}$$

$$Z_1^{kn}\Big|_{x_2=r_i-0} = Z_1^{kn}\Big|_{x_2=r_i+0}, \tag{5.25}$$

$$\left.\frac{dZ_1^{kn}}{dx_2}\right|_{x_2=r_i-0} = \frac{\epsilon_i}{\epsilon_{i+1}}\left.\frac{dZ_1^{kn}}{dx_2}\right|_{x_2=r_i+0}, \tag{5.26}$$

where λ is a parameter; $\xi_{kn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2$, $k = 1,2,\ldots$; $n = 0,1,...$; $i = 1,2,...,N-1$.

$$c^2(x_2)\frac{d^2 Z_2^{kn}(x_2)}{dx_2^2} + \left(\lambda - c^2(x_2)\xi_{kn}\right)Z_2^{kn}(x_2) = 0, \quad x_2 \in (0,r_1)\cup\ldots\cup(r_{N-1},b_2), \tag{5.27}$$

$$Z_2^{kn}\Big|_{x_2=0} = 0, \quad Z_2^{kn}\Big|_{x_2=b_2} = 0, \tag{5.28}$$

$$Z_2^{kn}\Big|_{x_2=r_i-0} = \frac{\mu_{i+1}}{\mu_i}\, Z_2^{kn}\Big|_{x_2=r_i+0}, \tag{5.29}$$

$$\left.\frac{dZ_2^{kn}}{dx_2}\right|_{x_2=r_i-0} = \left.\frac{dZ_2^{kn}}{dx_2}\right|_{x_2=r_i+0}, \tag{5.30}$$

where λ is a parameter; $\xi_{kn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2$, $k,n = 0,1,...$; $i = 1,2,...,N-1$.

$$c^2(x_2)\frac{d^2 Z_3^{kn}(x_2)}{dx_2^2} + \left(\lambda - c^2(x_2)\xi_{kn}\right)Z_3^{kn}(x_2) = 0, \quad x_2 \in (0,r_1)\cup\ldots\cup(r_{N-1},b_2), \tag{5.31}$$

$$\left.\frac{dZ_3^{kn}}{dx_2}\right|_{x_2=0} = 0, \quad \left.\frac{dZ_3^{kn}}{dx_2}\right|_{x_2=b_2} = 0, \tag{5.32}$$

$$Z_3^{kn}\Big|_{x_2=r_i-0} = Z_3^{kn}\Big|_{x_2=r_i+0}, \tag{5.33}$$

$$\left.\frac{dZ_3^{kn}}{dx_2}\right|_{x_2=r_i-0} = \frac{\epsilon_i}{\epsilon_{i+1}}\left.\frac{dZ_3^{kn}}{dx_2}\right|_{x_2=r_i+0}, \tag{5.34}$$

where λ is a parameter; $\xi_{kn} = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2$, $k = 0,1,\ldots$; $n = 1,2,\ldots$; $i = 1,2,\ldots,N-1$.

The problems (5.23)-(5.26), (5.27)-(5.30) (5.31)-(5.34) are eigenvalue-eigenfunction problems for the operators $L_{kn} = c^2(x_2)\left[-\frac{d^2}{dx_2^2} + \xi_{kn}\right]$, $k,n = 0,1,\ldots$.

Remark 5.1.1. The operator $L_{kn} = c^2(x_2)\left[-\frac{d^2}{dx_2^2} + \xi_{kn}\right]$ is positive-definite and symmetric operator for all ξ_{kn}, $k,n = 0,1,\ldots$. By the general theory of positive-definiteness and symmetry of operators (see, for example, (Zauderer, 1988, Vladimirov, 1971)), this implies that all eigenvalues $\lambda = \lambda_{kmn}$ of the operator L_{kn} are real positive numbers.

Proof. We give the proof only for the case $j = 2$. Let $\nu(x_2)$ be a piecewise constant function defined on $[0,b_2]$, such that $\nu(x_2) = [\mu_i]^2\epsilon_i$, for $x_2 \in (r_{i-1},r_i)$, $i = 1,2,\ldots,N$; $Z_2^{kn}(x_2)$ and $Y_2^{kn}(x_2)$ be square integrable functions on $[0,r_1)\cup\ldots\cup(r_{N-1},b_2]$

$$\int_0^{b_2} \upsilon(x_2)\left[Z_2^{kn}\right]^2\, dx_2 = \int_0^{r_1} (\mu_1)^2\epsilon_1\left[Z_2^{kn}\right]^2\, dx_2 + \ldots + \int_{r_{N-1}}^{b_2} (\mu_N)^2\epsilon_N\left[Z_2^{kn}\right]^2\, dx_2 < \infty,$$

satisfying the boundary conditions (5.28) and matching conditions (5.29),(5.30).

We want to show that the operator L_{kn} satisfies the followings:

$$\int_0^{b_2} \nu c^2(x_2)\left[-\frac{d^2Z_2^{kn}}{dx_2^2}+\xi_{kn}Z_2^{kn}\right]Y_2^{kn}\,dx_2 = \int_0^{b_2} \nu Z_2^{kn}c^2(x_2)\left[-\frac{d^2Y_2^{kn}}{dx_2^2}+\xi_{kn}Y_2^{kn}\right]dx_2, \quad (5.35)$$

$$\int_0^{b_2} \nu(x_2)c^2(x_2)\left[-\frac{d^2Z_2^{kn}}{dx_2^2}+\xi_{kn}Z_2^{kn}\right]Z_2^{kn}\,dx_2 > 0. \quad (5.36)$$

Under the above assumptions, applying the integration by parts and using the indicated boundary and matching conditions, the integral of the left side of (5.35) equals

$$\int_0^{b_2} \nu(x_2)c^2(x_2)\left[\frac{dZ_2^{kn}}{dx_2}\frac{dY_2^{kn}}{dx_2}+\xi_{kn}Z_2^{kn}Y_2^{kn}\right]dx_2. \quad (5.37)$$

Similarly, applying the integration by parts to the integral above gives

$$\int_0^{b_2} \nu(x_2)Z_2^{kn}c^2(x_2)\left[-\frac{d^2Y_2^{kn}}{dx_2^2}+\xi_{kn}Y_2^{kn}\right]dx_2.$$

Thus, the equation (5.35), implies the symmetry property of the operator L_{kn}, is satisfied.

Now, consider the equality (5.37). Replacing the term Y_2^{kn} by Z_2^{kn} in (5.37), and using $\nu(x_2) = (\mu_i)^2\,\epsilon_i > 0$ and $c(x_2) = c_i > 0$ for $x_2 \in (r_{i-1}, r_i)$, $i = 1, 2, \ldots, N$, we find

$$\int_0^{b_2} \nu(x_2)c^2(x_2)\left[-\frac{d^2Z_2^{kn}}{dx_2^2}+\xi_{kn}Z_2^{kn}\right]Z_2^{kn}dx_2 = \int_0^{b_2} \nu(x_2)c^2(x_2)\left[\left(\frac{dZ_2^{kn}}{dx_2}\right)^2+\xi_{kn}\left(Z_2^{kn}\right)^2\right]dx_2 > 0.$$

That is, the inequality (5.36), implies positive-definiteness of the operator L_{kn}.

Similar proof can be repeated for the cases $j = 1$ and $j = 3$ by taking $\nu(x_2) = \mu_i$ for $x_2 \in (r_{i-1}, r_i)$, $i = 1, 2, \ldots, N$. □

5.1.1 Constructing a Solution of the Problem (5.27)-(5.30)

Let $Z_2^{kn}(x_2)$ be any nonzero continuous function on $[0, r_1) \cup \ldots \cup (r_{N-1}, b_2]$ satisfying (5.27)-(5.30). By Remark 5.1.1, we know that the eigenvalues $\lambda = \lambda_{kmn}$ are real and positive for fixed k and n. Thus

$$0 < \lambda_{kmn} \int_0^{b_2} \nu(x_2) c^2(x_2) \left(Z_2^{kn}\right)^2 \, dx_2 = \int_0^{b_2} \nu(x_2) c^2(x_2) \left[-\frac{d^2 Z_2^{kn}}{dx_2^2} + \xi_{kn} Z_2^{kn} \right] Z_2^{kn} \, dx_2.$$

Using the last equality we have

$$\int_0^{b_2} \nu(x_2) \left[\lambda_{kmn} - c^2(x_2)\xi_{kn}\right] \left(Z_2^{kn}\right)^2 \, dx_2 = \int_0^{b_2} \nu(x_2) \left[-c^2(x_2) \frac{d^2 Z_2^{kn}}{dx_2^2} \right] dx_2.$$

Since the operator $-c^2(x_2)\frac{d^2}{dx_2^2}$ is positive-definite and symmetric for $c^2(x_2) > 0$, $x_2 \in (0, r_1) \cup ... \cup (r_{N-1}, b_2)$, then $\lambda_{kmn} - c_i^2 \xi_{kn} < 0$, $i = 1, 2, \ldots, N$ are not valid, simultaneously.

Without any loss of generality assume that $c_1 > c_2 > ... > c_N$ which implies $c_1^2 \xi_{kn} > c_2^2 \xi_{kn} > ... > c_N^2 \xi_{kn}$, for fixed k and n. We consider the case $\lambda_{kmn} > c_1^2 \xi_{kn} > ... > c_N^2 \xi_{kn}$ to solve the eigenvalue-eigenfunction problem (5.27)-(5.30).

For each $k, n = 0, 1, ...$ we find general solution of the ordinary differential equation (5.27) for $x_2 \in (0, r_1) \cup ... \cup (r_{N-1}, b_2)$ in the form

$$Z_2^{kn}(x_2) = A_i^{kn} \cos\left(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}}\, x_2\right) + B_i^{kn} \sin\left(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}}\, x_2\right), \tag{5.38}$$

$x_2 \in (r_{i-1}, r_i)$, with arbitrary constants A_i^{kn}, B_i^{kn}, $i = 1, 2, ..., N$. We use (5.28)-(5.30) to determine the constants A_i^{kn}, B_i^{kn}. Setting $B_1^{kn} = 1$ and using the conditions (5.28)-(5.30) we get $A_1^{kn} = 0$ and the following algebraic system

$$[\mathbf{Q}(\lambda)] \cdot \begin{bmatrix} 1 & A_2^{kn} & B_2^{kn} & \ldots & A_{N-1}^{kn} & B_{N-1}^{kn} & A_N^{kn} & B_N^{kn} \end{bmatrix}^T = \mathbf{0}, \tag{5.39}$$

where $\mathbf{0}$ is a zero column vector, and $\mathbf{Q}$ is $(2N-1) \times (2N-1)$ block matrix of the form

$$\begin{pmatrix}
[P_1(\lambda)] & [F_1(\lambda)] & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & [P_2(\lambda)] & [F_2(\lambda)] & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & [P_3(\lambda)] & [F_3(\lambda)] & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & [P_4(\lambda)] & [F_4(\lambda)] & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \ddots & \ddots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & 0 & [P_{N-2}(\lambda)] & [F_{N-2}(\lambda)] & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & [P_{N-1}(\lambda)] & [F_{N-1}(\lambda)] \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & [P_N(\lambda)]
\end{pmatrix},$$

$[P_i(\lambda)]$ and $[F_j(\lambda)]$ are submatrices defined by

$$[P_1(\lambda)] = \begin{bmatrix} \sin\left(\sqrt{\lambda/c_1^2 - \xi_{kn}}r_1\right) \\ \sqrt{\lambda/c_1^2 - \xi_{kn}}\cos\left(\sqrt{\lambda/c_1^2 - \xi_{kn}}r_1\right) \end{bmatrix}_{2\times 1};$$

$$[P_i(\lambda)] = \begin{bmatrix} \cos\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) & \sin\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) \\ -\sqrt{\lambda/c_i^2 - \xi_{kn}}\sin\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) & \sqrt{\lambda/c_i^2 - \xi_{kn}}\cos\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) \end{bmatrix}_{2\times 2},$$

for $i = 2,3,...,N-1$;

$$[P_N(\lambda)] = \begin{bmatrix} \cos\left(\sqrt{\lambda/c_N^2 - \xi_{kn}}b_2\right) & \sin\left(\sqrt{\lambda/c_N^2 - \xi_{kn}}b_2\right) \end{bmatrix}_{1\times 2};$$

$$[F_j(\lambda)] =$$

$$\begin{bmatrix} -\frac{\mu_{j+1}}{\mu_j}\cos\left(\sqrt{\lambda/c_{j+1}^2 - \xi_{kn}}r_j\right) & -\frac{\mu_{j+1}}{\mu_j}\sin\left(\sqrt{\lambda/c_{j+1}^2 - \xi_{kn}}r_j\right) \\ \sqrt{\lambda/c_{j+1}^2 - \xi_{kn}}\sin\left(\sqrt{\lambda/c_{j+1}^2 - \xi_{kn}}r_j\right) & -\sqrt{\lambda/c_{j+1}^2 - \xi_{kn}}\cos\left(\sqrt{\lambda/c_{j+1}^2 - \xi_{kn}}r_j\right) \end{bmatrix}_{2\times 2},$$

for $j = 1,2,...,N-1$.

Hence finding a nonzero function $Z_2^{kn}(x_2)$ satisfying (5.27)-(5.30) is reduced to determining the constants A_i^{kn}, B_i^{kn}, $i = 2,3,...,N$, satisfying (5.39). The system (5.39) is consistent if and only if the determinant of the matrix $[\mathbf{Q}]$ is equal to zero. Moreover, the roots of the characteristic equation of $\det[\mathbf{Q}(\lambda)] = 0$ are the eigenvalues of (5.27)-(5.30). The roots of $\det[\mathbf{Q}(\lambda)] = 0$ can be computed by MATLAB tools.

Let us assume that the roots $\lambda = \lambda_{kmn}$, $k,m,n = 0,1,...$ of $\det[\mathbf{Q}(\lambda)] = 0$ have been derived. Substituting $\lambda = \lambda_{kmn}$, $A_1^{kmn} = 0$, $B_1^{kmn} = 1$ into (5.39), we obtain the following relations:

$$[F_1(\lambda_{kmn})] \cdot \begin{bmatrix} A_2^{kmn} \\ B_2^{kmn} \end{bmatrix} = -[P_1(\lambda_{kmn})],$$

$$[F_i(\lambda_{kmn})] \cdot \begin{bmatrix} A_{i+1}^{kmn} \\ B_{i+1}^{kmn} \end{bmatrix} = -[P_i(\lambda_{kmn})] \cdot \begin{bmatrix} A_i^{kmn} \\ B_i^{kmn} \end{bmatrix}, \quad i = 2,3,...,N-1.$$

We note that $\det[F_i(\lambda_{kmn})] = \dfrac{\mu_{i+1}}{\mu_i}\sqrt{\mu_{i+1}\epsilon_{i+1}\lambda_{kmn} - \xi_{kn}} \neq 0$, for all $i = 1,2,...,N-1$. This is so $\dfrac{\mu_{i+1}}{\mu_i} \neq 0$ and by assumption $\lambda_{kmn} > \dfrac{\xi_{kn}}{\mu_i\epsilon_i}$, $i = 1,2,...,N$. Then for finding the values of A_i^{kmn}, B_i^{kmn}, $i = 2,3,...,N$ we can use the following recurrence relations:

$$\begin{bmatrix} A_2^{kmn} \\ B_2^{kmn} \end{bmatrix} = -[F_1(\lambda_{kmn})]^{-1} \cdot [P_1(\lambda_{kmn})],$$

$$\begin{bmatrix} A_{i+1}^{kmn} \\ B_{i+1}^{kmn} \end{bmatrix} = -[F_i(\lambda_{kmn})]^{-1} \cdot [P_i(\lambda_{kmn})] \cdot \begin{bmatrix} A_i^{kmn} \\ B_i^{kmn} \end{bmatrix}, \quad i = 2,3,...,N-1.$$

Substituting the obtained values A_i^{kmn}, B_i^{kmn}, $i = 1,2,...,N$ into (5.38) we get the explicit formula for the eigenfunction $Z_2^{kmn}(x_2)$, of (5.27)-(5.30) as follows:

$$Z_2^{kmn}(x_2) = A_i^{kmn} \cos\Big(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}} x_2 \Big) + B_i^{kmn} \sin\Big(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}} x_2 \Big), \qquad (5.40)$$

where $x_2 \in (r_{i-1}, r_i)$, $i = 1,2,...,N$; $k,m,n = 0,1,....$

5.1.2 Constructing Solutions of the Problems (5.23)-(5.26), (5.31)-(5.34)

In this section we solve the problems (5.23)-(5.26), (5.31)-(5.34). For construction of the solutions of these problems we follow the way that has been used for solving (5.27)-(5.30) in the last section. By repeating the calculations of the solution steps we find solutions of the problems (5.23)-(5.26), (5.31)-(5.34), respectively, for $x_2 \in (0,r_1) \cup ... \cup (r_{N-1}, b_2)$ as

$$Z_1^{kmn}(x_2) = A_i^{kmn} \cos\Big(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}} x_2 \Big) + B_i^{kmn} \sin\Big(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}} x_2 \Big), \qquad (5.41)$$

$k = 1,2\ldots,\ n = 0,1,\ldots$ for $x_2 \in (r_{i-1}, r_i),\ \ i = 1,2,\ldots,N,$

$$Z_3^{kmn}(x_2) = A_i^{kmn} \cos\Big(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}} x_2 \Big) + B_i^{kmn} \sin\Big(\sqrt{\lambda_{kmn}/c_i^2 - \xi_{kn}} x_2 \Big), \qquad (5.42)$$

$k = 0,1\ldots,\ n = 1,2,\ldots$ for $x_2 \in (r_{i-1}, r_i),\ \ i = 1,2,\ldots,N.$

Here $\lambda = \lambda_{kmn}$ are roots of the equation $\det[\mathbf{Q}(\lambda)] = 0$. The form of the block matrix $\mathbf{Q}$ is given in the last section. For this case, the submatrices $[P_i(\lambda)]$ and $[F_j(\lambda)]$ are defined as follows:

$$[P_1(\lambda)] = \begin{bmatrix} \cos\Big(\sqrt{\lambda/c_1^2 - \xi_{kn}} r_1 \Big) \\ -\sqrt{\lambda/c_1^2 - \xi_{kn}} \sin\Big(\sqrt{\lambda/c_1^2 - \xi_{kn}} r_1 \Big) \end{bmatrix}_{2\times 1};$$

$$[P_i(\lambda)] = \begin{bmatrix} \cos\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) & \sin\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) \\ -\sqrt{\lambda/c_i^2 - \xi_{kn}}\sin\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) & \sqrt{\lambda/c_i^2 - \xi_{kn}}\cos\left(\sqrt{\lambda/c_i^2 - \xi_{kn}}r_i\right) \end{bmatrix}_{2\times 2},$$

for $i = 2,3,...,N-1$;

$$[P_N(\lambda)] = \begin{bmatrix} -\sqrt{\eta_{kn}^N/c_N^2}\sin\left(\sqrt{\eta_{kn}^N/c_N^2}b_2\right) & \sqrt{\eta_{kn}^N/c_N^2}\cos\left(\sqrt{\eta_{kn}^N/c_N^2}b_2\right) \end{bmatrix}_{1\times 2},$$

where $\eta_{kn}^N = \lambda - c_N^2\xi_{kn}$;

$$[F_j(\lambda)] = \begin{bmatrix} -\cos\left(\sqrt{\eta_{kn}^j/c_{j+1}^2}r_j\right) & -\sin\left(\sqrt{\eta_{kn}^j/c_{j+1}^2}r_j\right) \\ \frac{\epsilon_j}{\epsilon_{j+1}}\sqrt{\eta_{kn}^j/c_{j+1}^2}\sin\left(\sqrt{\eta_{kn}^j/c_{j+1}^2}r_j\right) & -\frac{\epsilon_j}{\epsilon_{j+1}}\sqrt{\eta_{kn}^j/c_{j+1}^2}\cos\left(\sqrt{\eta_{kn}^j/c_{j+1}^2}r_j\right) \end{bmatrix}_{2\times 2},$$

where $\eta_{kn}^j = \lambda - c_{j+1}^2\xi_{kn}$, $j = 1,2,...,N-1$.

The roots of $\det[\mathbf{Q}(\lambda)] = 0$ can be computed by MATLAB tools.

Let us assume that the roots $\lambda = \lambda_{kmn}$, $k,m,n = 0,1,...$ of $\det[\mathbf{Q}(\lambda)] = 0$ have been derived. Then the values of A_i^{kmn}, B_i^{kmn}, $i = 2,3,...,N$ can be found by the use of the following recurrence relations:

$$A_1^{kmn} = 1, \quad B_1^{kmn} = 0,$$

$$\begin{bmatrix} A_2^{kmn} \\ B_2^{kmn} \end{bmatrix} = -[F_1(\lambda_{kmn})]^{-1} \cdot [P_1(\lambda_{kmn})],$$

$$\begin{bmatrix} A_{i+1}^{kmn} \\ B_{i+1}^{kmn} \end{bmatrix} = -[F_i(\lambda_{kmn})]^{-1} \cdot [P_i(\lambda_{kmn})] \cdot \begin{bmatrix} A_i^{kmn} \\ B_i^{kmn} \end{bmatrix}, \quad i = 2,3,...,N-1,$$

$\det[F_i(\lambda_{kmn})] \neq 0$, $i = 1,2,\ldots,N-1$.

As a result, substituting the explicit formulas for $Z_j^{kmn}(x_2)$ and $W_j^{kn}(x_3)$ into the equation (5.20) for each $j = 1,2,3$, we construct the eigenfunctions of (5.2)-(5.7), (5.8)-(5.13), (5.14)-(5.19) as

$$U_1^{kmn}(x_1,x_2,x_3) = W_1^{kmn}(x_1,x_3)Z_1^{kmn}(x_2), \tag{5.43}$$

$k = 1,2,\ldots$, $m = 0,1,\ldots$, $n = 0,1,\ldots$,

$$U_2^{kmn}(x_1,x_2,x_3) = W_2^{kmn}(x_1,x_3)Z_2^{kmn}(x_2), \tag{5.44}$$

$k = 0,1,\ldots$, $m = 0,1,\ldots$, $n = 0,1,\ldots$,

$$U_3^{kmn}(x_1,x_2,x_3) = W_3^{kmn}(x_1,x_3)Z_3^{kmn}(x_2), \tag{5.45}$$

$k = 0,1,\ldots$, $m = 0,1,\ldots$, $n = 0,1,\ldots$, for $x_1 \in [0,b_1]$, $x_2 \in [0,r_1) \cup ... \cup (r_{N-1},b_2]$, $x_3 \in [0,b_3]$.

5.1.2.1 Orthogonality Property of the Eigenfunctions

The set of the functions $U_j^{kmn}(x)$, $k,m,n = 0,1,...$, and $U_j^{k'm'n'}(x)$, $k',m',n' = 0,1,...$ satisfy the orthogonality property, that is,

$$\int_0^{b_1}\int_0^{b_3}\left[\int_0^{b_2} \nu(x_2)U_j^{kmn}(x)U_j^{k'm'n'}(x)\,dx_2\right]dx_3\,dx_1 =$$

$$\begin{cases} 0, & \text{if } k \neq k' \text{ or } m \neq m' \text{ or } n \neq n', \\ \kappa_{kmn}, & \text{if } k = k',\ m = m',\ n = n', \end{cases}$$

where $\kappa_{kmn} > 0$, $j = 1,2,3$.

For the proof see, Appendix A7.

5.2 Regularization of the Dirac Delta Function

In this section regularization of the 1D, 2D and 3D Dirac delta functions are given by means of the classical functions. On comparing the classical functions with the Dirac delta function, it can be seen that the Dirac delta function does not have point-wise values and so it can not be drawn. Regularization, that is, presentation of it by means of classical functions gives opportunity to graph it.

Regularization of the 1D Dirac Delta Function and its Derivative

Let $\delta(x_1 - x_1^0)$, $\delta(x_2 - x_2^0)$, $\delta(x_3 - x_3^0)$ be 1D Dirac delta functions concentrated at $x_1 = x_1^0$, $x_2 = x_2^0$, $x_3 = x_3^0$, respectively. Let us consider the classical functions $\sin\left(\frac{k\pi}{b_1}x_1\right)$, $\cos\left(\frac{k\pi}{b_1}x_1\right)$, $\sin\left(\frac{n\pi}{b_3}x_3\right)$, $\cos\left(\frac{n\pi}{b_3}x_3\right)$, and the functions $Z_1^{kmn}(x_2)$, $Z_2^{kmn}(x_2)$, $Z_3^{kmn}(x_2)$ that are defined as in (5.41), (5.40), (5.42), respectively, for $x_1 \in [0,b_1]$, $x_2 \in [0,r_1) \cup (r_1,r_2) \cup \ldots \cup (r_{N-1},b_2]$, $x_3 \in [0,b_3]$, $k,m,n = 0,1,\ldots$. Here, we show the regularization (approximation) of 1D Dirac delta function by means of the classical function mentioned above.

Example 1. Regularization of the Dirac delta function $\delta(x_1 - x_1^0)$ by means of the functions $\cos\left(\dfrac{k\pi}{b_1}x_1\right)$, $k = 0, 1, \ldots$ and $\sin\left(\dfrac{k\pi}{b_1}x_1\right)$, $k = 1, 2, \ldots$ for $x_1 \in [0, b_1]$ (see,(Arfken & Weber, 2005)).

