Minisuperspace models: Universe and Antiuniverse Pair Creation and Annihilation

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This is to certify that I have examined this copy of a master's thesis by

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[To my parents, Flutur and Kujtim, without whom I could not have embarked upon this journey.]

ABSTRACT

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Cosmology is an ancient area of science and has reached many advancements recently due to the discoveries in quantum physics and the improvements in observational astronomy. There have been many attempts to explain the origin of the universe. The cosmological models with homogeneous scalar fields seem to be fine tuned with the observations and therefore enabling us to build a decent mathematical model of our cosmos. Nevertheless, there are many problems we encounter in cosmology which we will discuss below. We start with a toy model with a homogeneous scalar field and try to connect with the observations. The mathematical of the model with the scalar field will be mostly constructed theoretically, and some analytical solutions will be presented. We will consider a simple model in cosmology but that can be extended to more difficult models as the principles are significantly the same. This thesis will discuss The No-Boundary proposal and a simple derivation of the Wheeler-DeWitt equation for our minisuperspace model. Although there is no quantum gravity theory yet, we introduce the Wheeler-DeWitt equation and developing a spinor representation of our model. Cosmological solutions will be plotted and will be studied independently. A potential will be chosen so that we will have the case of the harmonic and anti-harmonic oscillator. In the late steps of the work, the classical time will be discussed. We will also obtain a pair creation and annihilation of the universe at the time of creation from a preferred vacuum state if one considers that our universe was CPT symmetric at the very early times of the universe. We will also discuss the entanglement entropy of the created multiverse and a general formula for the quantized spinor solutions of the Dirac-type Wheeler-DeWitt equations in a minisuperspace model.

ÖZETÇE

Minisuperuzay modelleri: Evren- antievren çift yaratımı ve yok edilmesi
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Kozmoloji, eski bir bilim alanıdır ve kuantum fiziğindeki keşifler ve gözlemsel astronomideki gelişmeler sayesinde son zamanlarda birçok ilerleme kaydetmiştir. Evrenin kökenini açıklamak için birçok girişimde bulunuldu ve homojen skaler alanlara sahip kozmolojik modeller, gözlemlerle oluşmuş, ince detaylı görünüyor. Bu nedenle kozmosumuzun düzgün bir matematiksel modelini inşa etmemizi sağlıyor. Yine de aşağıda tartışacağımız kozmolojide karşılaştığımız birçok sorun var. Homojen skaler alanlı basit bir tasarım modeli ile başlıyoruz ve gözlemlerle bağlantı kurmaya çalışıyoruz. Skaler alanlı modelin matematiği çoğunlukla teorik olarak oluşturulacak ve bazı analitik çözümler sunulacaktır. Henüz bir kuantum yerçekimi teorisi olmamasına rağmen, Wheeler-DeWitt denklemini tanıtarak başlıyoruz ve modelimizin bir spinor temsilini geliştiriyoruz. Kozmolojik çözümler çizilecek ve bağımsız olarak incelenecektir. Potansiyel, harmonik ve anti-harmonik osilatör durumuna sahip olacağı şekilde seçilecektir. Çalışmanın ilerleyen aşamalarında klasik zaman ele alınacaktır. Ayrıca, evrenin çok erken zamanlarında, evrenimizin CPT simetrik olduğu düşünülürse tercih edilen bir boşluk durumundan yaratılış sırasında evrenin bir çift yaratılışı ve yok oluşunu elde edeceğiz. Ayrıca oluşturulan çoklu evrenin dolaşma entropisini ve bir minisuperspace modelinde Dirac tipi Wheeler-DeWitt denklemlerinin nicelenmiş spinor çözümleri için genel bir formülü tartışacağız.

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ABBREVIATIONS

WDW	Wheeler-deWitt
CMB	Cosmic microwave background
RW	Robertson–Walker
NB	No Boundary
FLRW	Friedmann–Lemaître–Robertson–Walker
PU oscillator	Pais-Uhlenbeck oscillator
CPT symmetry	Charge, parity, and time reversal symmetry

Chapter 1

INTRODUCTION TO ROBERTSON-WALKER METRIC AND ITS SIGNATURE DYNAMICS

A successful cosmological model needs at least three criteria to be satisfied:

- Dynamical laws and a complete set of variables which will determine the dynamics and the evolution of the universe.
- The correct solution of these dynamical laws which will predict and explain this dynamics.
- Fine-tuning and the consistency of the theory in the observable universe so that we have a better theoretical model of the cosmos.

In this project, the classical models of gravitation interacting with scalar fields whose solutions are based on degenerate metrics will be investigated. These cosmological solutions will demonstrate signature change which in turn are related to the transition between the Euclidean domain to Lorentzian spacetime. We consider Einstein's theory of general relativity coupled to a dilaton scalar ϕ with a suitably chosen potential $V(\phi)$. In a cosmological context with the Robertson-Walker metric[Kamenshchik et al., 1997]:

$$g = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{1.1}$$

The corresponding variational field equations reduce to a 2-dimensional system, called the Wheeler-De Witt Equation, in minisuperspace coordinates (a, ϕ)[Halliwell, 1988]. We will show that at the time of Big Bang the time coordinate transforms into space coordinate. To do so, we will build the corresponding Hilbert subspaces which defines a distinct quantum cosmology in which the parameters of the scalar potential are required to obey a particular quantization condition. we will discuss the class of two-dimensional dilaton-gravity models which are imposed by the classical and quantum constraints that arise in the canonical Hamilton formulation. Next, we will find some analytical solutions to the Wheeler De-Witt equation, which, in some domains they coincide with the solutions of classical cosmologies. The Hamiltonian constraint as a spinor wave equation in minisuperspace will be implemented. wE will also develop some spinor models in quantum cosmology as well as the spinor solutions to our model.

1.1 The Model

We start with the parameterized metric in terms of lapse function β and the scale function $\bar{R}(\beta)$ given by,

$$g = -\beta d\beta \otimes \beta + \frac{\bar{R}^2(\beta)}{[1+\frac{kr^2}{4}]^2} dx^i \otimes dx^i = -\beta d\beta \otimes \beta + \frac{\bar{R}^2(\beta)}{[1+\frac{kr^2}{4}]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{1.2}$$

where $r^2 = x^i x^i$. The above metric is degenerate on some three-dimensional hypersurface that partitions the manifold into two regions. The geometry of one region will be Lorentzian with signature $(-+++)$ and the geometry of the second region will be Euclidean with signature $(++++)$.

The next step is to find the scalar curvature and obtain Einstein field equations. To do so, let the co-frame 1-forms be calculated as:

$$e^0 = dt \qquad e^1 = \frac{R}{1+\frac{kr^2}{4}} dr \qquad e^2 = \frac{Rr}{1+\frac{kr^2}{4}} d\theta \qquad e^3 = \frac{Rr\sin\theta}{1+\frac{kr^2}{4}} d\phi. \tag{1.3}$$

The connection 1-forms with vanishing torsion can be determined by Maurer- Cartan structure equations

$$de^a + {\omega^a}_b \wedge e^b = 0. \tag{1.4}$$

Based on the co-frame 1-forms, the Maurer-Cartan structure equations may be expressed as:

$$de^1 + {\omega^1}_0 \wedge e^0 + {\omega^1}_2 \wedge e^2 + {\omega^1}_3 \wedge e^3 = \frac{\dot{R}}{R} e^0 \wedge e^1 + {\omega^1}_0 \wedge e^0 + {\omega^1}_2 \wedge e^2 + {\omega^1}_3 \wedge e^3 = 0, \tag{1.5}$$

$$de^2 + {\omega^2}_0 \wedge e^0 + {\omega^2}_1 \wedge e^1 + {\omega^2}_3 \wedge e^3 = \tfrac{\dot{R}}{R} e^0 \wedge e^2 + \tfrac{1-\frac{kr^2}{4}}{Rr} e^1 \wedge e^2 + {\omega^2}_0 \wedge e^0$$
$$+{\omega^2}_1 \wedge e^1 + {\omega^2}_3 \wedge e^3 = 0, \tag{1.6}$$

$$de^3 + {\omega^3}_0 \wedge e^0 + {\omega^3}_1 \wedge e^1 + {\omega^3}_2 \wedge e^2 =$$
$$\frac{\dot{R}}{R} e^0 \wedge e^3 + \frac{1-\frac{kr^2}{4}}{Rr} e^1 \wedge e^3 + \frac{(1-\frac{kr^2}{4})\cot\theta}{Rr} e^2 \wedge e^3 + {\omega^3}_0 \wedge e^0 + {\omega^3}_1 \wedge e^1 + {\omega^3}_2 \wedge e^2 = 0. \tag{1.7}$$

By comparing side by side, the non-vanishing connection 1-forms are identified as,

$$ {\omega^1}_0 = \tfrac{\dot{R}}{R} e^1 \qquad {\omega^2}_0 = \tfrac{\dot{R}}{R} e^2 \qquad {\omega^3}_0 = \tfrac{\dot{R}}{R} e^3 \qquad {\omega^1}_3 = -\tfrac{1-\frac{kr^2}{4}}{Rr} e^3$$
$${\omega^2}_3 = -\frac{(1-\frac{kr^2}{4})\cot\theta}{Rr} e^3 \qquad {\omega^1}_2 = -\frac{1-\frac{kr^2}{4}}{Rr} e^2. \tag{1.8}$$

Next, one can easily calculate the curvature 2-forms by using the second set of Cartan structure equation which are given by the formula

$$R^{ab} = d\omega^{ab} + {\omega^a}_c \wedge \omega^{cb}, \tag{1.9}$$

$$R^{10} = d\omega^{10} + {\omega^1}_c \wedge \omega^{c0} = \tfrac{\ddot{R}}{R} e^1 \wedge e^0 \qquad R^{20} = d\omega^{20} + {\omega^2}_c \wedge \omega^{c0} = \tfrac{\ddot{R}}{R} e^2 \wedge e^0$$
$$R^{30} = d\omega^{30} + {\omega^3}_c \wedge \omega^{c0} = \tfrac{\ddot{R}}{R} e^3 \wedge e^0 \qquad R^{13} = d\omega^{13} + {\omega^1}_c \wedge \omega^{c3} = (\tfrac{k}{R^2} + (\tfrac{\dot{R}}{R})^2) e^1 \wedge e^3$$
$$R^{23} = d\omega^{13} + {\omega^2}_c \wedge \omega^{c3} = (\frac{k}{R^2} + (\frac{\dot{R}}{R})^2) e^2 \wedge e^3. \tag{1.10}$$

After all these is done, it is easy to find the Ricci curvature one-form by the formula

$$P^a = \iota_b R^{ba}. \tag{1.11}$$

Imposing this equation, we calculate the curvature one-forms,

$$P^0 = \iota_0 R^{00} + \iota_1 R^{10} + \iota_2 R^{20} + \iota_3 R^{30} = \iota_1(\frac{\ddot{R}}{R} e^1) \wedge e^0 + \iota_2(\frac{\ddot{R}}{R} e^2 \wedge e^0) + \iota_3(\frac{\ddot{R}}{R} e^3 \wedge e^0) = 3\frac{\ddot{R}}{R} e^0,$$

$$P^1 = \iota_0 R^{01} + \iota_1 R^{11} + \iota_2 R^{21} + \iota_3 R^{31} = \iota_0(\frac{\ddot{R}}{R}) e^0 \wedge e^1 + \iota_2(\frac{k}{R^2} + (\frac{\dot{R}}{R})^2) e^2 \wedge e^1 + \iota_3(\frac{k}{R^2}$$
$$+(\frac{\dot{R}}{R})^2) e^3 \wedge e^1 = \left(\frac{\ddot{R}}{R} + \frac{2k}{R^2} + 2(\frac{\dot{R}}{R})^2\right) e^1,$$

$$P^2 = \iota_0 R^{02} + \iota_1 R^{12} + \iota_2 R^{22} + \iota_3 R^{32} = \iota_0(\frac{\ddot{R}}{R}) e^0 \wedge e^2 + \iota_1(\frac{k}{R^2} + (\frac{\dot{R}}{R})^2) e^1 \wedge e^2 + \iota_3(\frac{k}{R^2}$$
$$+(\frac{\dot{R}}{R})^2) e^3 \wedge e^2 = \left(\frac{\ddot{R}}{R} + \frac{2k}{R^2} + 2(\frac{\dot{R}}{R})^2\right) e^2,$$

$$P^3 = \iota_0 R^{03} + \iota_1 R^{13} + \iota_2 R^{23} + \iota_3 R^{33} = \iota_0(\frac{\ddot{R}}{R})e^0 \wedge e^3 + \iota_2(\frac{k}{R^2} + (\frac{\dot{R}}{R})^2)e^2 \wedge e^3$$
$$+\iota_1(\frac{k}{R^2} + (\frac{\dot{R}}{R})^2)e^1 \wedge e^3 = \left(\frac{\ddot{R}}{R} + \frac{2k}{R^2} + 2(\frac{\dot{R}}{R})^2\right) e^3.$$

