ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MATRIX SEQUENCES OF SPECIAL CASES OF GENERALIZED TRIBONACCI NUMBERS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

CANAN KOÇ

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ADVISOR: Prof. Dr. Yüksel SOYKAN

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

Canan KOÇ

ABSTRACT

Master of Science Thesis

MATRIX SEQUENCES OF SPECIAL CASES OF GENERALIZED TRIBONACCI NUMBERS

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The matrix sequences have taken so much interest for different type of numbers. In this thesis, we consider the matrix sequences of special cases of the generalized 3-step Fibonacci (generalized Tribonacci) numbers. We summarize the chapters as follows.

In chapter 1, we present some basic important definitions and some results used throughout the thesis.

In chapter 2, we define and investigate the Tribonacci and Tribonacci-Lucas matrix sequences. We give Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we present some identities and matrices related with these sequences.

ABSTRACT (continued)

In chapter 3, we introduce and investigate the adjusted Tribonacci-Lucas matrix sequence. We present Binet's formula, generating function, and the summation formulas for this sequence. Moreover, we give some identities and matrices related with this sequence.

In chapter 4, we define and investigate the third-order Pell and third-order Pell-Lucas matrix sequences. We give Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we present some identities and matrices related with these sequences.

In chapter 5, we define and investigate the Narayana matrix and Narayana-Lucas matrix sequences. We present Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

In chapter 6, we introduce and investigate the generalized Narayana matrix sequence and we deal with, in detail, three special cases of this sequence which we call them Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences. We present Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there always exist interrelation between generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences. This chapter contains our original work.

Keywords: Tribonacci numbers, Tribonacci matrix, Narayana numbers, Narayana matrix.

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

GENELLEŞTİRİLMİŞ TRİBONACCİ SAYILARININ ÖZEL DURUMLARININ MATRİS DİZİLERİ

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Matris dizileri, farklı sayı türleri için çok fazla ilgi görmüştür. Bu tezde, genelleştirilmiş 3 adımlı Fibonacci (genelleştirilmiş Tribonacci) sayılarının özel durumlarının matris dizilerini ele alıyoruz. Bölümleri aşağıda olduğu gibi özet olarak verebiliriz.

Bölüm 1'de, tez boyunca kullanılacak olan bazı temel önemli tanımları ve sonuçları sunuyoruz.

Bölüm 2'de, Tribonacci ve Tribonacci-Lucas matris dizilerini tanımlıyor ve araştırıyoruz. Bu diziler için Binet formüllerini, üreteç fonksiyonlarını ve toplama formüllerini veriyoruz. Ayrıca, bu dizilerle ilgili bazı özdeşlikler ve matrisler sunuyoruz.

Bölüm 3'de, düzeltilmiş Tribonacci-Lucas matris dizisini tanıtıyor ve araştırıyoruz. Bu dizi için Binet formülünü, üreteç fonksiyonunu ve toplama formüllerini sunuyoruz. Ayrıca, bu diziyle ilgili bazı özdeşlikler ve matrisler veriyoruz.

ÖZET (devam ediyor)

Bölüm 4'de, üçüncü mertebeden Pell ve üçüncü mertebeden Pell-Lucas matris dizilerini tanımlıyor ve araştırıyoruz. Bu diziler için Binet formüllerini, üreteç fonksiyonlarını ve toplam formüllerini veriyoruz. Ayrıca, bu dizilerle ilgili bazı özdeşlikler ve matrisler sunuyoruz.

Bölüm 5'de, Narayana ve Narayana-Lucas matris dizilerini tanımlıyor ve araştırıyoruz. Bu diziler için Binet formüllerini, üreteç fonksiyonlarını ve toplam formüllerini sunuyoruz. Ayrıca, bu dizilerle ilgili bazı özdeşlikler ve matrisler veriyoruz.

Bölüm 6'da, genelleştirilmiş Narayana matris dizisini tanıtıyor ve araştırıyoruz ve bu dizinin, Narayana, Narayana-Lucas ve Narayana-Perrin matris dizileri olarak adlandırdığımız üç özel durumunu ayrıntılı olarak ele alıyoruz. Bu diziler için Binet formüllerini, üreteç fonksiyonlarını ve toplam formüllerini sunuyoruz. Ayrıca bu dizilerle ilgili bazı özdeşlikler ve matrisler veriyoruz. Ayrıca, genelleştirilmiş Narayana, Narayana, Narayana-Lucas ve Narayana-Perrin matris dizileri arasında her zaman karşılıklı ilişki olduğunu gösteriyoruz. Bu bölüm, orijinal çalışmamızı içermektedir.

Anahtar Kelimeler: Tribonacci sayıları, Tribonacci matrisi, Narayana sayıları, Narayana matrisi.

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

T_n : n-th Fibonacci number

K_n : n-th Lucas number

P_n : n-th third-order Pell number

Q_n : n-th third-order Pell-Lucas number

J_n : n-th third-order Jacobsthal number

j_n : n-th third-order Jacobsthal-Lucas number

N_n : n-th Narayana number

U_n : n-th Narayana-Lucas number

CHAPTER 1

INTRODUCTION

In this chapter, we present some basic important definitions and some results used throughout the thesis.

1.1 GENERALIZED 3-STEP FIBONACCI NUMBERS

The generalized 3-step Fibonacci sequence (also called the generalized Tribonacci sequence or generalized (r,s,t)-sequence)

$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n. For more information on generalized 3-step Fibonacci numbers, see for example Soykan [22].

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned}
\alpha &= \alpha(r, s, t) = \frac{r}{3} + A + B, \\
\beta &= \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 B, \\
\gamma &= \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega B,
\end{aligned}$$

where

$$
\begin{aligned}
A &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{1/3}, \\
B &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{1/3}, \\
\Delta &= \Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4}, \\
\omega &= \frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).
\end{aligned}
$$

Note that we have the following identities

$$
\begin{aligned}
\alpha+\beta+\gamma &= r, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -s, \\
\alpha\beta\gamma &= t.
\end{aligned}
$$

If $\Delta(r,s,t)>0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized 3-step Fibonacci numbers (generalized Tribonacci numbers) can be expressed, for all integers n, using Binet's formula

$$
W_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.3}
$$

where

$$
\begin{aligned}
b_1 &= W_2-(\beta+\gamma)W_1+\beta\gamma W_0, \\
b_2 &= W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0, \\
b_3 &= W_2-(\alpha+\beta)W_1+\alpha\beta W_0.
\end{aligned}
$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [13]. This result of Howard and Saidak [13] is even true in the case of higher-order recurrence relations as the following theorem shows.

Theorem 1.1 *[13] Let $\{w_n\}$ be a sequence such that*

$$
\{w_n\}=a_1w_{n-1}+a_2w_{n-2}+...+a_kw_{n-k}
$$

for all integers n, with arbitrary initial conditions $w_0, w_1, ..., w_{k-1}$. Assume that each a_i and the initial conditions are complex numbers. Write

$$
\begin{aligned}
f(x) &= x^k - a_1x^{k-1} - a_2x^{k-2} - ... - a_{k-1}x - a_k \qquad (1.4)\\
&= (x-\alpha_1)^{d_1}(x-\alpha_2)^{d_2}...(x-\alpha_h)^{d_h}
\end{aligned}
$$

with $d_1 + d_2 + ... + d_h = k$, and $\alpha_1, \alpha_2, ..., \alpha_k$ distinct. Then

(a) *For all n,*

$$w_n = \sum_{m=1}^{k} N(n,m)(\alpha_m)^n \tag{1.5}$$

where

$$N(n,m) = A_1^{(m)} + A_2^{(m)}n + ... + A_{r_m}^{(m)}n^{r_m-1} = \sum_{u=0}^{r_m-1} A_{u+1}^{(m)}n^u$$

with each $A_i^{(m)}$ a constant determined by the initial conditions for $\{w_n\}$. Here, equation (1.5) is called the Binet form (or Binet formula) for $\{w_n\}$. We assume that $f(0) \neq 0$ so that $\{w_n\}$ can be extended to negative integers n.

If the zeros of (1.4) are distinct, as they are in our examples, then

$$w_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + ... + A_k(\alpha_k)^n.$$

(b) *The Binet form for $\{w_n\}$ is valid for all integers n.*

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.2 *Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized 3-step Fibonacci sequence (generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.6}$$

We next present Binet formula of generalized 3-step Fibonacci numbers (generalized Tribonacci numbers) $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.3 *(Binet formula of generalized* 3*-step Fibonacci numbers (generalized Tribonacci numbers))*

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.7}$$

where

$$\begin{aligned} d_1 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ d_2 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ d_3 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Note that from (1.3) and (1.7) we have

$$\begin{aligned} W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

1.2 GENERALIZED TRIBONACCI NUMBERS

In this section, we consider the case $r = 1$, $s = 1, t = 1$ and in this case we write $V_n = W_n$ and in this case we also call the sequence V_n as generalized Tribonacci sequence. So a generalized Tribonacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} \tag{1.8}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.
The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.8) holds for all integer n. For more information on generalized Tribonacci numbers, see for example Soykan [21].
(1.3) can be used to obtain Binet formula of generalized Tribonacci numbers. Binet formula of generalized Tribonacci numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.9}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, \quad (1.10)$$

$$b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \quad (1.11)$$

$$b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \quad (1.12)$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x^2 - x - 1 = 0.$$

Moreover

$$\begin{aligned}
\alpha &= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},\\
\beta &= \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},\\
\gamma &= \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3},
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma &= 1,\\
\alpha\beta+\alpha\gamma+\beta\gamma &= -1,\\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Tribonacci sequence$\{T_n\}_{n\geq 0}$, Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$, and adjusted Tribonacci-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \quad (1.13)$$

$$K_{n+3} = K_{n+2} + K_{n+1} + K_n, \qquad K_0 = 3, K_1 = 1, K_2 = 3, \quad (1.14)$$

$$H_{n+3} = H_{n+2} + H_{n+1} + H_n, \qquad H_0 = 4, H_1 = 2, H_2 = 0, \quad (1.15)$$

The sequences $\{T_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \quad (1.16)$$

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \quad (1.17)$$

$$H_{-n} = -H_{-(n-1)} - H_{-(n-2)} + H_{-(n-3)}, \quad (1.18)$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.13)-(1.15) hold for all integer n. T_n is the sequence A000073 in [17] and K_n is the sequence A001644 in [17].

We can give some relations between $\{T_n\}$ and $\{K_n\}$ as

$$K_n = 3T_{n+1} - 2T_n - T_{n-1} \tag{1.19}$$

and

$$K_n = T_n + 2T_{n-1} + 3T_{n-2} \tag{1.20}$$

and also

$$K_n = 4T_{n+1} - T_n - T_{n+2}. \tag{1.21}$$

For all integers n, Tribonacci, Tribonacci-Lucas and adjusted Tribonacci-Lucas numbers (using initial conditions in (1.34)-(1.12)) can be expressed using Binet's formulas as

$$\begin{aligned}
T_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\
K_n &= \alpha^n + \beta^n + \gamma^n, \\
H_n &= (\alpha-1)^2\alpha^n + (\beta-1)^2\beta^n + (\gamma-1)^2\gamma^n,
\end{aligned}$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1.4 *Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - x^3}. \tag{1.22}$$

The previous lemma gives the following results as particular examples.

Corollary 1.1 *Generated functions of Tribonacci, Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n x^n &= \frac{x}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} K_n x^n &= \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{4 - 2x - 6x^2}{1 - x - x^2 - x^3},
\end{aligned}$$

respectively.

1.3 GENERALIZED THIRD-ORDER PELL NUMBERS

In this section, we consider the case $r = 2$, $s = t = 1$ and in this case we write $V_n = W_n$. A generalized third order Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} \tag{1.23}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.
The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - 2V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.23) holds for all integer n. For more information on generalized third-order Pell numbers, see for example Soykan [31].
(1.3) can be used to obtain Binet formula of generalized third order Pell numbers. Binet formula of generalized third order Pell numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2-(\beta+\gamma)V_1+\beta\gamma V_0,\ b_2 = V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0,\ b_3 = V_2-(\alpha+\beta)V_1+\alpha\beta V_0. \tag{1.24}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - 2x^2 - x - 1 = 0$. Moreover

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma &= 2, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Now we define three special case of the sequence $\{V_n\}$. Third-order Pell sequence$\{P_n^{(3)}\}_{n\geq 0}$, third-order Pell-Lucas sequence $\{Q_n^{(3)}\}_{n\geq 0}$ and modified third-order Pell sequence $\{E_n^{(3)}\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$
\begin{aligned}
P_{n+3}^{(3)} &= 2P_{n+2}^{(3)} + P_{n+1}^{(3)} + P_n^{(3)}, \qquad P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2, && (1.25)\\
Q_{n+3}^{(3)} &= 2Q_{n+2}^{(3)} + Q_{n+1}^{(3)} + Q_n^{(3)}, \qquad Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6 && (1.26)\\
E_{n+3}^{(3)} &= 2E_{n+2}^{(3)} + E_{n+1}^{(3)} + E_n^{(3)}, \qquad E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1. && (1.27)
\end{aligned}
$$

The sequences $\{P_n^{(3)}\}_{n\geq 0}$, $\{Q_n^{(3)}\}_{n\geq 0}$ and $\{E_n^{(3)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$
\begin{aligned}
P_{-n}^{(3)} &= -P_{-(n-1)}^{(3)} - 2P_{-(n-2)}^{(3)} + P_{-(n-3)}^{(3)} && (1.28)\\
Q_{-n}^{(3)} &= -Q_{-(n-1)}^{(3)} - 2Q_{-(n-2)}^{(3)} + Q_{-(n-3)}^{(3)} && (1.29)\\
E_{-n}^{(3)} &= -E_{-(n-1)}^{(3)} - 2E_{-(n-2)}^{(3)} + E_{-(n-3)}^{(3)} && (1.30)
\end{aligned}
$$

for $n = 1, 2, 3, ...$ respectively.

In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(3)}, Q_n^{(3)}$ and $E_n^{(3)}$, respectively.

Note that P_n is the sequence A077939 in [17] associated with the expansion of $1/(1 - 2x - x^2 - x^3)$, Q_n is the sequence A276225 in [17] and Q_n is the sequence A077997 in [17]. For all integers n, third-order Pell, Pell-Lucas and modified Pell numbers can be expressed using Binet's formulas as

$$
\begin{aligned}
P_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},\\
Q_n &= \alpha^n + \beta^n + \gamma^n,\\
E_n &= \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},
\end{aligned}
$$

respectively.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1.5 *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized third-order Pell sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2}{1 - 2x - x^2 - x^3}. \tag{1.31}$$

The previous Lemma gives the following results as particular examples.

Corollary 1.2 *Generated functions of third-order Pell, Pell-Lucas and modified Pell numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n x^n &= \frac{x}{1-2x-x^2-x^3}, \\
\sum_{n=0}^{\infty} Q_n x^n &= \frac{3-4x-x^2}{1-2x-x^2-x^3}, \\
\sum_{n=0}^{\infty} E_n x^n &= \frac{x-x^2}{1-2x-x^2-x^3},
\end{aligned}$$

respectively.

1.4 GENERALIZED NARAYANA NUMBERS

In this section, we consider the case $r = 1$, $s = 0$, $t = 1$ and in this case we write $V_n = W_n$. A generalized Narayana sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-3} \tag{1.32}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.32) holds for all integer n.For more details on generalized Narayana numbers, see Soykan [18]. Binet formula of generalized Narayana numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.33}$$

where

$$b_1 = V_2-(\beta+\gamma)V_1+\beta\gamma V_0,\ b_2 = V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0,\ b_3 = V_2-(\alpha+\beta)V_1+\alpha\beta V_0. \tag{1.34}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x^2 - 1 = 0$. Moreover

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma &= 1, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Now we define three special case of the sequence $\{V_n\}$. Narayana sequence $\{N_n\}_{n\geq 0}$, Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ and Narayana-Perrin sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}
N_{n+3} &= N_{n+2} + N_n, \qquad N_0 = 0, N_1 = 1, N_2 = 1, &(1.35)\\
U_{n+3} &= U_{n+2} + U_n, \qquad U_0 = 3, U_1 = 1, U_2 = 1, &(1.36)\\
H_{n+3} &= H_{n+2} + H_n, \qquad H_0 = 3, H_1 = 0, H_2 = 2, &(1.37)
\end{aligned}$$

The sequences $\{N_n\}_{n\geq 0}, \{U_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
N_{-n} &= -N_{-(n-2)} + N_{-(n-3)}, \\
U_{-n} &= -U_{-(n-2)} + U_{-(n-3)}, \\
H_{-n} &= -H_{-(n-2)} + H_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.35)- (1.37) hold for all integer n. For more details on generalized Narayana numbers, see [18].

