

ADAPTIVE LYAPUNOV REDESIGN OF MODEL PREDICTIVE POWERED
DESCENT GUIDANCE

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DESCENT GUIDANCE**

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ABSTRACT

ADAPTIVE LYAPUNOV REDESIGN OF MODEL PREDICTIVE POWERED DESCENT GUIDANCE

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A new era of space systems is initiated with the development of reusable rocket systems. The experience gained from interplanetary landing missions shed a light upon the design of reusable rocket stages with vertical precision landing capabilities. For two decades, the descent guidance problem of interplanetary landers and reusable rocket stages are modeled and handled as computational optimal control problems which are usually solved as parametric optimization problems. Nevertheless, the precision of descent guidance highly relies on the robustness of the used solution procedure against unexpected subsystem faults occurring in sensors, thrusters and so on.

In many engineering applications, particularly in descent guidance problems, the subsystem faults are inevitable due to drastically changing environments and subjecting to physical stress. There are plenty of precautions used for fault detection in such missions, however, very few of them consist of approximating the faults or, in control theoretic terms, the uncertainties in an online manner. On the other hand, there exists a well-established literature of adaptive control theory which particularly focuses on the estimation and approximation of physical uncertainties.

In this thesis, the descent guidance problem of reusable rocket stages is solved via receding horizon optimal control strategy known as model predictive control in the presence of various physical uncertainty factors such as sensor faults, thruster losses and modelling errors. First, the formulation of model predictive control is explained with the various aspects such as estimation-based approach and feasibility issues. As a preliminary to estimation-based model predictive control, the globally adopted methods of Kalman filtering and the theory of quadratic programming are presented as the state estimator and the optimization solver used, respectively. Then, the state estimator used in optimal control formulation is redesigned to be adaptive to the disruptive effects of uncertainties, in the sense of discrete-time model reference adaptive control. Meanwhile, the approximate functional representations of uncertainties acting on the subsystems can be obtained.

In the major part of the thesis, the proposed redesign methodology on the state estimation part of optimal control is given and proven with the mathematical analysis and control theoretical perspectives. In the proposed methods, the state estimator is redesigned with a regressor structure (e.g. neural networks) that learns to mimic the physical uncertainty up to some adequacy. The online tuning law (or training) of regressor is obtained from the necessary and sufficient conditions of Lyapunov stability theory and it is shown that the proposed methods are input-to-state stable when the structure of uncertainty is known. On the other hand, it is also shown that the methods are semiglobally practically asymptotically stable in the presence of bounded approximation errors when used along the convex projections. In either case, the proposed online tuning laws consist of some particular Kalman filter terms which build a novel bridge between optimal filtering and Lyapunov-based adaptive control.

Keywords: adaptive control, model predictive control, Kalman filter, descent guidance, vertical landing, function approximation, quadratic programming, convex projection

ÖZ

MODEL ÖNGÖRÜLÜ DİKEY İNİŞ GÜDÜMÜNÜN UYARLAMALI LYAPUNOV-TABANLI TEKRAR TASARIMI

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Tekrar kullanılabilir roket sistemlerinin geliştirilmesi ile uzay sistemlerinde yeni bir çağ başlamış oldu. Dünya dışı gezegen ve uydulara yapılan keşif aracı görevlerinden edinilen tecrübeler ile dikey iniş yapabilen tekrar kullanılabilir roket kademelerinin tasarımına ışık düşürülmüş oldu. Yaklaşık yirmi yıldır, keşif araçları ve tekrar kullanılabilir roket kademelerinin dikey iniş güdümü, parametrik eniyileme problemleri şeklinde çözülen hesaplamalı eniyilemeli denetim problemleri olarak modellenmekte ve ele alınmaktadır. Bununla birlikte iniş güdümünün hassasiyeti, sensör, itici ve benzeri alt sistemlerde oluşan beklenmedik hatalara karşın, kullanılan çözüm prosedürünün gürbüzlüğüne dayanmaktadır.

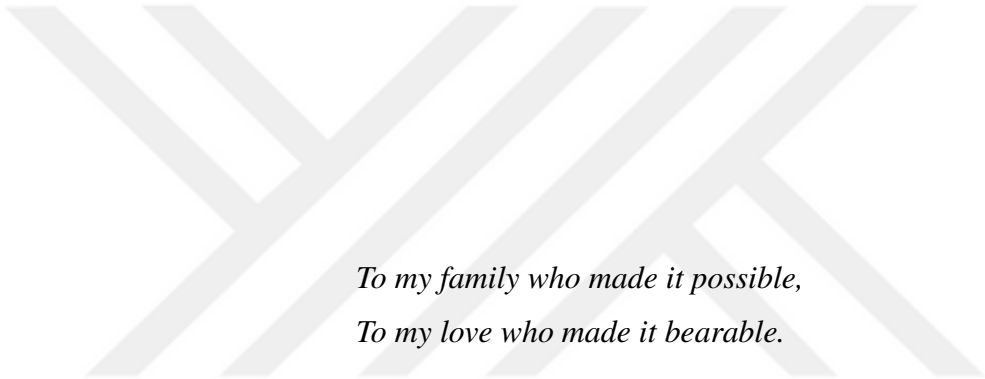
Özellikle dikey iniş güdümü problemleri olmak üzere mühendislikteki pek çok uygulamada, alt sistem hataları, büyük ölçüde değişen ortam koşulları ve fiziksel baskılar nedeniyle kaçınılmaz bir hal almaktadır. Bu tip görevlerde, hata tespiti için kullanılan pek çok önlem mevcuttur, ancak pek az bir bölümü bu hataların yani, kontrol teorisi tabiri ile, belirsizliklerin gerçek-zamanlı yakınsamasını incelemektedir. Diğer taraftan, uyarlamalı denetim teorisinin, özellikle fiziksel belirsizliklerin kestirim ve

tahminine odaklanan köklü bir literatürü mevcuttur.

Bu tezde, tekrar kullanılabilir roket kademelerinin iniş güdümü problemi, sensör hataları, itici kayıpları ve modelleme hataları gibi çeşitli belirsizlik faktörleri altında, model öngörülü denetim olarak bilinen uzaklaşan ufuk eniyilemeli denetim stratejisi ile çözülmektedir. İlk olarak, model öngörülü denetimin formülasyonu, kestirim-tabanlı yaklaşım ve fizibilite gibi çeşitli konular da ele alınarak açıklanmaktadır. Kestirim-tabanlı model öngörülü denetime ön hazırlık olarak, dünyaca kabul görmüş Kalman filtresi yöntemleri ve ikinci derece programlama teorisi, sırasıyla kullanılan durum kestirimcisi ve eniyileme çözücüsü olarak sunulmaktadır. Daha sonra, eniyilemeli denetim formülasyonunda yer alan durum kestirimcisinin, bozucu belirsizlik etkilerine karşı kendini uyarlayabilmesi adına, ayrık-zamanlı model referanslı uyarlamalı denetim mantığı ile tekrar tasarımı yapılmaktadır. Bu sırada, alt sistemlere etki eden belirsizliklerin tahmini fonksiyonel temsilleri de elde edilebilmektedir.

Tezin büyük bir bölümünde, matematiksel analiz ve kontrol teorisi bakış açılarıyla, eniyilemeli denetimdeki durum kestirimcisinin, öne sürülen yöntemler ile tekrar tasarımına ve bu yöntemlerin ispatlarına yer verilmektedir. Öne sürülen yöntemlerde durum kestirimcisinin, fiziksel belirsizlikleri mümkün mertebe taklit etmeyi öğrenen bir regresör yapısı (örn. yapay sinir ağları) ile tekrar tasarımı yapılmaktadır. Regresörün gerçek-zamanlı öğrenme kanunu (ya da eğitimi), Lyapunov kararlılık teorisinin gerekli ve yeterli koşullarından türetilmekte ve öne sürülen yöntemlerin, belirsizliğin yapısı bilindiği takdirde, girdiden-duruma kararlı olduğu gösterilmektedir. Diğer taraftan yöntemlerin, dışbükey izdüşümler ile birlikte kullanıldığında, sınırlandırılmış tahmin hatalarının mevcudiyetinde dahi yarı-evrensel asimptotik kararlı olduğu gösterilmektedir. Her iki durumda da, önerilen gerçek-zamanlı öğrenme kanununun Kalman filtresine özgü terimler içermesi, eniyilemeli kestirim ve Lyapunov-tabanlı uyarlamalı denetim arasında yeni bağlantılar kurma niteliği taşımaktadır.

Anahtar Kelimeler: uyarlamalı denetim, model öngörülü denetim, Kalman filtresi, iniş güdümü, dikey iniş, fonksiyon yaklaşımı, ikinci derece programlama, dışbükey izdüşüm



*To my family who made it possible,
To my love who made it bearable.*

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LIST OF NOTATIONS

$\mathbb{R}^n$	Real vector space in n -dimensions
$\mathbb{R}, \mathbb{C}$	Real and complex fields
$\mathbb{R}_+$	Set of positive real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}^{n \times m}$	Set of real matrices of $(n \times m)$ -dimensions
$\mathbb{S}^n$	Set of n -dimensional symmetric matrices
s.t.	<i>Subject to or such that</i>
$\ \cdot\ _p, \ \cdot\ _F$	p -norm and Frobenius norm of given vector or matrix argument
$\langle \cdot, \cdot \rangle$	Inner product of given two arguments
$\mathcal{B}_\delta(\cdot)$	Euclidean ball with radius δ , centered at given argument
$\det(\cdot), \text{rank}(\cdot)$	Determinant and rank of given matrix argument
$\kappa(\cdot), \text{tr}\{\cdot\}$	Condition number and trace of given matrix argument
$\text{diag}(\cdot, \cdot)$	Diagonal matrix construction with given arguments
$\lambda_{\max}(\cdot), \lambda_{\min}(\cdot)$	Maximum and minimum modulus of eigenvalues of given matrix argument
$\mathcal{A}, \mathcal{I}, \mathcal{X}, \mathcal{U}, \Omega$	Various set notations
$\Omega_1 \times \Omega_2$	Cartesian product of sets Ω_1 and Ω_2
$\text{Int}(\Omega), \partial\Omega$	Interior and boundary of set Ω
$\max, \min$	Maximum and minimum
$\arg \max, \arg \min$	Argument of maximum and argument of minimum
$\sup, \inf$	Supremum (i.e. least upper bound) and infimum (i.e. greatest lower bound)
$\text{dom}(\cdot), \text{ran}(\cdot)$	Domain and range of given argument
$\limsup, \liminf$	Limit superior and limit inferior

∇, Δ	Gradient and difference operators
$\mathcal{O}(\cdot)$	Order notation in terms of given argument
$\text{Proj}_\Gamma(\cdot, \cdot)$	Γ -Projection operator between given arguments
$\mathcal{N}(\cdot, \cdot)$	Normal distribution with given mean and variance arguments
$\mathcal{C}(X, Y)$	Class of functions acting between spaces X and Y
$\mathcal{C}^n$	Class of n -times differentiable functions
$\mathcal{P}, \mathcal{P}_s$	Classes of positive definite and positive semidefinite functions
$\mathcal{K}, \mathcal{K}_\infty$	Classes of strictly increasing and strictly increasing unbounded functions
$\mathcal{L}$	Class of strictly decreasing functions vanishing at infinity
L^p, ℓ^p	Space of Lebesgue measurable functions and sequences
L^1, ℓ^1	Space of absolutely integrable functions and sequences
L^2, ℓ^2	Space of square integrable functions and sequences
L^∞, ℓ^∞	Space of essentially bounded functions and sequences



CHAPTER 1

INTRODUCTION

In recent years, the aerospace industries from all around the world dedicate themselves to work on commercial interplanetary and orbital missions with affordable costs. In the pursuit of affordable commercial space missions, a new era of reusable launch vehicles with vertical take off and landing capabilities is begun. The vertical take off is the natural way of launching the rockets and missiles, on the other hand, the vertical landing is a completely new technological development compared to take off. In the light of the engineering experience gained from Lunar and Mars interplanetary landing missions, the vertical landing problem of reusable rockets are started to be handled as *computational guidance and control problems*.

Like the any other control problem, the descent guidance of reusable rockets is also prone to *uncertainties* encountered during the modeling or operational phase of system. The uncertainty sources like sensor faults, actuator faults, modeling errors or environmental changes deteriorate the performance of pinpoint landing or, even worse, completely destabilizes it. In order to recover the landing performance, these disruptive effects should be *eliminated from the guidance and control design*. Thus, two major approaches arise where in the first one, all the possible uncertainties should be somehow modeled and the algorithms are tested in the presence of them. In the second approach, the uncertainties should be somehow *estimated by using available flight data* on the operational phase and should be *eliminated algorithmically*. The online estimation of uncertainties bring us the possibility to utilize adaptive control methods. Particularly in the branch called *model reference adaptive control* (MRAC), it is possible to design controllers that adapting themselves to the time varying uncertainties while guaranteeing some certain stability and performance metrics.

1.1 Powered Descent Guidance

Descent guidance term stands for the guided descending motion of aerospace vehicles, particularly during the landing or diving phases of the mission profile. The distinction of *powered descent guidance* (PDG) is landing in the presence of controlled thrust generation rather than landing like the gliding motion of fixed-wing aircraft or the free fall motion of ballistic missiles. In the PDG phase, the aerospace vehicles such as rockets, rocket stages, shuttles or rover carriers are planned to land to a site determined by the mission requirements while satisfying a certain set of technical constraints. These technical constraints are determined by the design of both mission profile and airframe. Some constraints are given to maintain the safety of landing mission such that specifying terminal velocity or angle constraints. Some others are imposed to maintain the minimum fuel consumption, minimum reaching time and so on. In all aspects, the descent guidance is a challenging engineering problem that requires huge amount of effort and prominent technological developments on both algorithmic and system modeling infrastructure of autonomous landing systems.

In recent years, the reusable rockets take the attention of aerospace industry particularly in the civilian space transport applications. Both governmental organizations and commercial firms headed to develop reusable rocket stages with autonomous vertical take-off and landing capabilities as the carrier vehicles in order to reduce the tremendous costs of space transport. On the other hand, there were successful Mars and Lunar missions where the surface exploration vehicles had accomplished the descent phase repeatedly. Although there exist successful interplanetary landing missions for couple of decades, the promising methods that provide pinpoint capabilities for PDG on interplanetary landing have begun to arise in early 2000s [44, 62, 7, 11]. Based on the experience gained from Mars and Lunar landing missions, the aerospace industry has started to apply these methods in reusable rocket technology.

In the literature of computational guidance and control, the descent guidance problems are extensively solved as *trajectory optimization* or *optimal control* problems. In either case, certain constraints on the flight path or the system inputs are reflected to the parametric optimization problem [26, 32, 12].

In [3], the seminal work of Açıkmeşe & Ploen (2007) showed that Mars pinpoint landing problem can be formulated as a finite-dimensional *convex* trajectory optimization problem with *nonconvex* input constraints. Particularly with *second order cone programming* (SOCP) framework, it is also shown that the convex trajectory optimization can be efficiently solved. In the essence of the study, the nonconvex input constraints are related with the thruster physical considerations and the *convexification* procedure is discussed for the sake of SOCP solutions. Moreover, the optimization problem presented in the study pursues a solution to the *minimum-fuel* type optimal control that, as the name implies, optimizes the thruster inputs to consume least amount of fuel in the presence of convexified constraints.

As an extension to the study in [3], Bai et al. (2019) [4] proposed that the minimum-fuel optimal landing problem can be solved in the presence of multiple protruding cone-shaped obstacles. They have employed a simple *linearization* procedure for the nonconvex obstacle constraints in order to make them convex.

Focusing on the real-time operability of PDG, Reynolds et al. (2020) [45] investigated the nonconvex optimal control of a 6-degrees-of-freedom (6DOF) landing problem. The study is built upon the previous work on the real-time convexification procedures proposed by [58, 35, 9]. In the aforementioned work, a case study of 6DOF PDG with theoretically known optimal solution is conducted and it is shown that the preferred *penalized thrust region* (PTR) method is promising in the sense of real-time solution capabilities.

In [33], Liu (2019) presented a fuel-optimal vertical landing with blended aerodynamic forces and thrust forces. It is reported that the simultaneous presence of these forces makes optimal control problem to hard to handle. In the study of Liu, PDG is again transcribed as a convex optimization problem which is solved as a sequence of SOCP problems. In the proposed method, the convexification of nonconvex constraints and dynamics are intensively addressed.

A benchmark study on reusable launch vehicles and recovery guidance is conducted by Simplicio et al. (2019) [51]. A detailed 6DOF model to reusable launcher is developed where the vehicle is controlled via *thrust vector control*, *cold gas thrusters* and *aerodynamic fins*. Moreover, the designed launch vehicle is guided to land with

proposed *descending over extended envelopes using successive convexification-based optimization* (DESCENDO) algorithm which compromises the efficiency and optimality. The proposed DESCENDO algorithm is applied to PDG problem under the prescribed terminal constraints (i.e. boundary conditions), input constraints on thrust magnitude and direction, and input rate constraints.

Bojun et al. (2019) [8] proposed an *adaptive* entry guidance algorithm for high-precision vertical landing. The algorithm consists of the application of *predictor-corrector* type guidance and the identification of *aerodynamic deviations* via curve fitting approach. In other words, the proposed algorithm combines all kind of aerodynamic uncertainties to correct the predictions which provides some sort of adaptivity to the PDG. In the study, it is reported that the precision of landing is enhanced with the repeated adaptation of onboard prediction model via proposed aerodynamic fitting method.

1.2 Receding Horizon Optimal Control

The *receding horizon optimal control* (RHOC) is a widely-used control strategy that particularly suitable for the problems that require re-planning of the optimal input and state trajectories under given constraints. The term *receding time horizon* is considered as repeatedly solving an optimal control problem over a finite interval of time (i.e. horizon) and shifting this time horizon in each time instant.

In the literature, RHOC is usually referred to as model-based RHOC strategy called *model predictive control* (MPC). The MPC is formulated based on a mathematical model to system and, by using this model, the future states (or outputs) are predicted along the finite time horizon called *prediction horizon*. Utilizing this prediction, an optimal sequence of future control inputs with respect to some certain *performance index* or *objective function* is found. When the input sequence is optimized, some part (usually the first one only) of the optimal input sequence is exerted to the actual controlled system, then the whole procedure is repeated via shifting the time horizon.

In the optimization step of input sequence, the constraints on the input and state (or output) variables can be naturally handled. The constrained formulation of RHOC

optimization step brings us the advantage of reflecting physical considerations to the system while still considering the stability and the optimality of the closed-loop system. The disadvantages of RHOC strategy are discussed mainly on the computational complexity of prediction and optimization steps in each time instant. In the early ages of RHOC, these methods were only suitable for the systems with slowly varying dynamics or large sampling periods due to technological limits in processing units. Over time, the technological developments rendered possible to use RHOC methods in systems with fast dynamics such as aerospace and automotive systems.

The RHOC has a well-developed literature since 1980s since it is adopted by both the industry and the academy. There are lots of seminal works on the theoretical development of MPC as discussed in Mayne et al. (2000) [36]. Since the RHOC has an extensive literature, the works in line with PDG are mentioned for the sake of clarity of thesis work.

An inspiring work of Lee & Mesbahi (2017) [31] on Mars pinpoint landing showed that the model predictive control can be used as the receding horizon solution to fuel-optimal PDG problems. Utilizing the algebraic properties of *dual quaternions*, the 6DOF motion of lander spacecraft is parametrized along the geometric constraints defined by a *landing cone*. It is also shown that the dual quaternions enable the development of representation of motion in the convex subsets of vehicle *configuration space*. Moreover, the stability and the reachability issues of proposed model predictive control are investigated.

Ma et al. (2019) [34] proposed that the optimal PDG problem of reusable rockets can be solved via *receding horizon* strategy and *finite-element collocation* methods. In their formulation, the trajectory optimization problem is addressed as a *nonlinear programming* (NLP) problem after discretizing the state and input variables with collocation approach. In the study, the disruptive effects of atmospheric re-entry, modeling uncertainties and disturbances are considered, and NLP is solved over the specified guidance periods.

Another work on model predictive missile guidance and aircraft landing rather than vertical rocket landing is presented by Hong et al. (2019) [18]. In their study, a new approach called *model predictive convex programming* (MPCP) is proposed to

solve constrained optimal guidance problems in real-time. The constraints on input and state variables are considered along with the convexified *collision avoidance* constraints. Then, the nonlinear system dynamics are linearized in order to solve the convex optimization problem. By using their proposed MPCP approach, two main benchmark problems, namely *impact-angle-constrained* missile guidance and the landing trajectory optimization of aerodynamically controlled aircraft, are solved and the simulation studies are conducted.

1.3 Uncertainty Estimation and Adaptive Control

Mathematical models that represent the behaviour of dynamical systems are insufficient to describe the actual behaviour of systems almost all the time. Under some certain assumptions, the mathematical models might be capable to describe motion, however, due to *unmodeled* or *unexpected* physical phenomena these assumptions are usually violated. Particularly dealing with aerospace systems, using these mathematical models is prone to have physical uncertainties. Therefore, the one who designs the guidance and control architectures should be aware of them when working with model-based designs.

In the vast literature of guidance and control, there are plenty of paradigms that provide the *estimation* or *approximation* of these physical uncertainties. The term *uncertainty estimation* usually refers to gathering the approximate information or values for the uncertainties concurrently, during the operation. Similarly, the term *uncertainty approximation* refers to *represent* the uncertainty with a suitable kind of available knowledge. By a small abuse of terminology, these two terms are used interchangeably along the thesis. There are two paradigms emerged in order to estimate or approximate the uncertainties which are discussed in the following chapters.

The adaptive control theory is, generally speaking, interested in to *adapt* the control design to the internal or external variations in the system parameters. While doing so, it is aimed that the stability and performance of closed-loop system is preserved or recovered up to certain level. The adaptive control designs are usually classified in two groups, namely *direct methods* and *indirect methods* [59, 40]. In direct methods,

the estimated controller parameters (i.e. gains) are directly used. In many occasions, the controller parameters are updated to *match* the nominal design given by a *reference model* and the approach is called *model reference adaptive control* (MRAC) [14, 52, 27, 49]. The indirect methods usually cover the estimation of unknown system parameters, also known as *system identification*, and the approximation of uncertain part of the system in terms of *known* functions. The stability and performance of MRAC are usually analyzed with *Lyapunov's Stability Theorem* and its extensions to *systems with inputs* and *systems with uncertainties* [55, 19, 30].

In the literature of nonlinear and adaptive control, the term *Lyapunov redesign* is used to indicate that a stabilizing feedback control design can be constructed based on the knowledge of *Lyapunov function* that suffices the stability of nominal system [25, 13]. Alternatively, without changing the baseline control parameters, it is possible to augment an *uncertainty matching* control law to the baseline control law, in order to steer the behavior of uncertain system to the nominal one. In general, the *adaptive Lyapunov redesign* and the indirect MRAC designs are indistinguishable.

The literature of adaptive control has become extensive over the years although the emergence of industrial applications are relatively new. Particularly in aerospace industry, the MRAC designs became popular and are adopted by many control practitioners since they give promising results in real-life uncertainty estimation problems. However, the application of MRAC based guidance and control designs to PDG problems is still an open topic and the literature is started to expand recently.

Based on the uncertainty estimation and adaptive Lyapunov redesign, the foundations of this thesis are proposed in Altıntaş & Turgut (2019) [2]. In their study, the uncertainty factors on sensor readings are estimated with online indirect adaptive methods. In the paper, a traditional *Kalman filtering* (KF) framework is extended by wrapping a *neural network* based adaptive controller to the *correction* step of KF. A novel parameter update law that consists of KF terms is proposed to update neural network parameters while preserving the stability in the sense of Lyapunov. As a result, the uncertainty on measurements is eliminated from the state feedback signal which is used by an estimation-based controller. Although the proposition of parameter update law foresees the asymptotic stability, some major points are omitted in the pa-

perwork. In the simulation studies, the sensor characteristics are supposed to have sudden changes and the calibrations are deviated. By using the proposed method, the online calibration of sensors are maintained utilizing an external source and the tracking performance of estimation-based controller is recovered.

1.4 The Outline of the Thesis

The organization of this thesis consists of seven chapters. In the first chapter, the introductory information about powered descent guidance, receding horizon optimal control and uncertainty estimation with adaptive control are given along with the literature review.

In chapter 2, the background knowledge and some theoretical aspects about Kalman filtering, model predictive control and quadratic programming are given in order to clarify the used methods in descent guidance.

The chapter 3 explains how the disturbance states can be augmented to the classical model predictive control formulation. In this chapter, the estimation of input uncertainties are described as the augmented state estimates of Kalman filter.

In chapter 4, the approximation of output uncertainties are discussed. The output uncertainties of closed-loop system is considered as the input uncertainty to the correction dynamics of Kalman filter and it is stated that the functional approximation of these uncertainties can be done by using universal regressor structures. Moreover, the possibility to eliminate output uncertainty by adaptive Lyapunov redesign is introduced.

The chapter 5 extensively discusses the theoretical aspects of adaptive Lyapunov redesign on the Kalman filter. In this chapter, it is shown that when the structure of output uncertainty is known, it is possible to match it completely by a regressor trained with the proposed online parameter adaptation law. Moreover, utilizing the Lyapunov's necessary and sufficient conditions on the stability, the closed-loop system is proven to be input-to-state stable with proposed law. In the second half of chapter, the structure of uncertainty is supposed to be unknown and the effect of mis-

matched portion of uncertainties are investigated. By using convex projections on the proposed adaptation law, it is shown that the closed-loop system attains semiglobal practical asymptotic stability properties.

In chapter 6, the powered descent guidance problem is proposed along with the modeling of vertical landing rocket stage, physical uncertainties, flight path constraints and thrust constraints. In this chapter, the simulation setup is presented and the simulation studies on ideal case, nonideal case without learning and nonideal case with learning are conducted.

In the last chapter, the findings on estimation-based model predictive control with adaptive Lyapunov redesign are concluded and it is claimed that the proposed methods are capable to solve real-time descent guidance problems in the presence of physical uncertainties.



CHAPTER 2

PRELIMINARIES

2.1 Introduction

In this section, the general theory behind *recursive state estimation*, *quadratic programming* (QP) and *model predictive control* (MPC) are restated for the sake of clarity of the thesis. These three topics provide baseline knowledge to formulate and solve linear MPC problems with augmented state estimation.

The theory behind QP is studied for almost a century and now it is well-established with plenty of solution methods. On the other hand, MPC methods are parametric optimization problems of *discretize-then-optimize* type in optimal control theory, and are still advancing. The parametric optimization step of MPC can be directly solved as a QP and, in this section, the transcription of MPC formulation as a QP problem is explained with solution procedures. Moreover, in the case of full state information or disturbance estimation is required by the preferred MPC formulation, the recursive state estimation procedure with Kalman filtering is also explained.

In the following chapters 3, 4, and 5, the main idea of thesis work is built upon the structural properties of state estimator which are explained as a preliminary. Particularly in chapter 3, the traditional MPC method is extended with the augmentation of disturbance states and it is shown that the linear formulation of MPC remains unchanged. In chapter 6, the case studies of proposed method are modeled and simulated with the described QP solution strategies.

2.2 State Estimation with Kalman Filtering

Consider a discrete-time (or discretized) stochastic dynamical system with additive Gaussian white noise.

$$\begin{aligned}\mathbf{x}_k &= f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}) + \mathbf{w}_k \\ \mathbf{y}_k &= h(\mathbf{x}_k) + \mathbf{v}_k\end{aligned}\tag{2.1}$$

where the system states $\mathbf{x} \in \mathbb{R}^n$, the system inputs $\mathbf{u} \in \mathbb{R}^m$ and the system outputs $\mathbf{y} \in \mathbb{R}^l$. The functions $f \in \mathcal{C}^1(X, \mathbb{R}^n)$ and $h \in \mathcal{C}^1(Y, \mathbb{R}^l)$ represent system dynamics and output dynamics with domains $X \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, respectively. The additive Gaussian white noise in the system dynamics is represented by $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, Q_k)$ and in the measurement is represented by $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k)$ with multivariate normal distribution $\mathcal{N}$ of given *mean* and *variance* arguments. The symmetric positive semi-definite matrices $Q \succeq 0 \in \mathbb{S}^n$ and $R \succeq 0 \in \mathbb{S}^l$ are the covariance matrices of process and measurement noise, respectively.

The original formulation of Kalman filtering consists of linear stochastic dynamics to iterate both mean values of states and state prediction covariance under Gaussian noise assumption [23]. In each prediction step, the new states are computed from previous states & inputs by passing through system (i.e. process) dynamics while the prediction covariance is iterated and dispersed based on the linear process dynamics and the covariance of process noise. Moreover, in each correction (update) step, the error between predicted outputs from linear model Algorithm 1 line (2) and the measured outputs from actual dynamics is decreased by a normalizing *Kalman gain*. The normalizing gain is computed based on whether the predictions or measurements are statistically more reliable by putting a weight on prediction and measurement covariances.

Algorithm 1 Kalman Filter Algorithm.

Require: initial $\hat{\mathbf{x}}_{0|0} \in \mathbb{R}^n$, $P_{0|0} \in \mathbb{S}^n$, $Q_0 \in \mathbb{S}^n$, $R_0 \in \mathbb{S}^l$ and given $F \in \mathbb{R}^{n \times n}$,

$$G \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{l \times n}$$

- 1: **for** $k = 1, \dots, N$ **do**
 - 2: $\hat{\mathbf{x}}_{k|k-1} \leftarrow F\hat{\mathbf{x}}_{k-1|k-1} + G\mathbf{u}_{k-1}$ ▷ Project state means
 - 3: $P_{k|k-1} \leftarrow FP_{k-1|k-1}F^\top + Q_k$ ▷ Project prediction covariance
 - 4: $K_k \leftarrow P_{k|k-1}C^\top (CP_{k|k-1}C^\top + R_k)^{-1}$ ▷ Compute Kalman gain
 - 5: $\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k-1} + K_k(\mathbf{y}_k - C\hat{\mathbf{x}}_{k|k-1})$ ▷ Correct state means
 - 6: $P_{k|k} \leftarrow (I - K_kC)P_{k|k-1}$ ▷ Correct prediction covariance
 - 7: $k \leftarrow k + 1$
 - 8: **end for**
-

where the identity matrix is $I \in \mathbb{R}^{n \times n}$ and the symmetric positive definite matrix $P \succ 0 \in \mathbb{S}^n$ denotes the prediction error covariance matrix.

The linearity assumption of Kalman filter is a restrictive assumption since almost all physical systems are nonlinear and operate under possibly non-Gaussian probability distributions. This assumption makes the state estimates of linear Kalman filter to deteriorate from their true values since the Gaussian assumption on probability distributions is no longer valid and the corrections become degenerate. Furthermore, since the linear system dynamics can only represent the nonlinear equivalent on a certain neighborhood, the predictions also become unreliable.

The assumptions on the linearity and Gaussian distributions of Kalman filter can be overcome by various celebrated approaches such as extended Kalman filter (EKF) and unscented Kalman filter (UKF). In summary, the EKF formulation uses the same update rules of Algorithm 1 and instead of using a single linear model, it uses a point-wise *linearization* of (2.1) in each iteration step. By doing so, the Gaussian noise assumption becomes relatively reliable since the actual non-Gaussian probability distribution is caught up by point-wise Gaussian distributions [60]. The formulation of UKF is also shares the idea of linearization but in the statistical sense. Instead of linearizing (2.1), UKF uses *unscented transform* and samples a set of so-called *sigma points*. Then, passes them through nonlinear system dynamics and recomputes the Gaussian distribution, and updates the estimates [22].