The scalar curvature will eventually be calculated by the following formula

$$\mathcal{R} = \iota_b P^b. \tag{1.12}$$

As a result, one can calculate the scalar curvature as follows,

$$\mathcal{R} = \iota_0 P^0 + \iota_1 P^1 + \iota_2 P^2 + \iota_3 P^3 = \iota_0(3\frac{\ddot{R}}{R}e^0) + \iota_1(\frac{\ddot{R}}{R} + \frac{2k}{R^2} + 2(\frac{\dot{R}}{R})^2)e^1 + \iota_2(\frac{\ddot{R}}{R} + \frac{2k}{R^2}$$
$$+2(\frac{\dot{R}}{R})^2)e^2 + \iota_3(\frac{\ddot{R}}{R} + \frac{2k}{R^2} + 2(\frac{\dot{R}}{R})^2)e^3.$$

Finally the scalar curvature is calculated as

$$\mathcal{R} = 6\frac{\ddot{R}R + \dot{R}^2 + k}{R^2}. \tag{1.13}$$

1.2 Derivation of the Equations of Motion

Now that we have calculated the scalar curvature for the given metric, we need to plug it in the following action four-form[Dereli and Tucker, 1993],

$$I[g, \phi] = \int_M \left(\frac{\mathcal{R}}{2\kappa} - \frac{1}{2}g(d\phi, d\phi) - U(\phi)\right) * 1 = \int_M \Lambda. \tag{1.14}$$

Einstein field equations can be derived in the context of Lagrangian dynamics. Here, Λ takes the form,

$$\Lambda = Le^0 \wedge e^1 \wedge e^2 \wedge e^3 = LR^3 dt \wedge dx \wedge dy \wedge dz = \mathcal{L} dt \wedge dx \wedge dy \wedge dz. \tag{1.15}$$

The explicit form of Lagrangian would be ($\kappa = 1$ for simplicity):

$$L = \frac{3R\ddot{R}}{R^2} - 3\frac{\dot{R}^2}{R^2} + 6\frac{\dot{R}^2}{R^2} + 3\frac{k}{R^2} + \frac{1}{2}\dot{\phi} - U(\phi) = 3\left(\frac{\dot{R}}{R}\right)^{\cdot} + 6\frac{\dot{R}^2}{R^2} + 3\frac{k}{R^2} + \frac{\dot{\phi}^2}{2} - U(\phi). \tag{1.16}$$

Let $\mathcal{L} = R^3 L$. Therefore, the Lagrangian takes the following form below,

$$\mathcal{L} = 3R^2\ddot{R} + 3R\dot{R}^2 + 3kR + \left(\frac{\phi^2}{2} - U(\phi)\right) R^3. \tag{1.17}$$

We examine the Lagrangian 1-form given as,

$$\mathcal{L}^{'}dt = \left(-3R\dot{R}^2 + 3kR + (\frac{1}{2}\dot{\phi}^2 - U(\phi))R^3\right)dt. \tag{1.18}$$

Now, in order to obatin the equations of motion directly, I make a parameterizaton of t as a function of another evolution variable which is called lapse function. Let $t = F(\gamma), dt = F^{'}d\gamma$. After substituting dt in the Lagrangian 1-form we get,

$$\bar{\mathcal{L}}d\gamma = \left(-3\frac{R}{F'^2}\frac{dR}{d\gamma} + 3kR + (\frac{1}{2F'^2}(\frac{d\phi}{d\gamma})^2 - U(\phi))R^3\right)F^{'}d\gamma =$$

$$= \left(\frac{1}{F'}\left(-3RR^{'2} + \frac{1}{2}R^3\phi^{'2}\right) + F^{'}\left(3kR - R^3U(\phi)\right)\right)d\gamma. \tag{1.19}$$

We can obtain Einstein field equations by applying the Lagrangian equations of motion. In the chosen coordinates this becomes,

$$\frac{d}{d\gamma}\left(\frac{\partial\bar{\mathcal{L}}}{\partial x_i^{'}}\right) - \frac{\partial\bar{\mathcal{L}}}{\partial x_i} = 0. \tag{1.20}$$

Here, dynamical variables are: (R, ϕ, F) and prime notation means derivative with respect to γ. We can calculate the equations of motion,

$$\frac{d}{d\gamma}\left\{\frac{\partial}{\partial R'}\left(\frac{1}{F'}\left(-3RR^{'2} + \frac{1}{2}R^3\phi^{'2}\right) + F^{'}\left(3kR - R^3U(\phi)\right)\right)\right\} -$$

$$\frac{\partial}{\partial R}\left\{\frac{1}{F'}\left(-3RR^{'2} + \frac{1}{2}R^3\phi^{'2}\right) + F^{'}\left(3kR - R^3U(\phi)\right)\right\} = 0. \tag{1.21}$$

This will reduce to,

$$\frac{d}{d\gamma}\left\{-6\frac{RR^{'}}{F'}\right\} + 3\frac{R^{'2}}{F'} - \frac{3}{2}R^2\phi^{'2} - 3F^{'}k + 3F^{'}R^2U(\phi) = 0. \tag{1.22}$$

Note that: $(RR^{'})^{'} = R^{'2} + RR^{''}$, so that $RR^{''} = (RR^{'})^{'} - R^{'2}$,

$$-2\frac{R^{'2}}{F'} - 2\frac{RR^{''}}{F'} + \frac{R^{'2}}{F'} - \frac{1}{2}R^2\phi^{'2} - kF^{'} + F^{'}R^2U(\phi) = 0,$$

$$-2\frac{R^{'2}}{F'} - 2\left(\frac{RR^{'}}{F'}\right)^{'} + 3\frac{R^{'2}}{F'} - \frac{1}{2}R^2\phi^{'2} - kF^{'} + F^{'}R^2U(\phi) = 0.$$

Arranging the above equations, we obtain,

$$-\frac{R^{'2}}{F'} + 2\left(\frac{RR^{'}}{F'}\right)^{'} + \frac{1}{2}R^2\phi^{'2} + kF^{'} - F^{'}R^2U(\phi) = 0. \tag{1.23}$$

Variation with respect to ϕ yields,

$$\frac{d}{d\gamma}\left\{\frac{\partial}{\partial\phi'}\left(\frac{1}{F'}\left(-3RR'^2+\frac{1}{2}R^3\phi'^2\right)\right)+F'\left(3kR-R^3U(\phi)\right)\right\}-\frac{\partial}{\partial\phi}\left\{\frac{1}{F'}((-3RR'^2\right.$$

$$\left.+\frac{1}{2}R^3\phi'^2+F'\left(3kR-R^3U(\phi)\right)\right\}=\frac{d}{d\gamma}(\frac{R^3\phi'}{F'})+\frac{\partial}{\partial\phi}(F'R^3U(\phi))=0$$

Hence, the second Einstein Field equation is derived as,

$$\left(\frac{R^3\phi'}{F'}\right)'+\left(F'R^3\right)\frac{\partial U}{\partial\phi}=0. \tag{1.24}$$

The remaining equation can be obtained by varying the Lagrangian with respect to $F^{'}$,

$$\frac{d}{d\gamma}\left\{\frac{\partial}{\partial F'}\left(\frac{1}{F'}\left(-3RR^{'2}+\frac{1}{2}R^3\phi^{'2}\right)+F^{'}\left(3kR-R^3U(\phi)\right)\right)\right\}$$
$$-\frac{\partial}{\partial F}\left\{\frac{1}{F'}\left(-3RR^{'2}+\frac{1}{2}R^3\phi^{'2}\right)+F^{'}\left(3kR-R^3U(\phi)\right)\right\}=0,$$
$$\frac{d}{d\gamma}\left\{-\frac{1}{F^{'2}}\left(-3RR^{'2}+\frac{1}{2}R^3\phi^{'2}\right)+\left(3kR-R^3U(\phi)\right)\right\}=0.$$

Since the derivative of the expression with respect to γ is equal to zero, that means that the expression is a constant of motion in γ,

$$\frac{1}{F^{'2}}\left(-3RR^{'2}+\frac{1}{2}R^3\phi^{'2}\right)-\left(3kR-R^3U(\phi)\right)=constant. \tag{1.25}$$

If we let the lapse function $F^{'}$ be equal to 1 we can obtain the Einstein field equations as presented below after substitution. Before that, we may express $\mathcal{L}^{'}dt$ as,

$$\mathcal{L}^{'}dt=\mathcal{L}^{'}F^{'}d\gamma=L^{'}F^{'}d\gamma L^{'}d\gamma. \tag{1.26}$$

This allows us to write the derivative with respect to γ as the derivative with respect to time t, given by,

$$-\dot{R}^2+2\left(R\dot{R}\right)^{\cdot}+\frac{1}{2}R^2\phi^{'2}+k-R^2U(\phi)=0.$$

Adding and substracting $\dot{R}^2$ we get,

$$-2\dot{R}^2+2R\ddot{R}+3\dot{R}^2+\frac{1}{2}R^2\phi^{'2}+k-R^2U(\phi)=0.$$

This is equal to,

$$2\left(\frac{\dot{R}}{R}\right)^{\cdot}+3\left(\frac{\dot{R}}{R}\right)^2+\frac{k}{R^2}=-\frac{\dot{\phi}^2}{2}+U(\phi). \tag{1.27}$$

For equation (2) we have,

$$\left(R^3\dot{\phi}\right)^{\cdot}+R^3\frac{\partial U}{\partial\phi}=0.$$

In terms of the previous equation, one has

$$\ddot{\phi}+3\frac{\dot{R}}{R}\dot{\phi}+\frac{\partial U}{\partial\phi}=0. \tag{1.28}$$

For equation (1.25) we get:

$$-3RR^{'2} + \frac{1}{2}R^3\phi^{'2} - \big(3kR - R^3U(\phi)\big) = constant$$

After dividing by R^3 and letting the constant go to zero, the above equation reduces to:

$$3\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}\right] = \frac{\dot{\phi}^2}{2} + U(\phi) \qquad (1.29)$$

The equations (1.27), (1.28), (1.29) are so called the Einstein field equations for the given cosmological model we have chosen. Our next step is to solve the above differential equations, or at least numerically to see how the universe evolves.

1.3 Canonical Transformation of Field Equations.

For $-\infty < \phi < \infty$, $0 \leq R < \infty$, $\alpha^2 = \frac{3}{8}$, we introduce a new set of coordinates X, Y in terms of R, and ϕ we have [Dereli and Tucker, 1993],

$$X = R^{\frac{3}{2}}\cosh\alpha\phi$$

$$Y = R^{\frac{3}{2}}\sinh\alpha\phi. \qquad (1.30)$$

The derivative of X and Y with respect to t is found as,

$$\dot{X} = \frac{3}{2}R^{\frac{1}{2}}\dot{R}\cosh\alpha\phi + \alpha\dot{\phi}R^{\frac{3}{2}}\sinh\alpha\phi$$

$$\dot{Y} = \frac{3}{2}R^{\frac{1}{2}}\dot{R}\sinh\alpha\phi + \alpha\dot{\phi}R^{\frac{3}{2}}\cosh\alpha\phi \qquad (1.31)$$

As a result, one can notice that:
$\dot{X}^2 - \dot{Y}^2 = \frac{9}{4}R\dot{R}^2 - \alpha^2\dot{\phi}^2R^3 = \frac{9}{4}R\dot{R}^2 - \frac{3}{8}\dot{\phi}^2R^3$ and
$X^2 - Y^2 = R^3$.
Now let us return back to the following Lagrangian one-form,

$$\mathcal{L}' dt = \left(-3R\dot{R}^2 + 3kR + (\frac{1}{2}\dot{\phi}^2 - U(\phi))R^3\right) dt = (\frac{9R\dot{R}^2}{4(-2)\frac{3}{8}} + \frac{\frac{9}{8}kR}{\frac{3}{8}} + \frac{\frac{3}{8}\dot{\phi}^2}{\frac{6}{8}}R^3$$

$$-U(\phi)R^3)dt = \left(\frac{9R\dot{R}^2}{4(-2)\alpha^2} + \frac{\frac{9}{8}k(X^2 - Y^2)^{\frac{1}{3}}}{\alpha^2} + \frac{\alpha^2\dot{\phi}^2}{2\alpha^2}R^3 - U(\phi)(X^2 - Y^2)\right) dt.$$

Multiplying both sides by $-2\alpha^2$ we get,

$$-2\alpha^2\mathcal{L}' dt = \left(\frac{9R\dot{R}^2}{4} - \alpha^2\dot{\phi}^2 R^3 - \frac{9}{4}k(X^2 - Y^2)^{\frac{1}{3}} + 2\alpha^2 U(\phi)(X^2 - Y^2)\right) dt. \quad (1.32)$$