We can give some relations between $\{N_n\}$ and $\{U_n\}$ as

$$U_n = 3N_{n+4} - 5N_{n+3} + 2N_{n+2}, \tag{1.38}$$
$$U_n = -2N_{n+3} + 2N_{n+2} + 3N_{n+1}, \tag{1.39}$$
$$U_n = 3N_{n+1} - 2N_n, \tag{1.40}$$
$$U_n = N_n + 3N_{n-2}, \tag{1.41}$$

and

$$31N_n = -3U_{n+4} + U_{n+3} + 11U_{n+2}, \tag{1.42}$$
$$31N_n = -2U_{n+3} + 11U_{n+2} - 3U_{n+1}, \tag{1.43}$$
$$31N_n = 9U_{n+2} - 3U_{n+1} - 2U_n, \tag{1.44}$$
$$31N_n = 6U_{n+1} - 2U_n + 9U_{n-1}, \tag{1.45}$$
$$31N_n = 4U_n + 9U_{n-1} + 6U_{n-2}. \tag{1.46}$$

Note that all the above identities hold for all integers n (for more details, see [18]).

Note that N_n is the sequence A000930 in [17] associated with the Narayana's cows sequence and the sequence A078012 in [17] associated with the expansion of $(1-x)/(1-x-x^3)$ and U_n is the sequence A001609 in [17].

For all integers n, Narayana, Narayana-Lucas and Narayana-Perrin numbers (using initial conditions in (1.34)) can be expressed using Binet's formulas as

$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.47}$$
$$U_n = \alpha^n + \beta^n + \gamma^n, \tag{1.48}$$
$$H_n = \frac{(3+2\alpha)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3+2\beta)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3+2\gamma)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.49}$$

respectively.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1.6 *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Narayana sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{1 - x - x^3}. \tag{1.50}$$

The previous lemma gives the following results as particular examples.

Corollary 1.3 *Generated functions of Narayana, Narayana-Lucas and Narayana-Perrin numbers are*

$$\sum_{n=0}^{\infty} N_n x^n = \frac{x}{1-x-x^3}, \quad \sum_{n=0}^{\infty} U_n x^n = \frac{3-2x}{1-x-x^3}, \quad \sum_{n=0}^{\infty} H_n x^n = \frac{3-3x+2x^2}{1-x-x^3}, \tag{1.51}$$

respectively.

1.5 MATRIX SEQUENCES

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam (generalized Fibonacci) numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. On the other hand, the matrix sequences have taken so much interest for different type of numbers. We present some works on matrix sequences of the numbers in the following Table 1.1.

Table 1.1. A Few Special Study On The Matrix Sequences Of The Numbers.

Name of sequence	work on the matrix sequences of the numbers
Generalized Fibonacci	[6,7,11,32,33,34,35,36,39]
Generalized Tribonacci	[4,23,24,27,28,37,38]
Generalized Tetranacci	[25]

CHAPTER 2

ON MATRIX SEQUENCES OF TRIBONACCI AND TRIBONACCI-LUCAS NUMBERS

In this chapter, we introduce and investigate Tribonacci and Tribonacci-Lucas matrix sequences and investigate their properties. We give Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we present some identities and matrices related with these sequences. We use Soykan [23].

2.1 THE MATRIX SEQUENCES OF TRIBONACCI AND TRIBONACCI-LUCAS NUMBERS

In this section we define Tribonacci and Tribonacci-Lucas matrix sequences and investgate their properties.

Definition 2.1 *For any integer $n \geq 0$, the Tribonacci matrix $(\mathcal{T}_n)$ and Tribonacci-Lucas matrix $(\mathcal{K}_n)$ are defined by*

$$\mathcal{T}_n = \mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3}, \tag{2.1}$$

$$\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + \mathcal{K}_{n-3,} \tag{2.2}$$

respectively, with initial conditions

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{T}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & -1 \\ -1 & 4 & -1 \end{pmatrix}, \mathcal{K}_1 = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix}, \mathcal{K}_2 = \begin{pmatrix} 7 & 4 & 3 \\ 3 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

The sequences $\{\mathcal{T}_n\}_{n\geq 0}$ and $\{\mathcal{K}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{T}_{-n} = -\mathcal{T}_{-(n-1)} - \mathcal{T}_{-(n-2)} + \mathcal{T}_{-(n-3)}$$

and

$$\mathcal{K}_{-n} = -\mathcal{K}_{-(n-1)} - \mathcal{K}_{-(n-2)} + \mathcal{K}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.1) and (2.2) hold for all integers n.

We list some properties of the Tribonacci and Tribonacci-Lucas matrix sequences.

- For any integer $n \geq 0$, we have the following formulas of the matrix sequences:

$$\mathcal{T}_n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \tag{2.3}$$

$$\mathcal{K}_n = \begin{pmatrix} K_{n+1} & K_n + K_{n-1} & K_n \\ K_n & K_{n-1} + K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-2} + K_{n-3} & K_{n-2} \end{pmatrix}. \tag{2.4}$$

- For every integer n, the Binet formulas of the Tribonacci and Tribonacci-Lucas matrix sequences are given by

$$\mathcal{T}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \tag{2.5}$$

$$\mathcal{K}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n. \tag{2.6}$$

where

$$A_1 = \frac{\alpha\mathcal{T}_2 + \alpha(\alpha-1)\mathcal{T}_1 + \mathcal{T}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{T}_2 + \beta(\beta-1)\mathcal{T}_1 + \mathcal{T}_0}{\beta(\beta-\gamma)(\beta-\alpha)},$$

$$C_1 = \frac{\gamma\mathcal{T}_2 + \gamma(\gamma-1)\mathcal{T}_1 + \mathcal{T}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}, A_2 = \frac{\alpha\mathcal{K}_2 + \alpha(\alpha-1)\mathcal{K}_1 + \mathcal{K}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)},$$

$$B_2 = \frac{\beta\mathcal{K}_2 + \beta(\beta-1)\mathcal{K}_1 + \mathcal{K}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{K}_2 + \gamma(\gamma-1)\mathcal{K}_1 + \mathcal{K}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.$$

- For $m > j \geq 0$, we have

$$\sum_{i=0}^{n-1} \mathcal{T}_{mi+j} = \frac{\mathcal{T}_{mn+m+j} + \mathcal{T}_{mn-m+j} + (1-K_m)\mathcal{T}_{mn+j}}{K_m - K_{-m}} - \frac{\mathcal{T}_{m+j} + \mathcal{T}_{j-m} + (1-K_m)\mathcal{T}_j}{K_m - K_{-m}} \tag{2.7}$$

and

$$\sum_{i=0}^{n-1} \mathcal{K}_{mi+j} = \frac{\mathcal{K}_{mn+m+j} + \mathcal{K}_{mn-m+j} + (1-K_m)\mathcal{K}_{mn+j}}{K_m - K_{-m}} - \frac{\mathcal{K}_{m+j} + \mathcal{K}_{j-m} + (1-K_m)\mathcal{K}_j}{K_m - K_{-m}}. \tag{2.8}$$

- The generating function for the Tribonacci and Tribonacci-Lucas matrix sequences are given as

$$\sum_{n=0}^{\infty} \mathcal{T}_n x^n = \frac{1}{1-x-x^2-x^3} \begin{pmatrix} 1 & x+x^2 & x \\ x & 1-x & x^2 \\ x^2 & x-x^2 & 1-x-x^2 \end{pmatrix}$$

and

$$\sum_{n=0}^{\infty} \mathcal{K}_n x^n = \frac{1}{1-x-x^2-x^3} \begin{pmatrix} 1+2x+3x^2 & 2+2x-2x^2 & 3-2x-x^2 \\ 3-2x-x^2 & -2+4x+4x^2 & -1+4x-x^2 \\ -1+4x-x^2 & 4-6x & -1+5x^2 \end{pmatrix}$$

respectively.

2.2 RELATION BETWEEN TRIBONACCI AND TRIBONACCI-LUCAS MATRIX SEQUENCES

Now, we list some properties of Tribonacci and Tribonacci-Lucas matrix sequences.

- For the matrix sequences $\{\mathcal{T}_n\}$ and $\{\mathcal{K}_n\}$, we have the following identities.

(a) $\mathcal{K}_n = 3\mathcal{T}_{n+1} - 2\mathcal{T}_n - \mathcal{T}_{n-1}$.

(b) $\mathcal{K}_n = \mathcal{T}_n + 2\mathcal{T}_{n-1} + 3\mathcal{T}_{n-2}$.

(c) $\mathcal{K}_n = 4\mathcal{T}_{n+1} - \mathcal{T}_n - \mathcal{T}_{n+2}$.

(d) $\mathcal{K}_n = -\mathcal{T}_{n+2} + 4\mathcal{T}_{n+1} - \mathcal{T}_n$.

(e) $\mathcal{T}_n = \frac{1}{22}(5\mathcal{K}_{n+2} - 3\mathcal{K}_{n+1} - 4\mathcal{K}_n)$.

- For all non-negative integers m and n, we have the following identities.

(a) $\mathcal{K}_0\mathcal{T}_n = \mathcal{T}_n\mathcal{K}_0 = \mathcal{K}_n$.

(b) $\mathcal{T}_0\mathcal{K}_n = \mathcal{K}_n\mathcal{T}_0 = \mathcal{K}_n$.

- The following relations hold:

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\ A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0), \\ A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0). \end{aligned}$$

- For all non-negative integers m and n, we have the following identities.

(a) $\mathcal{T}_m\mathcal{T}_n = \mathcal{T}_{m+n} = \mathcal{T}_n\mathcal{T}_m$.

(b) $\mathcal{T}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{T}_m = \mathcal{K}_{m+n}$.

(c) $\mathcal{K}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{K}_m = 9\mathcal{T}_{m+n+2} - 12\mathcal{T}_{m+n+1} - 2\mathcal{T}_{m+n} + 4\mathcal{T}_{m+n-1} + \mathcal{T}_{m+n-2}$.

(d) $\mathcal{K}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{K}_m = \mathcal{T}_{m+n} + 4\mathcal{T}_{m+n-1} + 10\mathcal{T}_{m+n-2} + 12\mathcal{T}_{m+n-3} + 9\mathcal{T}_{m+n-4}$.

(e) $\mathcal{K}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{K}_m = \mathcal{T}_{m+n} - 8\mathcal{T}_{m+n+1} + 18\mathcal{T}_{m+n+2} - 8\mathcal{T}_{m+n+3} + \mathcal{T}_{m+n+4}$.

- For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{T}_n^m = \mathcal{T}_{mn}$.

(b) $\mathcal{T}_{n+1}^m = \mathcal{T}_1^m\mathcal{T}_{mn}$.

(c) $\mathcal{T}_{n-r}\mathcal{T}_{n+r} = \mathcal{T}_n^2 = \mathcal{T}_2^n$.

- For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{K}_{n-r}\mathcal{K}_{n+r} = \mathcal{K}_n^2$.

(b) $\mathcal{K}_n^m = \mathcal{K}_0^m\mathcal{T}_{mn}$.

CHAPTER 3

ON MATRIX SEQUENCE OF ADJUSTED TRIBONACCI-LUCAS NUMBERS

In this chapter, we present and investigate the adjusted Tribonacci-Lucas matrix sequence. We present Binet's formula, generating function, and the summation formulas for this sequence. Moreover, we give some identities and matrices related with this sequence. We use [28].

3.1 THE MATRIX SEQUENCES OF ADJUSTED TRIBONACCI-LUCAS NUMBERS

In this section, we define adjusted Tribonacci-Lucas matrix sequence and investigate its properties.

Definition 3.1 *For any integer $n \geq 0$, the adjusted Tribonacci-Lucas matrix (H_n) is defined by*

$$\mathcal{H}_n = \mathcal{H}_{n-1} + \mathcal{H}_{n-2} + \mathcal{H}_{n-3}, \tag{3.1}$$

with initial conditions

$$\mathcal{H}_0 = \begin{pmatrix} 2 & -2 & 4 \\ 4 & -2 & -6 \\ -6 & 10 & 4 \end{pmatrix}, \mathcal{H}_1 = \begin{pmatrix} 0 & 6 & 2 \\ 2 & -2 & 4 \\ 4 & -2 & -6 \end{pmatrix}, \mathcal{H}_2 = \begin{pmatrix} 6 & 2 & 0 \\ 0 & 6 & 2 \\ 2 & -2 & 4 \end{pmatrix}.$$

The sequence $\{\mathrm{H}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{H}_{-n} = -\mathcal{H}_{-(n-1)} - \mathcal{H}_{-(n-2)} + \mathcal{H}_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrences (3.1) holds for all integers n.
We list some properties of the adjusted Tribonacci-Lucas matrix sequence.

- For any integer $n \geq 0$, we have the following formula of the matrix sequence:

$$\mathcal{H}_n = \begin{pmatrix} H_{n+1} & H_n + H_{n-1} & H_n \\ H_n & H_{n-1} + H_{n-2} & H_{n-1} \\ H_{n-1} & H_{n-2} + H_{n-3} & H_{n-2} \end{pmatrix}. \tag{3.2}$$

- For every integer n, the Binet formula of the adjusted Tribonacci-Lucas matrix sequence is given by

$$\mathcal{H}_n = A_6\alpha^n + B_6\beta^n + C_6\gamma^n \tag{3.3}$$

where

$$\begin{aligned} A_6 &= \frac{\alpha\mathcal{H}_2 + \alpha(\alpha-1)\mathcal{H}_1 + \mathcal{H}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, \\ B_6 &= \frac{\beta\mathcal{H}_2 + \beta(\beta-1)\mathcal{H}_1 + \mathcal{H}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \\ C_6 &= \frac{\gamma\mathcal{H}_2 + \gamma(\gamma-1)\mathcal{H}_1 + \mathcal{H}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}. \end{aligned}$$

- For all integers m and j, we have

$$\sum_{k=0}^{n} \mathcal{H}_{mk+j} = \frac{\mathcal{H}_{mn+m+j} + \mathcal{H}_{mn-m+j} + (1-K_{-m})\mathcal{H}_{mn+j} - \mathcal{H}_{m+j} - \mathcal{H}_{j-m} + (K_m-1)\mathcal{H}_j}{K_m - K_{-m}}. \tag{3.4}$$

- The generating function for the adjusted Tribonacci-Lucas matrix sequence is given as

$$\sum_{n=0}^{\infty} \mathcal{H}_n x^n = \frac{1}{1-x-x^2-x^3} \begin{pmatrix} 4x^2-2x+2 & -2x^2+8x-2 & -6x^2-2x+4 \\ -6x^2-2x+4 & 10x^2-2 & 4x^2+10x-6 \\ 4x^2+10x-6 & -10x^2-12x+10 & 6x^2-10x+4 \end{pmatrix}.$$

3.2 RELATION BETWEEN TRIBONACCI, TRIBONACCI-LUCAS AND ADJUSTED TRIBONACCI-LUCAS MATRIX SEQUENCES

Now, we list some properties of the Tribonacci and adjusted Tribonacci-Lucas matrix sequences.

- For the matrix sequences $\{\mathcal{T}_n\}$ and $\{\mathrm{H}_n\}$, we have the following identities.

  (a) $44\mathcal{T}_n = -3\mathcal{H}_{n+4} + 4\mathcal{H}_{n+3} + 9\mathcal{H}_{n+2}$.

  (b) $44\mathcal{T}_n = \mathcal{H}_{n+3} + 6\mathcal{H}_{n+2} - 3\mathcal{H}_{n+1}$.