Let $\psi(x_1, x_1^0) = \delta(x_1 - x_1^0)$, and K be a fixed natural number. Then the functions

$$\psi^K = \sum_{k=0}^{K} \psi^k(x_1^0) \cos\left(\frac{k\pi}{b_1}x_1\right), \quad \psi^K = \sum_{k=1}^{K} \psi^k(x_1^0) \sin\left(\frac{k\pi}{b_1}x_1\right)$$

can be taken as regularization of $\delta(x_1 - x_1^0)$ by means of $\cos\left(\frac{k\pi}{b_1}x_1\right)$, $k = 0, 1, \ldots$ and $\sin\left(\frac{k\pi}{b_1}x_1\right)$, $k = 1, 2, \ldots$, respectively. Here, the coefficients $\psi^k(x_1^0)$ can be computed as follows:

$$\psi^k(x_1^0) = \int_0^{b_1} \delta(x_1 - x_1^0) \cos\left(\frac{k\pi}{b_1}x_1\right) dx_1 = \cos\left(\frac{k\pi}{b_1}x_1^0\right), \quad k = 0, 1, \ldots,$$

$$\psi^k(x_1^0) = \int_0^{b_1} \delta(x_1 - x_1^0) \sin\left(\frac{k\pi}{b_1}x_1\right) dx_1 = \sin\left(\frac{k\pi}{b_1}x_1^0\right), \quad k = 1, 2, \ldots.$$

Hence, we can write

$$\delta(x_1 - x_1^0) \sim \sum_{k=0}^{K} \cos\left(\frac{k\pi}{b_1}x_1^0\right) \cos\left(\frac{k\pi}{b_1}x_1\right),$$

$$\delta(x_1 - x_1^0) \sim \sum_{k=1}^{K} \sin\left(\frac{k\pi}{b_1}x_1^0\right) \sin\left(\frac{k\pi}{b_1}x_1\right).$$

Similarly,

$$\sum_{n=0}^{K} \cos\left(\frac{n\pi}{b_3}x_3^0\right) \cos\left(\frac{n\pi}{b_3}x_3\right),$$

$$\sum_{n=1}^{K} \sin\left(\frac{n\pi}{b_3}x_3^0\right) \sin\left(\frac{n\pi}{b_3}x_3\right)$$

can be taken as regularization of $\delta(x_3 - x_3^0)$ by means of $\cos\left(\frac{n\pi}{b_3}x_3\right)$, $n = 0, 1, \ldots$ and $\sin\left(\frac{n\pi}{b_3}x_3\right)$, $n = 1, 2, \ldots$, respectively, for $x_3 \in [0, b_3]$.

Example 2. Regularization of the Dirac delta function $\delta(x_2 - x_2^0)$ by means of the functions $Z_1^{kmn}(x_2)$ defined by (5.41), $k = 1, 2, \ldots$, $m, n = 0, 1, \ldots$; $Z_2^{kmn}(x_2)$ defined by (5.40), $k, m, n = 0, 1, \ldots$; $Z_3^{kmn}(x_2)$ defined by (5.42), $k, m = 0, 1, \ldots$, $n = 1, 2, \ldots$, for $x_2 \in [0, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{N-1}, b_2]$.

Let $\psi(x_2, x_2^0) = \delta(x_2 - x_2^0)$, and K be fixed natural number. Then the function

$$\psi^K(x_2, x_2^0) = \sum_{m=0}^{K} \psi^m(x^0) Z_2^{kmn}(x_2), \tag{5.46}$$

can be taken as a regularization of $\delta(x_2 - x_2^0)$ by means of $Z_2^{kmn}(x_2)$ for fixed k and fixed n, $k, m, n = 0, 1, \ldots$, The coefficients $\psi^m(x^0)$ can be found by the formula

$$\begin{aligned}
\psi^m(x_2^0) &= \int_0^{b_2} \nu(x_2)\delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2 \\
&= \int_0^{r_1} \epsilon_1(\mu_1)^2 \delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2 + \ldots + \int_{r_{N-1}}^{b_2} \epsilon_N(\mu_N)^2 \delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2 \\
&= \nu(x_2^0) Z_2^{kmn}(x_2^0),
\end{aligned}$$

with $\nu(x_2^0) = \left[\mu(x_2^0)\right]^2 \epsilon(x_2^0)$, for $x_2^0 \in [0, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{N-1}, b_2]$.
For example, if we choose $x_2^0 \in (r_1, r_2)$ then the formula above takes the form

$$\begin{aligned}
\psi^m(x_2^0) &= \underbrace{\int_0^{r_1} \epsilon_1(\mu_1)^2 \delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2}_{=0} + \int_{r_1}^{r_2} \epsilon_2(\mu_2)^2 \delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2 \\
&\quad + \underbrace{\ldots}_{=0} + \underbrace{\int_{r_{N-1}}^{b_2} \epsilon_N(\mu_N)^2 \delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2}_{=0} \\
&= \int_{r_1}^{r_2} \epsilon_2(\mu_2)^2 \delta(x_2 - x_2^0) Z_2^{kmn}(x_2)\, dx_2 \\
&= \epsilon_2(\mu_2)^2 Z_2^{kmn}(x_2^0) \\
&= \nu(x_2^0) Z_2^{kmn}(x_2^0).
\end{aligned}$$

Hence, we can write

$$\delta(x_2 - x_2^0) \sim \sum_{m=0}^{K} \nu(x_2^0) Z_2^{kmn}(x_2^0) Z_2^{kmn}(x_2),$$

for fixed k and fixed n; $x_2 \in [0, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{N-1}, b_2]$, $x_2^0 \in [0, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{N-1}, b_2]$.

Similarly,

$$\sum_{m=0}^{K} \nu(x_2^0) Z_1^{kmn}(x_2^0) Z_1^{kmn}(x_2),$$
$$\sum_{m=0}^{K} \nu(x_2^0) Z_3^{kmn}(x_2^0) Z_3^{kmn}(x_2)$$

can be taken as regularization of $\delta(x_2 - x_2^0)$ by means of the functions $Z_1^{kmn}(x_2)$ and $Z_3^{kmn}(x_2)$, respectively, for $x_2 \in [0, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{N-1}, b_2]$, $x_2^0 \in [0, r_1) \cup \ldots \cup (r_{N-1}, b_2]$. For these cases the piecewise constant function $\nu(x_2^0)$ need to be taken as $\nu(x_2^0) = \mu(x_2^0)$.

Regularization of the 2D Dirac Delta Function

Here we find regularization of the 2D Dirac delta functions $\delta(x_1 - x_1^0)\delta(x_2 - x_2^0)$, $\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$ by means of the functions $\sin\left(\frac{k\pi}{b_1} x_1\right)\cos\left(\frac{n\pi}{b_2} x_2\right)$, $\sin\left(\frac{k\pi}{b_1} x_1\right) Z_2^{kmn}(x_2)$, $\cos\left(\frac{k\pi}{b_1} x_1\right) Z_2^{kmn}(x_2)$. Using a similar way that is used in the regularization of the 1D Dirac delta function we can find

$$\delta(x_1 - x_1^0)\delta(x_2 - x_2^0) \sim \sum_{k=1}^{K} \sum_{n=0}^{K} \sin\left(\frac{k\pi}{b_1} x_1^0\right) \cos\left(\frac{n\pi}{b_2} x_2^0\right) \sin\left(\frac{k\pi}{b_1} x_1\right) \cos\left(\frac{n\pi}{b_2} x_2\right),$$

$$\delta(x_1 - x_1^0)\delta(x_2 - x_2^0) \sim \sum_{k=1}^{K} \sum_{m=0}^{K} \nu(x_2^0) \sin\left(\frac{k\pi}{b_1} x_1^0\right) Z_2^{kmn}(x_2^0) \sin\left(\frac{k\pi}{b_1} x_1\right) Z_2^{kmn}(x_2),$$

$$\delta(x_2 - x_2^0)\delta(x_3 - x_3^0) \sim \sum_{k=0}^{K} \sum_{m=0}^{K} \nu(x_2^0) \cos\left(\frac{k\pi}{b_3} x_3^0\right) Z_2^{kmn}(x_2^0) \cos\left(\frac{k\pi}{b_3} x_3\right) Z_2^{kmn}(x_2).$$

These double series with finite number of terms can be taken as regularizations of the 2D Dirac delta functions. Here K is a fixed natural number (parameter of regularization).

Regularization of the 3D Dirac Delta Function

Let us consider the 3D Dirac delta function $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$ concentrated at $x_1 = x_1^0$, $x_2 = x_2^0$, $x_3 = x_3^0$. Generalizing the way that is used for

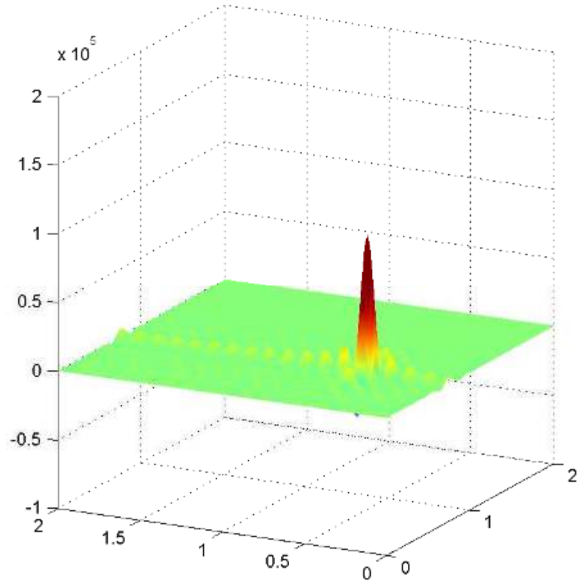

Figure 5.1 The 3D surface plot for the approximate function of $\delta(x - x^0)$ with $K = 20$.

regularizing (approximating) the 1D and 2D Dirac delta functions, we can make regularization of the function $\delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_2 - x_2^0)$ by means of the function $U_2^{kmn}(x)$ defined in (5.44), $k, m, n = 0, 1, \ldots,$ as follows

$$\delta(x - x^0) \sim \sum_{k=0}^{K} \sum_{m=0}^{K} \sum_{n=0}^{K} v(x_2^0) U_2^{kmn}(x^0) U_2^{kmn}(x), \tag{5.47}$$

here K is a fixed natural number (parameter of regularization).

Let us do a computational experiment to show the applicability of this approximation and the selection of the parameters of the regularization. For computation we consider the eigenfunctions $U_2^{kmn}(x)$, $k, m, n = 0, 1, \ldots,$ which are solutions of the problem (5.8)-(5.13). Using these eigenfunctions the values of the approximate function of $\delta(x - x^0)$ has been computed with different K by the formula (5.47) for the case $N = 2$ (two layered case) with

$$\mu_1 = 2, \mu_2 = 6, \epsilon_1 = 4, \epsilon_2 = 8.$$

The result of the computation for two layered case where $(x_1^0, x_2^0, x_3^0) = (0.5, 0.75, 0.75)$ and $(x_1, x_2, x_3) \in [0, 2] \times [0, 1.25) \cup (1.25, 2] \times [0, 1]$ is presented in Figures 5.1 and 5.2.

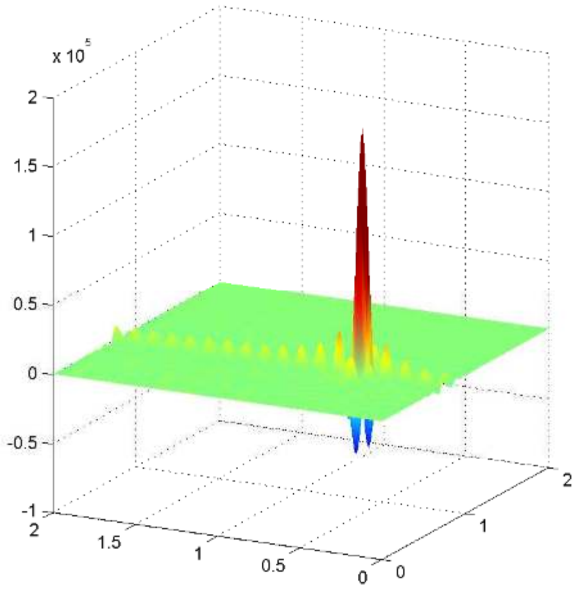

Figure 5.2 The 3D surface plot for the approximate function of $\delta(x - x^0)$ with $K = 36$.

The result of analysis of Figure 5.1 and Figure 5.2 indicate that the choice $K = 36$ is better than the choice $K = 20$, because the values of the approximate function for $K = 36$ are small or vanishing outside the small neighbourhood of the point $(x_1^0, x_2^0, x_3^0) = (0.5, 0.75, 0.75)$ but inside this neighbourhood the values of the approximate function are large. Therefore, the approximate function defined by (5.47) with the chosen parameter K can be taken as the approximation of the Dirac delta function $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$.

5.3 Summary of Chapter Five

A procedure for construction of the eigenvalues and corresponding eigenfunctions of the scalar differential operator $c^2(x_2)\Delta_x$ in a multi-layered rectangular parallelepiped with perfect conducting boundary, has been done. Here $c(x_2)$ is a function of one variable x_2 with piecewise constant values. It has been shown that eigenvalues are roots of a characteristic equation of a coefficient matrix. The eigenvalues can be computed by using symbolic calculation tools such as MATLAB. The orthogonality property of the eigenfunctions has been proved. The regularization of 1D, 2D and 3D Dirac delta functions and their derivatives have been done. The results of this chapter will be used in Chapter 6.

CHAPTER SIX
THE APPROXIMATE COMPUTATION OF TIME DEPENDENT MAGNETIC AND ELECTRIC MATRIX GREEN'S FUNCTIONS IN LAYERED RECTANGULAR PARALLELEPIPED

A method for an approximate computation of columns of time-dependent magnetic and electric Green's functions in multi-layered rectangular parallelepiped with the perfect conducting boundary is described. This method is a generalization of the method described in Chapter 3. We show that the adaption of the Fourier series meta-approach can be done for the approximate computation of columns of time-dependent magnetic Green's function in multi-layered parallelepiped and then electric Green's function can be computed approximately.

6.1 Statement of the Problem

Let b_1, b_2, b_3, r_i, $i = 1,2,...,N$ be positive reals with $0 = r_0 < r_1 < ... < r_N = b_2$, $x \in \mathbf{R}^3$;

$$D = \left\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad 0 < x_2 < b_2, \quad 0 < x_3 < b_3\right\}$$

be a rectangular parallelepiped, Γ be the boundary of D; and D_i, $i = 1,2,...,N$ be the parallelepipeds

$$D_i = \left\{x \in \mathbf{R}^3 : \quad 0 < x_1 < b_1, \quad r_{i-1} < x_2 < r_i, \quad 0 < x_3 < b_3\right\};$$

$D^0 = \bigcup\limits_{i=1}^{N} D_i$.

Let μ_i, ϵ_i, $i = 1,2,\ldots,N$ are given positive numbers; $\mu(x_2)$, $\epsilon(x_2)$ be two piecewise constant functions defined on $[0,b_2]$ such that

$$\mu(x_2) = \mu_i, \quad \epsilon(x_2) = \epsilon_i, \text{ for } x_2 \in (r_{i-1}, r_i), \; i = 1,2,\ldots,N.$$

Find the 3×3 matrices

$$\mathbf{G}^H(x,t,x^0) = \begin{pmatrix} H_1^1(x,t,x^0) & H_1^2(x,t,x^0) & H_1^3(x,t,x^0) \\ H_2^1(x,t,x^0) & H_2^2(x,t,x^0) & H_2^3(x,t,x^0) \\ H_3^1(x,t,x^0) & H_3^2(x,t,x^0) & H_3^3(x,t,x^0) \end{pmatrix},$$

$$\mathbf{G}^E(x,t,x^0) = \begin{pmatrix} E_1^1(x,t,x^0) & E_1^2(x,t,x^0) & E_1^3(x,t,x^0) \\ E_2^1(x,t,x^0) & E_2^2(x,t,x^0) & E_2^3(x,t,x^0) \\ E_3^1(x,t,x^0) & E_3^2(x,t,x^0) & E_3^3(x,t,x^0) \end{pmatrix},$$

whose columns $\mathbf{H}^s(x,t,x^0)$, $\mathbf{E}^s(x,t,x^0)$ satisfy

$$curl_x \mathbf{H}^s = \epsilon(x_2)\frac{\partial \mathbf{E}^s}{\partial t} + \delta(x - x^0)\delta(t)\vec{\mathbf{e}^s}, \;\; x \in D^0, \;\; t \in \mathbf{R}^+, \tag{6.1}$$

$$curl_x \mathbf{E}^s = -\mu(x_2)\frac{\partial \mathbf{H}^s}{\partial t}, \tag{6.2}$$

initial data

$$\mathbf{H}^s\big|_{t<0} = 0, \tag{6.3}$$

$$\mathbf{E}^s\big|_{t<0} = 0, \tag{6.4}$$

perfect conducting boundary conditions

$$(\mathbf{E}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \;\; (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0, \tag{6.5}$$

and matching (interface) conditions

$$\left(\mathbf{H}^s\big|_{x_2=r_i-0} - \mathbf{H}^s\big|_{x_2=r_i+0}\right) \times \vec{\nu} = 0, \quad \left(\mu_i\, \mathbf{H}^s\big|_{x_2=r_i-0} - \mu_{i+1}\, \mathbf{H}^s\big|_{x_2=r_i+0}\right) \cdot \vec{\nu} = 0, \tag{6.6}$$

$$\left(\mathbf{E}^s\big|_{x_2=r_i-0} - \mathbf{E}^s\big|_{x_2=r_i+0}\right) \times \vec{\nu} = 0, \quad \left(\epsilon_i\, \mathbf{E}^s\big|_{x_2=r_i-0} - \epsilon_{i+1}\, \mathbf{E}^s\big|_{x_2=r_i+0}\right) \cdot \vec{\nu} = 0. \tag{6.7}$$

Here $\vec{\mathbf{e}^s}$ are the basis vectors from $\mathbf{R}^3$; $x^0 = (x_1^0, x_2^0, x_3^0)^T \in D^0$, t is the time variable; $\vec{\mathbf{n}}$ is an outward unit normal vector on Γ; $\vec{\nu} = (0,1,0)$. The matrices $\mathbf{G}^H(x,t,x^0)$, $\mathbf{G}^E(x,t,x^0)$ are called time-dependent magnetic and electric matrix Green's functions for $x \in D^0$, $t \in \mathbf{R}^+$.

Remark 6.1.1. $div_x \mathbf{H}^s = 0$, $x \in D^0$ follows from (6.2), (6.3). This is the necessary condition that $\mathbf{H}^s$ satisfies (6.1)-(6.7) for $x \in D^0$.

6.1.1 Equivalent Form of Equations (6.1)-(6.7)

Applying $curl_x$ to (6.1) and using (6.2) we find

$$\frac{\partial^2 \mathbf{H}^s}{\partial t^2} + c^2(x_2) curl_x curl_x \mathbf{H}^s = c^2(x_2) curl_x \left[\delta(x - x^0)\vec{\mathbf{e}^s}\right]\delta(t), \quad x \in D^0, \ t \in \mathbf{R}.$$

It follows from (6.2)-(6.4) that

$$\mathbf{H}^s\Big|_{t<0} = 0, \quad \frac{\partial \mathbf{H}^s}{\partial t}\Big|_{t<0} = 0.$$

Taking the cross product of both sides of the equation (6.1) and $\vec{\mathbf{n}}$, and using the conditions in (6.5) we get

$$\left(curl_x \mathbf{H}^s \times \vec{\mathbf{n}}\right)\Big|_\Gamma = \epsilon(x_2)\frac{\partial}{\partial t}\left(\mathbf{E}^s \times \vec{\mathbf{n}}\right)\Big|_\Gamma + \left(\delta(x - x^0)\vec{\mathbf{e}^s}\delta(t) \times \vec{\mathbf{n}}\right)\Big|_\Gamma = 0.$$

Let us divide both sides of (6.1) by $\epsilon(x_2)$, and consider the new equation for each layer separately such that

$$\frac{1}{\epsilon_i}\, curl_x \mathbf{H}^s\Big|_{x_2=r_i-0} = \frac{\partial \mathbf{E}^s}{\partial t}\Big|_{x_2=r_i-0} + \frac{1}{\epsilon_i}\delta(x - x^0)\Big|_{x_2=r_i-0}\delta(t)\vec{\mathbf{e}^s}, \tag{6.8}$$

$$\frac{1}{\epsilon_{i+1}}\, curl_x \mathbf{H}^s\Big|_{x_2=r_i+0} = \frac{\partial \mathbf{E}^s}{\partial t}\Big|_{x_2=r_i+0} + \frac{1}{\epsilon_{i+1}}\delta(x - x^0)\Big|_{x_2=r_i+0}\delta(t)\vec{\mathbf{e}^s}, \tag{6.9}$$

subtracting (6.9) from (6.8) and then taking the cross product of both sides of the new equation and $\vec{\nu}$ and using (6.7) we obtain

$$\left(\frac{1}{\epsilon_i}\, curl_x \mathbf{H}^s\Big|_{x_2=r_i-0} - \frac{1}{\epsilon_{i+1}}\, curl_x \mathbf{H}^s\Big|_{x_2=r_i+0}\right) \times \vec{\nu} = 0.$$

By writing the equation (6.1) for each layer separately and making subtraction each equation from the consecutive one and taking dot product of both sides of these equations and ν, using the second condition in (6.7) we obtain

$$\left(curl_x \mathbf{H}^s\Big|_{x_2=r_i-0} - curl_x \mathbf{H}^s\Big|_{x_2=r_i+0}\right) \cdot \vec{\nu} = 0.$$

Let $\mathcal{E}^s$ be defined as

$$\mathcal{E}^s = \mathbf{E}^s + \frac{1}{\epsilon(x_2)}\delta(x - x^0)\Theta(t)\vec{\mathbf{e}^s},$$

differentiating both sides of this equation with respect to t we obtain

$$\frac{\partial \mathcal{E}^s}{\partial t} = \frac{1}{\epsilon(x_2)} curl_x \mathbf{H}^s, \quad x \in D^0, \ t \in \mathbf{R}.$$

Moreover, for $t < 0$ the equation of $\mathcal{E}^s$ gives

$$\mathcal{E}^s\Big|_{t<0} = \mathbf{0}.$$

6.1.2 Equations for Time-dependent Magnetic Green's Function

For finding columns of the magnetic Green's function we use the following equalities

$$\frac{\partial^2 \mathbf{H}^s}{\partial t^2} + c^2(x_2) curl_x curl_x \mathbf{H}^s = c^2(x_2) curl_x \left[\delta(x - x^0)\vec{\mathbf{e}^s}\right]\delta(t), \quad x \in D^0,\ t \in \mathbf{R}, \tag{6.10}$$

$$\mathbf{H}^s\big|_{t<0} = 0, \quad \frac{\partial \mathbf{H}^s}{\partial t}\bigg|_{t<0} = 0, \tag{6.11}$$

$$(curl_x \mathbf{H}^s \times \vec{\mathbf{n}})\big|_\Gamma = 0, \quad (\mathbf{H}^s \cdot \vec{\mathbf{n}})\big|_\Gamma = 0, \tag{6.12}$$

$$\left(\mathbf{H}^s\big|_{x_2=r_i-0} - \mathbf{H}^s\big|_{x_2=r_i+0}\right)\times\vec{\nu} = 0,\ \left(\mu_i \mathbf{H}^s\big|_{x_2=r_i-0} - \mu_{i+1}\mathbf{H}^s\big|_{x_2=r_i+0}\right)\cdot\vec{\nu} = 0, \tag{6.13}$$

$$\left(\frac{1}{\epsilon_i}\, curl_x \mathbf{H}^s\big|_{x_2=r_i-0} - \frac{1}{\epsilon_{i+1}}\, curl_x \mathbf{H}^s\big|_{x_2=r_i+0}\right)\times\vec{\nu} = 0, \tag{6.14}$$

$$\left(curl_x\, \mathbf{H}^s\big|_{x_2=r_i-0} - curl_x\, \mathbf{H}^s\big|_{x_2=r_i+0}\right)\cdot\vec{\nu} = 0, \tag{6.15}$$

where $s = 1,2,3;\ i = 1,2,...,N-1$.

Remark 6.1.2. $div_x \mathbf{H}^s = 0$, $x \in D^0$ follows from (6.10)-(6.11) and this is the necessary condition for (6.10)-(6.15).

6.1.3 Equations for the Time-dependent Electric Green's Function

Let $\mathbf{H}^s(x, t, x^0)$ be the s-column of the magnetic Green's function, which we assume to be computed.

$$\frac{\partial \mathcal{E}^s}{\partial t} = \frac{1}{\epsilon(x_2)} curl_x \mathbf{H}^s, \quad x \in D^0,\ t \in \mathbf{R}, \tag{6.16}$$

$$\mathcal{E}^s\big|_{t<0} = 0, \tag{6.17}$$

$$\mathcal{E}^s = \mathbf{E}^s + \frac{1}{\epsilon(x_2)}\delta(x - x^0)\Theta(t)\vec{\mathbf{e}^s}, \tag{6.18}$$

where $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t < 0$, that is, $\Theta(t)$ is the Heaviside step function.

(6.16)-(6.18) will be used for the computation of the time-dependent electric Green's function if the magnetic Green's function is known.

Remark 6.1.3. If $\mathbf{H}^s$, $\mathcal{E}^s$ satisfy (6.10)-(6.15) and (6.16)-(6.18) then $\mathbf{H}^s$ and $\mathbf{E}^s$ also satisfy equations (6.1)-(6.7).

Proof. Differentiating both sides of (6.18) with respect to t and using (6.16) we get (6.1). Checking $curl_x\mathbf{H}^s$ from this equation and substituting into (6.10) we have $\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{H}^s}{\partial t} + \frac{1}{\mu(x_2)}curl_x\mathbf{E}^s\right) = 0$. Integrating this equality with respect to t from $-\infty$ to t and using (6.11),(6.17) we find that $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (6.2). Equation (6.3) follows from (6.11). (6.4) follows from (6.17),(6.18). Taking the cross product of (6.16) and $\vec{\mathbf{n}}$, letting $x \in \Gamma$ and using (6.12) we find $\frac{\partial}{\partial t}(\mathcal{E}^s \times \vec{\mathbf{n}})|_\Gamma = 0$. Integrating the obtained equality with respect to t from $-\infty$ to t and using (6.17) we obtain $(\mathcal{E}^s \times \vec{\mathbf{n}})|_\Gamma = 0$. Taking the cross product of (6.18) and $\vec{\mathbf{n}}$, letting $x \in \Gamma$ and using $(\mathcal{E}^s \times \vec{\mathbf{n}})|_\Gamma = 0$ we have the first condition of (6.5). (6.6) follows from (6.13). Using (6.16),(6.18) we have found (6.1). Now, considering this equation for each layer separately and making subtraction and taking the cross product both sides of the new equation and $\vec{\nu}$, and using (6.14) we find $\frac{\partial}{\partial t}\left(\mathbf{E}^s|_{x_2=r_i-0} - \mathbf{E}^s|_{x_2=r_i+0}\right) \times \vec{\nu} = 0$. Integrating the obtained equality with respect to t from $-\infty$ to t and using (6.4) (which we have found) we get the first condition of (6.7). Similarly, considering (6.1) for each layer and making subtraction and taking the dot product both sides of the new equation and $\vec{\nu}$, and using (6.15) and integrating the obtained equality with respect to t from $-\infty$ to t and using (6.4) (which we have found) we get the second condition of (6.7).