In thesis, EKF is preferred over other filtering techniques since the point-wise linearization procedure of process dynamics is highly compatible with the linear MPC formulation. Noting that EKF requires linearized model of (2.1) to project one-step ahead and to update prediction covariance while MPC requires it to project N_p -step ahead in order to create a *prediction model* which will be explained later.

Algorithm 2 Extended Kalman Filter Algorithm.

Require: initial $\hat{\mathbf{x}}_{0|0} \in \mathbb{R}^n$, $P_{0|0} \in \mathbb{S}^n$, $Q_0 \in \mathbb{S}^n$, $R_0 \in \mathbb{S}^l$ and given $f \in \mathcal{C}^1(X, \mathbb{R}^n)$,

$h \in \mathcal{C}^1(Y, \mathbb{R}^l)$ with domains $X \subseteq \mathbb{R}^n \times \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$

```

1: for  $k = 1, \dots, N$  do
2:    $F_k \leftarrow \partial f / \partial \mathbf{x} |_{\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1}}$  ▷ Compute Jacobian matrices
3:    $C_k \leftarrow \partial h / \partial \mathbf{x} |_{\hat{\mathbf{x}}_{k-1|k-1}}$ 
4:    $\hat{\mathbf{x}}_{k|k-1} \leftarrow f(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1})$  ▷ Project state means
5:    $P_{k|k-1} \leftarrow F_k P_{k-1|k-1} F_k^\top + Q_k$  ▷ Project prediction covariance
6:    $K_k \leftarrow P_{k|k-1} C_k^\top (C_k P_{k|k-1} C_k^\top + R_k)^{-1}$  ▷ Compute Kalman gain
7:    $\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k-1} + K_k (\mathbf{y}_k - h(\hat{\mathbf{x}}_{k|k-1}))$  ▷ Correct state means
8:    $P_{k|k} \leftarrow (I - K_k C_k) P_{k|k-1}$  ▷ Correct prediction covariance
9:    $k \leftarrow k + 1$ 
10: end for

```

where the linearization of process dynamics gives Jacobian matrices $F_k \in \mathbb{R}^{n \times n}$ and $C_k \in \mathbb{R}^{l \times n}$.

The alternative way of updating prediction covariance is given by *Joseph form*. In this form of covariance update, the definiteness of prediction covariance matrix is restricted to be strictly positive such as

$$P_{k|k} = (I - K_k C_k) P_{k|k-1} (I - K_k C_k)^\top + K_k R_k K_k^\top. \quad (2.2)$$

The Joseph form of prediction update is computationally less efficient than the standard form Algorithm 1 line (6), however, it guarantees to preserve the positive definiteness (i.e. $P_{k|k} \succ 0, \forall k \in \mathbb{N}$) in each filter recursion.

2.3 Quadratic Programming with Linear Constraints

Problems with quadratic objective functions are the special type of optimization problems which may or may not deal with constraints. In quadratic problems the constraints are either equality or inequality type linear constraints. The QPs have a special place in optimization theory since they are intuitive and arise in many problem formulations just like *linear programming* (LP) counterparts. Furthermore, the nonlinear optimization problems can be also represented via *local* QPs that are ought to be solved *sequentially* to find extrema of nonlinear problem. This approach is one of promoters of the general concept of nonlinear programming methods such as *sequential quadratic programming* (SQP) and *trust region methods* [42]. The general form of quadratic minimization type problem is defined as follows.

Definition 2.3.1. Consider the minimization type optimization problem with quadratic objective function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$

$$\begin{aligned} &\text{minimize} && f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \mathbf{f}^\top \mathbf{x} + c \\ &\text{subject to} && \mathbf{x} \in \Omega \end{aligned} \tag{2.3}$$

where $\mathbf{x} \in \mathbb{R}^n$, $Q \in \mathbb{S}^n$, $\mathbf{f} \in \mathbb{R}^n$, $c \in \mathbb{R}$, and the feasible set defined by the intersection of half-spaces is

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, E\mathbf{x} = \mathbf{h}\}$$

with $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $E \in \mathbb{R}^{l \times n}$, $\mathbf{h} \in \mathbb{R}^l$. Then, the problem is defined as quadratic minimization with the gradient $\nabla f(\mathbf{x}) = Q\mathbf{x} + \mathbf{f}$ and the Hessian matrix $\nabla^2 f(\mathbf{x}) = Q$.

The QPs can be solved directly or iteratively in the presence of equality type or inequality type linear constraints, respectively. Depending on the type of constraints the solution strategies vary and this brings us the definition of *active* and *inactive* constraints.

Remark 2.3.1. It is trivial to find the extremum point $\mathbf{x}^* \in \mathbb{R}^n$ of an unconstrained quadratic problem such that $\nabla f(\mathbf{x}^*) = \mathbf{0}$ from which can be concluded that $Q\mathbf{x}^* + \mathbf{f} = \mathbf{0}$ and $\mathbf{x}^* = -Q^{-1}\mathbf{f}$ for all nonsingular Q .

Definition 2.3.2. Consider the quadratic problem defined in (2.3), then, the inequality constraint at $\mathbf{x}$ is called active if $A_1\mathbf{x} = \mathbf{b}_1$ where A_1 and $\mathbf{b}_1$ are the submatrices generated by the active constraints. Moreover, the inequality constraint at $\mathbf{x}$ is called inactive if it is not active.

2.3.1 Equality Constrained QP

Minimizer to quadratic problem defined in (2.3) with equality constraints or the case where all the constraints are active can be analytically found by solving *Karush-Kuhn-Tucker* (KKT) system.

Theorem 2.3.1 (KKT Condition for QPs). Let $\mathbf{x}^*$ be a regular point to (2.3). Also let it be a local optimum point for (2.3) as well. Then, $\exists \lambda \in \mathbb{R}^l$ and $\exists \mu \in \mathbb{R}^m$ with $\mu \geq \mathbf{0}$ such that

$$\begin{aligned} Q\mathbf{x}^* + \mathbf{f} + \lambda^\top E + \mu^\top A &= \mathbf{0} && \text{(stationarity)} \\ A\mathbf{x}^* - \mathbf{b} \leq \mathbf{0}, E\mathbf{x}^* - \mathbf{h} &= \mathbf{0} && \text{(primal feasibility)} \\ \mu &\geq \mathbf{0} && \text{(dual feasibility)} \\ \mu^\top (A\mathbf{x}^* - \mathbf{b}) &= 0 && \text{(complementary slackness).} \end{aligned} \tag{2.4}$$

Now, consider the equality constrained or all-active constraints case where $E\mathbf{x}^* = \mathbf{h}$, then the KKT system to be solved becomes

$$\begin{bmatrix} Q & E^\top \\ E & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{f} \\ \mathbf{h} \end{bmatrix}. \tag{2.5}$$

Consequently, the optimum solution $\mathbf{x}^*$ and corresponding *Lagrange multipliers* λ to the equality constrained QP in the form of (2.3) can be found by solving (2.5) via direct matrix inversion, elimination or factorization methods [15].

2.3.2 Inequality Constrained QP

Quadratic problems with inequality constraints cannot be handled by solving KKT conditions directly as in the case of equality constraints. Nevertheless, there exist well-established and adopted methodologies such as, generally speaking, searching

optimum on the constraint boundaries or starting from a feasible point and searching optimum in the interior of feasible region.

Returning back to scope of thesis, the optimization part of the MPC problem consists linear constraints. The constraints arising on input and state variables are usually inequality-type and, particularly, they are bound constraints (e.g $|\mathbf{x}| \leq \mathbf{x}_{max}$). It is reasonable since many engineering applications include minimum or maximum allowable levels or safety tolerances on design parameters. Moreover, solution procedure to QPs in MPC should not be computationally inefficient since MPC optimizes the input law in receding horizon fashion which will be discussed later. Thus, considering the inequality constrained QPs in online optimal control problem, an efficient and fast algorithm should be implemented.

2.3.2.1 Active-set Methods

Active-set methods utilize the fact that the optimum is attained on the constraint boundaries if the unconstrained optimum does not belong to the interior of feasible region. Consequently, in mixed type or inequality type constraints case, this fact dictates some constraints to become *active* in the optimum. The set of active constraints is called *active set* and frequently denoted by $\mathcal{A}$.

The algorithm starts with $\mathbf{x}_0$ inside the feasible region. Noting that, for an arbitrary initial point, a *Phase I* linear program can be solved to find a *basic feasible solution* [5] by using A and $\mathbf{b}$. Then, in each iteration step it decides adding or removing constraints from $\mathcal{A}_k$ in order to decrease objective function. In other words, for each step it solves the equality constrained QP with the elements of current $\mathcal{A}_k$ and the direction $\mathbf{d}_k$ is computed. When $\mathbf{d}_k = \mathbf{0}$ is attained, the Lagrange multipliers of equality constrained QP are checked and if all of them are nonnegative then $\mathbf{x}^*$ is returned as the optimizer. Otherwise, the negative Lagrange multiplier and corresponding constraint is removed from $\mathcal{A}_k$ and the procedure is repeated [39].

Algorithm 3 Active-set method.

Require: feasible initial $\mathbf{x}_0 \in \mathbb{R}^n$ and active-set $\mathcal{A}_0$

```
1: for  $k = 0, \dots, N$  do
2:    $\mathbf{x}^* \leftarrow$  solve equality constrained QP in (2.5) with  $\mathcal{A}_k$ 
3:   compute direction  $\mathbf{d}_k \leftarrow \mathbf{x}^* - \mathbf{x}_k$ 
4:   if  $\mathbf{d}_k = \mathbf{0}$  then
5:     if Lagrange multipliers  $\lambda_i \geq 0, \forall i$  then
6:       return  $\mathbf{x}_k$ 
7:     else
8:       remove first  $\lambda_i < 0$  from  $\mathcal{A}_k$ 
9:        $\mathbf{x}_{k+1} \leftarrow \mathbf{x}_k$ 
10:    end if
11:  else
12:     $\alpha_k \leftarrow \min \left\{ 1, \min_{i \notin \mathcal{A}_k} \frac{b_i - \mathbf{a}_i \mathbf{x}_k}{\mathbf{a}_i \mathbf{d}_k} \right\}$ 
13:     $\mathbf{x}_{k+1} \leftarrow \mathbf{x}_k + \alpha_k \mathbf{d}_k$ 
14:    if blocking constraint found then
15:      add it to  $\mathcal{A}_{k+1}$ 
16:    else
17:       $\mathcal{A}_{k+1} \leftarrow \mathcal{A}_k$ 
18:    end if
19:  end if
20:   $k \leftarrow k + 1$ 
21: end for
```

The major drawback of active-set methods is it handles the constraints (i.e. adds or removes from $\mathcal{A}_k$) once at a time and, hence, in the occurrence of many inequality constraints a large number of iterations is needed. This situation arises particularly when the method is initiated from a point where few constraints are active and there are many active constraints in optimum point. Moreover, active-set method solves KKT system in (2.5) with currently active constraints at each iteration which makes the solution cumbersome in the presence of large number of constraints.

2.3.2.2 Primal-Dual Methods

Most active-set methods are called *primal methods* since the solutions are based on primal variables and the active constraints are identified along the optimum primal variables. On the other hand, *dual methods* intensively used for the systematic identification of inactive constraints along Lagrange multipliers as the dual variables.

Few definitions about primal and dual problems are restated in order to clarify the connections between primal active-set methods and *Hildreth's QP procedure* which is, in fact, a dual active-set method explained in the following section.

Definition 2.3.3. Consider a general minimization problem (called as primal problem) with objective function $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and constraints $h : \text{dom}(f) \rightarrow \mathbb{R}^m$, $g : \text{dom}(f) \rightarrow \mathbb{R}^l$

$$\begin{aligned} & \text{minimize} && f(\mathbf{x}) \\ & \text{subject to} && h_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & && g_j(\mathbf{x}) = 0, \quad j = 1, \dots, l. \end{aligned} \tag{2.6}$$

A function $L : \text{dom}(f) \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}$ is called Lagrangian function such that

$$L(\mathbf{x}, \mu, \lambda) = f(\mathbf{x}) + \mu^\top h(\mathbf{x}) + \lambda^\top g(\mathbf{x})$$

where $\mu \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^l$ are called Lagrange multipliers.

Lemma 2.3.2. At each feasible $\mathbf{x} \in \text{dom}(f)$

$$f(\mathbf{x}) = \sup_{\substack{\mu \geq 0 \\ \lambda = 0}} L(\mathbf{x}, \mu, \lambda)$$

and the supremum is taken if and only if $\mu \geq 0$ satisfying $\mu_i h_i(\mathbf{x}) = 0$ for $i = 1, \dots, m$.

Corollary 2.3.2.1. The optimum value of the primal problem $f(\mathbf{x}^*)$ satisfies

$$f(\mathbf{x}^*) = \inf_{\mathbf{x}} \sup_{\substack{\mu \geq 0 \\ \lambda = 0}} L(\mathbf{x}, \mu, \lambda).$$

Then, utilizing the Corollary 2.3.2.1 to Lemma 2.3.2 the dual problem can be defined as a maximization counterpart of the minimization type problem in (2.6).

Definition 2.3.4. Given primal problem (2.6) the Lagrangian dual problem is defined as

$$\begin{aligned} & \text{maximize} && \theta(\mu, \lambda) \\ & \text{subject to} && \mu \geq 0 \end{aligned}$$

and the optimum value of the dual problem $\theta(\mu^*, \lambda^*)$ satisfies

$$\theta(\mu^*, \lambda^*) = \sup_{\substack{\mu \geq 0 \\ \lambda = 0}} \inf_{\mathbf{x}} L(\mathbf{x}, \mu, \lambda).$$

Now, consider the QP in (2.3) with inequality constraints and utilize the aforementioned definitions. Suppose that there exist feasible solutions such that $A\mathbf{x} \leq \mathbf{b}$. Then, the dual problem to the QP can be formulated as

$$\max_{\mu \geq 0} \min_{\mathbf{x}} L(\mathbf{x}, \mu) = \frac{1}{2} \mathbf{x}^\top Q \mathbf{x} + \mathbf{f}^\top \mathbf{x} + \mu^\top (A\mathbf{x} - \mathbf{b}). \quad (2.7)$$

Substituting the unconstrained minimum given by $\mathbf{x}^* = -Q^{-1}(\mathbf{f} + A^\top \mu)$ into (2.7), then the dual problem is written as

$$\begin{aligned} \max_{\mu \geq 0} \theta(\mu^*) &= -\frac{1}{2} \mu^\top (AQ^{-1}A^\top) \mu - \mu^\top (\mathbf{b} + AQ^{-1}\mathbf{f}) \\ &\quad - \frac{1}{2} \mathbf{b}^\top Q^{-1} \mathbf{b} \\ \max_{\mu \geq 0} \theta(\mu^*) &= -\frac{1}{2} \mu^\top H \mu - \mu^\top \mathbf{g} - \frac{1}{2} \mathbf{b}^\top Q^{-1} \mathbf{b} \end{aligned} \quad (2.8)$$

where $H := AQ^{-1}A^\top$ and $\mathbf{g} := \mathbf{b} + AQ^{-1}\mathbf{f}$. Noting that since the constraints are simpler than the ones of primal problem, solving the dual problem is relatively much easier. Finally, primal and dual variables in optimum can be achieved by

$$\begin{aligned} \mathbf{x}^* &= -Q^{-1}\mathbf{f} - Q^{-1}A_i^\top \mu_i^* \\ \mu_i^* &= -(A_i Q^{-1} A_i^\top)^{-1} (\mathbf{b}_i + A_i Q^{-1} \mathbf{f}), \quad \text{for } i \in \mathcal{I} \end{aligned} \quad (2.9)$$

where $\mathcal{I}$ is the index set of corresponding active constraints.

2.3.2.3 Hildreth's Procedure

In order to solve the inequality constrained primal problem (2.3) via its dual problem (2.8), a simple yet efficient algorithm called *Hildreth's QP procedure* with linear convergence rate was proposed [20].

In the algorithm, the direction vectors were selected as basis vectors and each Lagrange multiplier μ_i can be varied individually. In each k^{th} iteration, all minimizing μ_i^k 's are found and if any of them is negative, then, it is set to zero. The procedure is repeated for each component until the stopping condition $\|\mu_i^{k+1} - \mu_i^k\| < \epsilon$ is satisfied and the next iteration is started.

Algorithm 4 Hildreth's QP procedure.

Require: initial Lagrange multipliers μ_i^0

- 1: find unconstrained minimum $\mathbf{x}^0 \leftarrow -Q^{-1}\mathbf{f}$
 - 2: **if** $A_i\mathbf{x}_i^0 \leq \mathbf{b}_i, \forall i = 1, \dots, m$ **then**
 - 3: $\mathbf{x}^* \leftarrow \mathbf{x}^0$
 - 4: $\mu^* \leftarrow \mu^0$
 - 5: **return** $\mathbf{x}^*, \mu^*$
 - 6: **end if**
 - 7: form matrices $H \leftarrow A Q^{-1} A^\top$ and $\mathbf{g} \leftarrow \mathbf{b} + A Q^{-1} \mathbf{f}$
 - 8: **for** $k = 1, \dots, N$ **do**
 - 9: find minimizing and nonnegative μ_i^{k+1}
 - 10: $w_i^{k+1} \leftarrow -\frac{1}{H_{ii}} \left\{ \mathbf{g}_i + \sum_{j=1}^{i-1} H_{ij} \mu_j^{k+1} + \sum_{j=i+1}^m H_{ij} \mu_j^k \right\}$
 - 11: $\mu_i^{k+1} \leftarrow \max \{0, w_i^{k+1}\}$
 - 12: **if** stopping condition is satisfied $\forall i = 1, \dots, m$ **then**
 - 13: $\mu^* \leftarrow \mu^{k+1}$
 - 14: $\mathbf{x}^* \leftarrow \mathbf{x}^0 - Q^{-1} A^\top \mu^{k+1}$
 - 15: **return** $\mathbf{x}^*, \mu^*$
 - 16: **else**
 - 17: $\mu^{k+1} \leftarrow \mu^k$
 - 18: $k \leftarrow k + 1$
 - 19: **end if**
 - 20: **end for**
-

Although Hildreth's method is not a recently found solution technique it is quite popular around the optimization practitioners and still one the best alternative while solving inequality constrained medium-to-large scale QPs due its simplicity. Since the method is totally based on element-wise search, it does not require to execute any matrix operation when updating the dual variables.

Another major advantage of Hildreth's procedure is the algorithm does not fall into the ill-conditioned situations where conflicts between constraints occur. Numerical advantages and having a simple formulation make method plausible to use in real-time optimization problems like receding horizon MPC case and, hence, Hildreth's QP procedure is preferred over the other inequality constrained QP solution methods such as active-set or *interior point methods* [61] in this study.

2.4 Linear Model Predictive Control

The MPC formulation is based on a system model which predicts the future behavior of the actual system and optimizes the future sequence of control inputs to minimize some certain performance index (i.e. cost function). The prediction of future system states is usually done by sequential simulations over the discrete-time linear system model known as the *prediction model*.

Utilizing the linearity between the current states and future outputs, finding the optimal sequence of controls turns into a QP which needs to be solved at each step. Moreover, linear constraints on input variables and state variables may make the problem inequality-type QPs. In summary, in the each step of linear MPC problem, the prediction of future states (or outputs) is done via linear discrete-time representation of nonlinear system, then the input sequence is optimized with respect to desired quadratic performance index by solving the corresponding QP. Furthermore, if there exist limits on controls or states, the limits are turned to be the inequality constraints of QP.

2.4.1 Prediction Model

Consider a discrete-time linear time-invariant (LTI) system in the following form

$$\begin{aligned}\mathbf{x}_{k+1} &= F\mathbf{x}_k + G\mathbf{u}_k \\ \mathbf{y}_k &= C\mathbf{x}_k + D\mathbf{u}_k\end{aligned}\tag{2.10}$$

where the system states $\mathbf{x} \in \mathbb{R}^n$, system inputs $\mathbf{u} \in \mathbb{R}^m$ and system outputs $\mathbf{y} \in \mathbb{R}^l$. In addition, the corresponding state-space matrices are $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m}$,

$C \in \mathbb{R}^{l \times n}$ and $D \in \mathbb{R}^{l \times m}$. Then, by using the states at k^{th} sample and a sequence of control inputs, the future outputs within *prediction horizon*, N_p can be obtained as follows.

$$\begin{aligned}
\mathbf{y}_k &= C\mathbf{x}_k + D\mathbf{u}_k \\
\mathbf{y}_{k+1} &= C(F\mathbf{x}_k + G\mathbf{u}_k) + D\mathbf{u}_{k+1} \\
\mathbf{y}_{k+2} &= CF(F\mathbf{x}_k + G\mathbf{u}_k) + CG\mathbf{u}_{k+1} + D\mathbf{u}_{k+2} \\
&\vdots \\
\mathbf{y}_{k+N_p} &= CF^{N_p-1}(F\mathbf{x}_k + G\mathbf{u}_k) + CF^{N_p-2}G\mathbf{u}_{k+1} + \dots + D\mathbf{u}_{k+N_p}
\end{aligned}$$

Then, the pattern of the forward solution map leads to the following sequential simulation matrices.

$$\begin{aligned}
\xi_k &= \begin{bmatrix} \mathbf{y}_k \\ \mathbf{y}_{k+1} \\ \mathbf{y}_{k+2} \\ \vdots \\ \mathbf{y}_{k+N_p} \end{bmatrix}, \quad \eta_k = \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k+1} \\ \mathbf{u}_{k+2} \\ \vdots \\ \mathbf{u}_{k+N_p} \end{bmatrix}, \quad \Phi = \begin{bmatrix} C \\ CF \\ CF^2 \\ \vdots \\ CF^{N_p} \end{bmatrix}, \\
\Gamma &= \begin{bmatrix} D & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ CG & D & \mathbf{0} & \dots & \mathbf{0} \\ CFG & CG & D & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CF^{N_p-1}G & CF^{N_p-2}G & CF^{N_p-3}G & \dots & D \end{bmatrix}
\end{aligned} \tag{2.11}$$

In a compact form the prediction model can be expressed as

$$\xi_k = \Phi\mathbf{x}_k + \Gamma\eta_k \tag{2.12}$$

where the input and output variables of the prediction model are $\eta_k \in \mathbb{R}^{mN_p}$ and $\xi_k \in \mathbb{R}^{lN_p}$, respectively.

2.4.2 Optimal Control Problem

In the MPC formulation, the synthesis of future input sequence is, in fact, a parametric optimization problem such that the minimum of some certain cost function is searched over the feasible region and the minimizer is applied to the actual system. Then,

the whole parametric optimization procedure is repeated with the new iterates. This approach brings us the *receding horizon* concept or, in other words, the minimization of a running cost over a shifting fixed-length finite time horizon.

Consider a linear quadratic (LQ) optimal control problem such that a quadratic objective is minimized over the prediction horizon for LTI system in (2.10), then, the LQ receding horizon optimal control problem (i.e. linear MPC problem) can be presented as follows.

Theorem 2.4.1. *Let $J \in \mathcal{C}^2(S, \mathbb{R})$ with $S = \mathcal{U} \times \mathcal{X} \times \mathbb{R}^l$, then N_p -step optimal control problem with quadratic objective is*

$$\begin{aligned} \text{minimize} \quad & J(\mathbf{u}_k, \mathbf{x}_k, \mathbf{y}_k^d) = \sum_{i=0}^{N_p} \|\mathbf{y}_{k+i} - \mathbf{y}_{k+i}^d\|_{R_1}^2 + \|\mathbf{u}_{k+i}\|_{R_2}^2 \\ \text{subject to} \quad & \mathbf{x}_{k+1} = F\mathbf{x}_k + G\mathbf{u}_k \\ & \mathbf{y}_k = C\mathbf{x}_k + D\mathbf{u}_k \\ & \mathbf{u}_k \in \mathcal{U}, \mathbf{x}_k \in \mathcal{X} \end{aligned} \tag{2.13}$$

where $\mathbf{y}^d \in \ell^\infty([0, \infty), \mathbb{R}^l)$ is the bounded desired set-points on output variables, $\|\cdot\|_R$ is the standard weighted Euclidean norm with $R_1 \succeq 0 \in \mathbb{S}^l$, $R_2 \succ 0 \in \mathbb{S}^m$, and the feasible set of admissible controls $\mathcal{U} = \{\mathbf{u}_k \in \mathbb{R}^m : W_1\mathbf{u}_k \leq \mathbf{z}_1\}$ and feasible set of states $\mathcal{X} = \{\mathbf{x}_k \in \mathbb{R}^n : W_2\mathbf{x}_k \leq \mathbf{z}_2\}$. Furthermore, using the prediction model (2.11)-(2.12), the original optimal control problem (2.13) is transcribed as an inequality constrained QP such that

$$\begin{aligned} \text{minimize} \quad & \tilde{J}(\eta_k; \mathbf{x}_k, \xi_k^d) = \frac{1}{2}\eta_k^\top Q\eta_k + \mathbf{f}_k^\top \eta_k \\ \text{subject to} \quad & \eta_k \in \Omega_k \end{aligned} \tag{2.14}$$

where $Q := 2(\Gamma^\top \tilde{R}_1 \Gamma + \tilde{R}_2)$ and $\mathbf{f}_k := 2\Gamma^\top \tilde{R}_1(\Phi\mathbf{x}_k - \xi_k^d)$ with $\xi_k^d := \mathbf{1}_{N_p} \otimes \mathbf{y}_k^d$, $\tilde{R}_1 := I_{N_p} \otimes R_1$ and $\tilde{R}_2 := I_{N_p} \otimes R_2$. Then, the control sequence at k^{th} step $\eta_k^* = \arg \min_{\eta \in \Omega} \tilde{J}(\eta_k; \mathbf{x}_k, \xi_k^d)$ is solution to the optimal control problem in (2.13) for the LTI system given in (2.10).

In the operation of MPC, the QP defined in (2.14) is required to be solved at each iteration and the minimizing input sequence η_k^* is exerted to the actual system at each step with only the first element of the sequence.

There are several issues are reported over the years about the *feasibility* and the *stability* of MPC while solving in the form of a constrained QP. From another point of view, the feasibility and optimality of each MPC subproblem can be reduced to the *second order necessary conditions* (SONC) for optimality of QP in the sense that definitions from classical optimization theory is applied.

Definition 2.4.1 (Feasibility of Linear MPC). *Let Ω_k be a region defined by the input and state constraints, then solution η_k to problem (2.14) is called feasible if $\eta_k \in \Omega_k$. Moreover, a vector $\mathbf{p}_k$ is called feasible direction at η_k if $\exists \alpha_0 > 0$ such that $\eta_k + \alpha \mathbf{p}_k \in \Omega_k \forall \alpha \in [0, \alpha_0]$.*

Theorem 2.4.2 (SONC for Linear MPC). *Let $\Omega_k \subset \mathbb{R}^{mN_p}$, $J_k \in \mathcal{C}^2(\Omega_k)$ and η_k^* is a local minimizer to problem (2.14). Then,*

1. $\langle Q\eta_k^* + \mathbf{f}_k, \mathbf{p}_k \rangle \geq 0$
2. if $\langle Q\eta_k^* + \mathbf{f}_k, \mathbf{p}_k \rangle = 0$ then $\langle \mathbf{p}_k, Q\mathbf{p}_k \rangle \geq 0$

for any $\mathbf{p}_k$ feasible at η_k^* .

Noting that for the interior case (i.e. $\eta_k \in \text{Int}(\Omega_k)$), the MPC problem is feasible $\forall \mathbf{p}_k \in \mathbb{R}^{mN_p}$ and reduced to unconstrained minimization-type QP such that the optimality necessitates $R_2 \succ 0$ and $Q \succ 0$ which also implies the *Legendre-Clebsch Condition* for the optimal control problem [47].

The stability and convergence of MPC are complicated functions of design parameters N_p, R_1, R_2 and constraint boundaries. The following theorem and the sketch of proof [6] are presented in order to clarify the convergence of MPC in terms of design parameters.

Theorem 2.4.3 (Bemporad, Chisci, & Mosca, 1994). *Consider LTI system in (2.10) and the MPC control law based on optimizing*

$$\begin{aligned}
 J^*(\mathbf{x}_k) &= \min_{\mathbf{u}_k} \sum_{i=0}^{N_p-1} \|\mathbf{x}_{k+i}\|_{R_1}^2 + \|\mathbf{u}_{k+i}\|_{R_2}^2 \\
 \text{s.t. } \mathbf{x}_{k+1} &= F\mathbf{x}_k + G\mathbf{u}_k \\
 \mathbf{x}_k &\in \mathcal{X}, \mathbf{u}_k \in \mathcal{U}, \mathbf{x}_{N_p} = \mathbf{0}
 \end{aligned}$$

with $R_1, R_2 \succ 0$. If the optimization problem is feasible at $k = 0$, then, $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{0}$ and $\lim_{k \rightarrow \infty} \mathbf{u}_k = \mathbf{0}$ and the constraints are satisfied $\forall k \in \mathbb{N}$ and $\forall R_1, R_2 \succ 0$.

Proof of Theorem 2.4.3. Let $\eta_k = [\mathbf{u}_k \quad \mathbf{u}_{k+1} \quad \dots \quad \mathbf{u}_{k+N_p-1}]^\top$ be the input sequence at k^{th} iteration. Then, by the construction $\eta_{k+1} = [\mathbf{u}_{k+1} \quad \mathbf{u}_{k+2} \quad \dots \quad \mathbf{0}]^\top$ is feasible at $k + 1^{st}$ iteration. Applying the principle of optimality, the cost of sequence η_{k+1} gives

$$J^*(\mathbf{x}_k) \geq \langle \mathbf{x}_k, R_1 \mathbf{x}_k \rangle + \langle \mathbf{u}_k, R_2 \mathbf{u}_k \rangle + J^*(\mathbf{x}_{k+1}) \quad \forall k \in \mathbb{N}$$

Since $J^*(\mathbf{x}_k)$ is monotonically decreasing and bounded from below, there exists a limit point such that $\lim_{k \rightarrow \infty} J^*(\mathbf{x}_k) < \infty$. Moreover,

$$J^*(\mathbf{x}_k) - J^*(\mathbf{x}_{k+1}) \geq \langle \mathbf{x}_k, R_1 \mathbf{x}_k \rangle + \langle \mathbf{u}_k, R_2 \mathbf{u}_k \rangle \geq \epsilon > 0 \quad \forall k \in \mathbb{N}$$

Thus, $\mathbf{x}_k \rightarrow \mathbf{0}$ and $\mathbf{u}_k \rightarrow \mathbf{0}$ as $k \rightarrow \infty$ and $\epsilon \rightarrow 0$ since $R_1, R_2 \succ 0$. $\square$

2.4.3 Constraints & Extensions

Linear inequality constraints may or may not be the part of the solution of QP in (2.14) and in the unconstrained or interior case the optimal input sequence is given by $\nabla \tilde{J}(\eta_k^*; \mathbf{x}_k, \mathbf{r}_k) = \mathbf{0}$, with $\tilde{R}_1 \succeq 0$ and $\tilde{R}_2 \succ 0$, hence,

$$\begin{aligned} \eta_k^* &= -Q^{-1} \mathbf{f}_k \\ \eta_k^* &= -(\Gamma^\top \tilde{R}_1 \Gamma + \tilde{R}_2)^{-1} \Gamma^\top \tilde{R}_1 (\Phi \mathbf{x}_k - \xi_k^d) \end{aligned} \tag{2.15}$$

which is nothing but the solution to discrete-time *Riccati equation* of the corresponding LQ problem. In other words, the linear MPC problem is equivalent to *linear quadratic regulator* (LQR) problem at interior case with polyhedral feasible region constructed around the origin by linear constraints [50].

