PSEUDOSPECTRA OF MATRICES

by
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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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ABSTRACT

PSEUDOSPECTRA OF MATRICES

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In this study, we investigate the spectral and pseudospectral points of the quadratic and cubic polynomials with matrix coefficients (pencils) $\mathrm{P}(\lambda)=\mathrm{P}_1(\lambda)\mathrm{P}_2(\lambda)$ and $\mathrm{P}(\lambda)=\mathrm{P}_1(\lambda)\mathrm{P}_2(\lambda)\mathrm{P}_3(\lambda)$. Where $\mathrm{P}_1(\lambda)$, $\mathrm{P}_2(\lambda)$, $\mathrm{P}_3(\lambda)$ are linear pencils. The $\varepsilon-$pseudospectrum for these types of quadratic and cubic matrix polynomials can be detailed by means of the linear pencils above mentioned. But this can be possible for matrix polynomials of small dimensions. The spectrum of linear pencil coincides with the spectrum of matrix of dimension NxN, in the usual sense. Shortly, the pseudospectral properties of polynomials with matrix coefficients can be studied by means of linear pencils factors.

In the second chapter, we survey the qualitative properties concerning boundedness and connected components of pseudospectra for general matrix polynomials.

This study presents computed examples for the relation between the pseudospectra of matrices and quadratic and cubic pencils with using Mathematica programs and applications of numerical analysis.

Keywords: Matrix polynomial, eigenvalue, perturbation, $\varepsilon-$pseudospectrum.

ÖZET

MATRİSLERİN PSEUDOSPEKTRUMLARI

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Bu çalışmada, $P(\lambda) = P_1(\lambda)\ P_2(\lambda)$ ve $P(\lambda) = P_1(\lambda)\ P_2(\lambda)\ P_3(\lambda)$ şeklindeki kuadratik ve kübik matris katsayılı polinomların spektral ve pseudospektral noktalarını araştırıyoruz. Burada $P_1(\lambda),\ P_2(\lambda),\ P_3(\lambda)$ polinomları birer lineer demettir. Bu tipte verilen kuadratik ve kübik matris polinomlarının pseudospektrumu lineer demetleri yardımıyla bulunabilir. Bu küçük boyutlu matris polinomları için mümkün olmaktadır. Bir lineer demetin spectrumu, genel anlamda, NxN boyutlu matrisin spektrumu ile aynıdır. Kısaca, matris katsayılı polinomların pseudospektrumu, ele alınan matris katsayılı polinomun lineer çarpanlar şeklinde yazılışı mümkünse, lineer demetlerin spektrumu ve pseudospektrumu yardımıyla bulunabilir.

İkinci bölümde ise, genel matris polinomları için pseudospektrumun bağlı bileşenleri ve sınırlılığına ilişkin niteliksel özellikleri inceliyoruz.

Bu çalışmada, matrislerle kuadratik ve kübik demetlerin pseudospektrumu arasındaki ilişki ile ilgili örneklerin araştırılmasında, Mathematica programlarından ve nümerik analiz tekniklerinden yararlanılmıştır.

Anahtar Kelimeler: Matris polinomu, özdeğer, öteleme, $\varepsilon-$pseudospektrum.

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CHAPTER 1

INRODUCTION

1.1 Introduction

Let $\mathbb{C}^{N\times N}$ be the algebra of all $N\times N$ complex matrices and consider the matrix polynomial,

$$\mathrm{P}(\lambda)=\mathrm{A}_m\lambda^m+\mathrm{A}_{m-1}\lambda^{m-1}+...+\mathrm{A}_1\lambda+\mathrm{A}_0, \tag{1.1}$$

where λ is a complex variable and $\mathrm{A}_j\in\mathbb{C}^{N\times N},\quad j=0,1,2,...,m$ with $\det\mathrm{A}_m\neq 0$. If $m=2$, then $\mathrm{P}(\lambda)$ is called *quadratic matrix polynomial* or *quadratic pencil* and is written in the following form,

$$\mathrm{P}(\lambda)=\mathrm{A}_2\lambda^2+\mathrm{A}_1\lambda+\mathrm{A}_0. \tag{1.2}$$

If $m=3,$ it is called *cubic matrix polynomial* or cubic pencil and is written in the following form,

$$\mathrm{P}(\lambda)=\mathrm{A}_3\lambda^3+\mathrm{A}_2\lambda^2+\mathrm{A}_1\lambda+\mathrm{A}_0. \tag{1.3}$$

The study of polynomials with matrix coefficients has a long history, especially with regard to their spectral analysis. The one of these studies belongs to F. R. Gantmacher [1]. The Russian name of the book consisting his studies is *Teoriya*

Matrits. In this book, treatment of the characteristic and minimal polynomial and functions of matrices were searched.

In 1982, the book which was written by I. Gohberg, P. Lancaster and L. Rodman provides a comprehensive treatment of the theory of matrix polynomials higher degrees, and forms an important new part linear algebra for which the main concepts and results have been arrived since 1977 [2].

The operator polynomials of the form,

$$\mathrm{A}(\lambda) = \mathrm{A}_0 + \lambda \mathrm{A}_1 + \ldots + \lambda^n \mathrm{A}_n,$$

were investigated by A. S. Markus [3]. Here λ is a spectral parameter and $\mathrm{A}_0,\ \mathrm{A}_1, \ldots, \mathrm{A}_n$ are linear operators acting in a Hilbert space. We encounter some applications of the results to the spectral theory of differential operators in A. S. Markus works.

F. Tisseur and N. J. Higham [4] defined the *weighted pseudospectrum* which is mentioned in this thesis. However, they obtained a graphical representation of $\sigma_{\varepsilon,w}(\mathrm{P})$ by evaluating $s_{\min}(\mathrm{P}(z))$ on a grid of points in the complex plane. The $\varepsilon-$pseudospectrum of the scalar polynomial of the form

$$p(\lambda) = a_m \lambda^m + a_{m-1} \lambda^{m-1} + \ldots + a_1 \lambda + a_0\,,$$

was introduced by R. G. Mosier [5]. The $\varepsilon-$pseudozero set of $p(\lambda)$ was investigated by K. C. Toh and L. N. Trefethen [6].

The case for further development of algorithms for matrix polynomials rests largely on the pervasive second (and higher) degree polynomials used in the analysis of vibrating systems. Efficiencies are gained by avoiding linearizations and, as with problem areas already developed, pseudospectra give valuable insights into the ensitivities of spectra and, particularly, can be expected to clarify the effects of clustered and multiple eigenvalues.

1.2 History of Pseudospectra

In this section we attempt to survey the history of the subject of pseudospectra. In the context of Hermitian and near-Hermitian systems, mathematical physicists have for years spoken of *quasimodes,* which are the same as what we would call *pseumodes* or *pseudoeigenvectors.*

In the nonnormal context, the earliest definition of pseudospectra we have encountered is given in J.M.Varah's 1967, thesis at Stanford University, *The Computation of Bounds for the Invariant Subspaces of a General Matrix Operator*. Varah introduced the notion of an $\varepsilon-$pseudo-eigenvalue under the name r-approximate eigenvalue, as shown in Definition 1.1. Motivated by his analysis of the accuracy of eigenpairs produced by computer implementations of the inverse iteration algorithm, Varah incorporated a parameter, η_1, in his definition to describe floating-point precision.

Definition 1.1 $\begin{Bmatrix} \lambda \\ y \end{Bmatrix}$ is an r-approximate $\begin{Bmatrix} eigenvalue \\ eigenvector \end{Bmatrix}$ of A if there exists a matrix E with $\|\mathrm{E}\|_2 = r.\eta_1$ such that $\begin{Bmatrix} \lambda \\ y \end{Bmatrix}$ is an exact $\begin{Bmatrix} eigenvalue \\ eigenvector \end{Bmatrix}$ of $\mathrm{A}+\mathrm{E}$.

This definition is first definition from Varah's unpublished 1967 thesis [7]. Landau applied this concept to the theory of Toeplitz matrices and associated integral operators.

Definition 1.2 λ is an ε-approximate eigenvalue of A, if there exists $\varphi \in D(rQ)$, with $\|\varphi\| = 1$, such that $\|A_r\varphi - \lambda\varphi\| \le \varepsilon$. We call φ an ε-approximate eigenfunction corresponding to λ.

This definition is first published definition of pseudospectra from Landau, 1975 [8].

In Novosibirsk, S. K. Godunov and his colleagues conducted research related to pseudospectra throughout the 1980s. Those involved included A. G. Antonov, A. Y. Bulgakov (later Haydar Bulgak), O. P. Kirilyuk, V. I. Kostin, A. N. Malyshev, and S. I. Razzakov. Godunov, together with Ryabenkii and others, had made significant contributions in the 1960s to the study of how nonnormality affects the numerical stability of discretized differential equations. In fact, in their 1962 monograph, Godunov and Ryabenkii introduce the *spectrum of the family of operators* $\{R_h\}$, indexed by a mesh parameter h [9]. A point $z \in \mathbb{C}$ is in this set if, for every $\varepsilon > 0$, z is an $\varepsilon-$pseudoeigenvalue of R_h for all sufficiently small h.

The Novosibirsk group defined the $\varepsilon-$*spectrum* by relative rather than absolute perturbations:

$$\sigma_\varepsilon(A) = \{\ z \in \mathbb{C}:\ \|(z-A)^{-1}\| \ge (\varepsilon\|A\|)^{-1}\ \}$$

In [10] and [11], computed plots of pseudospectra were presented and called 'spectral portraits of matrices' containing various 'patches of spectrum', and computations based on both contour plotting and curve tracing were discussed. A sketch of two pseudospectra had appeared earlier in 1985 a paper by Kostin and Razzakov [12].

Pseudospectra were investigated in several papers by J. W. Demmel in the mid-1980s, appearing with the labels $S(A,\varepsilon)$ in [13] and $\sigma(\varepsilon, A)$ in [14]. The former paper contains the first published computer plot.

Beginning in the mid-1980s, D. Hinrichsen and A. J. Pritchard wrote a number of papers on the *stability radius* of matrix, i.e., the distance to the set of unstable matrices. In a 1992 paper [15] they introduce the term *spectral value set* and notation $\sigma(A,\rho)$ to denote the real structured $\varepsilon-$pseudospectrum of a nonnormal matrix. Another paper by Hinrichsen and Kelb in 1993 extended these ideas to complex perturbations and also to more general structured perturbations [16].

Trefethen's first publications that mentioned pseudospectra were [17] and [18] in 1990 (the first use the term $\varepsilon-$*approximate eigenvalues*). These were an outgrowth of a 1987 paper by Trefethen and Trummer, who found eigenvalues that were extraordinarily sensitive to perturbations but failed fully to appreciate their significance [19]. A crucial collaborator in this work was Trefethen's student Satish Reddy, who began to work with pseudospectra in 1988 and made many contributions after that date. Reddy's early work on these topics was summarized in his 1991 thesis at MIT [20].

In 1992 Trefethen's paper "Pseudospectra of Matrices" appeared, which presented the idea of pseudospectra and exhibited thirteen examples [21]. It was after this point that the idea began to be widely known. A later companion paper, “Pseudospectra of Linear Operators”, presented ten examples involving operators on infinite-dimensional spaces [22].

The papers by Demmel and Wilkinson mentioned above cite each other, and they both cite paper of Varah [23]. Apart from these cases, none of the papers discussed here published before 1990 cite any of the others. These data suggest that pseudospectra have been invented at least five times.

J. M. Varah 1967, $r-$approximate eigenvalues, 1979 $\varepsilon-$spectrum

H. J. Landau 1975 $\varepsilon-$approximate eigenvalues

S. K. Godunov et. al. 1982 spectral portraits

L. N. Trefethen 1990 $\varepsilon-$pseudospectrum

D. Hinrichsen and A. J. Pritchard 1992 spectral value

1.3 Pseudospectra of Matrices

Eigenvalues (more generally, spectra) have been a standard tool of the mathematical sciences for a century and a half. Since the arrival of computers, they have also been a standard tool of scientific computing.

In the simplest context of matrices, the definition are as follows. Let A be an $N \times N$ matrix with real or complex coefficient; we write $\mathrm{A} \in \mathbb{C}^{N\times N}$. Let v be a nonzero real or complex column vector of length N, and let λ be a real or complex scalar; we write $v \in \mathbb{C}^{N}$, and $\lambda \in \mathbb{C}$. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A and v is an eigenvector its corresponding eigenvalue, if

$$\mathrm{A}v = \lambda v\,.$$

The set of all eigenvalues of A is the spectrum of A, a nonempty subset of the complex plane $\mathbb{C}$ that we denote by $\sigma(\mathrm{A})$,

$$\sigma(\mathrm{A}) = \{\ \lambda \in \mathbb{C} :\ |\lambda \mathrm{I} - \mathrm{A}| = 0\ \}$$

where I is identity matrix. The spectrum can also be defined as the set of points $\lambda \in \mathbb{C}$ where the resolvent matrix

$$(\lambda - \mathrm{A})^{-1}$$

does not exist. $(\lambda - \mathrm{A})$ is shorthand for $(\lambda \mathrm{I} - \mathrm{A})$, where I is identity matrix [24]. The complement of the spectrum of A is called resolvent set of A and it is denoted by $\rho(A)$. Namely, $\rho(A) = \mathbb{C} - \sigma(A)$.

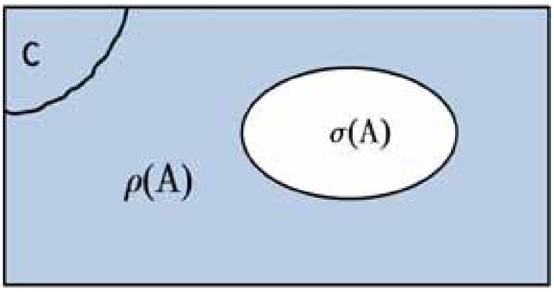


Figure 1.1: Where $\mathbb{C} = \rho(\mathrm{A}) \cup \sigma(\mathrm{A})$ and $\rho(\mathrm{A}) \cap \sigma(\mathrm{A}) = \varnothing$

Definition 1.3 The Euclidean norm of a vector $x = [x_1, x_2, x_3, ..., x_n] \in \mathbb{C}^N$ is:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}. \tag{1.4}$$

Definition 1.4 The Euclidean norm of a matrix $\mathrm{A}_{m \times n}$ is:

$$\|\mathrm{A}\|_2 = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}, \tag{1.5}$$

where a_{ij} is the element of i -th row and j -th column of the matrix A.