Therefore we obtain that if $\mathbf{H}^s$, $\mathcal{E}^s$ satisfy (6.10)-(6.15) and (6.16)-(6.18) then $\mathbf{H}^s$ and $\mathbf{E}^s$ also satisfy equations (6.1)-(6.7), that is, $\mathbf{H}^s$ and $\mathbf{E}^s$ are s-columns of time-dependent magnetic and electric Green's functions. □

6.2 Computation of the Magnetic Green's Function

In this section we find solution of the problem (6.10)-(6.15) for each $s = 1,2,3$, that is, the column elements of the magnetic Green's function. For solving we use the following steps. Firstly, using the equations (6.10)-(6.15), we write equations for finding each element of the magnetic Green's function, $H_j^s(x,t,x^0)$, separately. Secondly, we solve the problem for $H_2^s(x,t,x^0)$, $s = 1,2,3$, and compute the approximate solution $H_2^{s,K}(x,t,x^0)$ for fixed K, in the form of a series with a finite number of terms. Thirdly, using the explicit formulas for $H_2^{s,K}$, we write the equations for $H_1^{s,K}$, $H_3^{s,K}$

and solve each problem separately. As a result, we find each element $H_j^{s,K}(x,t,x^0)$, $j = 1,2,3$ of the magnetic Green's function explicitly in the form of series with a finite number of terms.

Now, let us consider the equations (6.10)-(6.15). Then the equations for $H_1^s(x,t,x^0)$, $H_2^s(x,t,x^0)$, and $H_3^s(x,t,x^0)$, $s = 1,2,3$, follow as

$$\frac{\partial^2 H_1^s}{\partial t^2} - c^2(x_2)\Delta_x H_1^s = c^2(x_2)\left\{curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]\right\}_1 \delta(t),\ \ x \in D^0,\ \ t \in \mathbf{R}^+, \tag{6.19}$$

$$H_1^s\big|_{t<0} = 0,\ \ \frac{\partial H_1^s}{\partial t}\bigg|_{t<0} = 0, \tag{6.20}$$

$$H_1^s\big|_{x_1=0} = 0, \quad H_1^s\big|_{x_1=b_1} = 0, \tag{6.21}$$

$$\frac{\partial H_1^s}{\partial x_2}\bigg|_{x_2=0} = 0, \quad \frac{\partial H_1^s}{\partial x_2}\bigg|_{x_2=b_2} = 0, \tag{6.22}$$

$$\frac{\partial H_1^s}{\partial x_3}\bigg|_{x_3=0} = 0, \quad \frac{\partial H_1^s}{\partial x_3}\bigg|_{x_3=b_3} = 0, \tag{6.23}$$

$$H_1^s\big|_{x_2=r_i-0} = H_1^s\big|_{x_2=r_i+0}, \tag{6.24}$$

$$\frac{\partial H_1^s}{\partial x_2}\bigg|_{x_2=r_i-0} = \frac{\epsilon_i}{\epsilon_{i+1}}\frac{\partial H_1^s}{\partial x_2}\bigg|_{x_2=r_i+0} + \frac{\partial H_2^s}{\partial x_1}\bigg|_{x_2=r_i-0} - \frac{\epsilon_i}{\epsilon_{i+1}}\frac{\partial H_2^s}{\partial x_1}\bigg|_{x_2=r_i+0}, \tag{6.25}$$

$$\frac{\partial^2 H_2^s}{\partial t^2} - c^2(x_2)\Delta_x H_2^s = c^2(x_2)\left\{curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]\right\}_2 \delta(t),\ \ x \in D^0,\ \ t \in \mathbf{R}^+, \tag{6.26}$$

$$H_2^s\big|_{t<0} = 0,\ \ \frac{\partial H_2^s}{\partial t}\bigg|_{t<0} = 0, \tag{6.27}$$

$$\frac{\partial H_2^s}{\partial x_1}\bigg|_{x_1=0} = 0, \quad \frac{\partial H_2^s}{\partial x_1}\bigg|_{x_1=b_1} = 0, \tag{6.28}$$

$$H_2^s\big|_{x_2=0} = 0, \quad H_2^s\big|_{x_2=b_2} = 0, \tag{6.29}$$

$$\frac{\partial H_2^s}{\partial x_3}\bigg|_{x_3=0} = 0, \quad \frac{\partial H_2^s}{\partial x_3}\bigg|_{x_3=b_3} = 0, \tag{6.30}$$

$$\mu_i H_2^s\big|_{x_2=r_i-0} = \mu_{i+1} H_2^s\big|_{x_2=r_i+0}, \tag{6.31}$$

$$\frac{\partial H_2^s}{\partial x_2}\bigg|_{x_2=r_i-0} = \frac{\partial H_2^s}{\partial x_2}\bigg|_{x_2=r_i+0}, \tag{6.32}$$

$$\frac{\partial^2 H_3^s}{\partial t^2} - c^2(x_2)\Delta_x H_3^s = c^2(x_2)\left\{curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]\right\}_3 \delta(t), \;\; x \in D^0, \;\; t \in \mathbf{R}^+, \tag{6.33}$$

$$H_3^s\big|_{t<0} = 0, \;\; \frac{\partial H_1^s}{\partial t}\bigg|_{t<0} = 0, \tag{6.34}$$

$$\frac{\partial H_3^s}{\partial x_1}\bigg|_{x_1=0} = 0, \quad \frac{\partial H_3^s}{\partial x_1}\bigg|_{x_1=b_1} = 0, \tag{6.35}$$

$$\frac{\partial H_3^s}{\partial x_2}\bigg|_{x_2=0} = 0, \quad \frac{\partial H_3^s}{\partial x_2}\bigg|_{x_2=b_2} = 0, \tag{6.36}$$

$$H_3^s\big|_{x_3=0} = 0, \quad H_3^s\big|_{x_3=b_3} = 0, \tag{6.37}$$

$$H_3^s\big|_{x_2=r_i-0} = H_3^s\big|_{x_2=r_i+0}, \tag{6.38}$$

$$\frac{\partial H_3^s}{\partial x_2}\bigg|_{x_2=r_i-0} = \frac{\epsilon_i}{\epsilon_{i+1}}\frac{\partial H_3^s}{\partial x_2}\bigg|_{x_2=r_i+0} + \frac{\partial H_2^s}{\partial x_3}\bigg|_{x_2=r_i-0} - \frac{\epsilon_i}{\epsilon_{i+1}}\frac{\partial H_2^s}{\partial x_3}\bigg|_{x_2=r_i+0}, \tag{6.39}$$

where $s = 1,2,3$, $i = 1,2,\ldots,N$. Here $c(x_2)$ is a piecewise function as in (5.1).

6.2.1 Approximate Computation of $H_2^s(x,t,x^0)$, $s = 1,2,3$

In this section, we compute the second elements of the column vectors of the magnetic Green's function, $H_2^s(x,t,x^0)$, $s = 1,2,3$, using (6.26)-(6.32). For an approximate solution of this problem we use the following steps:

Eigenvalue-Eigenfunction Problem Solving:

Let us consider the problem of finding all values of λ (eigenvalues) for which there exists a nonzero scalar function $U_2(x)$, satisfying

$$\Delta_x U_2(x) + \frac{1}{c^2(x_2)}\lambda U_2(x) = 0, \quad x \in D^0,$$

$$\frac{\partial U_2}{\partial x_1}\bigg|_{x_1=0} = 0, \quad \frac{\partial U_2}{\partial x_1}\bigg|_{x_1=b_1} = 0,$$

$$U_2(x)|_{x_2=0} = 0, \quad U_2(x)|_{x_2=b_2} = 0,$$

$$\frac{\partial U_2}{\partial x_3}\bigg|_{x_3=0} = 0, \quad \frac{\partial U_2}{\partial x_3}\bigg|_{x_3=b_3} = 0,$$

$$
\begin{aligned}
U_2|_{x_2=r_i-0} &= \frac{\mu_{i+1}}{\mu_i}\, U_2|_{x_2=r_i+0}\,,\\
\left.\frac{\partial U_2}{\partial x_2}\right|_{x_2=r_i+0} &= \left.\frac{\partial U_2}{\partial x_2}\right|_{x_2=r_i-0},
\end{aligned}
$$

$i = 1,2,\ldots,N-1$.

Construction of the eigenvalues ($\lambda = \lambda_{kmn}$) and the corresponding eigenfunctions ($U_2^{kmn}(x)$) of the problem above was described in Chapter 5. The eigenfunctions given in (5.44) are of the form

$$
U_2^{kmn}(x_1,x_2,x_3) = W_2^{kmn}(x_1,x_3)Z_2^{kmn}(x_2),
$$

for $x_1 \in [0,b_1]$, $x_2 \in [0,r_1) \cup ... \cup (r_{N-1},b_2]$, $x_3 \in [0,b_3]$, $k = 0,1,\ldots$, $m = 0,1,\ldots$, $n = 0,1,\ldots$. For further discussion, see Section 5.1.

Approximate Solution of (6.26)-(6.32):

Let us consider the problem of finding approximate solution of (6.26)-(6.32). Firstly, we replace the terms $\left\{curl_x\left[\delta(x-x^0)\mathbf{e}^{\vec{s}}\right]\right\}_2 = -\left\{curl_{x^0}\left[\delta(x-x^0)\mathbf{e}^{\vec{s}}\right]\right\}_2$ in (6.26) for each $s = 1,2,3$, by

$$
\begin{aligned}
\psi_2^{1,K}(x,x^0) &= \sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} \nu(x_2^0)Z_2^{kmn}(x_2^0)\frac{\partial W_2^{kn}(x_1^0,x_3^0)}{\partial x_3^0}Z_2^{kmn}(x_2)W_2^{kn}(x_1,x_3)\\
&= \sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} \left[\mu(x_2^0)\right]^2 \epsilon(x_2^0)\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}U_2^{kmn}(x),\\
&\quad \psi_2^{2,K}(x,x^0) = 0,\\
\psi_2^{3,K}(x,x^0) &= -\sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} \nu(x_2^0)Z_2^{kmn}(x_2^0)\frac{\partial W_2^{kn}(x_1^0,x_3^0)}{\partial x_1^0}Z_2^{kmn}(x_2)W_2^{kn}(x_1,x_3)\\
&= \sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} \left[\mu(x_2^0)\right]^2 \epsilon(x_2^0)\frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0}U_2^{kmn}(x).
\end{aligned}
$$

The equations above can be taken as the regularization of the generalized functions $\left\{curl_x\left[\delta(x-x^0)\mathbf{e}^{\vec{s}}\right]\right\}_2$, $s = 1,2,3$. Here K is a fixed natural number. Throughout Chapter 6, K denotes the parameter of regularization. For details, see Section 5.2.

Moreover, equations (6.26)-(6.32) are replaced by the following equations

$$\frac{\partial^2 H_2^{s,K}}{\partial t^2} - c^2(x_2)\Delta_x H_2^{s,K} = -c^2(x_2)\psi_2^{s,K}(x,x^0)\delta(t), \tag{6.40}$$

$$H_2^{s,K}\big|_{t<0} = 0, \quad \frac{\partial H_2^{s,K}}{\partial t}\bigg|_{t<0} = 0, \tag{6.41}$$

$$\frac{\partial H_2^{s,K}}{\partial x_1}\bigg|_{x_1=0} = 0, \quad \frac{\partial H_2^{s,K}}{\partial x_1}\bigg|_{x_1=b_1} = 0, \tag{6.42}$$

$$H_2^{s,K}\big|_{x_2=0} = 0, \quad H_2^{s,K}\big|_{x_2=b_2} = 0, \tag{6.43}$$

$$\frac{\partial H_2^{s,K}}{\partial x_3}\bigg|_{x_3=0} = 0, \quad \frac{\partial H_2^{s,K}}{\partial x_3}\bigg|_{x_3=b_3} = 0, \tag{6.44}$$

$$\mu_i\, H_2^{s,K}\,|_{x_2=r_i-0} = \mu_{i+1}\, H_2^{s,K}\big|_{x_2=r_i+0}, \tag{6.45}$$

$$\frac{\partial H_2^{s,K}}{\partial x_2}\bigg|_{x_2=r_i-0} = \frac{\partial H_2^{s,K}}{\partial x_2}\bigg|_{x_2=r_i+0}, \tag{6.46}$$

for $s = 1,2,3, \quad i = 1,2,\ldots,N-1,\ x \in D^0,\ t \in \mathbf{R}^+$.

Then we find solution of (6.40)-(6.46) for $x \in D^0$, $t \in \mathbf{R}^+$ in the form

$$H_2^{s,K}(x,t,x^0) = \sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} T_2^{s,kmn}(t,x^0)U_2^{kmn}(x), \tag{6.47}$$

Here $T_2^{s,kmn}(t,x^0)$ are unknown functions and $U_2^{kmn}(x)$ are eigenfunctions of the eigenvalue-eigenfunction problem stated above.

By substituting (6.47) into (6.40), (6.41) and using the orthogonality of the set of functions $U_2^{kmn}(x)$, we obtain the following initial value problems of $T_2^{s,kmn}(t,x^0)$:

$$\frac{d^2}{dt^2}T_2^{s,kmn}(t,x^0) + \lambda_{kmn}T_2^{s,kmn}(t,x^0) = -c^2(x_2)\psi_2^{s,kmn}(x^0)\delta(t),\ t \in \mathbf{R}^+, \tag{6.48}$$

$$T_2^{s,kmn}(0,x^0) = 0,\ \frac{d}{dt}T_2^{s,kmn}(0,x^0) = 0, \tag{6.49}$$

for $s = 1,2,3;\ k = 0,1,...;\ m = 0,1,...;\ n = 0,1,....$

By solving them for each $s = 1,2,3$, we get the explicit formula for the solutions of (6.48)-(6.49) in the form:

$$T_2^{s,kmn}(t,x^0) = -c^2(x_2)\frac{\sin\left(\sqrt{\lambda_{kmn}}t\right)}{\sqrt{\lambda_{kmn}}}\psi_2^{s,kmn}(x^0),\ x^0 \in D^0,\ t \in \mathbf{R}^+. \tag{6.50}$$

Using the formula

$$\psi_2^{s,kmn}(x^0) = \int_0^{b_1}\int_0^{b_3}\left[\sum_{i=1}^{N}\int_{r_{i-1}}^{r_i}[\mu_i]^2\epsilon_i\psi_2^s(x,x^0)U_2^{kmn}(x)\,dx_2\right]dx_3\,dx_1,\ x^0 \in D^0, \tag{6.51}$$

we find the values of $\psi_2^{s,kmn}(x^0)$, for each $s = 1,2,3$, as:

$$\psi_2^{1,kmn}(x^0) = [\mu_i]^2 \epsilon_i \frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0}, \quad \psi_2^{2,kmn}(x^0) = 0, \quad \psi_2^{3,kmn}(x^0) = -[\mu_i]^2 \epsilon_i \frac{\partial U_2^{kmn}(x^0)}{\partial x_1^0},$$

$i = 1,2,\ldots,N$.

Finally, substituting the formula (6.50) for $T_2^{s,kmn}(t,x^0)$ into (6.47) we obtain the approximate solution of (6.40)-(6.46) as

$$H_2^{s,K}(x,t,x^0) = -\sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} c^2(x_2)\frac{\sin(\sqrt{\lambda_{kmn}}t)}{\sqrt{\lambda_{kmn}}}\psi_2^{s,kmn}(x^0)U_2^{kmn}(x), \tag{6.52}$$

$s = 1,2,3$, for $x \in D^0$, $t \in \mathbf{R}^+$; $x^0 \in D^0$.

Remark 6.2.1. Let $H_2^{s,K}(x,t,x^0)$, $s = 1,2,3$, be the approximate solution of (6.26)-(6.32) for a fixed K. We will use the symbols $f_i^{s,K}$ and $g_i^{s,K}$ to denote

$$f_i^{s,K}(x_1,x_3,t,x^0) = \left.\frac{\partial H_2^{s,K}}{\partial x_1}\right|_{x_2=r_i-0} - \frac{\epsilon_i}{\epsilon_{i+1}}\left.\frac{\partial H_2^{s,K}}{\partial x_1}\right|_{x_2=r_i+0}, \tag{6.53}$$

$$g_i^{s,K}(x_1,x_3,t,x^0) = \left.\frac{\partial H_2^{s,K}}{\partial x_3}\right|_{x_2=r_i-0} - \frac{\epsilon_i}{\epsilon_{i+1}}\left.\frac{\partial H_2^{s,K}}{\partial x_3}\right|_{x_2=r_i+0}, \tag{6.54}$$

$x_1 \in (0,b_1)$, $x_3 \in (0,b_3)$, $x^0 \in D^0$; $t \in \mathbf{R}^+$; $i = 1,2\ldots,N-1$.

We note that, if $H_2^{s,K}(x,t,x^0)$ are constructed then $f_i^{s,K}(x_1,x_3,t,x^0)$, $g_i^{s,K}(x_1,x_3,t,x^0)$ are given functions for $s = 1,2,3$; $i = 1,2,\ldots,N-1$.

6.2.2 Approximate Computation of $H_1^s(x,t,x^0)$, $s = 1,2,3$

In this section, we construct the first elements of each column of the magnetic Green's function, that is, $H_1^s(x,t,x^0)$ satisfying the equations (6.19)-(6.25). For finding the approximate solution of (6.19)-(6.25) we follow the way described as follows:

Firstly, the terms $\left\{curl_x\left[\delta(x-x^0)\vec{\mathbf{e}^s}\right]\right\}_1$, $s = 1,2,3$, in (6.19), denoted by $\psi_1^s(x,x^0)$, are replaced by their regularization given as

$$\psi_1^{s,K}(x,x^0) = \sum_{k=1}^{K}\sum_{n=0}^{K}\psi_1^{s,kn}(x_2,x^0)W_1^{kn}(x_1,x_3), \tag{6.55}$$

for fixed natural number K. Here $W_1^{kn}(x_1, x_3)$ are given functions as in (5.21) and $\psi_1^{s,kn}(x_2, x^0)$ are the Fourier coefficients which can be computed by

$$\psi_1^{s,kn}(x_2, x^0) = \int_0^{b_1}\int_0^{b_3} \psi_1^s(x, x^0) W_1^{kn}(x_1, x_3)\, dx_3 dx_1,$$

$k = 1, 2, \ldots; n = 0, 1, \ldots$.

We now consider the term $\left.\frac{\partial H_2^s}{\partial x_1}\right|_{x_2=r_i-0} - \frac{\epsilon_i}{\epsilon_{i+1}} \left.\frac{\partial H_2^s}{\partial x_1}\right|_{x_2=r_i+0}$, $i = 1, 2, \ldots, N-1$ in (6.25). As a regularization of it we can take the function $f_i^{s,K}(x_1, x_3, t, x^0)$, defined in (6.53). Using the set of functions $W_1^{kn}(x_1, x_3)$ we can present $f_i^{s,K}(x_1, x_3, t, x^0)$ in a form similar to that in (6.55) as

$$f_i^{s,K}(x_1, x_3, t, x^0) = \sum_{k=1}^{K}\sum_{n=0}^{K} f_i^{s,kn}(t, x^0) W_1^{kn}(x_1, x_3), \tag{6.56}$$

where

$$f_i^{s,kn}(t, x^0) = \int_0^{b_1}\int_0^{b_3} f_i^{s,K}(x_1, x_3, t, x^0) W_1^{kn}(x_1, x_3)\, dx_3 dx_1,$$

$k = 1, 2, \ldots;\ n = 0, 1, \ldots$.

Thus, we can replace the term mentioned above by (6.56) in (6.25) .

Moreover, we replace the equations (6.19)-(6.25) by

$$\frac{\partial^2 H_1^{s,K}}{\partial t^2} - c^2(x_2)\Delta_x H_1^{s,K} = c^2(x_2)\psi_1^{s,K}(x, x^0)\delta(t),\ \ x \in D^0,\ \ t \in \mathbf{R}^+, \tag{6.57}$$

$$H_1^{s,K}\big|_{t<0} = 0,\ \ \left.\frac{\partial H_1^{s,K}}{\partial t}\right|_{t<0} = 0, \tag{6.58}$$

$$H_1^{s,K}\big|_{x_1=0} = 0, \quad H_1^{s,K}\big|_{x_1=b_1} = 0, \tag{6.59}$$

$$\left.\frac{\partial H_1^{s,K}}{\partial x_2}\right|_{x_2=0} = 0, \quad \left.\frac{\partial H_1^{s,K}}{\partial x_2}\right|_{x_2=b_2} = 0, \tag{6.60}$$

$$\left.\frac{\partial H_1^{s,K}}{\partial x_3}\right|_{x_3=0} = 0, \quad \left.\frac{\partial H_1^{s,K}}{\partial x_3}\right|_{x_3=b_3} = 0, \tag{6.61}$$

$$H_1^{s,K}\big|_{x_2=r_i-0} = H_1^{s,K}\big|_{x_2=r_i+0}, \tag{6.62}$$

$$\left.\frac{\partial H_1^{s,K}}{\partial x_2}\right|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\left.\frac{\partial H_1^{s,K}}{\partial x_2}\right|_{x_2=r_i-0} - \frac{\epsilon_{i+1}}{\epsilon_i} f_i^{s,K}, \tag{6.63}$$

for $s = 1, 2, 3,\ i = 1, 2, \ldots, N-1$.

We look for a solution of (6.57)-(6.63) in the form

$$H_1^{s,K}(x,t,x^0) = \sum_{k=1}^{K}\sum_{n=0}^{K} H_1^{s,kn}(x_2,t,x^0)W_1^{kn}(x_1,x_3), \tag{6.64}$$

here $H_1^{s,kn}(x_2,t,x^0)$ are unknown functions.

Substituting (6.55), (6.56), (6.64) into (6.57)-(6.63) and using the orthogonality property of the set of functions $W_1^{kn}(x_1,x_3)$ we get the equations for finding $H_1^{s,kn}$ for $x \in D^0$, $t \in \mathbf{R}^+$ as

$$\frac{\partial^2 H_1^{s,kn}}{\partial t^2} - c^2(x_2)\left[\frac{\partial^2 H_1^{s,kn}}{\partial x_2^2} - \xi_{kn}^2 H_1^{s,kn}\right] = c^2(x_2)\psi_1^{s,kn}(x_2,x^0)\delta(t), \tag{6.65}$$

$$H_1^{s,kn}\Big|_{t<0} = 0, \quad \frac{\partial H_1^{s,kn}}{\partial t}\Big|_{t<0} = 0, \tag{6.66}$$

$$\frac{\partial H_1^{s,kn}}{\partial x_2}\Big|_{x_2=0} = 0, \quad \frac{\partial H_1^{s,kn}}{\partial x_2}\Big|_{x_2=b_2} = 0, \tag{6.67}$$

$$H_1^{s,kn}\Big|_{x_2=r_i-0} = H_1^{s,kn}\Big|_{x_2=r_i+0}, \tag{6.68}$$

$$\frac{\partial H_1^{s,kn}}{\partial x_2}\Big|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\frac{\partial H_1^{s,kn}}{\partial x_2}\Big|_{x_2=r_i-0} - \frac{\epsilon_{i+1}}{\epsilon_i} f_i^{s,kn}(t,x^0), \tag{6.69}$$

where $\xi_{kn}^2 = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2$, $i = 1,2,\ldots,$ $k = 1,2,\ldots;$ $n = 0,1,\ldots$.

Let $h_1^{s,kn}(x_2,t,x^0) = H_1^{s,kn}(x_2,t,x^0) - v_1^{s,kn}(x_2,t,x^0)$ with $v_1^{s,kn}(x_2,t,x^0)$ for $(x_2,t) \in (r_{i-1},r_i)\times\mathbf{R}^+$, $i = 1,2,\ldots,N$, satisfying

$$\frac{\partial^2 v_1^{s,kn}}{\partial t^2} - c^2(x_2)\frac{\partial^2 v_1^{s,kn}}{\partial x_2^2} = 0, \tag{6.70}$$

$$v_1^{s,kn}\Big|_{t=0} = 0, \quad \frac{\partial v_1^{s,kn}}{\partial t}\Big|_{t=0}, \tag{6.71}$$

$$\frac{\partial v_1^{s,kn}}{\partial x_2}\Big|_{x_2=0} = 0, \quad \frac{\partial v_1^{s,kn}}{\partial x_2}\Big|_{x_2=b_2} = 0, \tag{6.72}$$

$$v_1^{s,kn}\Big|_{x_2=r_i+0} = v_1^{s,kn}\Big|_{x_2=r_i-0}, \tag{6.73}$$

$$\frac{\partial v_1^{s,kn}}{\partial x_2}\Big|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\frac{\partial v_1^{s,kn}}{\partial x_2}\Big|_{x_2=r_i-0} - \frac{\epsilon_{i+1}}{\epsilon_i} f_i^{s,kn}(t,x^0), \tag{6.74}$$

where $s = 1,2,3;$ $i = 1,2,\ldots,N-1;$ $k = 1,2,\ldots,$ $n = 0,1,\ldots$.

Remark 6.2.2. The problem (6.70)-(6.74) is an IBVP for the 1D wave equation with piecewise constant coefficients. The solution of (6.70)-(6.74), that is, the explicit formula for $v_1^{s,kn}(x_2,t,x^0)$, can be found by the method described in Chapter 4.