Focusing on the MPC with inequality constraints, the performance index is minimized over the sequence of input variables η_k only and all the constraints must be defined in the form of $A\eta_k \leq \mathbf{b}$. However, it is not a handicap for MPC since the large variety of constraints can be imposed by using different combinations of states and controls. Common linear inequality-type constraint implementations in the QP part of MPC are exemplified as follows.

1. Input magnitude constraints which prevent the input signals to exceed the actuator physical limits, $|\mathbf{u}_k| \leq u_{max}$,

$$\Omega_k = \left\{ \eta_k \in \mathbb{R}^{mN_p} : \begin{bmatrix} I_{N_p} \\ -I_{N_p} \end{bmatrix} \eta_k \leq \mathbf{1} u_{max} \right\}.$$

2. Output (or state if $C = I_n$) magnitude constraints which prevent the system outputs to exceed the physical limits of the actual system, $|\mathbf{y}_k| \leq y_{max}$. By using (2.12),

$$\Omega_k = \left\{ \eta_k \in \mathbb{R}^{mN_p} : \begin{bmatrix} \Gamma \\ -\Gamma \end{bmatrix} \eta_k \leq \mathbf{1} y_{max} + \begin{bmatrix} -I_{N_p} \\ I_{N_p} \end{bmatrix} \Phi \mathbf{x}_k \right\}.$$

Furthermore, besides the constraints, the performance index can be also manipulated to minimize the rate of change of input or state variables as well. Let $\Delta \mathbf{u}_k = \mathbf{u}_k - \mathbf{u}_{k-1}$ be the change in control commands and consider the addition of a quadratic term which penalizes the control effort in each step. Then,

$$J_1(\mathbf{u}_k, \mathbf{x}_k, \mathbf{y}_k^d) := J(\mathbf{u}_k, \mathbf{x}_k, \mathbf{y}_k^d) + \sum_{i=0}^{N_p} \rho \|\Delta \mathbf{u}_{k+i}\|^2 \quad (2.16)$$

where $\rho > 0$ is some scalar to weight the cost of control efforts. Then, the QP in (2.14) is slightly modified with the new Hessian matrix

$$Q_1 := Q + 2\rho D^\top D \quad \text{s.t.} \quad D := \begin{bmatrix} -\mathbf{1} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{1} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -\mathbf{1} \end{bmatrix} \quad (2.17)$$

where $D \in \mathbb{R}^{mN_p \times mN_p}$ is the matrix of first backward differences with suitable dimensions of each entry.



CHAPTER 3

ESTIMATION-BASED MODEL PREDICTIVE CONTROL

3.1 Introduction

Standard MPC formulation requires the full state information about dynamical system while solving QP defined in (2.14). In many practical applications having a full state information is, unfortunately, not possible since some of the states can be directly measured via physical devices while some of them cannot be measured. Therefore, in order to have a full state vector the missing ones need to be estimated. There are plenty of methods which estimate the states and feed them to the MPC optimization step. However, even if the states are available or they are estimated, the uncertainties or disturbances on the states degrade the performance of the controller as expected.

The state estimation problem can be solved by using KF or variants as discrete-time *state observers*. As described in section 2.2, the KF requires a vector of available measurements to correct the model predictions and estimate the full state vector. Under some certain assumptions, this brings us the idea of including the vector of (input) disturbances to the unmeasured portion of *extended* full state vector which includes both the system states and the disturbance states. Then, by using a suitable state observer (e.g. KF) the disturbance states acting on the system can be estimated with the knowledge of measured states in the statistical sense.

3.2 Linear MPC with Augmented State Estimation

Consider the discrete-time system in (2.1) with the system states $\mathbf{x} \in \mathbb{R}^n$, the system inputs $\mathbf{u} \in \mathbb{R}^m$ and the system outputs $\mathbf{y} \in \mathbb{R}^l$. Again consider the functions $f \in$

$\mathcal{C}^1(X, \mathbb{R}^n)$ and $h \in \mathcal{C}^1(Y, \mathbb{R}^l)$ represent system dynamics and output dynamics with domains $X \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, respectively. Let the additive input and output uncertainties are locally Lipschitz-continuous functions of available states such that $d^i : X \rightarrow \mathbb{R}^m$ and $d^o : Y \rightarrow \mathbb{R}^l$, respectively.

$$\begin{aligned}\mathbf{x}_k &= f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1} + d^i(\mathbf{x}_{k-1}, \mathbf{u}_{k-1})) + \mathbf{w}_k \\ \mathbf{y}_k &= h(\mathbf{x}_k) + d^o(\mathbf{x}_k) + \mathbf{v}_k\end{aligned}\tag{3.1}$$

where the additive Gaussian white noise in the system dynamics is represented by $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, Q_k)$ and in the measurement is represented by $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k)$ once again.

Now, consider a linear prediction model of the above system with Jacobian matrices in the sense of a linear system in the form (2.10) and let $\nu_k := \mathbf{u}_k + d^i(\mathbf{x}_{k-1})$ where the input uncertainty is assumed to be a function of system states only. Thus,

$$F = \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\mathbf{x}_0, \mathbf{u}_0}, \quad G = \left. \frac{\partial f}{\partial \nu} \frac{\partial \nu}{\partial \mathbf{u}} \right|_{\mathbf{x}_0, \nu_0}, \quad C = \left. \frac{\partial h}{\partial \mathbf{x}} \right|_{\mathbf{x}_0}$$

and noting that $\partial \nu / \partial \mathbf{u} = I_m$. Suppose that $\hat{\mathbf{d}}_k^i \in \text{ran}(d^i)$ and $\hat{\mathbf{d}}_k^o \in \text{ran}(d^o)$ are the estimated images of d^i and d^o at k^{th} sample instance, respectively.

$$\begin{aligned}\hat{\mathbf{x}}_{k|k-1} &= F\hat{\mathbf{x}}_{k-1|k-1} + G(\mathbf{u}_{k-1} + \hat{\mathbf{d}}_{k-1|k-1}^i) \\ \hat{\mathbf{d}}_{k|k-1}^i &= \hat{\mathbf{d}}_{k-1|k-1}^i \\ \hat{\mathbf{y}}_k &= C\hat{\mathbf{x}}_{k|k-1}\end{aligned}\tag{3.2}$$

Here, the input uncertainty is assumed to be constant at each prediction step while the output uncertainty still remains as an uncertainty. Then, the estimate of input uncertainty $\hat{\mathbf{d}}^i$ can be augmented as the *unmeasured* portion of state estimates and the new prediction model can be written as follows.

$$\begin{aligned}\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{d}}^i \end{bmatrix}_{k|k-1} &= \begin{bmatrix} F & G \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{d}}^i \end{bmatrix}_{k-1|k-1} + \begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} \mathbf{u}_{k-1} \\ \mathbf{y}_k &= \begin{bmatrix} C & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{d}}^i \end{bmatrix}_{k|k-1}\end{aligned}\tag{3.3}$$

Moreover, the augmented prediction model can be also written in a compact form

with $\hat{\mathbf{z}}_{k|k-1} := \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{d}}^i \end{bmatrix}_{k|k-1}^\top$ such that

$$\begin{aligned}\hat{\mathbf{z}}_{k|k-1} &= \tilde{F}\hat{\mathbf{z}}_{k-1|k-1} + \tilde{G}\mathbf{u}_{k-1} \\ \hat{\mathbf{y}}_k &= \tilde{C}\hat{\mathbf{z}}_{k|k-1}.\end{aligned}\tag{3.4}$$

Noting that the design of MPC and the KF for discrete-time linear dynamical system (3.4) necessitates the *controllability* and *observability* notions of control theory.

Definition 3.2.1. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{l \times n}$, and the controllability matrix be

$$W_c(A, B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}.$$

Then, the pair (A, B) is controllable if and only if $\text{rank}(W_c(A, B)) = n$, in a similar sense, the pair (A, C) is observable if and only if $\text{rank}(W_c(A^\top, C^\top)) = n$.

The augmentation of disturbance states arises a question that whether the disturbance states are observable through the available portion of augmented state vector.

Lemma 3.2.1 (Hautus Criteria). Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{l \times n}$ the following are equivalent.

1. The pair (A, B) is controllable.
2. For all $\lambda \in \mathbb{C}$ that are eigenvalues of A it holds that $\text{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} = n$.

In a similar sense, the following are also equivalent.

1. The pair (A, C) is observable.
2. For all $\lambda \in \mathbb{C}$ that are eigenvalues of A it holds that $\text{rank} \begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n$.

Utilizing Lemma 3.2.1 [54], it can be shown that the system (3.4) is observable if and only if $\text{rank}(W_c(\tilde{F}^\top, \tilde{C}^\top)) = n + m$. Equivalently, assume that the pair (F, C) is observable, then, the augmented system is also observable with the eigenvalues of associated augmented states $\lambda_{n+1} = \lambda_{n+2} = \dots = \lambda_{n+m} = 1$ such that

$$\text{rank} \begin{bmatrix} \lambda_j I - \tilde{F} \\ \tilde{C} \end{bmatrix} = n + m \quad \text{for } j = n + 1, n + 2, \dots, n + m.$$

The same filter recursions can be made to estimate input uncertainties as well as the system states by using the augmented prediction error covariance matrix and the process noise covariance matrix $P \succ 0 \in \mathbb{S}^{n+m}$ and $Q \succeq 0 \in \mathbb{S}^{n+m}$, and the estimation of uncertainty requires an initial guess on the distribution of the augmented states.

Standard MPC formulation can be unified with this augmented estimation procedure to make controller more robust to input uncertainties. This is, in fact, using MPC with the full state information augmented with uncertainty states which are estimated by KF.

Remark 3.2.1. *The synthesis of control sequence via (2.14) is unchanged except the N_p -step prediction model generated by (3.4) such that*

$$\hat{\xi}_k = \tilde{\Phi} \hat{\mathbf{z}}_{k|k-1} + \tilde{\Gamma} \eta_k. \quad (3.5)$$

Then, the optimal control sequence can be found by solving a similar constrained QP with the augmented prediction model matrices.

$$\eta_k^* = \arg \min_{\eta \in \Omega} \frac{1}{2} \eta_k^\top H \eta_k + \tilde{\mathbf{f}}_k^\top \eta_k \quad (3.6)$$

where $H := 2(\tilde{\Gamma}^\top \tilde{R}_1 \tilde{\Gamma} + \tilde{R}_2)$ and $\tilde{\mathbf{f}}_k := 2\tilde{\Gamma}^\top \tilde{R}_1(\tilde{\Phi} \hat{\mathbf{z}}_{k|k-1} - \xi_k^d)$ with $\xi_k^d := \mathbf{1}_{N_p} \otimes \mathbf{y}_k^d$, $\tilde{R}_1 := I_{N_p} \otimes R_1$ and $\tilde{R}_2 := I_{N_p} \otimes R_2$ as previously stated.

The aforementioned estimation-based control procedure uses the estimates of input uncertainties and, then, optimizing the control sequence based on the approximate knowledge of input uncertainties.

A similar approach can be used to estimate the output uncertainties (e.g. augmenting $\hat{\mathbf{d}}_k^o$ to the original states), however there are several issues that make the problem to be handled more extensively which are also discussed in [46].

CHAPTER 4

FUNCTIONAL APPROXIMATION OF UNCERTAINTIES

4.1 Introduction

There are two paradigms used while estimating or predicting the disturbances. The first approach assumes the disturbances as an augmented state and estimates it via standard state estimation procedures [46]. While doing so, the disturbances are presumed to have some dynamics which are augmented to the system dynamics.

In the second approach, a suitable *representation* of these disturbances are tried to be found in terms of *known* functions in the sense of function interpolation or regression. The functional approximation of uncertainty is a well-studied topic and can be done by various regression methods such as *least-squares methods*, *polynomial expansions*, *series expansions*, *splines* and as universal approximators the *neural networks* according to the *structure* of uncertainty. However, these methods are valid only for specific cases where the disturbances or uncertainties are assumed to be acting on the inputs or outputs separately.

Consider the correction step of conventional KF where the probability distribution of predicted states is combined with the probability distribution of measurements and normalized with the Kalman gain. Consequently, both the mean and variance of predicted states are updated according to the information gathered from *external measurements*. Then, an *uncompensated* output uncertainty appears on the information coming from external measurements and degrades the correction performance (as well as the estimation performance) of the filter. Consequently, this degradation causes both state and disturbance estimates (in extended state estimation case) to be biased [2] or leads to instability problems in the estimator. Naturally, the uncertainty

in measurements which are involved in the correction step of KF (i.e. output uncertainty of the system (3.1)) should be eliminated.

4.2 Approximation of Uncertainty Functions in State Estimation

Consider the update step of KF for the estimation of extended state information such that the input disturbance states are augmented to the system states and also suppose the bounded uncertain external measurements $\mathbf{y}_k \in \ell^\infty(\mathbb{R}^q)$ with $q \leq l$ such that

$$\hat{\mathbf{z}}_{k|k} = \hat{\mathbf{z}}_{k|k-1} + K_k (\mathbf{y}_k - C\hat{\mathbf{z}}_{k|k-1}) \quad (4.1)$$

and convert it to a discrete-time dynamical system using the associative rule as

$$\hat{\mathbf{z}}_{k|k} = (I - K_k C) \hat{\mathbf{z}}_{k|k-1} + K_k \mathbf{y}_k. \quad (4.2)$$

Suppose that the output uncertainty $d^o \in \ell^\infty(U, \mathbb{R}^q)$ is a function of augmented state estimates which is locally summable and bounded (i.e. in $L^\infty(\mathbb{N})$ with counting *measure*), within the domain $U \subset \mathbb{R}^{n+m}$, i.e.

$$\sup_{k \in \mathbb{N}} |d^o(\hat{\mathbf{z}}_k)| < \infty.$$

Let $\mathbf{y}_k^* \in \mathbb{R}^q$ be the *true measurements*, then, (4.2) can be written in terms of true measurements and additive bounded uncertainty function as

$$\hat{\mathbf{z}}_{k|k} = (I - K_k C) \hat{\mathbf{z}}_{k|k-1} + K_k (\mathbf{y}_k^* + d^o(\hat{\mathbf{z}}_k)). \quad (4.3)$$

Noting that for the system defined by (4.3), it is possible to create an analogy between input uncertainty case such that d^o is the additive input uncertainty to the correction dynamics of KF. Now, consider a *reference model* for the correction step which consists of *true augmented state estimates* $\hat{\mathbf{z}}_{k|k}^* \in \mathbb{R}^{n+m}$ and true measurements as well.

$$\hat{\mathbf{z}}_{k|k}^* = (I - K_k C) \hat{\mathbf{z}}_{k|k-1}^* + K_k \mathbf{y}_k^* \quad (4.4)$$

Remark 4.2.1. *If the functional approximation of system output uncertainty d^o is obtained, then, it can be conjectured that the actual augmented state estimates $\hat{\mathbf{z}}_{k|k}$ can be steered to the true augmented state estimates $\hat{\mathbf{z}}_{k|k}^*$.*

The idea behind approximation of d^o is, basically, expanding it on a space of known *basis functions* $\Phi : U \rightarrow \mathbb{R}^N$ and superposing them via *regression coefficients* (or weights) $\Theta \in \mathbb{R}^{N \times l}$ which are ought to be found. As a universal approximator *Radial Basis Function Neural Networks* (RBFNN) can be used to approximate [38, 43] the unknown disturbances or uncertainty functions in an *unstructured* sense. In order to clarify the universal function approximators, some mathematical definitions are given as follows.

Definition 4.2.1. Let the Gaussian function $\phi : \mathbb{R}^p \rightarrow \mathbb{R}$ be in the form of

$$\phi(\mathbf{x}; \mathbf{c}, R) = \exp \left\{ -\|\mathbf{x} - \mathbf{c}\|_R^2 \right\} \quad (4.5)$$

with the input argument $\mathbf{x} \in \mathbb{R}^p$, the given arguments $\mathbf{c} \in \mathbb{R}^p$, $R \in \mathbb{S}^p$ and the weighed Euclidean norm $\|\cdot\|_R$, also suppose that $\phi \in L^1(\mathbb{R}^p)$, then, it is called *Radial Basis Function (RBF)*. Furthermore, let the linear combination of RBFs be

$$\begin{aligned} \Theta^\top \Phi(\mathbf{x}) &= \sum_{i=1}^N \theta_i \phi_i(\mathbf{x}; \mathbf{c}_i, R_i) + \mathbf{b} \\ &= \begin{bmatrix} \theta^\top & \mathbf{b} \end{bmatrix} \begin{bmatrix} \phi_1(\mathbf{x}; \mathbf{c}_1, R_1) \\ \vdots \\ \phi_N(\mathbf{x}; \mathbf{c}_N, R_N) \\ 1 \end{bmatrix} \end{aligned} \quad (4.6)$$

where $\Phi \in \mathbb{R}^{N+1}$ is the array of regressors, $\Theta \in \mathbb{R}^{(N+1) \times q}$ are the weights, $\mathbf{b} \in \mathbb{R}^q$ are the bias terms with the centers of reception $\mathbf{c}_i$, then, the combination is called *Radial Basis Function Neural Network (RBFNN)*.

Definition 4.2.2. Consider a Gaussian function ϕ with the weight matrix $R_i = 1/2\sigma_i^2$ where σ_i denotes the width of i^{th} basis function. Then, the RBF is called *isotropic* if the width of Gaussian is fixed according to the spread of centers $\mathbf{c}_i$ with $\sigma_i = d_{max}/\sqrt{2N}$ and is in the form

$$\phi_i(\mathbf{x}) = \exp \left\{ -\frac{N}{d_{max}^2} \|\mathbf{x} - \mathbf{c}_i\|^2 \right\} \quad (4.7)$$

where N is the number of centers and d_{max} is the maximum distance between centers.

Using the definitions of RBFNNs, the *Universal Approximation Theorem for RBFNNs* [43] can be adapted to the functional approximation of output uncertainty with isotropic RBFNNs.

Theorem 4.2.1. *Let $\phi \in L^\infty(U, \mathbb{R})$ with $U \subseteq \mathbb{R}^p$ be a bounded and integrable function, assume that*

$$\int_U \phi(\mathbf{x}; \mathbf{c}, R) d\mathbf{x} \neq 0.$$

Then, for any function $d \in \mathcal{C}^1(U)$ which represents the uncertainty and ball with radius $\delta > 0$, there exists an isotropic RBFNN structure with the approximation tolerance $\varepsilon > 0$ of order $\mathcal{O}(N^{-1/2n})$

$$\hat{d}(\mathbf{x}) = \sum_{i=1}^N \theta_i \phi_i(\mathbf{x}; \mathbf{c}_i, R_i) = \Theta^\top \Phi(\mathbf{x})$$

with $R_i = d_{\max}/\sqrt{2N}I_n$ for $i = 1, \dots, N$ such that

$$\int_{\mathcal{B}_\delta} |d(\mathbf{x}) - \hat{d}(\mathbf{x})|^2 d\mathbf{x} \leq \varepsilon.$$

The theoretical basis of function approximation and regression with RBFs is provided by the following theorem [38]. It assures that it is always possible to approximate function d^o on a finite set of grid points by using linear combination of RBFs.

Theorem 4.2.2 (Micchelli, 1984). *Let $\phi_{ij} := \phi(\mathbf{x}_i; \mathbf{x}_j)$ be the Gaussian, the multi-quadratics, or the inverse multiquadratics function. Let $\{\mathbf{x}_i\}_{i=1}^N$ be a set of distinct points in $\mathbb{R}^p$. Then, the $N \times N$ interpolation matrix Φ , whose elements are ϕ_{ij} , is nonsingular.*

Returning back to the correction dynamics (4.2) with output uncertainty d^o , define a *Lyapunov redesign* term [25, 13, 2] which either completely represents the *structured* uncertainty or approximates the *unstructured* uncertainty within a bounded approximation tolerance.

$$\hat{\mathbf{z}}_{k|k} = (I - K_k C) \hat{\mathbf{z}}_{k|k-1} + K_k (\mathbf{y}_k - \hat{\Theta}_k^\top \Phi(\hat{\mathbf{z}}_{k|k-1})) \quad (4.8)$$

where $\hat{\Theta}_k \in \mathbb{R}^{(N+1) \times l}$ denotes the estimates of the true regression coefficients which are initially unknown.

CHAPTER 5

ESTIMATION WITH ADAPTIVE LYAPUNOV REDESIGN

5.1 Introduction

In this chapter *adaptive* methods that provide the functional approximation of output disturbances are proposed and the asymptotic stability of methods are proven. The proposed methods can be considered as the examples of *model reference adaptive control* (MRAC) design on the discrete-time state estimators with *true* aiding reference measurements which construct the reference model.

In the first section, the notion of *Lyapunov redesign* in the adaptive sense is realized on the uncertain discrete-time state estimator (i.e. KF with uncertainties on measurements). The measured variables used in the correction step of KF is considered as a discrete-time dynamical system with *uncertain inputs* which are, in fact, the output uncertainties acting on the actual dynamical system. Then, these uncertainties are imposed to be *completely matched* via generic regressor structure and the optimum regression weights are ought to be found in the sense of online parameter adaptation. It is shown that if there exists an aiding reference measurement to the KF with uncertain correction dynamics, then an online parameter *tuning law* can be found such that completely matches the uncertain part of measurements and eliminates it. Moreover, it is also shown that the existence of such tuning law provides the asymptotic convergence of both regression weights and adaptation errors to their optimum values and to zero, respectively.

In the second section, the assumption of completely matching the uncertainty is relaxed and the condition of parameter adaptation in the presence of *bounded* approximation errors is investigated. In this section, it is aimed to be presented that using a

sufficiently small regressor structure can also provide promising stability condition. Utilizing the notion of *convex* functions and sets, it is also shown that the proposed tuning law can be modified and be made robust to the errors encountered during the approximation of uncertainty.

5.2 Adaptive Lyapunov Redesign on State Estimator

Let the output uncertainty $d^o \in \mathcal{C}^1(U)$ is a function of augmented state estimates over a domain $U \subseteq \mathbb{R}^{n+m}$. Also let it be unstructured with an unknown *a priori* knowledge of basis and suppose $\sup_{k \in \mathbb{N}} |d^o(\hat{\mathbf{z}}_k)| < \infty$. Then, it can be expanded in terms of basis functions of a universal approximator structure (e.g. RBFNNs) *within a sufficiently small* approximation radius (see Theorem 4.2.1) such that

$$d^o(\hat{\mathbf{z}}_k) = \sum_{i=1}^N \theta_i^* \phi_i(\hat{\mathbf{z}}_{k|k-1}) + \mathbf{b}_k^* + \varepsilon_k$$

$$\text{s.t. } \|d^o(\hat{\mathbf{z}}_k) - \Theta^{*\top} \Phi(\hat{\mathbf{z}}_{k|k-1})\|_2 \leq \varepsilon_k$$

where $\Theta^* \in \mathbb{R}^{(N+1) \times l}$ are the *true regression parameters* (or optimum weights) which are initially unknown.

Remark 5.2.1. *Let the output uncertainty $d^o \in \mathcal{C}^1(U)$ is a bounded function of augmented state estimates over a domain $U \subseteq \mathbb{R}^{n+m}$. Assuming that the structure of d^o is known (i.e. there exists a suitable and known basis which completely represents), then it can be represented via regressor functions and regression parameters such that $d^o(\hat{\mathbf{z}}_k) = \Theta^{*\top} \Phi(\hat{\mathbf{z}}_{k|k-1})$.*

Consider the new measurement law defined in (4.8) and note that the actual measurements (i.e. uncertain measurements) can be written in terms of uncertainty function and the true measurements. Thus,

$$\hat{\mathbf{z}}_{k|k} = (I - K_k C) \hat{\mathbf{z}}_{k|k-1} + K_k \left(\mathbf{y}_k^* + d^o(\hat{\mathbf{z}}_k) - \hat{\Theta}_k^\top \Phi(\hat{\mathbf{z}}_{k|k-1}) \right). \quad (5.1)$$

Suppose now the true correction dynamics (4.4) is a reference model to (5.1) where the *tracking error* $\hat{\mathbf{e}}_{k|k} := \hat{\mathbf{z}}_{k|k} - \hat{\mathbf{z}}_{k|k}^*$ and the *parameter adaptation error* $\tilde{\Theta}_k := \Theta^* - \hat{\Theta}_k$, and let $\Phi_{k|k-1} := \Phi(\hat{\mathbf{z}}_{k|k-1})$. Assuming that d^o is a structured uncertainty and $\Phi_{k|k-1}$ is consist of known suitable basis functions, the *error dynamics* relating

the uncertain augmented state estimates to the true augmented state estimates with the functional approximation of d^o can be defined as

$$\hat{\mathbf{e}}_{k|k} = (I - K_k C) \hat{\mathbf{e}}_{k|k-1} + K_k \tilde{\Theta}_k^\top \Phi_{k|k-1}. \quad (5.2)$$

In order to approximate the output uncertainty function while estimating the augmented states which also include the input uncertainties, the *stability* and *convergence* of estimator must be preserved in the theoretical sense. Therefore, a proper stability and convergence analysis should be maintained.

First, the following mathematical definitions related to stability of dynamical systems [30] are redefined and adapted to discrete-time systems in order to create a base for the proofs of the stability of parameter adaptation and state estimation along the chapter.

Definition 5.2.1. Consider the state equations in the form $\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1})$. Suppose that $\mathbf{u}$ is set to be a constant value $\mathbf{u}^*$. Then, $\mathbf{x}^*$ is called as an equilibrium point to system if $f(\mathbf{x}^*, \mathbf{u}^*) = \mathbf{0}$ for all $k \in \mathbb{N}$.

Definition 5.2.2. The equilibrium point $\mathbf{x}^* = \mathbf{0}$ of the system (2.1) with unforced dynamics is stable if for any $\epsilon > 0$ and $k_0 \geq 0$ there exists $\delta(\epsilon, k_0) > 0$ such that for any k_0 the solution trajectories starting within the ball $\mathcal{B}_\delta = \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}_0\| < \delta\}$ keep stay bounded within another ball $\mathcal{B}_\epsilon = \{\mathbf{x} \in \mathbb{R}^p, k \in \mathbb{N} : \|\mathbf{x}_k\| < \epsilon\}$ for all $k \geq k_0 \geq 0$. Moreover, the equilibrium is uniformly stable if it is stable and the radius δ is independent of k_0 , and the equilibrium is unstable if it is not stable.

Definition 5.2.3. The equilibrium point $\mathbf{x}^* = \mathbf{0}$ of the system (2.1) is asymptotically stable if it is stable and the solution trajectories stay bounded within the ball $\mathcal{B}_\delta = \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}_0\| < \delta(k_0)\}$ such that $\mathbf{x}_k \rightarrow \mathbf{0} \forall \mathbf{x}_k \in \mathcal{B}_\delta$ as $k \rightarrow \infty \forall k \in \mathbb{N}$.

The equilibrium point $\mathbf{x}^*$ is said uniformly asymptotically stable if it is uniformly stable and the ball radius δ is independent of k_0 .

Definition 5.2.4. A scalar function $V : \mathbb{R}^p \rightarrow \mathbb{R}$ of argument $\mathbf{x} \in \mathbb{R}^p$ is called locally positive semidefinite if $V(\mathbf{0}) = 0$, and there exists a scalar constant $\epsilon > 0$ such that $V(\mathbf{x}) \geq 0$, for all nonzero $\mathbf{x}$ within the ball $\mathcal{B}_\epsilon = \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\| \leq \epsilon\}$. Moreover, $V(\mathbf{x})$ is called positive definite if the inequalities become strict and is said to be globally positive semidefinite if $\mathcal{B}_\epsilon = \mathbb{R}^p$.

The scalar function $V : \mathbb{R}^p \rightarrow \mathbb{R}$ is called locally (globally) negative semidefinite if $-V(\mathbf{x})$ is locally (globally) positive semidefinite.

In the stability theory of dynamical systems, particularly dealing with nonlinear systems, the notion of *input-to-state stability* (ISS) [54] is extensively used to prove the stability of solution trajectories in the presence of bounded inputs. Moreover, utilizing ISS, it is also possible to extend the concept of stability margins in linear systems theory to the *asymptotic gains* of nonlinear systems.

The following definition [17, 53, 24] is particularly useful when dealing with multiple classes of functions and given to improve the brevity of mathematical definitions and so the thesis.

Definition 5.2.5 (Comparison Functions). *Let $\mathcal{C}(X, Y)$ be a space of continuous functions acting on metric spaces from X to Y , then,*

$$\mathcal{P} := \{f \in \mathcal{C}(X, \mathbb{R}_+) : f \text{ is positive definite}\}$$

$$\mathcal{P}_s := \{f \in \mathcal{C}(X, \mathbb{R}_+) : f \text{ is positive semidefinite}\}$$

$$\mathcal{K} := \{f \in \mathcal{P} : f \text{ is strictly increasing}\}$$

$$\mathcal{K}_\infty := \{f \in \mathcal{K} : f \text{ is unbounded}\}$$

$$\mathcal{L} := \{f \in \mathcal{C}(X, \mathbb{R}_+) : f \text{ is strictly decreasing with } \lim_{k \rightarrow \infty} f(k) = 0\}$$

$$\mathcal{KL} := \{f \in \mathcal{C}(X \times \mathbb{R}_+, \mathbb{R}_+) : f(\cdot, k) \in \mathcal{K} \forall k \geq 0, f(\mu, \cdot) \in \mathcal{L} \forall \mu > 0 \in X\}.$$

Let the discrete-time system be

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}) \quad \forall k \in \mathbb{N} \quad (5.3)$$

with $f \in \mathcal{C}^1(X, \mathbb{R}^n)$, over the domain $X \subseteq \mathbb{R}^n \times \mathbb{R}^m$, and suppose $\mathbf{u} \in \ell^\infty([0, \infty), \mathbb{R}^m)$. Then, by utilizing the comparison functions the ISS concept can be defined as follows.