Some Basic Definitions for Pseudospectra of Matrices

Definition 1.5 Let $A \in \mathbb{C}^{N \times N}$ and $\varepsilon > 0$ be arbitrary. The $\varepsilon-$pseudospectrum $\sigma_\varepsilon(A)$ of A is the set of $\lambda \in \mathbb{C}$ such that,

$$\left\|(\lambda I - A)^{-1}\right\| > \varepsilon^{-1}. \tag{1.6}$$

Definition 1.6 $\sigma_\varepsilon(A)$ is the set of $\lambda \in \mathbb{C}$ such that,

$$\lambda \in \sigma(A + E), \tag{1.7}$$

for some $E \in \mathbb{C}^{N \times N}$ with $\|E\| \leq \varepsilon$.

In the words, the $\varepsilon-$pseudospectrum is the set of numbers that are eigenvalues of some perturbed matrix $A + E$ with $\|E\| < \varepsilon$. The examples in this thesis will be based on the use of the 2-norm, defined by (1.4) for a vector x and by (1.5) for a matrix A.

Definition 1.7 $\sigma_\varepsilon(A)$ is the set of $\lambda \in \mathbb{C}$ such that,

$$\|(\lambda - A)v\| < \varepsilon, \tag{1.8}$$

for some $v \in \mathbb{C}^N$ with $\|v\| = 1$.

The number λ in (1.6) (or equivalently in any of our definitions) is an $\varepsilon-$pseudo-eigenvalue of A, and v is a corresponding $\varepsilon-$pseudo-eigenvector. In words, the $\varepsilon-$pseudospectrum is the set of $\varepsilon-$pseudo-eigenvalues.

Definition 1.8 For $\| \; \| = \| \; \|_2$, $\sigma_\varepsilon(A)$ is the set of $\lambda \in \mathbb{C}$ such that,

$$s_{\min}(\lambda - A) < \varepsilon, \tag{1.9}$$

where $s_{\min}(\lambda - \mathrm{A})$ denotes the smallest singular value of $\lambda - \mathrm{A}$. If $\| \; \| = \| \; \|_2$, the norm of a matrix is its largest singular value and the norm of the inverse is the inverse of the smallest singular value. In particular,

$$\left\|(\lambda - \mathrm{A})^{-1}\right\|_2 = [s_{\min}(\lambda - \mathrm{A})]^{-1}. \tag{1.10}$$

From (1.10) it is clear that (1.9) is equivalent to (1.6).

CHAPTER 2

PSEUDOSPECTRA OF MATRIX POLYNOMIALS

2.1 Basic Definitions

Definition 2.1 (Eigenvalue, Eigenvector, Spectrum of a Matrix Polynomial) A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of the matrix polynomial $\mathrm{P}(\lambda)$ in (1.1) , if the system

$$\mathrm{P}(\lambda_0)v_0 = 0, \tag{2.1}$$

has a nonzero solution $v_0 \in \mathbb{C}^N$. This solution v_0 is known as an *eigenvector* of $\mathrm{P}(\lambda)$ corresponding to λ_0. The set of all eigenvalues of $\mathrm{P}(\lambda)$ is the *spectrum* of $\mathrm{P}(\lambda)$

$$\sigma(\mathrm{P}) = \{\ \lambda \in \mathbb{C}:\ \det(\mathrm{P}(\lambda)) = 0,\ (\mathrm{P}(\lambda)) \in \mathbb{C}^{N\times N}\ \}. \tag{2.2}$$

Since $\det \mathrm{A}_m \neq 0$ (1.1), $\sigma(\mathrm{P})$ contains no more than Nm distinct eigenvalues.

Definition 2.2 (Matrix Norm) We consider matrix norms on ($\mathbb{C}^{m\times n}, \mathbb{C}$). All results hold for $(\mathbb{R}^{m\times n}, \mathbb{R})$. A function $\|\ \| : \mathbb{C}^{m\times n} \to \mathbb{R}$ is called a matrix norm on $\mathbb{C}^{m\times n}$ if for all $\mathrm{A}, B \in \mathbb{C}^{m\times n}$ and $\alpha \in \mathbb{C}$.

M1) $\|A\| \geq 0$ (positivity),

M2) $\|A\| = 0 \Leftrightarrow A = 0$,

M3) $\|\alpha A\| = |\alpha|\ \|A\|$ (homogeneity),

M4) $\|A + B\| \le \|A\| + \|B\|$ (subadditivity).

A matrix norm is simply a vector norm on the finite dimensional vector spaces $(\mathbb{C}^{m\times n}, \mathbb{C})$ of mxn matrices.

Definition 2.3 (Subordinate Matrix Norm) A matrix norm $\|\ \|$ on $\mathbb{C}^{m\times n}$ is subordinate to the vector norms $\|\ \|_\alpha$ on $\mathbb{C}^n$ and $\|\ \|_\beta$ on $\mathbb{C}^m$ if,

$$\|\mathrm{A}x\|_\beta \le \|\mathrm{A}\|\ \|x\|_\alpha, \qquad \text{for all } \mathrm{A} \in \mathbb{C}^{m\times n} \text{ and } x \in \mathbb{C}^n.$$

Definition 2.4 (Spectral Norm) Let A^{T} be the conjugate transpose of the square matrix A, so that $(a_{ij})^t = (\overline{a_{ji}})$, then the spectral norm is defined as the square root of the maximum eigenvalue of $\mathrm{A}^T\mathrm{A}$, i.e.,

$$\|\mathrm{A}\|_2 = \text{(maximum eigenvalue of } \mathrm{A}^{\mathrm{T}}\mathrm{A})^{1/2} = \max_{\|x\|_2 \ne 0} \frac{\|\mathrm{A}x\|_2}{\|x\|_2}$$

Definition 2.5 (Weighted Pseudospectrum) Consider the pseudospectrum of perturbations of the matrix polynomial $\mathrm{P}(\lambda)$ in (1.1) of the form

$$\mathrm{P}_\Delta(\lambda) = (\mathrm{A}_m + \Delta_m)\lambda^m + (\mathrm{A}_{m-1} + \Delta_{m-1})\lambda^{m-1} + \ldots + (\mathrm{A}_1 + \Delta_1)\lambda + \mathrm{A}_0 + \Delta_0, \qquad (2.3)$$

where the matrices $\Delta_0,\ \Delta_1, \ldots, \Delta_m \in \mathbb{C}^{N\times N}$ are arbitrary. A weighted pseudospectrum is defined as follows: For a given $\varepsilon > 0$ and a given set of nonnegative weights $w = \{\omega_0, \omega_1, \ldots, \omega_m\}$, the $\varepsilon-$pseudospectrum of $\mathrm{P}(\lambda)$ with respect to w is defined to be

$$\sigma_{\varepsilon,w}(\mathrm{P}) = \{\ \lambda \in \mathbb{C}:\ \det(\mathrm{P}_{\Delta}(\lambda)) = 0,\ \left\|\Delta_j\right\| \le \varepsilon\omega_j\ \}, \quad j = 0,1,...,m \tag{2.4}$$

where $\|\ \|$ is any subordinate matrix norm.

The parameters $\omega_0, \omega_1, ..., \omega_m \ge 0$ allow freedom in how perturbations are measured; for example, in an absolute sense, when $\omega_0 = \omega_1 = ... = \omega_m = 1$, or, in a relative sense when $\omega_j = \left\|\mathrm{A}_j\right\|$ ($j = \{0,1,2,...,m\}$). Note also that, when $\varepsilon = 0$, $\sigma_{0,w}(\mathrm{P}) = \sigma(\mathrm{P})$.

Definition 2.6 (Associated Compact Set of Perturbations) The associated compact set of perturbations of $\mathrm{P}(\lambda)$ in (1.1) can be defined in the form

$$\mathrm{B(P,}\varepsilon\mathrm{,w)} = \{\ \mathrm{P}_{\Delta}(\lambda):\ \left\|\Delta_j\right\| \le \varepsilon w_j;\ j = 0,1,2,...,m\ \}, \tag{2.5}$$

where $\mathrm{P}_{\Delta}(\lambda)$ as defined in (2.3). So, the $\varepsilon-$pseudospectrum of $\mathrm{P}(\lambda)$ can also expressed in the form

$$\sigma_{\varepsilon,w}(\mathrm{P}) = \{\ \lambda \in \mathbb{C}:\ \det(\mathrm{P}_{\Delta}(\lambda)) = 0,\ \mathrm{P}_{\Delta}(\lambda) \in \mathrm{B}(\mathrm{P},\varepsilon,w)\ \}.$$

Definition 2.7 (Self-Adjoint Matrix Polynomial) If

$$\mathrm{A}_j^* = \mathrm{A}_j, \qquad\qquad j = 0,1,2,...,m$$

i.e., if all the coefficients of $\mathrm{P}(\lambda)$ in (1.1) are hermitian, then $\mathrm{P}(\lambda)$ is said to be self-adjoint matrix polynomial. Here, A_j^* is conjugate transpose of the matrix A.

Definition 2.8 (Root Neighborhoods of a Polynomial) The root neigborhoods of $\mathrm{P}(\lambda)$, a scalar polynomial over the complex field, are the sets of complex numbers that are the roots of polynomials which are near to $\mathrm{P}(\lambda)$. The term "near" means that the coefficients of the polynomials are within some fixed magnitude of the coefficients of $\mathrm{P}(\lambda)$.

Definition 2.9 ($\varepsilon-$Pseudozero Set of $p(\lambda)$) The $\varepsilon-$pseudozero set $Z_{\varepsilon}(p)$ is the set of zeros of all polynomials $\hat{p}$ obtained by coefficientwise perturbations of p of size $\leq \varepsilon$; where p is a scalar polynomial over the complex field.

Definition 2.10 (Compact Set) A set S of real numbers is called compact if every sequence in S has a subsequence that converges to an element again contained in S.

Definition 2.11 (Topological Space) A topology on a set X is a collection $\Im$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\Im$.

(2) The union of the elements of any subcollection of $\Im$ is in $\Im$.

(3) The intersection of the elements of any finite subcollection of $\Im$ is in $\Im$.

A set X for which a topology $\Im$ has been specified is called a topological space.

Definition 2.12 (Connected Component) A topological space decomposes into its connected components. The connectedness relation between two pairs of points satisfies transitivity, i.e., if $a \sim b$ and $b \sim c$ then $a \sim c$. Hence, being in the same component is an equivalence relation, and the equivalence classes are the connected components. Using connectedness, the connected component containing $x \in \mathrm{X}$ is the set of all y connected to x. That is, it is the set of y such that there is a continuous path from x to y.

Definition 2.13 (Closed Set) In a topological space, a set is closed if and only if it coincides with its closure. Equivalently, a set is closed if and only if it contains all of its limit points.

Definition 2.14 (Connected Set) A connected set is a set that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Definition 2.15 (Convex Set) Let C be a subset in a real or complex vector space. C is said to be a convex if, for all x and y in C and all t in the interval [0,1], the point $(1-t)\ x+ty$ is in C. In other words, every point on the line segment connecting x and y is in C.

Definition 2.16 (Toeplitz Matrix) Any $N \times N$ matrix A of the form,

$$\mathrm{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & a_2 & \vdots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ddots & \ddots & \vdots \\ \vdots & a_{-2} & \ddots & \ddots & a_1 & a_2 \\ a_{2-n} & \vdots & \ddots & \ddots & a_0 & a_1 \\ a_{1-n} & a_{2-n} & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix} \tag{2.6}$$

is a Toeplitz Matrix. The i,j-th element of A is denoted by $\mathrm{A}_{ij} = a_{j-i}$. A Toeplitz matrix has constant entries on each diagonal.

2.2 Basic Boundedness Properties

We define the scalar polynomial,

$$q_w(\lambda) = \omega_m \lambda^m + \omega_{m-1}\lambda^{m-1} + \ldots + \omega_1 \lambda + \omega_0 ,$$

where $w = \{\omega_0, \omega_1, \ldots, \omega_m\}$ nonnegative weights.

Lemma 2.1 If the pseudospectrum is defined in terms of the spectral norm, then

$$\sigma_{\varepsilon,w}(\mathrm{P}) = \{\ \lambda \in \mathbb{C}:\ s_{\min}(\mathrm{P}(\lambda))\ \leq\ \varepsilon\, q_w(|\lambda|)\ \}. \tag{2.7}$$

Here, $s_{\min}(.)$ denotes the minimum singular value of a complex matrix. The *spectral norm* is the matrix norm subordinate to the Euclidean vector norm.

As the eigenvalues of $\mathrm{P}_\Delta(\lambda)$ in (2.3) are continuous with respect to the entries of the coefficient matrices, it follows from the lemma that the boundary of the $\varepsilon-$pseudospectrum can be written in the form,

$$\partial\sigma_{\varepsilon,w}(\mathrm{P}) = \{\ \lambda \in \mathbb{C}: s_{\min}(\mathrm{P}(\lambda)) = \varepsilon\, q_w(|\lambda|)\ \} \tag{2.8}$$

and $\partial\sigma_{\varepsilon,w}(\mathrm{P})$ depends continuously on ε.

If there is a perturbation $\mathrm{P}_\Delta(\lambda) \in \mathrm{B}(\mathrm{P},\varepsilon,w)$ with identically zero determinant, then $\sigma_{\varepsilon,w}(\mathrm{P})$ coincides with the whole complex plane so, a priori, the pseudospectrum may be unbounded. On the other hand, since the leading coefficient A_m is nonsingular, ($\det \mathrm{A}_m \neq 0$), the matrix polynomial $\mathrm{P}(\lambda)$ has exactly Nm (finite) eigenvalues, counting multiplicities, so that for ε sufficiently small, the pseudospectrum must be bounded and consist of no more than Nm connected components.

Theorem 2.2 Let $\mathrm{P}(\lambda)$ be an $N \times N$ matrix polynomial as in (1.1). Then the pseudospectrum is bounded if and only if the $\varepsilon\omega_m-$pseudospectrum of the leading coefficient A_m of $\mathrm{P}(\lambda)$ doesn't contain the origin.