This is the so called Lagrangian one-form and it may be expressed as,

$$-2\alpha^2\mathcal{L}' dt = \left(\dot{X}^2 - \dot{Y}^2 - \frac{9}{4}k(X^2 - Y^2)^{\frac{1}{3}} + 2\alpha^2(X^2 - Y^2)U(\phi)\right) dt \quad (1.33)$$

To make calculations easier, we consider cosmologies with $k = 0$ and a potential of the form:

$$2\alpha^2(X^2 - Y^2)U(\phi) = a_1X^2 + a_2Y^2 + 2bXY.$$

In this representation, our Lagrangian takes the form,

$$\mathcal{L}' = \dot{X}^2 - \dot{Y}^2 - a_1X^2 - a_2Y^2 - 2bXY. \quad (1.34)$$

Sometimes this is called Dereli-Tucker (1993) Lagrangian function.
It would be useful to use the following identity for the sake of simplicity,

$$\cosh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}.$$

The above property of hyperbolic functions can be used to modify the potential U into a more useful and fancy form,

$$2\alpha^2(X^2 - Y^2)U(\phi) = a_1X^2 + a_2Y^2 + 2bXY = a_1X^2 - a_1Y^2 + a_1Y^2 + a_2Y^2$$

$$+2bR^3 \sinh\alpha\phi\cosh\alpha\phi. \quad (1.35)$$

Therefore, this will take the form,

$$2\alpha^2R^3U(\phi) = a_1R^3 + (a_1 + a_2)R^3\left(\frac{1}{2}\cosh 2\alpha\phi - \frac{1}{2}\right) + 2bR^3\sinh\alpha\phi\cosh\alpha\phi.$$

The potential $U(\phi)$ reduces to,

$$U(\phi) = \frac{a_1 - a_2}{4\alpha^2} + \frac{a_1 + a_2}{4\alpha^2}\cos 2\alpha\phi + \frac{b}{2\alpha^2}\sinh 2\alpha\phi. \tag{1.36}$$

Let us define the physical parameters as below[Dereli and Tucker, 1993],

$$\lambda = U(\phi = 0) = \frac{a_1 - a_2}{4\alpha^2} + \frac{a_1 + a_2}{4\alpha^2} = \frac{a_1}{2\alpha^2},$$

$$m^2 = \frac{\partial^2 U}{\partial \phi^2}|_{\phi=0} = a_1 + a_2. \tag{1.37}$$

As it can be seen, λ is the potential of the minisuperspace where ϕ vanishes. In this context, the potential $U(\phi)$ at any value arbitrary in ϕ will be,

$$U(\phi) = \frac{a_1 - a_2}{4\alpha^2} + \frac{a_1 + a_2}{4\alpha^2}(2\cosh^2\alpha\phi - 1) + \frac{b}{2\alpha^2}\sinh 2\alpha\phi =$$

$$= \frac{a_1}{2\alpha^2} + \frac{a_1 + a_2}{2\alpha^2} + \frac{b}{2\alpha^2}\sinh 2\alpha\phi.$$

As a result, one has,

$$U(\phi) = \lambda + \frac{m^2}{2\alpha^2}\sinh^2\alpha\phi + \frac{b}{2\alpha^2}\sinh 2\alpha\phi. \tag{1.38}$$

This choice of potential besides the sinh-Gordon scalar interaction, includes another term which breaks the symmetry of U under $\phi \to -\phi$ and this term is responsible for the signature-changing properties of the solutions. The potential (1.38) has a minimum value at $\phi = -\frac{1}{2\alpha}\tanh^{-1}\left(\frac{2b}{m^2}\right)$ and its value for $|\frac{2b}{m^2}| < 1$ is,

$$\lambda + 4m^2\alpha^2\left(\sqrt{1 - \frac{4b^2}{m^4}} - 1\right). \tag{1.39}$$

Below, in Figure 1.1, we plot the graph for the potential as a function of ϕ for the eternal cosmology with a single transition from a flat Euclidean metric to a flat Lorentzian metric at $\beta = 0$.

Now, let $\xi = \begin{bmatrix} X \\ Y \end{bmatrix}$. We require that the 'zero-energy' condition to be satisfied. We know that,

$$\dot{X}^2 - \dot{Y}^2 = a_1X^2 + a_2Y^2 + 2bXY.$$

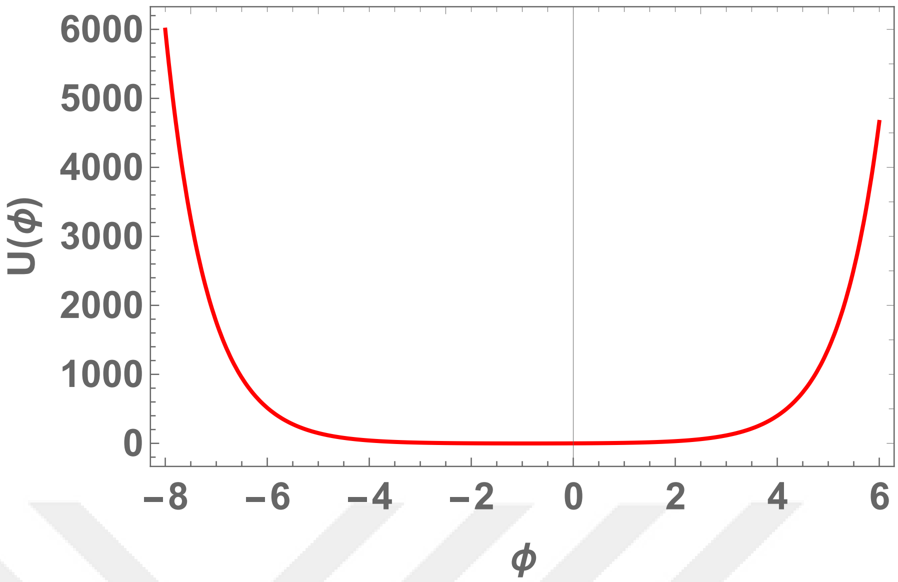


Figure 1.1: Potential versus scalar field for $m^2 = 5$, $b = 2$, and $\lambda = 0$.

The expression above can be written as a symplectic transformation in the of matrix multiplication,

$$\dot{\xi}^T J \dot{\xi} = \xi^T J M \xi,$$

where M=$\begin{pmatrix} a_1 & b \\ -b & -a_2 \end{pmatrix}$, and J=$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Before we give the solution, we first make the following transformation of $\xi(t)$,

$$\xi(t) = S\alpha(t).$$

So, we require solutions for α in terms of eigenvalues of matrix M from the obtained equation,

$$\dot{\alpha}^T J \dot{\alpha} = \alpha^T J M \alpha.$$

The solutions may be expressed as,

$$\alpha(t) = A \begin{pmatrix} e^{\sqrt{\lambda_+}t} & 0 \\ 0 & e^{\sqrt{\lambda_-}t} \end{pmatrix} + B \begin{pmatrix} e^{-\sqrt{\lambda_+}t} & 0 \\ 0 & e^{-\sqrt{\lambda_-}t}. \end{pmatrix}$$

The eigenvalues of the matrix M can be calculated by the following equation,

$$\lambda_{\pm} = \frac{TrM}{2} \pm \sqrt{\left(\frac{TrM}{2}\right)^2 - DetM} = \frac{a_1 - a_2}{2} \pm \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 - (b^2 - a_1 a_2)}$$

As a result, the eigenvalues $\lambda_\pm$ are,

$$\lambda_\pm = 2\lambda\alpha^2 - \frac{m^2}{2} \pm \frac{1}{2}\sqrt{m^4 - 4b^2}. \tag{1.40}$$

Therefore, the implicit form of the solutions for α are,

$$\alpha_1(t) = A_1 e^{\sqrt{\lambda_+ t}} + B_1 e^{-\sqrt{\lambda_+ t}},$$

$$\alpha_2(t) = A_2 e^{\sqrt{\lambda_- t}} + B_2 e^{-\sqrt{\lambda_- t}},$$

If we take $\dot{\alpha}_1(0) = \dot{\alpha}_2(0) = 0$ we obtain,

$$\sqrt{\lambda_+} A_1 e^{\sqrt{\lambda_+ t}} - \sqrt{\lambda_+} B_1 e^{-\sqrt{\lambda_+ t}} = 0,$$

$$\sqrt{\lambda_-} A_2 e^{\sqrt{\lambda_- t}} - \sqrt{\lambda_-} B_2 e^{-\sqrt{\lambda_- t}} = 0.$$

It is inferred that $A_1 = B_1$, and $A_2 = B_2$. Hence, the solutions will be,

$$\alpha_1(t) = 2A_1 \cosh \sqrt{\lambda_+} t$$

$$\alpha_2(t) = 2A_2 \cosh \sqrt{\lambda_-} t$$

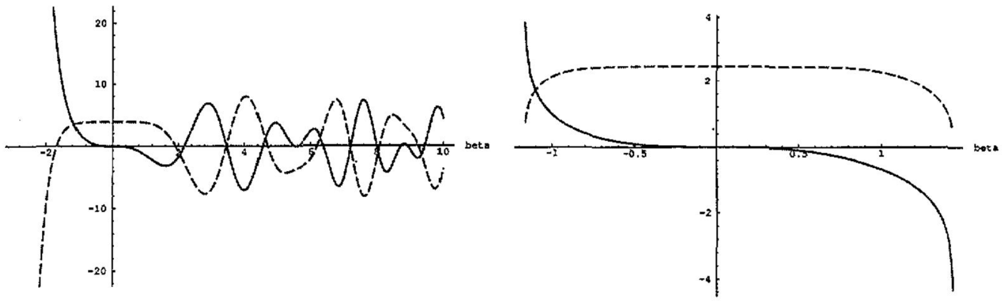
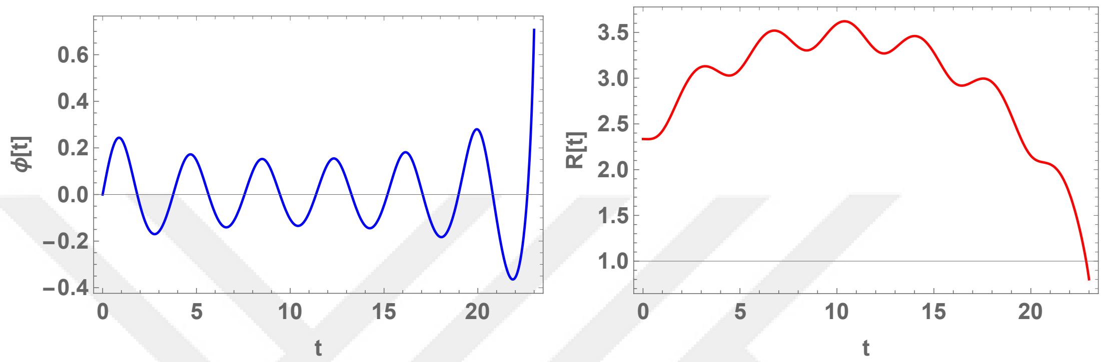


Figure 1.2: Potential versus scalar field for $m^2 = 5$, $b = 2$, and $\lambda = 0$ [Dereli and Tucker, 1993].

1.3.1 Numerical calculations for the Einstein field equations for our model

So far, we have obtained a potential for model of our choice. We have performed some numerical calculations to see how $R(t)$ and $\phi(t)$ change with time. We can plot their behavior in the figure 1.3. I have numerically solved the two coupled



(a) (b)

Figure 1.3: (a) the oscillating ϕ, (b) the behavior of the scale factor "R".

differential equations (1.27), and (1.28). The potential used is the one in equation (1.38).

Table 1.1: The constants and initial conditions assumed while doing the above numerical calculations.

Constants	Initial conditions
m=1	$\phi(t=0)$=0
b=1	$\dot{R}(t=0)$
λ=0	R(0)=2.3347
κ=0	$\dot{\phi}(t=0) = \frac{1}{2.3347}$

1.3.2 Comment

It can be easily seen that the singular behaviour of the scalar field at the birth and death of the universe is reflected in the behaviour of the scalar curvature $\mathcal{R}$.

Each solution describes a geometry in which the covariant metric tensor smoothly undergoes a transition from a Euclidean signature $(++++)$ to a Lorentzian signature $(-+++)$. This conclusion is equally equivalent for the case where $k=0$. The derivations and results obtained so far, will be used in the next chapter and we will reveal the importance of these conclusion in quantum cosmology.