Definition 5.2.6. *The system (5.3) is ISS if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each input $\mathbf{u} \in \ell^\infty([0, \infty), \mathbb{R}^m)$ and $\mathbf{x}_0 \in \mathbb{R}^n$, it holds that*

$$\|\mathbf{x}_k\| \leq \beta(\|\mathbf{x}_0\|, k) + \gamma(\sup_{k \in \mathbb{N}} \|\mathbf{u}_{k-1}\|) \quad \forall k \in \mathbb{N}.$$

Theorem 5.2.1. *Let $\mathbf{x}^* = \mathbf{0}$ be an equilibrium point of the discrete-time system in (5.3) with zero input (i.e. $\mathbf{u}_k \equiv \mathbf{0}$) and $\mathbf{x}_0, \mathbf{x}^* \in \mathcal{D} \subseteq \mathbb{R}^p$, $k_0 = 0$. Suppose that*

on the domain $\mathcal{D}$ there exists a continuously differentiable locally (globally) positive definite Lyapunov function $V \in \mathcal{P}(\mathcal{D})$ whose backward difference along the system trajectories is locally (globally) negative semidefinite (i.e. $-\Delta V_k \in \mathcal{P}_s$), such that

$$\begin{aligned} \Delta V_k &= V_k - V_{k-1} \leq 0 \\ V(f(k, \mathbf{x}_{k-1})) - V(\mathbf{x}_{k-1}) &\leq 0 \quad \forall \mathbf{x} \in \mathcal{D}, \forall k \in \mathbb{N}. \end{aligned} \tag{5.4}$$

Then, $\mathbf{x}^*$ is locally (globally) uniformly stable and if (5.4) is negative definite, then, it is locally (globally) uniformly asymptotically stable.

There is a natural connection between the global asymptotic stability and the ISS property of discrete-time dynamical system (5.3) [55, 21] which is restated after the following theorems and definitions that clarify this connection.

Definition 5.2.7. *The system (5.3) is called globally asymptotically stable at zero (0-GAS) if it is GAS with $\mathbf{u}_k \equiv \mathbf{0}$. Moreover, the system (5.3) possesses the asymptotic gain (AG) property if there exist $\gamma \in \mathcal{K}$ such that for any initial value $\mathbf{x}_0 \in \mathbb{R}^n$ and any admissible control $\mathbf{u}_k$ it holds that*

$$\limsup_{k \rightarrow \infty} \|\mathbf{x}_k\| \leq \gamma(\|\mathbf{u}_{k-1}\|_\infty) \quad \forall k \in \mathbb{N}.$$

The existence of such Lyapunov difference function is necessary and sufficient condition for the stability of unforced (i.e. $K_k \tilde{\Theta}_k^\top \Phi_{k|k-1} \equiv \mathbf{0}$) discrete-time dynamics in (5.2). In order to extend the Lyapunov stability in the presence of bounded inputs, the following Corollary 5.2.2.1 to Theorem 5.2.2 is utilized.

Theorem 5.2.2. *Let $V \in \mathcal{P}(\mathbb{R}^n)$, if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(\|\mathbf{x}_k\|) \leq V(\mathbf{x}_k) \leq \alpha_2(\|\mathbf{x}_k\|)$$

and it holds that

$$V(f(k, \mathbf{x}_{k-1})) - V(\mathbf{x}_{k-1}) \leq -\alpha_3(\|\mathbf{x}_k\|) + \rho(\|\mathbf{u}_k\|)$$

for some $\alpha_3 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$, then V is an ISS-Lyapunov function.

Corollary 5.2.2.1. *If the discrete-time dynamical system in (5.3) admits an ISS-Lyapunov function, then it is ISS.*

Theorem 5.2.3 (Sontag & Wang, 1996). *The system in (5.3) is ISS if and only if it is 0-GAS with AG property.*

Now, consider the error dynamics obtained in (5.4), then, in order to have a stable and convergent parameter adaptation which steers the uncertain correction dynamics to the true ones, the following proposed theorem must hold. Moreover, if the theorem holds then the concurrent estimation of the sequences of output uncertainty $\{\hat{\mathbf{d}}_k^o\}_{k=0}^\infty$ and input uncertainty $\{\hat{\mathbf{d}}_k^i\}_{k=0}^\infty$ should converge to their actual values which are initially unknown in a sufficiently large domain.

Theorem 5.2.4. *For the correction step of the state estimator as an uncertain discrete-time dynamical system*

$$\hat{\mathbf{z}}_{k|k} = (I - K_k C) \hat{\mathbf{z}}_{k|k-1} + K_k \mathbf{y}_k$$

with the measurement law

$$\mathbf{y}_k = \mathbf{y}_k^* + d^o(\hat{\mathbf{z}}_k) - \hat{\Theta}_k^\top \Phi_{k|k-1}$$

and the parameter adaptation law

$$\hat{\Theta}_k = \hat{\Theta}_{k-1} + \Gamma_\Theta \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k$$

where $\Gamma_\Theta = \Gamma_\Theta^\top \succ 0$ is the diagonal learning rate matrix and $P_{k|k}$, K_k are the prediction error covariance and Kalman gain matrices, the uncertain augmented state estimates $\hat{\mathbf{z}}_{k|k}$ asymptotically track the true augmented state estimates $\hat{\mathbf{z}}_{k|k}^*$ of the reference model

$$\hat{\mathbf{z}}_{k|k}^* = (I - K_k C) \hat{\mathbf{z}}_{k|k-1}^* + K_k \mathbf{y}_k^*$$

driven by any true measurement $\mathbf{y}_k^* \in \ell^\infty(\mathbb{R}^l)$, while all the signals in the closed-loop system remain uniformly bounded. Moreover, the closed-loop tracking error dynamics

$$\hat{\mathbf{e}}_{k|k} = (I - K_k C) \hat{\mathbf{e}}_{k|k-1} + K_k \tilde{\Theta}_k^\top \Phi_{k|k-1}$$

are uniformly input-to-state stable.

Proof of Theorem 5.2.4. Consider a positive definite discrete-time Lyapunov function candidate $V \in \mathcal{P}(\mathcal{D})$ where $\mathcal{D} \subseteq \mathbb{R}^{n+m} \times \mathbb{R}^{(N+1) \times l}$ in terms of error dynamics of adaptation.

$$V(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) = \hat{\mathbf{e}}_{k|k}^\top M_{k|k} \hat{\mathbf{e}}_{k|k} + \text{tr}\{\tilde{\Theta}_k^\top \Gamma_\Theta^{-1} \tilde{\Theta}_k\} \quad (5.5)$$

where $\text{tr}\{\cdot\}$ is the matrix trace operator, $\Gamma_\Theta \in \mathbb{S}^{N+1}$ is a constant diagonal learning rate matrix and supposing that there exists $M_{k|k} \in \mathbb{S}^{n+m}$ which is symmetric and positive definite.

Let the abbreviations of terms be

$$\begin{aligned} V_k &:= V(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) \\ A_k &:= I - K_k C \\ \Psi_{k|k-1} &:= K_k \tilde{\Theta}_k^\top \Phi_{k|k-1} \end{aligned}$$

for the sake of brevity of proofs. Then, substituting them into (5.4) the Lyapunov function candidate becomes

$$V_k = (A_k \hat{\mathbf{e}}_{k|k-1} + \Psi_{k|k-1})^\top M_{k|k} (A_k \hat{\mathbf{e}}_{k|k-1} + \Psi_{k|k-1}) + \text{tr}\{\tilde{\Theta}_k^\top \Gamma_\Theta^{-1} \tilde{\Theta}_k\}. \quad (5.6)$$

There are few key assumptions need to be made in order to reach a proper Lyapunov stability definition as follows.

1. The recursion of correction dynamics can be written as $A_k \hat{\mathbf{e}}_{k|k-1} = \hat{\mathbf{e}}_{k|k} - \Psi_{k|k-1}$, alternatively.
2. True regression coefficients *do not change* (i.e. $\Delta\Theta^* = \Theta_k^* - \Theta_{k-1}^* = \mathbf{0}$) over the iterations as long as the uncertainty function stays the same. Then, it leads to $\Delta\tilde{\Theta}_k = \Delta\Theta^* - \Delta\hat{\Theta}_k = -\Delta\hat{\Theta}_k$.
3. The matrix $M_{k|k-1}$ stays symmetric positive definite over the iterations.

Moreover, consider the following corollary in [63], which is adapted to our case to be able to state the boundedness of Kalman gain.

Theorem 5.2.5 (Zhang, 2017). *Let the real matrices F_k, G_k, C_k, R_k of appropriate sizes be bounded and let F_k be invertible. Also, let R_k be a symmetric positive definite matrix with a strictly positive lower bound, and the pair (F_k, C_k) be observable $\forall k \in \mathbb{N}$. Then, the prediction error covariance $P_{k|k-1}$ is symmetric positive definite and upper bounded, and the gain matrix K_k is bounded.*

Consequently, the backward difference of Lyapunov function is obtained simply by using (5.6) and (5.5) with one correction step behind which is

$$V_{k-1} = \hat{\mathbf{e}}_{k|k-1}^\top M_{k|k-1} \hat{\mathbf{e}}_{k|k-1} + \text{tr}\{\tilde{\Theta}_{k-1}^\top \Gamma_\Theta^{-1} \tilde{\Theta}_{k-1}\}. \quad (5.7)$$

Then, under the aforementioned assumptions and few algebraic manipulations, the Lyapunov difference function becomes

$$\begin{aligned}
\Delta V_k &= \hat{\mathbf{e}}_{k|k-1}^\top (A_k^\top M_{k|k} A_k - M_{k|k-1}) \hat{\mathbf{e}}_{k|k-1} \\
&\quad + \hat{\mathbf{e}}_{k|k}^\top M_{k|k} \Psi_{k|k-1} + \Psi_{k|k-1}^\top M_{k|k} \hat{\mathbf{e}}_{k|k} \\
&\quad - \Psi_{k|k-1}^\top M_{k|k} \Psi_{k|k-1} \\
&\quad - \text{tr}\{\Delta \hat{\Theta}_k^\top \Gamma_\Theta^{-1} \tilde{\Theta}_k + \tilde{\Theta}_k^\top \Gamma_\Theta^{-1} \Delta \hat{\Theta}_k\}.
\end{aligned} \tag{5.8}$$

By the assumption, since the matrix $M_{k|k}$ is positive definite the quadratic form $\Psi_{k|k-1}^\top M_{k|k} \Psi_{k|k-1}$ is positive semidefinite, which gives

$$-\Psi_{k|k-1}^\top M_{k|k} \Psi_{k|k-1} \leq 0, \quad \forall \Psi_{k|k-1} \in \mathbb{R}^{n+m}.$$

The Lyapunov difference function in (5.8) imparts a familiar expression which is, in fact, a special kind of *discrete-time Lyapunov equation* (DLE) which is defined as follows.

Definition 5.2.8. Let $A \in \mathbb{R}^{p \times p}$, then given any matrix $Q \succ 0 \in \mathbb{S}^p$ there exists a unique solution $P \succ 0 \in \mathbb{S}^p$ satisfying $A^\top P A - P = -Q$ which is called *discrete-time Lyapunov equation* if and only if the corresponding discrete-time linear system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ is globally asymptotically stable with $\lambda_{\max}(A) < 1$.

In order to show the boundedness of error dynamics the quadratic form defined by the following inequality should also hold

$$\hat{\mathbf{e}}_{k|k-1}^\top (A_k^\top M_{k|k} A_k - M_{k|k-1}) \hat{\mathbf{e}}_{k|k-1} \leq 0, \quad \forall k \in \mathbb{N}$$

which requires

$$A_k^\top M_{k|k} A_k - M_{k|k-1} \stackrel{!}{=} -N_{k|k-1} \tag{5.9}$$

where $N_{k|k-1}$ is some symmetric positive definite matrix of the same size with $M_{k|k-1}$ by the assumption. Utilizing the definition of DLE one may conjecture that unique positive definite $N_{k|k-1}$ exists as long as A_k is discrete-time stable. However, the sample indices of DLE solution M are not the same and the standard stability definition by DLE needs to be extended for this particular *recursive* case.

Consider the following lemma [10] which is adapted to our case.

Lemma 5.2.6 (Schur Complement). *Suppose a block matrix Σ_k of given matrices A_k , $M_{k|k}$ and $M_{k|k-1}$, and let $M_{k|k}$ is invertible*

$$\Sigma_k := \begin{bmatrix} M_{k|k}^{-1} & A_k \\ A_k^\top & M_{k|k-1} \end{bmatrix}.$$

Then, the Schur complement of block $M_{k|k}^{-1}$ in matrix Σ_k is given by

$$\Sigma_k / M_{k|k}^{-1} := M_{k|k-1} - A_k^\top M_{k|k} A_k. \quad (5.10)$$

Moreover, Σ_k is positive definite if and only if $M_{k|k}^{-1}$ and the Schur complement $\Sigma_k / M_{k|k}^{-1}$ are both positive definite.

Using the Lemma 5.2.6 the negative definiteness of DLE in the form of (5.9) can be proven by the positive definiteness of block matrix Σ_k . In other words, the positive definiteness of Σ_k is necessary and sufficient condition to show that the equality (5.9) holds.

The block matrix Σ_k is a real and symmetric matrix, and the positive definiteness can be assessed by using the following lemma [37] as a necessary and sufficient condition.

Lemma 5.2.7 (Sylvester's Criterion). *A real symmetric matrix Σ_k is positive definite if and only if all the leading principal minors are positive. Thus,*

$$\Sigma_k \succ 0 \iff \det(M_{k|k}^{-1}) > 0 \text{ and } \det(\Sigma_k) > 0.$$

According to the criterion, in order to show that Σ_k is positive definite which leads to the negative definiteness of (5.9), the determinants $\det(M_{k|k}^{-1})$ and $\det(\Sigma_k)$ need to be proven to be positive.

Considering the block matrix Σ_k , since the block $M_{k|k-1}$ is nonsingular by the assumption, by using another Schur complement $\Sigma_k / M_{k|k-1}$ the determinant of block matrix becomes

$$\begin{aligned} \det(\Sigma_k) &= \det(M_{k|k-1}) \det(\Sigma_k / M_{k|k-1}) \\ &= \det(M_{k|k-1}) \det(M_{k|k}^{-1} - A_k M_{k|k-1}^{-1} A_k^\top) \end{aligned} \quad (5.11)$$

where the determinant of $M_{k|k-1}$ is positive since the matrix is positive definite.

Now, focusing on the second determinant which is, interestingly, in the form of another recursive DLE. Let $P = M^{-1}$ where P is the prediction error covariance matrix previously defined in KF equations. Then, it follows that the DLE in (5.11) is, in fact, the Joseph form of prediction covariance update in (2.2) and it restricts the positive definiteness of $P_{k|k}$ such that

$$P_{k|k} - (I - K_k C)P_{k|k-1}(I - K_k C)^\top = K_k R K_k^\top \quad (5.12)$$

where $R = R^\top \succ 0$ is the measurement noise covariance matrix by the definition. Thus, the choice of $P_{k|k}^{-1}$ as $M_{k|k}$ leads to an immediate result that

1. $\det(\Sigma_k/M_{k|k-1}) > 0$ since $K_k R K_k^\top$ is positive definite for all nonzero K_k .
2. $\det(\Sigma_k) > 0$ since both determinants $\det(M_{k|k-1}) > 0$ and $\det(\Sigma_k/M_{k|k-1}) > 0$ by using (5.11).
3. Σ_k is positive definite by using Lemma 5.2.7.
4. $\Sigma_k/M_{k|k}^{-1}$ is positive definite by using Lemma 5.2.6.

Then, the recursive DLE in (5.9) is proven to be negative definite and the symmetric positive definite matrix $N_{k|k-1} \in \mathbb{S}^{n+m}$ exists and unique such that

$$A_k^\top M_{k|k} A_k - M_{k|k-1} \prec 0, \quad \exists! N_{k|k-1} \succ 0. \quad (5.13)$$

Consider again the original Lyapunov difference function in (5.8) where the first and third lines of the function are proven to be negative semidefinite and, hence, contribute to the stability of the system (5.2).

The terms in Lyapunov difference function given in (5.8) can be reordered as

$$\begin{aligned} \Delta V_k &= -\hat{\mathbf{e}}_{k|k-1}^\top N_{k|k-1} \hat{\mathbf{e}}_{k|k-1} - \Psi_{k|k-1}^\top M_{k|k} \Psi_{k|k-1} \\ &\quad + 2 \hat{\mathbf{e}}_{k|k}^\top M_{k|k} \Psi_{k|k-1} - 2 \operatorname{tr}\{\tilde{\Theta}_k^\top \Gamma_\Theta^{-1} \Delta \hat{\Theta}_k\}. \end{aligned}$$

Definition 5.2.9 (Trace Identity). *For any two vectors $\mathbf{u}$ and $\mathbf{v}$ with compatible dimensions, the trace identity relation is defined as $\mathbf{u}^\top \mathbf{v} = \operatorname{tr}\{\mathbf{v} \mathbf{u}^\top\}$.*

Furthermore, by using $\Psi_{k|k-1} = K_k \tilde{\Theta}_k^\top \Phi_{k|k-1}$ and the trace identity, (5.8) becomes

$$\begin{aligned} \Delta V_k &= -\hat{\mathbf{e}}_{k|k-1}^\top N_{k|k-1} \hat{\mathbf{e}}_{k|k-1} - \Psi_{k|k-1}^\top M_{k|k} \Psi_{k|k-1} \\ &\quad - 2 \operatorname{tr}\{\tilde{\Theta}_k^\top (\Gamma_\Theta^{-1} \Delta \hat{\Theta}_k - \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top M_{k|k} K_k)\}. \end{aligned}$$

Noting that $M_{k|k} = P_{k|k}^{-1}$ and, now, suppose the following parameter update law

$$\Delta \hat{\Theta}_k := \Gamma_{\Theta} \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^{\top} P_{k|k}^{-1} K_k \quad (5.14)$$

and substitute to the previous expression, then, the Lyapunov difference function is enforced to become negative semidefinite such as

$$\begin{aligned} \Delta V_k &= -\hat{\mathbf{e}}_{k|k-1}^{\top} N_{k|k-1} \hat{\mathbf{e}}_{k|k-1} \\ &\quad - \Psi_{k|k-1}^{\top} M_{k|k} \Psi_{k|k-1} \leq 0, \quad \forall k \in \mathbb{N} \end{aligned} \quad (5.15)$$

which immediately implies that the candidate (5.5) is a proper Lyapunov function with $-\Delta V_k \in \mathcal{P}_s$ and by using Theorem 5.2.1 the unforced (i.e. $K_k \tilde{\Theta}_k^{\top} \Phi_{k|k-1} \equiv 0$) error dynamics are locally uniformly stable [2].

Definition 5.2.10. Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuously differentiable function. Then, $V(\mathbf{x})$ is said to be radially unbounded if $V(\mathbf{x}) \rightarrow \infty$ as $\|\mathbf{x}\| \rightarrow \infty$.

In addition, it is also possible to extend the local stability definition to global since (5.5) is radially unbounded. Thus, simply observing that $V_k \rightarrow \infty$ as both $\|\hat{\mathbf{e}}_{k|k}\| \rightarrow \infty$ and $\|\tilde{\Theta}_k\| \rightarrow \infty$, then it can be concluded that, since V_k is radially unbounded the unforced error dynamics are globally uniformly stable.

In order to show that the error dynamics are not only uniformly stable but also uniformly asymptotically stable, some celebrated theorems [13, 30] can be utilized.

Lemma 5.2.8 (Barbalat). Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a uniformly continuous function of t . Suppose that the limit $\lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau < \infty$ exists, then, $\lim_{t \rightarrow \infty} f(t) = 0$.

Suppose that there exists function $W \in \mathcal{P}_s(\mathcal{D})$, and

$$V(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) - V(\hat{\mathbf{e}}_{k-1|k-1}, \tilde{\Theta}_{k-1}) \leq -W(\hat{\mathbf{e}}_{k-1|k-1}, \tilde{\Theta}_{k-1}) \leq 0. \quad (5.16)$$

Since $W(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k)$ is not strictly positive definite, the origin is not necessarily asymptotically stable. Define a set of points $\mathcal{E}$ in $\mathcal{D}$ where W is zero.

$$\mathcal{E} := \left\{ (\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) \in \mathcal{D} : W(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) = 0 \right\} \quad (5.17)$$

The Lyapunov function $V \in \mathcal{P}(\mathcal{D})$ is bounded from below and $-\Delta V_k \in \mathcal{P}_s(\mathcal{D})$, hence it has a limit as $k \rightarrow \infty$ and both the tracking and parameter adaptation errors $\hat{\mathbf{e}} \in \ell^\infty([0, \infty), \mathbb{R}^{n+m})$ and $\tilde{\Theta} \in \ell^\infty([0, \infty), \mathbb{R}^{(N+1) \times l})$, uniformly. Consider the

evolution of both sides of inequality (5.16) starting from $k = 1$.

$$\begin{aligned}
V(\hat{\mathbf{e}}_{1|1}, \tilde{\Theta}_1) - V(\hat{\mathbf{e}}_{0|0}, \tilde{\Theta}_0) &\leq -W(\hat{\mathbf{e}}_{0|0}, \tilde{\Theta}_0) \\
V(\hat{\mathbf{e}}_{2|2}, \tilde{\Theta}_2) - V(\hat{\mathbf{e}}_{1|1}, \tilde{\Theta}_1) &\leq -W(\hat{\mathbf{e}}_{1|1}, \tilde{\Theta}_1) \\
&\vdots \\
V(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) - V(\hat{\mathbf{e}}_{k-1|k-1}, \tilde{\Theta}_{k-1}) &\leq -W(\hat{\mathbf{e}}_{k-1|k-1}, \tilde{\Theta}_{k-1})
\end{aligned}$$

Then, telescoping the summations on the both sides gives

$$V(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) - V(\hat{\mathbf{e}}_{0|0}, \tilde{\Theta}_0) \leq -\sum_{n=0}^{k-1} W(\hat{\mathbf{e}}_{n|n}, \tilde{\Theta}_n) \leq 0. \quad (5.18)$$

Moreover, by using the descending behavior of V_k the inequality can be rearranged in such a way that

$$\begin{aligned}
\sum_{n=0}^{k-1} W(\hat{\mathbf{e}}_{n|n}, \tilde{\Theta}_n) &\leq V(\hat{\mathbf{e}}_{0|0}, \tilde{\Theta}_0) - V(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) \\
\sum_{n=0}^{k-1} W(\hat{\mathbf{e}}_{n|n}, \tilde{\Theta}_n) &\leq V(\hat{\mathbf{e}}_{0|0}, \tilde{\Theta}_0) < \infty.
\end{aligned}$$

Then, consequently the sequence generated by W_k converges to a finite value, i.e.

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{k-1} W(\hat{\mathbf{e}}_{n|n}, \tilde{\Theta}_n) < \infty. \quad (5.19)$$

Now, consider the error dynamics (5.2) again, since $\hat{\mathbf{e}}_{k|k}$ and $\tilde{\Theta}_k$ are uniformly bounded and, in addition, the regressor vector $\Phi_{k|k-1}$ and the reference measurements $\mathbf{y}_k^*$ are also bounded by the definition, the continuous function W on $\mathcal{D}$ is uniformly continuous in k . Then, by using Lemma 5.2.8 for the discrete-time case on domain $\mathcal{D}$, the limit exists and $\lim_{k \rightarrow \infty} W(\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) = 0$ which implies that the trajectories of error dynamics asymptotically approach to set $\mathcal{E}$ uniformly in each iteration. The following theorem concludes the asymptotic convergence proof of (5.2) in the case of zero input.

Theorem 5.2.9 (LaSalle-Yoshizawa). *Starting anywhere in a domain $\mathcal{D}$, all trajectories of the unforced nonautonomous dynamics (5.2), with a Lyapunov function satisfying (5.15), uniformly asymptotically approach the set $\mathcal{E}$.*

Consequently, by using Theorem 5.2.9 with Lemma 5.2.8, the error dynamics (5.2) are 0-GAS, uniformly.

Remark 5.2.2. *The boundedness of adaptation parameters (i.e. $\tilde{\Theta}_k \in \ell^\infty$) cannot be achieved directly in the case of unstructured uncertainties via adaptive law proposed in (5.14), no matter how small the approximation tolerance is. The convergence should be maintained either by a robust modification (see section 5.3) on the adaptive law [30, 19, 59] or a persistently exciting regressor vector $\Phi_{k|k-1}$ [19, 40, 59].*

Now, consider the presence of bounded inputs in discrete-time system (5.2) and let $\mathbf{u}_k := \tilde{\Theta}_k^\top \Phi_{k|k-1}$, then the system becomes

$$\hat{\mathbf{e}}_{k|k} = A_k \hat{\mathbf{e}}_{k|k-1} + K_k \mathbf{u}_k. \quad (5.20)$$

Noting that the matrix A_k is discrete stable with eigenvalues lie strictly inside the unit circle, such that $\lambda_{\max}(A_k) < 1 \forall i = 1, 2, \dots, n \forall k \in \mathbb{N}$. Then, there exist constants $c \geq 0$ and $\sigma \in [0, 1)$ such that $\|A^k\| \leq c\sigma^k, \forall k \in \mathbb{N}$ [28]. Now, consider the forward solution map of the system (5.20) up to k^{th} step starting from $k = 0$ and, suppose for now that, $A := A_k$ and $K := K_k$ for all k for the brevity. Then, by utilizing a pattern, there exists an upper bound that

$$\begin{aligned} \|\hat{\mathbf{e}}_{1|1}\| &\leq c\sigma\|\hat{\mathbf{e}}_{0|0}\| + \|K\|\|\mathbf{u}_0\| \\ \|\hat{\mathbf{e}}_{2|2}\| &\leq c\sigma^2\|\hat{\mathbf{e}}_{0|0}\| + c\sigma\|K\|\|\mathbf{u}_0\| + \|K\|\|\mathbf{u}_1\| \\ \|\hat{\mathbf{e}}_{3|3}\| &\leq c\sigma^3\|\hat{\mathbf{e}}_{0|0}\| + c\sigma^2\|K\|\|\mathbf{u}_0\| + c\sigma\|K\|\|\mathbf{u}_1\| + \|K\|\|\mathbf{u}_2\| \\ &\vdots \\ \|\hat{\mathbf{e}}_{k|k}\| &\leq c\sigma^k\|\hat{\mathbf{e}}_{0|0}\| + \sup_{k \in \mathbb{N}} |\mathbf{u}_{k-1}| \sum_{j=0}^{k-1} c\sigma^j \|K\|. \end{aligned}$$

Now, considering the Theorem 5.2.5 (i.e. boundedness of Kalman gain), it is possible to claim that $\|A_k\| \leq c\sigma_*, \forall k \in \mathbb{N}$, where $\sigma_* := 1 - \inf_{k \in \mathbb{N}} |K_k C_k|$, and $K_* := \sup_{k \in \mathbb{N}} |K_k|$. Therefore, an upper bound to error sequence is found as

$$\|\hat{\mathbf{e}}_{k|k}\| \leq c\sigma_*^k \|\hat{\mathbf{e}}_{0|0}\| + \sup_{k \in \mathbb{N}} |\mathbf{u}_{k-1}| \sum_{j=0}^{k-1} c\sigma_*^j K_*.$$

Then, by letting $k \rightarrow \infty$ and using the property of convergent geometric series (i.e. $\sigma_* < 1$), there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\beta(\|\hat{\mathbf{e}}_{0|0}\|, k) := c\sigma_*^k \|\hat{\mathbf{e}}_{0|0}\|$ and

$$\gamma(\sup_{k \in \mathbb{N}} |\tilde{\Theta}_k^\top \Phi_{k|k-1}|) := \sum_{j=0}^{\infty} c\sigma_*^j K_* \sup_{k \in \mathbb{N}} |\mathbf{u}_{k-1}| = \frac{cK_*}{1 - \sigma_*} \sup_{k \in \mathbb{N}} |\tilde{\Theta}_k^\top \Phi_{k|k-1}|.$$

Furthermore, the supremum exists since $\tilde{\Theta} \in \ell^\infty([0, \infty), \mathbb{R}^{(N+1) \times l})$ and $\Phi_{k|k-1} \in \ell^\infty(U, \mathbb{R}^{N+1})$ within the set $U \subseteq \mathbb{R}^{n+m}$. Then, it gives an immediate result that error dynamics in (5.2)-(5.20) admit AG since $\exists \gamma \in \mathcal{K}$. Consequently, the error dynamics are 0-GAS with AG property, hence, by using Theorem 5.2.3 the discrete-time system in (5.2) is ISS. Moreover, by Corollary 5.2.2.1 to Theorem 5.2.2, the Lyapunov candidate function in (5.5) is a proper ISS-Lyapunov function, such that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ for some $\alpha_3 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$, which completes the proof of Theorem 5.2.4. □

5.3 Parameter Adaptation within Approximation Tolerances

In the previous section, the output uncertainty function d^o is assumed to be structured and completely matched via some suitable set of known basis functions. On the other hand, *a priori* knowledge on the structure of uncertainty brings us a restrictive assumption and it may not always be possible to achieve this information all the time in a practical sense.

The tolerance of approximation may cause regression coefficients to grow unbounded while the tracking errors are still bounded at all times, hence, the parameter adaptation process suffers from lack of robustness. This phenomenon is named as *parameter drift* and there are several well-adopted techniques in adaptive control literature such as *σ -modification*, *e -modification*, *dead-zone modification* and *projection-based learning* to prevent parameter drifts [30, 19, 59]. These modifications, in essence, modifies the original parameter adaptation law in such a way that the regression coefficients stay uniformly bounded in the presence of bounded approximation errors (or external disturbances).

Consider the RBFNN expansion of d^o with a bounded *approximation error* (or tolerance) $\varepsilon \in \ell^\infty([0, \infty), \mathbb{R}^{n+m})$ and a constant upper bound $\varepsilon_{max} > 0$. Then, the function d^o can be represented via RBFNN structure as

$$d^o(\hat{\mathbf{z}}_k) = \sum_{i=1}^N \theta_i^* \phi_i(\hat{\mathbf{z}}_{k|k-1}) + \mathbf{b}_k^* + \varepsilon_k \quad \text{s.t.} \quad \sup_{k \in \mathbb{N}} |\varepsilon_k| \leq \varepsilon_{max}. \quad (5.21)$$

By using (5.21) within (5.1) in the form of $d^o(\hat{\mathbf{z}}_k) = \Theta^{*\top} \Phi(\hat{\mathbf{z}}_{k|k-1}) + \varepsilon_k$, the error dynamics become slightly different and the bounded approximation error term arises.

$$\hat{\mathbf{e}}_{k|k} = (I - K_k C) \hat{\mathbf{e}}_{k|k-1} + K_k \tilde{\Theta}_k^\top \Phi_{k|k-1} + K_k \varepsilon_k \quad (5.22)$$

Now, focus on how to sustain the uniform boundedness of both tracking error and the parameter adaptation error via modifying Theorem 5.2.4 by *projection-based* tuning of regression parameters.