Proof $\Leftarrow$) For fixed $\varepsilon > 0$ and $w = \{\omega_0, \omega_1, ..., \omega_m\} \geq 0$, suppose that $0 \notin \sigma_{\varepsilon\omega_m}(\mathrm{A}_m)$. Then $\det(\mathrm{A}_m + \Delta_m) \neq 0$ whenever $\|\Delta_m\| \leq \varepsilon\omega_m$, and

$$\xi_\varepsilon = \min \ \{ \ \left|\det(\mathrm{A}_m + \Delta_m)\right| \ : \ \left\|\Delta_m\right\| \le \varepsilon\omega_m \ \} > 0 .$$

Since the set $\mathrm{B}(\mathrm{P},\varepsilon,w)$ is the compact, there is an $\mathrm{M}_\varepsilon > 0$ such that for any perturbation of $\mathrm{P}(\lambda)$,

$$\mathrm{P}_\Delta(\lambda) = (\mathrm{A}_m + \Delta_m)\lambda^m + ... + (\mathrm{A}_1 + \Delta_1)\lambda + \mathrm{A}_0 + \Delta_0 \in \mathrm{B}(\mathrm{P},\varepsilon,w),$$

and for any $\lambda \in \mathbb{C}$ with $|\lambda| > \mathrm{M}_\varepsilon$, we have

$$\left|\det \mathrm{P}_\Delta(\lambda) - \det(\mathrm{A}_m + \Delta_m)\lambda^{Nm}\right| \ < \ \xi_\varepsilon \left|\lambda^{Nm}\right| \ \le \ \left|\det(\mathrm{A}_m + \Delta_m)\lambda^{Nm}\right|$$

(keeping in mind that $\det \mathrm{P}_\Delta(\lambda)$ is a scalar polynomial with leading term $\det(\mathrm{A}_m + \Delta_m)\lambda^{Nm}$). Hence, $\det \mathrm{P}_\Delta(\lambda) \neq 0$, i.e., $\sigma_{\varepsilon,w}(\mathrm{P}) \ \subseteq \ \{ \ \lambda \in \mathbb{C} : |\lambda| \le \mathrm{M}_\varepsilon \ \}$; $\sigma_{\varepsilon,w}(\mathrm{P})$ is bounded.

$\Rightarrow$) To prove the converse, assume that $\sigma_{\varepsilon,w}(\mathrm{P})$ is bounded but there is a

$$\mathrm{P}_{\hat{\Delta}}(\lambda) = (\mathrm{A}_m + \hat{\Delta}_m)\lambda^m + ... + (\mathrm{A}_1 + \hat{\Delta}_1)\lambda + \mathrm{A}_0 + \hat{\Delta}_0 \ \in \ \mathrm{B}(\mathrm{P},\varepsilon,w)$$

with $\det(\mathrm{A}_m + \hat{\Delta}_m) = 0$. Then at least one of the coefficients of the scalar polynomial $\det \mathrm{P}_{\hat{\Delta}}(\lambda)$ is nonzero; otherwise $\sigma_{\varepsilon,w}(\mathrm{P}) = \mathbb{C}$, a contradiction. Let the coefficient of λ^τ ($\tau \in \{0,1,...,Nm-1\}$) in $\det \mathrm{P}_{\hat{\Delta}}(\lambda)$ be nonzero and denote it by β_τ. Construct a sequence $\{\hat{\Delta}_{m,k}\}_{k\in\mathbb{N}} \subset \mathbb{C}^{N\times N}$ such that $\lim_{k\to\infty} \hat{\Delta}_{m,k} = \hat{\Delta}_m$ and $\det(\mathrm{A}_m + \hat{\Delta}_{m,k}) \neq 0$ and $\left\|\hat{\Delta}_{m,k}\right\| \le \varepsilon\omega_m \quad (k = 1,2,...)$.

Clearly, for a fixed $\delta > 0$, $\left|\det(\mathrm{A}_m + \hat{\Delta}_{m,k})\right| < \delta$ for all sufficiently large k. Since $\sigma_{\varepsilon,w}(\mathrm{P})$ is bounded, the $(Nm-\tau)$th elementary symmetric function of the roots of $\det[(\mathrm{A}_m + \hat{\Delta}_{m,k})\lambda^m + ... + (\mathrm{A}_1 + \hat{\Delta}_1)\lambda + \mathrm{A}_0 + \hat{\Delta}_0]$, which is equal to

$\pm\beta_\tau / \det(\mathrm{A}_m + \hat{\Delta}_{m,k})$, is bounded for all k. This contradicts the construction of the sequence $\{\hat{\Delta}_{m,k}\}_{k\in\mathbb{N}}$. □

Theorem 2.3 If $\sigma_{\varepsilon,w}(\mathrm{P})$ is bounded, then it has no more than Nm connected components, and any $\mathrm{P}_\Delta(\lambda) \in \mathrm{B}(\mathrm{P},\varepsilon,w)$ has an eigenvalue in each one of these components. Furthermore, $\mathrm{P}_\Delta(\lambda)$ and $\mathrm{P}(\lambda)$ have the same number of eigenvalues, counting multiplicities, in each connected component of $\sigma_{\varepsilon,w}(\mathrm{P})$.

Proof Suppose $\sigma_{\varepsilon,w}(\mathrm{P})$ is bounded. It follows from Theorem 2.2 that, for any perturbation of $\mathrm{P}(\lambda)$,

$$\mathrm{P}_\Delta(\lambda) = (\mathrm{A}_m + \Delta_m)\lambda^m + \ldots + (\mathrm{A}_1 + \Delta_1)\lambda + \mathrm{A}_0 + \Delta_0 \in \mathrm{B}(\mathrm{P},\varepsilon,w)$$

$\det(\mathrm{A}_m + \Delta_m) \neq 0$, i.e., the leading coefficient of the polynomial $\det \mathrm{P}_\Delta(\lambda)$ is nonsingular. As a sequence, $\mathrm{P}_\Delta(\lambda)$ has exactly Nm eigenvalues, counting multiplicities, as does every number of the family of matrix polynomials

$$\mathrm{P}_{\Delta,t}(\lambda) = (1-t)\ \mathrm{P}(\lambda) + t\ \mathrm{P}_\Delta(\lambda); \qquad t \in [0,1].$$

Moreover, for any $t \in [0,1]$, $\mathrm{P}_{\Delta,t}(\lambda)$ belongs to $\mathrm{B}(\mathrm{P},\varepsilon,w)$ and its eigenvalues lie in $\sigma_{\varepsilon,w}(\mathrm{P})$.

The coefficients of the scalar polynomial $\det \mathrm{P}_{\Delta,t}(\lambda)$ are continuous functions of $t \in [0,1]$. Hence, by the continuity of the zeros of $\det \mathrm{P}_{\Delta,t}(\lambda)$ with respect to its coefficients, as t varies from 0 to 1, the eigenvalues of $\mathrm{P}_{\Delta,t}(\lambda)$ trace continuous paths from the eigenvalues of $\mathrm{P}(\lambda)\,(=\mathrm{P}_{\Delta,0}(\lambda))$ to the eigenvalues of $\mathrm{P}_\Delta(\lambda)\,(=\mathrm{P}_{\Delta,1}(\lambda))$. Thus, if $\mathrm{P}(\lambda)$ has k eigenvalues (counting multiplicities) in a connected component Ω of $\sigma_{\varepsilon,w}(\mathrm{P})$ and its $Nm-k$ remaining eigenvalues are isolated in $\sigma_{\varepsilon,w}(P)\backslash\Omega,$ then this is true for the eigenvalues of every $\mathrm{P}_{\Delta,t}(\lambda),\ t\in[0,1].$ Consequently, $\mathrm{P}_{\Delta}(\lambda)$ has exactly k eigenvalues in $\Omega,$ counting multiplicities.

Finally, note that each bounded connected component of $\sigma_{\varepsilon,w}(\mathrm{P})$ contains at least one eigenvalue of $\mathrm{P}(\lambda),$ and by the above discussion, it contains at least one eigenvalue of perturbation $\mathrm{P}_{\Delta}(\lambda)$. Hence, $\sigma_{\varepsilon,w}(\mathrm{P})$ cannot have more than Nm connected compenents. □

Since the origin lies in $\sigma_{\varepsilon,w_m}(\mathrm{A}_m)$ if and only if $s_{\min}(\mathrm{A}_m)\le\varepsilon w_m,$ it also follows that:

Corollary 2.4 For any $\varepsilon>0$ such that $\varepsilon w_m<s_{\min}(\mathrm{A}_m),$ $\sigma_{\varepsilon,w}(\mathrm{P})$ consists of no more than Nm bounded connected components.

2.3 Matrix Polynomials with Bounded Numerical Range

Definition 2.17 (Numerical Range): The *numerical range* of the matrix polynomial $\mathrm{P}(\lambda)$ is defined and denoted by

$$W(\mathrm{P})=\{\ \lambda\in\mathbb{C}:\ v^{*}\mathrm{P}(\lambda)v=0,\ v\in\mathbb{C}^{N},\ v^{*}v=1\ \}. \tag{2.9}$$

Definition 2.18 (Classical Numerical Range) For the linear pencil $\mathrm{I}\lambda-\mathrm{A},$ $\mathrm{A}\in\mathbb{C}^{N\times N},$ $W(\mathrm{I}\lambda-\mathrm{A})$ coincides with the *classical numerical range* (also known as the *field of values*) of the matrix $\mathrm{A},$

$$F(\mathrm{A})=\{v^{*}\mathrm{A}v:\ v\in\mathbb{C}^{N},\ v^{*}v=1\}, \tag{2.10}$$

which is always compact and convex [25].

Some basic geometrical properties of $W(\mathrm{P})$ are given below:

i) The numerical range $W(\mathrm{P})$ is not always connected, and it is bounded if and only if $0 \notin F(\mathrm{A}_m)$. In this case, $W(\mathrm{P})$ has no more than m (bounded) connected components [26]. It is always closed and contains $\sigma(\mathrm{P})$.

ii) Suppose Ω is a bounded connected component of $W(\mathrm{P})$ and for a unit vector $v \in \mathbb{C}^N$, the scalar polynomial $v^*\mathrm{P}(\lambda)v$ has exactly $c(\Omega)$ roots in Ω, counting multiplicities. Then the number $c(\Omega)$ doesn't depend on v [27], i.e., it is constant. $\mathrm{P}(\lambda)$ has exactly $Nc(\Omega)$ eigenvalues in Ω, counting multiplicities [28].

iii) For any $\mu \in \partial W(\mathrm{P})$, the origin is a boundary point of $F(\mathrm{P}(\mu))$.

iv) If μ is a corner of $W(\mathrm{P})$ and there is a unit $v_\mu \in \mathbb{C}^N$ such that μ is a simple root of the polynomial $v_\mu^*\mathrm{P}(\lambda)v_\mu$, then μ is an eigenvalue of $\mathrm{P}(\lambda)$. Here, the precise definition of a "corner" is that of a sharp point: a point w on the boundary of a set $S \subset \mathbb{C}$ is called a *sharp point* of S if there exist two angles θ_1 and θ_2 with $0 \le \theta_1 < \theta_2 < 2\pi$ such that for all $z \in S$ and for all $\theta \in (\theta_1, \theta_2)$ the inequality $\operatorname{Re} e^{i\theta} w \ge \operatorname{Re} e^{i\theta} z$ holds.

The *inner numerical radius* of A is defined by

$$\hat{r}(\mathrm{A}) = \min\{|\lambda| : \lambda \in \partial F(\mathrm{A})\}. \tag{2.11}$$

The $\varepsilon-$pseudospectrum of A lies in the region

$$F(\mathrm{A}) + S(0,\varepsilon) = \{\lambda \in \mathbb{C} : \mathit{dist}(\lambda, F(\mathrm{A})) \le \varepsilon\}, \tag{2.12}$$

where $dist(\lambda, F(\mathrm{A}))$ denotes the distance between the point λ and the numerical range $F(\mathrm{A})$. Thus, $F(\mathrm{A})+S(0,\varepsilon)$ is often used as an initial region for the estimation of $\sigma_\varepsilon(\mathrm{A})$.

Theorem 2.5 Let $\mathrm{P}(\lambda)=\mathrm{A}_m\lambda^m+...+\mathrm{A}_1\lambda+\mathrm{A}_0$ be an $N\times N$ matrix polynomial with bounded numerical range that consists of ξ $(\le m)$ connected components $\Omega_1,\ \Omega_2,...,\Omega_\xi$. Then for given $\varepsilon>0$ and $w\ge 0$,

$$\sigma_{\varepsilon,w}(\mathrm{P}) \subseteq \left\{\lambda\in\mathbb{C}:\ \prod_{j=1}^{\xi} dist(\lambda,\Omega_j)^{c(\Omega_j)} \le \frac{\varepsilon q_w(|\lambda|)}{\hat{r}(\mathrm{A}_m)}\right\},$$

where it is assumed that $dist(\lambda,\Omega_j)=0$ when $\lambda\in\Omega_j$.