Chapter 2

COSMOLOGICAL MODEL WITH SCALAR FIELD AND SPINOR SOLUTIONS IN QUANTUM COSMOLOGY

In this chapter, we will discuss the 2-dimensional Wheeler-DeWitt equation as the first step into quantum cosmology. In order to have a more complete theory of our universe, one needs also to consider the quantum effects, which would be hard to be implemented in the macroscopic scales. It has been observed mathematically that when the universe is small, the scalar field provides an effective negative pressure which prevents the universe from getting back to the big crunch or so-called singularity by bouncing and followed by another expansion area. Many research has been done in this model where we have a fractal set of perpetually bouncing universe. In the appendix B, I gave a short derivation of PU operators, and one can easily recognize the resemblance between this work and the coordinate transformations we made in the Lagrangian obtained in (1.32). One of the most essential goals in quantum cosmology is to find a wave function for the universe. In the first part of this chapter, we will discuss The No-Boundary Proposal[Hawking, 1993] which is one of the most outstanding achievements from the collaboration of Stephen Hawking and James Hartle. This proposal has its own cons, but, first we will see its implications. After that, we will derive the two-dimensional WDheeler-DeWitt equation, which will be a Klein-Gordon type in nature[Önder and Tucker, 1993]. We know the problem that arises in the Klein-Gordon equation, which is the case with positive and negative probability densities. In order to avoid this discrepancy, we search for a spinor model in this minisuperspace model, and we will make sure that the positive probability density is obtained. In a two-dimensional space, the WDW equation coincides with the Klein-Gordon equation, and the spinor model I will discuss here can be easily iterated into the KG equation. At the end of this chapter, the classical solutions of the Dirac equation will be obtained.

2.1 The No-Boundary Proposal

The history of Physics recognizes a lot of triumphs, and many more developments are being published. However, there is one problem where all laws of Physics up to date cease to explain what happens at singularity. The main issue here is that the boundary conditions here are insufficient to determine the evolution of the universe. One way to deal with this problem is to adopt the Euclidean approach and evaluate the path integral over positive definite metrics. This way, there will be no need to specify these boundary conditions[Misner, 1969]. It was noticed that this point of view would be able to explain why the cosmological constant is close to zero, which in turn suggests that our spatially flat universe was in thermal equilibrium at early times[Vilenkin, 1995]. In this thesis, we will not talking about Boltzmann brains and how this model cannot overcome this trouble. If we extrapolate back in time, we will get a universe that is shrinking and getting smaller and smaller. It is natural ta ask about the quantum effects that arise when the universe is smaller than the size of an atom. One of the first attempts to build a cosmology based on Quantum Theory was pioneered by Stephen Hawking, Thomas Hertog, and James Hartle and their work was named, No Boundary proposal[Hartle and Hawking, 1983]. This idea was able to explain the reason why there is an arrow of time, the possibilities for a multiverse, and what is more important, dealing with the Big Bang singularity. They worked out the physical conditions that the early universe must satisfy if spacetime had no boundary in the past. The No Boundary Proposal (NBS) says that when we go back towards the beginning of our universe, the spacetime becomes fuzzy and cap off, somewhat like the North Pole on the surface of the Earth. We discussed this signature dynamics in the first chapter but in the No Boundary Proposal, this phenomenon is treated a different way. In the No Boundary state there is a smooth finite surface that does not have a single point of origin. Simply said, it treats the universe quantum mechanically! The starting point is based on the Richard Feynman's sum-over-histories formulation of Quantum Theory[Cheng, 1972]. A path

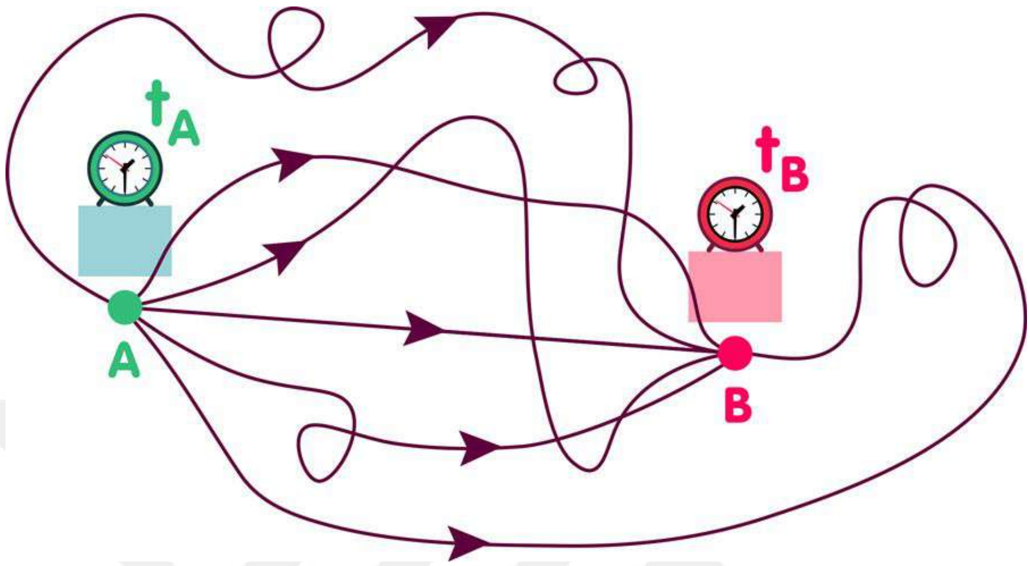
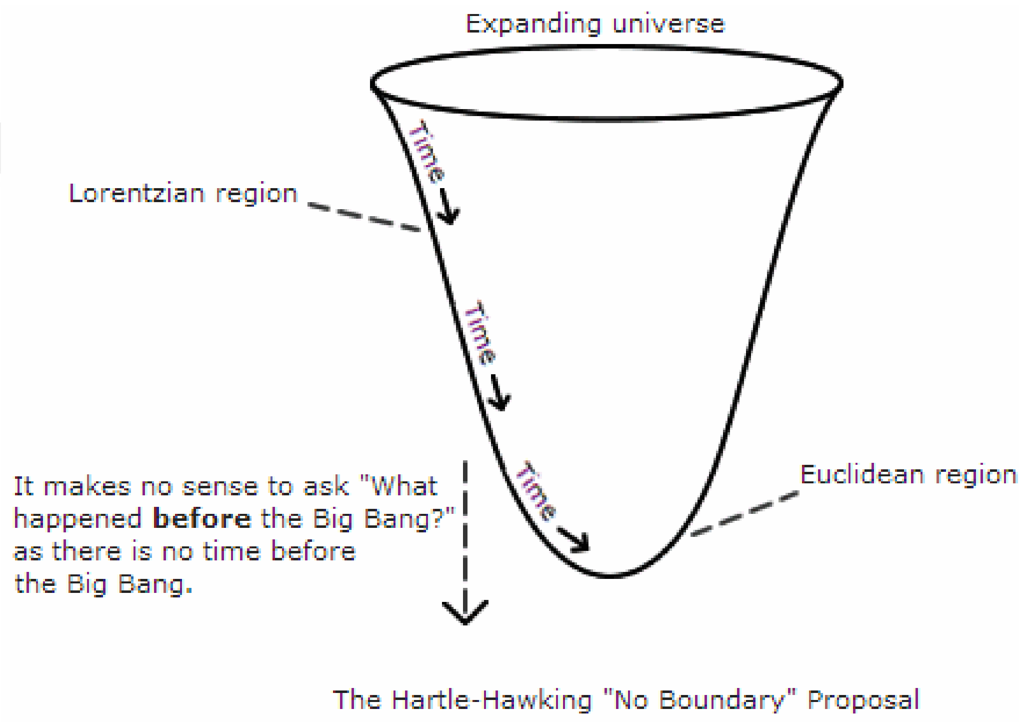


Figure 2.1: The path integral formulation of quantum theory. it considers all possibilities for the particle travelling from A to B and the path with the highest possibility is the classical path.

integral assigns a weighting to different possible histories of the universe. The good part is that the No Boundary wave function describes probabilities to different kinds of universes, or in another word a multiverse consisting of an ensemble of different universes. The NBP is a selection principle that selects the subset of histories of the universe that closed off in the past and therefore allowed us to determine which histories to the universe have the highest contribution to the classical history. The results of their pioneering work revealed that the most significant contribution would come from a geometry that is Euclidean on the bottom of the shuttlecock and an expanding de-Sitter universe on the top of the shuttlecock! Here, Euclidean means that the geometry has four spacetime dimensions and no time dimension, whereas Lorentzian means that the geometry has three space dimensions and one-time dimension. Consider the configuration or a universe (h_{ij}, ϕ). Mathematically, the No Boundary condition is expressed as:

$$\Psi[h_{ij}] = \int_{NB} \delta g_{\mu\nu} \delta\phi \; e^{-I_E[g_{\mu\nu},\phi/]\hbar} \tag{2.1}$$

As it can be seen, the No Boundary wave function describes different probabilities to different histories of the universe by associating each history to a geometric construction, the same as the shuttlecock picture. The No Boundary wave function also



(a)

Figure 2.2: The shuttlecock picture represent the No Boundary Proposal where it can be seen that the geometry is Euclidean on the bottom of the shuttlecock.

describes an ensemble of different histories where the past and the future are both probabilistic, and it gives us an amplitude. There is no notion of time at this level. There is a special condition that asserts that the histories contribute to the path integral that has no other boundary except the one at which we are evaluating the wave function. The classical evolution emerges from the No Boundary wave function when the universe gets sufficiently large. One of the most exciting aspects that this idea suggests is that the only way to get a classical universe with a deterministic notion of time is via a period of inflation. The possibilities for different histories might be grounds for the tiny irregularities in the cosmic microwave background or abbreviated as CMB.

2.2 *Quantisation of the Two Dimensional Dilaton-Gravity Models*

To begin with, let us consider the following Lagrangian density [Önder and Tucker, 1994],

$$\mathcal{L}(x,\dot{x},x',y,\dot{y},y') = \frac{1}{\mathcal{N}}\left(\dot{x}^2-\dot{y}^2+2\mathcal{V}(x'\dot{x}-y'\dot{y})+\mathcal{V}'(x+y)(\dot{x}-\dot{y})\right)+\frac{\mathcal{V}^2}{\mathcal{N}}(x'^2-y'^2)$$
$$+\frac{\mathcal{V}\mathcal{V}'}{\mathcal{N}}(x+y)(x'-y')+\mathcal{N}\left(\frac{(x'-y')^2}{c^2(x^2-y^2)(x-y)^2}+c(x^2-y^2)U(x,y)\right)-$$
$$-\mathcal{N}'\frac{x'-y'}{c^2(x^2-y^2)(x-y)}. \tag{2.2}$$

Based on chart coordinate {s,t}, we have $x' = \frac{\partial}{\partial s}$ and $\dot{x} = \frac{\partial}{\partial t}$. Letting all factors to be independent of coordinate s in the Gauge $\mathcal{N}=1$ and $\mathcal{V}=0$ we get the following Lagrangian density,

$$\mathcal{L}(x,\dot{x},x',y,\dot{y},y') = \dot{x}^2-\dot{y}^2+c(x^2-y^2)U(x,y). \tag{2.3}$$

Now we need to do the following canonical transformation $(\Psi,\mathcal{X}) \longrightarrow (x,y) \longrightarrow (X,Y)$ such that,

$$x = \frac{1}{2\sqrt{c}}(\frac{e^{c\mathcal{X}}}{\mathcal{X}}+e^{-c\Psi}),$$
$$y = \frac{1}{2\sqrt{c}}(\frac{e^{c\mathcal{X}}}{\mathcal{X}}-e^{-c\Psi}),$$
$$X = x+\frac{\alpha}{2\Lambda_0\sqrt{c}},$$
$$Y = y-\frac{\alpha}{2\Lambda_0\sqrt{c}}. \tag{2.4}$$

The chosen canonical transformations suggest that the potential is of the form,

$$U(\Psi) = \Lambda_0+\alpha e^{c\Psi}. \tag{2.5}$$

From here we can infer that,

$$x-y = \frac{1}{2\sqrt{c}}(\frac{e^{c\mathcal{X}}}{\mathcal{X}}+e^{-c\Psi})-\frac{1}{2\sqrt{c}}(\frac{e^{c\mathcal{X}}}{\mathcal{X}}-e^{-c\Psi}) = \frac{e^{-c\Psi}}{\sqrt{c}}.$$

In terms of the previous notation, one has,

$$X+Y = x+\frac{\alpha}{2\Lambda_0\sqrt{c}}+y-\frac{\alpha}{2\Lambda_0\sqrt{c}} = x+y,$$

$$\dot{X} = \dot{x}, \qquad \dot{Y} = \dot{y},$$

$$x-y+\frac{\alpha}{\sqrt{c}\Lambda_0} = x+\frac{\alpha}{2\sqrt{c}\Lambda_0}-(y-\frac{\alpha}{2\sqrt{c}\Lambda_0}) = X-Y. \tag{2.6}$$

Given all this information, we get the following Lagrangian density,

$$\mathcal{L} = \dot{x}^2-\dot{y}^2+c\Lambda_0(x^2-y^2)(1+\frac{\alpha}{\sqrt{c}\Lambda_0(x-y)}) = \dot{x}^2-\dot{y}^2+c\Lambda_0(x+y)(x-y+\frac{\alpha}{\sqrt{c}\Lambda_0}). \tag{2.7}$$

All the above expression reduces to,

$$\mathcal{L} = \dot{X}^2-\dot{Y}^2+4\omega^2(X^2-Y^2), \tag{2.8}$$

where $4\omega^2 = \Lambda_0 c$.