Checking the term $H_1^{s,kn}(x_2,t,x^0)$ in $h_1^{s,kn}(x_2,t,x^0) = H_1^{s,kn}(x_2,t,x^0) - v_1^{s,kn}(x_2,t,x^0)$ and substituting it into (6.65)-(6.69) we get

$$\frac{\partial^2 h_1^{s,kn}}{\partial t^2} - c^2(x_2)\left[\frac{\partial^2 h_1^{s,kn}}{\partial x_2^2} - \xi_{kn}^2 h_1^{s,kn}\right] = c^2(x_2)\psi_1^{s,kn}(x_2,x^0)\delta(t) - c^2(x_2)\xi_{kn}^2 v_1^{s,kn}, \quad (6.75)$$

$$h_1^{s,kn}\Big|_{t=0} = 0, \quad \frac{\partial h_1^{s,kn}}{\partial t}\Big|_{t=0} = 0, \quad (6.76)$$

$$\frac{\partial h_1^{s,kn}}{\partial x_2}\Big|_{x_2=0} = 0, \quad \frac{\partial h_1^{s,kn}}{\partial x_2}\Big|_{x_2=b_2} = 0, \quad (6.77)$$

$$h_1^{s,kn}\Big|_{x_2=r_i+0} = h_1^{s,kn}\Big|_{x_2=r_i-0}, \quad (6.78)$$

$$\frac{\partial h_1^{s,kn}}{\partial x_2}\Big|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\frac{\partial h_1^{s,kn}}{\partial x_2}\Big|_{x_2=r_i-0}. \quad (6.79)$$

By a method similar to that is applied for solving (6.40)-(6.46), we find an approximate solution of (6.75)-(6.79) as follows

$$h_1^{s,kKn}(x_2,t,x^0) = \sum_{m=0}^{K} T_1^{s,kmn}(t,x^0)Z_1^{kmn}(x_2), \quad (6.80)$$

here the functions $Z_1^{kmn}(x_2)$ are eigenfunctions of the problem (5.23)-(5.26), which was considered in Section 5.1.2, corresponding to the eigenvalues λ_{kmn}; the unknown functions $T_1^{s,kmn}(t,x^0)$, $s = 1,2,3$, are solutions of the IVP for $t \in \mathbf{R}^+$, $x^0 \in D^0$

$$\frac{d^2 T_1^{s,kmn}}{dt^2} + \lambda_{kmn}T_1^{s,kmn} = \psi_1^{s,kmn}(t,x^0)\delta(t) - c^2(x_2)\xi_{kn}^2 v_1^{s,kmn}(x_2^0,t),$$

$$T_1^{s,kmn}(t,x^0)\Big|_{t=0} = 0, \quad \frac{dT_1^{s,kmn}}{dt}\Big|_{t=0} = 0,$$

where

$$\begin{aligned}\psi_1^{s,kmn}(x^0) &= \sum_{i=1}^{N}\int_{r_{i-1}}^{r_i} \mu_i\psi_1^{s,kn}(x_2,x^0)Z_1^{kmn}(x_2)\,dx_2 \\ &= \int_0^{b_1}\int_0^{b_3}\left[\sum_{i=1}^{N}\int_{r_{i-1}}^{r_i}\mu_i\psi_1^s(x,x^0)Z_1^{kmn}(x_2)W_{kn}(x_1,x_3)\,dx_2\right]dx_3\,dx_1,\end{aligned}$$

$$v_1^{s,kmn}(x_2^0,t) = \sum_{i=1}^{N} \int_{r_{i-1}}^{r_i} \mu_i v_1^{s,kn}(x_2,t) Z_1^{kmn}(x_2)\, dx_2.$$

The solution of the IVP of $T_1^{s,kmn}(t,x^0)$ can be found as

$$\begin{aligned} T_1^{s,kmn}(t,x^0) \quad &= \quad \frac{\sin\left(\sqrt{\lambda_{kmn}}t\right)}{\sqrt{\lambda_{kmn}}} \psi_1^{s,kmn}(x^0) \\ &\quad + c^2(x_2) \frac{\xi_{kn}^2}{\sqrt{\lambda_{kmn}}} \int_0^t \sin\left(\sqrt{\lambda_{kmn}}(t-\tau)\right) v_1^{s,kmn}(x_2^0,\tau)\, d\tau, \end{aligned}$$

$k = 1,2,\ldots,\ m = 0,1,\ldots,\ n = 0,1,\ldots\,.$

As a result, we find the approximate solution of the problem (6.19)-(6.25) in the explicit form as

$$\begin{aligned} H_1^{s,K}(x,t,x^0) = \sum_{k=1}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} \Bigg[\frac{\sin\left(\sqrt{\lambda_{kmn}}t\right)}{\sqrt{\lambda_{kmn}}} \psi_1^{kmn}(x^0) Z_1^{kmn}(x_2) W_1^{kn}(x_1,x_3) \\ + c^2(x_2) \frac{\xi_{kn}^2}{\sqrt{\lambda_{kmn}}} \int_0^t \sin\left(\sqrt{\lambda_{kmn}}(t-\tau)\right) v_1^{s,kmn}(x_2^0,\tau)\, d\tau Z_1^{kmn}(x_2) W_1^{kn}(x_1,x_3) \Bigg] \\ + \sum_{k=1}^{K}\sum_{n=0}^{K} v_1^{s,kn}(x_2,t,x^0) W_1^{kn}(x_1,x_3), \end{aligned}$$

$s = 1,2,3$, for $x \in D^0$, $t \in \mathbf{R}^+$; $x^0 \in D^0$.

6.2.3 Approximate Computation of $H_3^s(x,t,x^0)$*,* $s = 1,2,3$

Here we construct the third elements of the magnetic Green's function by solving the problem (6.33)-(6.39). The approximate solution of (6.33)-(6.39) follows by the same way as described in the last section.

Firstly, we replace the equations (6.33)-(6.39) for $x \in D^0,\ t \in \mathbf{R}^+$ by

$$\frac{\partial^2 H_3^{s,K}}{\partial t^2} - c^2(x_2)\Delta_x H_3^{s,K} = c^2(x_2)\psi_3^{s,K}(x,x^0)\delta(t), \tag{6.81}$$

$$H_3^{s,K}\Big|_{t<0} = 0, \quad \frac{\partial H_3^{s,K}}{\partial t}\Bigg|_{t<0} = 0, \tag{6.82}$$

$$\frac{\partial H_3^{s,K}}{\partial x_1}\Bigg|_{x_1=0} = 0, \quad \frac{\partial H_3^{s,K}}{\partial x_1}\Bigg|_{x_1=b_1} = 0, \tag{6.83}$$

$$\left.\frac{\partial H_3^{s,K}}{\partial x_2}\right|_{x_2=0} = 0, \quad \left.\frac{\partial H_3^{s,K}}{\partial x_2}\right|_{x_2=b_2} = 0, \tag{6.84}$$

$$H_3^{s,K}\big|_{x_3=0} = 0, \quad H_3^{s,K}\big|_{x_3=b_3} = 0, \tag{6.85}$$

$$H_3^{s,K}\big|_{x_2=r_i-0} = H_3^{s,K}\big|_{x_2=r_i+0}, \tag{6.86}$$

$$\left.\frac{\partial H_3^{s,K}}{\partial x_2}\right|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\left.\frac{\partial H_3^{s,K}}{\partial x_2}\right|_{x_2=r_i-0} - \frac{\epsilon_{i+1}}{\epsilon_i} g_i^{s,K}, \tag{6.87}$$

where

$$\psi_3^{s,K}(x,x^0) = \sum_{k=0}^{K}\sum_{n=1}^{K}\psi_3^{s,kn}(x_2,x^0)W_3^{kn}(x_1,x_3), \tag{6.88}$$

$$g_i^{s,K}(x_1,x_3,t,x^0) = \sum_{k=0}^{K}\sum_{n=1}^{K} g_i^{s,kn}(t,x^0)W_3^{kn}(x_1,x_3), \tag{6.89}$$

for $s = 1,2,3$, $i = 1,2,\ldots,N-1$.

We look for a solution of (6.81)-(6.87) in the form

$$H_3^{s,K}(x,t,x^0) = \sum_{k=0}^{K}\sum_{n=1}^{K} H_3^{s,kn}(x_2,t,x^0)W_3^{kn}(x_1,x_3), \tag{6.90}$$

here $H_3^{s,kn}(x_2,t,x^0)$ are the unknown functions and $W_3^{kn}(x_1,x_3)$ are given functions as in (5.22).

Substituting (6.88), (6.89), (6.90) into (6.81)-(6.87) we get the equations for finding $H_1^{s,kn}(x_2,t,x^0)$ for $x \in D^0$, $t \in \mathbf{R}^+$ as

$$\frac{\partial^2 H_3^{s,kn}}{\partial t^2} - c^2(x_2)\left[\frac{\partial^2 H_3^{s,kn}}{\partial x_2^2} - \xi_{kn}^2 H_3^{s,kn}\right] = c^2(x_2)\psi_3^{s,kn}(x_2,x^0)\delta(t), \ \ x \in D^0, \ \ t \in \mathbf{R}^+, \tag{6.91}$$

$$H_3^{s,kn}\big|_{t<0} = 0, \ \ \left.\frac{\partial H_3^{s,kn}}{\partial t}\right|_{t<0} = 0, \tag{6.92}$$

$$\left.\frac{\partial H_3^{s,kn}}{\partial x_2}\right|_{x_2=0} = 0, \quad \left.\frac{\partial H_3^{s,kn}}{\partial x_2}\right|_{x_2=b_2} = 0, \tag{6.93}$$

$$H_3^{s,kn}\big|_{x_2=r_i-0} = H_3^{s,kn}\big|_{x_2=r_i+0}, \tag{6.94}$$

$$\left.\frac{\partial H_3^{s,kn}}{\partial x_2}\right|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\left.\frac{\partial H_3^{s,kn}}{\partial x_2}\right|_{x_2=r_i-0} - \frac{\epsilon_{i+1}}{\epsilon_i} g_i^{s,kn}(t,x^0), \tag{6.95}$$

where $\xi_{kn}^2 = \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2$, $k = 0,1,\ldots$; $n = 1,2,\ldots$.

Let $h_3^{s,kn}(x_2,t,x^0) = H_3^{s,kn}(x_2,t,x^0) - v_3^{s,kn}(x_2,t,x^0)$ with $v_3^{s,kn}(x_2,t,x^0)$ for $(x_2,t) \in (r_{i-1},r_i)\times\mathbf{R}^+,\ i = 1,2,\ldots,N$, satisfying

$$\frac{\partial^2 v_3^{s,kn}}{\partial t^2} - c^2(x_2)\frac{\partial^2 v_3^{s,kn}}{\partial x_2^2} = 0, \tag{6.96}$$

$$v_3^{s,kn}\Big|_{t=0} = 0, \quad \frac{\partial v_3^{s,kn}}{\partial t}\Bigg|_{t=0}, \tag{6.97}$$

$$\frac{\partial v_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=0} = 0, \quad \frac{\partial v_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=b_2} = 0, \tag{6.98}$$

$$v_3^{s,kn}\Big|_{x_2=r_i+0} = v_3^{s,kn}\Big|_{x_2=r_i-0}, \tag{6.99}$$

$$\frac{\partial v_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\frac{\partial v_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=r_i-0} - \frac{\epsilon_{i+1}}{\epsilon_i} g_i^{s,kn}(t,x^0), \tag{6.100}$$

where $s = 1,2,3$; $i = 1,2,\ldots,N-1$; $k = 0,1,\ldots$, $n = 1,2,\ldots$.

Remark 6.2.3. The solution of (6.96)-(6.100) follows by the same method as in Chapter 4.

Then equations (6.91)-(6.95) can be written as

$$\frac{\partial^2 h_3^{s,kn}}{\partial t^2} - c^2(x_2)\left[\frac{\partial^2 h_3^{s,kn}}{\partial x_2^2} - \xi_{kn}^2 h_3^{s,kn}\right] = c^2(x_2)\psi_3^{s,kn}(x_2,x^0)\delta(t) - c^2(x_2)\xi_{kn}^2 v_3^{s,kn}, \tag{6.101}$$

$$h_3^{s,kn}\Big|_{t=0} = 0, \quad \frac{\partial h_3^{s,kn}}{\partial t}\Bigg|_{t=0} = 0, \tag{6.102}$$

$$\frac{\partial h_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=0} = 0, \quad \frac{\partial h_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=b_2} = 0, \tag{6.103}$$

$$h_3^{s,kn}\Big|_{x_2=r_i+0} = h_3^{s,kn}\Big|_{x_2=r_i-0}, \tag{6.104}$$

$$\frac{\partial h_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=r_i+0} = \frac{\epsilon_{i+1}}{\epsilon_i}\frac{\partial h_3^{s,kn}}{\partial x_2}\Bigg|_{x_2=r_i-0}. \tag{6.105}$$

Using the same method that is applied for solving (6.75)-(6.79) we find an approximate solution of (6.101)-(6.105) as

$$h_3^{s,kKn}(x_2,t,x^0) = \sum_{m=0}^{K} T_3^{s,kmn}(t,x^0)Z_3^{kmn}(x_2), \tag{6.106}$$

$k = 0,1,\ldots$, $n = 1,2,\ldots$. Here $Z_3^{kmn}(x_2)$ are given functions as defined in Section 5.1.2, and $T_3^{s,kmn}(t,x^0)$ are unknown functions which can be found by solving an IVP similar to the IVP for $T_1^{s,kmn}(t,x^0)$ as

$$\begin{aligned}
T_3^{s,kmn}(t,x^0) &= \frac{\sin\left(\sqrt{\lambda_{kmn}}t\right)}{\sqrt{\lambda_{kmn}}}\psi_3^{s,kmn}(x^0) \\
&\quad + c^2(x_2)\frac{\xi_{kn}^2}{\sqrt{\lambda_{kmn}}}\int_0^t \sin\left(\sqrt{\lambda_{kmn}}(t-\tau)\right)v_3^{s,kmn}(x_2^0,\tau)\,d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\psi_3^{s,kmn}(x^0) &= \sum_{i=1}^{N}\int_{r_{i-1}}^{r_i}\mu_i\psi_3^{s,kn}(x_2,x^0)Z_3^{kmn}(x_2)\,dx_2 \\
&= \int_0^{b_1}\int_0^{b_3}\left[\sum_{i=1}^{N}\int_{r_{i-1}}^{r_i}\mu_i\psi_3^{s}(x,x^0)Z_3^{kmn}(x_2)W_3^{kn}(x_1,x_3)\,dx_2\right]dx_3\,dx_1,
\end{aligned}$$

$$v_3^{s,kmn}(x_2^0,t) = \sum_{i=1}^{N}\int_{r_{i-1}}^{r_i}\mu_i v_3^{s,kn}(x_2,t)Z_3^{kmn}(x_2)\,dx_2,$$

$k = 0,1,\ldots$, $m = 0,1,\ldots$, $n = 1,2,\ldots$.

As a result, we find the approximate solution of the problem (6.33)-(6.39) in the explicit form as

$$\begin{aligned}
H_3^{s,K}(x,t,x^0) = \sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=1}^{K}\Bigg[&\frac{\sin\left(\sqrt{\lambda_{kmn}}t\right)}{\sqrt{\lambda_{kmn}}}\psi_3^{s,kmn}(x^0)Z_3^{kmn}(x_2)W_3^{kn}(x_1,x_3) \\
+c^2(x_2)\frac{\xi_{kn}^2}{\sqrt{\lambda_{kmn}}}&\int_0^t \sin\left(\sqrt{\lambda_{kmn}}(t-\tau)\right)v_3^{s,kmn}(x_2^0,\tau)\,d\tau Z_3^{kmn}(x_2)W_3^{kn}(x_1,x_3)\Bigg] \\
&+\sum_{k=0}^{K}\sum_{n=1}^{K}v_3^{s,kn}(x_2,t,x^0)W_3^{kn}(x_1,x_3).
\end{aligned}$$

6.3 Computation of the Electric Green's Function

Let columns of the magnetic Green's function be derived, that is, $\mathbf{H}^{s,K}(x,t,x^0)$, $s = 1,2,3$ be approximate solution of the problem (6.10)-(6.15). For computation of $\mathbf{E}^s(x,t,x^0)$, satisfying (6.16)-(6.18), we replace $curl_x\mathbf{H}^s(x,t,x^0)$ by $curl_x\mathbf{H}^{s,K}(x,t,x^0)$ and $\delta(x-x^0)\vec{\mathbf{e}^s}$ by its regularization in (6.16) for $s = 1,2,3$, respectively. Then we find an explicit formula for $\mathbf{E}^{s,K}(x,t,x^0)$ as follows

$$\mathbf{E}^{s,K}(x,t,x^0) = \frac{1}{\epsilon(x_2^0)}\left[\int_0^t curl_x \mathbf{H}^{s,K}(x,\tau,x^0)\, d\tau - \Phi^{s,K}(x,x^0)\right], \tag{6.107}$$

where the vector functions defined by

$$\Phi^{s,K}(x,x^0) = \sum_{k=0}^{K}\sum_{m=0}^{K}\sum_{n=0}^{K} \nu_s(x_2^0) U_s^{kmn}(x^0) U_s^{kmn}(x) \vec{\mathbf{e}^s}, \tag{6.108}$$

where $0 < x_1^0 < b_1,\ r_{i-1} < x_2^0 < r_i,\ i = 1,2,\ldots,N,\ 0 < x_3^0 < b_3$, are regularizations of $\delta(x - x^0)\vec{\mathbf{e}^s}$.

6.4 Computational Experiments

In this section, we present computational experiments to show applicability of the method. Here we consider a *two − layered* rectangular parallelepiped $D = \left\{x \in \mathbf{R}^3 :\ 0 < x_1 < 2,\ 0 < x_2 < 2,\ 0 < x_3 < 1\right\}$. This parallelepiped divided into two layers with respect to $x_2 - variable$ and each layer (parallelepiped) D_i, $i = 1,2$ is of the form

$$D_1 = \left\{x \in \mathbf{R}^3 :\ 0 < x_1 < 2, \quad 0 < x_2 < 1.25, \quad 0 < x_3 < 1\right\},$$
$$D_2 = \left\{x \in \mathbf{R}^3 :\ 0 < x_1 < 2, \quad 1.25 < x_2 < 2, \quad 0 < x_3 < 1\right\},$$

which contain dielectric material characterized by the electric permittivity $\epsilon(x_2) = \epsilon_0\bar{\epsilon}(x_2)$ (ϵ_0 is the vacuum permittivity, and $\bar{\epsilon}(x_2)$ is relative permittivity of the material) and the magnetic permeability $\mu(x_2) = \mu_0\bar{\mu}(x_2)$ (μ_0 is the vacuum permeability, and $\bar{\mu}(x_2)$ is relative permeability of the material). $\bar{\epsilon}(x_2)$ and $\bar{\mu}(x_2)$ are chosen as

$$\bar{\mu}(x_2) = \begin{cases} 2, & x_2 \in (0,1.25), \\ 6, & x_2 \in (1.25,2), \end{cases} \qquad \bar{\epsilon}(x_2) = \begin{cases} 4, & x_2 \in (0,1.25), \\ 8, & x_2 \in (1.25,2), \end{cases}$$

and $c(x_2) = c_0\bar{c}(x_2)$ ($c_0 = \frac{1}{\mu_0\epsilon_0}$)

$$\bar{c}(x_2) = \begin{cases} \frac{1}{2\sqrt{2}}, & x_2 \in (0,1.25), \\ \frac{1}{4\sqrt{3}}, & x_2 \in (1.25,2). \end{cases}$$

The boundary of D is a perfect conducting boundary on which the tangential components of the electric field and normal component of the magnetic field component vanish. There is an electric current $\mathbf{J}^s(x,t)$ inside D. This electric current produces the electric and magnetic fields. In our experiments we take $\mathbf{J}^s(x,t)$, $s = 1,2,3$ as

$$\mathbf{J}^s(x,t) = \vec{\mathbf{e}^s}\delta(x - x^0)\delta(t),$$

the 3D parameter x^0 is chosen as $x_1^0 = 0.5$; $x_2^0 = 0.75$; $x_3^0 = 0.75$.

The main goal of this experiment is the approximate computation of the time-dependent magnetic and electric matrix Green's functions in two-layered rectangular parallelepiped. Firstly, we find eigenvalues and corresponding eigenfunctions of (5.8)-(5.13) using the technique described in Chapter 5. Substituting $N = 2$ into the algebraic system (5.40) we find the coefficient matrix $\mathbf{Q}^{(2)}(\lambda)$ of the form

$$\mathbf{Q}^{(2)}(\lambda) = \begin{pmatrix} \sin\left(\sqrt{\frac{\eta_{kn}^1}{c_1^2}}r\right) & -\frac{\mu_2}{\mu_1}\cos\left(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}r\right) & -\frac{\mu_2}{\mu_1}\sin\left(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}r\right) \\ \sqrt{\frac{\eta_{kn}^1}{c_1^2}}\cos\left(\sqrt{\frac{\eta_{kn}^1}{c_1^2}}r\right) & \sqrt{\frac{\eta_{kn}^2}{c_2^2}}\sin\left(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}r\right) & -\sqrt{\frac{\eta_{kn}^2}{c_2^2}}\cos\left(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}r\right) \\ 0 & \cos\left(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}b_2\right) & \sin\left(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}b_2\right) \end{pmatrix},$$

where $\eta_{kn}^i = \lambda - c_i^2\xi_{kn}$, $i = 1,2$, $k,n = 0,1,\ldots$.

For each $k,n = 0,1,\ldots$, the roots of the equation $det(\mathbf{Q}^{(2)}(\lambda)) = 0$, that is,$\lambda_{kmn}$, $m = 0,1,\ldots$, have been computed using MATLAB. The found roots are eigenvalues of (5.8)-(5.13).

After computing the eigenvalues λ_{kmn}, we have constructed the corresponding eigenfunctions $U_2^{kmn}(x)$ using the way described in Section 5.1 as

$$U_2^{kmn}(x) = W_2^{kn}(x_1,x_3)Z_2^{kmn}(x_2), \tag{6.109}$$

for $x_1 \in [0,b_1]$, $x_2 \in [0,r_1) \cup (r_1,r_2]$, $x_3 \in [0,b_3]$. Here the functions $W_2^{kn}(x_1,x_3)$ given as in (5.22), and $Z_2^{kmn}(x_2)$ has been computed as

$Z_2^{kmn}(x_2) =$

$$\begin{cases} \sin(\sqrt{\frac{\eta_{kn}^1}{c_1^2}}x_2), \quad x_2 \in [0,r); \\ \left[\frac{1}{2}\left(\frac{\mu_1}{\mu_2} - \frac{\sqrt{\lambda_{kmn}/c_1^2 - \xi_{kn}}}{\sqrt{\lambda_{kmn}/c_2^2 - \xi_{kn}}}\right)\sin\left(\left(\sqrt{\frac{\eta_{kn}^1}{c_1^2}} + \sqrt{\frac{\eta_{kn}^2}{c_2^2}}\right)r\right) + \right. \\ \left. \frac{1}{2}\left(\frac{\mu_1}{\mu_2} + \frac{\sqrt{\lambda_{kmn}/c_1^2 - \xi_{kn}}}{\sqrt{\lambda_{kmn}/c_2^2 - \xi_{kn}}}\right)\sin\left(\left(\sqrt{\frac{\eta_{kn}^1}{c_1^2}} - \sqrt{\frac{\eta_{kn}^2}{c_2^2}}\right)r\right)\right]\cos(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}x_2) + \\ \left[-\frac{1}{2}\left(\frac{\mu_1}{\mu_2} - \frac{\sqrt{\lambda_{kmn}/c_1^2 - \xi_{kn}}}{\sqrt{\lambda_{kmn}/c_2^2 - \xi_{kn}}}\right)\cos\left(\left(\sqrt{\frac{\eta_{kn}^1}{c_1^2}} + \sqrt{\frac{\eta_{kn}^2}{c_2^2}}\right)r\right) + \right. \\ \left. \frac{1}{2}\left(\frac{\mu_1}{\mu_2} + \frac{\sqrt{\lambda_{kmn}/c_1^2 - \xi_{kn}}}{\sqrt{\lambda_{kmn}/c_2^2 - \xi_{kn}}}\right)\cos\left(\left(\sqrt{\frac{\eta_{kn}^1}{c_1^2}} - \sqrt{\frac{\eta_{kn}^2}{c_2^2}}\right)r\right)\right]\sin(\sqrt{\frac{\eta_{kn}^2}{c_2^2}}x_2) + \\ \qquad\qquad\qquad\qquad x_2 \in (r, b_2] \end{cases}$$

where $\eta_{kn}^i = \lambda - c_i^2\xi_{kn}$, $i = 1,2$, $k,n = 0,1,\ldots$.

6.4.1 Approximate Computation and Visualization of Elements of Matrix Green's Function for Two Layered Parallelepiped

The method described in the last two sections have been applied for the approximate computation of the time-dependent magnetic and electric Green's matrices in the $N-layered$ rectangular parallelepiped D. For computational experiments we take a two-layered rectangular parallelepiped as defined above. Computation of $H_j^{s,K}(x,t,x^0)$ has been implemented in $MATLAB$ and its graphs for $j=2$, $s=1$ are presented. Here, K is the parameter of the regularization and taken as $K=36$ (see, Section 5.2). Figs. 6.1-6.6 present the second component of the first column of the regularized magnetic matrix, $H_2^{1,K}(x,t,x^0)$, for different time data. The x_1 and x_2 are horizontal axes and $z-axis$ is the magnitude of $H_2^{1,K}$. In Figures 6.1 and 6.2 the fronts of the magnetic wave did not reach the boundary of the parallelepiped. In Figures 6.4, 6.5, and 6.6 we see the further propagation of the magnetic waves and the reflections and transmissions from the different part of the boundary.

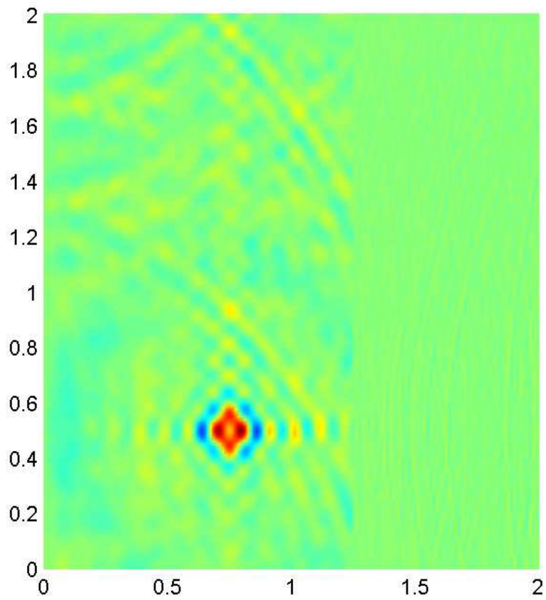

Figure 6.1 The map of the surface plot of $z = H_2^{1,K}(x_1, x_2, 0.75, t, x^0)$ for $K = 36$ at $t = 0.5/c_0$; source is placed at $x^0 = (0.5, 0.75, 0.75)^T$ position inside the parallelepiped; the horizontal axis is $x_2 - axis$ and the vertical axis is $x_1 - axis$.