  (c) $44\mathcal{T}_n = 7\mathcal{H}_{n+2} - 2\mathcal{H}_{n+1} + \mathcal{H}_n$.

  (d) $44\mathcal{T}_n = 5\mathcal{H}_{n+1} + 8\mathcal{H}_n + 7\mathcal{H}_{n-1}$.

  (e) $44\mathcal{T}_n = 13\mathcal{H}_n + 12\mathcal{H}_{n-1} + 5\mathcal{H}_{n-2}$.

  (f) $\mathcal{H}_n = 6\mathcal{T}_{n+4} - 2\mathcal{T}_{n+3} - 16\mathcal{T}_{n+2}$.

  (g) $\mathcal{H}_n = 4\mathcal{T}_{n+3} - 10\mathcal{T}_{n+2} + 6\mathcal{T}_{n+1}$.

  (h) $\mathcal{H}_n = -6\mathcal{T}_{n+2} + 10\mathcal{T}_{n+1} + 4\mathcal{T}_n$.

  (i) $\mathcal{H}_n = 4\mathcal{T}_{n+1} - 2\mathcal{T}_n - 6\mathcal{T}_{n-1}$.

  (j) $\mathcal{H}_n = 2\mathcal{T}_n - 2\mathcal{T}_{n-1} + 4\mathcal{T}_{n-2}$.

- For the matrix sequences $\{\mathcal{K}_n\}$ and $\{\mathrm{H}_n\}$, we have the following identities.

  (a) $2\mathcal{K}_n = \mathcal{H}_{n+3} - \mathcal{H}_{n+2}$.

  (b) $2\mathcal{K}_n = \mathcal{H}_{n+1} + \mathcal{H}_n$.

  (c) $2\mathcal{K}_n = 2\mathcal{H}_n + \mathcal{H}_{n-1} + \mathcal{H}_{n-2}$.

  (d) $\mathcal{H}_n = -3\mathcal{K}_{n+4} + 4\mathcal{K}_{n+3} + 3\mathcal{K}_{n+2}$.

  (e) $\mathcal{H}_n = \mathcal{K}_{n+3} - 3\mathcal{K}_{n+1}$.

  (f) $\mathcal{H}_n = \mathcal{K}_{n+2} - 2\mathcal{K}_{n+1} + \mathcal{K}_n$.

  (g) $\mathcal{H}_n = -\mathcal{K}_{n+1} + 2\mathcal{K}_n + \mathcal{K}_{n-1}$.

  (h) $\mathcal{H}_n = \mathcal{K}_n - \mathcal{K}_{n-2}$.

- For all non-negative integers m and n, we have the following identities.

  (a) $\mathrm{H}_0\mathcal{T}_n = \mathcal{T}_n\mathrm{H}_0 = \mathrm{H}_n$.

  (b) $\mathcal{T}_0\mathrm{H}_n = \mathrm{H}_n\mathcal{T}_0 = \mathrm{H}_n$.

- For all non-negative integers m and n, we have the following identity.

  $\mathcal{H}_{m+n} = \mathcal{T}_m\mathcal{H}_n = \mathcal{H}_n\mathcal{T}_m$.

- For all non-negative integers m and n, we have the following identities.

  (a) $\mathcal{H}_{m+n} = \frac{1}{44}(-3\mathcal{H}_{m+4} + 4\mathcal{H}_{m+3} + 9\mathcal{H}_{m+2})\mathcal{H}_n$.

  (b) $\mathcal{H}_{m+n} = \frac{1}{44}(\mathcal{H}_{m+3} + 6\mathcal{H}_{m+2} - 3\mathcal{H}_{m+1})\mathcal{H}_n$.

  (c) $\mathcal{H}_{m+n} = \frac{1}{44}(7\mathcal{H}_{m+2} - 2\mathcal{H}_{m+1} + \mathcal{H}_m)\mathcal{H}_n$.

  (d) $\mathcal{H}_{m+n} = \frac{1}{44}(5\mathcal{H}_{m+1} + 8\mathcal{H}_m + 7\mathcal{H}_{m-1})\mathcal{H}_n$.

  (e) $\mathcal{H}_{m+n} = \frac{1}{44}(13\mathcal{H}_m + 12\mathcal{H}_{m-1} + 5\mathcal{H}_{m-2})\mathcal{H}_n$.

  (f) $\mathrm{H}_m\mathrm{H}_n = \mathrm{H}_n\mathrm{H}_m = 36\mathcal{T}_{m+n+8} - 24\mathcal{T}_{m+n+7} - 188\mathcal{T}_{m+n+6} + 64\mathcal{T}_{m+n+5} + 256\mathcal{T}_{m+n+4}$.

  (g) $\mathrm{H}_m\mathrm{H}_n = \mathrm{H}_n\mathrm{H}_m = 16\mathcal{T}_{m+n+6} - 80\mathcal{T}_{m+n+5} + 148\mathcal{T}_{m+n+4} - 120\mathcal{T}_{m+n+3} + 36\mathcal{T}_{m+n+2}$.

  (h) $\mathrm{H}_m\mathrm{H}_n = \mathrm{H}_n\mathrm{H}_m = 36\mathcal{T}_{m+n+4} - 120\mathcal{T}_{m+n+3} + 52\mathcal{T}_{m+n+2} + 80\mathcal{T}_{m+n+1} + 16\mathcal{T}_{m+n}$.

  (i) $\mathrm{H}_m\mathrm{H}_n = \mathrm{H}_n\mathrm{H}_m = 16\mathcal{T}_{m+n+2} - 16\mathcal{T}_{m+n+1} - 44\mathcal{T}_{m+n} + 24\mathcal{T}_{m+n-1} + 36\mathcal{T}_{m+n-2}$.

  (j) $\mathrm{H}_m\mathrm{H}_n = \mathrm{H}_n\mathrm{H}_m = 4\mathcal{T}_{m+n} - 8\mathcal{T}_{m+n-1} + 20\mathcal{T}_{m+n-2} - 16\mathcal{T}_{m+n-3} + 16\mathcal{T}_{m+n-4}$.

CHAPTER 4

ON MATRIX SEQUENCES OF THIRD-ORDER PELL AND THIRD-ORDER PELL-LUCAS NUMBERS

In this chapter, we define and investigate the third-order Pell and third-order Pell-Lucas matrix sequences. We give Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we present some identities and matrices related with these sequences. We use Soykan [26].

4.1 THE MATRIX SEQUENCES OF THIRD-ORDER PELL AND THIRD-ORDER PELL-LUCAS NUMBERS

In this section, we define third-order Pell and third-order Pell-Lucas matrix sequences and investigate their properties.

Definition 4.1 *For any integer $n \geq 0$, the third-order Pell matrix $(\mathcal{P}_n)$ and third-order Pell-Lucas matrix $(\mathcal{Q}_n)$ are defined by*

$$\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2} + \mathcal{P}_{n-3}, \tag{4.1}$$

$$\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2} + \mathcal{Q}_{n-3,} \tag{4.2}$$

respectively, with initial conditions

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{P}_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{P}_2 = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{Q}_0 = \begin{pmatrix} 2 & 2 & 3 \\ 3 & -4 & -1 \\ -1 & 5 & -3 \end{pmatrix}, \ \mathcal{Q}_1 = \begin{pmatrix} 6 & 5 & 2 \\ 2 & 2 & 3 \\ 3 & -4 & -1 \end{pmatrix}, \mathcal{Q}_2 = \begin{pmatrix} 17 & 8 & 6 \\ 6 & 5 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

The sequences $\{\mathcal{P}_n\}_{n\geq 0}$ and $\{\mathcal{Q}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{P}_{-n} = -\mathcal{P}_{-(n-1)} - 2\mathcal{P}_{-(n-2)} + \mathcal{P}_{-(n-3)}$$

and

$$\mathcal{Q}_{-n} = -\mathcal{Q}_{-(n-1)} - 2\mathcal{Q}_{-(n-2)} + \mathcal{Q}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (4.1) and (4.2) hold for all integers n.

We list some properties of the third-order Pell and third-order Pell-Lucas matrix sequences.

- For any integer $n \geq 0$, we have the following formulas of the matrix sequences:

$$\mathcal{P}_n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} & P_{n-2} \end{pmatrix} \tag{4.3}$$

$$\mathcal{Q}_n = \begin{pmatrix} Q_{n+1} & Q_n + Q_{n-1} & Q_n \\ Q_n & Q_{n-1} + Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_{n-2} + Q_{n-3} & Q_{n-2} \end{pmatrix}. \tag{4.4}$$

- For every integer n, the Binet formulas of the third-order Pell and third-order Pell-Lucas matrix sequences are given by

$$\mathcal{P}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \tag{4.5}$$

$$\mathcal{Q}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n. \tag{4.6}$$

where

$$A_1 = \frac{\alpha\mathcal{P}_2 + \alpha(\alpha-2)\mathcal{P}_1 + \mathcal{P}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{P}_2 + \beta(\beta-2)\mathcal{P}_1 + \mathcal{P}_0}{\beta(\beta-\gamma)(\beta-\alpha)},$$

$$C_1 = \frac{\gamma\mathcal{P}_2 + \gamma(\gamma-2)\mathcal{P}_1 + \mathcal{P}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}, A_2 = \frac{\alpha\mathcal{Q}_2 + \alpha(\alpha-2)\mathcal{Q}_1 + \mathcal{Q}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)},$$

$$B_2 = \frac{\beta\mathcal{Q}_2 + \beta(\beta-2)\mathcal{Q}_1 + \mathcal{Q}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{Q}_2 + \gamma(\gamma-2)\mathcal{Q}_1 + \mathcal{Q}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.$$

- Let m and j be integers. Then we have

$$\sum_{i=0}^{n-1} \mathcal{P}_{mi+j} = \frac{\mathcal{P}_{mn+m+j} + \mathcal{P}_{mn-m+j} + (1-Q_m)\mathcal{P}_{mn+j}}{Q_m - Q_{-m}} - \frac{\mathcal{P}_{m+j} + \mathcal{P}_{j-m} + (1-Q_m)\mathcal{P}_j}{Q_m - Q_{-m}} \tag{4.7}$$

and

$$\sum_{i=0}^{n-1} \mathcal{Q}_{mi+j} = \frac{\mathcal{Q}_{mn+m+j} + \mathcal{Q}_{mn-m+j} + (1-Q_m)\mathcal{Q}_{mn+j}}{Q_m - Q_{-m}} - \frac{\mathcal{Q}_{m+j} + \mathcal{Q}_{j-m} + (1-Q_m)\mathcal{Q}_j}{Q_m - Q_{-m}}. \tag{4.8}$$

- The generating function for the third-order Pell and third-order Pell-Lucas matrix sequences are given as

$$\sum_{n=0}^{\infty} \mathcal{P}_n x^n = \frac{1}{1-2x-x^2-x^3} \begin{pmatrix} 1 & x^2+x & x \\ x & 1-2x & x^2 \\ x^2 & x-2x^2 & -x^2-2x+1 \end{pmatrix}$$

and

$$\sum_{n=0}^{\infty} \mathcal{Q}_n x^n = \frac{1}{1-2x-x^2-x^3} \begin{pmatrix} 3x^2+2x+2 & -4x^2+x+2 & -x^2-4x+3 \\ -x^2-4x+3 & 5x^2+10x-4 & -3x^2+5x-1 \\ -3x^2+5x-1 & 5x^2-14x+5 & 8x^2+5x-3 \end{pmatrix}$$

4.2 RELATION BETWEEN THIRD-ORDER PELL AND THIRD-ORDER PELL-LUCAS MATRIX SEQUENCES

Now, we list some properties of third-order Pell and third-order Pell-Lucas matrix sequences.

- For the matrix sequences $\{\mathcal{P}_n\}_{n\geq 0}$ and $\{\mathcal{Q}_n\}_{n\geq 0}$, we have the following identities

$$\begin{aligned}
\mathcal{Q}_n &= 8\mathcal{P}_{n+4} - 19\mathcal{P}_{n+3} - 3\mathcal{P}_{n+2}, \\
\mathcal{Q}_n &= -3\mathcal{P}_{n+3} + 5\mathcal{P}_{n+2} + 8\mathcal{P}_{n+1}, \\
\mathcal{Q}_n &= -\mathcal{P}_{n+2} + 5\mathcal{P}_{n+1} - 3\mathcal{P}_n, \\
\mathcal{Q}_n &= 3\mathcal{P}_{n+1} - 4\mathcal{P}_n - \mathcal{P}_{n-1}, \\
\mathcal{Q}_n &= 2\mathcal{P}_n + 2\mathcal{P}_{n-1} + 3\mathcal{P}_{n-2},
\end{aligned}$$

and

$$
\begin{aligned}
87\mathcal{P}_n &= 2\mathcal{Q}_{n+4} - 18\mathcal{Q}_{n+3} + 37\mathcal{Q}_{n+2}, \\
87\mathcal{P}_n &= -14\mathcal{Q}_{n+3} + 39\mathcal{Q}_{n+2} + 2\mathcal{Q}_{n+1}, \\
87\mathcal{P}_n &= 11\mathcal{Q}_{n+2} - 12\mathcal{Q}_{n+1} - 14\mathcal{Q}_{n}, \\
87\mathcal{P}_n &= 10\mathcal{Q}_{n+1} - 3\mathcal{Q}_{n} + 11\mathcal{Q}_{n-1}, \\
87\mathcal{P}_n &= 17\mathcal{Q}_{n} + 21\mathcal{Q}_{n-1} + 10\mathcal{Q}_{n-2},
\end{aligned}
$$

- For all non-negative integers m and n, we have the following identities.

  (a) $\mathcal{Q}_0\mathcal{P}_n = \mathcal{P}_n\mathcal{Q}_0 = \mathcal{Q}_n$.

  (b) $\mathcal{P}_0\mathcal{Q}_n = \mathcal{Q}_n\mathcal{P}_0 = \mathcal{Q}_n$.

- Then the following relations hold:

$$
\begin{aligned}
A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\
A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\
A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,.
\end{aligned}
$$

- For all non-negative integers m and n, we have the following identities.

  (a) $\mathrm{P}_m\mathrm{P}_n = \mathrm{P}_{m+n} = \mathrm{P}_n\mathrm{P}_m$.

  (b) $\mathrm{P}_m\mathcal{Q}_n = \mathcal{Q}_n\mathrm{P}_m = \mathcal{Q}_{m+n}$.

  (c) $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_n\mathcal{Q}_m = 9\mathcal{P}_{m+n+2} - 24\mathcal{P}_{m+n+1} + 10\mathcal{P}_{m+n} + 8\mathcal{P}_{m+n-1} + \mathcal{P}_{m+n-2}$.

  (d) $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_n\mathcal{Q}_m = 4\mathcal{P}_{m+n} + 8\mathcal{P}_{m+n-1} + 16\mathcal{P}_{m+n-2} + 12\mathcal{P}_{m+n-3} + 9\mathcal{P}_{m+n-4}$.

  (e) $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_n\mathcal{Q}_m = \mathcal{P}_{m+n+4} - 10\mathcal{P}_{m+n+3} + 31\mathcal{P}_{m+n+2} - 30\mathcal{P}_{m+n+1} + 9\mathcal{P}_{m+n}$.

- For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

  (a) $\mathcal{P}_n^m = \mathcal{P}_{mn}$.

  (b) $\mathcal{P}_{n+1}^m = \mathcal{P}_1^m\mathcal{P}_{mn}$.

  (c) $\mathcal{P}_{n-r}\mathcal{P}_{n+r} = \mathcal{P}_n^2 = \mathcal{P}_2^n$.

- For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

  (a) $\mathcal{Q}_{n-r}\mathcal{Q}_{n+r} = \mathcal{Q}_n^2$.

  (b) $\mathcal{Q}_n^m = \mathcal{Q}_0^m \mathcal{P}_{mn}$.

CHAPTER 5

ON MATRIX SEQUENCES OF NARAYANA AND NARAYANA-LUCAS NUMBERS

In this chapter, we define and investigate the Narayana matrix and Narayana-Lucas matrix sequences. We present Binet's formulas, generating functions, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. We use [27].