2.2.1 Pais–Uhlenbeck Oscillator

In this part we give a brief introduction to PU oscillators as well its consequences in cosmology. These higher-derivative theories are in many cases ghost-ridden, which may break unitarity, and that would be a big trouble for our cosmological model. However, under some circumstances there are higher derivative systems, the unitary can be maintained with "ghosts" . [Smilga, 2006]. Further research has been done in [Robert and Smilga, 2008]. To begin with, let the Lagrangian be,

$$L = \frac{1}{2}[\ddot{q}^2-(\Omega_1^2+\Omega_2^2)\dot{q}^2+\Omega_1^2\Omega_2^2q^2]+(nonlinear \ \ terms), \tag{2.9}$$

where we assume that $\Omega_1 \neq \Omega_2$. After some proper coordinate transformation, one obtains (see Appendix B),

$$H = \frac{P_1^2+\Omega_1^2X_1^2}{2}-\frac{P_2^2+\Omega_2^2X_2^2}{2}. \tag{2.10}$$

As it can be seen, there is a similarity between equation (2.10)[Smilga et al., 2009] and the Hamiltonian of the system we have found for our cosmological model,

$$H = p_X^2-p_Y^2+4\omega^2(X^2-Y^2). \tag{2.11}$$

In quantum mechanics we know that,

$$p_i = \frac{1}{i}\frac{\partial}{\partial x_i}.$$

Explicitly, the Hamiltonian is given by,

$$H = (-\partial_X^2 + \partial_Y^2) + W, \tag{2.12}$$

where $W = 4\omega^2(X^2 - Y^2)$. Since we are taking $\mathbb{R}^2$ as a minisuperspace with global coordinates {X, Y} with the Wheeler-DeWitt metric,

$$\mathcal{G} = \partial X \otimes \partial X - \partial Y \otimes \partial Y. \tag{2.13}$$

We can take

$$-\partial_X^2 + \partial_Y^2 = \hat{\Box}, \tag{2.14}$$

Here $\hat{\Box}$ is D'Alambert operator in the minisuperspace defined by the Wheeler De Witt metric as given above. The equation found resemble Klein Gordon equation and we can write it in differential form as below[Dereli et al., 1994],

$$d * d\Psi - W * \Psi = 0. \tag{2.15}$$

2.3 Wheeler-DeWitt Equation and Spinor Solutions

If one wants to understand the origins of the universe, one has to go back to the Planck era, when the universe was highly tiny[Hawking, 1985]. In this limit, there is no way to ignore the gravitational effects, and therefore, we need to include the gravitational field in the path integral[Chaudhuri and Minic, 1993]. In the case of the de Sitter metric, it is possible to rotate the time coordinate to imaginary values and a signature dynamics occurs where the metric becomes Euclidean from Lorentzian. Simply said, the Wheeler-DeWitt equation is the analog of the Schroedinger[Shestakova, 2018]. In the neighborhood of the surface "S" the metric of spacetime can be written locally in the (3+1) form,

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j, \tag{2.16}$$

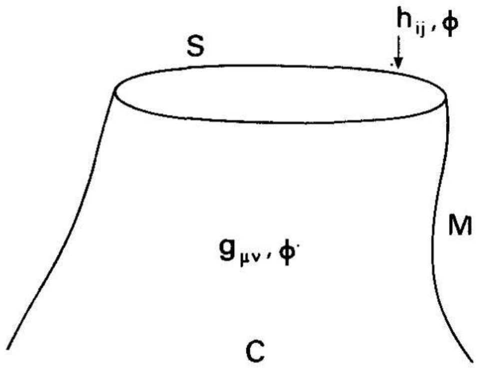


Figure 2.3: The three-metrics h_{ij} determines where S fits into the four-manifold M with metric $g_{\mu\nu}$. As a result, the wave function Ψ will be a functional only of the three-metric h_{ij} and the matter field configurations ϕ.

where $x^i(i = 1, 2, 3)$ are the coordinates on a three dimensional manifold S and indices i and j are lowered with the three-metric h_{ij}. The quantum state of a gravitational system can be described by a wavefunction Ψ which is a functional of the three metric h_{ij} and the matter field which in turn can be either scalar field or any other possible field. Here, Ψ is a function on the infinite dimensional manifold M called superspace which consists of all positive- definite three-metrics $h_{ij}(x)$ and the matter fields $\phi(x)$ on S[Boehmer et al., 2010].

In order to obtain the Wheeler-DeWitt equation, we use the following action form based on (2.13)[Hawking, 1993]:

$$I[g_{\mu\nu}, \phi] = \frac{m_P^2}{16\pi} \int d^3x d\tau h^{\frac{1}{2}} N \left[K_{ij}K^{ij} - K^2 -^3 R(h^{ij}) + 2\Lambda - 16\pi m_P^{-2} L(\phi), \right] \quad (2.17)$$

where the second fundamental form is,

$$K_{ij} = \frac{1}{N}\left[\frac{1}{2}\frac{\partial h_{ij}}{\partial \tau} + N_{(i|j)}\right]. \quad (2.18)$$

Since the wave function does not depend on time, by varying N at the surface one gets,

$$\int_{\mathcal{C}} d[g_{\mu\nu}]d[\phi]] \left[\frac{\delta I}{\delta N}\right] e^{-I[g_{\mu\nu},\phi} \quad (2.19)$$

The Hamiltonian constraint for general relativity is the field equation and it is given by,

$$H = -h^{\frac{1}{2}}(K_{ij}K^{ij} - K^2 -^3 R - 2\Lambda + 8\pi m_P^{-2} T_{nn}) = 0, \quad (2.20)$$

where T_{nn} is the Euclidean stress energy tensor. We require that the wavefunction Ψ obeys the zero-energy Schrödinger equation

$$\hat{H}\Psi = 0. \tag{2.21}$$

Because the lapse N and shift N_i are independent Lagrange multipliers, one can split the above equation into the Wheeler-DeWitt equation,

$$\hat{H}F = 0, \tag{2.22}$$

and the momentum constraints but the latest one is not important for us in this stage. However, it is very difficult to calculate the wavefunction of the universe for all the dynamical variables so we will mainly focus on minisuperspace models in which the infinite number of degrees of freedom of the gravitational and matter fields are restricted to a finite number. In this section we will show how spinor solutions of Wheeler-DeWitt equations admit Hermite polynomial solutions for some specific cases. Using the expression for the canonical momenta and expressing the second fundamental form in terms of the functional operator $\frac{\delta}{\delta h_{ij}}$, one obtains the following equation for the Wheeler-DeWitt equation,

$$\left[-G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - h^{\frac{1}{2}}\left({}^3R(h) - 2\Lambda + 8\pi m_P^{-2} T_{nn}(\frac{\delta}{\delta\phi},\phi)\right)\right]\Psi[h_{ij},\phi] = 0, \tag{2.23}$$

where m_P is the Planck's mass and G_{ijkl} is the metric on superspace with signature (-+++++). It is worth to state that the Wheeler-DeWitt equation is a hyperbolic equation on the superspace where the time coordinate is $h^{1/2}$.

2.3.1 The Hawking Massive Scalar Model

Stephen Hawking was able to derive the Wheeler-DeWitt equation for a massive scalar field model and the equation in (p+1) dimensions is,

$$\frac{1}{2}\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left[a^p\frac{\partial}{\partial a}\right] - \frac{\partial^2}{a^2\partial\phi^2} + V(a,\phi)\right]\Psi(a,\phi) = 0. \tag{2.24}$$

The configuration space can be divided into two regions based on the sign of V. Although V(potential) is generally a function of ϕ, here we also include the scale factor "a" for some reasons that are related to the fact of separating variables when one tries to solve the Wheeler-DeWitt equation, even though it seems hard not to say impossible to solve it[Hawking, 1984]. Notice that for $p = 0$ and for some smart coordinate change, the Wheeler-DeWitt equation will be reduced to the equation (2.10). The region is Euclidean for $V < 0$ and Lorentzian for $V > 0$. If $m\phi >> 1$ the wave function for the Euclidean solution of the field equation is,

$$\Psi(a) = N_0 e^{\frac{1}{3m^2a^2}(1-(1-m^2a^4)^{\frac{3}{2}})} \tag{2.25}$$

2.3.2 The Clifford algebra of minisuperspace

The model discussed can be presented by a (1,1) Clifford bundle structure[1]. The 1-forms dX and dY are represented by matrices which satisfy the following properties:

- dX $\vee$ dX= 1
- dY $\vee$ dY= -1
- dX $\vee$ dY+dY $\vee$ dX=0

We introduce the potentials V_1 and V_2 such that:

$$V_1 = 2i\omega(-Y + XdX \vee dY),$$

$$V_2 = 2i\omega(Y + XdX \vee dY),$$

We can observe that,

$$V_1 \vee V_2 = -4\omega^2(-Y + XdX \vee dY)(Y + XdX \vee dY) = -4\omega^2(-Y^2 + XYdX \vee dY -$$

$$XYdX \vee dY + X^2 dX \vee dY \vee dX \vee dY) = -4\omega^2(X^2 - Y^2).$$

Also letting,

$$\mathcal{D} = (d - \delta),$$

we get:

$$\mathcal{D}^2 = (d - \delta)(d - \delta) = d^2 + \delta^2 - d\delta - \delta d = -(d\delta + \delta d).$$

As it can be seen, $\mathcal{D}^2$ is the Laplace-Beltrami operator. Let Φ be a section of the Clifford bundle mentioned above, such that,

$$\Phi = \Phi_0 + \Phi_1 dX + \Phi_2 dY + \Phi_{12} dX \vee dY,$$

From $\mathcal{D}^2 - W$ we can obtain the following equation,

$$\mathcal{D}^2 - V_1 \vee V_2 = (\mathcal{D} + V_2)(\mathcal{D} + V_1).$$

Therefore:

$$\mathcal{D}\Phi + V_1 \vee \Phi = 0. \tag{2.26}$$

One can use equation (2.20) to construct the following Dirac type equation and the Clifford potential is chosen wisely so that to satisfy our previously chosen potential. Here I will introduce such two by two matrices that satisfy the above properties for the 1-forms. This will be equivalent to a 2-D Dirac equation,

$$i\gamma^a \nabla_a \Psi = m\Psi.$$

The gamma matrices here are two: $\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The basis of the Clifford algebra Cl(1,1) includes also the identity matrix and $\gamma^2 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A 2-spinor can be decomposed into its chiral components[Benn and Tucker, 1987]:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left(\frac{1+\gamma^2}{2}\right)\Psi + \left(\frac{1-\gamma^2}{2}\right)\Psi.$$

If one is interested and curious about how we reached from from the WDW equation into this form, it is easily observed that by iterating the Dirac equation, one can obtain the Klein-Gordon equation as follows,

$$(i\gamma^a\nabla_a - m)(i\gamma^a\nabla_a + m)\Psi = 0,$$

$$-(\gamma\nabla)^2 - m^2 = 0,$$

$$-(\Delta^2 + m^2) = 0.$$

By applying the above Dirac type equation one can get,

$$\left(i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_Y + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_X + \begin{pmatrix} 2\omega(X-Y) & 0 \\ 0 & 2\omega(X+Y) \end{pmatrix} \right) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0.$$

As a result, the coupled system of differential equations is:

$$\begin{cases} \partial_\xi F - 2i\omega\xi H = 0 \\ \partial_\eta H + 2i\omega\eta F = 0 \end{cases} . \tag{2.27}$$

If H, and F are solutions of the coupled system of differential equations, then the linear combination of both of them $U = \frac{H+F}{2}$ and $V = \frac{H-F}{2}$ will also be a solution. Conversely, $H = U+V$ and $F = U-V$ will also be a solution. Next, the requirement $\hat{H}F = 0$ indicates that:

$$(-\partial_X^2 + \partial_Y^2)F - 4\omega^2(X^2 - Y^2)F = 0. \tag{2.28}$$

After separating X and Y, we obtain:

$$\partial_X^2 F(X,Y) - 4\omega^2 X^2 F(X,Y) = \partial_Y^2 F(X,Y) - 4\omega^2 Y^2(X,Y), \tag{2.29}$$

Let $F = A(X)B(Y)$

$$\partial_X^2 A(X)B(Y) - 4\omega^2 X^2 A(X)B(Y) = \partial_Y^2 A(X)B(Y) - 4\omega^2 Y^2 A(X)B(Y),$$

$$B(Y)\partial_X^2 A(X) - 4\omega^2 X^2 A(X)B(Y) = A(X)\partial_Y^2 B(Y) - 4\omega^2 Y^2 A(X)B(Y). \tag{2.30}$$