Theorem 5.3.1. *For the correction step of the state estimator as an uncertain discrete-time dynamical system*

$$\hat{\mathbf{z}}_{k|k} = (I - K_k C) \hat{\mathbf{z}}_{k|k-1} + K_k \mathbf{y}_k$$

with the measurement law

$$\mathbf{y}_k = \mathbf{y}_k^* + d^o(\hat{\mathbf{z}}_k) - \hat{\Theta}_k^\top \Phi_{k|k-1}$$

and the projection-based parameter adaptation law

$$\hat{\Theta}_k = \hat{\Theta}_{k-1} + \text{Proj}_{\Gamma_\Theta}(\hat{\Theta}_k, \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k)$$

where $\text{Proj}_\Gamma(\cdot, \cdot)$ is the Γ -Projection operator and $P_{k|k}, K_k$ are the prediction error covariance and Kalman gain matrices, the uncertain augmented state estimates $\hat{\mathbf{z}}_{k|k}$ asymptotically track the true augmented state estimates $\hat{\mathbf{z}}_{k|k}^*$ outside the compact set

$$\Omega := \left\{ (\hat{\mathbf{e}}_{k|k}, \tilde{\Theta}_k) : \|\hat{\mathbf{e}}_{k|k-1}\| \leq c_1 \wedge \|\tilde{\Theta}_k\| \leq c_2 \right\}$$

where the positive scalars

$$c_1 := c\sigma_* K_* \frac{\lambda_{\max}(P_{k|k}^{-1})}{\lambda_{\min}(N_{k|k-1})} \varepsilon_{\max}, \quad c_2 := \frac{\kappa(K_k^\top P_{k|k}^{-1} K_k)^{1/2}}{\|\Phi_{k|k-1}\|} \varepsilon_{\max}.$$

Consequently, driven by any true measurement $\mathbf{y}_k^* \in \ell^\infty(\mathbb{R}^l)$, all the parameter adaptation errors remain uniformly bounded and asymptotically converge towards Ω within the set

$$\mathcal{A} := \left\{ \tilde{\Theta}_k : c_2 \leq \|\tilde{\Theta}_k\|_F \leq c_3 \right\}$$

with a positive upper bound

$$c_3 := \|\Theta^*\|_F + \left(\sum_{i=1}^l (\Theta_{\max}^i)^2 + \psi(\mathbf{p}_0^i; \gamma_\Theta, \gamma^i, \Theta_{\max}^i) \right)^{1/2} \quad \text{s.t. } \psi \in \ell^\infty(\mathbb{R}^{(N+1) \times l}, \mathbb{R})$$

where $\mathbf{p}_k^i := (\Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k)^i$, Θ_{max}^i is the maximum allowed parameter error, γ_Θ is the constant learning rate and γ^i is the projection tolerance for $i = 1, 2, \dots, l$.

Moreover, the closed-loop tracking error dynamics

$$\hat{\mathbf{e}}_{k|k} = (I - K_k C) \hat{\mathbf{e}}_{k|k-1} + K_k (\tilde{\Theta}_k^\top \Phi_{k|k-1} + \varepsilon_k)$$

are semiglobally uniformly practically asymptotically stable in the presence of bounded approximation error ε_k .

Proof of Theorem 5.3.1. Consider the same Lyapunov-based arguments from (5.5)-(5.8) in the case of bounded approximation error term ε_k and the abbreviation $\Psi_{k|k-1} = K_k \tilde{\Theta}_k^\top \Phi_{k|k-1}$. Then, the Lyapunov difference equation is rendered to the following form.

$$\begin{aligned} \Delta V_k = & -\hat{\mathbf{e}}_{k|k-1}^\top N_{k|k-1} \hat{\mathbf{e}}_{k|k-1} \\ & -\Phi_{k|k-1}^\top \tilde{\Theta}_k (K_k^\top M_{k|k} K_k) \tilde{\Theta}_k^\top \Phi_{k|k-1} \\ & -2 \operatorname{tr}\{\tilde{\Theta}_k^\top (\Gamma_\Theta^{-1} \Delta \hat{\Theta}_k - \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top M_{k|k} K_k)\} \\ & +2 \hat{\mathbf{e}}_{k|k-1}^\top A_k^\top M_{k|k} K_k \varepsilon_k \\ & +\varepsilon_k^\top (K_k^\top M_{k|k} K_k) \varepsilon_k \end{aligned} \quad (5.23)$$

In order to prove that the projection operator contributes to the negative semidefiniteness of (5.23) while ensuring the boundedness of parameter adaptation errors, few definitions need to be made about the projection [29].

Lemma 5.3.2 (Convex Projection). *Let $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ be a convex function. Given a constant $\delta > 0$ consider the subset*

$$\Omega_\delta = \{\theta \in \mathbb{R}^d : f(\theta) \leq \delta\} \subset \mathbb{R}^d.$$

Let $\theta^* \in \operatorname{Int}(\Omega_\delta)$ and assume that $f(\theta^*) \leq \delta$. Moreover, let $\theta \in \partial\Omega_\delta$ and assume that $f(\theta) = \delta$. Then, the inequality $\langle \theta^* - \theta, \nabla f(\theta) \rangle \leq 0$ holds.

Definition 5.3.1. *Let $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ be a convex function and $\Gamma \succ 0 \in \mathbb{S}^d$. Then, Γ -Projection is defined as*

$$\operatorname{Proj}_\Gamma(\theta, \mathbf{x}; f) := \begin{cases} \Gamma \mathbf{x} - \Gamma \frac{\nabla f(\theta) \nabla f(\theta)^\top}{\|\nabla f(\theta)\|_\Gamma^2} \Gamma \mathbf{x} f(\theta) & \text{if } f(\theta) > 0 \text{ and } \langle \mathbf{x}, \Gamma \nabla f(\theta) \rangle > 0 \\ \Gamma \mathbf{x} & \text{otherwise.} \end{cases}$$

By making few derivations from Lemma 5.3.2, it is shown in [29] that the projection operator forms a convex inequality presented in the following lemma.

Lemma 5.3.3. *Let $F = \begin{bmatrix} f_1 & f_2 & \dots & f_l \end{bmatrix}^\top \in \mathbb{R}^l$ be a convex function and $\Theta^* = \begin{bmatrix} \theta_1^* & \theta_2^* & \dots & \theta_l^* \end{bmatrix}^\top$, $\hat{\Theta} = \begin{bmatrix} \hat{\theta}_1 & \hat{\theta}_2 & \dots & \hat{\theta}_l \end{bmatrix}^\top$, $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_l \end{bmatrix}^\top$ with $\Theta, \hat{\Theta}, X \in \mathbb{R}^{(N+1) \times l}$, then,*

$$\text{tr}\{(\hat{\Theta} - \Theta^*)^\top (\Gamma^{-1} \text{Proj}_\Gamma(\hat{\Theta}, X; F) - X)\} \leq 0.$$

There is a special class of smooth functions where the spectrum of Hessian matrix (or second derivative for the univariate case) is uniformly bounded by some smoothness constant.

Definition 5.3.2. *Let $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ and $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$, and it holds that*

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{1}{2} \beta \|\mathbf{y} - \mathbf{x}\|^2$$

then, the function is called β -smooth with some smoothness constant $\beta > 0$, such that $\beta I - \nabla^2 f(\mathbf{x}) \succeq 0 \forall \mathbf{x}, \mathbf{y} \in \text{dom}(f)$.

Moreover, the existence of a uniform upper bound on the parameter adaptation errors can be deduced from the following lemma.

Lemma 5.3.4. *Let $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ be a convex β -smooth function. Consider a parameter update law*

$$\theta_k = \theta_{k-1} + \text{Proj}_\Gamma(\theta_k, \mathbf{p}_k)$$

where $\theta \in \mathbb{R}^d$ are the regression parameters and $\mathbf{p} \in \mathbb{R}^d$ is the update direction of parameters. Then, starting from any initial condition θ_0 within the set

$$\Omega_0 := \{\theta_k \in \mathbb{R}^d : f(\theta_k) \leq 0\}$$

the sequence generated by $\{\theta_k\}_{k=0}^\infty$ will remain in the set

$$\Omega_1 := \left\{ \theta_k \in \mathbb{R}^d : f(\theta_k) \leq 1 + \frac{\epsilon_k}{\langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle} \right\}$$

for some $\epsilon_k > 0$ and $\langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle \neq 0, \forall k \in \mathbb{N}$.

Proof of Lemma 5.3.4. Suppose convex function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth with parameter $\beta > 0$ at point $\theta_{k-1} \in \text{dom}(f)$, then by the *Mean Value Theorem* [48], the function f can be expanded as

$$f(\theta_k) = f(\theta_{k-1}) + \langle \nabla f(\theta_{k-1}), \theta_k - \theta_{k-1} \rangle + \frac{1}{2} \langle \theta_k - \theta_{k-1}, \nabla f(\nu)(\theta_k - \theta_{k-1}) \rangle$$

such that there exists $\nu \in [\theta_{k-1}, \theta_k]$. Let $\Delta f_k := f_k - f_{k-1}$ denote the first backward difference acting on f_k , then there exists an upper bound, $\epsilon > 0$, such that

$$\begin{aligned} \Delta f(\theta_k) &= \langle \nabla f(\theta_{k-1}), \Delta \theta_k \rangle + \frac{1}{2} \langle \Delta \theta_k, \nabla f(\nu) \Delta \theta_k \rangle \\ &\leq \langle \nabla f(\theta_{k-1}), \Delta \theta_k \rangle + \frac{1}{2} \beta \|\Delta \theta_k\|^2 \\ &\leq \langle \nabla f(\theta_{k-1}), \Delta \theta_k \rangle + \epsilon_k. \end{aligned}$$

Substitute the projection-based update and suppose $\Gamma = I$ such that

$$\Delta f(\theta_k) \leq \langle \nabla f(\theta_{k-1}), \text{Proj}_\Gamma(\theta_k, \mathbf{p}_k) \rangle + \epsilon_k.$$

Then, the piecewise definition of projection operator brings

$$\Delta f(\theta_k) = \begin{cases} \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle (1 - f(\theta_{k-1})) + \epsilon_k & \text{if } f(\theta_{k-1}) > 0 \text{ and } \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle > 0 \\ \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle + \epsilon_k & \text{otherwise} \end{cases}$$

which leads to an immediate result that

$$\Delta f(\theta_k) \begin{cases} > 0 & \text{if } 0 < f(\theta_{k-1}) < 1 + \mu_k \text{ and } \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle > 0 \\ = 0 & \text{if } f(\theta_{k-1}) = 1 + \mu_k \text{ and } \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle > 0 \\ \leq 0 & \text{if } f(\theta_{k-1}) \leq 0 \text{ and } \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle \leq -\epsilon_k. \end{cases}$$

It can be concluded that if $f(\theta_0) > 0$, then $f(\theta_k)$ increases monotonically for all k until it reaches to a positive valued barrier

$$\mu_k := \frac{\epsilon_k}{\langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle} \quad \text{s.t.} \quad \langle \nabla f(\theta_{k-1}), \mathbf{p}_k \rangle > 0.$$

Furthermore, if $f(\theta_0) \leq 0$, then $f(\theta_k)$ decreases monotonically for all k which implies that $f(\theta_k) \leq 1 + \mu_k$ for all k and concludes the proof of Lemma 5.3.4. $\square$

Choose the convex functions $f^i \in \mathcal{C}^2(\mathbb{R}^{N+1}, \mathbb{R})$ for the each i^{th} column of $\hat{\Theta}_k$ as

$$f_k^i := f(\hat{\Theta}_k^i) = \frac{(1 + \gamma^i) \|\hat{\Theta}_k^i\|^2 - (\Theta_{max}^i)^2}{\gamma^i (\Theta_{max}^i)^2} \quad (5.24)$$

where $\gamma^i > 0$ is the scalar *projection tolerance* and Θ_{max}^i is the *maximum allowable bound* on the parameters for the i^{th} column, then the gradient can be computed as

$$\nabla f_k^i = \frac{2(1 + \gamma^i)}{\gamma^i(\Theta_{max}^i)^2} \hat{\Theta}_k^i. \quad (5.25)$$

Remark 5.3.1. Let $\mathbf{x} \in \mathbb{R}^d$, then the functions $f(\mathbf{x}) = \|\mathbf{x}\|_p$ and $g(\mathbf{x}) = c$ are convex for all $1 \leq p \leq \infty$ and $c \in \mathbb{R}$. Noting that both the sum of convex functions and the square of nonnegative convex functions preserve the convexity, then $f^2 + g$ is also convex, and hence, the function in (5.24) is convex.

Consider the convex sets defined in Lemma 5.3.4 with functions (5.24)-(5.25), then Ω_0 for each column becomes

$$\Omega_0^i = \left\{ \hat{\Theta}_k^i \in \mathbb{R}^{N+1} : f(\hat{\Theta}_k^i) \leq 0 \right\} = \left\{ \hat{\Theta}_k^i \in \mathbb{R}^{N+1} : \|\hat{\Theta}_k^i\| \leq \frac{\Theta_{max}^i}{\sqrt{1 + \gamma^i}} \right\}. \quad (5.26)$$

The convex set Ω_1 for each column can be obtained without changing the structure of original convex function f_k^i by rewriting f_k^i inside the inequality in definition of Ω_1 in Lemma 5.3.4. Thus,

$$\frac{(1 + \gamma^i)\|\hat{\Theta}_k^i\|^2 - (\Theta_{max}^i)^2}{\gamma^i(\Theta_{max}^i)^2} \leq 1 + \epsilon_k^i \left(\frac{2(1 + \gamma^i)}{\gamma^i(\Theta_{max}^i)^2} (\hat{\Theta}_k^i)^\top \mathbf{p}_k^i \right)^{-1} \quad \text{s.t.} \quad (\hat{\Theta}_k^i)^\top \mathbf{p}_k^i > 0$$

where $\mathbf{p}_k^i := (\Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k)^i$ for our case and, here, $(\cdot)^i$ denotes the i^{th} column of a given matrix argument or element of a given vector argument. Then, after few manipulations it can be claimed that

$$\|\hat{\Theta}_k^i\|^2 - (\Theta_{max}^i)^2 \leq \epsilon_k^i \frac{(\gamma^i)^2 (\Theta_{max}^i)^4}{2(1 + \gamma^i)^2 (\hat{\Theta}_k^i)^\top \mathbf{p}_k^i}.$$

Noting that each function of choice f_k^i is β -smooth with smoothness constants $\beta^i = 2(1 + \gamma^i)/(\gamma^i(\Theta_{max}^i)^2)$ since f_k^i are quadratic functions of constant $\nabla^2 f_k^i$ matrices. Hence, the convex set Ω_1 for each column becomes

$$\Omega_1^i = \left\{ \hat{\Theta}_k^i \in \mathbb{R}^{N+1} : f(\hat{\Theta}_k^i) \leq 1 + \mu_k^i \right\} = \left\{ \hat{\Theta}_k^i \in \mathbb{R}^{N+1} : \|\hat{\Theta}_k^i\| \leq c^i \right\} \quad (5.27)$$

where the scalars can be found such that

$$\begin{aligned} c^i &= \sqrt{(\Theta_{max}^i)^2 + \epsilon_k^i \frac{(\gamma^i)^2 (\Theta_{max}^i)^4}{2(1 + \gamma^i)^2 (\hat{\Theta}_k^i)^\top \mathbf{p}_k^i}} \\ &= \sqrt{(\Theta_{max}^i)^2 + \frac{\gamma^i (\Theta_{max}^i)^2}{2(1 + \gamma^i)} \sup_{k \in \mathbb{N}} \frac{\|\Delta \hat{\Theta}_k^i\|^2}{(\hat{\Theta}_k^i)^\top \mathbf{p}_k^i}}. \end{aligned}$$

Here, the relaxations on Θ_{max}^i are denoted by $\epsilon_k^i = (1 + \gamma^i)/(\gamma^i(\Theta_{max}^i)^2)\|\Delta\hat{\Theta}_k^i\|^2$. Obtaining of parameter bounds, c^i , arises another question that whether a supremum exists for each i^{th} element. From the definition of projection operator, it follows that

$$\Gamma_\Theta \succ 0 \quad \text{and} \quad \Gamma_\Theta \frac{(\nabla f_k^i)(\nabla f_k^i)^\top}{\|\nabla f_k^i\|_{\Gamma_\Theta}^2} \Gamma_\Theta f_k^i \succ 0 \quad \text{s.t.} \quad f_k^i > 0, \langle \mathbf{p}_k^i, \Gamma_\Theta \nabla f_k^i \rangle > 0$$

and substituting the projection-based update law, the upper bound holds that

$$\sup_{k \in \mathbb{N}} \frac{\|\text{Proj}_{\Gamma_\Theta}(\hat{\Theta}_k^i, \mathbf{p}_k^i)\|^2}{(\hat{\Theta}_k^i)^\top \mathbf{p}_k^i} = \sup_{k \in \mathbb{N}} \frac{\|\Gamma_\Theta \mathbf{p}_k^i\|^2}{(\hat{\Theta}_k^i)^\top \mathbf{p}_k^i} \stackrel{!}{<} \infty \quad \text{s.t.} \quad (\hat{\Theta}_k^i)^\top \mathbf{p}_k^i > 0. \quad (5.28)$$

Then, consider the sequence of the inner product of parameter updates and the update directions.

$$\langle \hat{\Theta}_k^i, \mathbf{p}_k^i \rangle = \langle \hat{\Theta}_{k-1}^i, \mathbf{p}_k^i \rangle + \langle \text{Proj}_{\Gamma_\Theta}(\hat{\Theta}_k^i, \mathbf{p}_k^i), \mathbf{p}_k^i \rangle$$

Since $f_k^i > 0$ and $\langle \hat{\Theta}_k^i, \mathbf{p}_k^i \rangle > 0$, the projection is conditionally applied as follows.

$$\langle \hat{\Theta}_k^i, \mathbf{p}_k^i \rangle = \langle \hat{\Theta}_{k-1}^i, \mathbf{p}_k^i \rangle + \mathbf{p}_k^{i\top} \left(\Gamma_\Theta - \Gamma_\Theta \frac{(\hat{\Theta}_{k-1}^i)(\hat{\Theta}_{k-1}^i)^\top}{\|\hat{\Theta}_{k-1}^i\|_{\Gamma_\Theta}^2} \Gamma_\Theta f_k^i \right) \mathbf{p}_k^i.$$

Let the learning rate be $\Gamma_\Theta = \gamma_\Theta I$ with $\gamma_\Theta > 0$, then

$$\gamma_\Theta \mathbf{p}_k^{i\top} \left(I - \frac{(\hat{\Theta}_{k-1}^i)(\hat{\Theta}_{k-1}^i)^\top}{\text{tr}\{(\hat{\Theta}_{k-1}^i)(\hat{\Theta}_{k-1}^i)^\top\}} \frac{(1 + \gamma^i)\|\hat{\Theta}_{k-1}^i\|^2 - (\Theta_{max}^i)^2}{\gamma^i(\Theta_{max}^i)^2} \right) \mathbf{p}_k^i \leq 0$$

as long as $\|\hat{\Theta}_{k-1}^i\| \geq \Theta_{max}^i$ (i.e. parameters lie in the region of relaxation) and noting that the term has

$$\text{rank} \left(\frac{(\hat{\Theta}_{k-1}^i)(\hat{\Theta}_{k-1}^i)^\top}{\text{tr}\{(\hat{\Theta}_{k-1}^i)(\hat{\Theta}_{k-1}^i)^\top\}} \right) = 1$$

with a unitary non-zero eigenvalue. Consequently, there exists some $\eta_k^i > 0$ holds

$$\langle \hat{\Theta}_k^i, \mathbf{p}_k^i \rangle = \langle \hat{\Theta}_{k-1}^i, \mathbf{p}_k^i \rangle - \eta_k^i \|\mathbf{p}_k^i\|^2 \quad \text{s.t.} \quad \lim_{\|\hat{\Theta}_k^i\| \rightarrow \Theta_{max}^i} \eta_k^i = 0. \quad (5.29)$$

Since the inner product $\langle \hat{\Theta}_k^i, \mathbf{p}_k^i \rangle$ is monotonically decreasing and lower bounded, then it converges to a finite value by the *Monotone Convergence Theorem* [48]. Suppose there exists p_* such that $\|\mathbf{p}_k^i\| \geq p_* > 0$ (assuming $\lim_{k \rightarrow \infty} \|\hat{\Theta}_k^i\| \neq 0$), that holds

$$\lim_{\|\mathbf{p}_k^i\| \rightarrow p_*} \sup_{k \in \mathbb{N}} \frac{\gamma_\Theta \|\mathbf{p}_k^i\|^2}{\langle \hat{\Theta}_{k-1}^i, \mathbf{p}_k^i \rangle - \eta_k^i \|\mathbf{p}_k^i\|^2} < \infty \quad (5.30)$$

and the part of obtaining scalars $c^i = \sqrt{(\Theta_{max}^i)^2 + \psi(\mathbf{p}_0^i; \gamma_\Theta, \gamma^i, \Theta_{max}^i)}$ with $\psi^i \in \ell^\infty$ is complete.

Next, consider the Frobenius norm of parameter matrix $\hat{\Theta}_k \in \mathbb{R}^{(N+1) \times l}$ which is equivalent to

$$\|\hat{\Theta}_k\|_F^2 = \sum_{i=1}^l \text{tr}\{(\hat{\Theta}_k^i)(\hat{\Theta}_k^i)^\top\} \leq \sum_{i=1}^l (c^i)^2.$$

Let scalar c be $c := \left\| \begin{bmatrix} c^1 & c^2 & \dots & c^l \end{bmatrix} \right\|_F$, thus,

$$\|\hat{\Theta}_k\|_F \leq c. \quad (5.31)$$

Furthermore, consider the triangle inequality between the parameter estimates, true parameters and the parameter errors which gives

$$\|\tilde{\Theta}_k\|_F \leq \|\hat{\Theta}_k - \Theta^*\|_F \leq c + \|\Theta^*\|_F =: c_3.$$

Remembering that a projection-based adaptation law is used to update parameters.

$$\Delta \hat{\Theta}_k = \text{Proj}_{\Gamma_\Theta}(\hat{\Theta}_k, \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k). \quad (5.32)$$

Then, by using Lemma 5.3.3, the trace term in (5.23) is enforced to be negative semidefinite by the properties of projection operator.

$$-2 \text{tr}\{\tilde{\Theta}_k^\top (\Gamma_\Theta^{-1} \text{Proj}_{\Gamma_\Theta}(\hat{\Theta}_k, \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k) - \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k)\} \leq 0.$$

By using the parameter update law in (5.32), the Lyapunov difference function can be bounded from above such that

$$\begin{aligned} \Delta V_k &\leq -\lambda_{\min}(N_{k|k-1}) \|\hat{\mathbf{e}}_{k|k-1}\|^2 \\ &\quad - \lambda_{\min}(K_k^\top M_{k|k} K_k) \|\Phi_{k|k-1}\|^2 \|\tilde{\Theta}_k\|^2 \\ &\quad + 2 \lambda_{\max}(M_{k|k}) \|\hat{\mathbf{e}}_{k|k-1}\| \|A_k\| \|K_k\| \varepsilon_{\max} \\ &\quad + \lambda_{\max}(K_k^\top M_{k|k} K_k) \varepsilon_{\max}^2. \end{aligned} \quad (5.33)$$

Again considering the Theorem 5.2.5, it is possible to claim that $\|A_k\| \leq c\sigma_*$, $\forall k \in \mathbb{N}$, where $\sigma_* := 1 - \inf_{k \in \mathbb{N}} |K_k C_k|$, and $K_* := \sup_{k \in \mathbb{N}} |K_k|$. Then, the Lyapunov difference function is negative definite as long as

$$\begin{aligned} \|\hat{\mathbf{e}}_{k|k-1}\| &\geq 2 \frac{\lambda_{\max}(M_{k|k})}{\lambda_{\min}(N_{k|k-1})} c\sigma_* K_* \varepsilon_{\max} =: c_1 \\ \|\tilde{\Theta}_k\| &\geq \frac{1}{\|\Phi_{k|k-1}\|} \sqrt{\kappa(K_k^\top M_{k|k} K_k) \varepsilon_{\max}} =: c_2 \end{aligned} \quad (5.34)$$

where $\kappa(\cdot)$ is the condition number and $\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ are the maximum and minimum modulus of eigenvalues of a given matrix argument, respectively. Finally, inside the set $\mathcal{A}$, the Lyapunov difference function $-\Delta V_k \in \mathcal{P}(\mathbb{R}^{n+m} \times \mathcal{A})$ and by

using Lemma 5.3.4 starting anywhere inside the set Ω_0^i , the parameter trajectories stay inside the set Ω_1^i which proves both the tracking and parameter adaptation errors $\hat{e} \in \ell^\infty([0, \infty), \mathbb{R}^{n+m})$ and $\tilde{\Theta} \in \ell^\infty([0, \infty), \mathbb{R}^{(N+1) \times l})$, uniformly. Moreover, given any arbitrarily large compact Ω_0^i , starting anywhere from $\tilde{\Theta}_0^i \in \Omega_0^i$ and $\hat{e}_{0|0} \in \mathbb{R}^{n+m}$, the error dynamics (5.22) are semiglobally practically asymptotically stable [25, 16] which completes the proof of Theorem 5.3.1. $\square$



CHAPTER 6

POWERED DESCENT GUIDANCE WITH PROPOSED METHODS

6.1 Introduction

In some applications, PDG is handled as a *trajectory optimization* problem that finds the feasible set of waypoints and required kinematic variables (e.g. velocities, accelerations), and some crucial physical considerations are imposed as the constraints to the corresponding optimization problem. Another popular approach in PDG is setting up the problem as an *optimal control problem* with constraints that reflect the crucial physical considerations [58, 57]. In this formulation, the feasible sequence of physical inputs (e.g. thruster commands, fin deflections) are applied to the descending vehicle [56, 18]. Generally speaking, there are no clear distinctions between these two approaches and both can be used to solve PDG problems interchangeably.

In this chapter, the PDG is addressed as a real-time receding horizon optimal control problem. In the optimal control problem setup, the motion of reusable rocket stage is constituted with a simplified model and made available to be used in estimation-based MPC formulation. Meanwhile, constraints on the main/divert thrusters (i.e. input constraints) and constraints on the kinematic variables (i.e. state constraints) are imposed in order to provide the desired physical considerations. As one of the main scope of thesis, the accuracy and feasibility of PDG solutions are investigated in the presence of physical uncertainties applied on the different points of autonomous landing. Then, by the adaptive methods proposed in chapters 3, 4 and 5, the estimation and elimination of uncertainties encountered during descent phase are presented via computer simulations. In the simulation studies, by the elimination of uncertainties with proposed adaptive methods, it is observed that both performance and feasibility of PDG

solutions with estimation-based MPC are gradually recovered. Eventually, it is shown that the deteriorating effects of physical uncertainties in real-time PDG problems can be overcome by the notions coming from adaptive control and optimization theory.

In the first section of the chapter, the equations of motion for reusable rocket stage is provided in the presence of aerodynamic effects and mass depletion dynamics. Moreover, the modeling of physical uncertainties and external navigation aids (i.e. reference model for uncertain measurements) are presented. Moreover, input and state constraints for different phases of PDG are given based on physical considerations. In the latter sections, the simulation studies of ideal PDG and uncertain PDG are conducted and the results are presented where the adaptation is both disabled and enabled.

6.2 Modeling of Reusable Rocket System

In the thesis study, it is supposed that modeling the landing vehicle with 3-degrees-of-freedom (3DOF) point mass equations of motion is sufficient to show the contribution of proposed adaptive methods in PDG problems. Along the study, it is assumed that the motion of *vertical landing rocket stage* (VLRS) is controlled via equivalent thrust exerted on the center of gravity and all the aerodynamic forces are applied in the same manner.

6.2.1 Vertical Landing Rocket Stage Model

Consider a fixed, orthogonal and right-handed reference frame $\mathcal{F}_e(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with basis vectors $\mathbf{e}_i \in \mathbb{R}^3$ for $i = 1, 2, 3$ where all the kinematic variables of VLRS are assumed to be the representations of position map and its time derivatives on the coordinates of $\mathcal{F}_e$. Let the position map be $\mathbf{r} : [t_0, t_f] \subset \mathbb{R} \rightarrow \mathbb{R}^3$, then by utilizing the *Newton's laws of motion*, the differential equations that govern the motion can be expressed as follows.

$$m\ddot{\mathbf{r}}(t) = mg\mathbf{e}_3 + f_T(t) + f_A(t) \quad \forall t \in [t_0, t_f] \quad (6.1)$$

where $m \in \mathbb{R}_+$ is the mass of VLRS, $g \in \mathbb{R}_+$ is the magnitude of local gravitational acceleration and $f_T, f_A : [t_0, t_f] \subset \mathbb{R} \rightarrow \mathbb{R}^3$ are the mappings that represent the applied thrust and aerodynamic forces, respectively.

The VLRS is subjected to aerodynamic forces (and moments in 6DOF case) due to its motion in fluidic environment. The aerodynamic forces acting on VLRS can be modeled in terms of forces acting on $\mathbf{e}_i$ directions for $i = 1, 2, 3$ which are represented as follows.

$$f_A(t) = -\frac{1}{2}\rho_\infty S_A C_A \sum_{i=1}^3 |v_i(t)| v_i(t) \mathbf{e}_i \quad \text{for } i = 1, 2, 3 \quad (6.2)$$

where $v_i = (\mathbf{dr}/dt)_i$ denotes the i^{th} component of velocity vector, $\rho_\infty \in \mathbb{R}_+$ is the local density of the fluidic environment (i.e. air density), $S_A \in \mathbb{R}_+$ is the aerodynamic cross-sectional area of VLRS and $C_A = \text{diag}(C_X, C_Y, C_Z)$ is the matrix of corresponding aerodynamic force coefficients. A similar approach for simplified aerodynamics model can be found in [58].

The thrust forces acting on VLRS are all assumed to be applied by the equivalent effects of main and divert thrusters on VLRS center of gravity [56]. Therefore, the details of applying generated thrust forces from different locations on VLRS and transforming them to VLRS center of gravity are omitted for the sake of simplicity.

The applied thrust forces are supposed to be the system inputs which are commanded by the guidance system. On the other hand, the realization of these thrust forces are done by imposing physical thrust force limits (i.e. input saturations) such that $f_{min}^i \leq f_T^i(t) \leq f_{max}^i$ for all $i = 1, 2, 3$ throughout the simulation studies.

The mass depletion is another crucial physical consideration that directly affects to the overall mission cost and accuracy. Moreover, the mass depletion has dynamics which are proportional to the magnitude of applied thrust forces. By considering the atmospheric *back-pressure effects*, the change of mass can be modeled by the following differential equation [57].

$$\dot{m}(t) = -\frac{1}{I_{sp}g_0} \|f_T(t)\| - \frac{P_{amb}A_{noz}}{I_{sp}g_0} \quad \forall t \in [t_0, t_f] \quad (6.3)$$

where $I_{sp} \in \mathbb{R}_+$ denotes the *specific impulse*, g_0 denotes the g at sea level (i.e. standard gravity) and $A_{noz} \in \mathbb{R}_+$ denotes the *nozzle area* of VLRS while $P_{amb} \in \mathbb{R}_+$ is

the local ambient pressure. Noting that the mass depletion has monotonically decreasing functions as the solution family. Thus, as a physical consideration, the mass is supposed to vary between empty and full mass such that $0 < m_{dry} \leq m(t) \leq m_{full}$.

Consequently, by using (6.1)-(6.2)-(6.3), the nonlinear state equations which describe the motion and the mass depletion of VLRS are given by

$$\begin{aligned}
\dot{\mathbf{r}}(t) &= \mathbf{v}(t) \\
\dot{\mathbf{v}}(t) &= g\mathbf{e}_3 - \frac{1}{2m(t)}\rho_\infty S_A C_A \sum_{i=1}^3 |v_i(t)|v_i(t)\mathbf{e}_i + \frac{1}{m(t)}f_T(t) \\
\dot{m}(t) &= -\frac{P_{amb}A_{noz}}{I_{sp}g_0} - \frac{1}{I_{sp}g_0}\|f_T(t)\| \\
&\quad \forall t \in [t_0, t_f] \\
\text{s.t. } &f_{min}^i \leq f_T^i(t) \leq f_{max}^i \quad \text{for } i = 1, 2, 3 \\
&m_{dry} \leq m(t) \leq m_{full}.
\end{aligned} \tag{6.4}$$

Then, the state vector $\mathbf{x} \in \mathbb{R}^7$ of VLRS becomes $\mathbf{x}(t) = [\mathbf{r}(t)^\top \quad \mathbf{v}(t)^\top \quad m(t)]^\top$ while the inputs $\mathbf{u} \in \mathbb{R}^3$ are $\mathbf{u}(t) = f_T(t)$.