Proof Suppose that $\lambda_0\in\sigma_{\varepsilon,w}(\mathrm{P})$. Then there exists a perturbation of $\mathrm{P}(\lambda)$,

$$\mathrm{P}_\Delta(\lambda)=(\mathrm{A}_m+\Delta_m)\lambda^m+...+(\mathrm{A}_1+\Delta_1)\lambda+\mathrm{A}_0+\Delta_0 \ \in\ \mathrm{B}(\mathrm{P},\varepsilon,w),$$

and a unit vector $v_0\in\mathbb{C}^N$ such that $\mathrm{P}_\Delta(\lambda_0)v_0=0$. Hence,

$$v_0^*\ (\Delta_m\lambda_0^m+...+\Delta_1\lambda_0+\Delta_0)\ v_0 = -v_0^*\mathrm{P}(\lambda)v_0,$$

and consequently,

$$\sum_{j=0}^{m}\left|v_0^*\Delta_j v_0\right|\,|\lambda_0|^j \ \ge\ \left|\sum_{j=0}^{m}(v_0^*\Delta_j v_0)\lambda_0^j\right| \ =\ \left|v_0^*\mathrm{P}(\lambda_0)v_0\right|$$

$$=\left|v_0^*\mathrm{A}_m v_0\right|\ \prod_{j=1}^{m}\left|\lambda_0-\lambda_j(v_0)\right|,$$

where $\lambda_1(v_0),\ \lambda_2(v_0),...,\lambda_m(v_0)$ are the zeros of the polynomial $v_0^*\mathrm{P}(\lambda_0)v_0$. Since $\varepsilon\omega_j\ \ge\ \|\Delta_j\|\ \ge\ |v_0^*\Delta_j v_0|,\quad j=0,1,...,m,$ it follows that

$$\varepsilon q_w(|\lambda_0|) \ \geq \ \sum_{j=0}^{m} \|\Delta_j\| \ |\lambda_0|^j \ \geq \ \sum_{j=0}^{m} |v_0^* \Delta_j v_0| \ |\lambda_0|^j$$

$$\geq \hat{r}(\mathrm{A}_m) \prod_{j=1}^{m} |\lambda_0 - \lambda_j(v_0)| \ \geq \ \hat{r}(\mathrm{A}_m) \prod_{j=1}^{\xi} dist(\lambda_0, \Omega_j)^{c(\Omega_j)}. \ \square$$

CHAPTER 3

QUADRATIC AND CUBIC MATRIX POLYNOMIALS

3.1 Quadratic Matrix Polynomials with Toeplitz Matrix Coefficients

If $\mathrm{P}(\lambda) = \mathrm{I}\lambda - \mathrm{A}$ for some $\mathrm{A} \in \mathbb{C}^{N \times N}$, then $\mathrm{P}(\lambda)$ is called *linear pencil.* $\sigma(\mathrm{P})$ coincides with the spectrum of A, $\sigma(\mathrm{A}),$ in the usual sense. Furthermore, if we set $w = \{\omega_0, \omega_1\} = \{1, 0\}$, then $\sigma_{\varepsilon,w}(\mathrm{P})$ coincides with $\varepsilon-$pseudospectrum of the matrix A, that is,

$$\sigma_\varepsilon(\mathrm{A}) = \{\ \lambda \in \mathbb{C}:\ \lambda \in \sigma(\mathrm{A} + \Delta_0),\ \|\Delta_0\| \le \varepsilon\ \}. \tag{3.1}$$

Also, the special case $N = 1,$ the $\varepsilon-$pseudospectrum of the scalar polynomial $p(\lambda) = a_m\lambda^m + a_{m-1}\lambda^{m-1} + ... + a_1\lambda + a_0$ coincides with the root neighborhood of $p(\lambda)$ and the $\varepsilon-$pseudozero set of $p(\lambda).$

If the matrix polynomial $\mathrm{P}(\lambda)$ in (1.1) is real (i.e. all its coefficients are real matrices) or self- adjoint, then it is well known that the spectrum of $\mathrm{P}(\lambda)$ is symmetric with respect to the real axis. This symmetry also holds for the pseudospectra of matrix polynomials.

Proposition 3.1 Let $\mathrm{P}(\lambda)=\mathrm{A}_m\lambda^m+\mathrm{A}_{m-1}\lambda^{m-1}+\ldots+\mathrm{A}_1\lambda+\mathrm{A}_0$ be an $N\times N$ real or self-adjoint matrix polynomial. Then for any $\varepsilon>0$ and $w\geq 0$, the $\varepsilon-$pseudospectrum $\sigma_{\varepsilon,\omega}(\mathrm{P})$ is symmetric with respect to the real axis.

Proof Suppose $\mu\in\sigma_{\varepsilon,\omega}(\mathrm{P})$. Then there is a matrix polynomial $(\mathrm{A}_m+\Delta_m)\lambda^m+\ldots+(\mathrm{A}_1+\Delta_1)\lambda+\mathrm{A}_0+\Delta_0 \in \mathrm{B}(\mathrm{P},\varepsilon,w)$ such that,

$$\det\ [(\mathrm{A}_m+\Delta_m)\mu^m+\ldots+(\mathrm{A}_1+\Delta_1)\mu+\mathrm{A}_0+\Delta_0]=0\,.$$

If the coefficients of $\mathrm{P}(\lambda)$ are real, then

$$\det\ \overline{[(\mathrm{A}_m+\Delta_m)\mu^m+\ldots+(\mathrm{A}_1+\Delta_1)\mu+\mathrm{A}_0+\Delta_0]}=$$

$$\det\ [(\mathrm{A}_m+\overline{\Delta_m})\overline{\mu}^m+\ldots(\mathrm{A}_1+\overline{\Delta_1})\overline{\mu}+\mathrm{A}_0+\overline{\Delta_0}]=0\,,$$

where $(\mathrm{A}_m+\overline{\Delta_m})\lambda^m+\ldots+(\mathrm{A}_1+\overline{\Delta_1})\lambda+\mathrm{A}_0+\overline{\Delta_0}$ also lies in $\mathrm{B}(\mathrm{P},\varepsilon,w)$. Hence, $\overline{\mu}\in\sigma_{\varepsilon,w}(\mathrm{P})$.

Similarly, if all the coefficients of $\mathrm{P}(\lambda)$ are hermitian, then

$$\det\ [(\mathrm{A}_m+\Delta_m)\mu^m+\ldots+(\mathrm{A}_1+\Delta_m)\mu+\mathrm{A}_0+\Delta_0]^*=$$

$$\det\ [(\mathrm{A}_m+\Delta_m^*)\overline{\mu}^m+\ldots+(\mathrm{A}_1+\Delta_1^*)\overline{\mu}+\mathrm{A}_0+\Delta_0^*]=0\,,$$

where $(\mathrm{A}_m+\Delta_m^*)\lambda^m+\ldots+(\mathrm{A}_1+\Delta_1^*)\lambda+\mathrm{A}_0+\Delta_0^* \in \mathrm{B}(\mathrm{P},\varepsilon,w)$. Thus, $\overline{\mu}$ lies in $\sigma_{\varepsilon,w}(\mathrm{P})$. □

In this chapter we investigated the pseudospectrum of the quadratic and cubic matrix polynomials which were defined in (1.2) and (1.3). Here, we determined that quadratic and cubic matrix polynomials are equal to multiplicities of two and three linear pencils. Namely, the quadratic matrix polynomial can be written in the form

$$P(\lambda)=(\lambda I_n - A)\ (\lambda I_n - B)=\lambda^2+C\lambda+D, \qquad A,B,C,D\in\mathbb{C}^{N\times N}. \tag{3.2}$$

The matrices A and B were selected as tridiagonal Toeplitz matrices.

We investigated whether $\varepsilon-$pseudospectrum of $P(\lambda)$ in (3.3) can be fined by means of $\varepsilon-$pseudospectrum of the matrices A and B. Let's consider the quadratic matrix polynomial

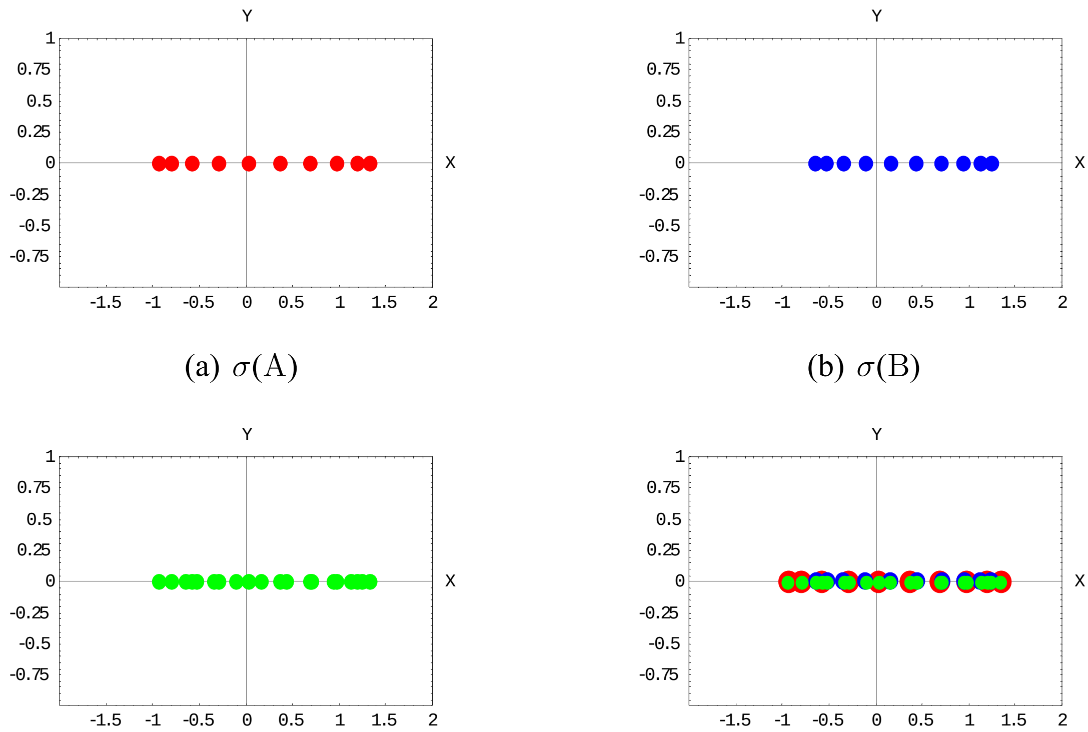
$$P(\lambda)=(\lambda I_n - A)\ (\lambda I_n - B), \tag{3.3}$$

with the tridiagonal Toeplitz matrices

$$A=\begin{bmatrix} 0.2 & 0.5 & & & \\ 0.7 & 0.2 & 0.5 & & \\ & 0.7 & \ddots & \ddots & \\ & & \ddots & \ddots & 0.5 \\ & & & 0.7 & 0.2 \end{bmatrix} \qquad B=\begin{bmatrix} 0.3 & 0.4 & & & \\ 0.6 & 0.3 & 0.4 & & \\ & 0.6 & \ddots & \ddots & \\ & & \ddots & \ddots & 0.4 \\ & & & 0.6 & 0.3 \end{bmatrix} \tag{3.4}$$

Firstly, we began with the tridiagonal Toeplitz matrices A and B of small dimension and than studied the matrices of larger size. The plots were obtained with Mathematica programming. Eigenvalues of A, B and $P(\lambda)$ were marked by red, blue and green points, respectively.

Suppose N=10, 25, 75, 100, 200, 300; for each one, the figures are shown as following:



(a) $\sigma(\mathrm{A})$

(b) $\sigma(\mathrm{B})$

(c) $\sigma(\mathrm{P})$

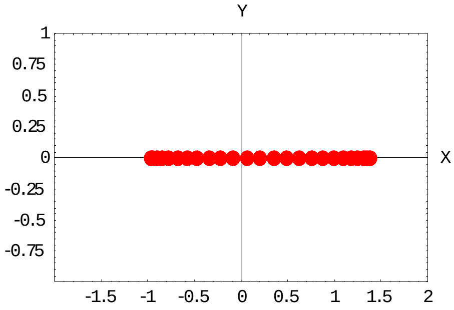
(d) $\sigma(\mathrm{A}),\ \sigma(\mathrm{B}),\ \sigma(\mathrm{P})$

Figure 3.1: (a) Spectral points of the tridiagonal matrix (3.4) A of dimension N=10. (b) Spectral points of the tridiagonal matrix (3.4) B of dimension N=10. (c) Spectral points of the quadratic matrix polynomial (3.3) $\mathrm{P}(\lambda)$ of dimension N=10 (d) All spectral points of them in the same region.

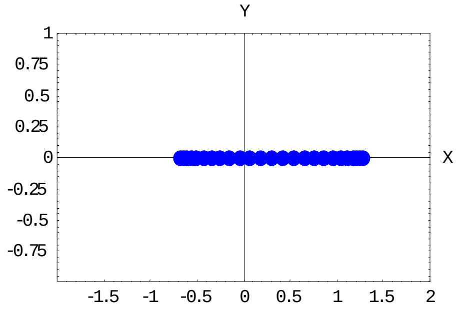
For N=10, it is seen that the spectrum of $\mathrm{P}(\lambda)$ is equal to union of spectrum of the matrices A and B. Namely,

$$\mathrm{P}(\lambda) = (\lambda \mathrm{I}_{10} - \mathrm{A})\ (\lambda \mathrm{I}_{10} - \mathrm{B})$$

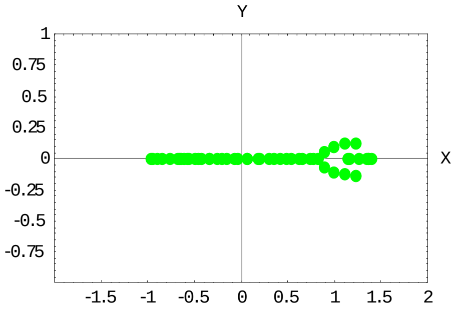
$$\sigma(\mathrm{P}) = \sigma(\mathrm{A}) \cup \sigma(\mathrm{B}).$$



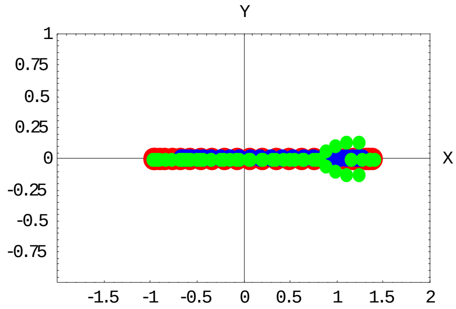
(a) $\sigma(\mathrm{A})$



(b) $\sigma(\mathrm{B})$



(c) $\sigma(\mathrm{P})$



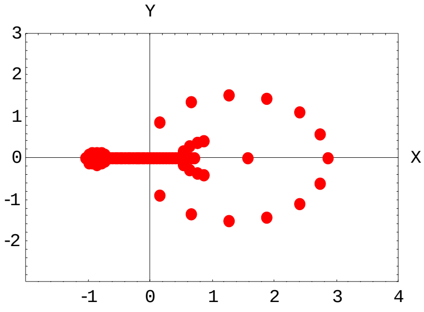
(d) $\sigma(\mathrm{A})$, $\sigma(\mathrm{B})$, $\sigma(\mathrm{P})$

Figure 3.2: (a) Spectral points of the tridiagonal matrix (3.4) A of dimension N=25. (b) Spectral points of the tridiagonal matrix (3.4) B of dimension N=25. (c) Spectral points of the quadratic matrix polynomial (3.3) $\mathrm{P}(\lambda)$ of dimension N=25 (d) All spectral points of them in the same region.