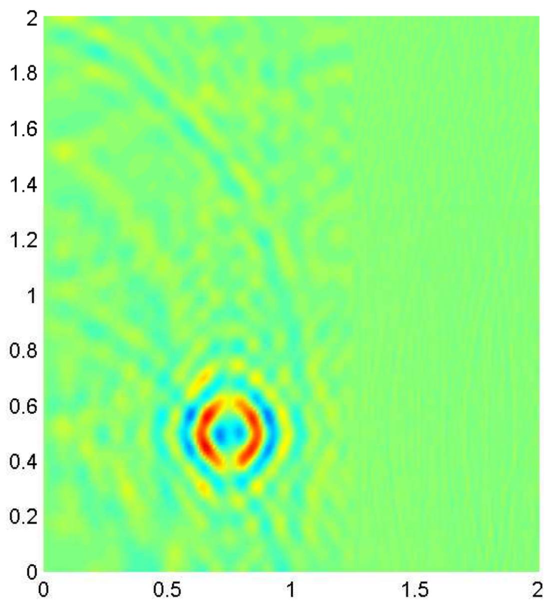

Figure 6.2 The map of the surface plot of $z = H_2^{1,K}(x_1, x_2, 0.75, t, x^0)$ for $K = 36$ at $t = 1/c_0$; source is placed at $x^0 = (0.5, 0.75, 0.75)^T$ position inside the parallelepiped; the horizontal axis is $x_2 - axis$ and the vertical axis is $x_1 - axis$.

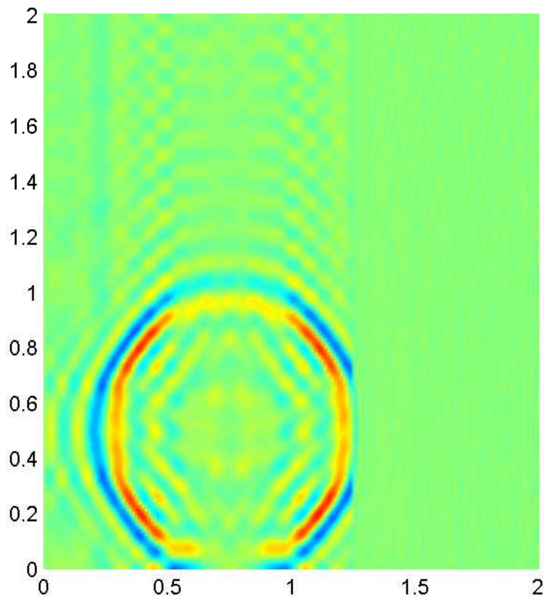

Figure 6.3 The map of the surface plot of $z = H_2^{1,K}(x_1, x_2, 0.75, t, x^0)$ for $K = N = 36$ at $t = 2/c_0$; source is placed at $x^0 = (0.5, 0.75, 0.75)^T$ position inside the parallelepiped; the horizontal axis is $x_2 - axis$ and the vertical axis is $x_1 - axis$.

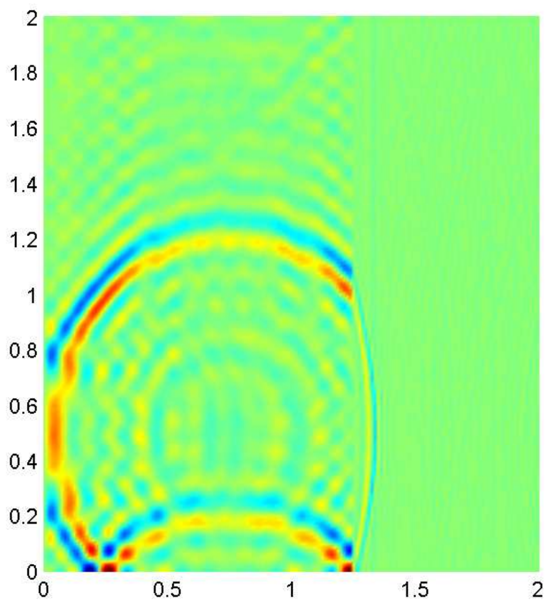

Figure 6.4 The map of the surface plot of $z = H_2^{1,K}(x_1, x_2, 0.75, t, x^0)$ for $K = 36$ at $t = 2.5/c_0$; source is placed at $x^0 = (0.5, 0.75, 0.75)^T$ position inside the parallelepiped; the horizontal axis is $x_2 - axis$ and the vertical axis is $x_1 - axis$.

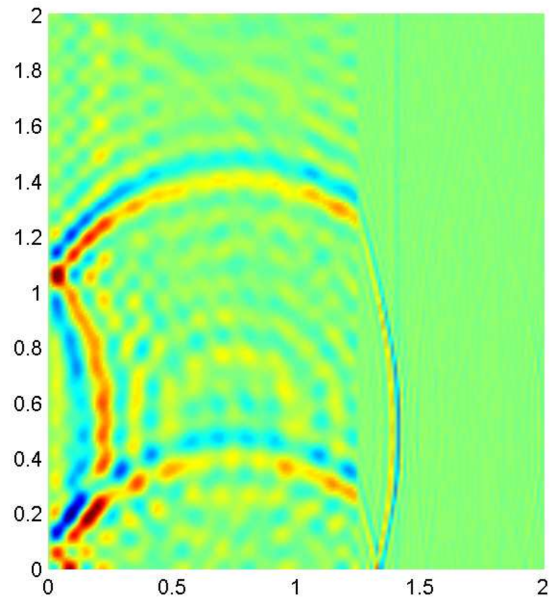

Figure 6.5 The map of the surface plot of $z = H_2^{1,K}(x_1, x_2, 0.75, t, x^0)$ for $K = 36$ at $t = 3/c_0$; source is placed at $x^0 = (0.5, 0.75, 0.75)^T$ position inside the parallelepiped; the horizontal axis is $x_2 - axis$ and the vertical axis is $x_1 - axis$.

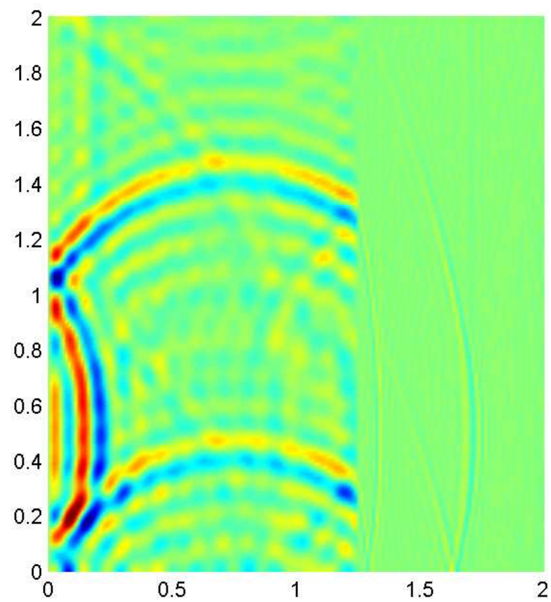

Figure 6.6 The map of the surface plot of $z = H_2^{1,K}(x_1, x_2, 0.75, t, x^0)$ for $K = 36$ at $t = 5/c_0$; source is placed at $x^0 = (0.5, 0.75, 0.75)^T$ position inside the parallelepiped; the horizontal axis is $x_2 - axis$ and the vertical axis is $x_1 - axis$.

6.5 Summary of Chapter Six

A new method for approximate computations of time dependent magnetic and electric matrix Green's functions in rectangular multi-layered parallelepiped with the perfect conducting boundary, is described. This method based on the Fourier series meta-approach. The problem has been decomposed into problems: approximate computation of the magnetic matrix Green's function and approximate computation of the electric Green's function. After calculation of the column elements of the magnetic matrix Green's function the approximate computation of the columns of the electric matrix Green's function has been given by explicit calculation of the differential operator $curl_x$ from each column of the magnetic Green's function with respect to 3D space variable x and then by integration with respect to time variable.

CHAPTER SEVEN
FINDING A SOLUTION OF IVP FOR THE ANISOTROPIC MAXWELL SYSTEM WITH DIFFERENTIABLE FUNCTION COEFFICIENTS

An analytical method for solving the initial value problem for the time-dependent Maxwell's equations in the general electrically and magnetically anisotropic vertical inhomogeneous media is described in this chapter. This method has the following steps. The Maxwell's equations are written as the symmetric hyperbolic system of the first order. This system is written in terms of the Fourier images with respect to lateral variables. The coefficients of this system are functions depending on vertical variable only. Applying method of integration along the characteristics the symmetric hyperbolic system is reduced to a system of the Volterra integral equations. Method of successive approximations is applied for solving the obtained system of the integral equations. Finally, the inverse Fourier transform is applied to find a solution of the initial value problem for the time-dependent Maxwell's equations.

7.1 Maxwell's Equations in Anisotropic Media

Let $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}$ be space and time variables, $\mu_{ij}(x_3) > 0$ and $\epsilon_{ij}(x_3) > 0$ are given functions depending on x_3, $\mathcal{M}(x_3) = (\mu_{ij}(x_3))_{3\times3}$ and $\mathcal{E}(x_3) = (\epsilon_{ij}(x_3))_{3\times3}$ are given 3×3 matrices, $\mathbf{J}(x,t) = (J_1(x,t), J_2(x,t), J_3(x,t))$ is a given vector function, $\mathbf{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))$ and $\mathbf{H}(x,t) = (H_1(x,t), H_2(x,t), H_3(x,t))$ are unknown vector functions satisfying the following initial value problem (IVP) for the anisotropic Maxwell's system

$$curl_x \mathbf{H}(x,t) = \mathcal{E}(x_3)\frac{\partial \mathbf{E}(x,t)}{\partial t} + \mathbf{J}(x,t), \tag{7.1}$$

$$curl_x \mathbf{E}(x,t) = -\mathcal{M}(x_3)\frac{\partial \mathbf{H}(x,t)}{\partial t}, \tag{7.2}$$

with the initial conditions

$$\mathbf{E}(x,t)\Big|_{t=0} = 0, \quad \mathbf{H}(x,t)\Big|_{t=0} = 0. \tag{7.3}$$

Here, $\mathbf{E}(x,t)$, $\mathbf{H}(x,t)$ denote the electric and magnetic fields, respectively; $\mathbf{J}(x,t)$ is the electric current density; $\mathcal{E}(x_3)$ is the electric permittivity and $\mathcal{M}(x_3)$ is the magnetic permeability.

7.1.1 Assumptions

We assume that $\mathcal{E}(x_3)$, $\mathcal{M}(x_3)$ are such symmetric positive definite matrices that there exist positive constants $\bar{\epsilon_0}$, $\bar{\mu_0}$ satisfying the following inequality

$$\langle \mathcal{E}(x_3)\cdot\xi,\xi\rangle \geq \bar{\epsilon_0}\cdot|\xi|^2, \ \ \forall\xi\in\mathbf{R}^3, \ \ \forall x_3\in\mathbf{R},$$
$$\langle \mathcal{M}(x_3)\cdot\xi,\xi\rangle \geq \bar{\mu_0}\cdot|\xi|^2, \ \ \forall\xi\in\mathbf{R}^3, \ \ \forall x_3\in\mathbf{R},$$

for an arbitrary vector $\xi\in\mathbf{R}^3$ and $x_3\in\mathbf{R}$.

Let T be a positive number, $a_0 = \max\{\bar{\epsilon_0},\bar{\mu_0}\}$, $M = \dfrac{1}{a_0}$. The triangle $\Delta(M,T)$ is defined by

$$\Delta(M,T) = \{(x_3,t): \ \ 0\leq t\leq T, \ \ -M(T-t)\leq x_3\leq M(T-t)\}. \tag{7.4}$$

We assume also that the elements of the matrices $\mathcal{E}(x_3)$, $\mathcal{M}(x_3)$ are twice continuously differentiable functions over $[-MT,MT]$. Moreover there exists the Fourier transform with respect to x_1 and x_2 of the components of $\mathbf{J}(x,t) = (J_1(x,t),J_2(x,t),J_3(x,t))$. The Fourier transformed image of the functions $J_k(x,t)$ denoted by $\widetilde{J_k}(\nu,x_3,t)$, are assumed to be from $C(C^1(\Delta(M,T));C_0(\mathbf{R}^2))$, $k = 1,2,3$. Here, $\nu = (\nu_1,\nu_2)$ is a parameter from $\mathbf{R}^2$. The class $C(C^1(\Delta(M,T);C_0(\mathbf{R}^2)))$ consists of all continuous functions, depending on (x_3,t,ν) ($(x_3,t)\in\Delta(M,T)$, $\nu\in\mathbf{R}^2$ that are differentiable with respect to x_3,t and have finite supports with respect to ν.

7.2 The Reduction of IVP (7.1)-(7.3) for a Symmetric Hyperbolic System

In this section, the initial value problem for the anisotropic Maxwell's system is written in the form of an initial value problem for a symmetric hyperbolic system.

Lemma 7.2.1. *Under the assumptions of Section 7.1.1, the initial value problem*

(7.1)-(7.3) can be written as an initial value problem of symmetric hyperbolic system

$$\boldsymbol{A}_0(x_3)\frac{\partial \vec{\mathbf{v}}(x,t)}{\partial t} + \sum_{j=1}^{3} \boldsymbol{A}_j \frac{\partial \vec{\mathbf{v}}(x,t)}{\partial x_j} = \vec{\mathbf{f}}(x,t), \quad x \in \mathbf{R}^3, \quad t \in \mathbf{R}^+, \tag{7.5}$$

$$\vec{\mathbf{v}}(x,t)|_{t=0} = \vec{\mathbf{v}}_0(x), \quad x \in \mathbf{R}^3, \tag{7.6}$$

here $\vec{\mathbf{v}}(x,t)$, $\vec{\mathbf{f}}(x,t)$ *are* 6×1 *vector functions,* $\boldsymbol{A}_0(x_3)$ *is a* 6×6 *symmetric positive definite matrix and* $\boldsymbol{A}_j$ *are* 6×6 *symmetric matrices* $j = 1,2,3$ *such that*

$$\vec{\mathbf{v}}(x,t) = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{H} \end{bmatrix}^*_{1\times 6} = \begin{bmatrix} E_1 & E_2 & E_3 & H_1 & H_2 & H_3 \end{bmatrix}^*,$$

$$\vec{\mathbf{f}}(x,t) = \begin{bmatrix} -\boldsymbol{J} & \boldsymbol{0} \end{bmatrix}^*_{1\times 6} = \begin{bmatrix} -J_1 & -J_2 & -J_3 & 0 & 0 & 0 \end{bmatrix}^*,$$

$$\vec{\mathbf{v}}_0(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^*,$$

$$\boldsymbol{A}_0(x) = \begin{pmatrix} \mathcal{E}(x)_{3\times 3} & 0 \\ 0 & \mathcal{M}(x)_{3\times 3} \end{pmatrix}_{6\times 6}, \quad \boldsymbol{A}_j = \begin{pmatrix} 0_{3\times 3} & A_j^1 \\ (A_j^1)^* & 0_{3\times 3} \end{pmatrix}_{6\times 6}, \tag{7.7}$$

$$\boldsymbol{A}_1^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}_{3\times 3}, \quad \boldsymbol{A}_2^1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{3\times 3}, \quad \boldsymbol{A}_3^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3},$$

where * *denotes the transposition.*

Proof. The initial value problem of anisotropic Maxwell's system in (7.1)-(7.3) can be written as an initial value problem of a first order symmetric hyperbolic system. (For further discussion about the reduction see Appendix A8). □

7.3 IVP (7.5)-(7.6) in Terms of the Fourier Images: the Reduction to the System of the Volterra Integral Equations

Applying Fourier transform with respect to x_1, x_2 to the equations (7.5), (7.6), the IVP for the symmetric hyperbolic system (7.5)-(7.6) can be written in the following form

$$\mathbf{A}_0(x_3)\frac{\partial \widetilde{\mathbf{v}}}{\partial t} + \mathbf{A}_3 \frac{\partial \widetilde{\mathbf{v}}}{\partial x_3} = \widetilde{\mathbf{B}}(\nu)\widetilde{\mathbf{v}} + \widetilde{\mathbf{f}}, \quad x \in \mathbf{R}^3, \quad t \in \mathbf{R}^+, \tag{7.8}$$

$$\widetilde{\mathbf{v}}(\nu, x_3, t)\Big|_{t=0} = \widetilde{\mathbf{v}}_0(\nu, x_3), \tag{7.9}$$

where

$$\widetilde{\mathbf{v}}(\nu, x_3, t) = \int\int_{\mathbf{R}^2} \vec{\mathbf{v}}(\bar{x}, x_3, t) e^{i\nu\bar{x}}\, d\bar{x}, \quad \bar{x} = (x_1, x_2),\ \nu = (\nu_1, \nu_2) \in \mathbf{R}^2,$$

$\widetilde{\mathbf{v}}(\nu, x_3, t)$, $\widetilde{\mathbf{f}}(\nu, x_3, t)$, $\widetilde{\mathbf{v}}_0(\nu, x_3)$ are the Fourier transformed image of $\vec{\mathbf{v}}(x,t)$, $\vec{\mathbf{f}}(x,t)$, $\vec{\mathbf{v}}_0(x,0)$. And we denote the matrix $\widetilde{\mathbf{B}}(\nu) = i\nu_1\mathbf{A}_1 + i\nu_2\mathbf{A}_2$.

7.3.1 Properties of the Coefficient Matrices A_0 and A_3

By the assumptions for $\mathcal{E}(x_3)$ and $\mathcal{M}(x_3)$ of the Section 7.1.1, we can see that the matrix $\mathbf{A}_0(x_3)$, defined in (7.7), is positive definite and and the elements of $\mathbf{A}_0(x_3)$ are from $C^2[-MT, MT]$. That is, for any vector $\psi \in \mathbf{R}^6$ and $x_3 \in \mathbf{R}$

$$(\mathbf{A}_0 \cdot \psi, \psi) \geq a_0 \cdot |\psi|^2, \quad a_0 = \max\{\varepsilon_0, \mu_0\}. \tag{7.10}$$

The matrix $\mathbf{A}_3$, defined in (7.7), is symmetric matrix. Thus, it is diagonalizable.

Remark 7.3.1. For the symmetric positive definite matrix $\mathbf{A}_0(x_3)$ and symmetric matrix $\mathbf{A}_3$ there exists a nonsingular matrix $\mathbf{T}(x_3)$ such that, $\mathbf{T}(x_3) = \mathbf{A}_0^{-\frac{1}{2}}(x_3)Q(x_3)$, and

$$\mathbf{T}^*(x_3)\mathbf{A}_0(x_3)\mathbf{T}(x_3) = \mathcal{I}_{6\times 6}, \quad \mathbf{T}^*(x_3)\mathbf{A}_3\mathbf{T}(x_3) = \mathbf{D}(x_3)_{6\times 6},$$

where, $\mathbf{Q}(x_3)$ is an orthogonal matrix and $\mathbf{D}(x_3) = (d_1(x_3), \ldots, d_6(x_3))$ is a diagonal matrix (see, Goldberg (1991)).

Lemma 7.3.2. *Let $x \in \mathbf{R}^3$, $t \in [0, \infty)$; $\boldsymbol{A}_0(x_3)$ be symmetric positive definite matrix and $\boldsymbol{A}_j$, $j = 1, 2, 3$ be symmetric matrices then the IVP (7.8)-(7.9) can be written in the following form*

$$\frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{D}(x_3)\frac{\partial \mathbf{V}}{\partial x_3} = B(\nu, x_3)\mathbf{V} + \boldsymbol{F}(\nu, x_3, t),\ \nu \in \mathbf{R}^2,\ x_3 \in \mathbf{R},\ t > 0, \tag{7.11}$$

$$\mathbf{V}|_{t=0} = \mathbf{V}_0(\nu, x_3), \ \ \nu \in \mathbf{R}^2, \ \ x_3 \in \mathbf{R}, \tag{7.12}$$

where, $\mathbf{V} = \mathbf{V}(\nu, x_3, t)$, $\widetilde{\mathbf{v}}(\nu, x_3, t) = \boldsymbol{T}(x_3)\mathbf{V}(\nu, x_3, t)$ and $\boldsymbol{B}(\nu, x_3) = \boldsymbol{T}^*(x_3)\widetilde{\boldsymbol{B}}(\nu, x_3)T(x_3) - \boldsymbol{T}^*(x_3)\boldsymbol{A}_3\frac{d\boldsymbol{T}}{dx_3}$ for an orthogonal matrix $\boldsymbol{T}(x_3)$.

Proof. By Remark 7.3.1, we suppose that there exists a function satisfying

$$\widetilde{\mathbf{v}}(\nu, x_3, t) = \mathbf{T}(x_3)\mathbf{V}(\nu, x_3, t). \tag{7.13}$$

Inserting (7.13) into the equations (7.8), (7.9) and multiplying the new equations by $\mathbf{T}^*(x_3)$ from the left we get the IVP (7.11)-(7.12), where

$$\mathbf{B}(\nu, x_3) = \nu_1 \mathbf{T}^*\mathbf{A}_1\mathbf{T} + \nu_2 \mathbf{T}^*\mathbf{A}_2\mathbf{T} + \mathbf{T}^*\mathbf{A}_3 \frac{d\mathbf{T}}{dx_3}$$

$$\mathbf{F}(\nu, x_3, t) = \mathbf{T}^*(x_3)\widetilde{\mathbf{f}}(\nu, x_3, t), \ \ \mathbf{V}_0(\nu, x_3) = \mathbf{T}^*(x_3)\widetilde{\mathbf{v}}_0(\nu, x_3).$$

□

7.4 Reduction of IVP (7.11)-(7.12) to Integral Equation System

Let us consider the IVP (7.11)-(7.12) in the following form

$$\frac{\partial V_j}{\partial t} + d_j(x_3)\frac{\partial V_j}{\partial x_3} = (B\mathbf{V})_j(\nu, x_3, t) + F_j(\nu, x_3, t), \ \nu \in \mathbf{R}^2, \ x_3 \in \mathbf{R}, \ t > 0, \tag{7.14}$$

$$V_j|_{t=0} = V_j^0(\nu, x_3), \ \ \nu \in \mathbf{R}^2, \ \ x_3 \in \mathbf{R}. \tag{7.15}$$

Here, $d_j(x_3) \in C^1[-MT, MT]$, $|d_j(x_3)| \leq M$ and $M = \frac{1}{a_0}$ for any $x_3 \in \mathbf{R}$ then

$$\frac{\partial \xi}{\partial \tau} = d_j(\xi(\tau)), \tag{7.16}$$

$$\xi(\tau)|_{\tau=t} = x_3, \tag{7.17}$$

where $0 \leq \tau \leq t \leq T$, $\xi = \xi(\tau; x_3, t)$.

The characteristics can be found as

$$\xi(\tau) = x_3 + \int_t^{\tau} d_j(\xi(\eta))d\eta = \varphi_j(\tau; x_3, t), \tag{7.18}$$

$\xi(\tau; x_3, t) \in [x_3 - M(t-\tau), x_3 + M(t-\tau)] \subset [-MT, MT]$.

Construction of the characteristics is discussed in the following section.

7.4.1 Construction of Characteristics

Let us consider the IVP $(7.14)-(7.15)$. If $d_j(x_3) \in C^1[-MT, MT]$ and $|d_j(x_3)| \leq M = \frac{1}{a_0}$ for any $x_3 \in \mathbf{R}$ then

$$\frac{d\xi}{d\tau} = d_j(\xi(\tau)), \quad 0 \leq \tau \leq t \leq T, \tag{7.19}$$

$$\xi(\tau)|_{\tau=t} = x_3. \tag{7.20}$$

The initial value problem (7.19)-(7.20) can be written in the form of Volterra type integral equation

$$\xi(\tau) = x_3 + \int_t^\tau d_j(\xi(\eta))d\eta. \tag{7.21}$$

To construct a solution of (7.21) we use the successive approximations.

Let us consider the sequence of functions

$$\xi_n(\tau) = x_3 + \int_t^\tau d_j(\xi_{n-1}(\eta))d\eta, \ \ n = 1, 2, \ldots,$$

$$\xi_0(\tau) = x_3, \ \ x_3 \in \mathbf{R},$$

here, $|x_3| < MT$, and $|d(x_3)| < M$. Then let us consider the Neumann series

$$x_3 + \sum_{n=0}^{\infty}[\xi_{n+1}(\tau) - \xi_n(\tau)];$$

and let $S_N = x_3 + \sum_{n=0}^{N}[\xi_{n+1}(\tau) - \xi_n(\tau)]$, that is,

$$S_N = x_3 + [\xi_1(\tau) - x_3] + [\xi_2(\tau) - \xi_1(\tau)] + \ldots + [\xi_{N+1}(\tau) - \xi_N(\tau)] = \xi_{N+1}(\tau).$$

Since, each term of the sequence $\xi_0(\tau)$, $\xi_1(\tau)$, ..., $\xi_n(\tau)$ is a continuous function, then the series $\sum_{n=0}^{\infty}[\xi_{n+1}(\tau) - \xi_n(\tau)]$ is continuous. Subtracting $\xi_n(\tau)$ from $\xi_{n+1}(\tau)$ we find

$$\xi_{n+1}(\tau) - \xi_n(\tau) = \int_t^\tau [d(\xi_n(\eta)) - d(\xi_{n-1}(\eta))]d\eta.$$

Remark 7.4.1. Let T and M be fixed numbers, if $|d_j(\xi_\tau)| \leq M$ and $d_j(\xi_\tau) \in C^1[-MT, MT]$ then for arbitrary $(x_3, t) \in \Delta(M, T)$

(i) $\xi_n(\tau, x_3, t) \in [x_3 - M(t - \tau), x_3 + M(t - \tau)] \subset [-MT, MT]$

(ii) then there exists a positive constant $K = K(M, T)$ such that

$$|\xi_{n+1}(\tau) - \xi_n(\tau)| \leq K \int_t^\tau |\xi_n(\eta) - \xi_{(n-1)}(\eta)|d\eta.$$

Proof. **(i)** $|\xi_n(\tau) - x_3| = \left|\int_t^\tau d(\xi_{n-1}(\eta))d\eta\right| \leq M(t - \tau)$ Hence,

$$|\xi_n(\tau) - x_3| \leq M(t - \tau) \quad \Rightarrow -M(t - \tau) \leq \xi_n(\tau) - x_3 \leq M(t - \tau).$$

So,

$$-MT \leq x_3 - M(t-\tau) \leq \xi_n(\tau) \leq x_3 + M(t-\tau) \leq MT$$

(ii) For fixed positive numbers T and M, if $|d_j(\xi_\tau)| \leq M$ and $d_j(\xi_\tau) \in C^1[-MT, MT]$ then for arbitrary $(x_3, t) \in \Delta(M, T)$ then we can find a constant $K = K(M, T)$ such that

$$|d_j'(x_3)| \leq K,$$

then

$$\begin{aligned}
|\xi_{n+1}(\tau) - \xi_n(\tau)| &= \left| \int_t^\tau \int_{\xi_{n-1}(\tau)}^{\xi_n(\tau)} d'(\xi(\eta)) d\xi d\eta \right| \\
&\leq \int_t^\tau \int_{\xi_{n-1}(\tau)}^{\xi_n(\tau)} |d'(\xi(\eta))| d\xi d\eta \\
&\leq K \int_t^\tau \int_{\xi_{n-1}(\tau)}^{\xi_n(\tau)} d\xi d\eta \\
&\leq K \int_t^\tau |\xi_n(\tau) - \xi_{n-1}(\tau)| d\eta
\end{aligned}$$

□

Let us denote $|z_n(\tau)| = |\xi_{n+1}(\tau) - \xi_n(\tau)|$, then using Remark7.4.1 we find,

$$|z_n(\tau)| \leq \left| \int_t^\tau |z_{n-1}(\eta)| d\eta \right|.$$

Since,

$$z_0(\tau) = |\xi_1(\tau) - \xi_0(\tau)| = |\xi_1(\tau) - x_3| \leq M(t-\tau), \quad 0 \leq \tau \leq t$$

by Remark 7.4.1 we have

$$|z_1| = \int_t^\tau |z_0(\eta)| d\eta \leq \int_t^\tau M(t-\eta) d\eta \leq KM \frac{(t-\tau)^2}{2!}.$$

Similarly,

$$|z_2| \leq K \int_\tau^t KM \frac{(t-\eta)^2}{2!} d\eta = M \frac{(K^2(t-\tau)^3)}{3!} \leq M \frac{(K^2 T^3)}{3!}.$$

Hence, each term of z_n is continuous, since $d_j(\xi(\tau)) \in C^1[-MT, MT]$; and

$$|z_n| \leq K \int_\tau^t KM \frac{(t-\eta)^2}{2!} d\eta = \frac{M}{K} \frac{(KT)^{(n+1)}}{(n+1)!}$$

for each $n = 1, 2 \ldots$.