The state equations in (6.4) can be represented in a compact form such that

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) \quad \forall t \in [t_0, t_f] \tag{6.5}$$

where $f \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^7)$ with $\mathcal{D} \subseteq \mathbb{R}^7 \times \mathbb{R}^3 \setminus \{\mathbf{0}, \mathbf{0}\}$.

In order to obtain the discrete-time LTI model for KF and MPC applications described in chapters 2 and 3, the nonlinear state equations in (6.5) can be linearized around a point $\{\mathbf{x}_e, \mathbf{u}_e\} \in \mathcal{D}$.

Remark 6.2.1. *The nonlinear state equations in (6.4)-(6.5) do not have an equilibrium point since m is monotonically decreasing over time. Alternatively, since $P_{amb}, A_{noz}, I_{sp}, g_0 > 0$ and $\|f_T(t)\| \geq 0$ for all $t \in [t_0, t_f]$, it holds that*

$$0 \neq -\frac{P_{amb}A_{noz}}{I_{sp}g_0} - \frac{1}{I_{sp}g_0}\|\mathbf{u}_e(t)\| \quad \forall \mathbf{u}_e \in \mathbb{R}^3.$$

Thus, the linearization point $\{\mathbf{x}_e, \mathbf{u}_e\} \in \mathcal{D}$ is not an equilibrium point.

A continuous-time LTI model to nonlinear state equations of VLRS in the form (6.5)

can be obtained by computing the following Jacobian matrices.

$$A = \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\mathbf{x}_e, \mathbf{u}_e}, \quad B = \left. \frac{\partial f}{\partial \mathbf{u}} \right|_{\mathbf{x}_e, \mathbf{u}_e}. \quad (6.6)$$

Moreover, assuming that all the position, velocity and mass variables can be measured by the onboard sensors of VLRS, hence $C = I_7$ and $D = 0$. Then, the continuous-time linear approximation around $\{\mathbf{x}_e, \mathbf{u}_e\}$ is given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(\mathbf{x}(t) - \mathbf{x}_e) + B(\mathbf{u}(t) - \mathbf{u}_e) + f(\mathbf{x}_e, \mathbf{u}_e) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \\ \text{s.t. } \forall \mathbf{x} &\in \mathcal{B}_\delta(\mathbf{x}_e), \forall t \in [t_0, t_f] \end{aligned} \quad (6.7)$$

where $\|\mathbf{x}(t) - \mathbf{x}_e\| < \delta$. Furthermore, the LTI system obtained from (6.7) can be *discretized* with any suitable discretization method. Consider the following discrete equivalents of pair (A, B) as (F, G) with a given *sampling time*, Δt and a given *series expansion order*, p .

$$F = I_7 + \sum_{i=1}^p \frac{A^i \Delta t^i}{i!}, \quad G = \left(\sum_{i=1}^p \frac{A^{i-1} \Delta t^i}{i!} \right) B \quad (6.8)$$

Consequently, the discrete-time equivalent of LTI system (6.7) without the *unsteady terms* (i.e. $f(\mathbf{x}_e, \mathbf{u}_e) \neq 0$) is obtained as

$$\begin{aligned} \mathbf{x}_k &= F\mathbf{x}_{k-1} + G\mathbf{u}_{k-1} \quad \forall k \in [k_0, k_f] \\ \mathbf{y}_k &= C\mathbf{x}_k. \end{aligned} \quad (6.9)$$

Noting that the continuous time interval is also discretized with the sampling time such that $[t_0, t_f] \rightarrow [k_0 \Delta t, k_f \Delta t]$.

6.2.2 Physical Uncertainties and Navigation Aids

The 3DOF model given for VLRS in (6.4) is supposed to be subjected to physical uncertainties which are likely to occur in real-life scenarios. There is a large variety of physical factors that cause difference between the mathematical system models and actual system models during real-time operations. Generally speaking, these uncertain factors are usually supposed to occur in specific subsystem parameters, signals or structures. As the first canonical example, there are physical uncertainties occurred in system inputs such as *actuator limits*, *backlash*, *elastic deformations* and *actuator*

delays. Another canonical example is the physical uncertainties occurring in measured variables such as *sensor noise, bias, delays* or any unexpected *trend* in data recordings. In addition, there are also physical uncertainties which may arise at the beginning of modeling phase or during the operations, due to *unsteady fluidic environments, unmodeled elastic modes* or having *insufficient information* about physical phenomenon.

Consider the following uncertainty factors that are used during the simulation studies:

- Flow instability and pressure loss effects on thrust generation via divert/main thrusters as *input uncertainties*.

$$\mathbf{u}_k = \mathbf{u}_k^* + \mathbf{w}_k + d^i(\mathbf{x}_k) \quad \forall k \in [k_0, k_f] \quad (6.10)$$

where $\mathbf{u}_k^* \in \mathbb{R}^3$ is the *nominal* thrust, $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_{thr})$ with $\Sigma_{thr} \in \mathbb{S}^3$ is the fluctuations on thrust generated and $d^i : \mathbb{R}^7 \rightarrow \mathbb{R}^3$ is a state-dependent function that represents deficiency on thrust generated.

- Inaccurate position and velocity measurements of onboard sensing devices (e.g. GPS modules, inertial measurement units) due to measurement noise and data trends as *output uncertainties*.

$$\mathbf{y}_k = \mathbf{y}_k^* + \mathbf{v}_k + d^o(\mathbf{z}_k) \quad \forall k \in [k_0, k_f] \quad (6.11)$$

where $\mathbf{y}_k^* \in \mathbb{R}^7$ is the *nominal* measurement provided by *navigation aids* or *redundant* sensors, $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_{meas})$ with $\Sigma_{meas} \in \mathbb{S}^7$ is the measurement noise and $d^o : \mathbb{R}^{10} \rightarrow \mathbb{R}^7$ is a function of augmented states (i.e. combination of system and input disturbance states) that represents the data trends or deficiencies on onboard measurements.

- Environmental variations due to altitude change such as variations on ambient pressure, density and gravitational acceleration as *additive modeling uncertainties*, and uncertain aerodynamic force coefficients as *multiplicative modeling*

uncertainties which are supposed as follows.

$$\begin{aligned}
P_{amb} &= P_{amb}^* + \Delta_{P_{amb}} \\
\rho_{\infty} &= \rho_{\infty}^* + \Delta_{\rho_{\infty}} \\
g &= g^* + \Delta_g \\
C_A &= C_A^*(I_3 + \Delta_{C_A}), \quad \Delta_{C_A} \in \mathbb{S}^3
\end{aligned} \tag{6.12}$$

where $(\cdot)^*$ denotes the assumed nominal value that used in control design and $\Delta_{(\cdot)}$ denotes the uncertainty factor on corresponding model parameters.

Along the simulation studies the uncertainties on environmental parameters arise *implicitly*. The control design is done based on VLRS mathematical model with constant environmental parameters while the VLRS simulation model is run with *gravity-altitude model* (GAM) [41] and *COESA atmosphere model* [1] that compute the instantaneous local environmental parameters. On the other hand, the uncertainties on aerodynamic force coefficients arise *explicitly* with supposed values in simulations.

The *navigation aids* are the reference measurements that can be provided by multiple internal and external sources. As an internal source, it can be supposed that *redundant sensors*, *data tables* or *onboard simulation models* can provide corresponding reference measurement in the presence of sensor faults and calibration drifts. Similarly, as an external source, a *data link* or *broadcast* can be supposed to provide reference kinematic variables which are measured externally by tracking radars.

In the simulation studies, it is assumed that the true or nominal reference measurements $\mathbf{y}_k^*$ are provided *ideally without losses* by either internal or external sources according to the measured variable and the provision methods are omitted since they are out of the scope of thesis.

6.2.3 Thrust and Flight Path Constraints

PDG problems are constrained optimization problems by the nature of physical capabilities of VLRS designs. The propulsion system that VLRS equipped has a limited capability of thrust generation on both main and divert thrusters. Moreover, it is also

possible to limit the generated thrust in order to be cautious about the fuel consumption. Therefore, the thrust commands generated by control system are limited to be generated within a predefined interval which brings us linear inequality-type *input constraints* used in PDG simulation studies.

$$\mathcal{U} = \left\{ \mathbf{u}_k \in \mathbb{R}^3 : \begin{bmatrix} -f_{div}^{max} \\ -f_{div}^{max} \\ 0 \end{bmatrix} \leq \mathbf{u}_k \leq \begin{bmatrix} f_{div}^{max} \\ f_{div}^{max} \\ f_{main}^{max} \end{bmatrix} \right\} \quad (6.13)$$

The flight path constraints are imposed to the kinematic variables of VLRS in order to maintain the safety of flight trajectory or to accomplish the goals within predefined accuracy margins. In either case, limiting the kinematic behavior of VLRS on a predefined interval brings us another linear inequality-type *state constraints* which are used in PDG simulation studies.

The receding horizon optimal control strategy (i.e. MPC) used in PDG study of the thesis necessitates to constrain input and states inside the *prediction horizon*. On the other hand, covering all the trajectory via single prediction horizon is computationally infeasible since long prediction horizons require more computer memory and increase QP solution time. In order to make MPC suitable for real-time operations, using sufficiently short prediction horizons are inevitable and, for this reason, the flight path constraints are chosen to be *state-triggered constraints* [58] via *approach distance*.

$$d_k^{app} = \|\hat{\mathbf{r}}_k - \mathbf{r}^{ls}\|_2 \quad \forall k \in [k_0, k_f] \quad (6.14)$$

where $\hat{\mathbf{r}}_k \in \mathbb{R}^3$ is the position estimates of VLRS and $\mathbf{r}^{ls} \in \mathbb{R}^3$ is the position of *landing site*. The three PDG phases that trigger the corresponding constraint set are described as follows:

- In *entry burn* phase, it is supposed that the VLRS approaches towards the landing site at high altitudes. Since VLRS is moving in relatively high speeds at high altitudes, it usually redirects its attitude by using divert thrusters. The entry burn constraints are active until the *mid-course* constraints are triggered.

$$\mathcal{X}_1 = \{ \mathbf{x}_k \in \mathbb{R}^7 : -\mathbf{x}_{eb}^{max} \leq \mathbf{x}_k \leq \mathbf{x}_{eb}^{max} \wedge d_k^{app} > d^{mc} \} \quad (6.15)$$

- The *mid-course* phase is triggered when the approach distance is between *mid-course distance*, d^{mc} , and *precision landing distance*, d^{pl} . In this phase, the

VLRS is substantially slowed down by main thruster while still approaching towards the landing site via divert thrusters.

$$\mathcal{X}_2 = \{\mathbf{x}_k \in \mathbb{R}^7 : -\mathbf{x}_{mc}^{max} \leq \mathbf{x}_k \leq \mathbf{x}_{mc}^{max} \wedge d^{pl} < d_k^{app} \leq d^{mc}\} \quad (6.16)$$

- The final phase *precision landing* is triggered when the approach distance is less than the *precision landing distance*, d^{pl} . In this phase, the VLRS velocity constraints are *tightened* and the velocity vector is forced to be almost-aligned with the upward directing basis vector, $\mathbf{e}_3$, in order to make a safe and plausible landing.

$$\mathcal{X}_3 = \{\mathbf{x}_k \in \mathbb{R}^7 : -\mathbf{x}_{pl}^{max} \leq \mathbf{x}_k \leq \mathbf{x}_{pl}^{max} \wedge d_k^{app} \leq d^{pl}\} \quad (6.17)$$

The feasibility of aforementioned input and state constraints relies on the accuracy of state measurements. It is expected that when the MPC of VLRS has an incorrect information about some portion of state vector, it is prone to satisfy the *false constraints* (if it can still stabilize the system) which may be catastrophic for PDG problems.

Remark 6.2.2. *As an example, consider a PDG scenario with velocity bias in sensor readings. Then, the guidance system will try to land VLRS in the presence of drifting positions since it is synthesizing control commands based on the available state information. Moreover, the MPC solutions will be still feasible although the biased velocity components are satisfying the false constraints.*

6.3 Simulation Studies on Reusable Rocket System

The simulation studies are presented in three subsections where the first one consists of the simulation of ideal PDG scenario with MPC and without any uncertainty source. In the second scenario, the uncertainty sources in (6.10)-(6.12) are included while the performance and feasibility of PDG with MPC are investigated. In the third scenario, the proposed *adaptive Lyapunov redesign* (ALR) terms in chapters 4, 5 are added to the state estimator of MPC and the simulation results of PDG with online uncertainty estimation are presented.

Problem 6.3.1. *The receding horizon estimation-based optimal control strategy used for PDG in simulation study is given as*

$$\text{minimize } J(\mathbf{u}_k, \hat{\mathbf{x}}_k, \mathbf{x}_k^d) = \sum_{i=0}^{N_p} \|\hat{\mathbf{x}}_{k+i} - \mathbf{x}_{k+i}^d\|_{R_1}^2 + \|\mathbf{u}_{k+i}\|_{R_2}^2 + \rho \|\Delta \mathbf{u}_{k+i}\|^2$$

$$\text{subject to } \hat{\mathbf{x}}_{k+1} = F\hat{\mathbf{x}}_k + G\mathbf{u}_k + K_k(\mathbf{y}_k - C\hat{\mathbf{x}}_k) \quad \text{in (6.9)}$$

$$\hat{\mathbf{x}}_k \in \begin{cases} \mathcal{X}_1 & d_k^{app} > d^{mc} \\ \mathcal{X}_2 & d^{pl} < d_k^{app} \leq d^{mc} \\ \mathcal{X}_3 & d_k^{app} \leq d^{pl} \end{cases} \quad \text{in (6.15) – (6.17)}$$

$$\mathbf{u}_k \in \mathcal{U} \quad \text{in (6.13)}$$

$$\forall k \in [k_0, k_f]$$

with the corresponding QP solution such that

$$\eta_k^* = \arg \min_{\eta \in \Omega} \left\{ \tilde{J}(\eta_k; \hat{\mathbf{x}}_k, \xi_k^d) = \frac{1}{2} \eta_k^\top H \eta_k + \mathbf{f}_k^\top \eta_k \right\} \quad \forall k \in [k_0, k_f]$$

where the matrices $H = 2(\Gamma^\top \tilde{R}_1 \Gamma + \tilde{R}_2)$ and $\mathbf{f}_k = 2\Gamma^\top \tilde{R}_1(\Phi \hat{\mathbf{x}}_k - \xi_k^d)$ are constructed via prediction model in (2.11)-(2.12) with $\xi_k^d = \mathbf{1}_{N_p} \otimes \mathbf{x}_k^d$, $\tilde{R}_1 = I_{N_p} \otimes R_1$ and $\tilde{R}_2 := I_{N_p} \otimes R_2$.

The overall PDG solution with ALR-MPC approach is demonstrated and summarized with the following algorithm.

Algorithm 5 ALR-MPC method for PDG.

Require: initial $\hat{\mathbf{x}}_{0|0} \in \mathbb{R}^7$, $\hat{\mathbf{d}}_{0|0}^i \in \mathbb{R}^3$, $\mathbf{u}_0 \in \mathbb{R}^3$, $P_{0|0} \in \mathbb{S}^{10}$, $\hat{\Theta}_0 \in \mathbb{R}^{(N+1) \times 3}$ and given $Q \in \mathbb{S}^{10}$, $R \in \mathbb{S}^7$, $R_1 \in \mathbb{S}^7$, $R_2 \in \mathbb{S}^3$, LTI representation (6.9) to VLRS system in (6.4), MPC design $N_p, \mathcal{U}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$, RBFNN design $N, d_{max}, \mathbf{c}$, adaptive law design $\gamma_\Theta, \gamma, \Theta_{max}$, navigational aids $\mathbf{y}_k^*$ and reference commands $\xi_k^d, \forall k \in \mathbb{N}$.

- 1: Construct prediction model matrices $\Gamma, \Phi, \tilde{R}_1, \tilde{R}_2$ as in (2.11) via (6.9).
 - 2: $H \leftarrow 2(\Gamma^\top \tilde{R}_1 \Gamma + \tilde{R}_2)$
 - 3: Form Ω with $\mathcal{U}, \mathcal{X}_1$ as in (6.13)-(6.15). ▷ Activate *entry burn* constraints.
 - 4: Construct input sequence η_1 via initial guess on.
 - 5: Gather first state estimates $\hat{\mathbf{z}}_{1|1}$ via (3.3) and *EKF* (Algorithm 2).
 - 6:

 - 7: **for** $k = 1, 2, \dots$, until termination criteria are met **do**
 - 8: $\hat{\mathbf{z}}_{k|k}^* \leftarrow (I - K_k C) \hat{\mathbf{z}}_{k|k-1}^* + K_k \mathbf{y}_k^*$ ▷ Compute reference state estimates.
 - 9: $\hat{\mathbf{e}}_{k|k} \leftarrow \hat{\mathbf{z}}_{k|k} - \hat{\mathbf{z}}_{k|k}^*$
 - 10: Compute RBFNN regressor vector $\Phi_{k|k-1}$ as in (4.5)-(4.6).
 - 11: $\hat{\Theta}_k \leftarrow \hat{\Theta}_{k-1} + \text{Proj}_{\Gamma_\Theta}(\hat{\Theta}_k, \Phi_{k|k-1} \hat{\mathbf{e}}_{k|k}^\top P_{k|k}^{-1} K_k)$
 - 12: $\mathbf{y}_k \leftarrow \mathbf{y}_k - \hat{\Theta}_k^\top \Phi_{k|k-1}$ ▷ Eliminate uncertainty from measurements.
 - 13: Gather corrected state estimates $\hat{\mathbf{x}}_{k|k}, \hat{\mathbf{d}}_{k|k}^i$ via (3.3) and *EKF* (Algorithm 2).
 - 14:

 - 15: $\mathbf{f}_k \leftarrow 2\Gamma^\top \tilde{R}_1(\Phi \hat{\mathbf{x}}_{k|k} - \xi_k^d)$
 - 16: **if** $d^{pl} < d_k^{app} \leq d^{mc}$ **then** ▷ Check *approach distance* computed via (6.14).
 - 17: Form Ω with $\mathcal{U}, \mathcal{X}_2$ as in (6.13)-(6.16). ▷ Trigger *mid-course* constraints.
 - 18: **else if** $d_k^{app} \leq d^{pl}$ **then**
 - 19: Form Ω with $\mathcal{U}, \mathcal{X}_3$ as in (6.13)-(6.17). ▷ Trigger *precision landing* constraints.
 - 20: **end if**
 - 21: $\eta_k^* \leftarrow \arg \min_{\eta \in \Omega} \tilde{J}(\eta_k; \hat{\mathbf{x}}_{k|k}, \xi_k^d)$ via *Hildreth's Procedure* (Algorithm 4).
 - 22: Choose $\mathbf{u}_k^*$ as the first element of input sequence η_k^* .
 - 23: $\mathbf{u}_k \leftarrow \mathbf{u}_k^* - \hat{\mathbf{d}}_{k|k}^i$ ▷ Eliminate uncertainty from inputs.
 - 24: $k \leftarrow k + 1$
 - 25: **end for**
-

Simulation studies on PDG are conducted with the following simulation settings and PDG algorithm parameters.

Table 6.1: Simulation settings & Problem 6.3.1 parameters.

Parameter	Value
Simulation stepsize, Δt ,	0.01
MPC stepsize, Δt_c	0.10
Prediction horizon, N_p	100
MPC error-to-cost weights, R_1	$\text{diag}(1, 1, 1, 50, 50, 100, 0)$
MPC input-to-cost weights, R_2	$0.1(G^\top G)$
MPC input rate weight, ρ	0.0
Divert thruster constraints, $f_{div}^{max} [kN]$	100
Main thruster constraints, $f_{main}^{max} [kN]$	250
Entry burn velocity constraints, $ \mathbf{v}_{eb}^{max} [ms^{-1}]$	(400, 400, 900)
Mid-course velocity constraints, $ \mathbf{v}_{mc}^{max} [ms^{-1}]$	(500, 500, 100)
Precision landing velocity constraints, $ \mathbf{v}_{pl}^{max} [ms^{-1}]$	(2, 2, 10)
Landing site position, $\mathbf{r}^{ls} [m]$	(0, 0, 0)
Mid-course engagement distance, $d^{mc} [m]$	1000
Precision landing engagement distance, $d^{pl} [m]$	100
EKF process noise covariance, Q	$10I_{10}$
EKF measurement noise covariance, R	$100I_7$
EKF initial prediction covariance, $P_{0 0}$	$\text{diag}(I_7, 100I_3)$
RBFNN neuron number, N	10
RBFNN learning rate, γ_Θ	2.0
RBFNN max. allowed weights, Θ_{max}	$(0.5\mathbf{1}_3, 0.5\mathbf{1}_3, 10) \times 10^3$
Projection tolerance, γ	$(0.1\mathbf{1}_3, 0.1\mathbf{1}_3, 1) \times 10^2$

The PDG problem is simulated with the VLRS mathematical model properties, uncertainty factors and environmental parameters which are adapted from [56] and given in the following table.

Table 6.2: Nominal VLRS model properties & environmental parameters.

Parameter	Value
Air density, $\rho_{\infty}^* [kgm^{-3}]$	1.225
Gravitational acceleration, $g^* [ms^{-2}]$	9.81
Ambient pressure, $P_{amb}^* [kPa]$	100
Mean Earth radius, $R_e [km]$	6371
Total mass, $m_{full} [kg]$	15×10^3
Empty mass, $m_{dry} [kg]$	10×10^3
Reference aerodynamic area, $S_A [m^2]$	10.0
Equivalent nozzle area, $A_{noz} [m^2]$	0.50
Specific impulse, $I_{sp} [s]$	300
Aerodynamic coefficients matrix, C_A^*	diag(1.0, 1.0, 0.5)

Remark 6.3.1. In Table 6.2, nominal VLRS and environmental properties for control design are given while the simulation model of VLRS is run with COESA atmosphere model and gravity-altitude model.

Remark 6.3.2. LTI representation (6.9) to VLRS is obtained at sea level operating point $\mathbf{x}_e = [\epsilon \ \epsilon \ \epsilon \ 0 \ 0 \ \epsilon \ m_{dry}]^T$ and $\mathbf{u}_e = [0 \ 0 \ \epsilon]$ where $|\epsilon| \ll 1$ and $m_{dry} = 10^4 [kg]$. In other words, the VLRS mathematical model is linearized around landing condition where the LTI representation becomes more adequate as the VLRS approaches to pinpoint landing phase.

In the following table, values and sources of the uncertainty factors used in simulations are given categorically.

Table 6.3: Values of the VLRS output uncertainty factors.

Uncertainty Factor	Value
Position measurement bias $[m]$	$(30, -50, 50)$
Velocity measurement bias $[ms^{-1}]$	$(-80, -120, 100)$
Mass measurement bias $[kg]$	20
Position measurement deviation (3σ) $[m]$	$(2.0, 2.0, 3.0)$
Velocity measurement deviation (3σ) $[ms^{-1}]$	$(1.0, 1.0, 1.0)$
Mass measurement deviation (3σ) $[kg]$	50

Table 6.4: Values of the VLRS input uncertainty factors.

Uncertainty Factor	Value
Divert thruster pressure loss effects $[kN]$	$(5.0, 5.0)$
Main thruster pressure loss effects $[kN]$	10.0
Divert thruster flow instability effects (3σ) $[kN]$	$(0.01, 0.01)$
Main thruster flow instability effects (3σ) $[kN]$	0.02

Table 6.5: Values of the VLRS modeling uncertainty factors.

Uncertainty Factor	Value/Source
Ambient pressure variation, $\Delta_{P_{amb}}$ $[kPa]$	implicit via COESA
Air density variation, $\Delta_{\rho_{\infty}}$ $[kgm^{-3}]$	implicit via COESA
Gravity variation, Δ_g $[ms^{-2}]$	implicit via GAM
Aerodynamic coefficient uncertainty, Δ_{C_A} $[\%]$	diag(40, 40, 40)

6.3.1 Case Study: Ideal PDG via receding horizon optimal control

In this simulation study, the PDG in Problem 6.3.1 is formulated ideally without any uncertainty on the thrust generation, measurements and environmental parameters. The problem is solved via proposed linear MPC approach in the presence of input constraints and state-triggered flight path constraints, then, the results are presented.

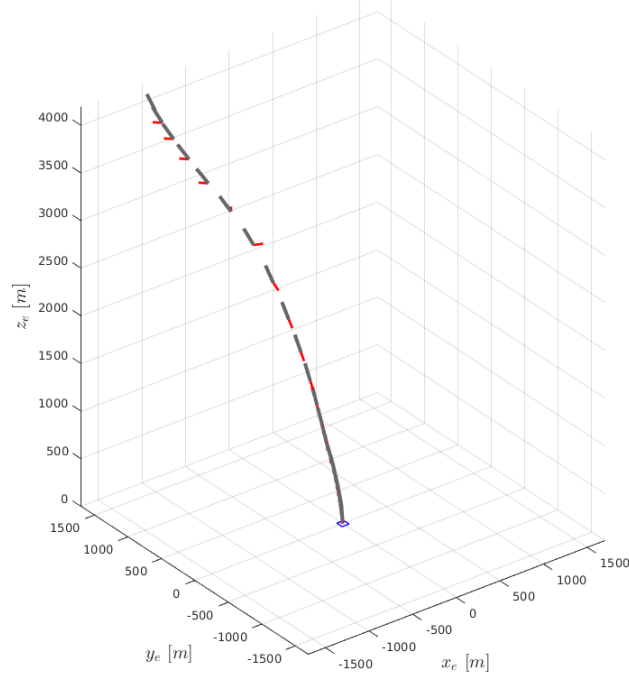


Figure 6.1: Illustration of ideal vertical landing with MPC. Starting from position $(-1000, 1500, 4000) [m]$ with initial velocity $(15, -20, -50) [ms^{-1}]$. The VLRs *direction* (gray), *equivalent thrust* (red) and *landing site* (blue) are denoted, respectively.

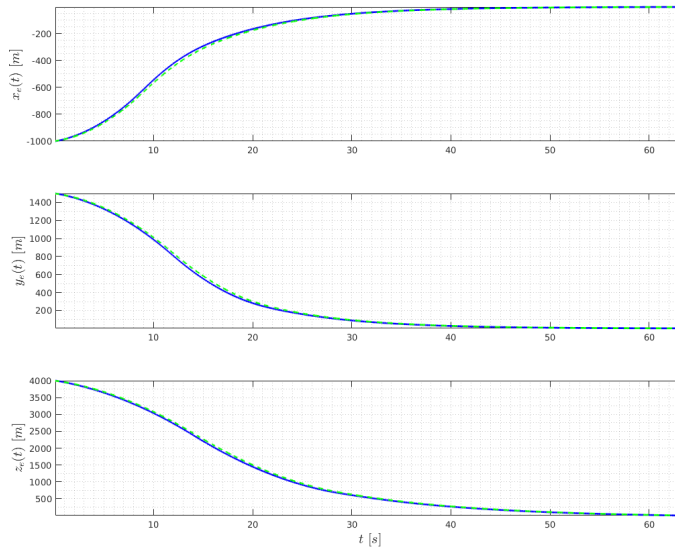


Figure 6.2: Evolution of VLRs *position* (blue, solid) and KF *position estimate* (green, dashed).

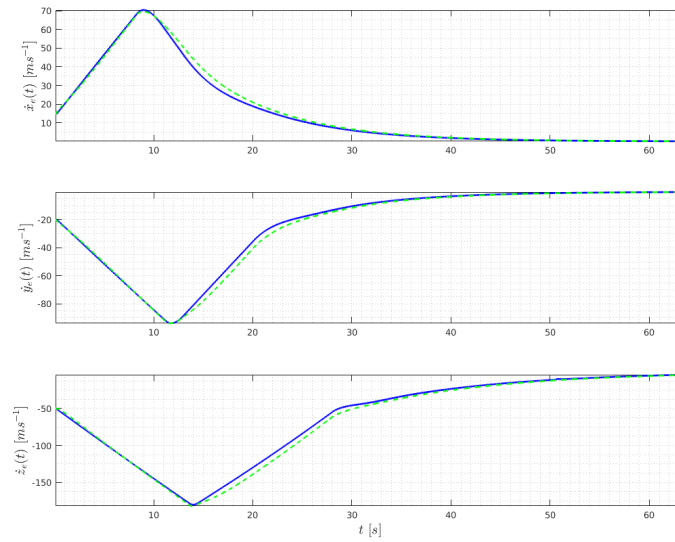


Figure 6.3: Evolution of VLRs *velocity* (blue, solid) and KF *velocity estimate* (green, dashed).

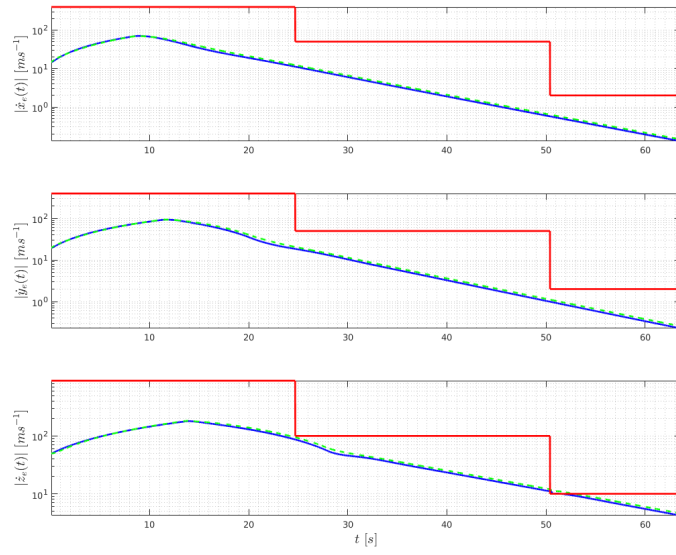


Figure 6.4: Evolution of VLRS *velocity magnitude* (blue, solid) and *KF velocity estimate magnitude*. The velocity constraints (red, solid) are satisfied for *entry burn*, *mid-course guidance* and *precision landing* phases, respectively.

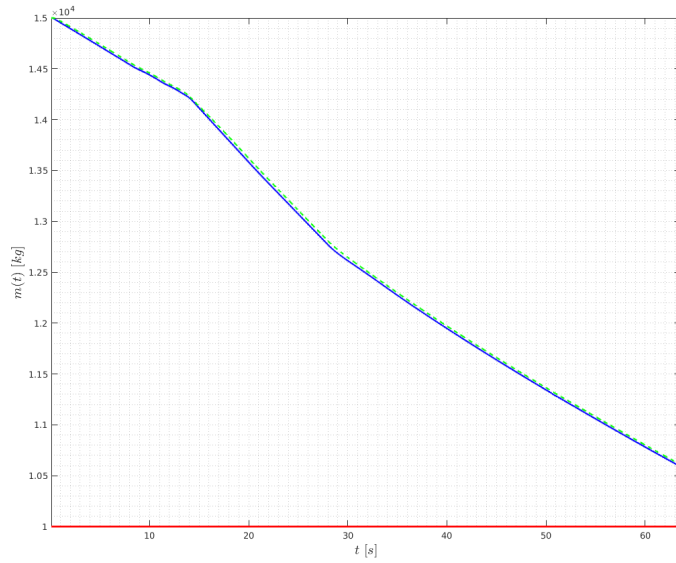


Figure 6.5: Evolution of VLRS *total mass* (blue, solid) and *KF total mass estimate* (green, dashed). The *total fuel consumption* does not exceed VLRS *dry mass requirement* (red, solid).