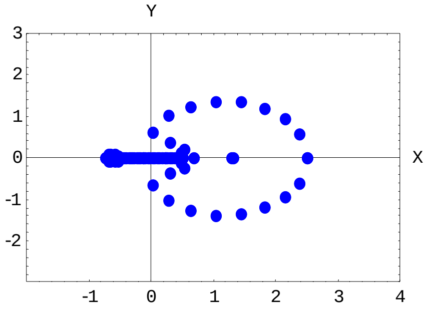
For N=25, it is seen that the spectrum of $\mathrm{P}(\lambda)$ is different from the union of spectrum of the matrices A and B. $\sigma(\mathrm{P})$ contains 50 distinct eigenvalues. It can be written in the form

$$\mathrm{P}(\lambda) = (\lambda \mathrm{I}_{25} - \mathrm{A})\ (\lambda \mathrm{I}_{25} - \mathrm{B})$$

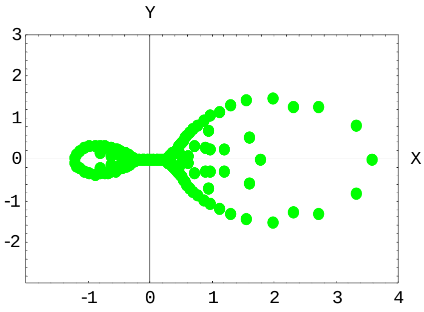
$$\sigma(\mathrm{P}) \approx \sigma(\mathrm{A}) \cup \sigma(\mathrm{B}).$$



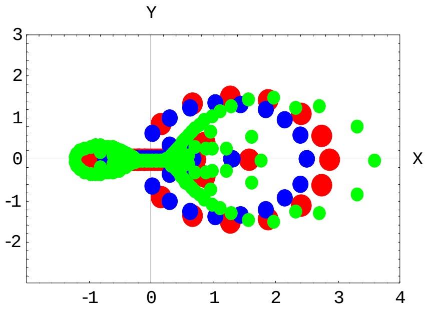
(a) $\sigma(\mathrm{A})$



(b) $\sigma(\mathrm{B})$



(c) $\sigma(\mathrm{P})$



(d) $\sigma(\mathrm{A})$, $\sigma(\mathrm{B})$, $\sigma(\mathrm{P})$

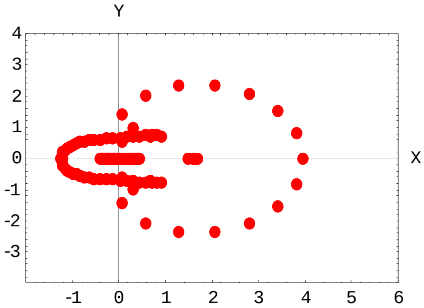
Figure 3.3: (a) Spectral points of the tridiagonal matrix (3.4) A of dimension N=75. (b) Spectral points of the tridiagonal matrix (3.4) B of dimension N=75. (c) Spectral points of the quadratic matrix polynomial (3.3) $\mathrm{P}(\lambda)$ of dimension N=75 (d) All spectral points of them in the same region.

It was re-indicated that $\sigma(\mathrm{P})$ is not equal to the union of $\sigma(\mathrm{A})$ and $\sigma(\mathrm{B})$. It contains 150 distinct eigenvalues. When N=75,

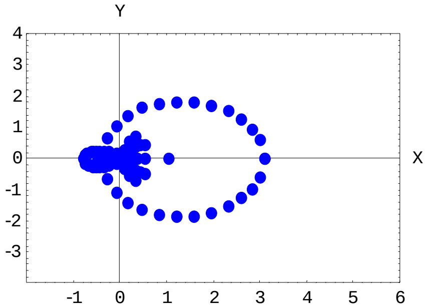
$$\mathrm{P}(\lambda) = (\lambda \mathrm{I}_{75} - \mathrm{A})\ (\lambda \mathrm{I}_{75} - \mathrm{B})$$

$$\sigma(\mathrm{P}) \approx \sigma(\mathrm{A}) \cup \sigma(\mathrm{B})$$

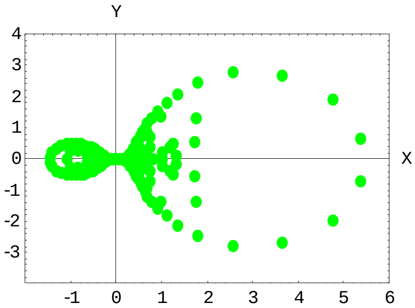
Similarly, we also obtained the same results for N=100, 200, 300.



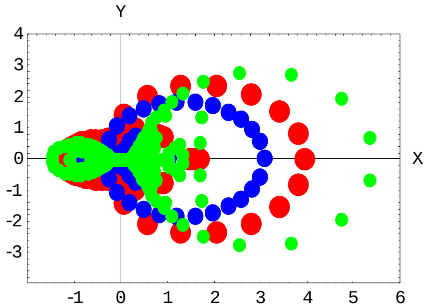
(a) $\sigma(\mathrm{A})$



(b) $\sigma(\mathrm{B})$



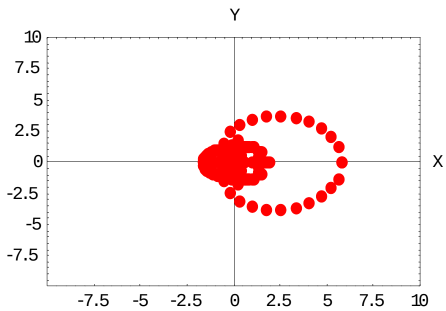
(c) $\sigma(\mathrm{P})$



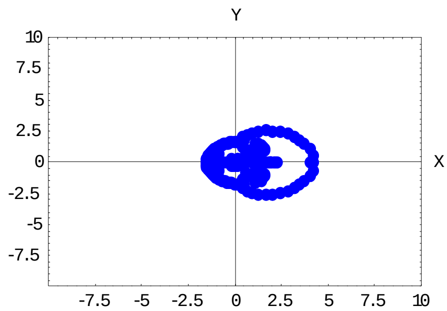
(d) $\sigma(\mathrm{A}),\ \sigma(\mathrm{B}),\ \sigma(\mathrm{P})$

Figure 3.4: (a) Spectral points of the tridiagonal matrix (3.4) A of dimension N=100. (b) Spectral points of the tridiagonal matrix (3.4) B of dimension N=100. (c) Spectral points of the quadratic matrix polynomial (3.3) $\mathrm{P}(\lambda)$ of dimension N=100 (d) All spectral points of them in the same region.

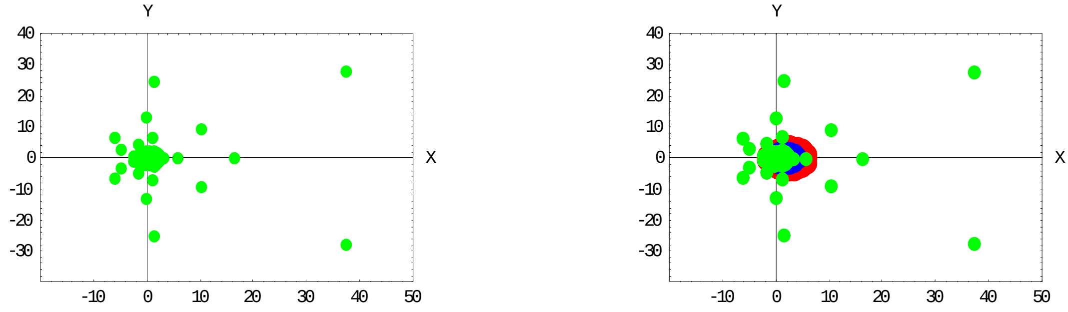
$$\mathrm{P}(\lambda) = (\lambda \mathrm{I}_{100} - \mathrm{A})\ (\lambda \mathrm{I}_{100} - \mathrm{B}) \qquad \sigma(\mathrm{P}) \approx \sigma(\mathrm{A}) \cup \sigma(\mathrm{B}).$$



(a) $\sigma(\mathrm{A})$

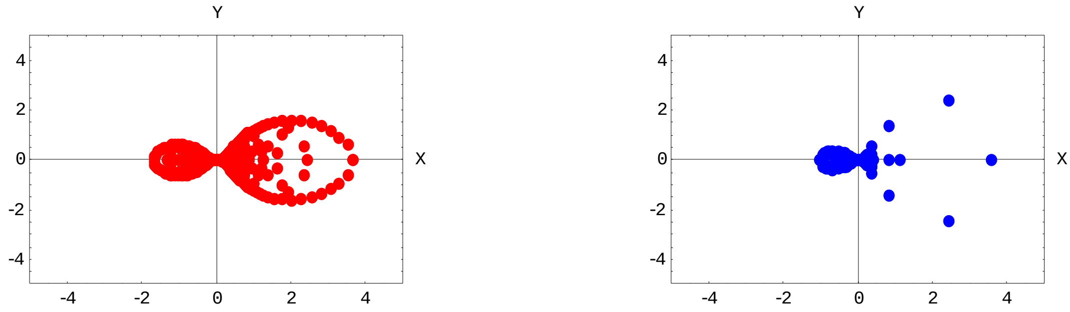


(b) $\sigma(\mathrm{B})$

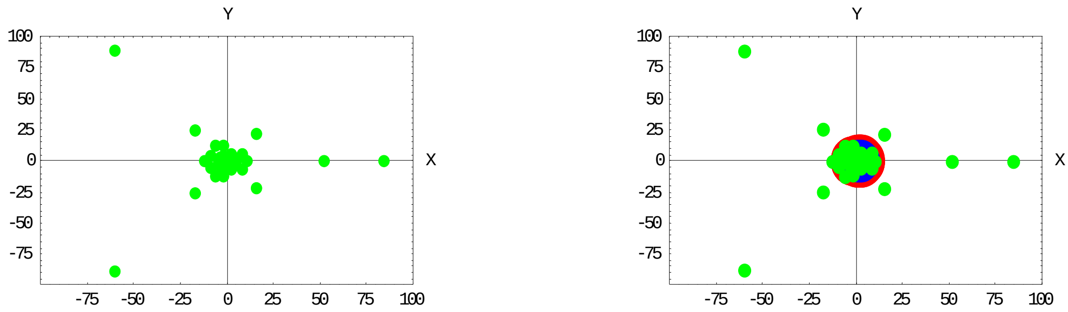


(c) σ(P) (d) σ(A), σ(B), σ(P)

Figure 3.5: (a) Spectral points of the tridiagonal matrix (3.4) A of dimension N=200. (b) Spectral points of the tridiagonal matrix (3.4) B of dimension N=200. (c) Spectral points of the quadratic matrix polynomial (3.3) P(λ) of dimension N=200 (d) All spectral points of them in the same region.



(a) σ(A) (b) σ(B)



(c) σ(P) (d) σ(A), σ(B), σ(P)

Figure 3.6: (a) Spectral points of the tridiagonal matrix (3.4) A of dimension N=300. (b) Spectral points of the tridiagonal matrix (3.4) B of dimension N=300. (c) Spectral points of the quadratic matrix polynomial (3.3) P(λ) of dimension N=300. (d) All spectral points of them in the same region.

The figures 3.1-6 are stated that as long as the dimensions of the matrices A and B become larger, the spectrum of $\mathrm{P}(\lambda)$ differ from the union of spectrum of A and B. Since the leading A_m coefficient is nonsingular, the matrix polynomial $\mathrm{P}(\lambda)$ has exactly Nm (finite) eigenvalues. The larger dimension of the matrix polynomial get the larger there becomes the degree of characteristic polynomial for the eigenvalues. When it is the larger degree of characteristic polynomial (n>3) , there is no mathematical process to find out eigenvalues. So, we can just reach the values nearer the roots. Here, dividing the process into sub-intervals inwhich there are the changes of function-sign and also narrowing the intervals, it is possible to figure out proper areas of roots.

Thus, the roots of characteristic polynomials corresponding to the matrices A and B are not the same as those of characteristic polynomial for $\mathrm{P}(\lambda)$. When we perturbed the matrices (3.4) A and B correspondig to $\|\Delta_0\| \leq \varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$; the below figures came out. The perturbation of the matrix polynomial $\mathrm{P}(\lambda)$ in (3.3) is written in this form,

$$\mathrm{P}_\Delta(\lambda) = (\lambda\mathrm{I} - (\mathrm{A} + \Delta_0))\ (\lambda\mathrm{I} - (\mathrm{B} + \Delta_0)) . \tag{3.5}$$

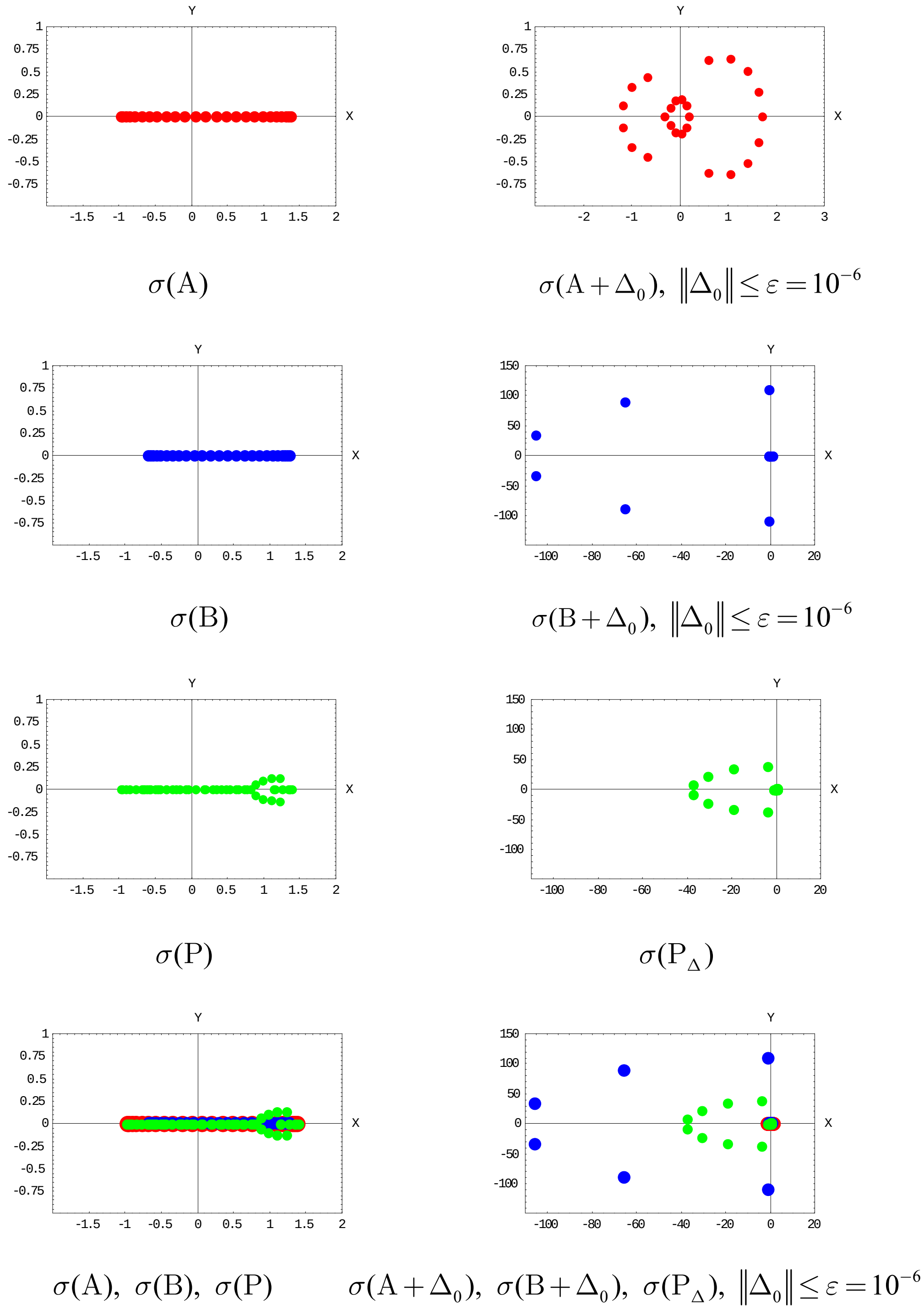


Figure 3.7: In the left four plots corresponding to N=25 show the spectral points of tridiagonal matrices A,B (3.4) and of quadratic matrix polynomial (3.3) $P(\lambda)$. In the right four plots corresponding to N=25, $\|\Delta_0\| \le \varepsilon = 10^{-6}$ show the pseudospectral points of randomly perturbed matrices $A+\Delta_0$, $B+\Delta_0$ and of perturbed matrix polynomial (3.5) $P_\Delta(\lambda)$.