Dividing both sides by $A(X)B(Y)$, one has,

$$\frac{\partial_X^2 A(X)}{A} - 4\omega^2 X^2 = \frac{\partial_Y^2 B(Y)}{B} - 4\omega^2 Y^2. \tag{2.31}$$

Let $\frac{\partial_Y^2 B(Y)}{B} - 4\omega^2 Y^2 = K^2$. The above differential equation is solved in a very straightforward way and the solution will be expressed in terms of Hermite polynomials.

$$\partial_X^2 A(X)A - 4\omega^2 X^2 A = kA.$$

This form has solutions similar to the case of the harmonic oscillator in one dimension. Letting $z_1 = \sqrt{2\omega}$ the solution includes a Hermite polynomial,

$$A(X) = ce^{-\frac{z_1^2}{2}} H_n(z_1).$$

Similarly, let $\frac{\partial_Y^2 B(Y)}{B} - 4\omega^2 Y^2 = M$. Hence the solution for B(Y) will be:

$$B(Y) = be^{-\frac{z_2^2}{2}} H_n(z_2),$$

where $z_2 = \sqrt{2\omega}Y$. Based on the derivations, we get,

$$F(X,Y) = A(X)B(Y) = bce^{-\frac{z_1^2}{2}} H_n(z_1) e^{-\frac{z_2^2}{2}} H_n(z_2).$$

More generally, the solution can be a linear combination for every value of $n \geq 0$:

$$F(X,Y) = \sum_{n=0}^{\infty} a_n e^{-\frac{z_1^2+z_2^2}{2}} H_n(z_1) H_n(z_2).$$

Solution for H(X, Y) will be the same with only coefficients different:

$$H(X,Y) = \sum_{n=0}^{\infty} b_n e^{-\frac{z_1^2+z_2^2}{2}} H_n(z_1) H_n(z_2).$$

Since the basis forms in Φ are all holonomic and hence all the components of Φ satisfy the equation,

$$d * d\Psi - W * \Psi = 0.$$

Now I decompose Φ as follows,

$$\Phi = \Phi \vee (1) = \Phi \vee (\frac{1}{2}(1 + dX + 1 - dX)) = \Phi \vee P_+ + \Phi \vee P_-,$$

where $\Phi \vee P_+$ is a solution of the first order differential equation above because Φ is also a solution. As a result,

$$\Psi = \Phi \vee P_+ \tag{2.32}$$

is a spinor solution for the first order differential equation. We want to remind that we have made use of the projector operators here in order to find a particular spinor solution. Our main focus is in the spinor solutions of (8) that are asymptotically well behaved as $|X|$ and $|Y|$ go to infinity. We ascertain that such solutions may be expressed in terms of a basis of Hermite functions. Taking Ψ as,

$$\Psi = (u - vdY) \vee P_+ \tag{2.33}$$

we get the coupled system of differential equations as written in (2.27). However, there is another way to obtain the above coupled differential equations by decomposing the Hamiltonian in the following way so that we can obtain a coupled system of partial differential equations:

$$\left(\frac{1}{2i\omega(Y-X)\frac{\sqrt{2}}{\sqrt{2}}}\frac{\partial_Y+\partial_X}{\sqrt{2}}(\partial_Y-\partial_X)+\frac{2i\omega(X+Y)}{\sqrt{2}}\right)F=0. \tag{2.34}$$

Here, I made sure that $X \neq Y$. I express functions X, abd Y in terms of the variables ξ and η such that:

$$\begin{aligned} Y(\xi,\eta) &= \frac{\xi+\eta}{\sqrt{2}}, \\ X(\xi,\eta) &= \frac{\eta-\xi}{\sqrt{2}}. \end{aligned} \tag{2.35}$$

From the above expression we infer the following identities: $\xi(X,Y) = \frac{Y-X}{\sqrt{2}}$ and $\eta(X,Y) = \frac{X+Y}{\sqrt{2}}$.

Since both ξ and η are both dependent on X, and Y, I will use the following partial differentiation property:

$$\begin{aligned} \frac{\partial}{\partial\xi} &= \frac{\partial}{\partial X}\frac{\partial X}{\partial \xi}+\frac{\partial}{\partial Y}\frac{\partial Y}{\partial \xi} = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial X}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial\eta} &= \frac{\partial}{\partial X}\frac{\partial X}{\partial \eta}+\frac{\partial}{\partial Y}\frac{\partial Y}{\partial \eta} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial X}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial Y}. \end{aligned} \tag{2.36}$$

From the above equations we have,

$$H = \frac{1}{2i\omega(Y-X)\frac{\sqrt{2}}{\sqrt{2}}}(\partial_Y-\partial_X)F = \frac{\partial_\xi F}{2i\omega\xi}.$$

In terms of H and F, one has,

$$\partial_\xi F - 2i\omega\xi H = 0. \tag{2.37}$$

The other part of the equation left is.

$$\frac{\partial_Y+\partial_X}{\sqrt{2}}H+\frac{2i\omega(X+Y)}{\sqrt{2}}F=0.$$

Using the definitions above it reduces to,

$$\partial_\eta H - 2i\omega\eta F = 0.$$

2.4 A typical coherent spinor state solution

Since we have chosen our potential to look similar to the potential in the harmonic oscillator, one might also ask about the coherent states. A coherent state is the specific quantum state of the quantum harmonic oscillator which describes describes a state in a system for which the ground-state wavepacket is displaced from the origin of the system. Let a coherent state for our cosmological model be expressed as:

$$H = \frac{i}{\omega}(\alpha + \beta)F$$
$$F = Ce^{-(\alpha(X^2+Y^2)-2\beta XY)}, \tag{2.38}$$

where α, β, and C are complex numbers taken as constants.

2.5 Classical time

In this section, we will mention some facts about the classical time because that is very important in order to talk about any probabilistic interpretation of our model. We are classical observers in a classical spacetime[Brout and Parentani, 1999]. Given as such, one needs to consider quantum states that are related to classical cosmologies, especially in our case to our model, which admits a degenerate spacetime metric as discussed in the beginning of this thesis. The Hamiltonian of the field equations needs to be considered in such a way that the classical and quantum degrees of freedom are put into correspondence. In our chosen model, this correspondence is given by a parametrized curve in our mini-superspace as[Dereli et al., 1994],

$$\tau \to X = X(\tau), Y = Y(\tau) \qquad \tau_0 < \tau < \tau_1. \tag{2.39}$$

In Lorentzian domain, τ can be considered as a classical time and an evolution parameter for our model. As a result, the density $\rho_K(X(\tau), Y(\tau))$ predicts the relative probability for finding the classical configurations X, Y from time τ_0 to time τ_1.

2.6 Spinor Solutions and the associated currents

Analytic expressions for a class of degenerate metric solutions to Einstein equations for a self-coupled scalar field have been investigated. We have obtained and studied the conformally flat metrics and the solutions are and will be related to spinor solutions in cosmology. In the chart X, Y we get a current,

$$j[\Psi] = uvdY - (\frac{1}{2}|u|^2 + \frac{1}{2}|v|^2)dX] \tag{2.40}$$

which shows that the density is positive and for a Killing vector field ∂_X we can express it as,

$$\rho_K(X, Y) = -j[\Psi](\partial_X). \tag{2.41}$$

The Killing field is chosen in such a way that enables the construction of the coherent state. This property has a great advantage because as we know, we can construct a spacetime in the vicinity of its peak provided that decoherence does not become a trouble. The peak is a classical solution for our model and we will show some examples and graphs to demonstrate such created spacetimes. But, before we do so, the research revealed that our spinor solution will eventually be:

$$\Psi_{s_n s_m} = \begin{pmatrix} s_n e^{c_n(X^2+Y^2)+2c_m XY} \\ -s_m e^{c_n(X^2+Y^2)+2c_m XY} \end{pmatrix}. \tag{2.42}$$

As a result, we get the following formula which we will use to plot the graphs and see where the classical solution is:

$$\Psi_{s_1 s_2} - \Psi_{s_3 s_4} = \begin{pmatrix} s_1 e^{c_2(X^2+Y^2)+2c_2 XY} - s_3 e^{c_3(X^2+Y^2)+2c_4 XY} \\ -s_2 e^{c_1(X^2+Y^2)+2c_2 XY - s_4 e^{c_3(X^2+Y^2)+2c_4 XY}} \end{pmatrix} \tag{2.43}$$

In this context, for c_1 and c_2 we have $c_1 = \frac{2i\omega s_1 s_2}{s_1^2 - s_2^2}$ and $c_2 = \frac{i\omega(s_1^2 - s_2^2)}{s_1^2 - s_2^2}$.

To better understand all that is said above, let us plot a graph below and see the classical loci. The peak represents the classical solutions of our model.

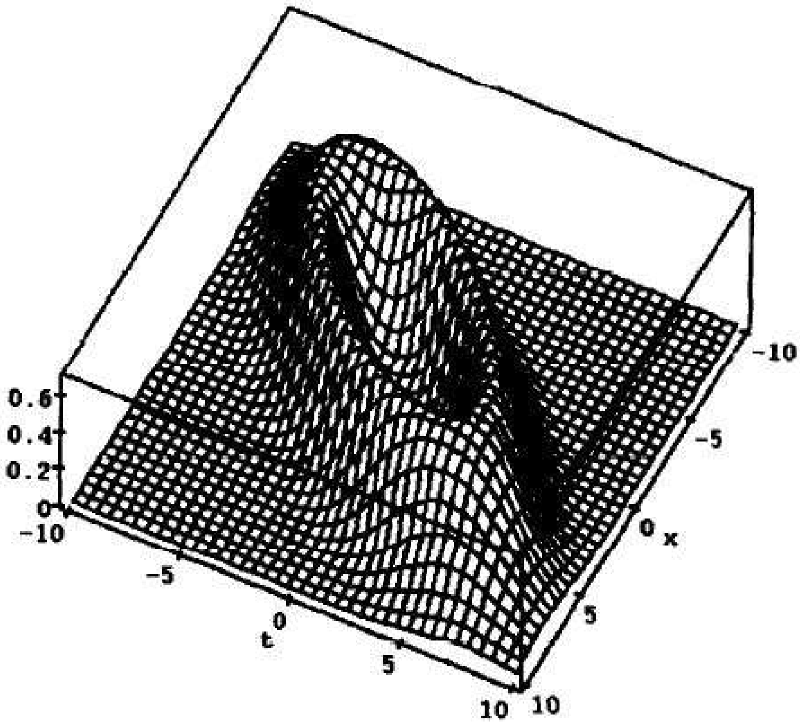


Figure 2.4: A plot of $|\Psi_{s_1,s_2} - \Psi_{s_3,s_4}|^2$ where for the two-dimensional dilaton gravity the constants are taken as $s_1 = 3+0.1i$, $s_2 = 1.3+0.1i$, $s_3 = 3.1+0.1i$, $s_4 = 1.4+0.1i$.

2.6.1 Discussion

Our model allows us to obtain a density from a given quantum state with maxima in the vicinity of the peak depicted in Figure 2.4. The divergence of the associated current around the peak in the graph is zero, and it has some implications in quantum cosmology, but not all models have associated currents that has a vanishing divergence.

Chapter 3

THIRD QUANTIZATION FOR THE SPINOR WAVE FUNCTIONS OF THE UNIVERSE IN AN EXTENDED MINISUPERSPACE

So far, we have considered a two-dimensional minisuperspace model with an oscillating property of the scalar field. In this chapter, we will try to quantize the spinor solutions and will obtain the pair creation and annihilation of the universes. To do so, we use the Eisenhart-Duval lift method by extending our minisuperspace from two-dimensional to a three-dimensional one. In the end, the added coordinate will require an additional constraint in our wave function Ψ in case one wants to reduce the equation to the original one. We will consider the third quantization of the Dirac-type equations in this extended minisuperspace. At the end of the chapter, we will see that the number density of universes is expressed by the Fermi-Dirac distribution, just like positrons and electrons. The entanglement entropy of the obtained multi-universe system will be discussed. In recent years, some research papers have been published from the Perimeter institute that claims that our universe's state does not violate CPT. We will briefly discuss this possibility and try to connect it to our cosmological model.