5.1 THE MATRIX SEQUENCES OF NARAYANA AND NARAYANA-LUCAS NUMBERS

In this section we define Narayana and Narayana-Lucas matrix sequences and investigate their properties.

Definition 5.1 *For any integer $n \geq 0$, the Narayana matrix $(\mathcal{N}_n)$ and Narayana-Lucas matrix $(\mathcal{U}_n)$ are defined by*

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, \tag{5.1}$$

$$\mathcal{U}_n = \mathcal{U}_{n-1} + \mathcal{U}_{n-3}, \tag{5.2}$$

respectively, with initial conditions

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{N}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{N}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{U}_0 = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & 0 \\ 0 & 3 & -2 \end{pmatrix}, \mathcal{U}_1 = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 3 \\ 3 & -2 & 0 \end{pmatrix}, \mathcal{U}_2 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

The sequences $\{\mathcal{N}_n\}_{n\geq 0}$ and $\{\mathcal{U}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$\mathcal{N}_{-n} = -\mathcal{N}_{-(n-2)} + \mathcal{N}_{-(n-3)}$

and

$$\mathcal{U}_{-n} = -\mathcal{U}_{-(n-2)} + \mathcal{U}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (5.1) and (5.2) hold for all integers n.

We list some properties of the Narayana and Narayana-Lucas matrix sequences.

- For any integer $n \geq 0$, we have the following formulas of the matrix sequences:

$$\begin{aligned} \mathcal{N}_n &= \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix} & (5.3) \\ \mathcal{U}_n &= \begin{pmatrix} U_{n+1} & U_{n-1} & U_n \\ U_n & U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-3} & U_{n-2} \end{pmatrix}. & (5.4) \end{aligned}$$

- For every integer n, the Binet formulas of the Narayana and Narayana-Lucas matrix sequences are given by

$$\begin{aligned} \mathcal{N}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n, & (5.5) \\ \mathcal{U}_n &= A_2\alpha^n + B_2\beta^n + C_2\gamma^n. & (5.6) \end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{\alpha\mathcal{N}_2 + \alpha(\alpha-1)\mathcal{N}_1 + \mathcal{N}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{N}_2 + \beta(\beta-1)\mathcal{N}_1 + \mathcal{N}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_1 = \frac{\gamma\mathcal{N}_2 + \gamma(\gamma-1)\mathcal{N}_1 + \mathcal{N}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)} \\ A_2 &= \frac{\alpha\mathcal{U}_2 + \alpha(\alpha-1)\mathcal{U}_1 + \mathcal{U}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_2 = \frac{\beta\mathcal{U}_2 + \beta(\beta-1)\mathcal{U}_1 + \mathcal{U}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{U}_2 + \gamma(\gamma-1)\mathcal{U}_1 + \mathcal{U}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}. \end{aligned}$$

- For all m, j we have

$$\sum_{k=0}^{n-1}\mathcal{N}_{mk+j} = \frac{\mathcal{N}_{mn+m+j} + \mathcal{N}_{mn-m+j} + (1-U_m)\mathcal{N}_{mn+j} - \mathcal{N}_{m+j} - \mathcal{N}_{j-m} + (U_m-1)\mathcal{N}_j}{U_m + (1-U_{-m}) - 1} \tag{5.7}$$

and

$$\sum_{k=0}^{n-1}\mathcal{U}_{mk+j}=\frac{\mathcal{U}_{mn+m+j}+\mathcal{U}_{mn-m+j}+(1-U_m)\mathcal{U}_{mn+j}-\mathcal{U}_{m+j}-\mathcal{U}_{j-m}+(U_m-1)\mathcal{U}_j}{U_m+(1-U_{-m})-1}. \tag{5.8}$$

- The generating function for the Narayana and Narayana-Lucas matrix sequences are given as

$$\sum_{n=0}^{\infty}\mathcal{N}_n x^n=\frac{1}{1-x-x^3}\begin{pmatrix}1 & x^2 & x\\ x & 1-x & x^2\\ x^2 & x-x^2 & 1-x\end{pmatrix}$$

and

$$\sum_{n=0}^{\infty}\mathcal{U}_n x^n=\frac{1}{1-x-x^3}\begin{pmatrix}3x^2+1 & 3x-2x^2 & 3-2x\\ 3-2x & 3x^2+2x-2 & 3x-2x^2\\ 3x-2x^2 & 2x^2-5x+3 & 3x^2+2x-2\end{pmatrix}$$

respectively.

5.2 RELATION BETWEEN NARAYANA AND NARAYANA-LUCAS MATRIX SEQUENCES

Now, we list some properties of the Narayana and Narayana-Lucas matrix sequences.

- For the matrix sequences $\{\mathcal{N}_n\}$ and $\{\mathcal{U}_n\}$, we have the following identities.

  (a) $\mathcal{U}_n=3\mathcal{N}_{n+4}-5\mathcal{N}_{n+3}+2\mathcal{N}_{n+2}$.

  (b) $\mathcal{U}_n=-2\mathcal{N}_{n+3}+2\mathcal{N}_{n+2}+3\mathcal{N}_{n+1}$.

  (c) $\mathcal{U}_n=3\mathcal{N}_{n+1}-2\mathcal{N}_n$.

  (d) $\mathcal{U}_n=\mathcal{N}_n+3\mathcal{N}_{n-2}$.

  (e) $31\mathcal{N}_n=-3\mathcal{U}_{n+4}+\mathcal{U}_{n+3}+11\mathcal{U}_{n+2}$.

  (f) $31\mathcal{N}_n=-2\mathcal{U}_{n+3}+11\mathcal{U}_{n+2}-3\mathcal{U}_{n+1}$.

  (g) $31\mathcal{N}_n=9\mathcal{U}_{n+2}-3\mathcal{U}_{n+1}-2\mathcal{U}_n$.

(h) $31\mathcal{N}_n = 6\mathcal{U}_{n+1} - 2\mathcal{U}_n + 9\mathcal{U}_{n-1}$.

(i) $31\mathcal{N}_n = 4\mathcal{U}_n + 9\mathcal{U}_{n-1} + 6\mathcal{U}_{n-2}$.

- For all non-negative integers m and n, we have the following identities.

(a) $\mathcal{U}_0 \mathrm{N}_n = \mathcal{N}_n \mathcal{U}_0 = \mathcal{U}_n$.

(b) $\mathcal{N}_0 \mathcal{U}_n = \mathcal{U}_n \mathcal{N}_0 = \mathcal{U}_n$.

- Then the following relations hold:

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\ A_1 B_1 &= B_1 A_1 = A_1 C_1 = C_1 A_1 = C_1 B_1 = B_1 C_1 = (0)\,, \\ A_2 B_2 &= B_2 A_2 = A_2 C_2 = C_2 A_2 = C_2 B_2 = B_2 C_2 = (0)\,. \end{aligned}$$

- For all non-negative integers m and n, we have the following identities.

(a) $\mathcal{N}_m \mathcal{N}_n = \mathcal{N}_{m+n} = \mathcal{N}_n \mathcal{N}_m$.

(b) $\mathcal{N}_m \mathcal{U}_n = \mathcal{U}_n \mathcal{N}_m = \mathcal{U}_{m+n}$.

(c) $\mathcal{U}_m \mathcal{U}_n = \mathcal{U}_n \mathcal{U}_m = 9\mathcal{N}_{m+n+8} - 30\mathcal{N}_{m+n+7} + 37\mathcal{N}_{m+n+6} - 20\mathcal{N}_{m+n+5} + 4\mathcal{N}_{m+n+4}$.

(d) $\mathcal{U}_m \mathcal{U}_n = \mathcal{U}_n \mathcal{U}_m = 4\mathcal{N}_{m+n+6} - 8\mathcal{N}_{m+n+5} - 8\mathcal{N}_{m+n+4} + 12\mathcal{N}_{m+n+3} + 9\mathcal{N}_{m+n+2}$.

(e) $\mathcal{U}_m \mathcal{U}_n = \mathcal{U}_n \mathcal{U}_m = 9\mathcal{N}_{m+n+2} - 12\mathcal{N}_{m+n+1} + 4\mathcal{N}_{m+n}$.

(f) $\mathcal{U}_m \mathcal{U}_n = \mathcal{U}_n \mathcal{U}_m = \mathcal{N}_{m+n} + 6\mathcal{N}_{m+n-2} + 9\mathcal{N}_{m+n-4}$.

- For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{N}_n^m = \mathcal{N}_{mn}$.

(b) $\mathcal{N}_{n+1}^m = \mathcal{N}_1^m \mathcal{N}_{mn}$.

(c) $\mathcal{N}_{n-r} \mathcal{N}_{n+r} = \mathcal{N}_n^2 = \mathcal{N}_2^n$.

- For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{U}_{n-r} \mathcal{U}_{n+r} = \mathcal{U}_n^2$.

(b) $\mathcal{U}_n^m = \mathcal{U}_0^m \mathcal{N}_{mn}$.

CHAPTER 6

ON MATRIX SEQUENCE OF GENERALIZED NARAYANA NUMBERS

In this chapter, we introduce and investigate the generalized Narayana matrix sequence and we deal with, in detail, three special cases of this sequence which we call them Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences. We present Binet's formulas, generating functions, and the summation formulas for these sequences. We present the proofs to indicate how these sum formulas, in general, were discovered. Of course, all the listed sum formulas may be proved by induction, but that method of proof gives no clue about their discovery. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there always exist interrelation between generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

We use our paper [29].

6.1 THE MATRIX SEQUENCES OF NARAYANA AND NARAYANA-LUCAS NUMBERS

In this section we define generalized Narayana matrix sequence and investigate its properties.

Definition 6.1 *For any integer $n \geq 0$, the generalized Narayana matrix $(\mathcal{V}_n)$ is defined by*

$$\mathcal{V}_n = \mathcal{V}_{n-1} + \mathcal{V}_{n-3} \tag{6.1}$$

with initial conditions

$$\mathcal{V}_0 = \begin{pmatrix} V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \end{pmatrix},$$

$$\mathcal{V}_1 = \begin{pmatrix} V_2 & V_0 & V_1 \\ V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \end{pmatrix},$$

$$\mathcal{V}_2 = \begin{pmatrix} V_0 + V_2 & V_1 & V_2 \\ V_2 & V_0 & V_1 \\ V_1 & V_2 - V_1 & V_0 \end{pmatrix}.$$

The sequence $\{\mathcal{V}_n\}_{n\geq 0}$ *can be extended to negative subscripts by defining*

$$\mathcal{V}_{-n} = -\mathcal{V}_{-(n-2)} + \mathcal{V}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ *respectively. Therefore, recurrence (6.1) holds for all integers* n.

Three special cases of generalized Narayana matrix sequence (take $V_n = N_n, V_n = U_n, V_n = H_n$, respectively) can be defined as follows.

Definition 6.2 *For any integer* $n \geq 0$, *the Narayana matrix* $(\mathcal{N}_n)$ *and Narayana-Lucas matrix* $(\mathcal{U}_n)$ *and Narayana-Perrin matrix* $(\mathcal{H}_n)$ *are defined by*

$$\begin{aligned} \mathcal{N}_n &= \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, \\ \mathcal{U}_n &= \mathcal{U}_{n-1} + \mathcal{U}_{n-3}, \\ \mathcal{H}_n &= \mathcal{H}_{n-1} + \mathcal{H}_{n-3}, \end{aligned}$$

respectively, with initial conditions

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{N}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{N}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{U}_0 = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & 0 \\ 0 & 3 & -2 \end{pmatrix}, \mathcal{U}_1 = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 3 \\ 3 & -2 & 0 \end{pmatrix}, \mathcal{U}_2 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix},$$

$$\mathcal{H}_0 = \begin{pmatrix} 0 & 2 & 3 \\ 3 & -3 & 2 \\ 2 & 1 & -3 \end{pmatrix}, \mathcal{H}_1 = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 3 & -3 & 2 \end{pmatrix}, \mathcal{H}_2 = \begin{pmatrix} 5 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

The sequences $\{\mathcal{N}_n\}_{n\geq 0}$, $\{\mathcal{U}_n\}_{n\geq 0}$ and $\{\mathcal{H}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
\mathcal{N}_{-n} &= -\mathcal{N}_{-(n-2)} + \mathcal{N}_{-(n-3)}, \\
\mathcal{U}_{-n} &= -\mathcal{U}_{-(n-2)} + \mathcal{U}_{-(n-3)}, \\
\mathcal{H}_{-n} &= -\mathcal{H}_{-(n-2)} + \mathcal{H}_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

The Narayana matrix $(\mathcal{N}_n)$ and Narayana-Lucas matrix $(\mathcal{U}_n)$ were defined and studied in [27].

The following theorem gives the nth general terms of the generalized Narayana matrix sequence.

Theorem 6.1 *For any integer n, we have the following formulas of the generalized Narayana matrix sequence:*

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} \tag{6.2}$$

Proof. Suppose that $n \geq 0$. We prove (6.2) by strong mathematical induction on n. If $n = 0$ then, since $V_{-1} = V_2 - V_1$, $V_{-2} = V_1 - V_0$, $V_{-3} = V_0 + V_1 - V_2$, we have

$$\mathcal{V}_0 = \begin{pmatrix} V_1 & V_{-1} & V_0 \\ V_0 & V_{-2} & V_{-1} \\ V_{-1} & V_{-3} & V_{-2} \end{pmatrix} = \begin{pmatrix} V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \end{pmatrix}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1$, we have

$$\begin{aligned}
\mathcal{V}_{k+1} &= \mathcal{V}_k + \mathcal{V}_{k-2} \\
&= \begin{pmatrix} V_{k+1} & V_{k-1} & V_k \\ V_k & V_{k-2} & V_{k-1} \\ V_{k-1} & V_{k-3} & V_{k-2} \end{pmatrix} + \begin{pmatrix} V_{k-1} & V_{k-3} & V_{k-2} \\ V_{k-2} & V_{k-4} & V_{k-3} \\ V_{k-3} & V_{k-5} & V_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{k-1}+V_{k+1} & V_{k-1}+V_{k-3} & V_k+V_{k-2} \\ V_k+V_{k-2} & V_{k-2}+V_{k-4} & V_{k-1}+V_{k-3} \\ V_{k-1}+V_{k-3} & V_{k-3}+V_{k-5} & V_{k-2}+V_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+2} & V_k & V_{k+1} \\ V_{k+1} & V_{k-1} & V_k \\ V_k & V_{k-2} & V_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+1+1} & V_{k+1-1} & V_{k+1} \\ V_{k+1} & V_{k+1-2} & V_{k+1-1} \\ V_{k+1-1} & V_{k+1-3} & V_{k+1-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on $k+1$, this proves (6.2).

For the case $n \leq 0$, similarly,(6.2) can be proved by strong mathematical induction on n.

□

The following theorem gives the nth general terms of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Corollary 6.1 *For any integer n, we have the following formulas of the matrix sequences:*

$$\begin{aligned}
\mathcal{N}_n &= \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}, \\
\mathcal{U}_n &= \begin{pmatrix} U_{n+1} & U_{n-1} & U_n \\ U_n & U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-3} & U_{n-2} \end{pmatrix}, \\
\mathcal{H}_n &= \begin{pmatrix} H_{n+1} & H_{n-1} & H_n \\ H_n & H_{n-2} & H_{n-1} \\ H_{n-1} & H_{n-3} & H_{n-2} \end{pmatrix}.
\end{aligned}$$

We now give the Binet's formula for the generalized Narayana matrix sequence.