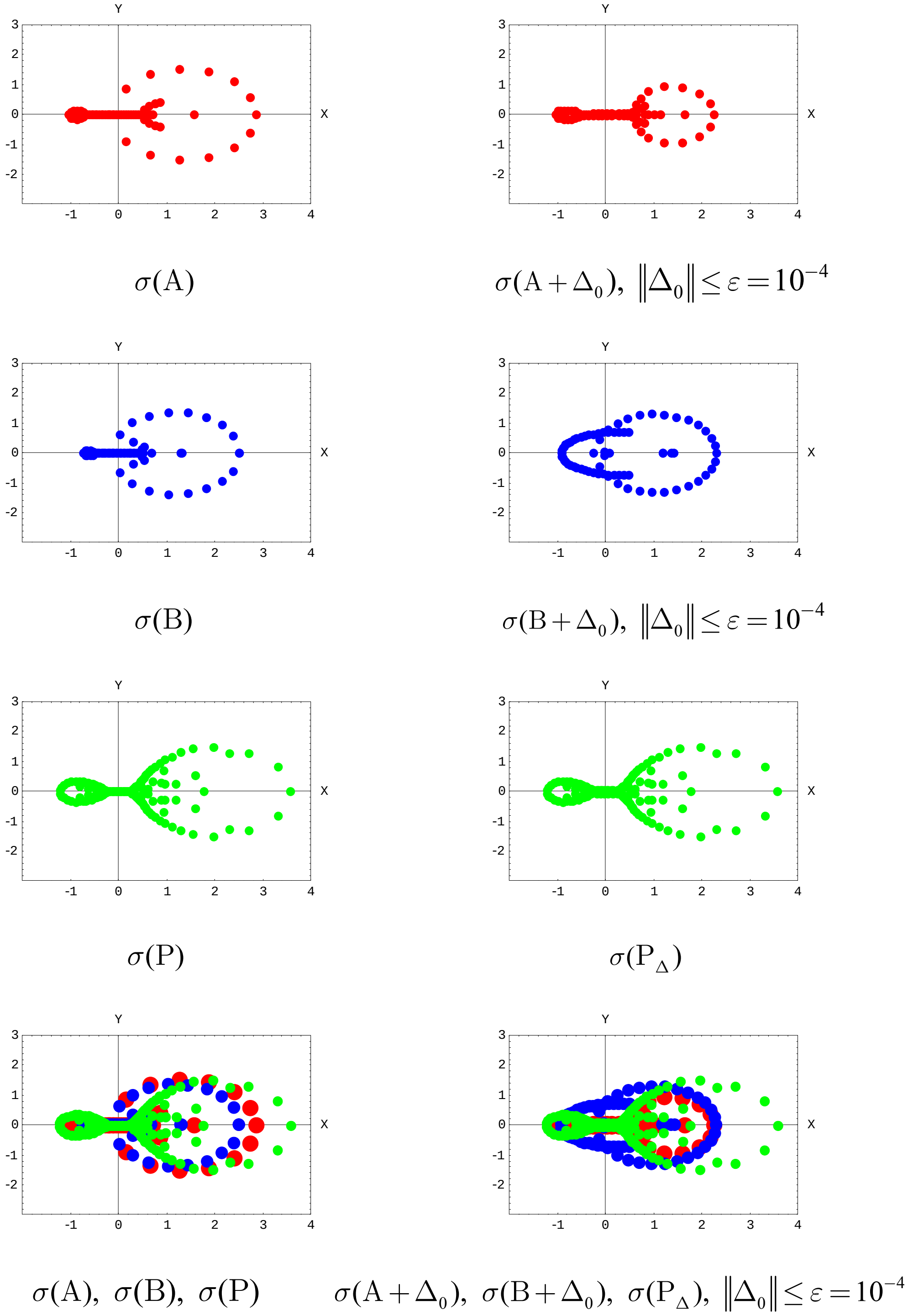
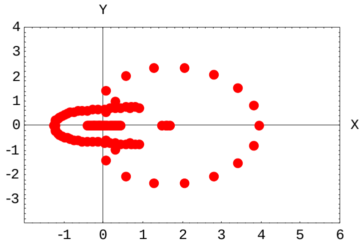
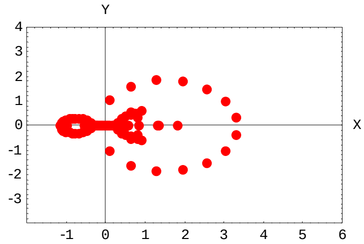


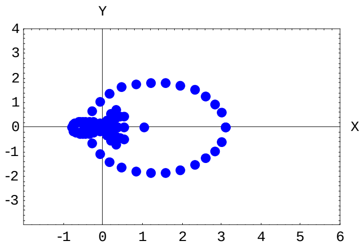
Figure 3.8: In the left four plots corresponding to N=75 show the spectral points of tridiagonal matrices (3.4) A, B and of quadratic matrix polynomial (3.3) $P(\lambda)$. In the right four plots corresponding to N=75, $\|\Delta_0\| \le \varepsilon = 10^{-4}$ show the pseudospectral points of randomly perturbed matrices $A+\Delta_0$, $B+\Delta_0$ and of perturbed matrix polynomial (3.5) $P_\Delta(\lambda)$.



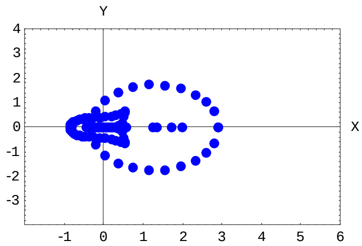
$\sigma(\mathrm{A})$



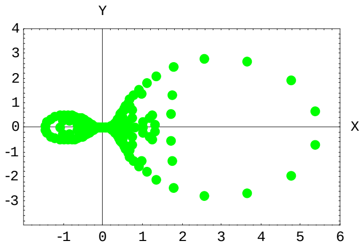
$\sigma(\mathrm{A}+\Delta_0),\ \|\Delta_0\| \le \varepsilon = 10^{-2}$



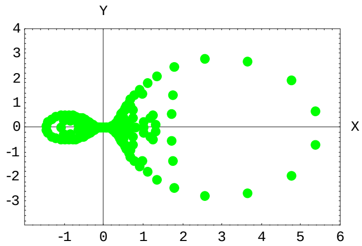
$\sigma(\mathrm{B})$



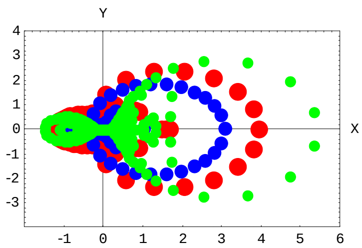
$\sigma(\mathrm{B}+\Delta_0),\ \|\Delta_0\| \le \varepsilon = 10^{-2}$



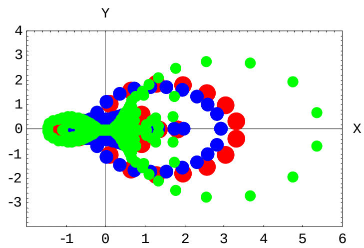
$\sigma(\mathrm{P})$



$\sigma(\mathrm{P}_\Delta)$

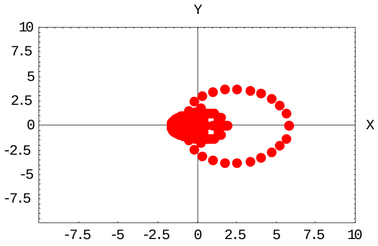


$\sigma(\mathrm{A}),\ \sigma(\mathrm{B}),\ \sigma(\mathrm{P})$

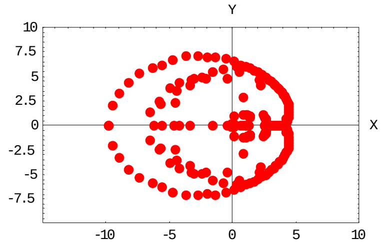


$\sigma(\mathrm{A}+\Delta_0),\ \sigma(\mathrm{B}+\Delta_0),\ \sigma(\mathrm{P}_\Delta),\ \|\Delta_0\| \le \varepsilon = 10^{-2}$

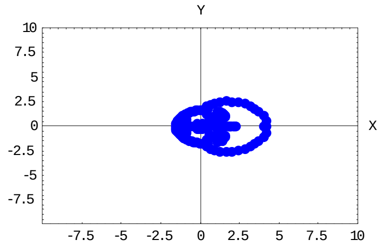
Figure 3.9: In the left four plots corresponding to N=100 show the spectral points of tridiagonal matrices (3.4) A, B and of quadratic matrix polynomial (3.3) $\mathrm{P}(\lambda)$. In the right four plots corresponding to N=100, $\|\Delta_0\| = 10^{-2}$ show the pseudospectral points of randomly perturbed matrices $\mathrm{A}+\Delta_0$, $\mathrm{B}+\Delta_0$ and of perturbed matrix polynomial (3.5) $\mathrm{P}_\Delta(\lambda)$.



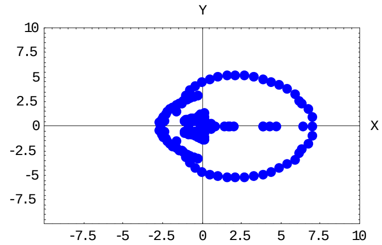
$\sigma(\mathrm{A})$



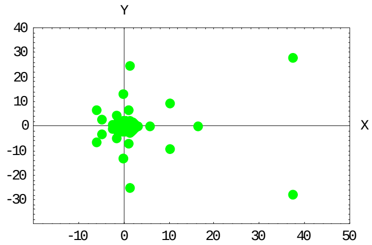
$\sigma(\mathrm{A}+\Delta_0),\ \|\Delta_0\|=10^{-6}$



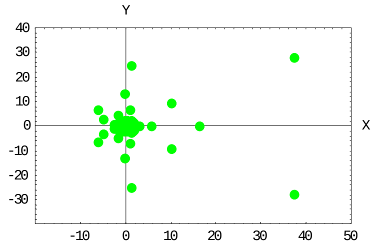
$\sigma(\mathrm{B})$



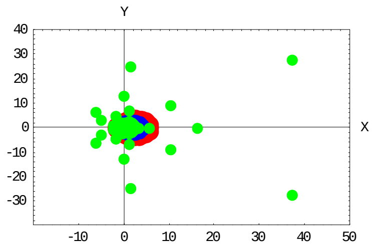
$\sigma(\mathrm{B}+\Delta_0),\ \|\Delta_0\|=10^{-6}$



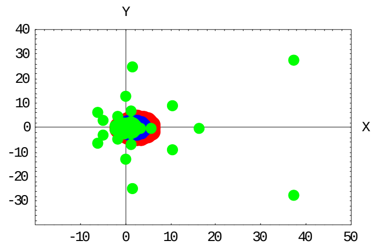
$\sigma(\mathrm{P})$



$\sigma(\mathrm{P}_\Delta)$



$\sigma(\mathrm{A}),\ \sigma(\mathrm{B}),\ \sigma(\mathrm{P})$



$\sigma(\mathrm{A}+\Delta_0),\ \sigma(\mathrm{B}+\Delta_0),\ \sigma(\mathrm{P}_\Delta),\ \|\Delta_0\|=10^{-6}$

Figure 3.10: In the left four plots corresponding to N=200 show the spectral points of tridiagonal matrices (3.4) A, B and quadratic matrix polynomial (3.3) $\mathrm{P}(\lambda)$. In the right part four plots corresponding to N=200, $\|\Delta_0\|=10^{-6}$ show the pseudospectral points of randomly perturbed matrices $\mathrm{A}+\Delta_0$, $\mathrm{B}+\Delta_0$ and of perturbed matrix polynomial (3.5) $\mathrm{P}_\Delta(\lambda)$.

If we investigate the figures 3.7-10, we see that the spectrum of $P_{\Delta}(\lambda)$ is not equal to the union of pseudospectrum of the matrices A and B but very nearer to it. Symmetry of the pseudospectra with respect to the real axis is apparent, comforming Proposition 3.1.