3.1 *Eisenhart-Duval lift for the Wheeler-DeWitt equation*

In this subsesction we will use an application of Eisenhart-Duval lift in our minisuperspace model and we start with our action of the gravitating scalar field ϕ as given in (1.14). Here the metric $g_{\mu\nu}$ is the FLRW metric with a flat space,

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{3.1}$$

here I have denoted the scale factor by a(t) instead of R(t). The scale factor is depended only on time and is spatially homogeneous[Kim and Oh, 1996]. Under some suitable coordinate transformation we obtained equation (2.10) but I will express the factor"W" below:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2Ve^{2nx}\right)\Phi(x,y) = 0, \tag{3.2}$$

where n in the equation can take different integer values based on the model we discuss. After introducing an auxiliary dimension z, one obtains the metric of the extended space as:

$$G_{MN}dX^M dX^N = 2Ve^{2nx}(-dx^2 + dy^2) + dz^2, \tag{3.3}$$

where $X^M = (x, y, z)$. In this extended minisuperspace[Kan et al., 2021a], the Laplace equation is[McGuigan, 1988]:

$$\frac{1}{\sqrt{-G}}\partial_M(\sqrt{-G}G^{MN}\partial_N\Phi) = \left[\frac{1}{2Ve^{2nx}}(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{\partial^2}{\partial z^2}\right]\Phi = 0, \tag{3.4}$$

where conventionally $G = -(2V)^2e^{4nx}$ is the determinant of the metric G_{MN}, and $\partial_M \equiv \frac{\partial}{\partial X^M}$. The constraint that arises in the extended minisuperspace is:

$$\frac{\partial^2\Phi}{\partial z^2} = -\Phi. \tag{3.5}$$

If one tries to solve the second order partial differential equation, one obtains a solution in terms of Bessel function of order ν [Kazama and Nakayama, 1985]. Since (3.4) is of second order, the two possible solutions are:

$$\Phi_\nu^{(1)}(x,y) \propto J_{-i\nu/n}(\sqrt{2V}e^{nx}/n)e^{i\nu y}, \tag{3.6}$$

$$\Phi_\nu^{(2)}(x,y) \propto J_{i\nu/n}(\sqrt{2V}e^{nx}/n)e^{i\nu y}. \tag{3.7}$$

In the previous chapter, we we performed the square-rooting of the WDW equation, and in the end we obtained the Dirac equation[Kiefer, 1988]. However, we need to make a slightly change here since we are considering the extended minisuperspace[Finn et al., 2018]. Therefore, the Dirac equation in this case is equivalent to[Robles-Pérez, 2021]:

$$\left[\sigma^1\left(\frac{\partial}{\partial x + \frac{n}{2}}\right) + i\sigma^2\frac{\partial}{\partial y} + i\sigma^3\sqrt{2V}e^{nx}\frac{\partial}{\partial z}\right]\Psi = 0, \tag{3.8}$$

($\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices and we have also used the following constraint:

$$\frac{1}{i}\frac{\partial}{\partial z}\Psi = \Psi. \tag{3.9}$$

After using the Pauli matrices, we obtain the following equation from (3.8):

$$\begin{pmatrix} -\sqrt{2V}e^{nx} & \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{n}{2} \\ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{n}{2} & \sqrt{2V}e^{nx} \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

The solutions of the equation above are found to be as[Kan et al., 2021a]:

$$\Psi^{(1)}_{\pm,\nu} \propto \pm J_{-\frac{i\nu}{n}\pm\frac{1}{2}}(\sqrt{2V}e^{nx}/n)e^{i\nu y}, \tag{3.11}$$

$$\Psi^{(2)}_{\pm,\nu} \propto \pm J_{\frac{i\nu}{n}\mp\frac{1}{2}}(\sqrt{2V}e^{nx}/n)e^{i\nu y}. \tag{3.12}$$

Our next important step will be to use the solutions (3. 11), (3. 12) and apply the third quantization method ad discussed below.

3.2 Universe-antiuniverse pair creation by third quantization method

In this section, we will quantize the solutions of the WDW equation obtained in (3. 11) and (2. 12) and then discuss the pair creation of our universe and antiuniverse. Consider an operator acting on the state vectors of a system of multiple universes, as in equation (3. 8). Let us expand the spinor wave function Φ as [Kan et al., 2021b]:

$$\Phi(x,y) = \int_0^\infty \left[b_\nu u_\nu(x,y) + d^\dagger_\nu u_{-\nu}(x,y) + b_{-\nu}u_{-\nu}(x,y) + d^\dagger_{-\nu}u_\nu(x,y) \right] d\nu, \tag{3.13}$$

where we have defined the inner product of two spinors by:

$$(f,g) = \sqrt{2V}e^{nx}\int f^\dagger g\, dy, \tag{3.14}$$

and the inner product is symmetric as suspected. The normalized mode functions are,

$$(u_\nu, u_{\nu'} = \delta(\nu,\nu'). \tag{3.15}$$

The most important part here is to define the anticommutators of the fermionic operators. Using the results from quantum field theory, we get,

$$\{b_\nu, b^\dagger_{\nu'}\} = \{d_\nu, d^\dagger_{\nu'}\} = \delta(\nu - \nu') \tag{3.16}$$

$$\{\Psi_\alpha(x,y), \Psi^\dagger_\beta(x,y')\} = \delta_{\alpha\beta}\delta(y-y'), \tag{3.17}$$

where α and β is either $+$ or $-$.

The spinor field can be expressed as:

$$\Psi(x,y) = \int_0^\infty \left[b^{in}_\nu u^{in}_\nu(x,y) + d^{in\dagger}_\nu u^{in}_{-\nu}(x,y) + b^{in}_{-\nu} v^{in}_{-\nu}(x,y) + d^{in\dagger}_{-\nu} v^{in}_\nu(x,y)\right] d\nu =$$

$$= \int_0^\infty \left[b^{out}_\nu u^{out}_\nu(x,y) + d^{out\dagger}_\nu u^{out}_{-\nu}(x,y) + b^{out}_{-\nu} v^{out}_{-\nu}(x,y) + d^{out\dagger}_{-\nu} v^{out}_\nu(x,y)\right] d\nu. \tag{3.18}$$

The in-mode and out-mode spinor functions are:

$$u^{in}_\nu(x,y) = \frac{1}{2\sqrt{n\cosh\frac{\pi\nu}{n}}}\begin{pmatrix} J_{-i\frac{\nu}{n}+\frac{1}{2}}(\sqrt{2V}e^{nx}/n) \\ -J_{-i\frac{\nu}{n}-\frac{1}{2}}(\sqrt{2V}e^{nx}/n)\end{pmatrix} e^{i\nu y} \tag{3.19}$$

$$v^{in}_\nu(x,y) = \frac{1}{2\sqrt{n\cosh\frac{\pi\nu}{n}}}\begin{pmatrix} J_{i\frac{\nu}{n}-\frac{1}{2}}(\sqrt{2V}e^{nx}/n) \\ J_{i\frac{\nu}{n}+\frac{1}{2}}(\sqrt{2V}e^{nx}/n)\end{pmatrix} e^{i\nu y} \tag{3.20}$$

$$u^{out}_\nu(x,y) = \frac{u^{in}_\nu(x,y)}{\sqrt{1+e^{-2\pi\nu/n}}} + i\frac{v^{in}_\nu(x,y)}{\sqrt{1+e^{2\pi\nu/n}}} \qquad \nu > 0 \tag{3.21}$$

$$u^{out}_{-\nu}(x,y) = \frac{u^{in}_{-\nu}(x,y)}{\sqrt{1+e^{-2\pi\nu/n}}} - i\frac{v^{in}_{-\nu}(x,y)}{\sqrt{1+e^{2\pi\nu/n}}} \qquad \nu > 0 \tag{3.22}$$

$$v^{out}_\nu(x,y) = i\frac{u^{in}_\nu(x,y)}{\sqrt{1+e^{2\pi\nu/n}}} + \frac{v^{in}_\nu(x,y)}{\sqrt{1+e^{-2\pi\nu/n}}} \qquad \nu > 0 \tag{3.23}$$

$$v^{out}_{-\nu}(x,y) = \frac{v^{in}_{-\nu}(x,y)}{\sqrt{1+e^{-2\pi\nu/n}}} - i\frac{u^{in}_{-\nu}(x,y)}{\sqrt{1+e^{2\pi\nu/n}}} \qquad \nu > 0. \tag{3.24}$$

Based on what we have obtained so far, we can conclude that there is a relation among the fermionic operators (in- and out- operators)[Kan et al., 2021b]:

$$b^{out}_\nu = \frac{1}{\sqrt{1+e^{-2\pi|\nu|/n}}} b^{in}_\nu - i\frac{1}{\sqrt{1+e^{2\pi|\nu|/n}}} d^{in\dagger}_{-\nu} \tag{3.25}$$

$$b^{out}_{-\nu} = \frac{1}{\sqrt{1+e^{-2\pi|\nu|/n}}} b^{in}_{-\nu} + i\frac{1}{\sqrt{1+e^{2\pi|\nu|/n}}} d^{in\dagger}_{\nu} \tag{3.26}$$

$$d^{out\dagger}_\nu = \frac{1}{\sqrt{1+e^{-2\pi|\nu|/n}}} d^{in\dagger}_\nu + i\frac{1}{\sqrt{1+e^{2\pi|\nu|/n}}} b^{in}_{-\nu} \tag{3.27}$$

$$d^{out\dagger}_{-\nu} = \frac{1}{\sqrt{1+e^{-2\pi|\nu|/n}}} d^{in\dagger}_{-\nu} - i\frac{1}{\sqrt{1+e^{2\pi|\nu|/n}}} b^{in}_{\nu}. \tag{3.28}$$

Now, consider a vacuum state $|0\rangle$. It is directly calculated that the average number of "fermionic" universes formed is:

$$N_\nu = \langle 0, in|\, b^{out\dagger}_\nu b^{out\dagger}_\nu \,|0, in\rangle = \frac{1}{e^{2\pi|\nu|/n}+1}, \tag{3.29}$$

and that of the "anti-universes" is:

$$\bar{N}_\nu = \langle 0, in| d_\nu^{out\dagger} d_\nu^{out\dagger} |0, in\rangle = \frac{1}{e^{2\pi|\nu|/n} + 1}. \tag{3.30}$$

As it can be seen, the number of spontaneously created universes and antiuniverses is the same and it obeys Fermi-Dirac distribution. In the next section, I will discuss a preferred vacuum state, a CPT symmetric state but before that let us briefly analyze the entanglement entropy of these multiple universes.

3.2.1 Entanglement entropy of multiple universes

Using the equations (3. 25), (3. 26), (2. 27), (3. 28), for a specific ν we get the following relation:

$$|0, in\rangle_\nu |\bar{0}, in\rangle_{-\nu} = a \left(|0, out\rangle_\nu |\bar{0}, out\rangle_{-\nu} - isgn(\nu) e^{-\pi|\nu|/n} |1, out\rangle_\nu |\bar{1}, out\rangle_{-\nu}\right), \tag{3.31}$$

where $a = \frac{1}{\sqrt{1+e^{-2\pi|\nu|/n}}}$.

The von Neumann entanglement entropy is defined by[Horodecki et al., 2009]:

$$S = -Tr\left[\rho_\nu \log_2 \rho_\nu\right], \tag{3.32}$$

whereρ_ν can be calculated by the following formula:

$$\rho_\nu = Tr_{-\nu}\left[|0, in\rangle_\nu |\bar{0}, in\rangle_{-\nu} \langle\bar{0}, in|_{-\nu} \langle 0, in|_\nu\right]. \tag{3.33}$$

Therefore, the von Neumann entropy is:

$$S = \log_2\left(\frac{1 + e^{-2\pi|\nu|/n}}{e^{-\frac{2\pi|\nu|/n}{1+e^{2\pi|\nu|/n}}}}\right) \tag{3.34}$$

The von Neumann entropy in plotted in figure (3.1):

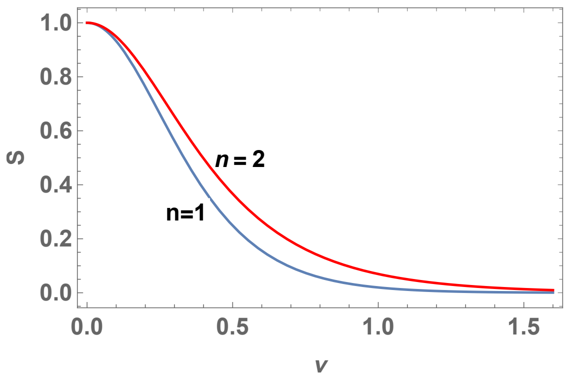


Figure 3.1: A plot for the calculated von Neumann entropy as a function of ν.