Theorem 6.2 *For every integer n, the Binet's formula of the generalized Narayana matrix sequence are given by*

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n$$

where

$$A = \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + \mathcal{V}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B = \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + \mathcal{V}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C = \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + \mathcal{V}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}$$

Proof. We need to prove the theorem only for $n \geq 0$. By the assumption, the characteristic equation of (6.1) is $x^3 - x^2 - x - 1 = 0$ and the roots of it are α, β and γ. So it's general solution is given by

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n.$$

Using initial condition which is given in Definition 6.1, and also applying lineer algebra operations, we obtain the matrices A, B, C as desired. This gives the formula for $\mathcal{V}_n$. $\square$
The following theorem gives the Binet's formulas of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Corollary 6.2 *For every integer n, the Binet formulas of the Narayana and Narayana-Lucas matrix sequences are given by*

$$\begin{aligned}
\mathcal{N}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \\
\mathcal{U}_n &= A_2\alpha^n + B_2\beta^n + C_2\gamma^n, \\
\mathcal{H}_n &= A_3\alpha^n + B_3\beta^n + C_3\gamma^n,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha\mathcal{N}_2 + \alpha(\alpha-1)\mathcal{N}_1 + \mathcal{N}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{N}_2 + \beta(\beta-1)\mathcal{N}_1 + \mathcal{N}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \\
C_1 &= \frac{\gamma\mathcal{N}_2 + \gamma(\gamma-1)\mathcal{N}_1 + \mathcal{N}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}, A_2 = \frac{\alpha\mathcal{U}_2 + \alpha(\alpha-1)\mathcal{U}_1 + \mathcal{U}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, \\
B_2 &= \frac{\beta\mathcal{U}_2 + \beta(\beta-1)\mathcal{U}_1 + \mathcal{U}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{U}_2 + \gamma(\gamma-1)\mathcal{U}_1 + \mathcal{U}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}, \\
A_3 &= \frac{\alpha\mathcal{H}_2 + \alpha(\alpha-1)\mathcal{H}_1 + \mathcal{H}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_3 = \frac{\beta\mathcal{H}_2 + \beta(\beta-1)\mathcal{H}_1 + \mathcal{H}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \\
C_3 &= \frac{\gamma\mathcal{H}_2 + \gamma(\gamma-1)\mathcal{H}_1 + \mathcal{H}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.
\end{aligned}$$

The well known Binet formulas for generalized Narayana numbers is given in (1.33). But, we will obtain these functions in terms of generalized Narayana matrix sequence as a consequence of Theorems 6.1 and 6.2. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 6.3 *For every integers n, the Binet's formulas for the generalized Narayana numbers is given as*

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.$$

Proof. From Theorem 6.2, we have

$$\begin{aligned}
\mathcal{V}_n &= A\alpha^n + B\beta^n + C\gamma^n \\
&= \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + \mathcal{V}_0}{\alpha\left(\alpha-\gamma\right)\left(\alpha-\beta\right)}\alpha^n + \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + \mathcal{V}_0}{\beta\left(\beta-\gamma\right)\left(\beta-\alpha\right)}\beta^n \\
&\quad + \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + \mathcal{V}_0}{\gamma\left(\gamma-\beta\right)\left(\gamma-\alpha\right)}\gamma^n \\
&= \frac{\alpha^{n-1}}{\left(\alpha-\gamma\right)\left(\alpha-\beta\right)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \alpha V_2 + \alpha\left(\alpha-1\right)V_1 + V_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\
&\quad + \frac{\beta^{n-1}}{\left(\beta-\gamma\right)\left(\beta-\alpha\right)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \beta V_2 + \beta\left(\beta-1\right)V_1 + V_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\
&\quad + \frac{\gamma^{n-1}}{\left(\gamma-\beta\right)\left(\gamma-\alpha\right)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \gamma V_2 + \gamma\left(\gamma-1\right)V_1 + V_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.
\end{aligned}$$

(we only write the 2nd row and 1st column entries of the matrices). By Theorem 6.1, we know that

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} & V_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned} V_n &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}(\alpha V_2+\alpha(\alpha-1)V_1+V_0) \\ &\quad +\frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}(\beta V_2+\beta(\beta-1)V_1+V_0) \\ &\quad +\frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}(\gamma V_2+\gamma(\gamma-1)V_1+V_0) \\ &= \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned}$$

where

$$b_1=V_2-(\beta+\gamma)V_1+\beta\gamma V_0,\ b_2=V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0,\ b_3=V_2-(\alpha+\beta)V_1+\alpha\beta V_0.$$

Note that

$$\begin{aligned} \alpha V_2+\alpha(\alpha-1)V_1+V_0 &= \alpha(V_2+(\alpha-1)V_1+\frac{1}{\alpha}V_0) \\ &= \alpha(V_2-(\beta+\gamma)V_1+\beta\gamma V_0)=\alpha b_1, \\ \beta V_2+\beta(\beta-1)V_1+V_0 &= \beta(V_2+(\beta-1)V_1+\frac{1}{\beta}V_0) \\ &= \beta(V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0)=\beta b_2, \\ \gamma V_2+\gamma(\gamma-1)V_1+V_0 &= \gamma(V_2+(\gamma-1)V_1+\frac{1}{\gamma}V_0) \\ &= \gamma(V_2-(\alpha+\beta)V_1+\alpha\beta V_0)=\gamma b_3. \end{aligned}$$

□

Now, we present summation formulas for the generalized Narayana matrix sequence.

Theorem 6.3 *For all integers m,j we have*

$$\sum_{k=0}^{n-1}\mathcal{V}_{mk+j}=\frac{\mathcal{V}_{mn+m+j}+\mathcal{V}_{mn-m+j}+(1-U_m)\mathcal{V}_{mn+j}-\mathcal{V}_{m+j}-\mathcal{V}_{j-m}+(U_m-1)\mathcal{V}_j}{U_m+(1-U_{-m})-1} \tag{6.3}$$

Proof. Note that

$$\begin{aligned} \sum_{i=0}^{n-1}\mathcal{V}_{mi+j} &= \sum_{i=0}^{n-1}(A\alpha^{mi+j}+B\beta^{mi+j}+C\gamma^{mi+j}) \\ &= A\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right)+B\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right)+C\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right) \end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (6.3) as required. □

As in Corollary 6.3, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices to obtain summation formula for the generalized Narayana sequence..

Corollary 6.4 *For all integers m, j we have*

$$\sum_{k=0}^{n-1} V_{mk+j} = \frac{V_{mn+m+j} + V_{mn-m+j} + (1-U_m)V_{mn+j} - V_{m+j} - V_{j-m} + (U_m - 1)V_j}{U_m + (1-U_{-m}) - 1}.$$

We now give generating functions of $\mathcal{V}_n$.

Theorem 6.4 *The generating function for the generalized Narayana matrix sequences is given as*

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{V}_n x^n &= \frac{\mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1)x^2}{1 - x - x^3} \\
&= \frac{1}{1 - x - x^3} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
a_{11} &= V_0 x^2 + (-V_1 + V_2)\, x + V_1 \\
a_{21} &= (-V_1 + V_2)\, x^2 + (-V_0 + V_1)\, x + V_0 \\
a_{31} &= (-V_0 + V_1)\, x^2 + (V_0 + V_1 - V_2)\, x - V_1 + V_2
\end{aligned}$$

$$\begin{aligned}
a_{12} &= (-V_0 + V_1)\, x^2 + (V_0 + V_1 - V_2)\, x - V_1 + V_2 \\
a_{22} &= (V_0 + V_1 - V_2)\, x^2 + (V_0 - 2V_1 + V_2)\, x - V_0 + V_1 \\
a_{32} &= V_0 + V_1 - V_2 + x^2 (V_0 - 2V_1 + V_2) + x\,(V_2 - 2V_0)
\end{aligned}$$

$$\begin{aligned}
a_{13} &= (-V_1 + V_2)\, x^2 + (-V_0 + V_1)\, x + V_0 \\
a_{23} &= (-V_0 + V_1)\, x^2 + (V_0 + V_1 - V_2)\, x - V_1 + V_2 \\
a_{33} &= (V_0 + V_1 - V_2)\, x^2 + (V_0 - 2V_1 + V_2)\, x - V_0 + V_1
\end{aligned}$$

Proof. Suppose that $g(x) = \sum_{n=0}^{\infty} \mathcal{V}_n x^n$ is the generating function for the sequence $\{\mathcal{V}_n\}_{n\geq 0}$. Using the definition of the matrix sequence of generalized Narayana numbers

(6.1), and substracting $x\sum_{n=0}^{\infty}\mathcal{V}_nx^n$ and $x^3\sum_{n=0}^{\infty}\mathcal{V}_nx^n$ from $\sum_{n=0}^{\infty}\mathcal{V}_nx^n$ we obtain

$$\begin{aligned}(1-x-x^3)\sum_{n=0}^{\infty}\mathcal{V}_nx^n &= \sum_{n=0}^{\infty}\mathcal{V}_nx^n - x\sum_{n=0}^{\infty}\mathcal{V}_nx^n - x^3\sum_{n=0}^{\infty}\mathcal{V}_nx^n \\ &= \sum_{n=0}^{\infty}\mathcal{V}_nx^n - \sum_{n=0}^{\infty}\mathcal{V}_nx^{n+1} - \sum_{n=0}^{\infty}\mathcal{V}_nx^{n+3} \\ &= \sum_{n=0}^{\infty}\mathcal{V}_nx^n - \sum_{n=1}^{\infty}\mathcal{V}_{n-1}x^n - \sum_{n=3}^{\infty}\mathcal{V}_{n-3}x^n \\ &= (\mathcal{V}_0+\mathcal{V}_1x+\mathcal{V}_2x^2)-(\mathcal{V}_0x+\mathcal{V}_1x^2)+\sum_{n=3}^{\infty}(\mathcal{V}_n-\mathcal{V}_{n-1}-\mathcal{V}_{n-3})x^n \\ &= \mathcal{V}_0+\mathcal{V}_1x+\mathcal{V}_2x^2-\mathcal{V}_0x-\mathcal{V}_1x^2 \\ &= \mathcal{V}_0+(\mathcal{V}_1-\mathcal{V}_0)x+(\mathcal{V}_2-\mathcal{V}_1)x^2.\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty}\mathcal{V}_nx^n = \frac{\mathcal{V}_0+(\mathcal{V}_1-\mathcal{V}_0)x+(\mathcal{V}_2-\mathcal{V}_1)x^2}{1-x-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{V}_nx^n$ in the Theorem. This completes the proof. $\square$

The following corollary gives the generating functions of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences.

Corollary 6.5 *The generating functions for the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences are given as*

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{N}_nx^n &= \frac{1}{1-x-x^3}\begin{pmatrix}1 & x^2 & x\\ x & 1-x & x^2\\ x^2 & x-x^2 & 1-x\end{pmatrix},\\ \sum_{n=0}^{\infty}\mathcal{U}_nx^n &= \frac{1}{1-x-x^3}\begin{pmatrix}3x^2+1 & 3x-2x^2 & 3-2x\\ 3-2x & 3x^2+2x-2 & 3x-2x^2\\ 3x-2x^2 & 2x^2-5x+3 & 3x^2+2x-2\end{pmatrix},\\ \sum_{n=0}^{\infty}\mathcal{H}_nx^n &= \frac{1}{1-x-x^3}\begin{pmatrix}3x^2+2x & -3x^2+x+2 & 2x^2-3x+3\\ 2x^2-3x+3 & x^2+5x-3 & -3x^2+x+2\\ -3x^2+x+2 & 5x^2-4x+1 & x^2+5x-3\end{pmatrix}.\end{aligned}$$

The well known generating function for generalized Narayana numbers is as in (1.50). However, we will obtain these functions in terms of generalized Narayana matrix sequences as a consequence of Theorem 6.4. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 6.4. Thus we have the following corollary.

Corollary 6.6 *The generating function for the generalized Narayana sequence* $\{V_n\}$ *is given as*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{1 - x - x^3}.$$

Using Theorem 6.1 and Corollary 6.1, we see that

$$\begin{aligned}
\mathcal{V}_{-1} &= \begin{pmatrix} V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \\ V_1 - V_0 & V_0 - 2V_1 + V_2 & V_0 + V_1 - V_2 \end{pmatrix}, \\
\mathcal{V}_{-2} &= \begin{pmatrix} V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \\ V_1 - V_0 & V_0 - 2V_1 + V_2 & V_0 + V_1 - V_2 \\ V_0 + V_1 - V_2 & V_2 - 2V_0 & V_0 - 2V_1 + V_2 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \mathcal{N}_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \\
\mathcal{U}_{-1} &= \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & -2 \\ -2 & 2 & 3 \end{pmatrix}, \mathcal{U}_{-2} = \begin{pmatrix} 0 & 3 & -2 \\ -2 & 2 & 3 \\ 3 & -5 & 2 \end{pmatrix}, \\
\mathcal{H}_{-1} &= \begin{pmatrix} 3 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 5 & 1 \end{pmatrix}, \mathcal{H}_{-2} = \begin{pmatrix} 2 & 1 & -3 \\ -3 & 5 & 1 \\ 1 & -4 & 5 \end{pmatrix}.
\end{aligned}$$

We now give generating functions of the generalized Narayana matrix sequence $\mathcal{V}_n$ for negative indices.

Theorem 6.5 *For negative indices, the generating function for the generalized Narayana matrix sequence is given as*

$$\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n = \frac{\mathcal{V}_0+(\mathcal{V}_0+\mathcal{V}_{-1})x+(\mathcal{V}_{-1}+\mathcal{V}_{-2})x^2}{1+x-x^3}$$

$$= \frac{1}{1+x-x^3}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$$b_{11} = (V_0-V_1+V_2)\,x^2+(V_0+V_1)\,x+V_1$$

$$b_{21} = (-V_0+V_2)\,x^2+(V_0-V_1+V_2)\,x+V_0$$

$$b_{31} = V_2-V_1-x\,(V_0-V_2)-x^2\,(V_2-2V_1)$$

and

$$b_{12} = V_2-V_1-x\,(V_0-V_2)-x^2\,(V_2-2V_1)$$

$$b_{22} = V_1-V_0-x\,(V_2-2V_1)-x^2\,(V_1-2V_0)$$

$$b_{32} = V_0+V_1-V_2-x\,(V_1-2V_0)-x^2\,(V_0+2V_1-2V_2)$$

and

$$b_{13} = (-V_0+V_2)\,x^2+(V_0-V_1+V_2)\,x+V_0$$

$$b_{23} = V_2-V_1-x\,(V_0-V_2)-x^2\,(V_2-2V_1)$$

$$b_{33} = V_1-V_0-x\,(V_2-2V_1)-x^2\,(V_1-2V_0)$$

Proof. Then, using Definition 6.1, and adding $xg(x)$ to $g(x)$ and also substracting $x^3g(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1+x-x^3)g(x) &= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + x\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n - x^3\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n \\
&= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+1} - \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+3} \\
&= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \sum_{n=1}^{\infty}\mathcal{V}_{-n+1}x^{n} - \sum_{n=3}^{\infty}\mathcal{V}_{-n+3}x^{n} \\
&= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2)+(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) \\
&\quad +\sum_{n=3}^{\infty}(\mathcal{V}_{-n}+\mathcal{V}_{-n+1}-\mathcal{V}_{-n+3})x^n \\
&= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2)+(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) \\
&= \mathcal{V}_0+(\mathcal{V}_0+\mathcal{V}_{-1})x+(\mathcal{V}_{-1}+\mathcal{V}_{-2})x^2
\end{aligned}$$

Rearranging above equation, we get

$$g(x) = \frac{\mathcal{V}_0 + (\mathcal{V}_0 + \mathcal{V}_{-1})x + (\mathcal{V}_{-1} + \mathcal{V}_{-2})x^2}{1 + x - x^3}$$

which equals the $\sum_{n=0}^{\infty} \mathcal{V}_{-n}x^n$ in the Theorem. $\square$

The following corollary gives the generating functions of the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences with negative indices .