3.2 Cubic Matrix Polynomial with Toeplitz Matrix Coefficients

We investigated spectrum and pseudospectrum of the cubic matrix polynomials with the tridiagonal Toeplitz matrix coefficients in this section. We consider the cubic matrix polynomial

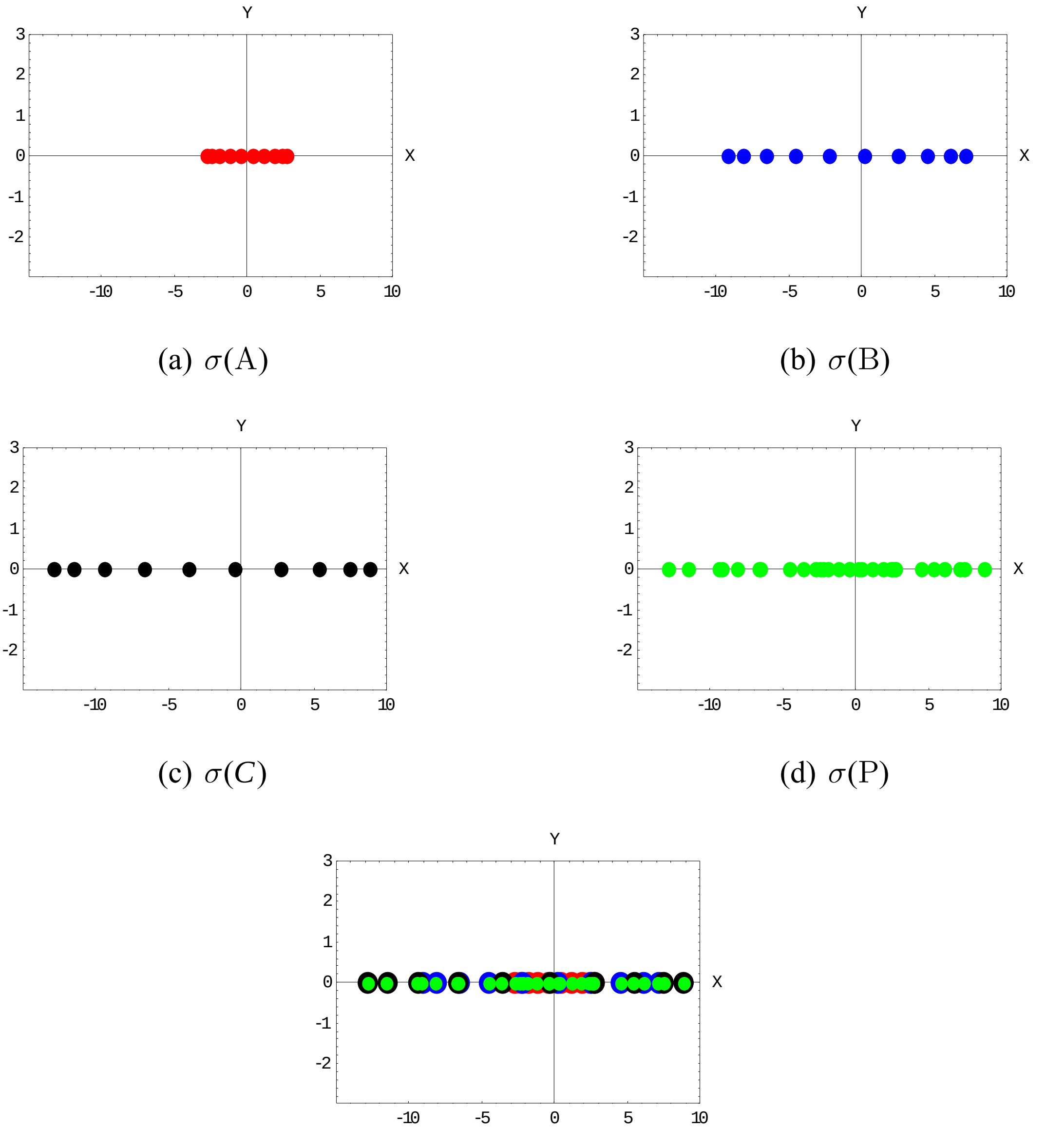
$$P(\lambda) = (\lambda I_n - A)\ (\lambda I_n - B)\ (\lambda I_n - C), \tag{3.6}$$

with the tridiagonal Toeplitz matrices

$$A = \begin{bmatrix} 0 & 1 & & & \\ 2 & 0 & 1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 3 & & & \\ 6 & -1 & 3 & & \\ & 6 & -1 & \ddots & \\ & & \ddots & \ddots & 3 \\ & & & 6 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} -2 & 4 & & & \\ 8 & -2 & 4 & & \\ & 8 & -2 & \ddots & \\ & & \ddots & \ddots & 4 \\ & & & 8 & -2 \end{bmatrix}. \tag{3.7}$$

Firstly, we searched the spectrum of cubic matrix polynomial (3.6) $P(\lambda)$. Firstly, we consider N=10.



(a) $\sigma(\mathrm{A})$ (b) $\sigma(\mathrm{B})$

(c) $\sigma(C)$ (d) $\sigma(\mathrm{P})$

(e) $\sigma(\mathrm{A}),\ \sigma(\mathrm{B}),\ \sigma(C),\ \sigma(\mathrm{P})$

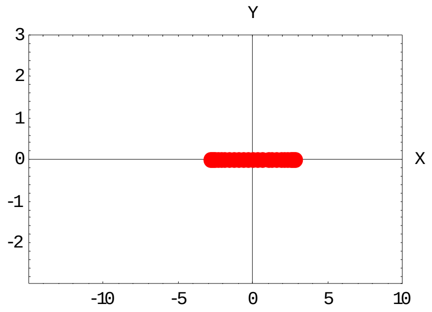
Figure 3.11: (a) Spectral points of the tridiagonal Toeplitz matrix (3.7) A of dimension N=10. (b) Spectral points of the tridiagonal Toeplitz matrix (3.7) B of dimension N=10. (c) Spectral points of the tridiagonal Toeplitz matrix (3.7) C of dimension N=10. (d) Spectral points of the cubic matrix polynomial (3.6) $\mathrm{P}(\lambda)$ of dimension N=10. (e) All spectral points of them in the same region.

For N=10, we see that spectral points of $\mathrm{P}(\lambda)$ are equal to the union of spectrum of matrices A, B and C. It is written the equality

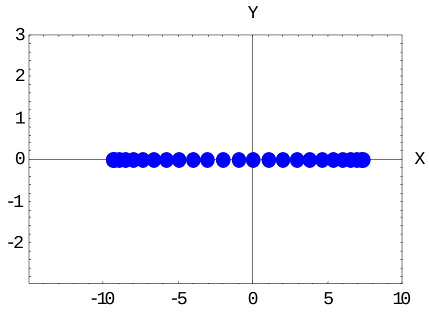
$$P(\lambda) = (\lambda I_{10} - A)\ (\lambda I_{10} - B)\ (\lambda I_{10} - C)$$

$$\sigma(P) = \sigma(A) \cup \sigma(B) \cup \sigma(C).$$

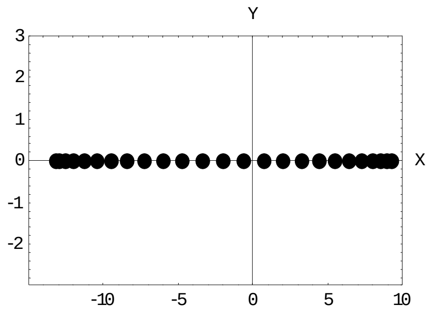
Let's research for N=25.



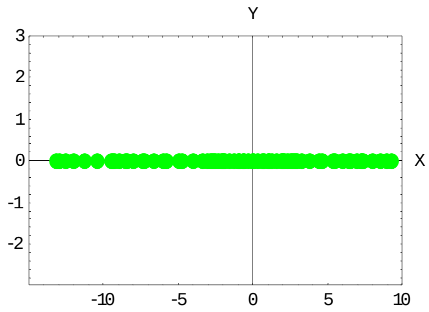
(a) $\sigma(A)$



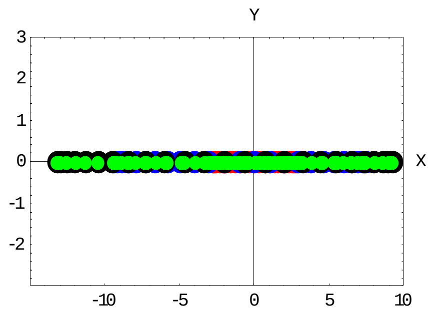
(b) $\sigma(B)$



(c) $\sigma(C)$



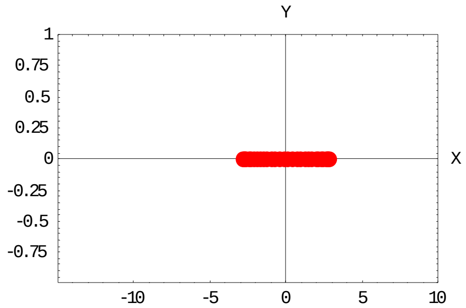
(d) $\sigma(P)$



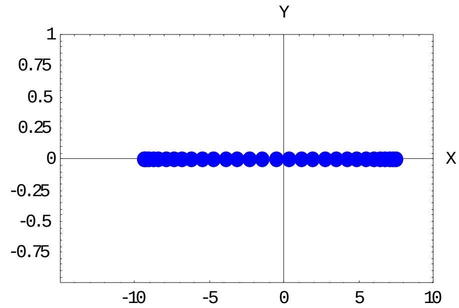
(e) $\sigma(A)$, $\sigma(B)$, $\sigma(C)$, $\sigma(P)$

Figure 3.12: (a) Spectral points of the tridiagonal Toeplitz matrix (3.7) A of dimension N=25. (b) Spectral points of the tridiagonal Toeplitz matrix (3.7) B of dimension N=25. (c) Spectral points of the tridiagonal Toeplitz matrix (3.7) C of dimension N=25. (d) Spectral points of the cubic matrix polynomial (3.6) $P(\lambda)$ of dimension N=25. (e) All spectral points of them in the same region.

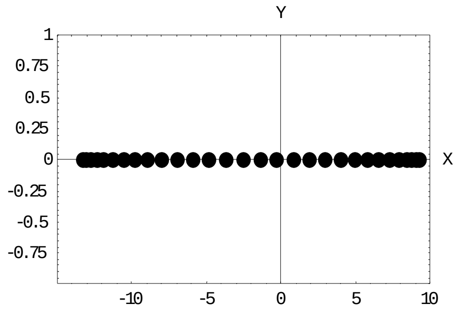
The spectrum of the cubic matrix polynomial (3.6) $\mathrm{P}(\lambda)$ is equal to the union of spectrum of the tridiagonal Toeplitz matrices (3.7) A, B and C. It has 75 distinct eigenvalues for N=25. We research the matrices A, B and C of dimensions 30.



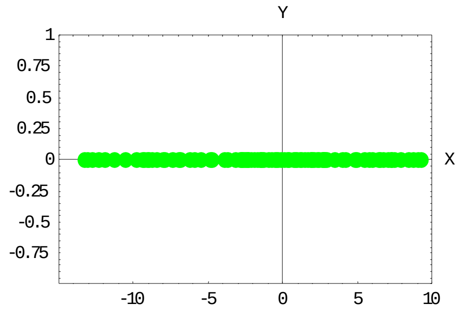
(a) $\sigma(\mathrm{A})$



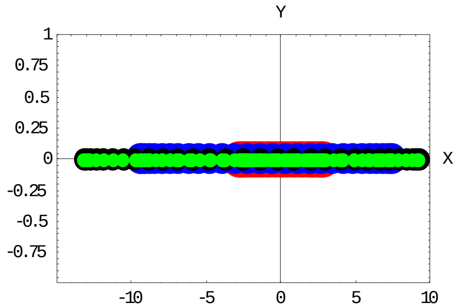
(b) $\sigma(\mathrm{B})$



(c) $\sigma(C)$



(d) $\sigma(\mathrm{P})$



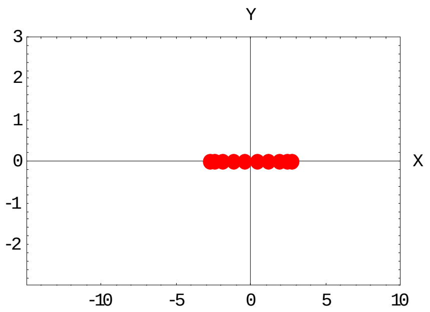
(e) $\sigma(\mathrm{A}),\ \sigma(\mathrm{B}),\ \sigma(C),\ \sigma(\mathrm{P})$

Figure 3.13: (a) Spectral points of the tridiagonal Toeplitz matrix (3.7) A of dimension N=30. (b) Spectral points of the tridiagonal Toeplitz matrix (3.7) B of dimension N=30. (c) Spectral points of the tridiagonal Toeplitz matrix (3.7) C of dimension N=30. (d) Spectral points of the cubic matrix polynomial (3.6) $\mathrm{P}(\lambda)$ of dimension N=30. (e) All spectral points of them in the same region.

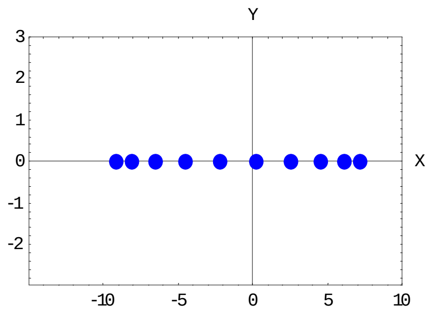
We see again that $\sigma(\mathrm{P})$ is equal to union of $\sigma(\mathrm{A})$, $\sigma(\mathrm{B})$ and $\sigma(C)$ for N=30. $\sigma(\mathrm{P})$ contains 90 distinct eigenvalues. The matrices (3.7) A, B and C of different dimensions were perturbed for $\|\Delta_0\| \le \varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$. Then the perturbed cubic matrix polynomial can be written in this form,

$$\mathrm{P}_{\Delta}(\lambda) = (\lambda\mathrm{I} - (\mathrm{A} + \Delta_0))\ (\lambda\mathrm{I} - (\mathrm{B} + \Delta_0))\ (\lambda\mathrm{I} - (C + \Delta_0)), \tag{3.8}$$

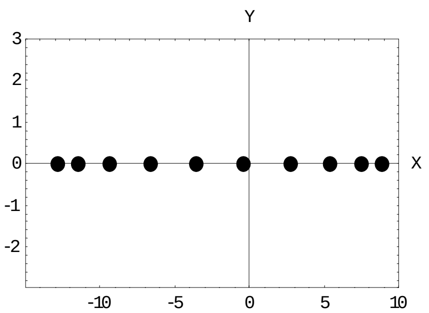
where $\Delta_0 \in \mathbb{C}^{N\times N}$, $\|\Delta_0\| \le \varepsilon$, $\varepsilon > 0$. We compared pseudospectral points of the matrices (3.7) A, B and C with pseudospectral points of $\mathrm{P}_{\Delta}(\lambda)$. Firstly, let's look at the plots corresponding to N=10 and $\|\Delta_0\| \le \varepsilon = 10^{-2}$.