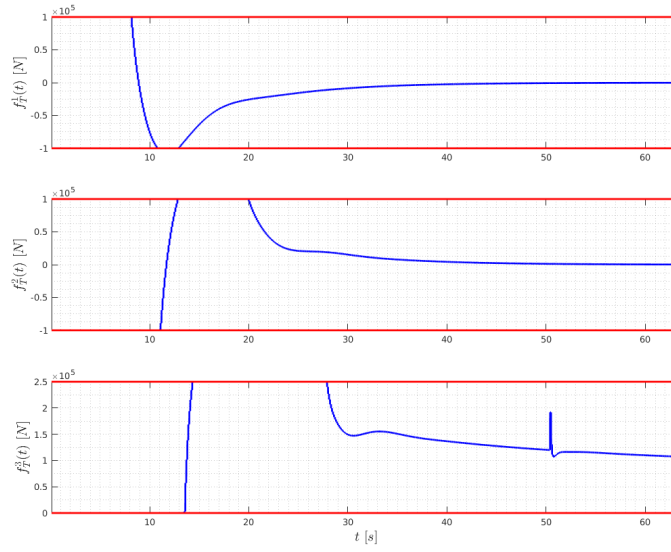


Figure 6.6: Evolution of applied VLRS *equivalent thrust forces* (blue). The input constraints (red) on *divert thrusters* and *main thruster* are satisfied, respectively.

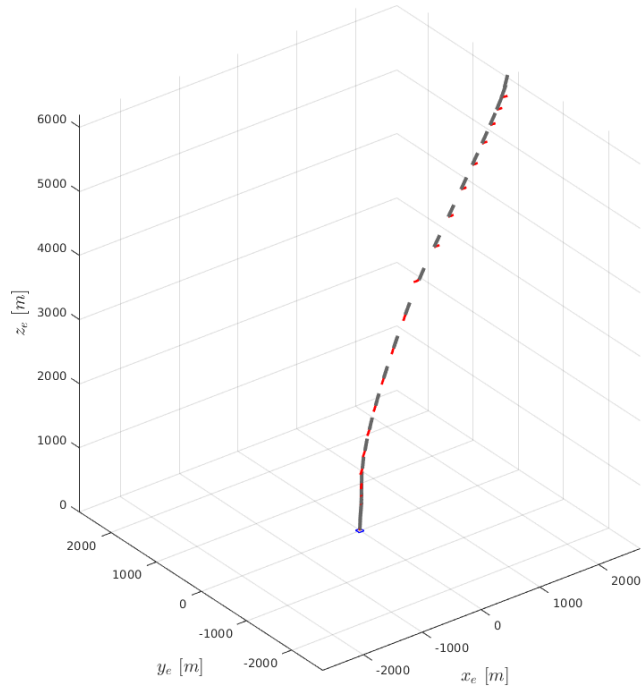


Figure 6.7: Illustration of ideal vertical landing with MPC. Starting from position $(2500, 50, 6000) [m]$ with initial velocity $(-25, -10, -50) [ms^{-1}]$. The VLRS *direction* (gray), *equivalent thrust* (red) and *landing site* (blue) are denoted, respectively.

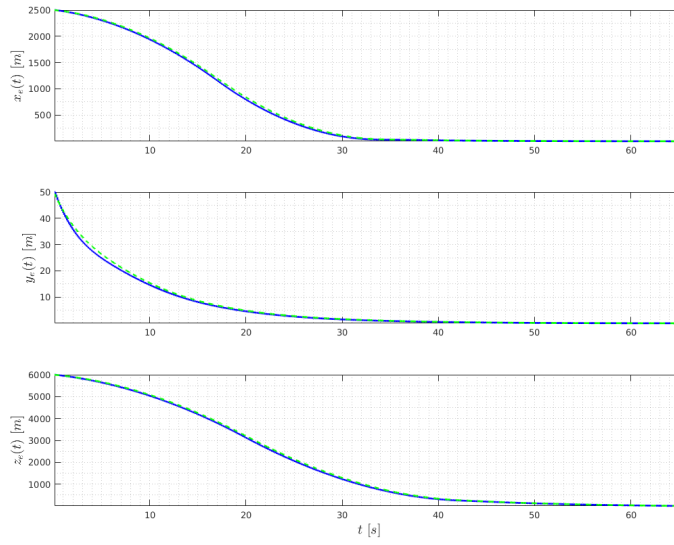


Figure 6.8: Evolution of VLRs *position* (blue, solid) and KF *position estimate* (green, dashed).

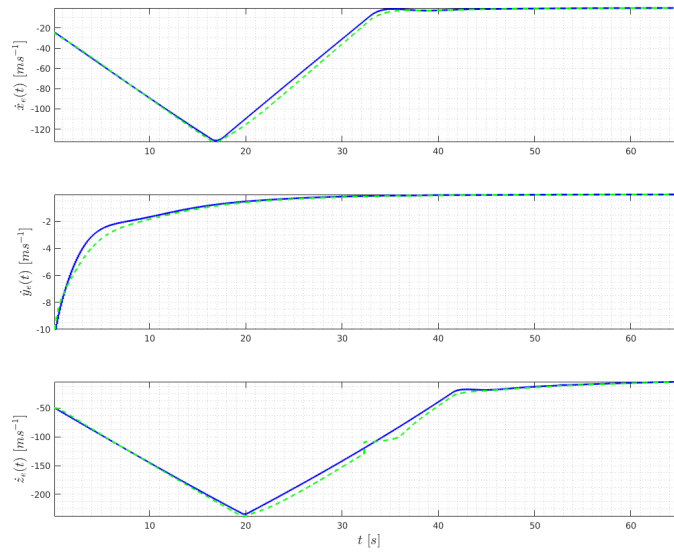


Figure 6.9: Evolution of VLRs *velocity* (blue, solid) and KF *velocity estimate* (green, dashed).

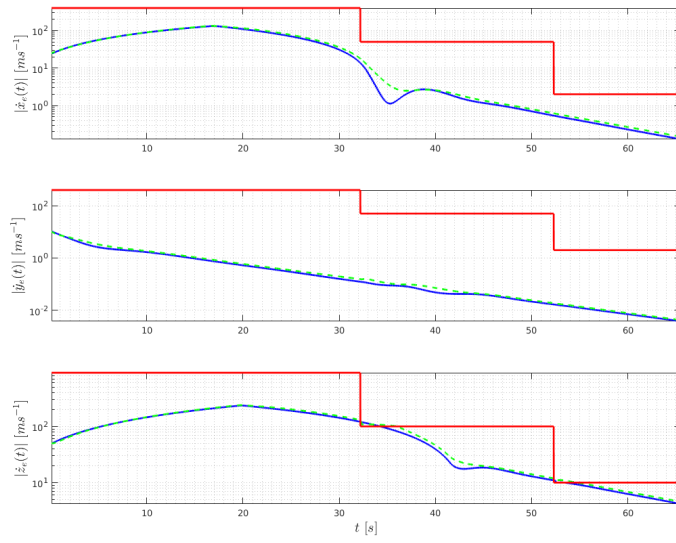


Figure 6.10: Evolution of VLRS *velocity magnitude* (blue, solid) and KF *velocity estimate magnitude*. The velocity constraints (red, solid) are satisfied for *entry burn*, *mid-course guidance* and *precision landing* phases, respectively.

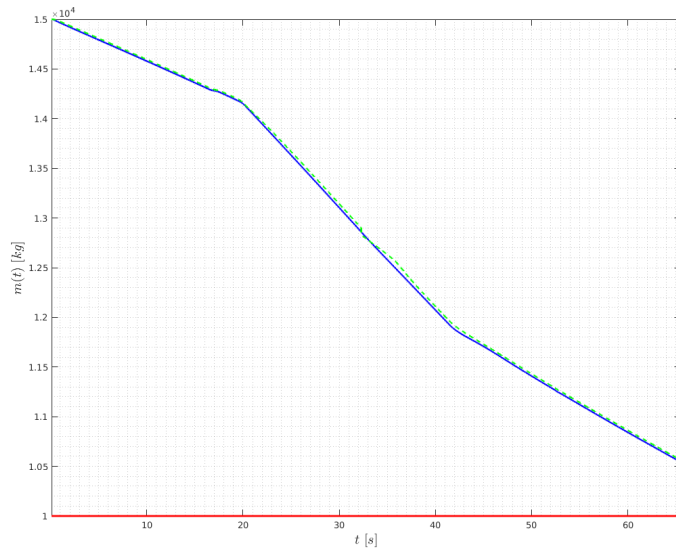


Figure 6.11: Evolution of VLRS *total mass* (blue, solid) and KF *total mass estimate* (green, dashed). The *total fuel consumption* does not exceed VLRS *dry mass requirement* (red, solid).

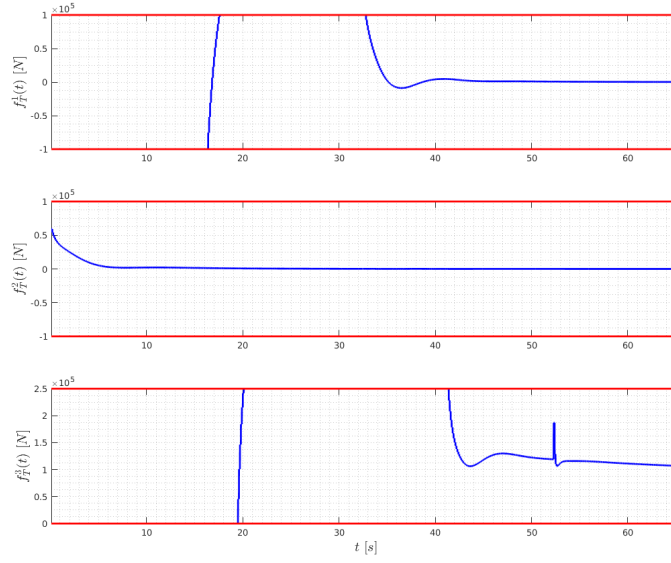


Figure 6.12: Evolution of applied VLRS *equivalent thrust forces* (blue). The input constraints (red) on *divert thrusters* and *main thruster* are satisfied, respectively.

6.3.2 Case Study: PDG via MPC in the presence of physical uncertainties

In this simulation study, the PDG in Problem 6.3.1 is formulated in the presence of thrust generation, measurement and modeling uncertainties with the values given in Tables 6.3-6.5. The PDG solutions obtained with linear MPC are highly deteriorated due to uncertainties which are presented.

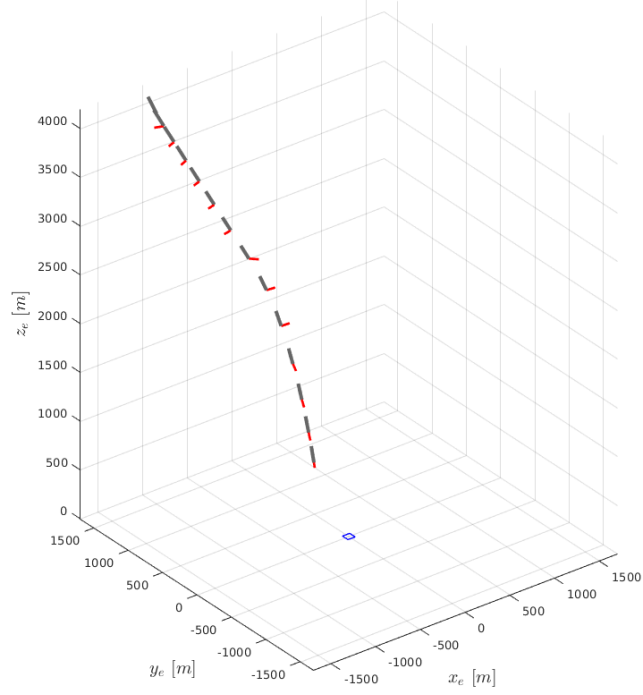


Figure 6.13: Illustration of unsuccessful vertical landing with MPC. Starting from position $(-1000, 1500, 4000)$ [m] with initial velocity $(15, -20, -50)$ [ms^{-1}]. The VLRs direction (gray), equivalent thrust (red) and landing site (blue) are denoted, respectively.

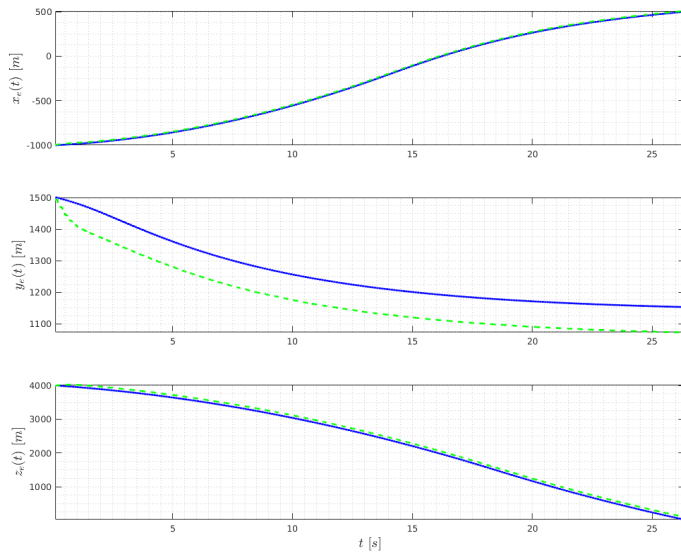


Figure 6.14: Evolution of VLRS *position* (blue, solid) and biased *KF position estimate* (green, dashed) due to multiple uncertainties.

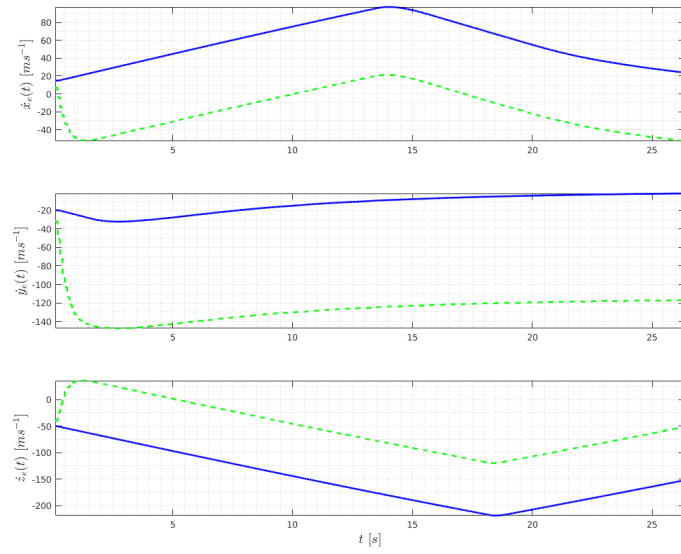


Figure 6.15: Evolution of VLRS *velocity* (blue, solid) and biased *KF velocity estimate* (green, dashed) due to multiple uncertainties.

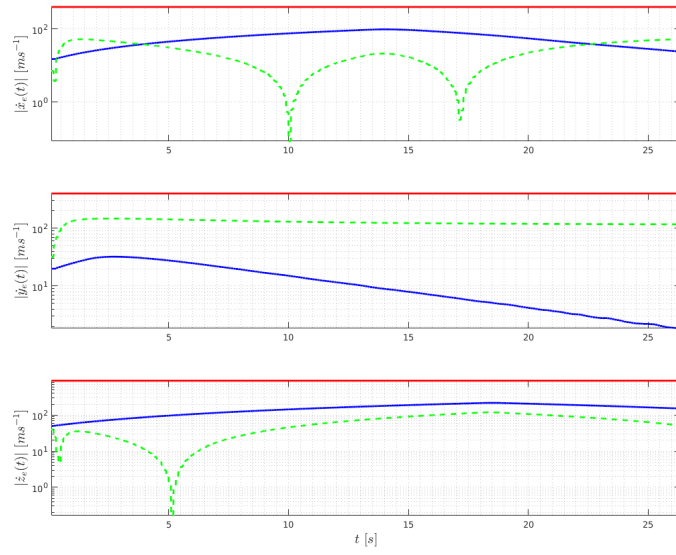


Figure 6.16: Evolution of VLRs *velocity magnitude* (blue, solid) and biased *KF velocity estimate magnitude*. The velocity constraints (red, solid) are satisfied for *entry burn* phase, however the VLRs landed to wrong site at catastrophic speeds.

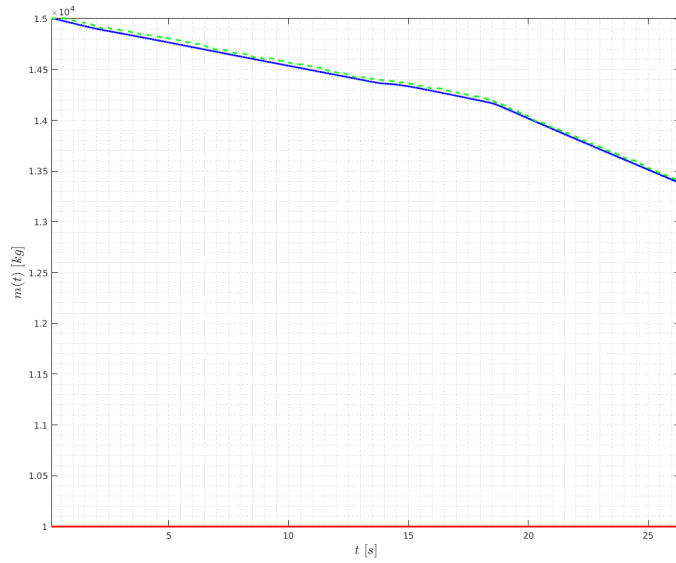


Figure 6.17: Evolution of VLRs *total mass* (blue, solid) and biased *KF total mass estimate* (green, dashed). The *total fuel consumption* does not exceed VLRs *dry mass* requirement (red, solid).

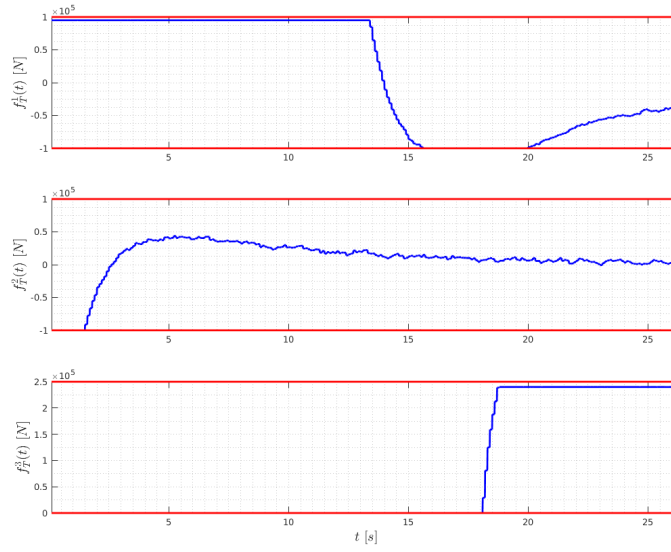


Figure 6.18: Evolution of applied VLRs *equivalent thrust forces* (blue). The input constraints (red) on *divert thrusters* and *main thruster* are satisfied, however due to input uncertainties, thrust levels are slightly deviated.

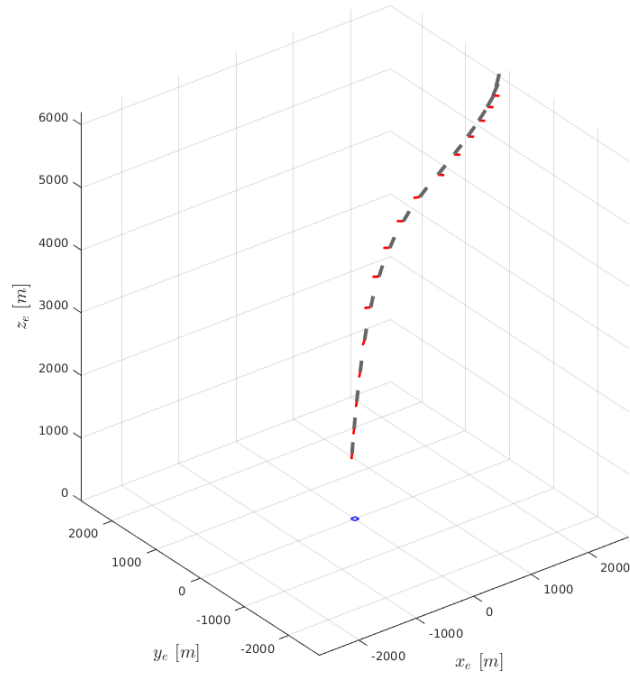


Figure 6.19: Illustration of unsuccessful vertical landing with MPC. Starting from position $(2500, 50, 6000)$ [m] with initial velocity $(-25, -10, -50)$ [m s⁻¹]. The VLRs direction (gray), equivalent thrust (red) and landing site (blue) are denoted, respectively.

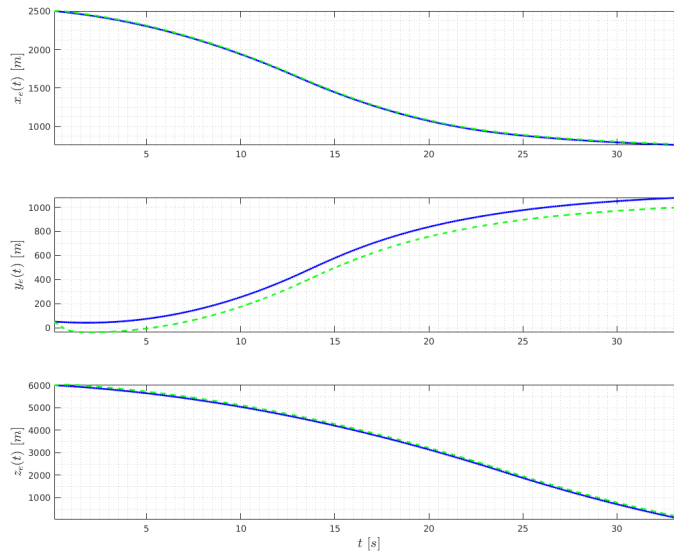


Figure 6.20: Evolution of VLRS *position* (blue, solid) and biased *KF position estimate* (green, dashed) due to multiple uncertainties.

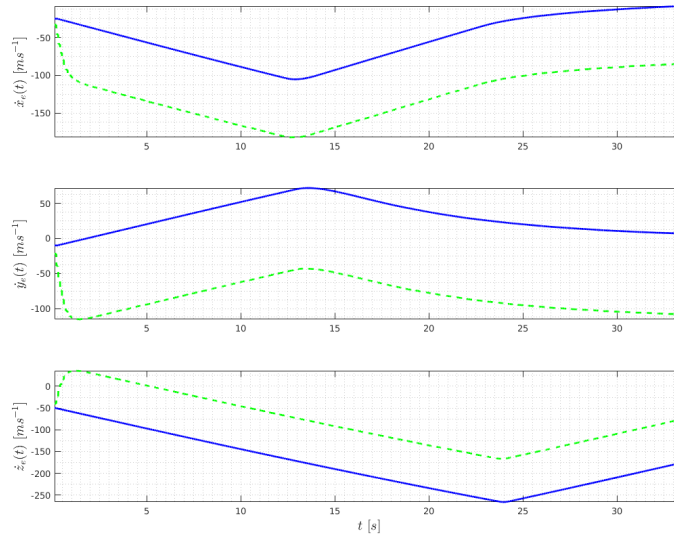


Figure 6.21: Evolution of VLRS *velocity* (blue, solid) and biased *KF velocity estimate* (green, dashed) due to multiple uncertainties.

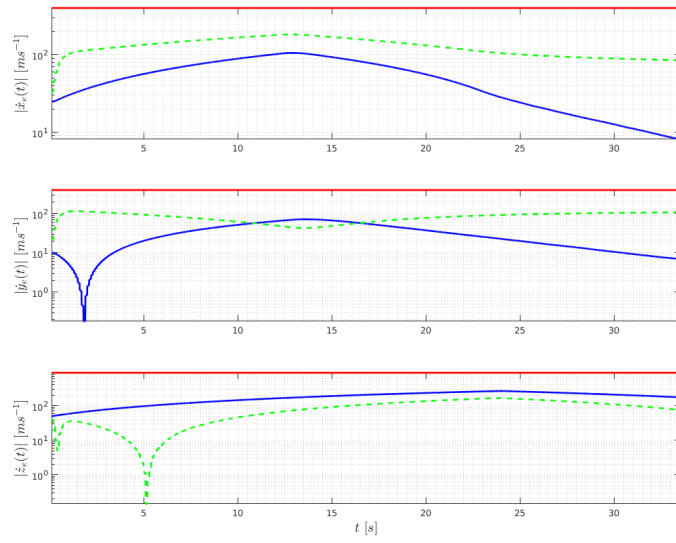


Figure 6.22: Evolution of VLRS *velocity magnitude* (blue, solid) and biased *KF velocity estimate magnitude*. The velocity constraints (red, solid) are satisfied for *entry burn* phase, however the VLRS landed to wrong site at catastrophic speeds.

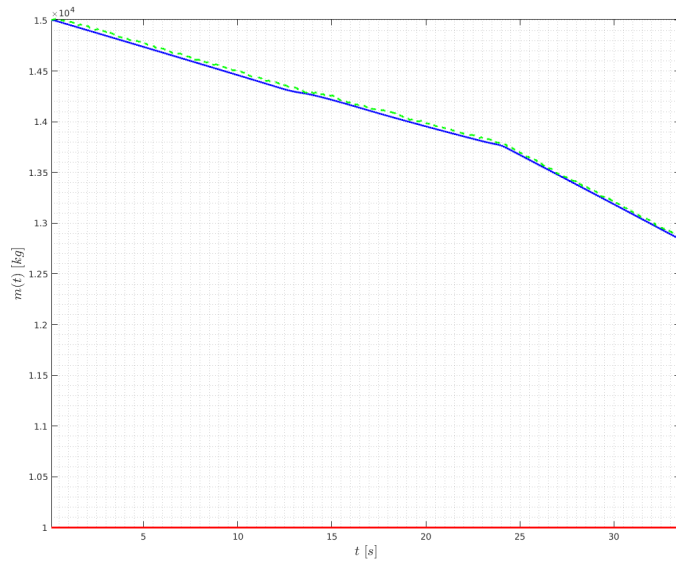


Figure 6.23: Evolution of VLRS *total mass* (blue, solid) and biased *KF total mass estimate* (green, dashed). The *total fuel consumption* does not exceed VLRS *dry mass* requirement (red, solid).

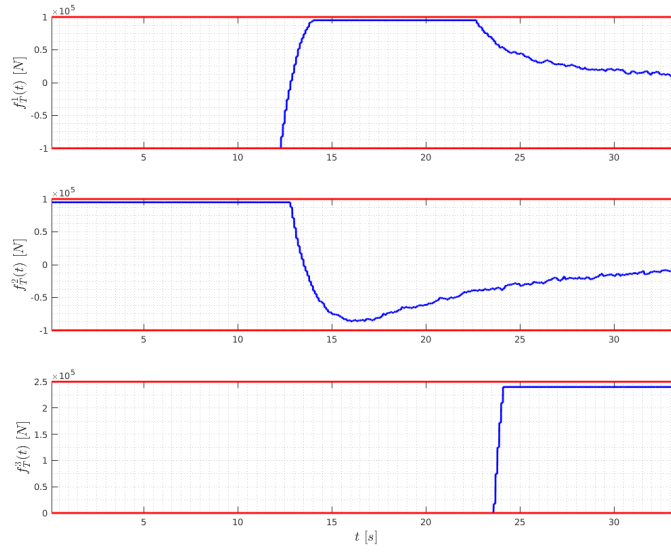


Figure 6.24: Evolution of applied VLRS *equivalent thrust forces* (blue). The input constraints (red) on *divert thrusters* and *main thruster* are satisfied, however due to input uncertainties, thrust levels are slightly deviated.

6.3.3 Case Study: PDG via proposed ALR-MPC approach with online uncertainty estimation

In this simulation study, the PDG in Problem 6.3.1 is formulated in the presence of thrust generation, measurement and modeling uncertainties with the values given in Tables 6.3-6.5. The proposed ALR extension is used along MPC in order to approximate the uncertainties acting on system. The recovery of MPC performance and feasibility in the presence of various uncertainty types is presented in simulation results.

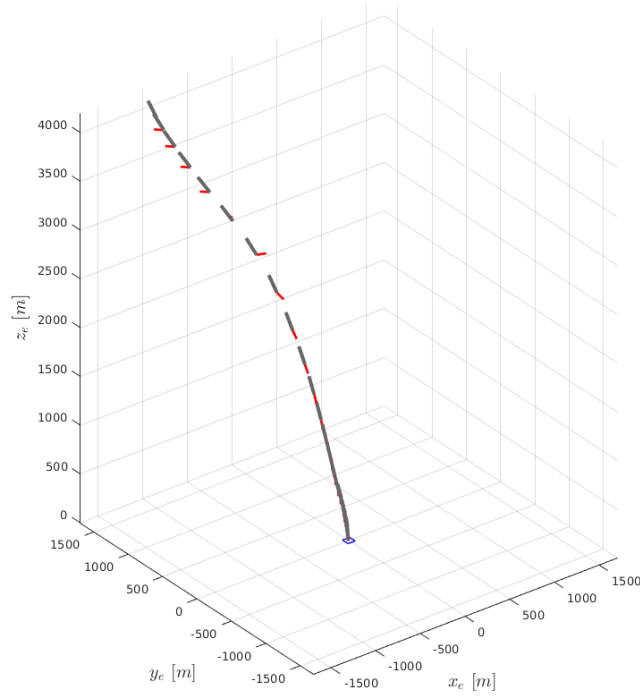


Figure 6.25: Illustration of recovered vertical landing with ALR-MPC. Starting from position $(-1000, 1500, 4000)$ [m] with initial velocity $(15, -20, -50)$ [$m s^{-1}$]. The VLRs direction (gray), equivalent thrust (red) and landing site (blue) are denoted, respectively.

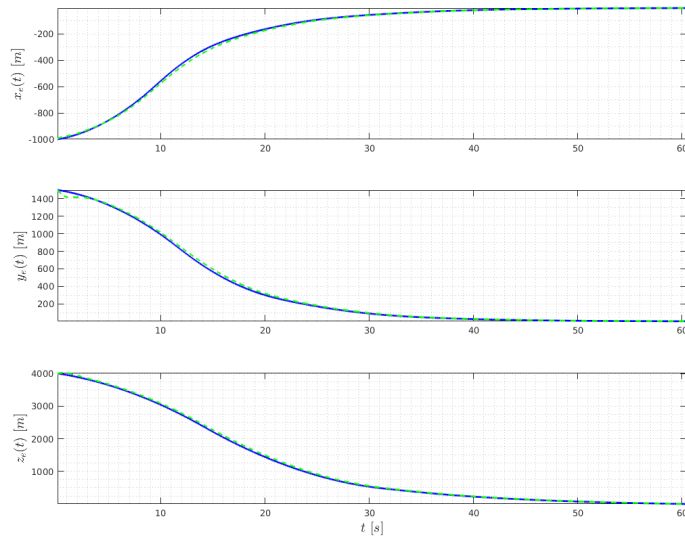


Figure 6.26: Evolution of VLRS *position* (blue, solid) and recovered *KF position estimate* (green, dashed) in the presence of multiple uncertainties.