Corollary 6.7 *The generating functions for the Narayana, Narayana-Lucas and Narayana-Perrin matrix sequences with negative indices are given as*

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{N}_{-n}x^n &= \frac{1}{1+x-x^3}\begin{pmatrix} x+1 & x^2+x & x^2 \\ x^2 & -x^2+x+1 & x^2+x \\ x^2+x & -x & -x^2+x+1 \end{pmatrix}, \\
\sum_{n=0}^{\infty} \mathcal{U}_{-n}x^n &= \frac{1}{1+x-x^3}\begin{pmatrix} 3x^2+4x+1 & x^2-2x & -2x^2+3x+3 \\ -2x^2+3x+3 & 5x^2+x-2 & x^2-2x \\ x^2-2x & -3x^2+5x+3 & 5x^2+x-2 \end{pmatrix}, \\
\sum_{n=0}^{\infty} \mathcal{H}_{-n}x^n &= \frac{1}{1+x-x^3}\begin{pmatrix} 5x^2+3x & -2x^2-x+2 & -x^2+5x+3 \\ -x^2+5x+3 & 6x^2-2x-3 & -2x^2-x+2 \\ -2x^2-x+2 & x^2+6x+1 & 6x^2-2x-3 \end{pmatrix},
\end{aligned}$$

respectively.

Now, we will obtain generating functions for generalized Narayana numbers in terms of generalized Narayana matrix sequences with negative indices as a consequence of Theorem 6.5. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 6.5. Thus we have the following corollary.

Corollary 6.8 *The generating functions for the generalized Narayana sequence* $\{V_{-n}\}_{n\geq 0}$ *is given as*

$$\sum_{n=0}^{\infty} V_{-n}x^n = \frac{V_0 + (V_0 - V_1 + V_2)x + (-V_0 + V_2)x^2}{1 + x - x^3}.$$

The previous corollary gives the following results as particular examples.

Corollary 6.9 *Generated functions of Narayana, Narayana-Lucas and Narayana-Perrin numbers with negative indices are*

$$\begin{aligned}
\sum_{n=0}^{\infty} N_{-n}x^n &= \frac{x^2}{1+x-x^3}, \\
\sum_{n=0}^{\infty} U_{-n}x^n &= \frac{3+3x-2x^2}{1+x-x^3}, \\
\sum_{n=0}^{\infty} H_{-n}x^n &= \frac{3+5x-x^2}{1+x-x^3},
\end{aligned}$$

respectively.

6.2 SOME IDENTITIES

In this section, we assume that m and n are arbitrary integers, unless otherwise mentioned. In this section, we obtain some identities of generalized Narayana and Narayana, Narayana-Lucas and Narayana-Perrin numbers. We need these identities in the next section. First, we can give a few basic relations between $\{V_n\}$ and $\{N_n\}$.

Lemma 6.6 *The following equalities are true:*

(a) $V_n = (V_0 + V_1 - V_2)N_{n+4} + (V_2 - 2V_0)N_{n+3} + (V_0 - 2V_1 + V_2)N_{n+2}$.

(b) $V_n = (V_1 - V_0)N_{n+3} + (V_0 - 2V_1 + V_2)N_{n+2} + (V_0 + V_1 - V_2)N_{n+1}$.

(c) $V_n = (V_2 - V_1)N_{n+2} + (V_0 + V_1 - V_2)N_{n+1} + (V_1 - V_0)N_n$.

(d) $V_n = V_0N_{n+1} + (V_1 - V_0)N_n + (V_2 - V_1)N_{n-1}$.

(e) $V_n = V_1N_n + (V_2 - V_1)N_{n-1} + V_0N_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). Writing

$V_n = a \times N_{n+4} + b \times N_{n+3} + c \times N_{n+2}$

and solving the system of equations

$$\begin{aligned}
V_0 &= a \times N_4 + b \times N_3 + c \times N_2 \\
V_1 &= a \times N_5 + b \times N_4 + c \times N_3 \\
V_2 &= a \times N_6 + b \times N_5 + c \times N_4
\end{aligned}$$

we find that $a = V_0 + V_1 - V_2, b = V_2 - 2V_0, c = V_0 - 2V_1 + V_2$. The other equalities can be proved similarly. $\square$

Note that all the identities in the above lemma can be proved by induction as well.

Next, we present a few basic relations between $\{N_n\}$ and $\{V_n\}$.

Lemma 6.7 *The following equalities are true:*

(a) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n = (V_2^2-V_1V_2-V_0V_1)V_{n+4}+(V_1^2+V_1V_2+V_0V_1-V_2^2-V_0V_2)V_{n+3}+(V_0^2+V_2V_0-V_1V_2)V_{n+2}$.

(b) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n = (V_1^2-V_0V_2)V_{n+3}+(V_0^2+V_2V_0-V_1V_2)V_{n+2}+(V_2^2-V_1V_2-V_0V_1)V_{n+1}$.

(c) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n = (V_0^2+V_1^2-V_2V_1)V_{n+2}+(V_2^2-V_1V_2-V_0V_1)V_{n+1}+(V_1^2-V_0V_2)V_n$.

(d) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n = (V_0^2-V_0V_1+V_1^2-2V_1V_2+V_2^2)V_{n+1}+(V_1^2-V_0V_2)V_n+(V_0^2+V_1^2-V_2V_1)V_{n-1}$.

(e) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)N_n = (V_0^2-V_0V_1-V_0V_2+2V_1^2-2V_1V_2+V_2^2)V_n+(V_0^2+V_1^2-V_2V_1)V_{n-1}+(V_0^2-V_0V_1+V_1^2-2V_1V_2+V_2^2)V_{n-2}$.

Now, we give a few basic relations between $\{V_n\}$ and $\{U_n\}$.

Lemma 6.8 *The following equalities are true:*

(a) $31V_n = (V_0-14V_1+11V_2)U_{n+4}+(10V_0+15V_1-14V_2)U_{n+3}+(10V_1-14V_0+V_2)U_{n+2}$.

(b) $31V_n = (11V_0+V_1-3V_2)U_{n+3}+(10V_1-14V_0+V_2)U_{n+2}+(V_0-14V_1+11V_2)U_{n+1}$.

(c) $31V_n = (11V_1-3V_0-2V_2)U_{n+2}+(V_0-14V_1+11V_2)U_{n+1}+(11V_0+V_1-3V_2)U_n$.

(d) $31V_n = (9V_2-3V_1-2V_0)U_{n+1}+(11V_0+V_1-3V_2)U_n+(11V_1-3V_0-2V_2)U_{n-1}$.

(e) $31V_n = (9V_0-2V_1+6V_2)U_n+(11V_1-3V_0-2V_2)U_{n-1}+(9V_2-3V_1-2V_0)U_{n-2}$.

Next, we present a few basic relations between $\{U_n\}$ and $\{V_n\}$.

Lemma 6.9 *The following equalities are true:*

(a) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n = (3V_1^2+2V_1V_2+2V_0V_1-2V_2^2-3V_0V_2)V_{n+4} + (3V_0^2-2V_0V_1+5V_0V_2-2V_1^2-5V_1V_2+2V_2^2)V_{n+3} + (-2V_0^2-2V_0V_2-3V_1V_0+3V_2^2-V_1V_2)V_{n+2}$.

(b) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n = (3V_0^2+V_1^2+2V_0V_2-3V_1V_2)V_{n+3} + (-2V_0^2-2V_0V_2-3V_1V_0+3V_2^2-V_1V_2)V_{n+2} + (3V_1^2-2V_2^2+2V_0V_1-3V_0V_2+2V_1V_2)V_{n+1}$.

(c) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n = (V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)V_{n+2}+(3V_1^2+2V_1V_2+2V_0V_1-2V_2^2-3V_0V_2)V_{n+1}+(3V_0^2+2V_2V_0+V_1^2-3V_2V_1)V_n$.

(d) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n = (V_0^2-V_0V_1-3V_0V_2+4V_1^2-2V_1V_2+V_2^2)V_{n+1}+(3V_0^2+2V_2V_0+V_1^2-3V_2V_1)V_n+(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)V_{n-1}$.

(e) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)U_n = (4V_0^2-V_0V_1-V_0V_2+5V_1^2-5V_1V_2+V_2^2)V_n+(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)V_{n-1}+(V_0^2-V_0V_1-3V_0V_2+4V_1^2-2V_1V_2+V_2^2)V_{n-2}$.

Now, we give a few basic relations between $\{V_n\}$ and $\{H_n\}$.

Lemma 6.10 *The following equalities are true:*

(a) $53V_n = (15V_2-11V_1-10V_0)H_{n+4}+(25V_0+V_1-11V_2)H_{n+3}+(25V_1-11V_0-10V_2)H_{n+2}$.

(b) $53V_n = (15V_0-10V_1+4V_2)H_{n+3}+(25V_1-11V_0-10V_2)H_{n+2}+(15V_2-11V_1-10V_0)H_{n+1}$.

(c) $53V_n = (4V_0+15V_1-6V_2)H_{n+2}+(15V_2-11V_1-10V_0)H_{n+1}+(15V_0-10V_1+4V_2)H_n$.

(d) $53V_n = (4V_1-6V_0+9V_2)H_{n+1}+(15V_0-10V_1+4V_2)H_n+(4V_0+15V_1-6V_2)H_{n-1}$.

(e) $53V_n = (9V_0-6V_1+13V_2)H_n+(4V_0+15V_1-6V_2)H_{n-1}+(4V_1-6V_0+9V_2)H_{n-2}$.

Next, we present a few basic relations between $\{H_n\}$ and $\{V_n\}$.

Lemma 6.11 *The following equalities are true:*

(a) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n = (2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)V_{n+4}+(V_0^2-5V_0V_1+4V_0V_2-3V_1^2-6V_1V_2+5V_2^2)V_{n+3}+(-3V_0^2-V_0V_1-5V_0V_2+2V_1^2+2V_1V_2+V_2^2)V_{n+2}$.

(b) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n = (3V_0^2+3V_0V_2-2V_1V_0+2V_2^2-5V_1V_2)V_{n+3}+(-3V_0^2-V_0V_1-5V_0V_2+2V_1^2+2V_1V_2+V_2^2)V_{n+2}+(2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)V_{n+1}$.

(c) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n = (2V_1^2-3V_1V_2-3V_0V_1+3V_2^2-2V_0V_2)V_{n+2}+(2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)V_{n+1}+(3V_0^2+3V_0V_2-2V_1V_0+2V_2^2-5V_1V_2)V_n$.

(d) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n = (2V_0^2-3V_2V_0+5V_1^2-2V_2V_1)V_{n+1}+(3V_0^2+3V_0V_2-2V_1V_0+2V_2^2-5V_1V_2)V_n+(2V_1^2-3V_1V_2-3V_0V_1+3V_2^2-2V_0V_2)V_{n-1}$.

(e) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)H_n = (5V_0^2-2V_0V_1+5V_1^2-7V_1V_2+2V_2^2)V_n+(2V_1^2-3V_1V_2-3V_0V_1+3V_2^2-2V_0V_2)V_{n-1}+(2V_0^2-3V_2V_0+5V_1^2-2V_2V_1)V_{n-2}$.

6.3 RELATION BETWEEN GENERALIZED NARAYANA MATRIX SEQUENCES AND IT'S SPECIAL CASES

In this section, we assume that m and n are arbitrary integers, unless otherwise mentioned. The following theorem shows that there always exist interrelation between generalized Narayana and Narayana matrix sequences.

Theorem 6.12 *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{N}_n\}$, we have the following identities.*

(a) $\mathcal{V}_n = (V_0+V_1-V_2)\mathcal{N}_{n+4}+(V_2-2V_0)\mathcal{N}_{n+3}+(V_0-2V_1+V_2)\mathcal{N}_{n+2}$.

(b) $\mathcal{V}_n = (V_1-V_0)\mathcal{N}_{n+3}+(V_0-2V_1+V_2)\mathcal{N}_{n+2}+(V_0+V_1-V_2)\mathcal{N}_{n+1}$.

(c) $\mathcal{V}_n = (V_2-V_1)\mathcal{N}_{n+2}+(V_0+V_1-V_2)\mathcal{N}_{n+1}+(V_1-V_0)\mathcal{N}_n$.

(d) $\mathcal{V}_n = V_0\mathcal{N}_{n+1} + (V_1 - V_0)\mathcal{N}_n + (V_2 - V_1)\mathcal{N}_{n-1}$.

(e) $\mathcal{V}_n = V_1N_n + (V_2 - V_1)N_{n-1} + V_0N_{n-2}$.

(f) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n = (V_2^2-V_1V_2-V_0V_1)\mathcal{V}_{n+4}+ (V_1^2 + V_1V_2 + V_0V_1 - V_2^2 - V_0V_2)\mathcal{V}_{n+3} + (V_0^2 + V_2V_0 - V_1V_2)\mathcal{V}_{n+2}$.

(g) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{N}_n = (V_1^2 - V_0V_2)\mathcal{V}_{n+3} + (V_0^2 + V_2V_0 - V_1V_2)\mathcal{V}_{n+2} + (V_2^2 - V_1V_2 - V_0V_1)\mathcal{V}_{n+1}$.

(h) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{N}_n = (V_0^2+V_1^2-V_2V_1)\mathcal{V}_{n+2}+ (V_2^2 - V_1V_2 - V_0V_1)\mathcal{V}_{n+1} + (V_1^2 - V_0V_2)\mathcal{V}_n$.

(i) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{N}_n = (V_0^2 - V_0V_1 + V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n+1} + (V_1^2 - V_0V_2)\mathcal{V}_n + (V_0^2 + V_1^2 - V_2V_1)\mathcal{V}_{n-1}$.

(j) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{N}_n = (V_0^2 - V_0V_1 - V_0V_2 + 2V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_n + (V_0^2 + V_1^2 - V_2V_1)\mathcal{V}_{n-1} + (V_0^2 - V_0V_1 + V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n-2}$.

Proof. From Lemmas 6.6 and 6.7, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized Narayana and Narayana-Lucas matrix sequences.

Theorem 6.13 *For the matrix sequences* $\{\mathcal{V}_n\}$ *and* $\{\mathcal{U}_n\}$*, we have the following identities.*

(a) $31\mathcal{V}_n = (V_0 - 14V_1 + 11V_2)\mathcal{U}_{n+4} + (10V_0 + 15V_1 - 14V_2)\mathcal{U}_{n+3} + (10V_1 - 14V_0 + V_2)\mathcal{U}_{n+2}$.

(b) $31\mathcal{V}_n = (11V_0 + V_1 - 3V_2)\mathcal{U}_{n+3} + (10V_1 - 14V_0 + V_2)\mathcal{U}_{n+2} + (V_0 - 14V_1 + 11V_2)\mathcal{U}_{n+1}$.

(c) $31\mathcal{V}_n = (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{n+2} + (V_0 - 14V_1 + 11V_2)\mathcal{U}_{n+1} + (11V_0 + V_1 - 3V_2)\mathcal{U}_n$.

(d) $31\mathcal{V}_n = (9V_2 - 3V_1 - 2V_0)\mathcal{U}_{n+1} + (11V_0 + V_1 - 3V_2)\mathcal{U}_n + (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{n-1}$.

(e) $31\mathcal{V}_n = (9V_0 - 2V_1 + 6V_2)\mathcal{U}_n + (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{n-1} + (9V_2 - 3V_1 - 2V_0)\mathcal{U}_{n-2}$.

(f) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{U}_n = (3V_1^2+2V_1V_2+2V_0V_1-2V_2^2-3V_0V_2)\mathcal{V}_{n+4}+(3V_0^2-2V_0V_1+5V_0V_2-2V_1^2-5V_1V_2+2V_2^2)\mathcal{V}_{n+3}+(-2V_0^2-2V_0V_2-3V_1V_0+3V_2^2-V_1V_2)\mathcal{V}_{n+2}$.

(g) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{U}_n = (3V_0^2+V_1^2+2V_0V_2-3V_1V_2)\mathcal{V}_{n+3}+(-2V_0^2-2V_0V_2-3V_1V_0+3V_2^2-V_1V_2)\mathcal{V}_{n+2}+(3V_1^2-2V_2^2+2V_0V_1-3V_0V_2+2V_1V_2)\mathcal{V}_{n+1}$.

(h) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{U}_n = (V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)\mathcal{V}_{n+2}+(3V_1^2+2V_1V_2+2V_0V_1-2V_2^2-3V_0V_2)\mathcal{V}_{n+1}+(3V_0^2+2V_2V_0+V_1^2-3V_2V_1)\mathcal{V}_n$.