The series $\sum_{n=0}^{\infty} \frac{M}{K} \frac{(KT)^{(n+1)}}{(n+1)!}$ converges uniformly to $\frac{M}{K} e^{KT}$. Then by Weierstrass Theorem the series

$$S_N = x_3 + \sum_{n=0}^{N} [\xi_{n+1}(\tau) - \xi_n(\tau)] = \xi_{N+1}(\tau)$$

converges uniformly to a continuous function $\xi(\tau)$. And

$$|d(\xi_n(\tau)) - d(\xi(\tau))| \leq K|\xi_n(\tau) - \xi(\tau)|$$

which is convergent, so $d(\xi_n(\tau))$ is also convergent. Then by the continuity of $d_j(\xi(\tau))$ and the Second Weierstrass Theorem

$$\begin{aligned}
\xi(\tau) = \lim_{N\to\infty} \xi_N(\tau) &= \lim_{N\to\infty} \left[x_3 + \int_t^{\tau} d_j(\xi_{n-1}(\eta)) d\eta \right] \\
&= \lim_{N\to\infty} x_3 + \lim_{N\to\infty} \int_t^{\tau} d_j(\xi_{n-1}(\eta)) d\eta \\
&= \lim_{N\to\infty} x_3 + \int_t^{\tau} \lim_{N\to\infty} d_j(\xi_{n-1}(\eta)) d\eta \\
&= \lim_{N\to\infty} x_3 + \int_t^{\tau} d_j(\lim_{N\to\infty} \xi_{n-1}(\eta)) d\eta \\
&= \lim_{N\to\infty} x_3 + \int_t^{\tau} d_j(\xi(\eta)) d\eta.
\end{aligned}$$

So, $\xi(\tau)$ is a solution of the integral equation (7.21) and equivalently a solution of the IVP (7.19)-(7.20). Therefore we construct the characteristics as follows

$$\xi(\tau) = x_3 + \int_t^{\tau} d_j(\xi(\eta)) d\eta.$$

Let $\varphi_j(\tau, x_3, t)$ be a function

$$\xi(\tau) = x_3 + \int_t^{\tau} d_j(\xi(\eta)) d\eta = \varphi_j(\tau; x_3, t)$$

and $\varphi_j(t, x_3, t) = x_3$.

7.5 The Reduction of IVP (7.14)-(7.15) to a System of the Volterra Integral Equations

Using the characteristics

$$\xi(\tau) = x_3 + \int_t^{\tau} d_j(\xi(\eta)) d\eta,$$

the partial differential equation (7.14) can be written in the form of an integral equation by the following lemma.

Lemma 7.5.1. *The IVP (7.11)-(7.12) can be written in the form of Volterra integral equation*

$$\mathbf{V}(\nu, x_3, t) = G(\nu, x_3, t) + \int_0^t (\boldsymbol{K}\mathbf{V})(\nu, x_3, t, \tau)d\tau, \tag{7.22}$$

where $\mathbf{V}$ *is the unknown vector function;* $\boldsymbol{K} = (K_1, K_2, K_3, K_4, K_5, K_6)$ *is the vector integral operator; the given vector function* $\boldsymbol{G} = (G_1, G_2, G_3, G_4, G_5, G_6)$ *is the inhomogeneous term of the integral equation and they are defined as*

$$(\boldsymbol{K}\mathbf{V})_j(\nu, x_3, t, \tau) = (B\mathbf{V})_j(\nu, \varphi_j(\tau; x_3, t), \tau), \quad j = 1, 2, \ldots, 6; \tag{7.23}$$

$$G_j(\nu, x_3, t) = \mathbf{V}_j^0(\nu, \varphi_j(0; x_3, t)) + \int_0^t F_j(\nu, \varphi_j(\tau; x_3, t), \tau)d\tau, \quad j = 1, 2, \ldots, 6. \tag{7.24}$$

Proof. The differential equation (7.14) along the characteristics can be written as

$$\frac{d\mathbf{V}_j}{d\tau}(\nu, \varphi_j(\tau; x_3, t), \tau) = (B\mathbf{V})_j(\nu, \varphi_j(\tau; x_3, t), \tau) + F_j(\nu, \varphi_j(\tau; x_3, t), \tau). \tag{7.25}$$

Integrating the last equation along the characteristics with respect to τ from 0 to t and using $\varphi_j(t, x_3, t) = x_3$ and the initial data (7.15) we find

$$\mathbf{V}_j(\nu, x_3, t) = \mathbf{V}_j^0(\nu, \varphi_j(0, x_3, t)) + \int_0^t \Big[(B\mathbf{V})_j + F_j\Big](\nu, \varphi_j(\tau; x_3, t), \tau)d\tau, \tag{7.26}$$

where $\mathbf{V}_j(\nu, \varphi_j(0, x_3, t), 0) = \mathbf{V}_j^0(\nu, \varphi_j(0, x_3, t))$.
Thus, the IVP (7.14)-(7.15) can be written in the form of Volterra integral equation

$$\mathbf{V}_j(\nu, x_3, t) = G_j(\nu, x_3, t) + \int_0^t (\mathbf{K}\mathbf{V})_j(\nu, x_3\tau)d\tau, \tag{7.27}$$

where

$$(\mathbf{K}\mathbf{V})_j(\nu, x_3, t, \tau) = (B\mathbf{V})_j(\nu, \varphi_j(\tau; x_3, t), \tau), \quad j = 1, 2, \ldots, 6,$$

$$G_j(\nu, x_3, t) = \mathbf{V}_j^0(\nu, \varphi_j(0; x_3, t)) + \int_0^t F_j(\nu, \varphi_j(\tau; x_3, t), \tau)d\tau, \quad j = 1, 2, \ldots, 6.$$

□

7.5.1 *Finding A Solution of the Integral Equation (7.22)*

Let ϵ be a positive number and $(x_3, t) \in \Delta(M, T)$, then let us consider the integral equation (7.22). To find a solution of this integral equation we apply method of successive approximations. For this we consider the sequence of the functions

$$\begin{aligned} \mathbf{V}^0(\nu, x_3, t) &= \mathbf{G}(\nu, x_3, t), \\ \mathbf{V}^n(\nu, x_3, t) &= \int_0^t (\mathbf{K}\mathbf{V}^{n-1})(\nu, x_3, \tau) d\tau, \quad n = 1, 2, \dots . \end{aligned}$$

Main aim of this method is to show that for $(x_3, t) \in \Delta(M, T)$, $|\nu| \le \epsilon$, the series

$$\sum_{n=0}^{\infty} \mathbf{V}^n(\nu, x_3, t) = \left(\sum_{n=0}^{\infty} V_1^n(\nu, x_3, t), \sum_{n=0}^{\infty} V_2^n(\nu, x_3, t), \dots, \sum_{n=0}^{\infty} V_6^n(\nu, x_3, t) \right)$$

converges uniformly to a continuous vector function

$$\mathbf{V}(\nu, x_3, t) = (V_1(\nu, x_3, t), V_2(\nu, x_3, t), \dots, V_6(\nu, x_3, t)) .$$

Then the vector function $\mathbf{V}(\nu, x_3, t)$ is a solution of the integral equation (7.22).

For this, we need to show firstly the following lemma.

Lemma 7.5.2. *Let us consider the Volterra integral equation* (7.22) *with the operator* $\boldsymbol{K}$ *and inhomogeneous term G defined in (7.23) and (7.24), respectively. Then the operator* $\boldsymbol{K}$ *satisfies the following properties*

(K1) $\displaystyle\int_0^t (\boldsymbol{K}\mathbf{V})(\nu, x_3, t, \tau) d\tau \in C(\Delta(M, T), \mathbf{R}^2)$, $\forall \mathbf{V}(\nu, x_3, t) \in C(C^1(\Delta(M, T)); C(\mathbf{R}^2))$,

(K2) *There exists a constant* $B_0 = B_0(M, T)$ *such that,*

$$\left|(\boldsymbol{K}\mathbf{V})(\nu, x_3, t, \tau)\right| \le B_0 \|\mathbf{V}\|(\tau), \quad \forall (x_3, t) \in \Delta(M, T),$$

where, $\displaystyle \|\mathbf{V}\|(\tau) = \max_{0 \le \eta \le \tau} \max_{(x_3, \tau) \in \Delta(M, T)} \left| \mathbf{V}(x_3, \eta) \right|.$

Proof. Firstly, let us check the linearity property of the operator **K** defined in (7.23).

Let α and β be arbitrary constants, $\mathbf{V}_1$ and $\mathbf{V}_2$ be arbitrary vector functions,

$$\begin{aligned}
K_j(\alpha\mathbf{V}_1+\beta\mathbf{V}_2)(\nu,x_3,t,\tau) &= [B(\alpha\mathbf{V}_1+\beta\mathbf{V}_2)]_j(\nu,\varphi_j(\tau;x_3,t),\tau)\\
&= [\alpha(B\mathbf{V}_1)+\beta(B\mathbf{V}_2)]_j(\nu,\varphi_j(\tau;x_3,t),\tau)\\
&= \alpha(K_j\mathbf{V}_1)(\nu,\varphi_j(\tau;x_3,t),\tau)+\beta(K_j\mathbf{V}_2)(\nu,\varphi_j(\tau;x_3,t),\tau).
\end{aligned}$$

Hence, the operator $\mathbf{K}$ is linear.

Let us check if the operator $\mathbf{K}$ satisfies property $(K1)$.

Since, the matrix $A_0(x_3)$ is continuous, $B(\nu,x_3)$ is also continuous over $\Delta(M,T)$. Therefore $(\mathbf{KV})(\nu,x_3,\tau)\in C(\Delta(M,T),\mathbf{R}^2)$ for a continuous function $\mathbf{V}(\nu,x_3,t)$ in $C(C^1(\Delta(M,T));C(\mathbf{R}^2))$ then

$$\int_0^t(\mathbf{KV})(\nu,x_3,t,\tau)d\tau\in C(\Delta(M,T),\mathbf{R}^2)$$

Now, let us check if the operator satisfies the property $(K2)$.

$$\begin{gathered}
|K_j\mathbf{V}(\nu,x_3,t,\tau)| = |(B\mathbf{V})_j(\nu,\varphi_j(\tau;x_3,t),\tau)|\\
\le\left(|\nu_1|\cdot\|T^*\|\cdot\|A_1\|\cdot\|T\|+|\nu_2|\cdot\|T^*\|\cdot\|A_2\|\cdot\|T\|+\|T^*\|\cdot\|A_3\|\cdot\|\tfrac{\partial T(x_3)}{\partial x_3}\|\right)\cdot\|\mathbf{V}\|(\tau)\\
\le\left(|\nu_1|\tfrac{1}{\sqrt{a_0}}\cdot 1\cdot\tfrac{1}{\sqrt{a_0}}+|\nu_2|\tfrac{1}{\sqrt{a_0}}\cdot 1\cdot\tfrac{1}{\sqrt{a_0}}+\tfrac{1}{\sqrt{a_0}}\cdot 1\cdot R_1\right)\cdot\|\mathbf{V}\|(\tau)\\
\le\left(\tfrac{|\nu_1|+|\nu_2|+R_1\sqrt{a_0}}{a_0}\right)\cdot\|\mathbf{V}\|(\tau)\le B_0(a_0,T)\cdot\|\mathbf{V}\|(\tau).
\end{gathered}$$

where $R_1=\|\dfrac{\partial T(x_3)}{\partial x_3}\|$ and $\|\mathbf{V}\|(\tau)=\max\limits_{0\le\eta\le\tau}\ \max\limits_{(x_3,\tau)\in\Delta(M,T)}|\mathbf{V}(x_3,\tau)|$.

For the constant $B_0(a_0,T)=\left(\dfrac{|\nu_1|+|\nu_2|+R_1\sqrt{a_0}}{a_0}\right)$, then the operator K satisfies the property $(K2)$. □

Theorem 7.5.3. *Let $G(\nu,x_3,t)$ be as defined in (7.24) and $\mathbf{K}$ be the operator satisfying the properties $(K1)$ and $(K2)$, then there exists a solution*

$$\mathbf{V}(\nu,x_3,t)\in C(C^1(\Delta(M,T));C(\mathbf{R}^2))$$

of the integral equation (7.22).

Here, the class $C(C^1(\Delta(M,T));C(\mathbf{R}^2))$ consists of all continuous functions of $(x_3,t) \in \Delta(M,T)$ and $v = (v_1,v_2) \in \mathbf{R}^2$ that is differentiable with respect to x_3 and t over $\Delta(M,T)$, into the class $C(\mathbf{R}^2)$.

Proof. The first step of the proof is to show that each term of the sequence $\mathbf{V}^n(v,x_3,t)$, $n = 0,1,2,\ldots$ is continuous.

Let us consider the sequence of functions

$$\mathbf{V}^0(v,x_3,t) = \mathbf{G}(v,x_3,t), \tag{7.28}$$

$$\mathbf{V}^1(v,x_3,t) = \int_0^t (\mathbf{K}\mathbf{V}^0)(v,x_3,\tau)d\tau, \tag{7.29}$$

$$\vdots$$

$$\mathbf{V}^n(v,x_3,t) = \int_0^t (\mathbf{K}\mathbf{V}^{n-1})(v,x_3,\tau)d\tau. \tag{7.30}$$

By the continuity of $\mathbf{G}(v,x_3,t)$, defined in (7.24), and the property $(K1)$ of the operator $\mathbf{K}$, each term of the sequence $\mathbf{V}^n(v,x_3,t)$, $n = 0,1,2,\ldots$ is continuous.

The second step is to show that each term of the sequence $\mathbf{V}^n(v,x_3,t)$ is bounded, that is,

$$||\mathbf{V}^n||(t) \le \frac{(B_0 t)^n}{n!}||\mathbf{G}||(t), \quad \forall n = 0,1,\ldots .$$

For proving we use mathematical induction and the second property $(K2)$ of the operator $\mathbf{K}$.

Let $D(M,T,t)$ be a domain of the form

$$D(M,T,t) = \{(x,\tau) : 0 \le \tau \le t,\ (x,\tau) \in \Delta(M,T)\}.$$

For $k = 0$,

$$\begin{aligned}
|\mathbf{V}^0(v,x_3,\tau)| &= |\mathbf{G}(v,\mathbf{x_3},\tau)|, \quad (x_3,\tau) \in \Delta(M,T) \\
&\le \max_{0\le\tau\le t} \max_{(x_3,\tau)\in\Delta(M,T)} |\mathbf{G}(v,\mathbf{x_3},\mathbf{t})|, \quad (x_3,\tau) \in \Delta(M,T) \\
&\le ||\mathbf{G}||(t), \quad (x_3,t) \in D(M,T,t)
\end{aligned}$$

where $||\mathbf{V}^0||(t) = \max\limits_{0\le\tau\le t} \max\limits_{(x,t)\in\Delta(M,T)} |\mathbf{V}^0(v,x_3,\tau)|$. Thus,

$$||\mathbf{V}^0||(t) \le ||\mathbf{G}||(t).$$

For $k = 1$,

$$\begin{aligned}
|\mathbf{V}^1(\nu,x_3,\tau)| &= \left|\int_0^\tau (\mathbf{K}\mathbf{V}^0)(\nu,x_3,z)dz\right|, \quad \forall(x_3,\tau)\in\Delta(M,T)\\
&\le \int_0^\tau |(\mathbf{K}\mathbf{V}^0)(\nu,x_3,z)|dz, \quad \forall(x_3,\tau)\in\Delta(M,T)\\
&\le \int_0^\tau B_0\|\mathbf{V}^0\|(z)dz \le \int_0^\tau B_0\|\mathbf{G}\|(z)dz\\
&\le B_0\|\mathbf{G}\|(\tau)\tau, \quad \forall(x,\tau)\in D(M,T,t)\\
&\le B_0\|\mathbf{G}\|(t)t,
\end{aligned}$$

then

$$\|\mathbf{V}^1\|(t) = \max_{(x_3,\tau)\in D(M,T,t)} |\mathbf{V}^1(x,\tau)| \le \max_{0\le\tau\le t} |B_0\|\mathbf{G}\|(\tau)\tau| \le B_0 t\|\mathbf{G}\|(t).$$

Thus,

$$\|\mathbf{V}^1\|(t) \le B_0\cdot t\cdot\|\mathbf{G}\|(t), \quad \forall(x,\tau)\in D(M,T,t).$$

Now, assuming that $\|\mathbf{V}^k\|(t) \le \frac{(B_0t)^k}{k!}\|\mathbf{G}\|(t),\ \forall k = 0,1,\ldots,n$, we prove $\|\mathbf{V}^{(k+1)}\|(t) \le \frac{(B_0t)^{(k+1)}}{(k+1)!}\|\mathbf{G}\|(t)$.

$$\begin{aligned}
|\mathbf{V}^{k+1}(\nu,x_3,\tau)| &= \left|\int_0^\tau (K\mathbf{V}^k)(\nu,x_3,z)dz\right|, \quad \forall(x,\tau)\in\Delta(M,T)\\
&\le \int_0^\tau |(K\mathbf{V}^k)(\nu,x_3,z)|dz, \quad \forall(x,\tau)\in\Delta(M,T)\\
&\le \int_0^\tau B_0\|\mathbf{V}^k\|(z)dz\\
&\le \int_0^\tau \frac{B_0^{k+1}}{k!}\cdot z^k\cdot\|\mathbf{G}\|(z)dz\\
&\le \frac{B_0^{k+1}}{(k+1)!}\|\mathbf{G}\|(\tau)\tau^{k+1}, \quad \forall(x,\tau)\in D(M,T,t)\\
&\le B_0^{k+1}\|\mathbf{G}\|(t)\frac{t^{k+1}}{(k+1)!}.
\end{aligned}$$

Thus, we conclude that $\|\mathbf{V}^n\|(t) \le \frac{(B_0t)^n}{n!}\|\mathbf{G}\|(t),\ \forall n = 0,1,\ldots$ and $\|\mathbf{V}^n\|(t) \le \frac{(B_0T)^n}{n!}\|\mathbf{G}\|(T)$.

We know that the numerical series $\sum_{n=0}^{\infty}\frac{(B_0T)^n}{n!}\|\mathbf{G}\|(T)$ converges to $e^{B_0\cdot T}\cdot\|\mathbf{G}\|(T)$. Then by the Weierstrass theorem (see, Weierstrass M-Test in Cohen (2003)), the functional series $\sum_{n=0}^{\infty}\mathbf{V}^n(x,t)$ is uniformly convergent to a continuous function $\mathbf{V}(\nu,x_3,t)$ over $(x_3,t)\in\Delta(M,T)$ i.e. $\mathbf{V}(\nu,x_3,t)\in C(\Delta(M,T))$.

The last step is to show that $\mathbf{V}(v,x_3,t)$ is a solution of the integral equation (7.22). Let us consider the summation of the sequence of functions in (7.30)

$$\sum_{n=0}^{p}\mathbf{V}^n(v,x_3,t) = \mathbf{G}(v,x_3,t) + \int_0^t \sum_{n=0}^{p-1}(K\mathbf{V}^n)(v,x_3,\tau)d\tau, \tag{7.31}$$

taking limit as $p \to \infty$ and using the Dominated Convergence Theorem (see, Evans (1998)) and linearity property of the operator $\mathbf{K}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbf{V}^n(v,x_3,t) &= \lim_{p\to\infty}\sum_{n=0}^{p}\mathbf{V}^n(x,t) = \lim_{p\to\infty}\left[\mathbf{G}(v,x_3,t) + \int_0^t \sum_{n=0}^{p-1}(\mathbf{K}\mathbf{V}^n)(v,x_3,\tau)d\tau\right] \\
&= \lim_{p\to\infty}\mathbf{G}(v,x_3,t) + \int_0^t \lim_{p\to\infty}\sum_{n=0}^{p-1}(\mathbf{K}\mathbf{V}^n)(v,x_3,\tau)d\tau \\
&= \lim_{p\to\infty}\mathbf{G}(v,x_3,t) + \int_0^t \sum_{n=0}^{p-1}(\mathbf{K}\lim_{p\to\infty}\mathbf{V}^n)(v,x_3,\tau)d\tau \\
&= \mathbf{G}(v,x_3,t) + \int_0^t \sum_{n=0}^{p-1}(\mathbf{K}\mathbf{V})(v,x_3,\tau)d\tau.
\end{aligned}$$

Hence, we get

$$\mathbf{V}(v,x_3,t) = \mathbf{G}(v,x_3,t) + \int_0^t (\mathbf{K}\mathbf{V})(v,x_3,\tau)d\tau,$$

that is, $\mathbf{V}(v,x_3,t)$ is a solution of the integral equation (7.22. □

Lemma 7.5.4. *Suppose that* $\mathbf{G}(v,x_3,t) \in C(\Delta(M,T))$ *is given function,* $\mathbf{K}$ *is a linear operator satisfying the properties* $(K1)$, $(K2)$, *then the solution* $\mathbf{V}(v,x_3,t)$ *of* (7.22) *is unique in* $C(\Delta(M,T))$.

Proof. Let $\mathbf{V}^1(v,x_3,t)$ and $\mathbf{V}^2(v,x_3,t)$ be two solutions of the integral equation (7.22), that is,

$$\begin{aligned}
\mathbf{V}^1(v,x_3,t) &= \mathbf{G}(v,x_3,t) + \int_0^t (\mathbf{K}\mathbf{V}^1)(v,x_3,\tau)d\tau, \\
\mathbf{V}^2(v,x_3,t) &= \mathbf{G}(v,x_3,t) + \int_0^t (\mathbf{K}\mathbf{V}^2)(v,x_3,\tau)d\tau.
\end{aligned}$$

Subtracting $\mathbf{V}^2(v,x_3,t)$ from $\mathbf{V}^1(v,x_3,t)$

$$\mathbf{V}^1(v,x_3,t) - \mathbf{V}^2(v,x_3,t) = \int_0^t [\mathbf{K}(\mathbf{V}^1 - \mathbf{V}^2)](v,x_3,t,\tau)d\tau,$$

and letting $\mathbf{W}(\nu, x_3, t) = \mathbf{V}^1(\nu, x_3, t) - \mathbf{V}^2(\nu, x_3, t)$, we have

$$\begin{aligned}
|\mathbf{W}(\nu, x_3, t)| &= \left| \int_0^\tau [\mathbf{K}(\mathbf{V}^1 - \mathbf{V}^2)](\nu, x_3, z) dz \right| \\
&\le \int_0^\tau |\mathbf{K}(\mathbf{V}^1 - \mathbf{V}^2)|(\nu, x_3, z) dz \\
&\le B_0 \int_0^\tau ||\mathbf{W}||(z) dz, \quad \forall (x_3, \tau) \in D(M, T, t),
\end{aligned}$$

then

$$||\mathbf{W}||(t) \le B_0 \int_0^t ||\mathbf{W}||(z) dz.$$

By differentiating and multiplying both sides with $e^{B_0 \cdot t}$, we get

$$\frac{d}{dt}\left[e^{B_0 \cdot t}||\mathbf{W}||(t)\right] \le 0 \quad \Rightarrow e^{B_0 \cdot t}||\mathbf{W}||(t) \le 0 \quad \Rightarrow ||\mathbf{W}||(t) = 0.$$

Hence, $\mathbf{V}^1(\nu, x_3, t) = \mathbf{V}^2(\nu, x_3, t)$. □

As a result, we proved that there exists a unique solution $\mathbf{V}(\nu, x_3, t) \in C(\Delta(M, T))$ of the integral equation (7.22).