3.3 CPT symmetric Universe

So far the classical Physics has been very successful in describing the structure and the evolution of the large scale structures of the universe and the entire cosmos. However, very little is known what happened at the very early times of the universe, the singularity and how it was formed. In this section, we will discuss a possible extension of the universe beyond the Big Bang[Volovik, 2019]. More specifically, we will focus on Euclidean signature universe and spontaneously broken CPT symmetry from the analytic continuation in terms of the physical proper time t. Consider the metric for the radiation dominated Universe[Boyle et al., 2018b]:

$$g_{\mu\nu} = dt^2 - a^2(t)d\mathbf{r}^2, \qquad a(t) \approx t. \tag{3.35}$$

Suppose $a(t) \approx \sqrt{t}$. Let us extend the scale factor analytically around the singularity. As a result, spacetime and antispacetime can be connected by a 2π rotation about the singularity and they both coexist for $t > 0$ as a quantum superposition of both of them. However, due to decoherence only one of the two states survives. In the case where $t < 0$ the metric above has Euclidean signature. At $t = 0$, there is a Z_2 symmetry between the spacetime and antispacetime is spontaneously broken. CPT seems likely to be a fundamental symmetry of the nature. It removes perturbation modes which would cause the spacetime to become singular at $\tau = 0$[Boyle et al., 2018a]. In lieu of this idea, there is still much work going on and further research papers will be published in the future.

Chapter 4

CONCLUSION

We stated initially that quantum cosmology is an idea that tries to find evidence of quantum gravity in the cosmos. It tries to explain how the universe was formed and what happened on very small scales up to Planck's scale. We began in Chapter 1 by describing the Robertson-Walker metric. It describes a homogeneous, isotropic, expanding (or contracting) universe path-connected but not necessarily simply connected. We considered a cosmological model with a simple scalar field which is only time-dependent, and for the simplicity of our research, we took a vanishing cosmological constant. Starting with a toy model would be a great idea in case one tries to understand or universe. In this chapter, we derived the equations of motion for our model and some simple numerical calculations associated with this model. The choice of the potential has been wisely chosen so that in the end we have a harmonic and ghost oscillator case. As it was observed, the potential is a function of only the scalar field ϕ. In Chapter 2, we discussed the No-Boundary Proposal and also introduced a spinor model for our minisuperspace. The problem with this proposal is that the path integral for real, Lorentzian four-geometries in Einstein gravity leads to uncontrolled fluctuations, which are still a problem in quantum cosmology, especially if one tries to avoid the Boltzmann brains. Wheeler-DeWitt equation has also been derived and related to the aim of our work. Later in this chapter, we also discussed the Clifford algebra of the minisuperspace, and at the end, we gave an idea about the classical time.

In Chapter 3, we focused on the third quantization method in order to quantize our spinor solutions, and we had a pair creation and annihilation of the universe at the time of the Big Bang. Some possible Feynman diagram of this processed is shown Figure 4.1. We also discuss the entanglement entropy of the multiple universes based on this idea. As it can can thought, the universe was created at the time of the Big

Bang and that there was a special singularity at t=0s. If one considers the vacuum state of the universe, there might be many choices of this state. However, there have been some claims recently that our universe might have been CPT symmetric at the very early times of creation. If that is true, then there might have been a preferred vacuum state according to quantum field theory in curved spacetime[Boyle et al., 2018b]. The application of this idea still needs further research and thus the research in the consisted models of quantum cosmology is highly motivated.

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Appendix A

DERIVATIONS OF EISENHART-DUVAL LIFT METHOD

So far we have obtained a general formula for the Wheeler-DeWitt equation in a minisuperspace quantum cosmology with a massive scalar field defined as:

$$\left[\frac{1}{a^{s+1}}\frac{\partial}{\partial a}a^{s}\frac{\partial}{\partial a}-\frac{1}{a^{3}}\frac{\partial^{2}}{\partial\phi^{2}}+2U(a,\phi)\right]\Psi(a,\phi)=0. \tag{A.1}$$

In the following "gauge choice" one gets,

$$G_{MN}=2U(a,\phi)\tilde{G}_{MN}=diag(-2Ua,2Ua^{3},1). \tag{A.2}$$

The extended WDW equation can therefore be written as:

$$\left[\frac{1}{\sqrt{-G}}\partial_{M}\sqrt{-G}G^{MN}\partial_{N}-\xi R\right]\Psi=0 \tag{A.3}$$

First, let us calculate the Ricci scalar and then derive the constraint that was implemented in the Eisenhart-Duval lift. Let the coframe coordinate basis be denoted by:

$$e^{0}=-(2Ua)^{1/2}da,\qquad e^{1}=(2Ua^{3})^{1/2}d\phi,\qquad e^{2}=d\chi. \tag{A.4}$$

Using the following identity:

$$\star(dx^{\mu})=\eta^{\mu\lambda}\epsilon_{\lambda\nu\rho}\frac{1}{2!}dx^{\nu}\wedge dx^{\rho}, \tag{A.5}$$

one gets,

$$\star d\star d=\frac{1}{2Ua^{2}}\frac{\partial}{\partial a}a\frac{\partial}{\partial a}-\frac{1}{2Ua^{3}}\frac{\partial^{2}}{\partial\phi^{2}}-\frac{\partial^{2}}{\partial\chi^{2}}, \tag{A.6}$$

where $U=a^{3}V-\frac{1}{2}Ka$ and the metric in this context is:

$$ds^{2}=-e^{0}\otimes e^{0}+e^{1}\otimes e^{1}+e^{2}\otimes e^{2}. \tag{A.7}$$

Next step we calculate de^a which are identified as,

$$de^0 = \frac{a^{7/2}V'}{(2U)^{1/2}} d\phi \wedge da = \left(\frac{a}{2U}\right)^{3/2} V' e^1 \wedge e^0, \tag{A.8}$$

where $V' = \frac{dV}{d\phi}$ and $\frac{dU}{da} = 3a^2V(\phi) - \frac{1}{2}K$. As a result,

$$de^1 = \frac{6a^5V - 2Ka^3}{(8U^3a^7)^{1/2}} e^0 \wedge e^1. \tag{A.9}$$

For the connection one-forms can be found by the following formula:

$$2\omega^a{}_b = \iota_b de^a - \iota^a de_b + (\iota^a \iota_b de_c)e^c. \tag{A.10}$$

The only surviving connection 1-form is,

$$2\omega^1{}_0 = \iota_0 de^1 - \iota^1 de_0 + (\iota^1 \iota_0 de_c)e^c = \frac{12a^5V - 4Ka^3}{(8U^3a^7)^{1/2}} e^1 + \left(\frac{a}{2U}\right)^{3/2} V' e^0. \tag{A.11}$$

Using the curvature 2-form equation:

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \tag{A.12}$$

one can easily obtain,

$$R^{01} = d\left[\frac{12a^5V - 4Ka^3}{(8U^3a^7)^{1/2}} e^1 + \left(\frac{a}{2U}\right)^{3/2} V' e^0\right] = \frac{a^5(V')^2 - a^2V"U - 2Ka^3V}{4U^3a^2} e^0 \wedge e^1. \tag{A.13}$$

The Ricci scalar is therefore found directly as:

$$\mathcal{R} = \frac{a^3(V')^2 - V"(a^3V - \frac{1}{2}Ka) - 2KaV}{2U^3}. \tag{A.14}$$

In the equation below,

$$(d \star d - \xi \star \mathcal{R})\,\Psi = 0, \tag{A.15}$$

by direct substitution we find that,

$$\left(\frac{1}{2Ua^2}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{1}{2Ua^3}\frac{\partial^2}{\partial \phi^2} - \frac{\partial^2}{\partial \chi^2} - \xi\frac{a^3(V')^2 - V"(a^3V - \frac{1}{2}Ka) - 2KaV}{2U^3}\right)\Psi = 0. \tag{A.16}$$

Multiplying both sides by $2U = 2a^3V - Ka$, we get,

$$\left(\tfrac{1}{a^2}\tfrac{\partial}{\partial a}a\tfrac{\partial}{\partial a} - \tfrac{1}{a^3}\tfrac{\partial^2}{\partial \phi^2} - (2a^3V - Ka)\tfrac{\partial^2}{\partial \chi^2} + 2\xi\tfrac{2a^3((V')^2 - V"V) + Ka(V" - 4V)}{(2a^3V - Ka)^2}\right)\Psi = 0 \tag{A.17}$$

If we want to get back the equation (A. 1) for s=1, we need to implement the following constraints, which happen to be the same as the one presented in the research paper.

$$\xi = 0, \qquad\qquad -\frac{\partial^2\Psi}{\partial\chi^2} = p^2\Psi, \tag{A.18}$$

here $p^2 = 1$ and the obtained equation is the same as (A. 1):

$$\left[\frac{1}{a^2}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{1}{a^3}\frac{\partial^2}{\partial\phi^2} + 2U(a,\phi)\right]\Psi(a,\phi) = 0. \tag{A.19}$$

As it can be seen, the contribution from the scalar curvature has been vanished by requiring $\xi = 0$ but if that were different from zero, then the WDW would become different from the conventional one. The value of ξ has been extensively studied in quantum field theory in curved spacetime. It has been proven that the conformal coupling in three dimensions is $\xi = \frac{1}{8}$ and in n-dimensions $\xi = \frac{n-2}{4(n-1)}$.

Appendix B

DERIVATION OF THE CANONICAL HAMILTONIAN CORRESPONDING TO THE PAIS-UHLENBECK OSCILLATOR

Consider the following Lagrangian of this class:

$$L = \frac{1}{2}[\ddot{q}^2 - (\Omega_1^2 + \Omega_2^2)\dot{q}^2 + \Omega_1^2\Omega_2^2 q^2] + (nonlinear \quad terms). \tag{B.1}$$

Hamiltonian can be found by the following equation,

$$H = \sum p_i \dot{q}_i - L. \tag{B.2}$$

Given that the canonical momenta as:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x}, \qquad p_q = \frac{\partial L}{\partial \dot{q}} = -(\Omega_1^2 + \Omega_2^2)x. \tag{B.3}$$

Hence, one can obtain the Hamiltonian as follows:

$$\sum p_i \dot{q}_i - L = \frac{p_x^2}{2} - \frac{1}{2}(\Omega_1^2 + \Omega_2^2)x^2 - \frac{1}{2}\Omega_1^2\Omega_2^2 q^2. \tag{B.4}$$

If for $H \equiv H(q_i, p_i)$ one considers $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, the Hamiltonian will take this form:

$$H = p_q x + \frac{1}{2}p_x^2 + \frac{(\Omega_1^2 + \Omega_2^2)x^2}{2} - \frac{1}{2}\Omega_1^2\Omega_2^2 q^2. \tag{B.5}$$

We see that can can find the Hamiltonian as a function of both generalized coordinates and canonical momenta as H$\equiv H(x, q, p_x, p_q)$. Let:

$$q = \frac{1}{\Omega_1}\frac{\Omega_1 X_2 - P_1}{\sqrt{\Omega_1^2 - \Omega_2^2}} \qquad x = \frac{\Omega_1 X_1 - P_2}{\sqrt{\Omega_1^2 - \Omega_2^2}}$$

$$p_q = \Omega_1 \frac{\Omega_1 P_2 - \Omega_2^2 X_1}{\sqrt{\Omega_1^2 - \Omega_2^2}} \qquad P_x = \frac{\Omega_1 P_1 - \Omega_2^2 X_2}{\sqrt{\Omega_1^2 - \Omega_2^2}}.$$

With some simple algebra, it is easy to find P_1, P_2, X_1, and X_2. The results are:

$$P_1 = \Omega_1 \frac{p_x + \Omega_2^2 q}{\sqrt{\Omega_1^2 - \Omega_2^2}} \qquad P_2 = \frac{p_q + \Omega_2^2 x}{\sqrt{\Omega_1^2 - \Omega_2^2}}$$

$$X_1 = \frac{1}{\Omega_1} \frac{p_q + \Omega_1^2 x}{\sqrt{\Omega_1^2 - \Omega_2^2}} \qquad X_2 = \frac{p_x + \Omega_1^2 q}{\sqrt{\Omega_1^2 - \Omega_2^2}}. \tag{B.6}$$

We have made the following coordinate transformations so that we can get a similar form for the Hamiltonian to the one we found by the other method. After substituting p_q, p_x, p, x in the Hamiltonian above, we get:

$$H = \frac{\Omega_1 P_2 - \Omega_2^2 X_1}{\sqrt{\Omega_1^2 - \Omega_2^2}} \frac{(\Omega_1^2 X_1 - P_2 \Omega_1)}{\sqrt{\Omega_1^2 - \Omega_2^2}} + \frac{1}{2} \left[\frac{\Omega_1 P_1 - \Omega_2^2 X_2}{\sqrt{\Omega_1^2 - \Omega_2^2}} \right]^2 - \frac{\Omega_1^2 \Omega_2^2}{2} \frac{(\Omega_1 X_2 - P_2)^2}{(\Omega_1^2 - \Omega_2^2)(\Omega_1^2)}$$
$$+ \frac{(\Omega_1^2 + \Omega_2^1)}{2} \frac{(\Omega_1 X_1 - P_2)^2}{\Omega_1^2 - \Omega_2^2}. \tag{B.7}$$

After doing the required simplifications we obtain:

$$H = \frac{P_1^2 + \Omega_1^2 X_1^2}{2} - \frac{P_2^2 + \Omega_2^2 X_2^2}{2}. \tag{B.8}$$