(i) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{U}_n = (V_0^2-V_0V_1-3V_0V_2+4V_1^2-2V_1V_2+V_2^2)\mathcal{V}_{n+1}+(3V_0^2+2V_2V_0+V_1^2-3V_2V_1)\mathcal{V}_n+(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)\mathcal{V}_{n-1}$.

(j) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{U}_n = (4V_0^2-V_0V_1-V_0V_2+5V_1^2-5V_1V_2+V_2^2)\mathcal{V}_n+(V_0^2-3V_0V_1+V_1^2-4V_1V_2+3V_2^2)\mathcal{V}_{n-1}+(V_0^2-V_0V_1-3V_0V_2+4V_1^2-2V_1V_2+V_2^2)\mathcal{V}_{n-2}$.

Proof. From Lemmas 6.8 and 6.9, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized Narayana and Narayana-Perrin matrix sequences.

Theorem 6.14 *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{H}_n\}$, we have the following identities.*

(a) $53\mathcal{V}_n = (15V_2-11V_1-10V_0)\mathcal{H}_{n+4}+(25V_0+V_1-11V_2)\mathcal{H}_{n+3}+(25V_1-11V_0-10V_2)\mathcal{H}_{n+2}$.

(b) $53\mathcal{V}_n = (15V_0-10V_1+4V_2)\mathcal{H}_{n+3}+(25V_1-11V_0-10V_2)\mathcal{H}_{n+2}+(15V_2-11V_1-10V_0)\mathcal{H}_{n+1}$.

(c) $53\mathcal{V}_n = (4V_0+15V_1-6V_2)\mathcal{H}_{n+2}+(15V_2-11V_1-10V_0)\mathcal{H}_{n+1}+(15V_0-10V_1+4V_2)\mathcal{H}_n$.

(d) $53\mathcal{V}_n = (4V_1-6V_0+9V_2)\mathcal{H}_{n+1}+(15V_0-10V_1+4V_2)\mathcal{H}_n+(4V_0+15V_1-6V_2)\mathcal{H}_{n-1}$.

(e) $53\mathcal{V}_n = (9V_0-6V_1+13V_2)\mathcal{H}_n+(4V_0+15V_1-6V_2)\mathcal{H}_{n-1}+(4V_1-6V_0+9V_2)\mathcal{H}_{n-2}$.

(f) $(V_0^3+V_0^2V_2+V_0V_1^2-3V_0V_1V_2+V_1^3+V_1^2V_2-2V_1V_2^2+V_2^3)\mathcal{H}_n = (2V_0^2+3V_0V_1-V_0V_2+3V_1^2+V_1V_2-3V_2^2)\mathcal{V}_{n+4}+(V_0^2-5V_0V_1+4V_0V_2-3V_1^2-6V_1V_2+5V_2^2)\mathcal{V}_{n+3}+(-3V_0^2-V_0V_1-5V_0V_2+2V_1^2+2V_1V_2+V_2^2)\mathcal{V}_{n+2}$.

(g) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (3V_0^2 + 3V_0V_2 - 2V_1V_0 + 2V_2^2 - 5V_1V_2)\mathcal{V}_{n+3} + (-3V_0^2 - V_0V_1 - 5V_0V_2 + 2V_1^2 + 2V_1V_2 + V_2^2)\mathcal{V}_{n+2} + (2V_0^2 + 3V_0V_1 - V_0V_2 + 3V_1^2 + V_1V_2 - 3V_2^2)\mathcal{V}_{n+1}$.

(h) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (2V_1^2 - 3V_1V_2 - 3V_0V_1 + 3V_2^2 - 2V_0V_2)\mathcal{V}_{n+2} + (2V_0^2 + 3V_0V_1 - V_0V_2 + 3V_1^2 + V_1V_2 - 3V_2^2)\mathcal{V}_{n+1} + (3V_0^2 + 3V_0V_2 - 2V_1V_0 + 2V_2^2 - 5V_1V_2)\mathcal{V}_n$.

(i) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (2V_0^2 - 3V_2V_0 + 5V_1^2 - 2V_2V_1)\mathcal{V}_{n+1} + (3V_0^2 + 3V_0V_2 - 2V_1V_0 + 2V_2^2 - 5V_1V_2)\mathcal{V}_n + (2V_1^2 - 3V_1V_2 - 3V_0V_1 + 3V_2^2 - 2V_0V_2)\mathcal{V}_{n-1}$.

(j) $(V_0^3 + V_0^2V_2 + V_0V_1^2 - 3V_0V_1V_2 + V_1^3 + V_1^2V_2 - 2V_1V_2^2 + V_2^3)\mathcal{H}_n = (5V_0^2 - 2V_0V_1 + 5V_1^2 - 7V_1V_2 + 2V_2^2)\mathcal{V}_n + (2V_1^2 - 3V_1V_2 - 3V_0V_1 + 3V_2^2 - 2V_0V_2)\mathcal{V}_{n-1} + (2V_0^2 - 3V_2V_0 + 5V_1^2 - 2V_2V_1)\mathcal{V}_{n-2}$.

Proof. From Lemmas 6.10 and 6.11, (a)-(j) follow. □

To prove the following Lemma 6.16 (c) we need the next lemma.

Lemma 6.15 *Let A, B, C as in Theorem 6.2 and $A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3$ as in Corollary 6.2. Then the following relations hold:*

$$\begin{aligned}
A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\
AB &= BA = AC = CA = CB = BC = (0)\,, \\
A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\
A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,, \\
A_3B_3 &= B_3A_3 = A_3C_3 = C_3A_3 = C_3B_3 = B_3C_3 = (0)\,.
\end{aligned}$$

Proof. Using $\alpha + \beta + \gamma = 1$, $\alpha\beta + \alpha\gamma + \beta\gamma = 0$ and $\alpha\beta\gamma = 1$, required equalities can be established by matrix calculations. See also [27]. □

Lemma 6.16 *For all integers m and n, we have the following identities.*

(a) $\mathcal{N}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_0 = \mathcal{V}_n$.

(b) $\mathcal{V}_0\mathcal{N}_n = \mathcal{N}_n\mathcal{V}_0 = \mathcal{V}_n$.

(c) $\mathcal{N}_m\mathcal{N}_n = \mathcal{N}_n\mathcal{N}_m = \mathcal{N}_{m+n}$.

(d) $\mathcal{N}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_m = \mathcal{V}_{m+n}$.

(e) $\mathcal{N}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{N}_m = \mathcal{U}_{m+n}$.

(f) $\mathcal{N}_m\mathcal{H}_n = \mathcal{H}_n\mathcal{N}_m = \mathcal{H}_{m+n}$.

(g) $\mathcal{V}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_0$.

(h) $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$.

(i) $\mathcal{N}_{-n} = (\mathcal{N}_n)^{-1}$.

(j) $\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^n$

Proof. Identities can be established easily.

(a) Since $\mathcal{N}_0$ is the identity matrix, (a) follows.

(b) It can be seen by using Lemma 6.6.

(c) (c) is given in [27]. We supply the proof for completeness. Using Lemma 6.15 we obtain

$$\begin{aligned}
\mathcal{N}_m\mathcal{N}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n) \\
&= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + A_1B_1\alpha^m\beta^n + B_1A_1\alpha^n\beta^m \\
&\quad + A_1C_1\alpha^m\gamma^n + C_1A_1\alpha^n\gamma^m + B_1C_1\beta^m\gamma^n + C_1B_1\beta^n\gamma^m \\
&= A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} \\
&= \mathcal{N}_{m+n}.
\end{aligned}$$

(d) From (b), we have

$$\mathcal{N}_m\mathcal{V}_n = \mathcal{N}_m\mathcal{N}_n\mathcal{U}_0.$$

Now from (c) and again from (b), we obtain $\mathcal{N}_m\mathcal{V}_n = \mathcal{N}_{m+n}\mathcal{V}_0 = \mathcal{V}_{m+n}$.

It can be shown similarly that $\mathcal{V}_n\mathcal{N}_m = \mathcal{V}_{m+n}$.

(e) Take $\mathcal{V}_n = \mathcal{U}_n$ in (d).

(f) Take $\mathcal{V}_n = \mathcal{H}_n$ in (d).

(g) After matrix multiplication, just compare the row and column entries of the matrices.

(h) Using (d) and (g) and (b) we get

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_n\mathcal{N}_m = \mathcal{V}_n\mathcal{V}_0\mathcal{N}_m = \mathcal{V}_n\mathcal{V}_m.$$

Again, using (d) and (g) and (b), we obtain

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_m\mathcal{N}_n = \mathcal{V}_m\mathcal{V}_0\mathcal{N}_n = \mathcal{V}_m\mathcal{V}_n.$$

This completes the proof of (h).

(i) Suppose first that $n \geq 0$. We prove by mathematical induction. If $n = 0$ then we have

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = (\mathcal{N}_0)^{-1}$$

which is true and

$$\mathcal{N}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = (\mathcal{N}_1)^{-1}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k + 1$, by using (c), we obtain

$$
\begin{aligned}
(\mathcal{N}_{k+1})^{-1} &= (\mathcal{N}_k\mathcal{N}_1)^{-1} = (\mathcal{N}_1)^{-1}(\mathcal{N}_k)^{-1} = \mathcal{N}_{-1}\mathcal{N}_{-k} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} N_{-k+1} & N_{-k-1} & N_{-k} \\ N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \\ N_{1-k} - N_{-k} & N_{-k-1} - N_{-k-2} & N_{-k} - N_{-k-1} \end{pmatrix} \\
&= \begin{pmatrix} N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \\ N_{-k-2} & N_{-k-4} & N_{-k-3} \end{pmatrix} \\
&= \begin{pmatrix} N_{-(k+1)+1} & N_{-(k+1)-1} & N_{-(k+1)} \\ N_{-(k+1)} & N_{-(k+1)-2} & N_{-(k+1)-1} \\ N_{-(k+1)-1} & N_{-(k+1)-3} & N_{-(k+1)-2} \end{pmatrix} \\
&= \mathcal{N}_{-(k+1)}
\end{aligned}
$$

Thus, by induction on n, this proves (g) for $n \geq 0$. Suppose now that $n \leq 0$. Say $m = -n$. Then (g) can be written as

$$\mathcal{N}_m = (\mathcal{N}_{-m})^{-1}$$

and we prove this. Since $m \geq 0$, from the first part of the proof, we have

$$\mathcal{N}_{-m} = (\mathcal{N}_m)^{-1}$$

and so

$$(\mathcal{N}_{-m})^{-1} = ((\mathcal{N}_m)^{-1})^{-1} = \mathcal{N}_m$$

which completes the proof.

(j) Taking $-n+1$ for m and 1 for n in $\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_m\mathcal{V}_n$ which is given in (h), we obtain that

$$\mathcal{V}_0\mathcal{V}_{-n} = \mathcal{V}_{-n+1}\mathcal{V}_{-1}. \tag{6.4}$$

If we multiply both side of the equation (6.4) with $\mathcal{V}_0$ we have the relation

$$\begin{aligned}\mathcal{V}_0\mathcal{V}_0\mathcal{V}_{-n} &= \mathcal{V}_0\mathcal{V}_{-n+1}\mathcal{V}_{-1} \\ &= \mathcal{V}_{-n+2}\mathcal{V}_{-1}\mathcal{V}_{-1}.\end{aligned}$$

Repeating this process we then obtain

$\mathcal{V}_0^{n-1}\mathcal{V}_{-n} = \mathcal{V}_{-1}^{n}.$

Thus, it follows that

$\mathcal{V}_{-n} = \mathcal{V}_0^{1-n}\mathcal{V}_{-1}^{n}.$

This completes the proof. □

Note that using Lemma 6.16 (j) and (d), we obtain

$\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^{n} = (\mathcal{V}_n\mathcal{N}_{-n})^{1-n}\mathcal{V}_{-1}^{n} = \mathcal{N}_{-n}^{1-n}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^{n}$

and then by Lemma (i), we get

$\mathcal{V}_{-n} = \mathcal{N}_n^{n-1}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^{n}.$

Using Lemma 6.16 and comparing matrix entries, we have next result.

Corollary 6.10 *For generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin numbers, we have the following identities:*

(a) $V_{m+n} = N_mV_{n+1} + N_{m-2}V_n + N_{m-1}V_{n-1} = N_{m+1}V_n + N_{m-1}V_{n-1} + N_mV_{n-2}.$

(b) $N_{m+n} = N_mN_{n+1} + N_{m-2}N_n + N_{m-1}N_{n-1} = N_{m+1}N_n + N_{m-1}N_{n-1} + N_mN_{n-2}.$

(c) $U_{m+n} = N_mU_{n+1} + N_{m-2}U_n + N_{m-1}U_{n-1} = N_{m+1}U_n + N_{m-1}U_{n-1} + N_mU_{n-2}.$

(d) $H_{m+n} = N_mH_{n+1} + N_{m-2}H_n + N_{m-1}H_{n-1} = N_{m+1}H_n + N_{m-1}H_{n-1} + N_mH_{n-2}.$

(e) $V_{m+1}V_n + V_{m-1}V_{n-1} + V_mV_{n-2} = V_mV_{n+1} + V_{m-2}V_n + V_{m-1}V_{n-1} = V_0V_{m+n+1} + (V_1 - V_0)V_{m+n} + (V_2 - V_1)V_{m+n-1}.$

(f) $N_{m+1}N_n + N_{m-1}N_{n-1} + N_mN_{n-2} = N_mN_{n+1} + N_{m-2}N_n + N_{m-1}N_{n-1} = N_{m+n}.$

(g) $U_{m+1}U_n+U_{m-1}U_{n-1}+U_mU_{n-2} = U_mU_{n+1}+U_{m-2}U_n+U_{m-1}U_{n-1} = 3U_{m+n+1}-2U_{m+n}$.

(h) $H_{m+1}H_n + H_{m-1}H_{n-1} + H_mH_{n-2} = H_mH_{n+1} + H_{m-2}H_n + H_{m-1}H_{n-1} = 3H_{m+n+1} - 3H_{m+n} + 2H_{m+n-1}$.

Proof. We prove (a) and (e) by using Lemma 6.16 (d) and (h). The others are special cases of (a) and (e). Lemma 6.16 (d), i.e., $\mathcal{N}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_m = \mathcal{V}_{m+n}$, can be writtten as

$$\begin{pmatrix} V_{m+n+1} & V_{m+n-1} & V_{m+n} \\ V_{m+n} & V_{m+n-2} & V_{m+n-1} \\ V_{m+n-1} & V_{m+n-3} & V_{m+n-2} \end{pmatrix} = \begin{pmatrix} N_{m+1} & N_{m-1} & N_m \\ N_m & N_{m-2} & N_{m-1} \\ N_{m-1} & N_{m-3} & N_{m-2} \end{pmatrix} \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} \begin{pmatrix} N_{m+1} & N_{m-1} & N_m \\ N_m & N_{m-2} & N_{m-1} \\ N_{m-1} & N_{m-3} & N_{m-2} \end{pmatrix}.$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (a).

Lemma 6.16 (h), i.e., $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$, can be writtten as

$$\begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} \begin{pmatrix} V_{m+1} & V_{m-1} & V_m \\ V_m & V_{m-2} & V_{m-1} \\ V_{m-1} & V_{m-3} & V_{m-2} \end{pmatrix}$$
$$= \begin{pmatrix} V_{m+1} & V_{m-1} & V_m \\ V_m & V_{m-2} & V_{m-1} \\ V_{m-1} & V_{m-3} & V_{m-2} \end{pmatrix} \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} V_1 & V_2 - V_1 & V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 \\ V_2 - V_1 & V_0 + V_1 - V_2 & V_1 - V_0 \end{pmatrix} \begin{pmatrix} V_{m+n+1} & V_{m+n-1} & V_{m+n} \\ V_{m+n} & V_{m+n-2} & V_{m+n-1} \\ V_{m+n-1} & V_{m+n-3} & V_{m+n-2} \end{pmatrix}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (e). □

As an application of Lemma 6.16 (i) and Corollary 6.10 (b), we present the following example.