Theorem 7.5.5. *Suppose that* $\mathbf{G}^1(\nu, x_3, t) \in C(\Delta(M, T))$, $\mathbf{G}^2(\nu, x_3, t) \in C(\Delta(M, T))$ *are given functions,* $\mathbf{K}$ *is a linear operator satisfying* $(K1)$, $(K2)$, *and* $\mathbf{V}^1(\nu, x_3, t)$, $\mathbf{V}^2(\nu, x_3, t)$ *are two solutions of the integral equation* (7.22) *with the inhomogeneous terms* $\mathbf{G}^1(\nu, x_3, t)$, $\mathbf{G}^2(\nu, x_3, t)$, *respectively. Then there exists a constant* $L = L(M, T)$ *such that*

$$|\mathbf{V}^1(\nu, x_3, t) - \mathbf{V}^2(\nu, x_3, t)| \le L \cdot |\mathbf{G}^1(\nu, x_3, t) - \mathbf{G}^2(\nu, x_3, t)|. \tag{7.32}$$

Proof. Let $\mathbf{V}^1(\nu, x_3, t)$, $\mathbf{V}^2(\nu, x_3, t)$ be two solutions of the integral equation (7.22) with the inhomogeneous terms $\mathbf{G}^1(\nu, x_3, t)$, $\mathbf{G}^2(\nu, x_3, t)$, respectively. Then

$$\mathbf{V}^1(\nu, x_3, t) = \mathbf{G}^1(\nu, x_3, t) + \int_0^t (\mathbf{K}\mathbf{V}^1)(\nu, x_3, \tau) d\tau,$$
$$\mathbf{V}^2(\nu, x_3, t) = \mathbf{G}^2(\nu, x_3, t) + \int_0^t (\mathbf{K}\mathbf{V}^2)(\nu, x_3, \tau) d\tau.$$

Subtracting $\mathbf{V}^2(\nu, x_3, t)$ from $\mathbf{V}^1(\nu, x_3, t)$, we get

$$\mathbf{V}^1(\nu, x_3, t) - \mathbf{V}^2(\nu, x_3, t) = \mathbf{G}^1(\nu, x_3, t) - \mathbf{G}^2(\nu, x_3, t) + \int_0^t [\mathbf{K}(\mathbf{V}^1 - \mathbf{V}^2)](\nu, x_3, \tau) d\tau,$$

let $\tilde{W}(v,x_3,t) = \mathbf{V}^1(v,x_3,t) - \mathbf{V}^2(v,x_3,t)$ and $\tilde{G}(v,x_3,t) = \mathbf{G}^1(v,x_3,t) - \mathbf{G}^2(v,x_3,t)$

$$\begin{aligned}
|\tilde{W}(v,x_3,t)| &= \left|\tilde{G}(v,x_3,t) + \int_0^\tau [\mathbf{K}(\tilde{W})](v,x_3,z)dz\right| \\
&\leq |\tilde{G}(v,x_3,t)| + \left|\int_0^\tau [(\mathbf{K}\tilde{W})](v,x_3,z)dz\right| \\
&\leq |\tilde{G}(v,x_3,t)| + \int_0^\tau |(\mathbf{K}\tilde{W})(v,x_3,z)|dz \\
&\leq |\tilde{G}(v,x_3,t)| + B_0\int_0^\tau ||\tilde{W}||(z)dz, \quad \forall (x_3,\tau) \in D(M,T,t)
\end{aligned}$$

then

$$||\tilde{W}||(t) \leq ||\tilde{G}||(t) + B_0\int_0^t ||\tilde{W}||(z)dz.$$

By the Gronwall's Lemma, we have

$$||\tilde{W}||(t) \leq ||\tilde{G}||(t)e^{B_0 t} \leq ||\tilde{G}||(t)e^{B_0 T}.$$

Hence, the inequality (7.32) is proved for a constant $L = e^{B_0 T}$. □

Theorem 7.5.6. *Let* $\mathbf{G}(v,x_3,t) \in C^1(\Delta(M,T))$ *is given function,* $\mathbf{K}$ *is a linear operator satisfying the properties* $(K1)$, $(K2)$*, and*

$$\int_0^t \frac{\partial}{\partial x_3}[(\mathbf{K}\mathbf{V})(v,x_3,\tau)]\,d\tau \in C(\Delta(M,T)), \quad \int_0^t \frac{\partial}{\partial t}[(K\mathbf{V})(v,x_3,\tau)]\,d\tau \in C(\Delta(M,T)).$$

Then

$$\frac{\partial \mathbf{V}}{\partial x_3}(x,t) \in C(\Delta(M,T)), \quad \frac{\partial \mathbf{V}}{\partial t}(x,t) \in C(\Delta(M,T)).$$

Proof. By the Theorem 7.5.3, there exists a solution $\mathbf{V}(v,x_3,t) \in C(\Delta(M,T))$ of the integral equation (7.22) in the form

$$\mathbf{V}(v,x_3,t) = \mathbf{G}(v,x_3,t) + \int_0^t (\mathbf{K}\mathbf{V})(v,x_3,\tau)d\tau.$$

Since, $\mathbf{V}(v,x_3,t) \in C(\Delta(M,T))$ and by assumption $\mathbf{G}(v,x_3,t) \in C^1(\Delta(M,T))$,

$$\int_0^t \frac{\partial}{\partial x_3}[(\mathbf{K}\mathbf{V})(v,x_3,\tau)]\,d\tau \in C(\Delta(M,T)), \quad \int_0^t \frac{\partial}{\partial t}[(\mathbf{K}\mathbf{V})(v,x_3,\tau)]\,d\tau \in C(\Delta(M,T)).$$

Additionally, the upper and lower limits of the integral is differentiable functions over $\Delta(M,T)$, so we can take the differentiation with respect to x_3 and t

$$\frac{\partial \mathbf{V}}{\partial t}(v,x_3,t) = \frac{\partial \mathbf{G}}{\partial t}(v,x_3,t) + (\mathbf{K}\mathbf{V})(v,x_3,t) + \int_0^t \frac{\partial}{\partial t}(\mathbf{K}\mathbf{V})(v,x_3,\tau)d\tau \in C(\Delta(M,T)),$$

$$\frac{\partial \mathbf{V}}{\partial x_3}(\nu, x_3, t) = \frac{\partial \mathbf{G}}{\partial x_3}(\nu, x_3, t) + \int_0^t \frac{\partial}{\partial x_3}(\mathbf{K}\mathbf{V})(\nu, x_3, \tau)d\tau \in C(\Delta(M,T)).$$

□

Corollary 7.5.7. *There exists a unique solution of the IVP* (7.14) – (7.15)

$$\mathbf{V}(\nu, x_3, t) \in C(C^1(\Delta(M,T)); C(\mathbf{R}^2)).$$

7.6 Application of the Inverse Fourier Transform and Finding A Solution of (7.1)-(7.3)

Lemma 7.6.1. *Under assumptions above,* $G_j(\nu, x_3, t) \in C(C^1(\Delta(M,T); C_0(\mathbf{R}^2)))$ *and has finite support with respect to* ν*, for all* $j = 1, 2, 3$.

Proof. Let $G_j(\nu, x_3, t)$ be defined as in (7.24) where

$$\mathbf{V}^0(\nu, x_3, t) = T^*(x_3)\widetilde{\mathbf{v}}_0(\nu, x_3, t) = T^*(x_3)F_{x_1,x_2}[\vec{\mathbf{v}}_0(\mathbf{x}, t)](\nu),$$

$$F(\nu, x_3, t) = T^*(x_3)\widetilde{\mathbf{f}}(\nu, x_3, t) = T^*(x_3)F_{x_1,x_2}[\vec{\mathbf{f}}(\mathbf{x}, t)](\nu),$$

and $F_{x_1,x_2}[\cdot](\nu)$ denotes the Fourier transform with respect to ν and

$$F_{x_1,x_2}[\vec{\mathbf{f}}] = \begin{pmatrix} -\widetilde{J}_1(\nu, x_3, t) \\ -\widetilde{J}_2(\nu, x_3, t) \\ -\widetilde{J}_3(\nu, x_3, t) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since, we assumed that $\widetilde{J}_k \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2))$, then the function $F(\nu, x_3, t) \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2))$. Hence, $\mathbf{G}(\nu, x_3, t) \in C(C^1(\Delta(M,T); C_0(\mathbf{R}^2)))$. □

Lemma 7.6.2. *Under the above assumptions and the linear operator K is, defined as in* (7.23)*, satisfying the properties* $(K1) - (K2)$ *then the unique solution* $\mathbf{V}(\nu, x_3, t)$ *has a finite support with respect to* ν*, that is,*

$$\mathbf{V}(\nu, x_3, t) \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2)).$$

Proof. The integral equation has the form

$$V_j(\nu, x_3, t) = G_j(\nu, x_3, t) + \int_0^t (K\mathbf{V})j(\nu, x_3, \tau)d\tau,$$

then

$$\begin{aligned}
|V_j(\nu, x_3, t)| &= \left| G_j(\nu, x_3, t) + \int_0^t (K\mathbf{V})j(\nu, x_3, \tau)d\tau \right| \\
&\leq |G_j(\nu, x_3, t)| + \int_0^t |(K\mathbf{V})j(\nu, x_3, \tau)|\, d\tau \\
&\leq |G_j(\nu, x_3, t)| + \int_0^t |(K\mathbf{V})j(\nu, x_3, \tau)|\, d\tau \\
&\leq \max_{(x_3,t)\in\Delta(M,T)} |G_j(\nu, x_3, t)| + B_0 \int_0^t \max_{(x_3,t)\in\Delta(M,T)} |V_j(\nu, x_3, \tau)| d\tau, \\
&\leq ||\mathbf{G}|| + B_0 \int_0^t ||\mathbf{V}||(\tau)d\tau, \quad \forall (x,t) \in \Delta(M,T).
\end{aligned}$$

Hence,

$$||\mathbf{V}||(\tau) \leq ||\mathbf{G}||(\tau) + B_0 \int_0^t ||\mathbf{V}||(\tau)d\tau,$$

and by Gronwall's Lemma,

$$||\mathbf{V}||(t) \leq ||\mathbf{G}||(t)e^{B_0 t} \leq ||\mathbf{G}||(T)e^{B_0 T}. \tag{7.33}$$

By Lemma 7.6.1 $G_j(\nu, x_3, t)$ has a finite support with respect to ν then there exists a constant A so that if $|\nu| > A$ then $|G_j(\nu, x_3, t)| \equiv 0$. Hence if we take $|\nu| > A$ we have by the equation (7.33),

$$||\mathbf{V}||(t) \leq 0 \quad \Rightarrow ||\mathbf{V}||(t) = 0, \text{ for } (x_3, t) \in \Delta(M,T) \text{ and } |\nu| > A$$

That means the solution $\mathbf{V}(\nu, x_3, t)$ of the problem $(7.14) - (7.15)$ has a finite support with respect to ν and considering the Lemma 7.5.2, the solution belongs to the class

$$\mathbf{V}(\nu, x_3, t) \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2)).$$

□

Lemma 7.6.3. *If* $G_j(\nu, x_3, t) \in C(C^1(\Delta(M,T); C_0(\mathbf{R}^2)))$ *is defined as in* (7.24)*, the linear operator* K *is defined as in* (7.23) *and satisfying the properties* $(K1) - (K2)$ *then there exists a unique solution*

$$\widetilde{\mathbf{v}} \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2))$$

of IVP (7.8) – (7.9).

Proof. By the Lemma 7.5.2 and Lemma 7.6.2 there exists a unique solution $\mathbf{V}(\nu, x_3, t)$ and

$$T^*(x_3)\widetilde{\mathbf{v}}(\nu, x_3, t) = \mathbf{V}(\nu, x_3, t) \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2)).$$

Thus,

$$\widetilde{\mathbf{v}}(\nu, x_3, t) \in C(C^1(\Delta(M,T)); C_0(\mathbf{R}^2)).$$

□

That is, $\widetilde{\mathbf{v}}(\nu, x_3, t)$ is a solution of IVP (7.8) – (7.9) and has finite support with respect to ν. So, by taking the Inverse Fourier transform with respect to $\nu = (\nu_1, \nu_2)$

$$\vec{\mathbf{v}}(x,t) = \iint_{\mathbf{R}^2} \widetilde{\mathbf{v}}(\nu, x_3, t) e^{i\bar{x}\nu} d\nu,$$

$$\vec{\mathbf{v}}_0(x) = \iint_{\mathbf{R}^2} \widetilde{\mathbf{v}}_0(\nu, x_3) e^{i\bar{x}\nu} d\nu,$$

where $\vec{\mathbf{v}}(x,t) = F_\nu[\widetilde{\mathbf{v}}(\nu, x_3, t)](\bar{x})$ and $\widetilde{\mathbf{v}}_0(x) = F_\nu[\widetilde{\mathbf{v}}_0(\nu, x_3)](\bar{x})$. Hence, there exists a unique solution

$$\vec{\mathbf{v}}(x,t) \in C^1(\Delta(M,T);\ PW(\mathbf{R}^2)).$$

We note that $C^1(\Delta(M,T);\ PW(\mathbf{R}^2))$ is the class of all continuous functions of $(x_3, t) \in \Delta(M,T)$ and $(x_1, x_2) \in \mathbf{R}^2$ that is differentiable with respect to x_3 and t over $\Delta(M,T)$ and belongs to $PW(\mathbf{R}^2)$ for $(x_1, x_2) \in \mathbf{R}^2$.

Here, $PW(\mathbf{R}^2)$ denotes the Paley Weiner Space of $\mathbf{R}^2$. The definition of Paley Wiener Space and the real version of Paley Wiener Theorem is included in Appendix A9.

7.7 Summary of Chapter Seven

An analytical method for solving the initial value problem for the time-dependent anisotropic Maxwell's system with differentiable function coefficients has been suggested. The initial value problem for the anisotropic Maxwell's system has been reduced to an initial value problem for the symmetric hyperbolic system of the first order. Then applying Fourier transform with respect to the variables x_1 and x_2 the initial value problem has been written in terms of Fourier images. Applying integration along characteristics to the obtained symmetric hyperbolic system has been reduced to the initial value problem for the system of Volterra integral equations. Applying method of successive approximations, a solution of this integral equation system has been found in the space $C(C^1(\Delta(M;T));C_0(\mathbf{R}^2))$. Finally, applying the inverse Fourier transform, a solution of the original initial value problem for the Maxwell's equations has been determined in the space $C^1(\Delta(M;T);PW(\mathbf{R}^2))$ (see, AppendixA9).

CHAPTER EIGHT
CONCLUSION

The main results of the thesis are the following:

- The method of the approximate computation of the magnetic and electric matrix Green's functions in the rectangular parallelepiped with the perfect conducting boundary and the multi-layered isotropic structures has been developed. This method based on the Fourier series expansion meta-approach. The method has been implemented in MATLAB and the error analysis of this method has been done by MATLAB tools. The computational experiments confirmed the robustness of the method. Moreover it has been shown that the method is applicable for the numerical computation of the electromagnetic wave propagation in rectangular parallelepiped arising from the pulse point sources. The modern computer tools allow us to observe the electromagnetic wave propagation.

- The method of solving the initial value problem for the time-dependent Maxwell's equations in the general electrically and magnetically anisotropic vertical inhomogeneous media has been found. This method based on the following. The Maxwell's equations are written as the symmetric hyperbolic system of the first order. This system is written in terms of the Fourier images with respect to lateral variables. The coefficients of this system are functions depending on vertical variable only. Applying method of integration along the characteristics the symmetric hyperbolic system is reduced to a system of the Volterra integral equations. Method of successive approximations is applied for solving the obtained system of the integral equations. Finally, the inverse Fourier transform is applied to find a solution of the initial value problem for the time-dependent Maxwell's equations.

The result of Chapter 2 were published in the work:

- V.Yakhno and S.Ersoy, Computing the electric and magnetic matrix Green's functions in a rectangular parallelepiped with a perfect conducting boundary, Abstract and Applied Analysis, Vol.(2014), 1-13, 2014 (doi:10.1155/2014/586370). (sci-expanded)

The work about the result of Chapter 3 is in press:

- V.G.Yakhno and Ş.Ersoy, The computation of the time-dependent magnetic and electric matrix Green's functions in a parallelepiped, Applied Mathematical Modelling, (2015), http://dx.doi.org/10.1016/j.apm.2015.01.057. (sci-expanded)

Conference papers are the followings:

- S. Ersoy (Kecelli) and V. Yakhno, A Boundary Value Problem of the Frequency Dependent Maxwell's System for Layered Materials. 14th International Congress on Computational and Applied Mathematics (ICCAM-2009), 99-100, Antalya, 2009.
- S. Ersoy and V.G. Yakhno, The Computation of the Frequency-Dependent Magnetic and Electric Green's Functions in Layered Rectangular Parallelepiped. 2nd International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2013), Sarajevo, 2013.

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APPENDICES

A1 Properties of the Eigenvalues and Eigenfunctions of (2.7)-(2.8)

Property 1. All eigenvalues of the problem (2.7)-(2.8) are real and nonnegative.

Proof. We have

$$\langle -\Delta_x \mathbf{I}_3 \mathbf{U}(x), \mathbf{U}(x)\rangle = -\sum_{j=1}^{3} \int\int_V\int \Delta U_j(x) U_j(x)\, dx, \tag{A1.1}$$

where $\langle .,. \rangle$ is the inner (dot) product in $\mathbf{R}^3$.

Using the first Green's identity (see, for example, Vladimirov (1971)) we find that

$$-\sum_{j=1}^{3} \int\int_V\int \Delta U_j(x) U_j(x) dx = \sum_{j=1}^{3} \int\int_V\int |\nabla_x U_j|^2 dx - \sum_{j=1}^{3} \int\int_\Gamma U_j \frac{\partial U_j}{\partial n} d\Gamma \tag{A1.2}$$

Using (2.9) we have from (A1.1), (A1.2)

$$\langle -\Delta_x \mathbf{I}_3 \mathbf{U}(x), \mathbf{U}(x)\rangle = \sum_{j=1}^{3} \int\int_V\int |\nabla_x U_j|^2\, dx. \tag{A1.3}$$

With (2.7), we find

$$\sum_{j=1}^{3} \int\int_V\int |\nabla_x U_j|^2\, dx = \lambda \int\int_V\int |\mathbf{U}(x)|^2\, dx,$$

or

$$\lambda = \sum_{j=1}^{3} \frac{\int\int_V\int |\nabla_x U_j|^2}{\int\int_V\int |\mathbf{U}(x)|^2\, dx} \geq 0. \tag{A1.4}$$

This means that eigenvalues of (2.7),(2.8) are real and nonnegative. □

Property 2. $\lambda = 0$ is not an eigenvalue of (2.7),(2.8).

Proof. Let $\lambda = 0$ be an eigenvalue of (2.7)-(2.8). Then we have

$$\langle -\Delta_x \mathbf{I}_3 \mathbf{U}(x), \mathbf{U}(x)\rangle = 0. \tag{A1.5}$$

Using (A1.3) and (A1.5) we have $|\nabla U_j|^2 = 0$, that is, $U_j(x) = constant$, $j = 1,2,3$. This means that, if $\lambda = 0$ is an eigenvalue of the problem (2.7),(2.8) then $\mathbf{U}(x) = constant$ is an eigenfunction corresponding to this value, and it needs to satisfy (2.7),(2.8). But, $\mathbf{U}(x) \equiv 0$ due to the second equation in (2.8). This is a contradiction because the eigenfunction should be different from zero. Therefore, $\lambda = 0$ is not an eigenvalue of (2.7),(2.8). □

A2 Properties of the Dirac Delta Function

Let x^0 be a 3D parameter from V, $\delta(x-x^0)=\delta(x_1-x_1^0)\delta(x_2-x_2^0)\delta(x_3-x_3^0)$ be the Dirac delta function concentrated at $x_1=x_1^0$, $x_2=x_2^0$, $x_3=x_3^0$, and $h(x)\in C_0^\infty(V)$. We have the following properties of the Dirac delta function:

Property 1.

$$\frac{d}{dx_j}\delta(x-x^0)=-\frac{d}{dx_j^0}\delta(x-x^0),\quad j=1,2,3. \tag{A2.6}$$

Proof. We can prove this property as follows

$$\begin{aligned}\left\langle\frac{\partial}{\partial x_j}\delta(x-x^0),h(x)\right\rangle &= -\left\langle\delta(x-x^0),\frac{\partial}{\partial x_j}h(x)\right\rangle=-\frac{\partial}{\partial x_j^0}h(x^0)\\ &= -\frac{\partial}{\partial x_j^0}\left\langle\delta(x-x^0),h(x)\right\rangle=\left\langle-\frac{\partial}{\partial x_j^0}\delta(x-x^0),h(x)\right\rangle,\end{aligned}$$

$j=1,2,3.$ □

Property 2.

$$curl_x\left[\delta(x-x^0)\vec{\mathbf{e}}^s\right]=-curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}}^s\right],\quad s=1,2,3.$$

Proof. Let for example, $s=1$, the vector distribution $curl_x\left[\delta(x-x^0)\vec{\mathbf{e}}^1\right]$ can be written as follows

$$curl_x\left[\delta(x-x^0)\vec{\mathbf{e}}^1\right]=0\cdot\vec{\mathbf{e}}^1+\frac{\partial\delta(x-x^0)}{\partial x_3}\vec{\mathbf{e}}^2-\frac{\partial\delta(x-x^0)}{\partial x_2}\vec{\mathbf{e}}^3.$$

Using the equality (A2.6) we get

$$\begin{aligned}curl_x\left[\delta(x-x^0)\vec{\mathbf{e}}^1\right] &= 0\cdot\vec{\mathbf{e}}^1+\frac{\partial\delta(x-x^0)}{\partial x_3}\vec{\mathbf{e}}^2-\frac{\partial\delta(x-x^0)}{\partial x_2}\vec{\mathbf{e}}^3\\ &= 0\cdot\vec{\mathbf{e}}^1-\frac{\partial\delta(x-x^0)}{\partial x_3^0}\vec{\mathbf{e}}^2+\frac{\partial\delta(x-x^0)}{\partial x_2^0}\vec{\mathbf{e}}^3\\ &= -curl_{x^0}\left[\delta(x-x^0)\vec{\mathbf{e}}^1\right].\end{aligned}$$

Similar arguments hold for $j=2,3.$ □

A3 Proof of the Lemma 2.3.1

Let $D^{\alpha} = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}}$ be the differential operator, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1, \alpha_2, \alpha_3$ are non-negative integers. We have

$$\int\int\int_V [D^{\alpha} h(x)]^2\, dx < \infty, \tag{A3.7}$$

for an arbitrary $h(x) \in C_0^{\infty}(V)$.

Using the Parseval's equality (2.16) we find

$$\int\int\int_V [D^{\alpha} h(x)]^2\, dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[h_{j,\alpha}^{kmn}\right]^2 < \infty. \tag{A3.8}$$

Here $h_{j,\alpha}^{kmn}$ denotes the Fourier coefficients of the function $D^{\alpha}h(x)$ relative to basis functions $\left\{U_j^{kmn}(x)\right\}$, $k, m, n = 0, 1, \ldots;\ j = 1, 2, 3$.

For even positive integers $\alpha_1, \alpha_2, \alpha_3$ we have

$$h_{j,\alpha}^{kmn} = (-1)^{|\alpha|/2}\left(\frac{k\pi}{b_1}\right)^{\alpha_1}\left(\frac{m\pi}{b_2}\right)^{\alpha_2}\left(\frac{n\pi}{b_3}\right)^{\alpha_3} h_j^{kmn}, \tag{A3.9}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Using (2.13),(A3.9) we have

$$\begin{aligned} \left|\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0} h_2^{kmn}(x)\right| &\le \left|\frac{n\pi}{b_3} B_{kmn} h_2^{kmn}\right| \le 2c_1\left|\frac{1}{k^{\alpha_1} m^{\alpha_2} n^{\alpha_3-1}} h_{2,\alpha}^{kmn}\right| \\ &\le c_1\left[\frac{1}{k^{2\alpha_1} m^{2\alpha_2} n^{2\alpha_3-2}} + \left|h_{2,\alpha}^{kmn}\right|^2\right], \end{aligned} \tag{A3.10}$$

where $c_1 = \pi^{1-|\alpha|}\sqrt{2/(b_1 b_2 b_3)}\, b_1^{\alpha_1} b_2^{\alpha_2} b_3^{\alpha_3-1}$.

The convergence of $\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |h_{2,\alpha}^{kmn}|^2$ follows from (A3.8). The series

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{k^{2\alpha_1} m^{2\alpha_2} n^{2\alpha_3-2}}$$

converges for $2\alpha_1 > 1$, $2\alpha_2 > 1$, $2(\alpha_3 - 1) > 1$. Therefore, taking even positive integers $\alpha_1, \alpha_2, \alpha_3$ and using equality (A3.10), we find that the series (2.21) is convergent for any fixed $x^0 \in V$ and $h(x) \in C_0^{\infty}(V)$. The convergence of (2.22) can be established similarly.

Lemma 2.3.1 is proved.

A4 Convergence of the series (2.40)

Let us consider the convergence of the following series for any fixed $x^0 \in V$ and any function $h(x) \in C_0^\infty(V)$:

$$-\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}\int\int_V\int U_1^{kmn}(x)\frac{\partial h(x)}{\partial x_3}dx, \tag{A4.11}$$

$$-\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_3^{kmn}(x^0)}{\partial x_1^0}\int\int_V\int U_3^{kmn}(x)\frac{\partial h(x)}{\partial x_1}dx, \tag{A4.12}$$

$$\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}U_2^{kmn}(x^0)h_2^{kmn}. \tag{A4.13}$$

Let $h(x)$ be an arbitrary function from $C_0^\infty(V)$ and $\phi(x) = \dfrac{\partial h(x)}{\partial x_3}$; $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where α_j, $j = 1,2,3$ are natural numbers, such that $2\alpha_1 > 1$, $2\alpha_2 > 1$, $2(\alpha_3 - 1) > 1$. Let ϕ_1^{kmn}, $\phi_{1,\alpha}^{kmn}$ be the Fourier coefficients of $\phi(x)$ and $D^\alpha\phi(x)$ relative to the basis $\{U_1^{kmn}(x)\}$, $k = 1,2,...;\ m,n = 0,1,2,3,....$ Then we have

$$\left|\frac{1}{\omega\epsilon(\omega^2\epsilon\mu-\lambda_{kmn})}\frac{\partial U_1^{kmn}(x^0)}{\partial x_3^0}\int\int_V\int U_1^{kmn}(x)\phi(x)dx\right| \le \left(\frac{A_{kmn}}{\omega\epsilon(\lambda_{kmn}-\omega^2\epsilon\mu)}\right)\frac{n\pi}{b_3}\left|\phi_1^{kmn}\right|$$
$$\le \frac{A_{kmn}}{\omega\epsilon}\frac{\pi n}{b_3}\left|\phi_1^{kmn}\right| \le c_2\left[\frac{1}{k^{2\alpha_1}m^{2\alpha_2}n^{2(\alpha_3-1)}}+\left|\phi_{1,\alpha}^{kmn}\right|^2\right],$$

where $c_2 = \frac{\pi^{1-|\alpha|}}{\omega\epsilon}\sqrt{2/(b_1b_2b_3)}\,b_1^{\alpha_1}b_2^{\alpha_2}b_3^{(\alpha_3-1)}$. Using the technique similar to Appendix A3 we find that $\sum\sum\sum c_2\left[\frac{1}{k^{2\alpha_1}m^{2\alpha_2}n^{2(\alpha_3-1)}}+\left|\phi_{1,\alpha}^{kmn}\right|^2\right]$ converges and as a result of it, the series (A4.11) converges absolutely for any fixed $x^0 \in V$ and any function $h(x) \in C_0^\infty(V)$. The convergence of the series (A4.12), (A4.13) for any fixed $x^0 \in V$ and an arbitrary function $h(x) \in C_0^\infty(V)$ can be established in a similar way. Convergence of series (A4.11), (A4.12), (A4.13) implies the absolute convergence the series in (3.40) for any fixed $x^0 \in V$ and arbitrary $h(x) \in C_0^\infty(V)$.