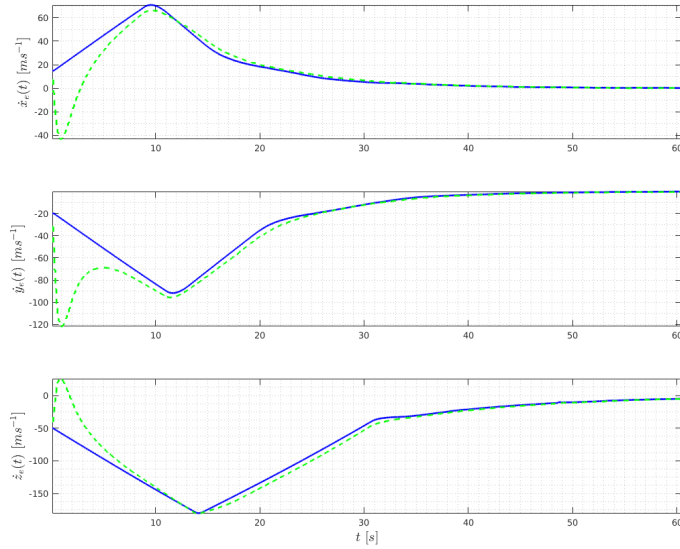


Figure 6.27: Evolution of VLRS *velocity* (blue, solid) and recovered *KF velocity estimate* (green, dashed) in the presence of multiple uncertainties.

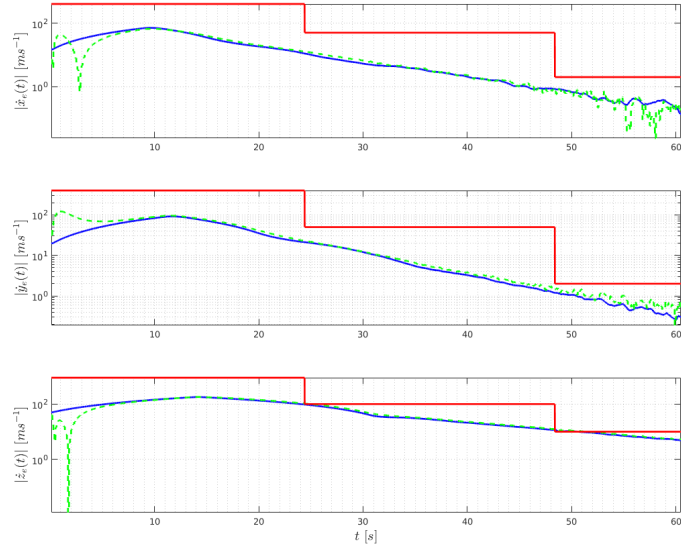


Figure 6.28: Evolution of VLRS *velocity magnitude* (blue, solid) and recovered *KF velocity estimate magnitude*. The velocity constraints (red, solid) are satisfied for all phases in spite of uncertainties.

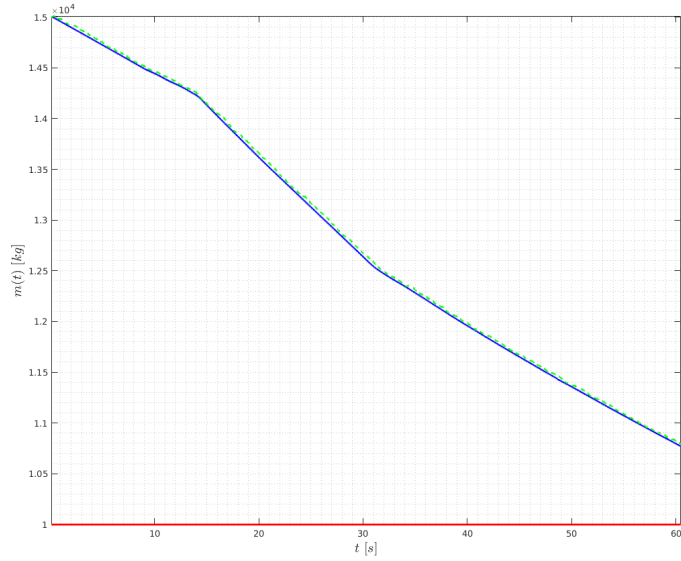


Figure 6.29: Evolution of VLRS *total mass* (blue, solid) and recovered *KF total mass estimate* (green, dashed). The *total fuel consumption* does not exceed VLRS *dry mass requirement* (red, solid).

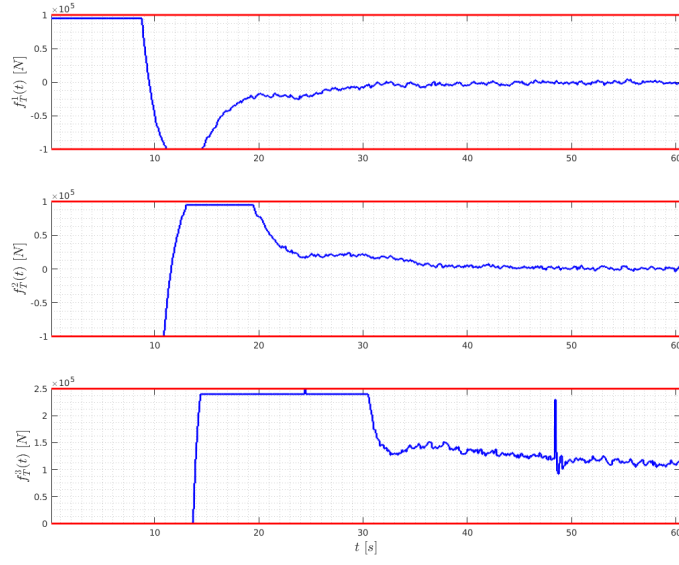


Figure 6.30: Evolution of applied VLRS *equivalent thrust forces* (blue). The input constraints (red) on *divert thrusters* and *main thruster* are satisfied, respectively.

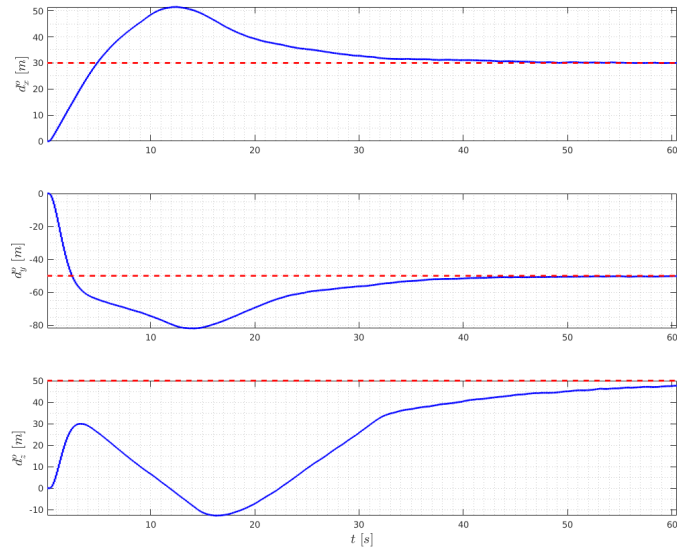


Figure 6.31: Evolution of VLRS *positional output uncertainties* (blue, solid). The uncertainty estimates converge to the prescribed position measurement bias values (red, dashed) gradually.

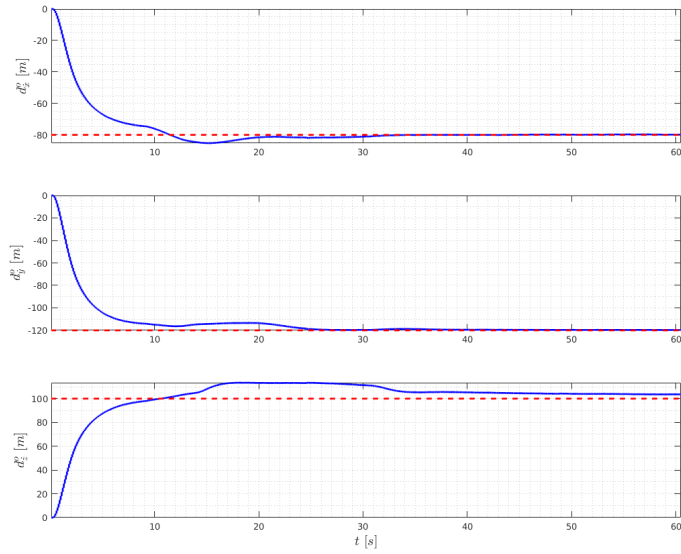


Figure 6.32: Evolution of VLRS *rate output uncertainties* (blue, solid). The uncertainty estimates converge to the prescribed velocity measurement bias values (red, dashed) gradually.

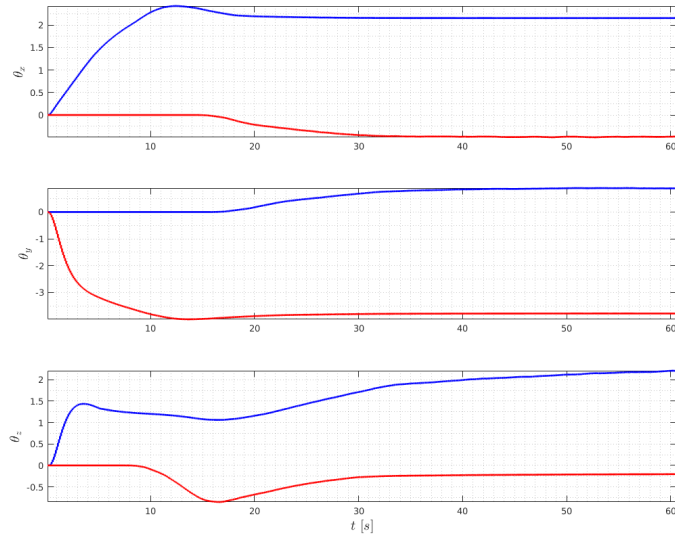


Figure 6.33: Evolution of maximum (blue) and minimum (red) value of RBFNN weights on positional uncertainties. The adaptation parameters converge towards finite values.

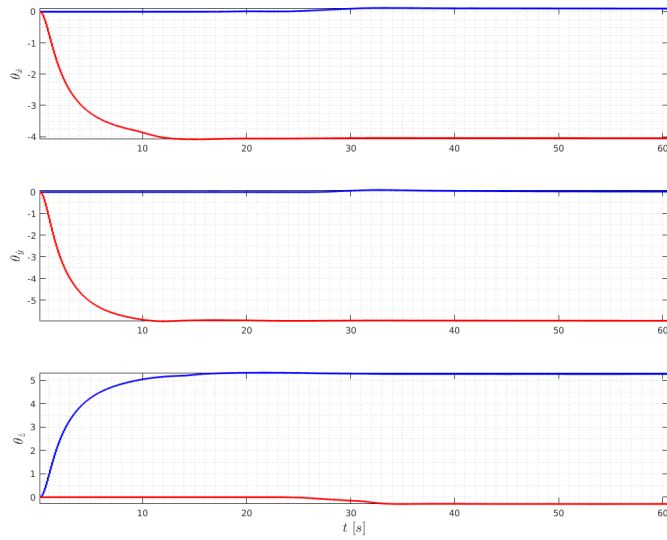


Figure 6.34: Evolution of maximum (blue) and minimum (red) value of RBFNN weights on velocity uncertainties. The adaptation parameters converge towards finite values.

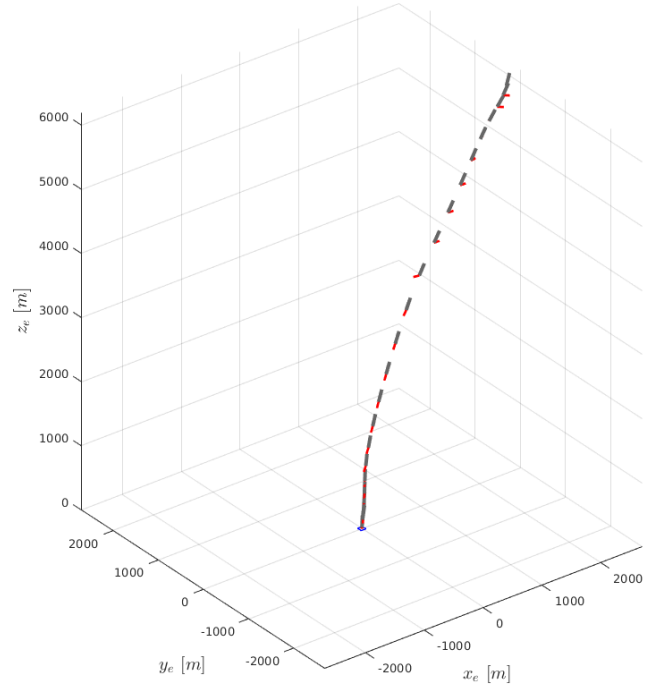


Figure 6.35: Illustration of recovered vertical landing with ALR-MPC. Starting from position $(2500, 50, 6000)$ [m] with initial velocity $(-25, -10, -50)$ [m s⁻¹]. The VLRs direction (gray), equivalent thrust (red) and landing site (blue) are denoted, respectively.

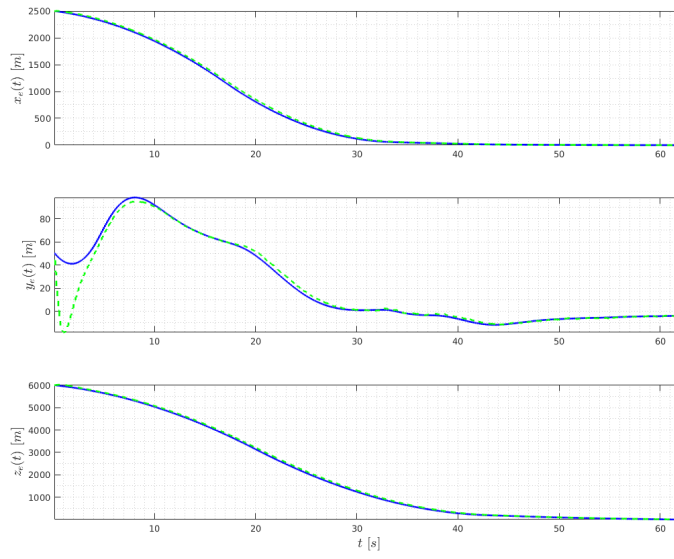


Figure 6.36: Evolution of VLRs *position* (blue, solid) and recovered *KF position estimate* (green, dashed) in the presence of multiple uncertainties.

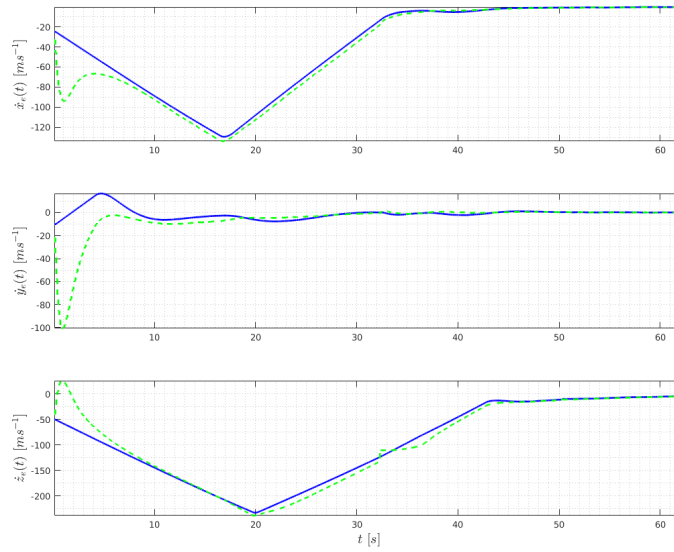


Figure 6.37: Evolution of VLRs *velocity* (blue, solid) and recovered *KF velocity estimate* (green, dashed) in the presence of multiple uncertainties.

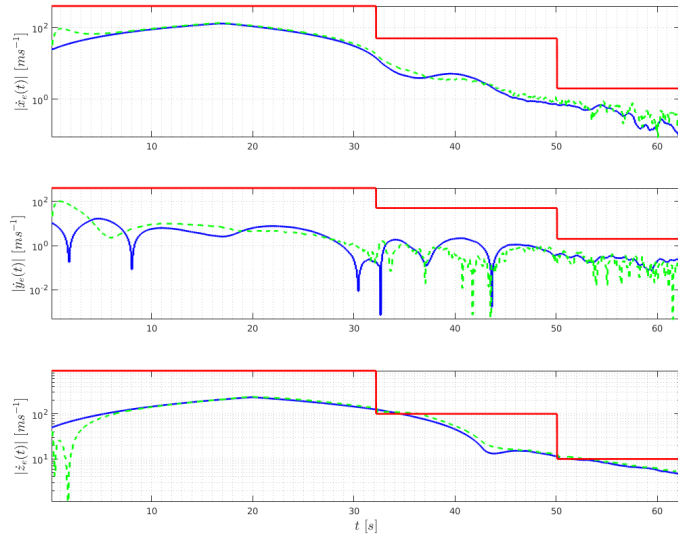


Figure 6.38: Evolution of VLRs *velocity magnitude* (blue, solid) and recovered *KF velocity estimate magnitude*. The velocity constraints (red, solid) are satisfied for all phases in spite of uncertainties.

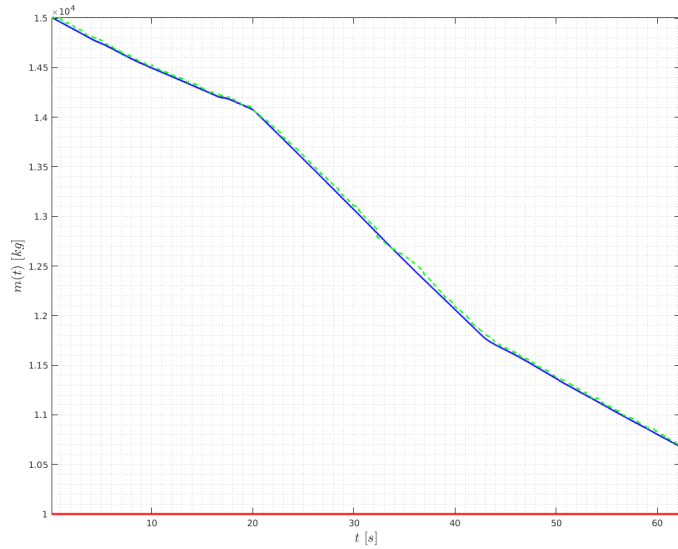


Figure 6.39: Evolution of VLRs *total mass* (blue, solid) and recovered *KF total mass estimate* (green, dashed). The *total fuel consumption* does not exceed VLRs *dry mass* requirement (red, solid).

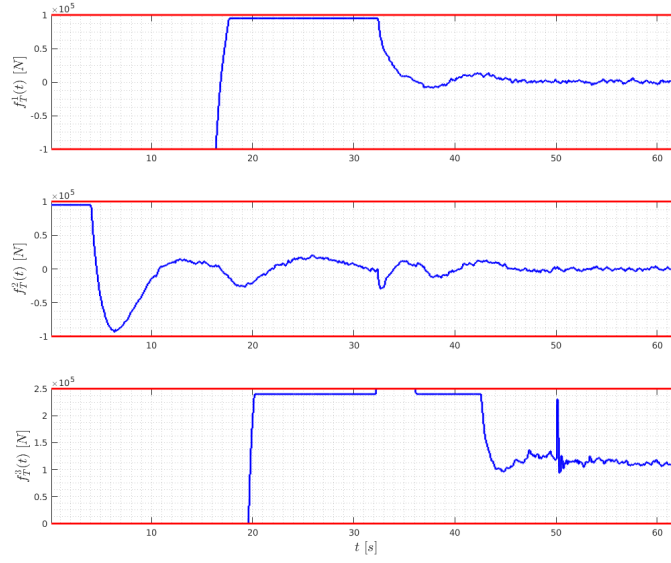


Figure 6.40: Evolution of applied VLRS *equivalent thrust forces* (blue). The input constraints (red) on *divert thrusters* and *main thruster* are satisfied, respectively.

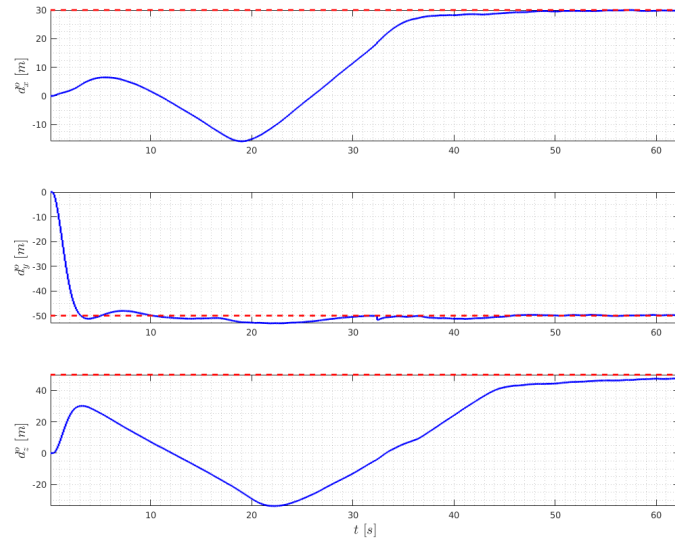


Figure 6.41: Evolution of VLRS *positional output uncertainties* (blue, solid). The uncertainty estimates converge to the prescribed position measurement bias values (red, dashed) gradually.

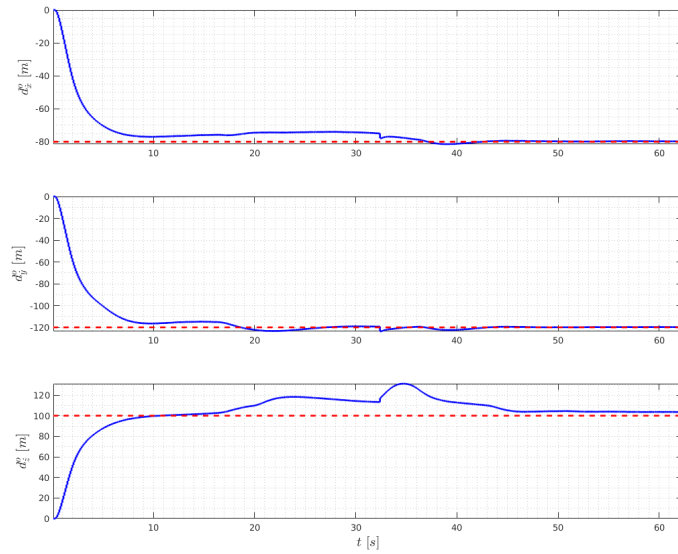


Figure 6.42: Evolution of VLRS *rate output uncertainties* (blue, solid). The uncertainty estimates converge to the prescribed velocity measurement bias values (red, dashed) gradually.

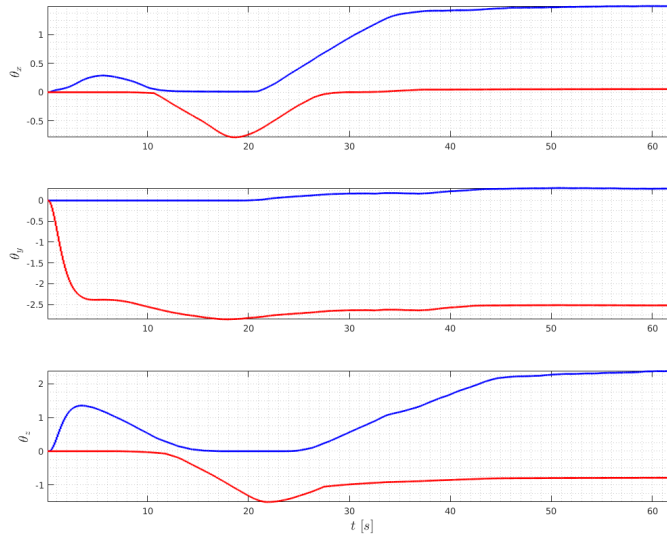


Figure 6.43: Evolution of maximum (blue) and minimum (red) value of RBFNN weights on positional uncertainties. The adaptation parameters converge towards finite values.

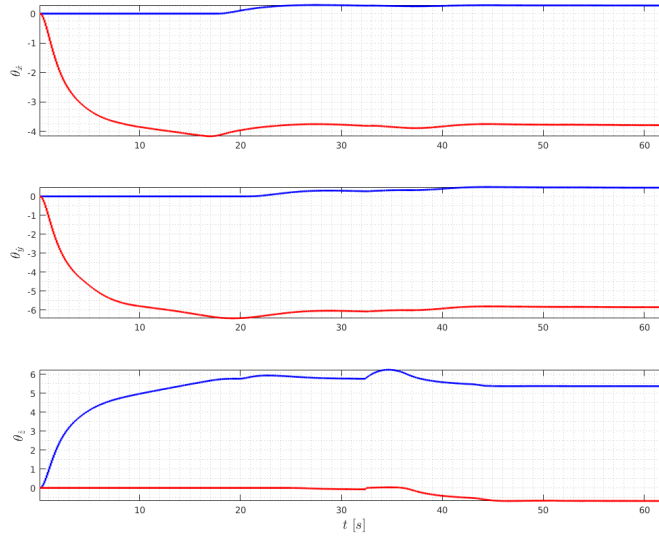


Figure 6.44: Evolution of maximum (blue) and minimum (red) value of RBFNN weights on velocity uncertainties. The adaptation parameters converge towards finite values.

6.3.4 Discussion of Results

In the ideal case, receding horizon control approach for PDG is sufficient to land VLRS successfully to landing site while satisfying the flight path constraints and thrust constraints. It is demonstrated that for two arbitrarily chosen initial conditions, the VLRS can be landed successfully within prescribed input and state constraints via MPC. Moreover, the state estimates closely track the actual states since there are no output uncertainties acting on the measurements used in KF. In Figures 6.4, 6.10, it is shown that the state-triggered constraints on velocity states are satisfied for all phases of flight. It is particularly important to satisfy *precision landing* constraints since the consequences of relatively high terminal velocities will be catastrophic. In both runs with different initial conditions, the landing is ended up within the safety margins.

In Figures 6.5, 6.11 it can be seen that the fuel consumption is acceptable and the fuel is not entirely consumed. This situation can be explained with the objective function choice in PDG Problem 6.3.1 since such objective leads to a *fuel-optimal* control problem where the finite summation of thrust inputs are minimized.

One another important point to be mentioned is the generated thrust profile. In Figures 6.6, 6.12, it can be recognized that the *bang-bang* patterns occur during *entry burn* phase in divert thrusters since the VLRS mainly diverts its attitude towards landing site in this phase. In addition, the main thruster is switched off almost throughout the *entry burn*, then banded on during *mid-course guidance* phase since the substantial slow-down of VLRS occurs in this phase. Then, the main thruster is used nearly in half throttle settings by MPC until the end of *precision landing*.

In the second case where the exaggerated physical uncertainties occur, the PDG scenarios are evidently unsuccessful. This unfavorable situation mainly occurs due to erroneous state estimates used in MPC. In Figures 6.14, 6.15, 6.20, 6.21, it can be seen that the KF estimates track the actual states with substantial bias due to uncertainties. Due to such high levels of uncertainty, the MPC cannot even steer the VLRS to *mid-course* or *precision landing* phases and the vehicle hit the ground (or sea level) with catastrophic speeds.

In the third case, the same uncertainty definitions used in the second case are included

in PDG scenarios and the elimination of these uncertainties are maintained by proposed ALR extension to the MPC. In Figures 6.26, 6.27, 6.36, 6.37, the biased KF estimates gradually start to closely track actual states while the online uncertainty estimation is active. The concurrent estimation and elimination of physical uncertainties recovers the feasibility of velocity constraints as can be seen in Figures 6.28, 6.38. In Figures 6.31, 6.32, 6.41, 6.42, it is demonstrated that the compound effects of multiple uncertainties are estimated throughout the *entry burn* and *mid-course guidance* phases. Then, the effect of output uncertainties become dominant in *precision landing* phase and the uncertainty estimates successfully converge towards the measurement bias values given in Table 6.3. During the online learning of uncertainties, the convergence and boundedness of adaptation parameters can be observed from Figures 6.33, 6.34, 6.43, 6.44.

CHAPTER 7

CONCLUSION

In this thesis, recently famous powered descent guidance problems are solved via receding horizon optimal control strategy robustified with adaptive Lyapunov redesign approach, in the presence of actuation, measurement and modeling uncertainties. The adaptive extension of model predictive control is maintained by a neuro-control based uncertainty approximation on the measurement channel of traditional Kalman filter framework.

It is shown that the powered descent guidance of reusable rocket stages, landers or similar kind of aerospace vehicles can be solved via constrained optimal control formulation in receding horizon fashion which is transcribed as a parametric optimization problem. Endowed with linearized vehicle dynamics and constraints, the quadratic fuel-optimal objective function brings us the model predictive control design on descent guidance. By using suitable sets of constraints on thrust levels and flight path variables, it is shown that the rocket stages can be landed vertically in prescribed safety margins. On the other hand, when the descent guidance scenarios start to getting away from idealized conditions, the success of precision landing is highly deteriorated. It is claimed that the aforementioned deteriorations can be concurrently estimated and eliminated from control design via online adaptive methods.

In the thesis, the online adaptive modification and redesign of state estimators are described in the sense of indirect model reference adaptive control. Utilizing the mathematical analysis and Lyapunov-based stability notions, it is proven that the existence of such modification brings us the input-to-state stability of state estimation part of control design in the case of structured physical uncertainties. Moreover, when the structure of uncertainties is not known, it is also proven that the proposed meth-

ods ensure semiglobal asymptotic stability in practical sense. The proposed methods are simulated in uncertain powered descent guidance and the promising results on pinpoint landing are demonstrated.

As the future extensions of this thesis, there are couple of points to be mentioned. First of all, the proposed methods require an external source as a reference measurement signal while estimating system uncertainties. In the thesis, this external source is supposed to be provided by a redundant set of sensors or broadcast signal. This assumption might ignite the idea of using the proximity broadcast of *another moving platform* (e.g. rocket stage) which leads to a fault-tolerant and collaborative autonomous *formation landing* of reusable stages. From this point of view, the proven input-to-state stability property of proposed method is promising since it brings us the notion of *passivity* and L^2 -*stability* of interconnected dynamical systems.

Another future work might be extending the studies on the state estimation part. Throughout the thesis, the traditional Kalman filter and its variants are considered as the state estimator. However, the proof on the positive definiteness properties of Kalman filter design can be extended to more generalized *Bayesian filtering* framework. In this case, the linearity and Gaussian probability density assumptions of Kalman filter can be removed. Instead of using prediction covariance and Kalman filter gain based adaptation law, a novel adaptation law that depends on different parametrizations of probability distributions might be proposed.

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