Example 6.1 *For all integers n, we have the following identities.*

$N_{-n} = N_{n-1}^2 - N_nN_{n-2}$

and

$$N_{n+1}^3 + N_n^3 + N_{n-1}^3 - 2N_nN_{n+1}^2 + (N_{n-1} + N_{n+1})N_n^2 - (3N_n - N_{n-1})N_{n+1}N_{n-1} = 1.$$

Solution. By comparing the 2nd rows and 1st columns entries of both sides of the relation $\mathcal{N}_{-n} = (\mathcal{N}_n)^{-1}$ which is given in Lemma 6.16 (i), we get

$$\begin{aligned} N_{-n} &= \frac{N_{n-1}^2 - N_nN_{n-2}}{N_{n-3}N_n^2 - 2N_nN_{n-1}N_{n-2} + N_{n-1}^3 - N_{n+1}N_{n-3}N_{n-1} + N_{n+1}N_{n-2}^2} \\ &= \frac{N_{n-1}^2 - N_nN_{n-2}}{N_{n+1}^3 + N_n^3 + N_{n-1}^3 - 2N_nN_{n+1}^2 + (N_{n-1} + N_{n+1})N_n^2 - (3N_n - N_{n-1})N_{n+1}N_{n-1}} \end{aligned} \tag{6.5}$$

where we used the identities

$$\begin{aligned} N_n &= N_{n-1} + N_{n-3} \Rightarrow N_n - N_{n-1} = N_{n-3}, \\ N_{n+1} &= N_n + N_{n-2} \Rightarrow N_{n+1} - N_n = N_{n-2}. \end{aligned}$$

Using (taking $m = n$ in) Corollary 6.10 (b), we get

$$N_{2n} = N_{n-1}^2 + N_nN_{n+1} + N_nN_{n-2}. \tag{6.6}$$

In [19, Corollary 12 (a)], the following formula is presented for N_{-n} :

$$N_{-n} = 2N_n^2 + N_{2n} - 3N_{n+1}N_n.$$

which (using (6.6)) can be written as

$$N_{-n} = N_nN_{n-2} - 2N_nN_{n+1} + 2N_n^2 + N_{n-1}^2. \tag{6.7}$$

Note that

$$\begin{aligned} N_{n-1}^2 - N_nN_{n-2} &= (N_nN_{n-2} - 2N_nN_{n+1} + 2N_n^2 + N_{n-1}^2) - 2N_n\,(N_n - N_{n+1} + N_{n-2}) \\ &= N_nN_{n-2} - 2N_nN_{n+1} + 2N_n^2 + N_{n-1}^2 \end{aligned}$$

because $N_n - N_{n+1} + N_{n-2} = 0$. So the rights sides of the equations (6.5) and (6.7) must be equal. This completes the solution. □

Theorem 6.17 *For all integers m and n, we have the following identities.*

(a) $\mathcal{V}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_m = (V_1 - V_2)^2\mathcal{N}_{m+n+4} - 2(V_1 - V_2)(V_0 + V_1 - V_2)\mathcal{N}_{m+n+3} + (V_2^2 - V_1^2 + V_0^2 + 4V_0V_1 - 4V_0V_2)\mathcal{N}_{m+n+2} + 2(V_1 - V_0)\,(V_0 + V_1 - V_2)\,\mathcal{N}_{m+n+1} + (V_0 - V_1)^2\mathcal{N}_{m+n}.$

(b) $\mathcal{V}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_m = (V_2 - V_1)\mathcal{V}_{m+n+2} + (V_0 + V_1 - V_2)\mathcal{V}_{m+n+1} + (V_1 - V_0)\mathcal{V}_{m+n}.$

(c) $\mathcal{N}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{N}_m = (V_2 - V_1)\mathcal{N}_{m+n+2} + (V_0 + V_1 - V_2)\mathcal{N}_{m+n+1} + (V_1 - V_0)\mathcal{N}_{m+n}.$

(d) $31\mathcal{N}_m\mathcal{V}_n = 31\mathcal{V}_n\mathcal{N}_m = (11V_1 - 3V_0 - 2V_2)\mathcal{U}_{m+n+2} + (V_0 - 14V_1 + 11V_2)\mathcal{U}_{m+n+1} + (11V_0 + V_1 - 3V_2)\mathcal{U}_{m+n}.$

(e) $53\mathcal{N}_m\mathcal{V}_n = 53\mathcal{V}_n\mathcal{N}_m = (4V_0 + 15V_1 - 6V_2)\mathcal{H}_{m+n+2} + (15V_2 - 11V_1 - 10V_0)\mathcal{H}_{m+n+1} + (15V_0 - 10V_1 + 4V_2)\mathcal{H}_{m+n}.$

Proof.

(a) It follows from Theorem 6.12 (c) and Lemma 6.16 (c).

(b) It follows from Theorem 6.12 (c) and Lemma 6.16 (d).

(c) It follows from Theorem 6.12 (c) and Lemma 6.16 (c).

(d) It follows from Theorem 6.13 (c) and Lemma 6.16 (e).

(e) It follows from Theorem 6.14 (c) and Lemma 6.16 (f). □

Note that in Theorem 6.17 we use (c)'s of Theorems 6.12, 6.13 and 6.14. Using (a),(b),(d),(e),(f),(g)(h),(i of Theorems 6.12, 6.13 and 6.14 we can establish other recurence relations.
Using Theorem 6.17 and comparing matrix entries, we have next result.

Theorem 6.18 *For generalized Narayana, Narayana, Narayana-Lucas and Narayana-Perrin numbers, we have the following identities:*

(a) $V_mV_{n+1} + V_{m-2}V_n + V_{m-1}V_{n-1} = V_{m+1}V_n + V_{m-1}V_{n-1} + V_mV_{n-2} = (V_1 - V_2)^2N_{m+n+4} + 2(V_2 - V_1)(V_0 + V_1 - V_2)N_{m+n+3} + (V_0^2 + 4V_0V_1 - 4V_0V_2 - V_1^2 + V_2^2)N_{m+n+2} + (2V_1 - 2V_0)(V_0 + V_1 - V_2)N_{m+n+1} + (V_0 - V_1)^2N_{m+n}.$

(b) $V_mV_{n+1} + V_{m-2}V_n + V_{m-1}V_{n-1} = V_{m+1}V_n + V_{m-1}V_{n-1} + V_mV_{n-2} = (V_2 - V_1)V_{m+n+2} + (V_0 + V_1 - V_2)V_{m+n+1} + (V_1 - V_0)V_{m+n}.$

(c) $N_mV_{n+1} + N_{m-2}V_n + N_{m-1}V_{n-1} = N_{m+1}V_n + N_{m-1}V_{n-1} + N_mV_{n-2} = (V_2 - V_1)N_{m+n+2} + (V_0 + V_1 - V_2)N_{m+n+1} + (V_1 - V_0)N_{m+n}.$

(d) $31(N_mV_{n+1}+N_{m-2}V_n+N_{m-1}V_{n-1}) = 31(N_{m+1}V_n+N_{m-1}V_{n-1}+N_mV_{n-2}) = (11V_1 - 3V_0 - 2V_2)U_{m+n+2} + (V_0 - 14V_1 + 11V_2)U_{m+n+1} + (11V_0 + V_1 - 3V_2)U_{m+n}$.

(e) $53(N_mV_{n+1}+N_{m-2}V_n+N_{m-1}V_{n-1}) = 53(N_{m+1}V_n+N_{m-1}V_{n-1}+N_mV_{n-2}) = (4V_0 + 15V_1 - 6V_2)H_{m+n+2} + (15V_0 - 10V_1 + 4V_2)H_{m+n} + (15V_2 - 11V_1 - 10V_0)H_{m+n+1}$.

Proof. By multiplying matrices and then by comparing the 2nd rows and 1st columns entries in Theorem 6.17 (a), we get the required identities in (a). The remaining of identities can be proved by considering again Theorem 6.17. □

Taking $V_n = N_n$ in Theorem 6.18, we obtain the following corollary.

Corollary 6.11 *For Narayana numbers, we have the following identities:*

(a) $N_mN_{n+1}+N_{m-2}N_n+N_{m-1}N_{n-1} = N_{m+1}N_n+N_{m-1}N_{n-1}+N_mN_{n-2} = N_{m+n}$.

(b) $31(N_mN_{n+1}+N_{m-2}N_n+N_{m-1}N_{n-1}) = 31(N_{m+1}N_n+N_{m-1}N_{n-1}+N_mN_{n-2}) = 9U_{m+n+2} - 3U_{m+n+1} - 2U_{m+n}$.

(c) $53(N_mN_{n+1}+N_{m-2}N_n+N_{m-1}N_{n-1}) = 53(N_{m+1}N_n+N_{m-1}N_{n-1}+N_mN_{n-2}) = 9H_{m+n+2} + 4H_{m+n+1} - 6H_{m+n}$.

Taking $V_n = U_n$ in Theorem 6.18, we get the following corollary.

Corollary 6.12 *For Narayana-Lucas numbers, we have the following identities:*

(a) $U_mU_{n+1}+U_{m-2}U_n+U_{m-1}U_{n-1} = U_{m+1}U_n+U_{m-1}U_{n-1}+U_mU_{n-2} = 9N_{m+n+2} - 12N_{m+n+1} + 4N_{m+n}$.

(b) $U_mU_{n+1}+U_{m-2}U_n+U_{m-1}U_{n-1} = U_{m+1}U_n+U_{m-1}U_{n-1}+U_mU_{n-2} = 3U_{m+n+1} - 2U_{m+n}$.

(c) $N_mU_{n+1}+N_{m-2}U_n+N_{m-1}U_{n-1} = N_{m+1}U_n+N_{m-1}U_{n-1}+N_mU_{n-2} = 3N_{m+n+1} - 2N_{m+n}$.

(d) $N_mU_{n+1}+N_{m-2}U_n+N_{m-1}U_{n-1} = N_{m+1}U_n+N_{m-1}U_{n-1}+N_mU_{n-2} = U_{m+n}$.

(e) $53(N_mU_{n+1}+N_{m-2}U_n+N_{m-1}U_{n-1}) = 53(N_{m+1}U_n+N_{m-1}U_{n-1}+N_mU_{n-2}) = 21H_{m+n+2} - 26H_{m+n+1} + 39H_{m+n}$.

Taking $V_n = H_n$ in Theorem 6.18, we obtain the following corollary.

Corollary 6.13 *For Narayana-Perrin numbers, we have the following identities:*

(a) $H_m H_{n+1} + H_{m-2} H_n + H_{m-1} H_{n-1} = H_{m+1} H_n + H_{m-1} H_{n-1} + H_m H_{n-2} = 4N_{m+n+4} + 4N_{m+n+3} - 11N_{m+n+2} - 6N_{m+n+1} + 9N_{m+n}$.

(b) $H_m H_{n+1} + H_{m-2} H_n + H_{m-1} H_{n-1} = H_{m+1} H_n + H_{m-1} H_{n-1} + H_m H_{n-2} = 2H_{m+n+2} + H_{m+n+1} - 3H_{m+n}$.

(c) $N_m H_{n+1} + N_{m-2} H_n + N_{m-1} H_{n-1} = N_{m+1} H_n + N_{m-1} H_{n-1} + N_m H_{n-2} = 2N_{m+n+2} + N_{m+n+1} - 3N_{m+n}$.

(d) $31(N_m H_{n+1} + N_{m-2} H_n + N_{m-1} H_{n-1}) = 31(N_{m+1} H_n + N_{m-1} H_{n-1} + N_m H_{n-2}) = -13U_{m+n+2} + 25U_{m+n+1} + 27U_{m+n}$.

(e) $N_m H_{n+1} + N_{m-2} H_n + N_{m-1} H_{n-1} = N_{m+1} H_n + N_{m-1} H_{n-1} + N_m H_{n-2} = H_{m+n}$.

The next two theorems provide us the convenience to obtain the powers of generalized Narayana, Narayana, Narayana-Lucas and Naraya-perrin matrix sequences.

Theorem 6.19 *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{N}_n^m = \mathcal{N}_{mn}$,

(b) $\mathcal{N}_{n+1}^m = \mathcal{N}_1^m \mathcal{N}_{mn}$,

(c) $\mathcal{N}_{n-r} \mathcal{N}_{n+r} = \mathcal{N}_n^2 = \mathcal{N}_2^n$.

Proof. We prove for $m, n, r \geq 0$. The other cases can be proved similarly.

(a) We can write $\mathcal{N}_n^m$ as

$$\mathcal{N}_n^m = \mathcal{N}_n \mathcal{N}_n ... \mathcal{N}_n \ (m \text{ times}).$$

Using Theorem 6.16 (c) iteratively, we obtain the required result:

$$\begin{aligned}
\mathcal{N}_n^m &= \underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m \text{ times}} \\
&= \mathcal{N}_{2n}\underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m-1 \text{ times}} \\
&= \mathcal{N}_{3n}\underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m-2 \text{ times}} \\
&\quad \vdots \\
&= \mathcal{N}_{(m-1)n}\mathcal{N}_n \\
&= \mathcal{N}_{mn}.
\end{aligned}$$

(b) As a similar approach in (a) we have

$$\mathcal{N}_{n+1}^m = \mathcal{N}_{n+1}.\mathcal{N}_{n+1}...\mathcal{N}_{n+1} = \mathcal{N}_{m(n+1)} = \mathcal{N}_m\mathcal{N}_{mn} = \mathcal{N}_1\mathcal{N}_{m-1}\mathcal{N}_{mn}.$$

Using Theorem 6.16 (c), we can write iteratively $\mathcal{N}_m = \mathcal{N}_1\mathcal{N}_{m-1}$, $\mathcal{N}_{m-1} = \mathcal{N}_1\mathcal{N}_{m-2}$, ..., $\mathcal{N}_2 = \mathcal{N}_1\mathcal{N}_1$. Now it follows that

$$\mathcal{N}_{n+1}^m = \underbrace{\mathcal{N}_1\mathcal{N}_1...\mathcal{N}_1}_{m \text{ times}}\mathcal{N}_{mn} = \mathcal{N}_1^m\mathcal{N}_{mn}.$$

(c) Theorem 6.16 (c) gives

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_{2n} = \mathcal{N}_n\mathcal{N}_n = \mathcal{N}_n^2$$

and also

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_{2n} = \underbrace{\mathcal{N}_2\mathcal{N}_2...\mathcal{N}_2}_{n \text{ times}} = \mathcal{N}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{V}_n$.

Theorem 6.20 *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{V}_{n-r}\mathcal{V}_{n+r} = \mathcal{V}_n^2$,

(b) $\mathcal{V}_n^m = \mathcal{V}_0^m\mathcal{N}_{mn}$.

Proof.

(a) We use Binet's formula of generalized Narayana sequence which is given in Theorem 6.2. So

$$
\begin{aligned}
& \mathcal{V}_{n-r}\mathcal{V}_{n+r} - \mathcal{V}_n^2 \\
= \;& (A\alpha^{n-r} + B\beta^{n-r} + C\gamma^{n-r})(A\alpha^{n+r} + B\beta^{n+r} + C\gamma^{n+r}) - (A\alpha^n + B\beta^n + C\gamma^n)^2 \\
= \;& AB\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + AC\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 + BC\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 \\
= \;& 0
\end{aligned}
$$

since $AB = AC = BC = 0$ (see Lemma 6.15). Now we get the result as required.

(b) By Theorem 6.19, we have

$$\mathcal{V}_0^m \mathcal{N}_{mn} = \underbrace{\mathcal{V}_0\mathcal{V}_0...\mathcal{V}_0}_{m \text{ times}}\underbrace{\mathcal{N}_n\mathcal{N}_n...\mathcal{N}_n}_{m \text{ times}}.$$

When we apply Lemma 6.16 (b) iteratively, it follows that

$$
\begin{aligned}
\mathcal{V}_0^m \mathcal{N}_{mn} &= (\mathcal{V}_0\mathcal{N}_n)(\mathcal{V}_0\mathcal{N}_n)...(\mathcal{V}_0\mathcal{N}_n) \\
&= \mathcal{V}_n\mathcal{V}_n...\mathcal{V}_n = \mathcal{V}_n^m.
\end{aligned}
$$

This completes the proof. □

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