A5 Elements of the Distribution Theory

Let $V \in \mathbf{R}^3$ be an open set. Similar to Vladimirov (2002), $\mathcal{D}(V)$ is the space of the test functions which consists of the infinitely differentiable functions with compact supports in V. The linear continuous functionals over $\mathcal{D}(V)$ are called distributions. We denote the space of all distributions as $\mathcal{D}'(V)$. A sequence of distributions $T_1, T_2, ..., T_k, ...$ in $\mathcal{D}'(V)$ converges to a distribution $T \in \mathcal{D}'(V)$ if for any test function $h \in \mathcal{D}(V)$

$$\langle T_k, h\rangle \to \langle T, h\rangle \;\; as \;\; k \to \infty.$$

This convergence is termed as weak convergence. Distributions determined by functions locally integrable in V are termed regular distributions. All other distributions are said to be singular. From the Du Bois Reymond lemma (see, Vladimirov (2002), pp. 15-16) it follows that any regular distribution in V is defined by a unique function (unique up to the values on a set of measure zero) that is locally integrable in V. Consequently, there is a one-to-one correspondence between locally integrable functions in V and regular distributions in V. For this reason we identify a function locally integrable in V with the regular distribution from $\mathcal{D}'(V)$ that is generated by it. In this sense, locally integrable functions in V are regular distributions from $\mathcal{D}'(V)$. We shall say that the regular distribution T belongs to the class of p-times continuously differentiable functions $C^p(V)$ if for any test function $h \in \mathcal{D}(V)$ the functional T coincides with the functional defined by the rule

$$\langle T, h\rangle = \int_V f_0(x)h(x)dx,$$

where $f_0(x)$ is a function of the class $C^p(V)$. It is clear from the definition of the singular distributions that they have no values at separate point $x \in V$ and their graphs can not be drawn. The simple example of a singular distribution is the Dirac delta function $\delta(x - x^0)$ defined for any test function $h \in \mathcal{D}(V)$ as the functional

$$\langle \delta(x - x^0), h\rangle = h(x^0).$$

The other example of a singular distribution is

$$\frac{\Theta(t)}{2\pi c}\delta(c^2t^2 - |x|^2), \tag{A5.14}$$

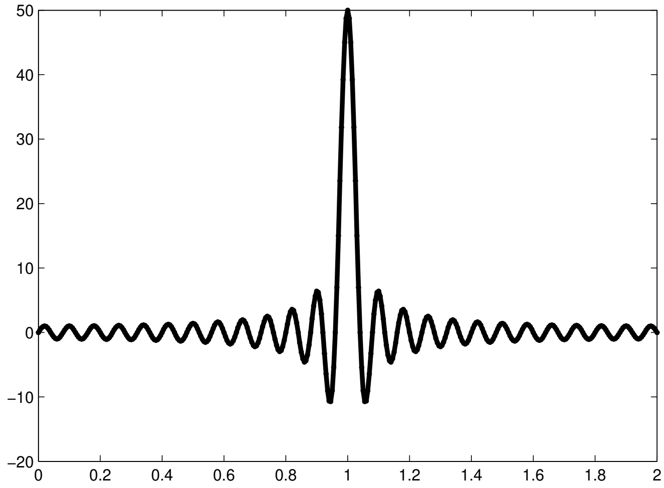

Figure A.1 The graph of the function $\delta_N(x, x^0)$ for $N = 100$, $x^0 = 1$.

where $x = (x_1, x_2, x_3)$ is the space variable, $|x|^2 = x_1^2 + x_2^2 + x_3^2$, t is the time variable, $c > 0$ is a constant, $\Theta(t)$ is the Heaviside step function (see Section 1.3), $\delta(t)$ is the Dirac delta function of one variable t (see, for example, Vladimirov (1971), pp. 149). This singular distribution is the fundamental solution of the wave operator $\frac{\partial^2}{\partial t^2} - c^2\Delta$. Generally speaking, generalized functions do not have values at separate points. Nevertheless, one may speak of the vanishing a generalized function in an open set. We say that a generalized function $T \in \mathcal{D}'(V)$ vanishes in an open set $V' \subset V$ if $\langle T, h\rangle = 0$ for any test function $h \in \mathcal{D}(V')$ and we write, formally, $T(x) = 0$ for $x \in V'$ (see, for example, Vladimirov (2002), pp. 13).

From the completeness of the space $\mathcal{D}'(V)$ (see, Vladimirov (2002), pp. 12-13) there follows: any weak limit of locally integrable functions in V is a distribution in $\mathcal{D}'(V)$. Therefore, it is possible to construct a theory of distributions by proceeding from weakly convergent sequences of ordinary locally integrable functions (see this approach in Antosik & Mikusinski (1973)). Completing the set of locally integrable functions by all weak limits of sequences of locally integrable functions, we obtain all distributions of $\mathcal{D}'(V)$.

If the sequence of distributions defined by continuous and locally integrable functions $\{U_k\}_{k=1,2,...}$ in V weakly converges to the distribution $T \in \mathcal{D}'(V)$ as $k \to \infty$,

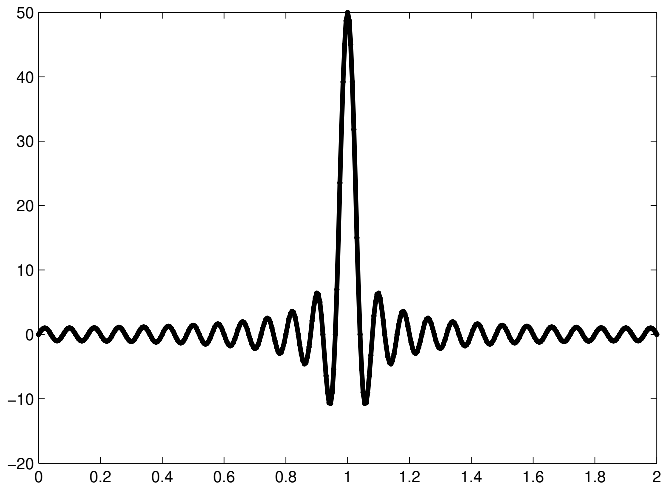

Figure A.2 The graph of the function $\frac{\partial \delta_N(x,x^0)}{\partial x^0}$ for $N = 100$, $x^0 = 1$.

then for fixed value of k, U_k is called the regularization (or approximation) of the distribution T and k is the parameter of this regularization. As the regularizations of the Dirac distribution $\delta(x - x^0) \in \mathcal{D}'(0, b)$ and its derivative $\frac{\partial \delta(x-x^0)}{\partial x^0} \in \mathcal{D}'(0, b)$, where $x^0 \in (0, b)$, we can take (see, for example, Kanwal (1983), pp. 65-66)) the following continuous and locally integrable functions

$$\delta_N(x, x^0) = \frac{2}{b} \sum_{n=1}^{N} \sin\left(\frac{n\pi}{b} x^0\right) \sin\left(\frac{n\pi}{b} x\right),$$

$$\frac{\partial \delta_N(x, x^0)}{\partial x^0} = \frac{2}{b^2} \sum_{n=1}^{N} n\pi \cos\left(\frac{n\pi}{b} x^0\right) \sin\left(\frac{n\pi}{b} x\right).$$

The graphs of the functions $\delta_N(x, x^0)$ and $\frac{\partial \delta_N(x,x^0)}{\partial x^0}$ ($N = 100$) are presented in Figures A.1, A.2. The generalized functions $\delta(x - x^0)$ and $\frac{\partial \delta(x-x^0)}{\partial x^0}$ are equal to zero for all $x \in R$ for which $x \neq x^0$ (see, for example, Vladimirov (2002)). At the same time their approximations $\delta_N(x, x^0)$, $\frac{\partial \delta_N(x,x^0)}{\partial x^0}$ have a ripple in x-axis. This ripple is an error of the approximation.

A6 Generalized Solution of the Wave Equation

Approximate Green's function for the wave equation in V:

Let $u(x,t,x^0)$ be a solution of

$$\frac{\partial^2 H_2^1}{\partial t^2} = c^2 \Delta_x H_2^1 - c^2 \frac{\partial}{\partial x_3^0}\delta(x-x^0)\delta(t), \tag{A6.15}$$

$$H_2^1\big|_{t\le 0} = 0, \quad \left.\frac{\partial H_2^1}{\partial t}\right|_{t\le 0} = 0, \tag{A6.16}$$

$$H_2^1\big|_{x_2=0} = 0, \; H_2^1\big|_{x_2=b_2} = 0, \quad 0 \le x_1 \le b_1, \; 0 \le x_3 \le b_3, \tag{A6.17}$$

$$\left.\frac{\partial H_2^1}{\partial x_1}\right|_{x_1=0} = 0, \; \left.\frac{\partial H_2^1}{\partial x_1}\right|_{x_1=b_1} = 0, \quad 0 \le x_2 \le b_2, \; 0 \le x_3 \le b_3, \tag{A6.18}$$

$$\left.\frac{\partial H_2^1}{\partial x_3}\right|_{x_3=0} = 0, \; \left.\frac{\partial H_2^1}{\partial x_3}\right|_{x_3=b_3} = 0, \quad 0 \le x_1 \le b_1, \; 0 \le x_2 \le b_2. \tag{A6.19}$$

when the inhomogeneous term $-c^2 \frac{\partial}{\partial x_3^0}\delta(x-x^0)\delta(t)$ is replaced by $\delta(x-x^0)\delta(t)$. This generalized function $u(x,t,x^0)$ is the Green's function of the wave equation in V. Applying the method of Section 3.3 we find an approximate Green's function in the form

$$u^N(x,t,x^0) = \Theta(t) \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} \frac{\sin(c\sqrt{\lambda_{kmn}}t)}{c\sqrt{\lambda_{kmn}}} U_2^{kmn}(x^0) U_2^{kmn}(x),$$

where λ_{kmn} and $U_2^{kmn}(x)$ are defined in Section 2. The graph of $u^N(x,t,x^0)$ (N=100) is presented in Figure A.3 for $t = 0.4$, $x_1^0 = x_2^0 = 1$, $x_3^0 = 0.5$, $c = 1$, $x_3 = 0.5$, $x_2 = 1$, $0 \le x_1 \le 3$. This graph has two peaks at $r = 0.6$ and $r = 1.4$ and ripple around zero for all r outside the neighborhood of these two points.

We note that the Green's function $u(x,t,x^0)$ coincides with the fundamental solution of the wave operator (see formula (A5.14)) for all times t when the wave front $|x-x^0| = ct$ does not reach the boundary Γ of the parallelepiped V. We note also that the fundamental solution of the wave operator is zero for $|x-x^0| > ct$ and the sequence $u^N(x,t,x^0)$ has a limit in the space of generalized functions $\mathcal{D}'(V \times \mathbf{R})$ for each parameter x^0 from V. This means that $u^N(x,t,x^0)$ is an approximation of $u(x,t,x^0)$.

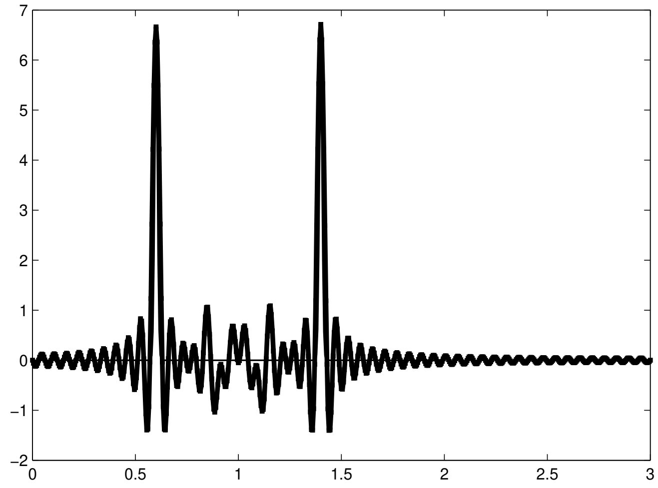

Figure A.3 The graph of approximate Green's function $u^N(x,t,x^0)$ of the wave equation for $N = 100$, $c = 1$, $x^0 = (1,1,0.5)$, $x_3 = 0.5$, $x_2 = 1$, $0 \leq x_3 \leq 3$ at times $t = 0.4$.

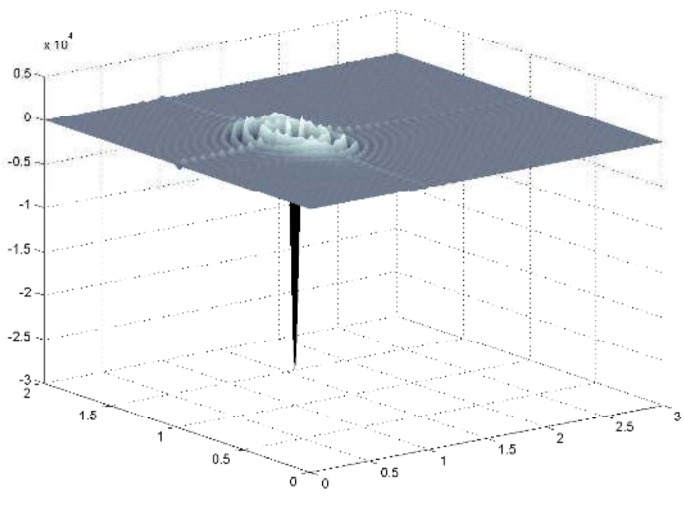

Figure A.4 The map of the surfaces $z = w^N(x,t,x^0)$ for $N = 100$, $x^0 = (1,1,0.5)$, $c = 1$, $x_3 = 0.5$ at times $t = 0.3$.

A generalized solution of the wave equation in V and its approximate computation:

Let $v(x,t,x^0)$ be a solution of (A6.15)-(A6.19) when the inhomogeneous term $-c^2\frac{\partial}{\partial x_3^0}\delta(x-x^0)\delta(t)$ is replaced by $\frac{\partial^2}{\partial x_3 \partial x_3^0}\delta(x-x^0)\delta(t)$ and $w(x,t,x^0)$ be defined by

$$w(x,x^0,t) = \int_{-\infty}^{t} v(x,x^0,\tau)d\tau.$$

Applying the approximate computation of $v(x,t,x^0)$ by the method of the Section 3.3 and then integrating from $-\infty$ to t we find an approximation of $w(x,t,x^0)$ in the form

$$w^N(x,t,x^0) = \Theta(t)\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left[\frac{\cos(c\sqrt{\lambda_{kmn}}t)-1}{c^2\lambda_{kmn}}\right]\frac{\partial^2 U_2^{kmn}(x^0)}{\partial x_3^0}\frac{\partial U_2^{kmn}(x)}{\partial x_3}.$$

The graphic presentation of the surface $z = w^N(x,t,x^0)$ ($N = 100$) for $x^0 = (1,1,0.5)$, $c = 1$, $x_3 = 0.5$ at times $t = 0.3$ is given in Figure A.4.

Using (Yakhno & Ersoy, 2014) we can show that $\lim_{N\to\infty} w^N = w$ in the space of generalized functions $\mathcal{D}'(V\times\mathbf{R})$ for any $x^0 \in V$ and therefore w^N ($N = 100$) is an approximation of w. Moreover, we can show that w is equal to the generalized function

$$\frac{\partial^2}{\partial x_3 \partial x_3^0}\left[\frac{\Theta(ct-|x-x^0|)}{4\pi c^2|x-x^0|}\right]$$

for all times t when the wave front $|x-x^0| = ct$ did not reach the boundary Γ of V. Hence w is zero outside the front sphere $|x-x^0| = ct$. In Figure A.4 there is a visible (especially in vertical and horizontal directions) ripple corresponding to points for which $|x-x^0| > ct$. This ripple is an error of approximation w by means of w^N ($N = 100$). We note that it is impossible to define a norm for singular generalized functions. For this reason we can not evaluate accuracy of the approximation in the classical sense. The comparison of generalized functions w and w^N can be made by the comparison of their integral characteristics similar to Section 3.5.1.

A7 Orthogonality Property of the Eigenfunctions

The set of the functions $U_j^{kmn}(x)$, $k,m,n = 0,1,...$, and $U_j^{k'm'n'}(x)$, $k',m',n' = 0,1,...$ satisfy the orthogonality property, that is,

$$\int_0^{b_1}\int_0^{b_3}\left[\int_0^{b_2} \nu(x_2)U_j^{kmn}(x)U_j^{k'm'n'}(x)\,dx_2\right]dx_3\,dx_1 =$$
$$\begin{cases} 0, & \text{if } k \neq k' \text{ } or \text{ } m \neq m' \text{ } or \text{ } n \neq n', \\ \kappa_{kmn}, & \text{if } k = k',\ m = m',\ n = n', \end{cases}$$

where $\kappa_{kmn} > 0$, $j = 1,2,3$.

Proof. For the proof we consider the case $j = 2$.

Let $(k,m,n) \neq (k',m',n')$, λ_{kmn} and $\lambda_{k'm'n'}$ be distinct eigenvalues and $U_2^{kmn}(x)$, $U_2^{k'm'n'}(x)$ be the corresponding eigenfunctions $k,m,n = 0,1,\ldots$; $k',m',n' = 0,1,\ldots$ of the eigenvalue-eigenfunction problem (5.8)-(5.13). Consider the following partial differential equations

$$c^2(x_2)\Delta_x U_2^{kmn}(x) + \lambda_{kmn}U_2^{kmn}(x) = 0,$$
$$c^2(x_2)\Delta_x U_2^{k'm'n'}(x) + \lambda_{k'm'n'}U_2^{k'm'n'}(x) = 0.$$

Multiplying the first equation by $U_2^{k'm'n'}(x)$ and the second one by $U_2^{kmn}(x)$ and making subtraction we get

$$c^2(x_2)\left[\Delta_x U_2^{kmn}(x)U_2^{k'm'n'}(x) - \Delta_x U_2^{k'm'n'}(x)U_2^{kmn}(x)\right] +$$
$$(\lambda_{kmn} - \lambda_{k'm'n'})U_2^{kmn}(x)U_2^{k'm'n'}(x) = 0.$$

Integrating the last relation with respect to x_1 from 0 to b_1, with respect to x_2 from 0 to b_2, and with respect to x_3 from 0 to b_3 we find

$$\int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \nu(x_2)c^2(x_2)\left[\Delta_x U_2^{kmn}(x)U_2^{k'm'n'}(x) - \Delta_x U_2^{k'm'n'}(x)U_2^{kmn}(x)\right]\,dx_1dx_2dx_3$$
$$+\int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \nu(x_2)\left(\lambda_{kmn} - \lambda_{k'm'n'}\right)U_2^{kmn}(x)U_2^{k'm'n'}(x)\,dx_1dx_2dx_3 = 0.$$

Applying integration by parts many times, using the boundary conditions (5.9), (5.10), (5.11) and matching conditions (5.12), (5.13) we get

$$(\lambda_{kmn} - \lambda_{k'm'n'}) \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \nu(x_2) U_2^{kmn}(x) U_2^{k'm'n'}(x)\, dx_1 dx_2 dx_3 = 0.$$

Since for $(k,m,n) \neq (k',m',n')$, $\lambda_{kmn} \neq \lambda_{k'm'n'}$ we obtain

$$\int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \nu(x_2) U_2^{kmn}(x) U_2^{k'm'n'}(x)\, dx_1 dx_2 dx_3 = 0.$$

We note that for $(k,m,n) = (k',m',n')$ the left hand side of the last integral should be different from zero since the eigenfunctions $U_2^{kmn}(x)$ are nonzero functions.

Letting

$$\kappa_{kmn} = \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \nu(x_2) \left[U_2^{kmn}(x)\right]^2 \, dx_1 dx_2 dx_3 \neq 0,$$

we conclude that

$$\int_0^{b_1}\int_0^{b_3}\left[\int_0^{b_2} \nu(x_2) U_2^{kmn}(x) U_2^{k'm'n'}(x)\, dx_2\right] dx_3\, dx_1 =$$

$$\begin{cases} 0, & \text{if } k \neq k' \text{ } or \text{ } m \neq m' \text{ } or \text{ } n \neq n', \\ \kappa_{kmn}, & \text{if } k = k',\ m = m',\ n = n', \end{cases}$$

where $\kappa_{kmn} > 0$.

The proof for U_2 and U_3 is similar. □

A8 Reduction of IVP of the Maxwell System to IVP for a Symmetric Hyperbolic System

Let us consider the IVP for the anisotropic Maxwell's system (7.1)-(7.3). Under the assumptions given in Section 7.1.1 and using the definition of $curl_x$ equation (7.1) can be rewritten in the following form:

$$\mathcal{E}\frac{\partial \mathbf{E}}{\partial t} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \frac{\partial \mathbf{H}}{\partial x_1} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{\partial \mathbf{H}}{\partial x_2} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial \mathbf{H}}{\partial x_3} = -\mathbf{J}.$$

Let us make the following denotations

$$A_1^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; A_2^1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; A_3^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then equation (7.1) takes the following form

$$\mathcal{E}\frac{\partial \mathbf{E}}{\partial t} + \sum_{j=1}^{3} \mathbf{A}_j^1 \frac{\partial \mathbf{H}}{\partial x_j} = -\mathbf{J}. \tag{A8.20}$$

Applying the same procedure to equation (7.2) we get

$$\mathcal{M}\frac{\partial \vec{H}}{\partial t} + \sum_{j=1}^{3} (A_j^1)^* \frac{\partial \vec{E}}{\partial x_j} = \mathbf{0}. \tag{A8.21}$$

Then equations (7.1) and (7.2) can be written as

$$\begin{pmatrix} \mathcal{E}(x_3)_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \mathcal{M}(x_3)_{3\times 3} \end{pmatrix}_{6\times 6} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_{6\times 1} + \sum_{j=1}^{3} A_j \frac{\partial}{\partial x_j} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_{6\times 1} = -\begin{pmatrix} \mathbf{J} \\ 0 \end{pmatrix}_{6\times 1}. \tag{A8.22}$$

Let

$$\vec{\mathbf{v}}(x,t) = \begin{bmatrix} \mathbf{E} & \mathbf{H} \end{bmatrix}_{1\times 6}^* = \begin{bmatrix} E_1 & E_2 & E_3 & H_1 & H_2 & H_3 \end{bmatrix}^*,$$

$$\vec{\mathbf{f}}(x,t) = \begin{bmatrix} -\mathbf{J} & \mathbf{0} \end{bmatrix}_{1\times 6}^* = \begin{bmatrix} -J_1 & -J_2 & -J_3 & 0 & 0 & 0 \end{bmatrix}^*,$$

$$\vec{\mathbf{v}}_0(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^*,$$

$$A_0(x) = \begin{pmatrix} \mathcal{E}(x)_{3\times 3} & 0 \\ 0 & \mathcal{M}(x)_{3\times 3} \end{pmatrix}_{6\times 6}, \quad A_j = \begin{pmatrix} 0_{3\times 3} & A_j^1 \\ (A_j^1)^* & 0_{3\times 3} \end{pmatrix}_{6\times 6},$$

then (A8.22) and the initial data (7.3) takes the form

$$\mathbf{A}_0(x_3)\frac{\partial \vec{\mathbf{v}}(x,t)}{\partial t} + \sum_{j=1}^{3} \mathbf{A}_j \frac{\partial \vec{\mathbf{v}}(x,t)}{\partial x_j} = \vec{\mathbf{f}}(x,t), \quad x \in \mathbf{R}^3, \quad t \in \mathbf{R}^+,$$

$$\vec{\mathbf{v}}(x,t)|_{t=0} = \vec{\mathbf{v}}_0(x), \quad x \in \mathbf{R}^3,$$

that is, the initial value problem of a symmetric hyperbolic system.

A9 Paley-Wiener Space and the Real Version of Paley-Wiener Theorem

The result here has been taken from the paper Andersen (2004) (see also Bang (1995), Tuan & Zayed (2002)).

It is well known that the classical Fourier transform F is an isomorphism of the Schwartz space $S(\mathbf{R}^k)$ onto itself (see Vladimirov (1971), Andersen (2004), Bang (1995), Tuan & Zayed (2002)). The space $C_0^\infty(\mathbf{R}^k)$ which consists of all smooth functions with compact support is dense in the Schwartz space $S(\mathbf{R}^k)$, and as the classical Paley Wiener theorem states that the Fourier transform image of $C_0^\infty(\mathbf{R}^k)$ is described as a rapidly decreasing function having a holomorphic extension to $\mathbf{C}^k$ of exponential type. Here, we give the definition of the Paley-Wiener space in order to consider the real version of the Paley-Wiener theorem.

Definition A9.1. The Paley-Wiener space $PW(\mathbf{R}^k)$ is the space of all functions $\varphi(x) \in C^\infty(\mathbf{R}^k)$ satisfying

(a) $(1+|x|)^m \Delta^n \varphi(x) \in \mathcal{L}_2(\mathbf{R}^2)$ for all $m, n \in \{0, 1, 2...\}$,

(b) $R_\varphi^\Delta = \lim_{n\to\infty} ||\Delta^n \varphi(x)||_2^{1/2n} < \infty$

where $\mathcal{L}_2(\mathbf{R}^2)$ is the space of square integrable functions with the norm $||\varphi||_2 = \left(\int_{\mathbf{R}^2} |\varphi(x)|^2\, dx\right)^{1/2}$ for any $\varphi(x) \in \mathcal{L}_2(\mathbf{R}^2)$; $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_k^2$ is the Laplacian on $\mathbf{R}^k$. Further, $PW_B(\mathbf{R}^k) = \left\{\varphi(x) \in PW(\mathbf{R}^k) : R_\varphi^\Delta = B\right\}$ for $B \geq 0$.

Theorem A9.2. *The inverse Fourier transform* F^{-1} *is a bijection on* $C_0^\infty(\mathbf{R}^k)$ *onto* $PW(\mathbf{R}^k)$, *mapping* $C_B^\infty(\mathbf{R}^k)$ *onto* $PW_B(\mathbf{R}^k)$. *Here* $C_B^\infty(\mathbf{R}^k)$ *is defined as*

$$C_B^\infty(\mathbf{R}^k) = \{\varphi(x) \in C_B^\infty(\mathbf{R}^k) : R_\varphi = B\},$$

where $R_\varphi = \sup_{x \in supp\varphi} |x|$ *is the radius of the support of* $\varphi(x)$.

Proof. The proof of the theorem is included in the paper Andersen (2004). □