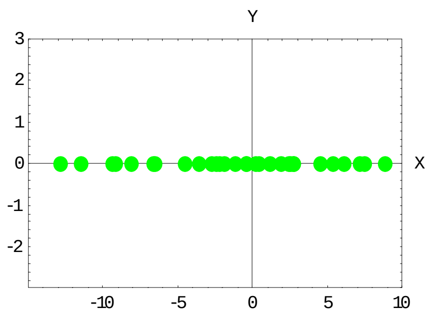
(a) $\sigma(\mathrm{A} + \Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$



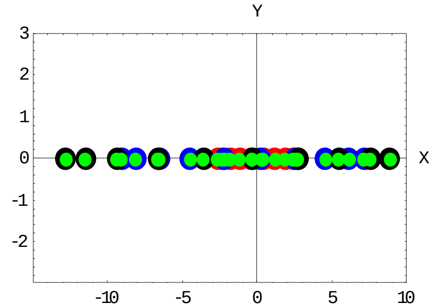
(b) $\sigma(\mathrm{B} + \Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$



(c) $\sigma(C + \Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$



(d) $\sigma(\mathrm{P}_{\Delta})$



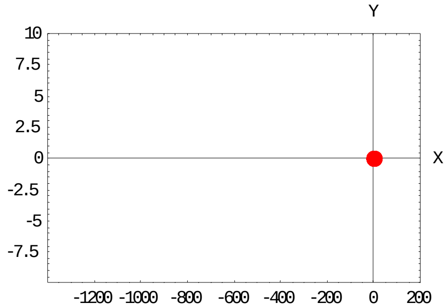
(e) $\sigma(\mathrm{A}+\Delta_0),\ \sigma(\mathrm{B}+\Delta_0),\ \sigma(C+\Delta_0),$

$\sigma(\mathrm{P}_\Delta),\ \|\Delta_0\| \le \varepsilon = 10^{-2}.$

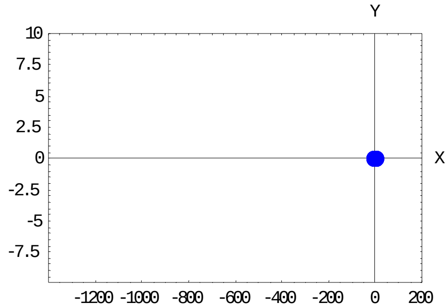
Figure 3.14: (a) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{A}+\Delta_0$ corresponding to N=10, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. (b) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{B}+\Delta_0$ corresponding to N=10, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. (c) Pseudospectral points of the perturbed matrix (3.7) $C+\Delta_0$ corresponding to N=10, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. (d) Pseudospectral points of the perturbed matrix polynomial (3.8) $\mathrm{P}_\Delta(\lambda)$. (e) All the pseudospectral points of them in the same region.

Figure 3.14 indicates that spectral points of the tridiagonal Toeplitz matrices (3.7) A, B and C are not much sensitive for N=10, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. However, we see that the pseudospectral points of $\mathrm{P}_\Delta(\lambda)$ are equal to the union of pseudospectral points of the matrices $\mathrm{A}+\Delta_0$, $\mathrm{B}+\Delta_0$ and $C+\Delta_0$.

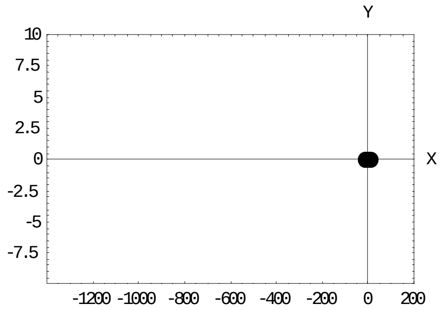
Let's take N=25, 30; $\|\Delta_0\| \le \varepsilon = 10^{-4}$ and $\|\Delta_0\| \le \varepsilon = 10^{-2}$.



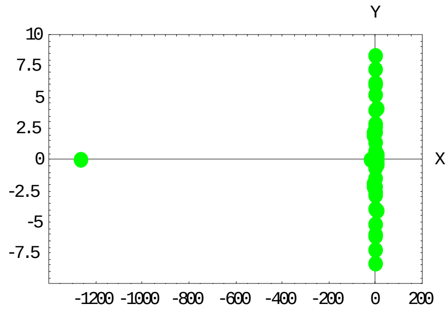
(a) $\sigma(\mathrm{A}+\Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-4}$.



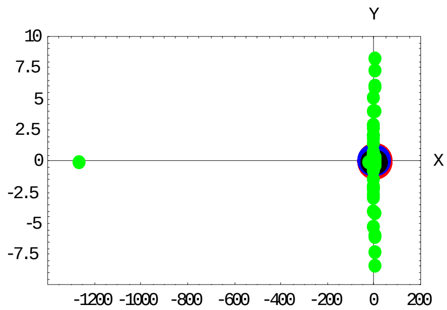
(b) $\sigma(\mathrm{B}+\Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-4}$.



(c) $\sigma(C+\Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-4}$.



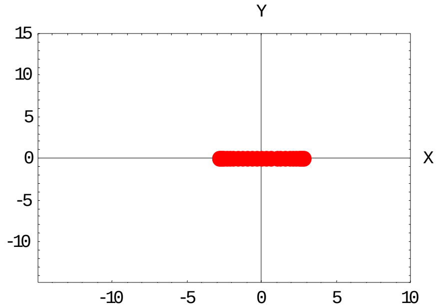
(d) $\sigma(\mathrm{P}_\Delta)$



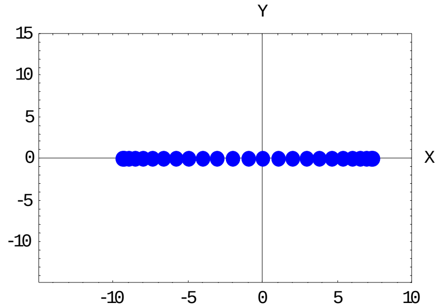
(e) $\sigma(\mathrm{A}+\Delta_0)$, $\sigma(\mathrm{B}+\Delta_0)$, $\sigma(C+\Delta_0)$

$\sigma(\mathrm{P}_\Delta)$, $\|\Delta_0\| \le \varepsilon = 10^{-4}$.

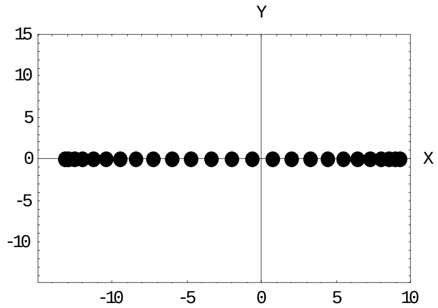
Figure 3.15: (a) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{A}+\Delta_0$ corresponding to N=25, $\|\Delta_0\| \le \varepsilon = 10^{-4}$. (b) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{B}+\Delta_0$ corresponding to N=25, $\|\Delta_0\| \le \varepsilon = 10^{-4}$. (c) Pseudospectral points of the perturbed matrix (3.7) $C+\Delta_0$ corresponding to N=25, $\|\Delta_0\| \le \varepsilon = 10^{-4}$. (d) Pseudospectral points of the perturbed matrix polynomial (3.8) $\mathrm{P}_\Delta(\lambda)$. (e) All pseudospectral points of them in the same region.



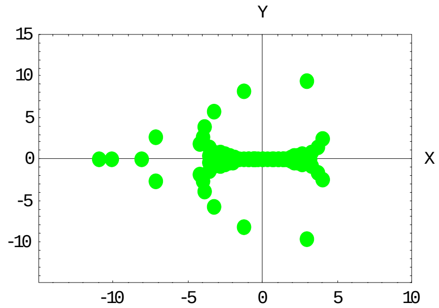
(a) $\sigma(\mathrm{A}+\Delta_0),\ \|\Delta_0\| \leq \varepsilon = 10^{-2}.$



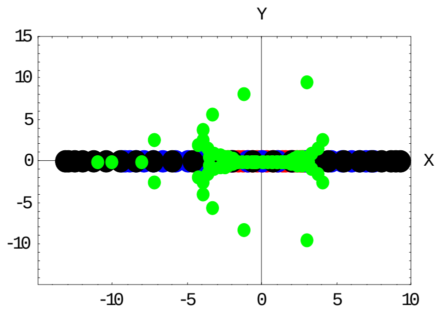
(b) $\sigma(\mathrm{B}+\Delta_0),\ \|\Delta_0\| \leq \varepsilon = 10^{-2}.$



(c) $\sigma(C+\Delta_0),\ \|\Delta_0\| \leq \varepsilon = 10^{-2}.$



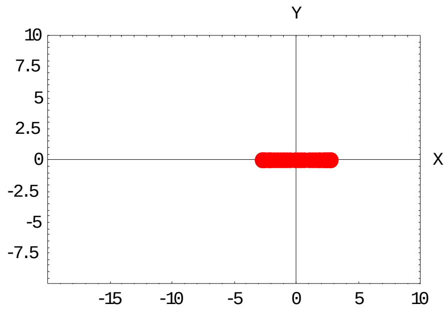
(d) $\sigma(\mathrm{P}_\Delta)$



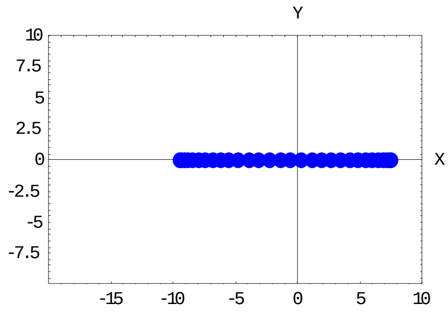
(e) $\sigma(\mathrm{A}+\Delta_0),\ \sigma(\mathrm{B}+\Delta_0),\ \sigma(C+\Delta_0)$

$\sigma(\mathrm{P}_\Delta),\ \|\Delta_0\| \leq \varepsilon = 10^{-2}.$

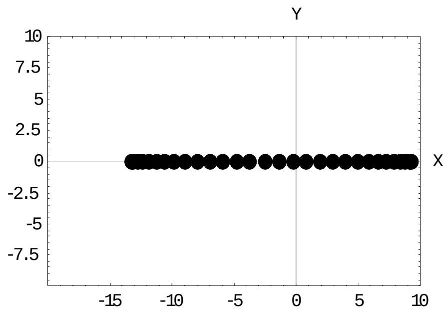
Figure 3.16: (a) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{A}+\Delta_0$ corresponding to N=25, $\|\Delta_0\| \leq \varepsilon = 10^{-2}$. (b) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{B}+\Delta_0$ corresponding to N=25, $\|\Delta_0\| \leq \varepsilon = 10^{-2}$. (c) Pseudospectral points of the perturbed matrix (3.7) $C+\Delta_0$ corresponding to N=25, $\|\Delta_0\| \leq \varepsilon = 10^{-2}$. (d) Pseudospectral points of the perturbed matrix polynomial (3.8) $\mathrm{P}_\Delta(\lambda)$. (e) All pseudospectral points of them in the same region.



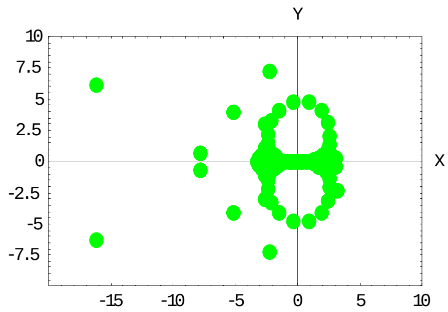
(a) $\sigma(\mathrm{A}+\Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$.



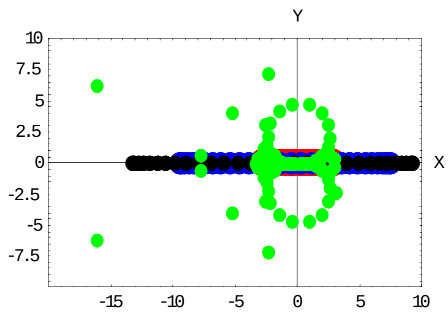
(b) $\sigma(\mathrm{B}+\Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$.



(c) $\sigma(C+\Delta_0)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$.



(d) $\sigma(\mathrm{P}_\Delta)$



(e) $\sigma(\mathrm{A}+\Delta_0)$, $\sigma(\mathrm{B}+\Delta_0)$, $\sigma(C+\Delta_0)$

$\sigma(\mathrm{P}_\Delta)$, $\|\Delta_0\| \le \varepsilon = 10^{-2}$.

Figure 3.17: (a) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{A}+\Delta_0$ corresponding to N=30, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. (b) Pseudospectral points of the perturbed matrix (3.7) $\mathrm{B}+\Delta_0$ corresponding to N=30, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. (c) Pseudospectral points of the perturbed matrix (3.7) $C+\Delta_0$ corresponding to N=30, $\|\Delta_0\| \le \varepsilon = 10^{-2}$. (d) Pseudospectral points of the perturbed matrix polynomial (3.8) $\mathrm{P}_\Delta(\lambda)$. (e) All pseudospectral points of them in the same region.

Figure 3.15-17 indicates that pseudospectral points of the perturbed cubic matrix polynomial (3.8) $P_{\Delta}(\lambda)$ differ from the union of the pseudospectral points of the perturbed matrices $A+\Delta_0$, $B+\Delta_0$ and $C+\Delta_0$ for N=25,30; $\|\Delta_0\| \leq \varepsilon = 10^{-4}, 10^{-2}$.

CONCLUSIONS

In this study we investigate the pseudospectral points of linear, quadratic and cubic polynomials (pencils) with matrix coefficients and we show that the pseudospectral properties of polynomials with matrix coefficients can be studied by means of linear pencils factors. Also in this study we write Mathematica programs to compute and plot the $\varepsilon-$pseudospectral region for pencils. We plot $\varepsilon-$pseudospectral points for linear, quadratic and cubic polynomials with matrix coefficients with dimensions N=10, 25 75, 100, 200, 300. We don’t plot $\varepsilon-$pseudospectrum for large N dimensions, for example for N=1000, 2000, 5000...Because we don’t have a computer with delicate CPU to compute our problems.

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